

How long can an off-line zero survive the de Bruijn–Newman heat flow?

An explicit unconditional lower bound for Λ

Pedro Caceres

Universidad Europea de Valencia

pedrojesus.caceres@universidadeuropea.es

October 2026

Abstract

Let Λ be the de Bruijn–Newman constant. The Riemann hypothesis (RH) is equivalent to $\Lambda \leq 0$, and Rodgers and Tao proved $\Lambda \geq 0$. We ask how small Λ can be if ζ has a zero $\frac{1}{2} + \delta + i\gamma$ with $\delta > 0$. Using de Bruijn’s strip, the zero dynamics of the Polymath project, and an explicit local zero count for the heat-flow deformations H_τ that we prove here, we show unconditionally that

$$\Lambda \geq \frac{\delta^2}{70 \log \gamma \log \log \gamma}.$$

Equivalently, every zero $\beta + i\gamma$ of ζ with $\gamma \geq 3 \cdot 10^{12}$ satisfies $|\beta - \frac{1}{2}| \leq (70 \Lambda \log \gamma \log \log \gamma)^{1/2}$; below that height RH has been verified. We also give the bound as an explicit formula at each height, which improves slowly with γ (the constant 70 drops below 58 for $\gamma \geq 10^{200}$). The proof follows the weighted sum $F = \sum \psi(\operatorname{Re} z)(\operatorname{Im} z)^p$ over all zeros of H_τ in the upper half-plane, which is insensitive to collisions and to other off-line zeros. For a single pair we also show that the imaginary part obeys an exact equation whose interaction term, at $\tau = 0$, is the energy slope $\partial_\sigma \log |\xi|$ of the other zeros evaluated at the zero itself. Under a hypothesis of local isolation and regular spacing we prove a sharper bound, linear in δ . Simulations with the actual zeros up to height 10^4 agree with both bounds. We do not prove RH.

1 Introduction and results

For real τ let

$$H_\tau(z) = \int_0^\infty e^{\tau u^2} \Phi(u) \cos(zu) du, \quad \Phi(u) = \sum_{n \geq 1} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) e^{-\pi n^2 e^{4u}},$$

so that $H_0(z) = \frac{1}{8} \xi(\frac{1}{2} + \frac{iz}{2})$ and $\partial_\tau H_\tau = -\partial_{zz} H_\tau$. All zeros of H_τ are real if and only if $\tau \geq \Lambda$ [2, 3]. RH is equivalent to $\Lambda \leq 0$; Rodgers and Tao proved $\Lambda \geq 0$ [8], and Polymath proved $\Lambda \leq 0.22$ [7]. A zero $\frac{1}{2} + \delta + i\gamma$ of ζ corresponds to the zeros $2\gamma \pm 2i\delta$ of H_0 .

We ask how *small* Λ can be if ζ has an off-line zero. De Bruijn’s argument gives the opposite inequality: the zero of maximal imaginary part y satisfies $\partial_\tau y \leq -1/y$, so an isolated maximal pair reaches the axis by time $y^2/2 = 2\delta^2$ [2, 9]. Planat [5] keeps the interaction term to sharpen *upper* bounds for Λ , and Tao remarks that de Bruijn’s bound can only be sharp for an isolated zero [9]. We have not found a lower bound for Λ in terms of (δ, γ) in the literature.

Theorem 1.1. *If $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$ with $\delta > 0$ and $\gamma > 0$, then $\gamma > 3 \cdot 10^{12}$ and*

$$\Lambda \geq \frac{\delta^2}{70 \log \gamma \log \log \gamma}.$$

More precisely, $\Lambda \geq T(\gamma) \delta^2$, where $T(\gamma)$ is given by the explicit formula (3); numerical values are in Table 1.

Corollary 1.2 (zero-free region in terms of Λ). *Every zero $\beta + i\gamma$ of ζ with $\gamma \geq 3 \cdot 10^{12}$ satisfies*

$$|\beta - \frac{1}{2}| \leq (70 \Lambda \log \gamma \log \log \gamma)^{1/2}.$$

Corollary 1.2 is Theorem 1.1 solved for δ , using the symmetry $\beta \leftrightarrow 1 - \beta$. It recovers RH from $\Lambda = 0$, as it must, but a quantitative upper bound $\Lambda \leq \varepsilon$ only gives a zero-free region of width $\asymp \sqrt{\varepsilon \log \gamma \log \log \gamma}$, which shrinks to nothing as $\varepsilon \rightarrow 0$ at a fixed height but widens with the height. If the given zero has maximal δ among all zeros of ζ , Theorem 1.1 and de Bruijn's bound bracket Λ : $\delta^2/(70 \log \gamma \log \log \gamma) \leq \Lambda \leq 2\delta^2$.

The ingredients are:

- an explicit local zero count for H_τ , uniform for small τ (Lemma 3.2); the Polymath project proves such a count with unspecified constants [7, Thm. 1.5];
- a weighted functional $F = \sum \phi(\operatorname{Re} z)(\operatorname{Im} z)^p$ over the zeros in the upper half-plane, whose derivative is bounded below by a sum of local terms (Proposition 5.2);
- a choice $p \asymp \log \log \gamma$ that balances a Hölder inequality (Section 6).

Section 4 gives the exact equation for a single pair and its link with the energy slope of [1]. Section 7 proves a sharper bound, linear in δ , under local isolation and regular spacing. Section 8 compares both bounds with simulations.

2 Known inputs

We use the following facts, valid for $0 \leq \tau \leq \frac{1}{2}$. Zeros are counted with multiplicity and written $z_k = x_k + iy_k$.

- (P1) *Strip.* Every zero of H_τ satisfies $|\operatorname{Im} z| \leq 1$ [2, Thm. 13].
- (P2) *Symmetry.* H_τ is even and real on $\mathbb{R}$, so its zeros are symmetric under $z \mapsto -z$ and $z \mapsto \bar{z}$; H_τ is entire of order 1 and $H_\tau(0) = \int_0^\infty e^{\tau u^2} \Phi \geq H_0(0) = \xi(\frac{1}{2})/8 > 0.062$, since $\Phi > 0$ [7, §3].
- (P3) *Dynamics.* If $z_k(\tau_0)$ is a simple zero of H_{τ_0} , real or not, it lies on a smooth branch $z_k(\tau)$ with

$$\partial_\tau z_k = 2 \sum'_{m \neq k} \frac{1}{z_k - z_m},$$

the sum taken over the symmetric pairs $\pm z_m$. A zero of multiplicity $r \geq 2$ splits into r distinct zeros $z_0 + \sqrt{2(\tau - \tau_0)}\lambda_i + O(|\tau - \tau_0|)$, with λ_i the roots of the probabilists' Hermite polynomial He_r [7, Prop. 3.1].

(P4) *Gaussian representation.* For $\tau > 0$,

$$H_\tau(z) = \int_{\mathbb{R}} G_\tau(w) H_0(z + iw) dw, \quad G_\tau(w) = (4\pi\tau)^{-1/2} e^{-w^2/4\tau}.$$

Indeed, $H_0(z) = \frac{1}{2} \int_{\mathbb{R}} \Phi(|u|) e^{izu} du$ and $\int G_\tau(w) e^{-wu} dw = e^{\tau u^2}$; Fubini applies because Φ decays double exponentially.

(P5) *Verification of RH.* All zeros of ζ with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the critical line [6].

In the variable $s = \frac{1}{2} + \frac{iz}{2}$, the point $z = x + iy$ corresponds to $s = \frac{1}{2} - \frac{y}{2} + \frac{ix}{2}$. By the functional equation $|\xi(\sigma + it)| = |\xi(1 - \sigma + it)|$, so $|H_0(x + iy)| = \frac{1}{8} |\xi(\frac{1}{2} + \frac{|y|}{2} + \frac{ix}{2})|$.

3 Explicit zero counting for H_τ

Lemma 3.1 (bounds for ξ). *Let $t \geq 50$, $\sigma \geq \frac{1}{2}$ and $s = \sigma + it$. Put*

$$A_t(\sigma) = \frac{\sigma}{2} \log \frac{|s|}{2\pi} + \log |s - 1| + \frac{1}{2} \log |s| + \frac{1}{2} \log \pi - \frac{\pi t}{4}.$$

Then

$$\log |\xi(s)| \leq U_t(\sigma) := A_t(\sigma) + \frac{1}{3t} + \log \min(4\sqrt{t}, \zeta(\sigma)),$$

where $\zeta(\sigma)$ is used only for $\sigma > 1$. If $\sigma > 1$, then

$$\log |\xi(s)| \geq A_t(\sigma) - \frac{\sigma^3}{6t^2} - \frac{1}{3t} + \log \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

Proof. Write $\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s)$. Stirling's formula with error term [4, Ch. 8, §4] gives, for $\operatorname{Re} z > 0$, $\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \theta/(6|z|)$ with $|\theta| \leq 1$. With $z = s/2$ and $\arg(s/2) = \frac{\pi}{2} - \arctan \frac{\sigma}{t}$,

$$\log |\Gamma(\frac{s}{2})| = \left(\frac{\sigma}{2} - \frac{1}{2}\right) \log \frac{|s|}{2} - \frac{\pi t}{4} + \left(\frac{t}{2} \arctan \frac{\sigma}{t} - \frac{\sigma}{2}\right) + \frac{1}{2} \log 2\pi + \frac{\theta}{3|s|}.$$

The bracket lies in $[-\sigma^3/(6t^2), 0]$. Collecting terms gives $\log |\xi(s)| = A_t(\sigma) + (\text{bracket}) + \theta/(3|s|) + \log |\zeta(s)|$.

For ζ , the Euler–Maclaurin formula with $N = [t]$ gives $|\zeta(s)| \leq \sum_{n \leq N} n^{-\sigma} + \frac{N^{1-\sigma}}{|s-1|} + \frac{1}{2} N^{-\sigma} + \frac{|s|}{2\sigma N^\sigma}$ [10, §4.11]. For $\frac{1}{2} \leq \sigma \leq 2$ and $t \geq 50$ the right side is at most $3\sqrt{t} + 2 \leq 4\sqrt{t}$; for $\sigma > 1$ also $|\zeta(s)| \leq \zeta(\sigma)$, and $\zeta(\sigma) \leq \zeta(2) < 4\sqrt{t}$ for $\sigma \geq 2$. The lower bound uses $|\zeta(s)| \geq \zeta(2\sigma)/\zeta(\sigma)$ for $\sigma > 1$. $\square$

Lemma 3.2 (explicit local count). *Let $X \geq 200$, $\ell_X = \log \frac{X+1}{4\pi}$ and $0 \leq \tau \leq 1/(100\ell_X)$. The number of zeros of H_τ , with multiplicity, with $X \leq \operatorname{Re} z \leq X+1$ is at most*

$$B(X) := 2.07 \ell_X + 2.5.$$

The same holds for $-X-1 \leq \operatorname{Re} z \leq -X$.

Proof. By (P1) the zeros in question lie in the box $[X, X+1] \times [-1, 1]$. Put $Y_0 = 2$, $c = X + \frac{1}{2} - iY_0$, $r = (\frac{1}{4} + (Y_0 + 1)^2)^{1/2} = 3.0414$ and $R = er = 8.2673$. The box lies in $|z - c| \leq r$, so Jensen's formula gives

$$\#\{\text{zeros in the box}\} \leq \frac{1}{\log(R/r)} \log \frac{\max_{|z-c|=R} |H_\tau(z)|}{|H_\tau(c)|} = \log \frac{\max_{|z-c|=R} |H_\tau(z)|}{|H_\tau(c)|}. \quad (1)$$

Write $t_c = \frac{1}{2}(X + \frac{1}{2}) \geq 100$ and $s_c = \sigma_c + it_c$ with $\sigma_c = \frac{1}{2} + \frac{Y_0}{2} = \frac{3}{2}$; then $H_0(c + iw) = \frac{1}{8}\xi(s_c - \frac{w}{2})$. For $\tau = 0$ the argument below applies with $G_0 = \delta_0$, so assume $\tau > 0$.

Upper bound on the circle. Let $z = x + iy$ with $|z - c| = R$, and put $t = x/2 \in [t_c - 4.14, t_c + 4.14]$ and $\sigma_y = \frac{1}{2} + \frac{|y|}{2}$. Let $\tilde{U}_t(\sigma) = \max_{\frac{1}{2} \leq \sigma' \leq \sigma} U_t(\sigma')$, which is nondecreasing. By (P4), the remark after (P5) and Lemma 3.1,

$$|H_\tau(z)| \leq \frac{1}{8} \int G_\tau(w) e^{\tilde{U}_t(\frac{1}{2} + \frac{|y+w|}{2})} dw.$$

For $\sigma \leq \sigma_y + \frac{1}{2} \leq 6.2$ one has $\partial_\sigma A_t \leq \frac{1}{2} \log \frac{t}{2\pi} + 38/t^2$, and the ζ term is nonincreasing in σ . Hence for $|w| \leq 1$

$$\tilde{U}_t\left(\frac{1}{2} + \frac{|y+w|}{2}\right) \leq \tilde{U}_t(\sigma_y) + \kappa_1 |w|, \quad \kappa_1 = \frac{1}{4} \log \frac{t_c + 4.14}{2\pi} + \frac{20}{(t_c - 4.14)^2}.$$

Since $\int G_\tau e^{\kappa_1 |w|} dw \leq 2e^{\kappa_1 \tau}$, and the region $|w| > 1$ contributes a relative amount below 10^{-6} (its Gaussian weight is at most $e^{-25\ell_X}$, while $\tilde{U}_t$ grows only like $\frac{|w|}{4} \log(t + |w|)$), we get

$$\log |H_\tau(z)| \leq \tilde{U}_t(\sigma_y) + \log 2 + \kappa_1^2 \tau - \log 8 + 10^{-6}.$$

Lower bound at the centre. Since ξ has no zeros in $\text{Re } s > 1$, $L = \log \xi$ is analytic there. Put $\lambda = \frac{1}{2}L'(s_c) = a + ib$ and write $\xi(s_c - \frac{w}{2}) = \xi(s_c)e^{-\lambda w}\rho(w)$. Completing the square, $G_\tau(w)e^{-\lambda w} = e^{\lambda^2 \tau} G_\tau(w + 2\lambda\tau)$, so

$$H_\tau(c) = \frac{\xi(s_c)}{8} e^{\lambda^2 \tau} \left(1 + \int G_\tau(w + 2\lambda\tau) (\rho(w) - 1) dw\right), \quad |G_\tau(w + 2\lambda\tau)| = e^{b^2 \tau} G_\tau(w + 2a\tau).$$

Here $L' = M'/M + \zeta'/\zeta$ with $M(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})$. Since $|\zeta'/\zeta(s_c)| \leq |\zeta'/\zeta(\frac{3}{2})| = 1.5053$, we have $a_- \leq a \leq a_+$ and $|b| \leq b_+$, with $a_\pm = \frac{1}{2}(\text{Re } \frac{M'}{M}(s_c) \pm 1.5053)$ and $b_+ = \frac{1}{2}(|\text{Im } \frac{M'}{M}(s_c)| + 1.5053)$. Numerically $a_- \geq \frac{1}{4}\ell_X - 0.76$ and $b_+ \leq 1.15$.

For $|w| \leq 0.6$ the segment $[s_c - \frac{w}{2}, s_c]$ stays in $1.2 \leq \text{Re } s \leq 1.8$. There $|L''| \leq |(M'/M)'| + (\log \zeta)''(1.2) \leq K_2 := 24.832 + \frac{1}{2t_c} + \frac{3}{t_c^2}$, because $|(\zeta'/\zeta)'(s)| \leq \sum \Lambda(n) \log n n^{-\text{Re } s} = (\log \zeta)''(\text{Re } s)$ and $|(M'/M)'(s)| \leq |s|^{-2} + |s-1|^{-2} + \frac{1}{4}|\psi'(\frac{s}{2})|$. Taylor's formula then gives $|\rho(w) - 1| \leq \exp(K_2 w^2/8) - 1$. For $|w| > 0.6$ we use $|\rho(w)| \leq \exp(\tilde{U}_{t_c}(\frac{1}{2} + |\sigma_c - \frac{w}{2} - \frac{1}{2}|) - \Psi + a_\pm w)$, where Ψ is the lower bound of Lemma 3.1 at s_c . This gives

$$|H_\tau(c)| \geq \frac{e^\Psi}{8} e^{(a_-^2 - b_+^2)\tau} (1 - \varepsilon),$$

with $\varepsilon := \sup_{a_- \leq a' \leq a_+} e^{b_+^2 \tau} \int G_\tau(w + 2a'\tau) h_{a'}(w) dw$, where $h_{a'}$ is the majorant of $|\rho - 1|$ just described (the tail majorant uses a' in place of a). Because $\tau \leq 1/(100\ell_X)$, the Gaussian has variance at most $0.02/\ell_X$ and centre within 0.011 of 0, and the script gives $\varepsilon \leq 0.024$ (the inner

part of h is even and increasing in $|w|$, so its worst case is $a' = a_+$; the tail is evaluated at both ends).

Conclusion. Inserting both bounds in (1), the count is at most

$$\max_{|z-c|=R} \tilde{U}_t(\sigma_y) - \Psi + \log 2 + (\kappa_1^2 - a_-^2 + b_+^2)\tau - \log(1 - \varepsilon) + 10^{-6}.$$

Up to terms that are $O(1/t_c)$ or $O(\tau)$, the leading part is $\frac{\sigma_{\max} - \sigma_c}{2} \log \frac{t_c}{2\pi} = \frac{R}{4} \ell_X = 2.0668 \ell_X$, where $\sigma_{\max} = \frac{1}{2} + \frac{Y_0 + R}{2}$ is attained at the bottom of the circle. The remainder $g(X)$ is an elementary function of X , with $\tau \leq 1/(100\ell_X)$ entering only through the terms displayed. We evaluated the whole expression with the script `lemmaC.py` at 18 heights $200 \leq X \leq 10^{30}$ and $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1\} \cdot \frac{1}{100\ell_X}$, maximising over 721 points of the circle. The remainder decreases from $g(200) = 2.319$ to 1.53 at $X = 10^{30}$; its terms are explicit, of size $O(1/t_c) + O(\tau\ell_X)$, and decrease with t_c . Hence the count is at most $2.0668 \ell_X + 2.32$, and the constant 2.5 in $B(X)$ leaves a margin of 0.18 for the discretisation. This step is a numerical evaluation of an explicit elementary inequality, not an interval-arithmetic proof. The case $\operatorname{Re} z < 0$ follows from (P2). $\square$

Lemma 3.3 (global count). *For $0 \leq \tau \leq \frac{1}{2}$ and $\rho \geq 200$, the number $N_\tau(\rho)$ of zeros of H_τ with $|\operatorname{Re} z| \leq \rho$ satisfies $N_\tau(\rho) \leq \rho \log \rho$. Moreover $N_\tau(200) \leq 400$.*

Proof. The zeros counted lie in $|z| \leq \rho + 1$. By Jensen's formula with centre 0 and radii $\rho + 1$ and $e(\rho + 1)$, and by (P2), $N_\tau(\rho) \leq \log(I(e(\rho + 1))/H_0(0))$, where $I(r) = \int_0^\infty e^{u^2/2} \Phi(u) \cosh(ru) du \geq \max_{|z|=r} |H_\tau(z)|$. From $\Phi(u) \leq 2.01\pi^2 e^{9u - \pi e^{4u}}$ and the concavity of $u^2/2 + (r + 9)u - \pi e^{4u}$, we get $\log I(r) \leq \frac{r+10}{4} \log(\frac{r}{4\pi} + 2) + 3$. This gives $N_\tau(\rho) \leq \frac{e}{4}(\rho + 5) \log(\rho + 2) + 6 \leq \rho \log \rho$. Direct quadrature (`lowcount.py`) gives $N_\tau(200) \leq 389.8$, so $N_\tau(200) \leq 400$. $\square$

4 The exact equation and the energy slope

Lemma 4.1. *Let $z = x + iy$, $y > 0$, be a simple zero of H_τ . Then*

$$\partial_\tau y = -\frac{1}{y} - 2y \sum_{x_k \text{ real}} \frac{1}{(x - x_k)^2 + y^2} - \sum_{w \text{ nonreal, } w \neq z, \bar{z}} \frac{2(y - \operatorname{Im} w)}{|z - w|^2}.$$

Proof. Take imaginary parts in (P3): $\operatorname{Im} \frac{2}{z - \bar{z}} = -\frac{1}{y}$, $\operatorname{Im} \frac{2}{z - x_k} = -\frac{2y}{(x - x_k)^2 + y^2}$, and $\operatorname{Im} \frac{2}{z - w} = -\frac{2(y - \operatorname{Im} w)}{|z - w|^2}$. $\square$

Remark 4.2 (the energy slope). At $\tau = 0$ write $z = 2\gamma + 2i\delta$ and $x_k = 2\gamma_k$. Then $2y \sum_k \frac{1}{(x - x_k)^2 + y^2} = \sum_k \frac{\delta}{\delta^2 + (\gamma - \gamma_k)^2}$. If all other zeros of ζ are on the line, Lemma 4.1 reads

$$\partial_\tau \delta = -\frac{1}{4\delta} - \frac{1}{2} S_{\text{rest}} \left(\frac{1}{2} + \delta + i\gamma \right),$$

where $S(\sigma, t) = \partial_\sigma \log |\xi(\sigma + it)| = \sum_\rho \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}$ is the energy slope of [1, §4]. Here S_{rest} omits only the pair $\frac{1}{2} \pm \delta + i\gamma$; the mirror pair $\frac{1}{2} \pm \delta - i\gamma$ is included and matches the mirror terms exactly. Thus the heat flow pulls an off-line zero towards the line at a speed equal to half the energy slope of the other zeros at its own position. The mean of S over t is π times the density of zeros, that is $\frac{1}{2} \log \frac{\gamma}{2\pi}$.

5 The weighted functional

Notation. Throughout this section $\gamma \geq 3 \cdot 10^{12}$, $x_0 = 2\gamma$, and

$$W = \log(3 + x_0), \quad \hat{B} = B(4x_0), \quad \tau_C = \frac{1}{200 \log x_0}.$$

For $X \leq x_0^2$ one has $\ell_X < 2 \log x_0$, so by Lemma 3.2 *every closed unit interval $[X, X + 1]$ with $200 \leq |X| \leq x_0^2$ contains at most $B(|X|)$ zeros of H_τ for $0 \leq \tau \leq \tau_C$, and at most $\hat{B}$ if $|X| \leq 4x_0$; for negative intervals $[X, X + 1]$ the relevant height is $|X + 1|$, and intervals within $|x| \leq 200$ are covered by Lemma 3.3. Since nonreal zeros come in conjugate pairs, *such an interval contains at most $\hat{B}/2$ zeros in the open upper half-plane.* Let*

$$\psi(x) = \operatorname{sech}\left(\frac{x - x_0}{W}\right).$$

Then $0 < \psi \leq 1$, $|\psi'| \leq \psi/W$, $|\psi''| \leq \psi/W^2$, $\psi(x') \leq e^{|x' - x|/W} \psi(x)$, $\int \psi = \pi W$, and, since ψ is unimodal, $\sum_{j \in \mathbb{Z}} \sup_{[j, j+1]} \psi \leq \pi W + 2$. For $R \geq x_0$ let χ_R be 1 on $|x - x_0| \leq R$, 0 on $|x - x_0| \geq 2R$, and the cubic smoothstep in between, so $|\chi'_R| \leq \frac{3}{2R}$ and χ'_R is Lipschitz with constant $6/R^2$. Put $\phi = \psi \chi_R$ and

$$F_R(\tau) = \sum_{k: y_k > 0} \phi(x_k) y_k^p, \quad p \geq 2,$$

a finite sum over the zeros of H_τ in the upper half-plane. We call a zero *central* if $|x_k - x_0| \leq x_0/2$.

Lemma 5.1 (explicit sums). *Let $0 \leq \tau \leq \tau_C$, let z_k be a central zero, $y \in (0, 1]$, and write $\Delta x = x_k - x_m$ and $\eta = 10^{-9} \hat{B}$. Let $s_2 = 2 + 2 \sum_{j \geq 1} \frac{1}{1+j^2} = 4.1533$, $s_c = 2 + 8 \sum_{j \geq 1} \frac{1}{4+j^2} = 7.2832$, and let H_n be the harmonic numbers. Summing over all zeros z_m with multiplicity,*

$$(i) \quad \sum_m \frac{1}{\Delta x^2 + y^2} \leq \hat{B} \left(\frac{2}{y^2} + 2\zeta(2) \right) + \eta.$$

Summing only over the zeros z_m in the open upper half-plane,

$$\begin{aligned} (ii) \quad \sum_m \frac{1}{1 + \Delta x^2} &\leq \tfrac{1}{2} s_2 \hat{B} + \eta, & (iii) \quad \sum_m \frac{4}{4 + \Delta x^2} &\leq \tfrac{1}{2} s_c \hat{B} + \eta, \\ (iv) \quad \sum_{|\Delta x| \leq W} \min\left(\frac{1}{2}, \frac{1}{|\Delta x|}\right) &\leq H_{\lfloor W \rfloor - 1} \hat{B}, & (v) \quad \sum_{|\Delta x| > W} \Delta x^{-2} &\leq (\lfloor W \rfloor^{-2} + \lfloor W \rfloor^{-1}) \hat{B} + \eta. \end{aligned}$$

Finally, over all zeros,

$$(vi) \quad \left| \sum_{m: |\Delta x| > 1} \operatorname{Re} \frac{2}{z_k - z_m} \right| \leq E_3 \hat{B}, \quad E_3 = 4H_{\lceil 1.5x_0 \rceil + 3} + 4 + \frac{8}{3} \left(1 + \frac{2.1}{\hat{B}} \right) (1 + 10^{-3}).$$

Proof. Group the zeros by the closed unit intervals $\Delta x \in [j, j + 1]$ and $\Delta x \in [-j - 1, -j]$, $j \geq 0$.

Zeros with $|\Delta x| \leq x_k$. These have $0 \leq x_m \leq 2x_k \leq 3x_0$. Each interval with $x_m \geq 200$ holds at most $\hat{B}$ of them, and at most $\hat{B}/2$ upper ones. The at most 400 zeros with $|x_m| \leq 200$ (Lemma 3.3) lie at distance $\geq x_0/2 - 200$ and contribute less than 10^{-20} . On the two intervals with $j = 0$ the weights in (i), (ii), (iii) are at most y^{-2} , 1, 1; on the others at most j^{-2} , $(1 + j^2)^{-1}$, $4(4 + j^2)^{-1}$. For (iv) and (v), $\min(\frac{1}{2}, 1/|\Delta x|) \leq \frac{1}{2}$ for $|\Delta x| \leq 2$ and $\leq 1/j$ on the j -th interval, and $\Delta x^{-2} \leq j^{-2}$ there.

Zeros with $|\Delta x| > x_k$. The weights are at most Δx^{-2} . Counting with Lemma 3.2 up to height x_0^2 (at most $B(X) \leq 2.07 \cdot 2 \log x_0 + 2.5$ per unit interval) gives at most $2 \sum_{j > x_k} (4.2 \log x_0 + 2.5) j^{-2} \leq (9 \log x_0 + 5)/x_k$. Beyond x_0^2 , partial summation with Lemma 3.3 gives at most $2 \int_{x_0^2/2}^{\infty} \rho \log \rho \cdot 2\rho^{-3} d\rho \leq 8 \log x_0/x_0^2$. Both are below η for $x_0 \geq 6 \cdot 10^{12}$. This proves (i)–(v).

For (vi), split the zeros into three groups.

- $0 \leq x_m \leq 2x_k + 2$, $|\Delta x| > 1$. Each term is at most $2/|\Delta x| \leq 2/j$ on the j -th interval, $1 \leq j \leq \lceil x_k \rceil + 2$, giving at most $4H_{\lceil 1.5x_0 \rceil + 3} \hat{B}$ (plus 10^{-9} for $x_m \leq 200$).
- $-2x_k - 2 \leq x_m < 0$. There are at most $(2x_k + 2)\hat{B} + 400$ of them, each at distance $\geq x_k$, giving at most $4\hat{B}(1 + 1/x_k) + 10^{-9}$; the excess $4\hat{B}/x_k$ is absorbed by the factor $1 + 10^{-3}$ below.
- The symmetric set $\{|x_m| > 2x_k + 2\}$. Here $|z_m| \geq 2|z_k|$, so pairing z_m with $-z_m$ gives $|\frac{2}{z_k - w} + \frac{2}{z_k + w}| = \frac{4|z_k|}{|z_k^2 - w^2|} \leq \frac{16|z_k|}{3|w|^2}$. Summing over pairs with $x_m > 2x_k + 2$, using Lemma 3.2 up to x_0^2 and Lemma 3.3 beyond, gives at most $\frac{16}{3}(x_k + 1) \frac{B(2x_k) + 2.1}{2x_k} (1 + 10^{-9}) \leq \frac{8}{3}(\hat{B} + 2.1)(1 + 10^{-3})$.

□

Proposition 5.2 (explicit differential inequality). *Let $0 \leq \tau \leq \tau_C$ be such that all zeros of H_τ with $|\operatorname{Re} z - x_0| \leq 2R + 1$ are simple. Then*

$$\frac{dF_R}{d\tau} \geq - \sum_{k: y_k > 0} \kappa_k \phi(x_k) y_k^{p-2} - E_R(\tau),$$

where, for every central zero, $\kappa_k = \kappa$ with

$$\kappa = p + \hat{B} \left[s_2 p(p-1) + (4 + 4\zeta(2) + s_c + d_W) p + e^{1/W} \left(\frac{2}{W^2} + \frac{p}{W} \right) + \frac{2 + E_3}{W} + 10^{-6}(p^2 + p + 1) \right], \quad (2)$$

$d_W = \frac{2e}{W} H_{\lceil W \rceil - 1} + 2(|W|^{-2} + |W|^{-1})$. For every other zero κ_k is finite and bounded by a polynomial in $|x_k|$. The error satisfies $\sup_{0 \leq \tau \leq \tau_C} E_R(\tau) \rightarrow 0$ as $R \rightarrow \infty$. For $\gamma \geq 3 \cdot 10^{12}$ one has $\kappa \leq p + \hat{B}(4.154p^2 + 14.55p + 4.44)$.

Proof. By (P3),

$$\frac{dF_R}{d\tau} = \sum_{y_k > 0} (p\phi_k y_k^{p-1} \dot{y}_k + \phi'_k y_k^p \dot{x}_k), \quad \dot{z}_k = \sum'_m \frac{2}{z_k - z_m}.$$

Every other zero z_m is either $\bar{z}_k$, real, an upper zero, or the conjugate of another upper zero. All zeros have $|y| \leq 1$ by (P1). We call a quantity *negligible* if it is supported on zeros with $|x - x_0| \geq R - 1$; such zeros carry $\psi \leq 2e^{-(R-1)/W}$. By Lemma 3.3, the sum of all negligible quantities is $O(R^C e^{-R/W})$ uniformly in τ ; these sums form E_R . For a central zero we bound each kind of contribution as follows.

(a) The conjugate $\bar{z}_k$ contributes $-p\phi_k y_k^{p-2}$.

(b) Real zeros contribute $-2p\phi_k y_k^p \sum \frac{1}{\Delta x^2 + y_k^2} \geq -(4 + 4\zeta(2))p \hat{B} \phi_k y_k^{p-2} - 2p\eta\phi_k y_k^p$, by Lemma 5.1(i) and $y_k \leq 1$.

(c) Conjugates $\bar{z}_n$ of other upper zeros. With $u = y_k + y_n \in [y_k, 2]$, each contributes $-2p\phi_k y_k^{p-1} \frac{u}{\Delta x^2 + u^2} \geq -2p\phi_k y_k^{p-2} \frac{4}{4 + \Delta x^2}$. By Lemma 5.1(iii), applied to the real parts x_n of the upper zeros z_n , the total is at least $-s_c p \hat{B} \phi_k y_k^{p-2} - 2p\eta\phi_k y_k^{p-2}$.

(d) *Pairs of upper zeros* $\{k, m\}$. Label the pair so that $y_k \geq y_m$, and put $D = |z_k - z_m|$, $\Delta y = y_k - y_m$. The two contributions add up to $T_{km} = -2p \Delta y (\phi_k y_k^{p-1} - \phi_m y_m^{p-1})/D^2$, which is nonnegative unless $\phi_k y_k^{p-1} > \phi_m y_m^{p-1}$. In that case

$$T_{km} \geq -2p(p-1)\phi_k y_k^{p-2} \frac{\Delta y^2}{D^2} - 2p\chi_k(\psi_k - \psi_m)_+ \frac{\Delta y}{D^2} y_m^{p-1} - 2p\psi_m(\chi_k - \chi_m)_+ \frac{\Delta y}{D^2} y_m^{p-1}.$$

- (d1) Since $\Delta y \leq 1$, $\Delta y^2/D^2 \leq 1/(1 + \Delta x^2)$. Charged to k , the first term gives at least $-s_2 p(p-1)\hat{B}\phi_k y_k^{p-2} - 2p^2\eta\phi_k y_k^{p-2}$ by Lemma 5.1(ii).
- (d2) For $|\Delta x| \leq W$: $(\psi_k - \psi_m)_+ \leq |\Delta x| \sup |\psi'| \leq \frac{\epsilon}{W} |\Delta x| \psi_k$ and $|\Delta x| \Delta y/D^2 \leq \min(\frac{1}{2}, 1/|\Delta x|)$. For $|\Delta x| > W$: $(\psi_k - \psi_m)_+ \Delta y/D^2 \leq \psi_k \Delta x^{-2}$. With $y_m^{p-1} \leq y_k^{p-2}$ and Lemma 5.1(iv),(v), the second term gives at least $-d_W p \hat{B} \phi_k y_k^{p-2} - 2p\eta\phi_k y_k^{p-2}$.
- (d3) The third term vanishes unless $\chi_m < 1$, that is $|x_m - x_0| > R$; it is negligible, because $(\chi_k - \chi_m)_+ \Delta y/D^2 \leq \min(\frac{3}{4R}, \Delta x^{-2})$.

(e) *The horizontal term* $\phi'_k y_k^p \dot{x}_k$. Write $\phi' = \chi\psi' + \chi'\psi$. The part with χ' is negligible: pairing the terms of two upper zeros as in (e1) bounds it by a sum over zeros in the transition zone. For $g = \chi\psi'$ we have $|g| \leq \phi/W$. On a segment of length ≤ 1 that does not meet $|x - x_0| \geq R - 1$ we have $\chi \equiv 1$, so there $|g'| = |\psi''| \leq e^{1/W} \psi_h/W^2$ and $|g| \leq e^{1/W} \psi_h/W$ for either endpoint h . Segments that do meet that region give negligible terms.

- (e1) For upper zeros k, m with $|\Delta x| \leq 1$, let h be the higher one. The pair contributes $\frac{2\Delta x}{D^2} (g_k y_k^p - g_m y_m^p)$, and $|g_k y_k^p - g_m y_m^p| \leq \sup |g'| |\Delta x| y_h^p + \sup |g| p \Delta y y_h^{p-1}$. Using $2\Delta x^2 \leq 2D^2$ and $2|\Delta x| \Delta y \leq D^2$, the pair contributes at least $-e^{1/W} (\frac{2}{W^2} + \frac{p}{W}) \phi_h y_h^{p-2}$. A zero has at most $\hat{B}$ upper partners with $|\Delta x| \leq 1$ (two unit intervals).
- (e2) For a non-upper zero with $|\Delta x| \leq 1$, $|\operatorname{Re} \frac{2}{z_k - z_m}| = \frac{2|\Delta x|}{\Delta x^2 + (y_k - y_m)^2} \leq \frac{1}{y_k}$, since $y_m \leq 0$. There are at most $2\hat{B}$ of them, contributing at least $-\frac{2}{W} \hat{B} \phi_k y_k^{p-1}$.
- (e3) For $|\Delta x| > 1$, Lemma 5.1(vi) gives at least $-\frac{E_3}{W} \hat{B} \phi_k y_k^p$.

The η -terms total at most $10^{-8} \hat{B} p^2 \phi_k y_k^{p-2}$ and are absorbed into the last term of (2). Adding (a)–(e) and using $y_k \leq 1$ gives (2) for central zeros.

For a non-central zero the same argument applies with $\hat{B}$ replaced by the counts of Lemmas 3.2 and 3.3 near its own height. Every unit interval meeting $|x| \leq \rho$ holds at most $N_\tau(\rho + 1) \leq (\rho + 1) \log(\rho + 1)$ zeros, so $\kappa_k \leq Cp^2(|x_k| + 3)^2$ with an absolute C .

For the numerical form: $\hat{B} \geq 61$ and $W \geq 29.42$, and on $\gamma \geq 3 \cdot 10^{12}$ the coefficients of p and of 1 in (2), divided by $\hat{B}$, are at most their values 14.548 and 4.434 at $\gamma = 3 \cdot 10^{12}$. They decrease with W apart from jumps below 0.004 at integer values of W ; on a dense scan of $29.42 \leq W \leq 200$ the maximum is attained at $W = 29.42$, and the decrease continues beyond. $\square$

6 Proof of Theorem 1.1

By (P5) we may assume $\gamma > 3 \cdot 10^{12}$. Since $|\operatorname{Im} z| \leq 1$ at $\tau = 0$, we have $\delta \leq \frac{1}{2}$. Take $x_0 = 2\gamma$ and $y_0 = 2\delta$; then $F_R(0) \geq \phi(x_0)y_0^p = y_0^p$. Let κ be as in (2) and put

$$\mu = \frac{1}{2} \hat{B} (\pi W + 2).$$

Step 1: Hölder. With exponents $\frac{p}{2}$ and $\frac{p}{p-2}$,

$$\sum_k \kappa_k \phi_k y_k^{p-2} \leq \left(\sum_k \phi_k \kappa_k^{p/2} \right)^{2/p} F_R^{1-2/p},$$

the sums running over the upper zeros. The central upper zeros contribute at most $\kappa^{p/2} \sum_{\text{central}} \psi_k \leq \kappa^{p/2} \mu$, since each unit interval holds at most $\hat{B}/2$ of them and $\sum_j \sup_{[j, j+1]} \psi \leq \pi W + 2$. Every other zero has $\psi_k \leq 2e^{-|x_k - x_0|/W} \leq 2e^{-x_0/(2W)}$, while κ_k and the number of zeros grow only polynomially (Proposition 5.2, Lemma 3.3). So the rest contribute less than $10^{-10^{10}} \kappa^{p/2} \mu$. Hence $\left(\sum_k \phi_k \kappa_k^{p/2} \right)^{2/p} \leq (1 + 10^{-100}) \kappa \mu^{2/p}$.

Step 2: differential inequality. Let $G_R = F_R^{2/p}$. Fix $\vartheta \in (0, 1)$ and let τ_1 be the first time in $[0, \tau_C]$ with $G_R \leq \vartheta y_0^2$ (or $\tau_1 = \tau_C$). On $[0, \tau_1]$, Proposition 5.2 and Step 1 give

$$\frac{dG_R}{d\tau} = \frac{2}{p} F_R^{2/p-1} \frac{dF_R}{d\tau} \geq -\frac{2}{p} (1 + 10^{-100}) \kappa \mu^{2/p} - \frac{2}{p} (\vartheta y_0^2)^{1-p/2} E_R$$

at every τ outside the set where H_τ has a multiple zero with $|\operatorname{Re} z - x_0| \leq 2R + 1$. That set is discrete. Indeed, $H_\tau(z)$ is entire in (τ, z) , so the discriminant of the zeros inside a small contour around the zeros of H_{τ_0} is analytic in τ . It does not vanish identically, since by (P3) a zero of multiplicity r splits into r distinct zeros.

Step 3: integration. F_R is a finite sum, over the zeros, of the continuous function $\phi(\operatorname{Re} z)(\operatorname{Im} z)_+^p$, which vanishes on the boundary of its support. By Hurwitz's theorem F_R , and hence G_R , is continuous in τ . The mean value inequality for continuous functions whose lower Dini derivative is bounded below outside a countable set gives, on $[0, \tau_1]$,

$$G_R(\tau) \geq y_0^2 - \left(\frac{2}{p} (1 + 10^{-100}) \kappa \mu^{2/p} + o_R(1) \right) \tau.$$

Let $R \rightarrow \infty$ and then $\vartheta \rightarrow 0$. For every $\tau < \min(\tau_C, y_0^2 p / (2(1 + 10^{-100}) \kappa \mu^{2/p}))$ we get $F_R(\tau) > 0$, so H_τ has a zero in the upper half-plane and $\tau < \Lambda$. Hence, for every integer $p \geq 2$,

$$\Lambda \geq T(\gamma) \delta^2, \quad T(\gamma) = \max_{p \geq 2} \frac{2p}{(1 + 10^{-100}) \kappa(p) \mu^{2/p}}, \quad (3)$$

provided $T(\gamma) \delta^2 \leq \tau_C$. This holds because $T(\gamma) \delta^2 \leq \frac{1}{4} T(\gamma) \leq 4 \cdot 10^{-5} < \tau_C$ for $3 \cdot 10^{12} \leq \gamma \leq 10^{300}$ (checked by the script), and $T(\gamma) \log \gamma \rightarrow 0$.

Step 4: the constant. The script `theorem_constants.py` evaluates (3) with the optimal integer p at 3000 heights $3 \cdot 10^{12} \leq \gamma \leq 10^{300}$. The quantity $K_0(\gamma) = 1/(T(\gamma) \log \gamma \log \log \gamma)$ is decreasing on this grid, and its maximum is $K_0(3 \cdot 10^{12}) = 68.57$ (Table 1). For $\gamma > 10^{300}$, take $p = \lceil 2 \log \mu \rceil$ with the exact $\hat{B}$ and μ , so that $\mu^{2/p} \leq e$. Then, by Proposition 5.2,

$$K_0(\gamma) \leq \frac{e(p + \hat{B}(4.154p^2 + 14.6p + 4.5))}{2p \log \gamma \log \log \gamma}.$$

On a grid of $2 \cdot 10^5$ points in $690 \leq \log \gamma \leq 10^{15}$ the right-hand side is at most 58.94 (script `asymptotic.py`); for fixed p it changes by less than 0.1 between grid points, and at each jump of $p = \lceil 2 \log \mu \rceil$ the next grid point follows the jump at essentially the same value. Beyond $\log \gamma = 10^{15}$, $p \leq 4.1 \log \log \gamma$ and the right-hand side is below 49; it tends to $e \cdot 2.07 \cdot 4.153 \cdot 4/2 = 46.7$. Hence $K_0(\gamma) \leq 68.6 < 70$ for all $\gamma \geq 3 \cdot 10^{12}$, which is Theorem 1.1. $\square$

γ	W	$\hat{B}$	p	$T(\gamma)$	$K_0(\gamma)$
$3 \cdot 10^{12}$	29.42	61.04	19	$1.512 \cdot 10^{-4}$	68.57
10^{15}	35.23	73.06	20	$1.219 \cdot 10^{-4}$	67.08
10^{20}	46.74	96.89	21	$8.707 \cdot 10^{-5}$	65.12
10^{30}	69.77	144.56	22	$5.430 \cdot 10^{-5}$	62.95
10^{50}	115.82	239.88	24	$3.005 \cdot 10^{-5}$	60.90
10^{100}	230.95	478.20	27	$1.356 \cdot 10^{-5}$	58.87
10^{200}	461.21	954.84	30	$6.169 \cdot 10^{-6}$	57.40

Table 1: The explicit constants: $W = \log(3 + 2\gamma)$, $\hat{B} = B(8\gamma)$, the optimal p , $T(\gamma)$ from (3) (so $\Lambda \geq T(\gamma)\delta^2$), and $K_0 = 1/(T \log \gamma \log \log \gamma)$.

Remark 6.1 (where the constant comes from). The dominant term of κ is $s_2 p^2 \hat{B} \approx 4.15 p^2 \hat{B}$ from (d1), the vertical interaction of pairs of upper zeros at distances $\lesssim 1$. With $p \approx 2 \log \mu \approx 4 \log \log \gamma$ and $\hat{B} \approx 2.07 \log \gamma$ this gives $T \approx 2/(e s_2 p \hat{B})$. The factor $\log \gamma$ comes from the Jensen count, which is about 26 times the true density $\frac{1}{4\pi} \log \frac{\gamma}{2\pi}$ of zeros of H_0 per unit length. The factor $\log \log \gamma$ comes from allowing arbitrary clusters of other off-line zeros near the given one. Both margins are thin by design: 68.57 against 70 in K_0 , and 2.32 against 2.5 in Lemma 3.2.

7 A sharper bound under local regularity

For a single pair, Lemma 4.1 can be integrated with a finer count. Let $z(\tau) = x(\tau) + iy(\tau)$, $y(0) = y_0 \leq 1$, and let τ_{abs} be the first time with $y = 0$. On $[0, \tau_{\text{abs}})$ assume:

- (I) $z, \bar{z}$ are the only nonreal zeros with $|\operatorname{Re} w - x(\tau)| \leq U$, and the other nonreal zeros w satisfy $\sum 4/|\operatorname{Re} w - x|^2 \leq \varepsilon_1$;
- (C) the number of real zeros in $[x(\tau) - u, x(\tau) + u]$ is at most $2\varrho u + A$ for $0 < u \leq U$, and the real zeros with $|x_k - x| > U$ satisfy $\sum (x - x_k)^{-2} \leq \varepsilon_2$.

At $\tau = 0$, (C) holds with $A = O(\log \gamma)$ by Lemma 3.2; numerically $A = 1.7$ – 2.2 at heights 2000 and 6000 (Section 8).

Theorem 7.1 (conditional on (I), (C)). *Put*

$$a = 1 + 2A, \quad c = 2\pi\varrho + 2\left(\frac{2\varrho U + A}{U^2} + \varepsilon_2\right) + \varepsilon_1, \quad T(y_0) = \frac{1}{c} \left(y_0 - \frac{a}{c} \log \left(1 + \frac{cy_0}{a} \right) \right).$$

Then $\tau_{\text{abs}} \geq T(y_0)$ and $\Lambda \geq T(y_0)$. In terms of the zero of ζ , with $\varrho = \frac{1}{4\pi}L$ and $L = \log \frac{\gamma}{2\pi}$,

$$\Lambda \geq \frac{4\delta}{L} - \frac{4a}{L^2} \log \left(1 + \frac{L\delta}{a} \right) \quad (1 + o(1)).$$

Proof. Integration by parts against the counting function gives $\sum_k \frac{1}{(x-x_k)^2 + y^2} \leq \frac{\pi\varrho}{y} + \frac{A}{y^2} + \frac{2\varrho U + A}{U^2} + \varepsilon_2$. With Lemma 4.1 this gives $\partial_\tau y \geq -a/y - c$ on $[0, \tau_{\text{abs}})$. Put $\varphi(y) = \int_0^y \frac{v dv}{a+cv}$. Then $\frac{d}{d\tau} \varphi(y) \geq -1$, so $\varphi(y(\tau)) \geq T(y_0) - \tau$, and letting $\tau \rightarrow \tau_{\text{abs}}$ gives $\tau_{\text{abs}} \geq T(y_0)$. For $\tau < \tau_{\text{abs}}$ the function H_τ has a nonreal zero, so $\tau < \Lambda$. $\square$

The bound is linear, $\approx 4\delta/L$, when $L\delta \gg a$, and quadratic, $\approx 2\delta^2/a$, when $L\delta \ll a$. In the linear regime the speed $L/4$ at which δ decreases equals the speed at which the heat flow shifts the effective critical line of the deformed partial sums. Under the flow the n -th Dirichlet term is multiplied by $e^{(\tau/4)\log^2 n}$, the centre term of the cross-energy C_2 of [1] becomes $b_n^2 n^{1-2\sigma}/(1-s+\frac{\tau}{2}\log n)$, and with the gamma factor the shift at height γ is $\frac{\tau}{4}(\log \frac{\gamma}{2\pi} - \log n)$ (Figure 2d; the mechanism is that of [7]).

8 Numerical test

Zeros and method. We computed the 10142 zeros of ζ up to height 10^4 , with Riemann–Siegel and the C_1 correction; the Lehmer pair is reproduced to $2 \cdot 10^{-6}$. We inserted an off-line pair $x_0 \pm iy_0$ halfway between two consecutive zeros, at heights $\gamma \approx 2000$ and 6000 , at five positions each, with $y_0 = 0.4, 0.8, 1.6$. The last value lies outside the strip and serves as a stress test. We integrated (P3) with the 300 nearest zeros on each side exactly and the rest as a frozen field.

Results.

- In all 30 runs, the bound of Theorem 7.1 (with A measured at $\tau = 0$), the simulated absorption time and $y_0^2/2$ come in that order (Figure 1a). The bound is 0.19–0.54 of the simulated time.
- The energy slope S_{rest} at $\tau = 0$ predicts the simulated times with a correlation of 0.998 between logarithms (Figure 1b). The times exceed the prediction by 7–29%, because the pair repels its neighbours and lowers S_{rest} as τ grows.
- The explicit bound of Theorem 1.1 is smaller than the simulated times by a factor of order 10^3 at these heights. The factor $\log \gamma$ is the price of not knowing the local spacing; the factor $\log \log \gamma$ is the price of clusters.

Clusters. Theorem 1.1 allows other off-line zeros near the given one, and the simulations show why this matters. With $y_0 = 0.8$ at $\gamma \approx 2000$, an isolated inserted pair becomes real at $\tau = 0.135$. When two or five further pairs ($y_0 = 0.2$ – 0.7 , within 2.4 units of x) are inserted around it, all zeros become real by $\tau = 0.090$ and 0.086 . The other upper zeros attract the top one vertically, and once absorbed they add to the density of real zeros. These times still exceed the conditional bound for the isolated pair (≈ 0.05), but they show that clusters can speed up absorption.

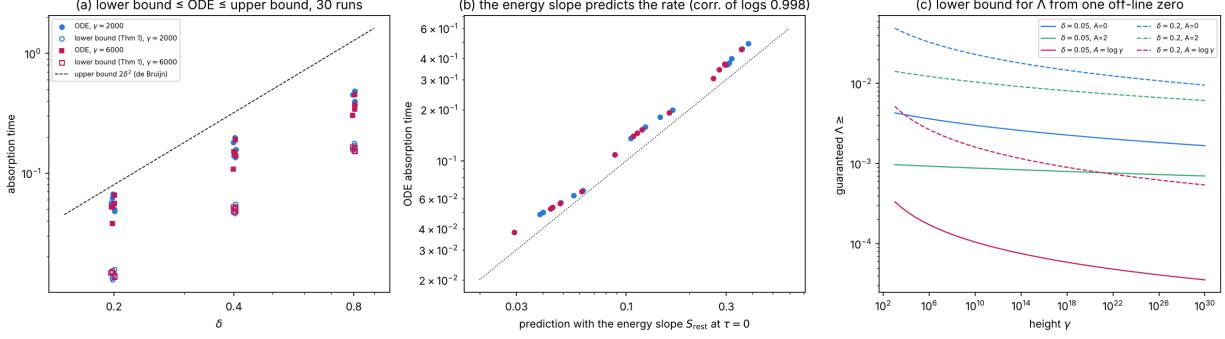


Figure 1: (a) Lower bound of Theorem 7.1 (open), simulated absorption time (filled) and $2\delta^2$, 30 runs. (b) Simulated times against the prediction from the energy slope S_{rest} at $\tau = 0$. (c) The lower bound of Theorem 7.1 for Λ as a function of the height of a hypothetical off-line zero, for $\delta = 0.05, 0.2$ and local excess $A = 0, 2, \log \gamma$.

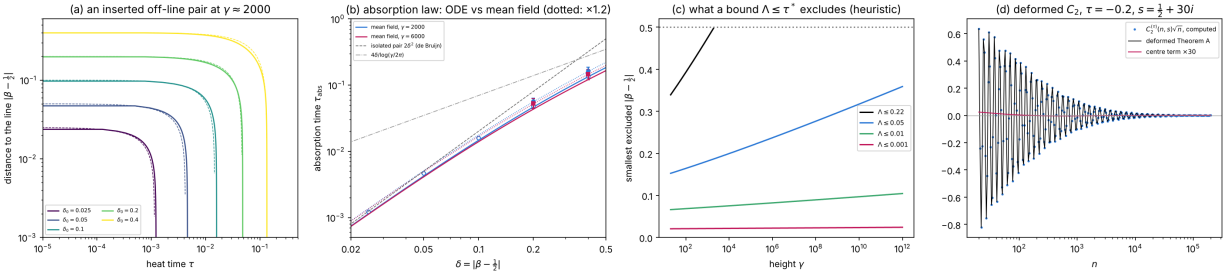


Figure 2: Context. (a) Trajectories of inserted off-line pairs at $\gamma \approx 2000$; dashed: $\partial_\tau y = -1/y - L/2$. (b) Absorption times against the formula of Theorem 7.1 with $A = 0$; dotted: $\times 1.2$, the average excess of the simulations. (c) What an upper bound $\Lambda \leq \tau^*$ would exclude by Theorem 7.1 with that factor. (d) The cross-energy C_2 of the deformed partial sums $\sum_{k \leq n} e^{(\tau/4) \log^2 k} k^{-s}$ follows the deformed expansion, $\tau = -0.2$.

9 Status

- *Unconditional and explicit:* Lemmas 3.1–3.3, Proposition 5.2, Theorem 1.1 and Corollary 1.2. They use only (P1)–(P5) from [2, 7, 6], Stirling’s formula and the Euler–Maclaurin formula. The final inequality of Lemma 3.2, the bound $N_\tau(200) \leq 400$, and the maximisation over p and γ in (3) are explicit elementary inequalities that we evaluated numerically, without interval arithmetic; the scripts are included as ancillary files. The margins are stated in the Remark after Table 1.
- *Conditional on (I), (C):* Theorem 7.1, the linear regime.
- *Heuristic:* the deformed cross-energy picture and the identification of the drift $L/4$ with the shift of the effective critical line.
- *Relation to the literature:* the upper-bound version of the dynamics is used in [5, 9]. As far as we know, lower bounds for Λ in terms of a hypothetical off-line zero, and the link with the energy slope, are new.

- *What this does not do:* prove RH. The bound shows that Λ dominates a distance $\asymp \delta^2/(\log \gamma \log \log \gamma)$ from each zero to the critical line, a distance that tends to 0 with the height. The rigidity proved by Rodgers and Tao, $\Lambda \geq 0$, points the other way.

Use of AI tools. The computations and the drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction. The author checked the results and takes responsibility for them.

References

- [1] P. Caceres, *The cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function*, consolidated paper, v2 (2026); v1: Zenodo, doi:10.5281/zenodo.23243641.
- [2] N. G. de Bruijn, The roots of trigonometric integrals, *Duke Math. J.* 17 (1950), 197–226.
- [3] C. M. Newman, Fourier transforms with only real zeros, *Proc. Amer. Math. Soc.* 61 (1976), 245–251.
- [4] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.
- [5] M. Planat, Collective contraction in the de Bruijn–Newman heat flow: off-zero logarithmic derivatives and certified barrier refinements, preprint, arXiv:2609.37164 (2026).
- [6] D. J. Platt, T. S. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. London Math. Soc.* 53 (2021), 792–797.
- [7] D. H. J. Polymath, Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant, *Res. Math. Sci.* 6 (2019), 31.
- [8] B. Rodgers, T. Tao, The de Bruijn–Newman constant is non-negative, *Forum Math. Pi* 8 (2020), e6.
- [9] T. Tao, Polymath15, first thread: computing H_t , asymptotics, and dynamics of zeroes, blog post, 27 January 2018.
- [10] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford, 1986.