

A 23-Point Counterexample to a Modified Half-Frequency Conjecture for Union–Intersection Systems

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Abstract

We consider a variant of Vince’s half-frequency conjecture for union–intersection systems in which an entire system must be nontrivial, while a system is declared trivial only when all its blocks are singletons or when its only block is the whole ground set; every such trivial system is stipulated to be irreducible. For nontrivial systems, irreducibility retains Vince’s two-part partition criterion, and complete irreducibility is tested on the system and all subsystems induced by actual blocks. We give a system on 23 points, generated by 17 three-element sets, whose union–intersection closure consists of 111 *distinct* blocks. Every point belongs to at most 55 blocks, yet the system is nontrivial and completely irreducible under the stipulated convention. A normal-form count, a membership-preserving quotient, and explicit certificates for the induced subsystems make the argument self-contained. This is a counterexample to the *explicitly modified* conjecture, not an assertion that the original wording of Vince’s Conjecture 5 has a unique settled interpretation.

Keywords. Union–intersection system; completely irreducible set system; frequent element; counterexample; induced subsystem.

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1 Statement of the modified question

A *union–intersection system* (UIS) is a pair $\mathcal{S} = (P, \mathcal{B})$, where P is a finite nonempty set and $\mathcal{B}$ is a nonempty family of distinct subsets of P , satisfying

$$A, B \in \mathcal{B}, \quad A \cap B \neq \emptyset \quad \implies \quad A \cup B \in \mathcal{B}. \quad (1)$$

For $x \in P$ put

$$N(\mathcal{S}) = |\mathcal{B}|, \quad N_x(\mathcal{S}) = |\{B \in \mathcal{B} : x \in B\}|, \quad R(\mathcal{S}) = \max_{x \in P} \frac{N_x(\mathcal{S})}{N(\mathcal{S})}.$$

All counts in this paper refer to *distinct subsets of the ground set*, not to choices of generators or their multiplicities.

Vince [1] introduced complete irreducibility and proposed a half-frequency statement for such systems (Conjecture 5). For clarity, we study the following *separate, explicit convention*, which is not identical to the printed definition of “trivial” in [1]. Write $n = |P|$. Define

$$\text{Triv}_{\text{pure}}(\mathcal{S}) \iff (\forall B \in \mathcal{B}, |B| = 1) \text{ or } (\forall B \in \mathcal{B}, |B| = n). \quad (2)$$

Since $\mathcal{B}$ is nonempty and consists of distinct sets, the second alternative is equivalent to $\mathcal{B} = \{P\}$. In particular, a system containing both singleton blocks and the whole set is *not* automatically trivial under (2).

Let $\text{Sep}(\mathcal{S})$ mean that there is a partition $P = P_1 \sqcup P_2$ into two nonempty sets such that every $B \in \mathcal{B} \setminus \{P\}$ lies entirely within one of P_1, P_2 . We define

$$\text{Irr}_{\text{pure}}(\mathcal{S}) \iff \text{Triv}_{\text{pure}}(\mathcal{S}) \text{ or } \neg \text{Sep}(\mathcal{S}). \quad (3)$$

Thus *every trivial system is declared irreducible*, while a nontrivial system is irreducible exactly when it has no separating partition of the type in Vince's Definition 2. For each actual block $Q \in \mathcal{B}$ define its *downward induced system* by

$$\mathcal{S}[Q] = (Q, \mathcal{B}(Q)), \quad \mathcal{B}(Q) = \{B \in \mathcal{B} : B \subseteq Q\}.$$

We say $\mathcal{S}$ is CI_{pure} if both $\text{Irr}_{\text{pure}}(\mathcal{S})$ and $\text{Irr}_{\text{pure}}(\mathcal{S}[Q])$ hold for *every* $Q \in \mathcal{B}$. The variant considered here asks whether

$$\mathcal{S} \text{ nontrivial and } \text{CI}_{\text{pure}} \implies R(\mathcal{S}) \geq \frac{1}{2}. \quad (4)$$

It is important that “nontrivial” is imposed on the *whole* system, not on all its induced subsystems.

Theorem 1. *There exists a nontrivial, CI_{pure} union–intersection system $\mathcal{S} = (P, \mathcal{B})$ satisfying*

$$|P| = 23, \quad |\mathcal{B}| = 111, \quad \max_{x \in P} N_x(\mathcal{S}) = 55.$$

Consequently $R(\mathcal{S}) = 55/111 < 1/2$, and (4) is false.

2 The explicit generators

All symbols in

$$P = \{u_1, \dots, u_{14}\} \sqcup \{c_1, c_2, c_3, c_4\} \sqcup \{v_{10}, \dots, v_{14}\}$$

denote distinct points. Define the following 17 three-element sets:

$$A_i = \{u_i, u_{i+1}, c_{\lceil i/2 \rceil}\}, \quad 1 \leq i \leq 8, \quad (5)$$

$$A_9 = \{u_9, u_{10}, v_{10}\}, \quad (6)$$

$$A_i = \{u_i, u_{i+1}, v_{i+1}\}, \quad 10 \leq i \leq 13, \quad (7)$$

$$B_i = \{v_i, u_{i+1}, v_{i+1}\}, \quad 10 \leq i \leq 13. \quad (8)$$

Let $\mathcal{E}$ be this family and let $\mathcal{B}$ be its *least* union–intersection closure, obtained by repeatedly adjoining $X \cup Y$ whenever X, Y already occur and $X \cap Y \neq \emptyset$. No singleton or empty set is adjoined by fiat. Every block of $\mathcal{B}$ is nonempty.

We prove the theorem using an auxiliary construction in which the blocks admit particularly simple normal forms.

3 Normal forms and exact counting

Introduce an auxiliary 27-point ground set

$$\hat{P} = \{u_1\} \sqcup \bigsqcup_{i=2}^{14} T_i, \quad T_i = \{u_i, v_i\}.$$

On $\widehat{P}$ take the generators

$$\widehat{A}_i = \{u_i\} \cup T_{i+1} \quad (1 \leq i \leq 13), \quad \widehat{B}_i = \{v_i\} \cup T_{i+1} \quad (10 \leq i \leq 13). \quad (9)$$

Let $\widehat{\mathcal{B}}$ be their least union–intersection closure. For $1 \leq i \leq 13$ set

$$\mathcal{L}_i = \begin{cases} \{\{u_i\}\}, & 1 \leq i \leq 9, \\ \{\{u_i\}, \{v_i\}, T_i\}, & 10 \leq i \leq 13. \end{cases}$$

For $1 \leq i < j \leq 14$ and $S \in \mathcal{L}_i$ put

$$Q(i, j, S) = S \cup \bigcup_{h=i+1}^j T_h. \quad (10)$$

Lemma 2. *The family $\widehat{\mathcal{B}}$ consists exactly of the distinct sets $Q(i, j, S)$ in (10). In particular,*

$$|\widehat{\mathcal{B}}| = \sum_{i=1}^9 (14-i) + 3 \sum_{i=10}^{13} (14-i) = 111. \quad (11)$$

Proof. All the generators in (9) are normal forms with $j = i + 1$. The two extra generators at each $i \geq 10$ give the two singleton choices at initial layer i ; their nonempty intersection T_{i+1} permits their union, yielding the $S = T_i$ option. Starting from any of these choices, successively adjoining $\widehat{A}_{i+1}, \dots, \widehat{A}_{j-1}$ yields $Q(i, j, S)$: at each step the current block contains the point needed to intersect the next generator. Thus every displayed normal form belongs to the closure.

Conversely, these normal forms are closed under union of intersecting members. If $Q(i, j, S)$ and $Q(i, j', S')$ have the same starting layer, the union is $Q(i, \max\{j, j'\}, S \cup S')$, with $S \cup S' \in \mathcal{L}_i$. If $i < h$, a normal form starting at i meets one starting at h if and only if $h \leq j$; in that case their union is $Q(i, \max\{j, k\}, S)$, where k is the terminal layer of the later form. This proves closure.

Finally, a normal-form set uniquely determines its first occupied layer i , its last occupied layer j , and its intersection S with the first layer. Hence no two normal forms represent the same subset. Counting choices proves (11). $\square$

Lemma 3. *In the auxiliary family, the point frequencies are*

$$N_{u_l} = l(15-l) - 1 \quad (1 \leq l \leq 9), \quad (12)$$

$$N_{v_l} = (l-1)(15-l) \quad (2 \leq l \leq 9), \quad (13)$$

$$N_{u_{10+t}} = N_{v_{10+t}} = 53 + 4t - 3t^2 \quad (0 \leq t \leq 4). \quad (14)$$

Proof. For $l \leq 9$, a block containing u_l either starts at an earlier layer and ends at or beyond l , or starts at layer l . The respective counts are $(l-1)(15-l)$ and $14-l$, giving (12). An early v_l cannot occur in the initial layer, so only the first contribution remains, giving (13).

For $l = 10 + t$, the number of choices of initial forms from earlier layers is $9 + 3t$; each of these forms has $5 - t$ choices of last layer. If $l < 14$, the two initial masks containing u_l (or v_l) are available at layer l with $4 - t$ terminal choices. Hence

$$N_{u_{10+t}} = N_{v_{10+t}} = (9 + 3t)(5 - t) + 2(4 - t) = 53 + 4t - 3t^2,$$

including $t = 4$, for which the initial contribution is zero. $\square$

4 A four-pair identification that preserves the closure

Identify four pairs of auxiliary points:

$$v_2 \sim v_3 \mapsto c_1, \quad v_4 \sim v_5 \mapsto c_2, \quad v_6 \sim v_7 \mapsto c_3, \quad v_8 \sim v_9 \mapsto c_4,$$

and leave every u_i and $v_{10}, \dots, v_{14}$ fixed. Write $\pi : \hat{P} \rightarrow P$ for this map and extend π to subsets by $\pi(X) = \{\pi(x) : x \in X\}$. For each *normal-form block* Q and every $2 \leq l \leq 9$,

$$v_l \in Q \iff \{u_{l-1}, u_l\} \subseteq Q. \quad (15)$$

Indeed, the only way to use v_l is to cross into full layer T_l from a strictly earlier initial layer; that condition is exactly the presence of both consecutive u -points. The right-hand side is recoverable after taking π , because all u -points are fixed.

Lemma 4. *For $Q, Q' \in \hat{\mathcal{B}}$,*

$$\pi(Q) \subseteq \pi(Q') \iff Q \subseteq Q', \quad (16)$$

$$\pi(Q) \cap \pi(Q') \neq \emptyset \iff Q \cap Q' \neq \emptyset. \quad (17)$$

Consequently π is one-to-one on $\hat{\mathcal{B}}$, and its block image $\pi(\hat{\mathcal{B}})$ is exactly the closure $\mathcal{B}$ of (5)–(8).

Proof. One implication in (16) is automatic. For the converse, all unmerged point memberships are preserved, while for an early v_l membership in Q can be recovered from the two u -points using (15). Hence $\pi(Q) \subseteq \pi(Q')$ implies $Q \subseteq Q'$. This also proves injectivity on blocks.

For (17), only the forward implication requires proof. If the image sets meet at a point not formed by an identification, then the original sets meet. If they meet only at c_j , the relevant preimages are among v_{2j}, v_{2j+1} . Either they share one of those points or one contains v_{2j} and the other v_{2j+1} . In the latter case, (15) shows that both contain u_{2j} . Thus their original intersection is nonempty.

Since π preserves unions, (17) implies that the image of the old closure is closed under every required union–intersection operation. It contains the images of the generators (9), which are precisely (5)–(8), so it contains their least closure. Conversely, every old normal form was obtained by successive legal unions of old generators, and these unions remain legal under π . Therefore each image block belongs to the new least closure, giving equality. $\square$

We now determine the frequencies of the newly identified points. For $1 \leq j \leq 4$, the normal forms satisfy

$$u_{2j} \in Q \iff v_{2j} \in Q \text{ or } v_{2j+1} \in Q. \quad (18)$$

To see this, if a block starts before layer $2j$ and reaches it, it contains v_{2j} ; if it starts at layer $2j$, it necessarily reaches layer $2j+1$ and contains v_{2j+1} . No other normal form contains u_{2j} . The reverse implication follows from (15). As π is injective on blocks, the frequency of c_j in the quotient equals the frequency of u_{2j} in the auxiliary family. The frequencies at every unmerged point are unchanged.

The resulting frequencies of *all 23 points* are:

Points (in listed order)	Frequencies (in the same order)
$u_1, \dots, u_9$	13, 25, 35, 43, 49, 53, 55, 55, 53
$u_{10}, \dots, u_{14}$	53, 54, 49, 38, 21
$v_{10}, \dots, v_{14}$	53, 54, 49, 38, 21
c_1, c_2, c_3, c_4	25, 43, 53, 55

The maximum 55 occurs precisely at u_7, u_8, c_4 . By Lemmas 2 and 4, $|\mathcal{B}| = 111$, so $R(\mathcal{S}) = 55/111 < 1/2$.

5 Complete irreducibility for the stipulated convention

We verify the whole system and *every* downward induced subsystem, without appealing to an unrestricted basis criterion.

Lemma 5. *Each of the 17 minimal blocks of $\mathcal{B}$ induces a copy of $O_3 = ([3], \{[3]\})$. Every other $Q \in \mathcal{B}$ has two proper blocks $U, V \subsetneq Q$ satisfying*

$$U \cap V \neq \emptyset, \quad U \cup V = Q. \quad (19)$$

Proof. Work first with the normal forms on $\hat{P}$. The 17 generator forms $Q(i, i+1, \{u_i\})$ ($1 \leq i \leq 13$) and $Q(i, i+1, \{v_i\})$ ($10 \leq i \leq 13$) are exactly the minimal blocks. For $i \geq 10$, the remaining form with $j = i+1$ is $Q(i, i+1, T_i)$, which is the union of the two distinct proper blocks with singleton initial masks. Their intersection contains T_{i+1} .

Every other nonminimal block has $j \geq i+2$ and satisfies

$$Q(i, j, S) = Q(i, j-1, S) \cup \hat{A}_{j-1}.$$

Both members on the right are proper subsets: the first omits the last layer T_j , while the second omits the earliest layer i . Their intersection contains u_{j-1} . These cases exhaust all normal forms other than the 17 generators.

The quotient π preserves the fact of being a proper subset, nonempty intersection, and union (Lemma 4). Hence all the stated certificates remain valid for the 23-point family. The minimal blocks remain exactly the 17 three-element generators. A minimal block Q has $\mathcal{B}(Q) = \{Q\}$, proving the first assertion. $\square$

Proof of Theorem 1. First, the whole system is *nontrivial*: it contains A_1 of size 3, strictly between 1 and $|P| = 23$. The whole ground set is itself a block, namely $\pi(Q(1, 14, \{u_1\}))$.

For each minimal block Q , Lemma 5 shows that $\mathcal{S}[Q] \cong O_3$. Such a system is trivial under (2), since its sole block is its entire ground set; hence it is irreducible under (3).

Let Q be nonminimal, and take the two proper intersecting blocks U, V from Lemma 5. The induced system $\mathcal{S}[Q]$ is nontrivial, since it contains its whole ground set Q and a proper nonsingleton block (at least one of U, V contains a three-element generator). If a separating partition $Q = Q_1 \sqcup Q_2$ existed, both U and V would have to lie in the same part, because they intersect. But $U \cup V = Q$, forcing that part to be all of Q , a contradiction. Thus $\neg \text{Sep}(\mathcal{S}[Q])$.

In particular, the argument applies to $Q = P$, so it also proves irreducibility of the whole system. Every induced subsystem is irreducible, as required. Together with the exact counts above, this proves the theorem. $\square$

6 Scope and reproducibility

The “pure triviality” convention (2) is *not* Vince’s printed definition: the latter also counts as trivial systems whose singleton and whole-ground-set blocks coexist [1, Section 2]. The present counterexample is therefore stated only for the deliberately modified implication (4). We make no claim here to determine a unique intended interpretation of every trivial induced system in the original Conjecture 5.

The mathematical proof above does not require computation. As a separate check, the 17 displayed generators may be closed under intersecting union, with identical subsets deduplicated; the resulting family has 111 distinct blocks. A direct block-by-block calculation confirms the frequency table and the proper-intersecting cover certificates. Such computations are corroboration, not formal verification or independent peer review.

AI assistance and verification status. This manuscript and supporting checks were prepared with substantial assistance from OpenAI’s ChatGPT (GPT-6) combined with Pharos, including mathematical exploration, proof drafting, and Python verification code. The work has not undergone formal proof verification or an independently certified expert review. The human contributor is responsible for reviewing the mathematical content, contributor metadata, and any public-deposit representations before submission.

References

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