

The Dancing Sand Theorem

The 2-part of the sandpile group of the hypercube, for every n

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Abstract

Let $K(Q_n)$ be the sandpile group (critical group, Jacobian) of the n -dimensional hypercube graph Q_n . Its odd part was determined by Bai in 2003; its 2-part has been open since then, and the latest work on it determines its $n - 1$ largest cyclic factors. **We determine the whole 2-part of $K(Q_n)$ for every n , by an explicit rule.** The group splits over the indecomposable tilting modules $T(\lambda)$ of SL_2 in characteristic 2 that occur in $V^{\otimes n}$, with their multiplicities. The contribution of $T(\lambda)$ at the shift $i = (n - \lambda)/2$ consists of explicit small cyclic factors and of $2^{|I|}$ large ones, where I is the set of binary digits of $\lambda + 1$ below the leading one. The large ones are obtained by pooling a sequence of explicit integers — binomial valuations and carries — into a non-increasing sequence, in the manner of isotonic regression. The proof reduces each contribution, by elimination inside explicit 2-adic lattices, to the Smith form of an explicit $2^{|I|} \times 2^{|I|}$ integer matrix, and computes that Smith form from the valuations of its entries. It needs a unique cheapest matching for an upper bound and an inductive potential for a lower bound. As consequences we recover Bai's counts and the theorems of Gao, Marx-Kuo, McDonald and Yuen on the largest factors, and we prove their Conjectures 4.14 and 5.4 and the formula for the $(n + 1)$ -th factor stated in the 2019 poster of three of them. **We also prove the fold law: the 2-part of the sandpile group of the folded cube $Q_n/\langle(1, \dots, 1)\rangle$ is $2 \cdot \mathrm{Syl}_2 K(Q_n)$,** the group obtained by lowering every exponent by one. With the results above this gives the whole 2-part of the sandpile group of the folded cube, and, with the known odd part, its whole sandpile group. **The whole proof has been checked in the proof assistant Lean 4 with the Mathlib library, with nothing cited.** This work has not been refereed by a human expert.

1. Introduction and main results

1.0 Summary of results

For $n \geq 1$, $K(Q_n)$ is the sandpile group of the n -cube, $\bar{K}(n)$ that of the folded n -cube $Q_n/\langle 1 \rangle$ (§1.3), and $\mathbb{Z}_2$ is the ring of 2-adic integers.

Result	What it says	Where
Main Theorem (the Dancing Sand Theorem)	$\mathrm{Syl}_2 K(Q_n) \oplus \mathbb{Z}_2 \cong \bigoplus_{\lambda} X(\lambda, (n - \lambda)/2)^{m_{\lambda}(n)}$, with $X(\lambda, i)$ given by an explicit rule	§1.2, proof in §8
Theorem D	each $X(\lambda, i)$ is a known group of small clocks plus $\mathrm{coker}(2^k M(\lambda, i))$, M an explicit $2^{ I } \times 2^{ I }$ matrix with closed-form entries	§4
Theorems O and F	the Smith form of $M(\lambda, i)$ is the pooled sequence of clocks: an upper bound through a unique cheapest matching, a lower bound through a potential built digit by digit	§6, §7
Theorem DW (the fold law, «dry and wet»)	$\mathrm{Syl}_2 \bar{K}(n) \cong 2 \cdot \mathrm{Syl}_2 K(Q_n)$; equivalently $\mathrm{Syl}_2 K(Q_n) \cong (\mathbb{Z}/2)^{a_n} \oplus (\mathrm{Syl}_2 \bar{K}(n)$ with every exponent raised by one), $a_n = 2^{n-2} - 2^{\lfloor (n-2)/2 \rfloor}$	§1.3, proof in §10

Result	What it says	Where
Corollary 1.4	the whole sandpile group of the folded cube	§1.3, §10.8
Corollary 1.5	Bai’s two counts, re-proved from the rule	§9.1
Corollary 1.6	the $n + 1$ largest cyclic factors of $\text{Syl}_2 K(Q_n)$: Gao et al.’s Theorems 4.1 and 4.2, their Conjecture 4.14 and the $(n + 1)$ -th factor of the 2019 poster [MGM19]	§9.2
Corollary 1.7	$\text{Syl}_2 K(Q_{2^e}) \cong \text{Syl}_2 K(Q_{2^e-1})^2 \oplus \mathbb{Z}/2^{2^e+e-1}$ (Gao et al.’s Conjecture 5.4)	§9.3

Every proof behind the table was re-derived by a reader other than its author. The chains of the Main Theorem and of Theorem DW were read cold before this text was written, by readers with no access to the audits. The first version of this text was read cold as a whole on 5 October 2026 by a reader with no access to the constructors’ reports, the audits or the earlier readings: «holds with gaps». It found one gap: the integer part of Lemma 7.2(b) was false as stated, and four proofs used it; no theorem was affected. The second version repaired it (§7.1, §7.4, §7.5, §9.2), and its changes were read cold the same day by another reader: **the repair holds**. That reader also found one false sentence, in a remark used nowhere (the author then found the same missing hypothesis in Lemma 10.6(a), harmless because its only use is for an even system), and found, with a two-line proof, that the fit of every house is integral (Lemma 7.9). The changes of each later version were read cold in turn, each time by a new reader; none of these readings found a gap or an error in the mathematics. Between them, they found false or inexact sentences in the record of the work and in §12, points of presentation, and defects of typesetting; the list of corrections of each version, with its sources, and every report are kept with the project (Appendix A). The whole proof has also been checked in Lean 4 with Mathlib, with nothing cited, and that check was read cold before release; §12.3 gives the Lean name of each result of the table except Theorems O and F, which are Theorems 6.2 and 7.8 among the 68 numbered statements it covers. No human expert has refereed this work. The words of our own are translated into standard terms in §1.7; what we found in the literature, and where we searched, is in §1.8.

1.1 The problem

For a finite connected graph G with Laplacian $L(G)$, the cokernel of $L(G)$ acting on $\mathbb{Z}^{V(G)}$ is $\mathbb{Z} \oplus K(G)$, where $K(G)$ is a finite abelian group of order the number of spanning trees of G (Kirchhoff). It is called the sandpile group, the critical group or the Jacobian of G ; see [Lor91], [Big99], [Kli18]. For the hypercube Q_n — vertices $\{0, 1\}^n$, edges between words that differ in one letter — the number of spanning trees is $2^{2^n-n-1} \prod_{j=1}^n j^{C(n,j)}$, so $K(Q_n)$ has order a power of 2 times $\prod j^{C(n,j)}$.

Reiner conjectured in 2001 the number of cyclic factors of $K(Q_n)$ and the number of its factors $\mathbb{Z}/2$, and Bai proved both and determined every odd Sylow subgroup [Bai03]: for an odd prime p , $\text{Syl}_p K(Q_n) \cong \text{Syl}_p \bigoplus_{j=1}^n (\mathbb{Z}/j)^{C(n,j)}$ [Bai03, Theorem 1.2]. About the 2-part he wrote: «The full structure of the Sylow-2 subgroup of the critical group of the n -cube is still unknown» [Bai03, p. 253]. The question was repeated by Ducey and Jalil [DJ14] («the full structure of the 2-primary component of both the critical group and the Smith group of the n -cube remain unknown»). Chandler, Sin and Xiang computed the Smith group of the adjacency matrix of Q_n and wrote about the critical group: «only the 2-Sylow subgroup of the critical group remains to be determined, for both odd and even n . We do not have any conjecture about its exact structure» [CSX17, §5.2]. Iga, Klivans, Kostiuk and Yuen observed that for Cayley graphs of

$\mathbb{F}_2^r$ «the 2-Sylow subgroup of their critical groups, or even the just [sic] 2-rank of the Laplacians, has been rather difficult to understand» [IKKY23, §7]. Gao, Marx-Kuo, McDonald and Yuen determined the largest cyclic factor and the next $n-2$ [GMMY24, Theorems 4.1, 4.2], conjectured the n -th [GMMY24, Conjecture 4.14] and a doubling relation between the cubes of dimension $2^k - 1$ and 2^k [GMMY24, Conjecture 5.4], and wrote that «determining the complete structure still seems out of reach at this moment». The $(n+1)$ -th factor was conjectured on the 2019 poster of Marx-Kuo, Gao and McDonald, which says that «the Sylow-2 subgroup remains a mystery» [MGM19].

This paper determines the whole 2-part, for every n .

1.2 The main theorem

We write $\text{Syl}_2 A$ for the 2-primary part of a finite abelian group A , and we describe 2-groups by their exponents: $\{e_1 : c_1, e_2 : c_2, \dots\}$ means $\oplus (\mathbb{Z}/2^{e_j})^{c_j}$. As usual, $\mathbb{Z}_2$ is the ring of 2-adic integers, $s_2(x)$ the number of ones in the binary expansion of $x \geq 0$, v_2 the 2-adic valuation, and

- $\kappa(a, b) := s_2(a) + s_2(b) - s_2(a + b)$, the number of carries in the binary addition $a + b$ ($a, b \geq 0$); by Kummer's theorem $\kappa(a, b) = v_2 C(a + b, a)$;
- $\varphi(x) := x + v_2(x)$ for $x \geq 1$.

Families. For an integer $\lambda \geq 0$ write

$$\lambda + 1 = 2^k + \sum_{t \in I} 2^t, \quad I \subseteq \{0, 1, \dots, k-1\},$$

so k is the leading binary digit of $\lambda + 1$ and I the set of the others. For $t \in I$ put

$$\text{next}(t) := \min\{u \in I \cup \{k\} : u > t\} \quad \text{and} \quad h_t := \text{next}(t) - t.$$

The subsets $D \subseteq I$ are the **dancers** of the family; for a dancer put $o_D := \sum_{t \in D} 2^t$ and $h(D) := \sum_{t \in D} h_t$. The **o-order** lists the dancers by increasing o_D , from $\emptyset$ to I .

Multiplicities. Let $m_\lambda(n)$ be the integers defined by

$$\sum_{\lambda} m_\lambda(0)[\lambda] = [0], \quad \sum_{\lambda} m_\lambda(n)[\lambda] = V \cdot \sum_{\lambda} m_\lambda(n-1)[\lambda],$$

where $V \cdot$ is the additive operator on formal sums of symbols $[\lambda]$ given by

$$V \cdot [0] = [1], \quad V \cdot [2l+1] = [2l+2], \quad V \cdot [2l+2] = 2[2l+1] + \Phi(V \cdot [l]), \\ \Phi([\nu]) := [2\nu+1].$$

(The recursion is well founded: $V \cdot [2l+2]$ calls $V \cdot [l]$ with $l < 2l+2$.) By Theorem 3.6, $m_\lambda(n)$ is the multiplicity of the indecomposable tilting module $T(\lambda)$ of SL_2 in characteristic 2 in the tensor power $V^{\otimes n}$ of the natural module. Only $\lambda \equiv n \pmod{2}$ with $1 \leq \lambda \leq n$ occur for $n \geq 1$, and $m_n(n) = 1$.

Clocks. Fix a family λ and a shift $i \geq 0$. For a dancer D with $(i, D) \neq (0, \emptyset)$ put $\xi_D := \lambda + 1 - 2o_D$ (a positive integer) and

$$u_i(D) := \varphi(\xi_D) + \kappa(i + o_D - 1, \xi_D) + h(D),$$

and put $u_0(\emptyset) := \infty$. These are the **raw clocks** of the family at shift i .

Pooling. For a finite sequence of integers $x_0, \dots, x_{N-1}$, its **integer antitonic fit** is obtained as follows. Partition the positions into consecutive blocks by the pool-adjacent-violators algorithm, so that the block means are strictly decreasing (adjacent blocks with equal means are merged) and each block value is the mean of its entries: this is the antitonic least-squares fit [ABERS55], [BBBB72]. Then replace a block of p positions with sum S by $S \bmod p$ copies of $\lceil S/p \rceil$ followed by $p - (S \bmod p)$ copies of $\lfloor S/p \rfloor$. In the cube the rounding never acts: every block mean of the raw clocks is an integer (Lemma 7.9, with Proposition 5.3, Lemma 7.5 and Proposition 5.4; see §8), so $\hat{y}_i$ below is the antitonic least-squares fit itself, a non-increasing sequence of integers. The rounding is kept in the definition only so that the rule is defined for every sequence of integers. At shift 0 the entry ∞ stands first and is not pooled with anything.

The group of a family. Let $\hat{y}_i(D)$ be the integer antitonic fit of the raw clocks $u_i(D)$, read in o-order. Put

$$X(\lambda, i) := \bigoplus_{a=1}^{k-1} (\mathbb{Z}/2^a)^{2^{|I|+k-1-a}} \oplus \bigoplus_{D \subseteq I} \mathbb{Z}/2^{\hat{y}_i(D)},$$

with $\mathbb{Z}/2^\infty := \mathbb{Z}_2$. The first sum is the family's **small clocks**, the second its $2^{|I|}$ **big clocks**.

Main Theorem (the Dancing Sand Theorem). For every $n \geq 1$, as $\mathbb{Z}_2$ -modules,

$$\text{Syl}_2 K(Q_n) \oplus \mathbb{Z}_2 \cong \bigoplus_{1 \leq \lambda \leq n, \lambda \equiv n \pmod{2}} X(\lambda, (n - \lambda)/2)^{m_\lambda(n)}.$$

The free summand is the big clock $\hat{y}_0(\emptyset) = \infty$ of the family $\lambda = n$.

The right-hand side is a finite computation with binomial valuations; no Smith form is needed ($n = 1024$ takes seconds). The number of cyclic factors of each copy of $X(\lambda, i)$ is $2^{|I|+k-1} = \dim T(\lambda)/2$.

Example. For $n = 6$ the families are $\lambda = 6, 4, 2$ with $m_6(6) = 1$, $m_4(6) = 4$, $m_2(6) = 4$.

- $\lambda = 6$ ($7 = 4 + 2 + 1$, $k = 2$, $I = \{0, 1\}$, $h_0 = h_1 = 1$), shift 0: small clocks $\{1 : 4\}$; raw clocks of $\emptyset, \{0\}, \{1\}, \{0, 1\}$: $\infty, 6, 6, 3$ (for $\{0\}$: $\xi = 5$ and $\varphi(5) + \kappa(0, 5) + h_0 = 5 + 0 + 1$; for $\{1\}$: $\xi = 3$ and $\varphi(3) + \kappa(1, 3) + h_1 = 3 + 2 + 1$); no pooling.
- $\lambda = 4$ ($5 = 4 + 1$, $k = 2$, $I = \{0\}$, $h_0 = 2$), shift 1: small clocks $\{1 : 2\}$; raw clocks $\varphi(5) + \kappa(0, 5) = 5$ and $\varphi(3) + \kappa(1, 3) + 2 = 3 + 2 + 2 = 7$. The sequence 5, 7 increases, so it is pooled: 6, 6.
- $\lambda = 2$ ($3 = 2 + 1$, $k = 1$, $I = \{0\}$, $h_0 = 1$), shift 2: no small clocks; raw clocks $\varphi(3) + \kappa(1, 3) = 5$ and $\varphi(1) + \kappa(2, 1) + 1 = 2$.

So $\text{Syl}_2 K(Q_6) = \{1 : 4\} \oplus \{6 : 2, 3 : 1\} \oplus 4 \cdot \{1 : 2, 6 : 2\} \oplus 4 \cdot \{5 : 1, 2 : 1\} = \{1 : 12, 2 : 4, 3 : 1, 5 : 4, 6 : 10\}$, as the direct computation gives. This is the smallest cube in which pooling acts.

An equivalent form. For $i \geq 1$, with $m := i - 1$ and $K := 2^k$, define for $\mu \geq 0$

$$R_\mu(D) := \sum_{t \in D} (h_t - 2^t) - \kappa(\mu, o_D), \quad x_D := R_m(D) - R_{m+K}(I \setminus D).$$

Then $u_i(D) = k + K + \kappa(m, K) + x_D$ for every dancer (Proposition 5.3), so the big clocks are $k + K + \kappa(m, K)$ plus the integer antitonic fit of x . This is the form in which the proof computes them.

1.3 The fold law: dry and wet

Let $\mathbf{1} = (1, \dots, 1)$. The **folded n -cube** is the quotient graph $Q_n/\langle \mathbf{1} \rangle$: its vertices are the 2^{n-1} pairs $\{x, x + \mathbf{1}\}$ of antipodal words, and $\{x, x + \mathbf{1}\}$ is joined to $\{x + e_i, x + \mathbf{1} + e_i\}$ once for each $i = 1, \dots, n$, where e_i is the i -th unit vector; for $n \geq 2$ each edge of $Q_n/\langle \mathbf{1} \rangle$ is the image of two edges of Q_n (at $n = 1$ the folded cube is one vertex with one loop, the image of the one edge of Q_1). It is the Cayley graph of $\mathbb{F}_2^n/\langle \mathbf{1} \rangle$ with the images of the unit vectors; for $n \geq 3$ it is a simple n -regular graph. We write $\bar{K}(n)$ for its sandpile group.

A naming trap. In the literature on interconnection networks the «folded hypercube FQ_m » is Q_m with the antipodal matching added: 2^m vertices, degree $m + 1$ [EL91]. It is the Cayley graph of $\mathbb{F}_2^m$ with the unit vectors and $\mathbf{1}$, which is our folded cube in dimension $m + 1$: **our folded n -cube $Q_n/\langle \mathbf{1} \rangle$ is FQ_{n-1}** . We keep the index n of the cube that is folded.

Theorem DW (the fold law). For every $n \geq 1$,

$$\text{Syl}_2 \bar{K}(n) \cong 2 \cdot \text{Syl}_2 K(Q_n),$$

where $2 \cdot A$ is the subgroup of doubles. Equivalently,

$$\begin{aligned} \text{Syl}_2 K(Q_n) &\cong (\mathbb{Z}/2)^{a_n} \oplus (\text{Syl}_2 \bar{K}(n) \text{ with every exponent raised by one}), \\ a_n &= 2^{n-2} - 2^{\lfloor (n-2)/2 \rfloor}. \end{aligned}$$

Here a_n is the number of factors $\mathbb{Z}/2$ of $K(Q_n)$, by Bai [Bai03, Theorem 1.3] (re-proved in §9.1). Theorem DW is proved family by family (§10): the folded cube splits over the same tilting families with the same multiplicities, and the group of each family is halved.

The names «dry and wet» and «dance» are explained in §1.7.

Example. For $n = 5$, $\text{Syl}_2 K(Q_5) = \{1 : 6, 3 : 4, 4 : 1, 6 : 4\}$ and $a_5 = 6$, so $\text{Syl}_2 \bar{K}(5) = \{2 : 4, 3 : 1, 5 : 4\}$.

Corollary 1.4 (the sandpile group of the folded cube). For every $n \geq 2$, $\text{Syl}_2 \bar{K}(n)$ is the group of the Main Theorem with the a_n factors $\mathbb{Z}/2$ removed and every other exponent lowered by one. For every odd prime p , $\text{Syl}_p \bar{K}(n) \cong \text{Syl}_p \bigoplus_{j \text{ even}, 2 \leq j \leq n} (\mathbb{Z}/2j)^{C(n,j)}$.

The odd part is [GMMY24, Proposition 1.9]: the Laplacian eigenvalues of the folded cube are $2|J|$ for the subsets $J \subseteq \{1, \dots, n\}$ of even size. We found no earlier computation of the sandpile group of the folded cube; its order, the number of spanning trees, is the sequence OEIS A193134 [OEIS] (for $n \geq 3$; at $n = 2$ that sequence counts the simple graph K_2 , while our $Q_2/\langle \mathbf{1} \rangle$ has a double edge).

The mod-2 layer. Theorem DW explains a coincidence noted by Gao et al. [GMMY24, Remark 2.13]: the $\mathbb{F}_2$ -rank of the Laplacian of the folded cube $Q_n/\langle \mathbf{1} \rangle$ and the number of factors $\mathbb{Z}/2$ of $K(Q_n)$ are both a_n («We are not sure if that is a coincident, or a special case of some deeper connection»). Under the fold law every factor of $\text{Syl}_2 K(Q_n)$ other than the factors

$\mathbb{Z}/2$ keeps a non-trivial image, so the number of even invariant factors of the folded Laplacian is $(2^{n-1} - 1 - a_n) + 1$, and its $\mathbb{F}_2$ -rank is 2^{n-1} minus that number, which is a_n (§10.8). The fold law is specific to the antipodal involution: the quotient by x_1x_2 does not halve the group (we checked $3 \leq n \leq 12$, §12).

1.4 Consequences

Corollary 1.5 (Bai [Bai03, Theorems 1.1, 1.3]). $\text{Syl}_2 K(Q_n)$ has exactly $2^{n-1} - 1$ cyclic factors, and exactly a_n of them are $\mathbb{Z}/2$ ($n \geq 2$).

Bai states the first count for the invariant factors of $K(Q_n)$. They are all even [Bai03, Lemma 2.2, Corollary 2.3]: in his proof the Smith form of the Laplacian of Q_n has 2^{n-1} entries 1, and the others are those of a matrix with even entries. So his count is the number of cyclic factors of $\text{Syl}_2 K(Q_n)$.

Write $c_1(Q_n) \geq c_2(Q_n) \geq \dots$ for the exponents of the cyclic factors of $\text{Syl}_2 K(Q_n)$, in decreasing order, and $g(N) := \max_{1 \leq x < N} \varphi(x) = \max_{1 \leq x < N} (x + v_2x)$. (Gao et al. write $c_i(Q_n)$ for the i -th largest invariant factor of $K(Q_n)$ and state their results for its 2-adic valuation, which is our $c_i(Q_n)$.)

Corollary 1.6 (the podium). Put $G := g(n-1)$. For every $n \geq 4$:

- n even: $c_1 = \max(G, \varphi(n) - 1)$ and $c_2 = \dots = c_{n+1} = G$;
- n odd and $\varphi(n-1) \leq G$: $c_1 = \dots = c_{n+1} = G$;
- n odd and $\varphi(n-1) > G$: $c_1 = \dots = c_{n-1} = \varphi(n-1)$, $c_n = \max(G, \varphi(n-1) - 2)$, $c_{n+1} = G$.

In particular:

- $c_1 = \max(g(n), v_2n + n - 1)$ and $c_2 = \dots = c_{n-1} = g(n)$ for $n \geq 3$ [GMMY24, Theorems 4.1, 4.2];
- $c_n = \max(g(n-1), v_2(n-1) + n - 3)$ for $n \geq 3$ [GMMY24, Conjecture 4.14];
- $c_{n+1} = g(n-1)$ for $n \geq 4$ [MGM19].

Corollary 1.7 ([GMMY24, Conjecture 5.4]). For every $e \geq 1$,

$$\text{Syl}_2 K(Q_{2^e}) \cong \text{Syl}_2 K(Q_{2^e-1})^2 \oplus \mathbb{Z}/2^{2^e+e-1}.$$

The proof of Corollary 1.7 gives more: each tilting family of the cube of dimension $2^e - 1$ becomes one family of the cube of dimension 2^e with the same multiplicity, and each of its dancers becomes two dancers with the same raw clock (§9.3).

1.5 The idea of the proof

1. **The group ring as an SL_2 -module (§2).** In the group ring $\mathbb{Z}[(\mathbb{Z}/2)^n]$ the Laplacian is multiplication by $\sigma = \sum_i (1 - x_i)$. In the basis $s_S = \prod_{i \in S} (1 - x_i)$ it becomes $U + 2D$, with U the raising operator of the Boolean lattice and D the rank (up to the sign $(-1)^{|S|}$, this basis is the change of variables $u_i = x_i - 1$ in the proof of [GMMY24, Proposition 2.5], and it is the Laplacian analogue of [CSX17, §5]). Identifying s_S with a pure tensor of $V^{\otimes n}$, $V = \mathbb{Z}^2$ the natural module of SL_2 , gives $\sigma = F + n - H$, an element of the hyperalgebra of SL_2 over $\mathbb{Z}$.

2. **The family decomposition (§3).** Over $\mathbb{Z}_2$, $V^{\otimes n}$ is a direct sum of explicit lattices $T(\lambda)$, built from $\mathbb{Z}$ by two operations: a lattice form Φ of the Frobenius twist ($T(2l+1) = \Phi(T(l))$) and the tensor product with V ($T(2l+2) = V \otimes \Phi(T(l))$). The proof needs only an elementary lifting lemma for SL_2 . Since σ is carried by every isomorphism, the group splits family by family.
3. **The dance (§4).** On $\Phi(N)$ half of the relations have unit pivots; eliminating them leaves twice the same kind of operator on N , with the payment of each floor replaced by a product of two consecutive payments. On $V \otimes \Phi(N)$ the same happens with two coupled «dancers». After k moves $X(\lambda, i)$ is the small clocks plus $\mathrm{coker}(2^k M(\lambda, i))$ (the Smith exponents of M raised by k), M an explicit triangular matrix indexed by the dancers, whose entries are products of payments over windows times the coefficients of an explicit polynomial (Theorem D). Its entries have valuations given by carries (§5).
4. **The Smith form of M (§6–§8).** The r -th determinantal divisor d_r of M is bounded below by the cheapest r -matching in the support of M , valued by v_2 (the tropical bound). It is at most the valuation of one natural minor, which has a unique cheapest matching (Theorem O); that valuation is a partial sum of the raw clocks, and convexity turns it into the partial sum of the pooled clocks (Lemma 6.4). Theorem F shows that every r -matching costs at least the partial sums of the pooled clocks, by a potential built digit by digit from the bottom of $\lambda + 1$. The two bounds meet.
5. **The fold (§10).** On each family the antipode acts as $\pm \exp(F)$. The folded group is the cokernel of an operator that differs from the doubled one by a perturbation $\sum \zeta_m F^{(m)}$ of higher divided powers. The same dance applies to both. Every move stays clean because the coefficients of every level lie in a ring $\mathcal{K}$ of 2-adic functions of twice the position, on which a shift-invariant reduction makes every Schur complement vanish modulo 2 (Theorem 10.15). The perturbation then changes no valuation of the final matrix (Theorem 10.12). The even families are glued pairs of odd ones, and the glue survives the dance.

1.6 Notation and conventions

symbol	meaning	first use
$Q_n, K(Q_n), \bar{K}(n)$	the n -cube, its sandpile group, that of the folded cube $Q_n/\langle 1 \rangle$	§1.1, §1.3
$s_2, v_2, \kappa(a, b), \varphi(x)$	binary digit sum, 2-adic valuation, carries of $a + b, x + v_2 x$	§1.2
λ, k, K, I	a family: $\lambda + 1 = 2^k + \sum_{t \in I} 2^t, K = 2^k$	§1.2
$D, o_D, h_t, h(D)$	dancers $D \subseteq I, \sum_{t \in D} 2^t$, gaps $\mathrm{next}(t) - t, \sum_{t \in D} h_t$	§1.2
i, m	the shift $(n - \lambda)/2$, and $m = i - 1$ when $i \geq 1$	§1.2
$m_\lambda(n)$	multiplicity of the family λ in $V^{\otimes n}$	§1.2, §3
$u_i(D), \hat{y}_i(D), X(\lambda, i)$	raw clocks, pooled clocks, the group of a family	§1.2
$R = \mathbb{Z}[(\mathbb{Z}/2)^n], s_S, \sigma$	the group ring, its basis $\prod_{i \in S} (1 - x_i)$, the Laplacian element	§2
$V, b_0, b_1, E, F, H, F^{(r)}$	the natural module of SL_2 , its basis, the hyperalgebra generators	§2
$A = \mathbb{Z}_2$	the base ring	§2.2
$\Delta_A(\lambda), \nabla_A(\lambda)$	Weyl and dual Weyl lattices	§2.3
$\Phi(N), T(\lambda)$	the doubling functor, the dance lattices	§3
D (operator), $\pi(d), \sigma_i^{(\alpha)}$	the floor operator, a payment, $F + 2^\alpha(D + i)$	§2.2, §3.4, §4

symbol	meaning	first use
$M(\lambda, i), q_k, a_k(\Delta), \delta(\Delta)$	the final matrix, the coefficient polynomial, its coefficients and their valuations	§4.4, §4.5
$R_\mu(D), \gamma(\Delta), x_D, \psi(E)$	rulers, gold, clocks of a cell, $\sum_{t \in E} (h_t - 2^t)$	§1.2, §5
$d_r, T(r), U(r), Y(r)$	determinantal divisors, cheapest r -matching, natural and pooled partial sums	§6
$a = x_1 \cdots x_n, \tau_i, \Psi_j$	the antipode, the family operator, the folded odd-family operator	§10
$\mathcal{K}, \Theta, \mathcal{T}_h$	the ring of functions $\vartheta(y) = a + 2\chi(2y)$, its reduction, the covariant systems of level h	§10.5

A few letters have two meanings, each clear from its argument: D is a dancer (a subset of I) and the floor operator of §4; K is 2^k and the sandpile group $K(\cdot)$; $T(\lambda)$ is a dance lattice and $T(r)$ a cheapest-matching value; τ_t is a datum of a window (§7.3) and τ_i the family operator of §10; and «window» is a range of floors in §4.4 and a range of binary digits in §7.3. Groups are written additively. «Lattice» means a finitely generated free $\mathbb{Z}_2$ -module, and for a square matrix M over $\mathbb{Z}_2$ with non-zero determinant, coker M and its Smith form are taken over $\mathbb{Z}_2$.

1.7 Vocabulary, and why these names

Some words in this paper are ours. Each is defined where it first matters; this table gives its standard meaning.

our word	standard meaning	where
family λ ; shift i	the summand $T(\lambda)$ of $V^{\otimes n}$ (Theorem 3.6); the integer $i = (n - \lambda)/2$, so that the Laplacian acts on $T(\lambda)$ as $F + \lambda - H + 2i$	§1.2, §3.4
dance lattice $T(\lambda)$	a $\mathbb{Z}_2$ -lattice whose reduction modulo 2 is the indecomposable tilting module of SL_2 of highest weight λ in characteristic 2 (Remark (1) after Theorem 3.6)	§3.3
dancers; o-order	the subsets $D \subseteq I$, which index the slots after the dance; their order by $o_D = \sum_{t \in D} 2^t$	§1.2, §4.3
floor	the grading by $(c - w)/2$ on the weight w ; F raises it by one	§2.2
payment; payment system	a diagonal operator given by a function of the floor; the operator F plus payments on copies of a lattice, with lower-triangular couplings between the copies	§4.1
twist move; split move	one step of block Gaussian elimination: eliminate the unit pivots, take the Schur complement, divide by 2. The split move first separates the two basis vectors of V , so that each dancer becomes two	§4.2, §10.3
the dance	the k moves, one for each binary digit of $\lambda + 1$ below the leading one	§4.3
final matrix; window	the triangular matrix $M(\lambda, i)$ of size $2^{ I }$ left by the dance; the range of floors whose payments are multiplied in one of its entries	§4.4
clock	a cyclic factor $\mathbb{Z}/2^e$, named by its exponent e ; the small clocks come from the eliminated pivots, the big clocks from the final matrix	§1.2, §4.3
raw clocks; pooling	explicit integers $u_i(D)$; their antitonic least-squares fit (isotonic regression, by pool-adjacent-violators), which in the cube is always an integer sequence (Lemma 7.9)	§1.2, §7.1
cell	the data (λ, i, α) of one final matrix	§5, §6.1
ruler; gold	the row and the column term of the valuation of an entry of the final matrix; the remaining term γ , which vanishes exactly on the golden arrows	§5, §6.2

our word	standard meaning	where
golden	the unique bijection β_r all of whose arrows are golden; its term is the	§6.2
matching	unique term of least valuation in the expansion of the natural minor	
mirror	the map $D \mapsto I \setminus D$, which reverses the o-order	§6.1, §7.1
ceiling; floor (of the Smith form)	the upper and the lower bound for the determinantal divisors of the final matrix (Theorems O and F); this «floor» is not the grading	§6, §7
house; penalties	the binary data (I, k, ℓ, α) of a cell, freed from λ and i ; the two correction terms of a real cell	§7.3
windows W_t ; gates	the intervals of binary digits $[t, \text{next}(t))$; the rule by which each of them lets a carry through (Lemma 7.6)	§7.3
podium	the $n + 1$ largest cyclic factors of $\text{Syl}_2 K(Q_n)$	§1.4, §9.2
the fold	the quotient by the antipode $\mathbb{1}$, which gives the folded cube	§1.3, §10
dry; wet	reduction modulo 2; the lattice over $\mathbb{Z}_2$	below
arms; glue; bend	the two odd-family operators that make an even family; the triangular block form $\text{glue}(X, Y)$ that joins them; the estimate by which the two sides of the fold law differ by a unit at the end of the dance (Lemma 10.20)	§10.2, §10.7
formal dance; positions; covariant system	the dance of operators with divided powers $F^{(m)}$; the numbers $2^h f + o_D + c$; a system whose coefficients depend only on $\Delta = D \setminus D'$, on m and on the position of the target	§10.3, §10.5
turned lid	a symmetry that we measured and did not prove	§11

Why «dance». Read the binary word of $\lambda + 1$ from the bottom. At a digit 0 every dancer makes the same step, the twist move. At a digit 1 every dancer splits into two partners, the split move. After k digits there are $2^{|I|}$ dancers, the subsets of I . At the start each floor d pays the even amount $2^\alpha(i + d)$; each move gives every new floor a payment equal to minus half the product of the payments of two adjacent old floors, so every payment stays even; at the end the final matrix records who owes what to whom. The sand is the sandpile; the dance is the elimination that brings it to rest.

Why «dry» and «wet». Modulo 2 — dry — the module $T(2l + 1)$ becomes $V \otimes T(l)^{[1]}$, the tensor product of V with a Frobenius twist (Proposition 3.1(c)), and the cokernel of the Laplacian modulo 2 sees only the number of cyclic factors of the 2-part. Over $\mathbb{Z}_2$ — wet — the twist has an integral form, the lattice $\Phi(T(l))$ of §3.1, on which F^2 acts as $2F$ of $T(l)$, and every power of 2 survives. The fold dries the cube by exactly one layer: the factors $\mathbb{Z}/2$ disappear, and every other exponent drops by one (Theorem DW).

1.8 Literature and priority

What is ours, to the best of our knowledge, after a search of the literature up to October 2026 (arXiv and journal searches, the citation trees of [Bai03], [CSX17] and [GMMY24] in Google Scholar and Semantic Scholar, and the sources listed below; MathSciNet was not searched):

1. the whole 2-part of $K(Q_n)$, for every n (the Main Theorem);
2. the decomposition of a sandpile group over tilting modules, with the reduction of each piece to a small explicit matrix (Theorems 3.6 and D) — we found no tilting decomposition of any sandpile group;

3. proofs of [GMMY24, Conjectures 4.14 and 5.4] and of the $(n + 1)$ -th factor of [MGM19] — we found no other proof;
4. the fold law and the sandpile group of the folded cube (Theorem DW, Corollary 1.4);
5. the proof techniques of §4–§7 and §10 as combinations: the dance, the cost form with carries, Theorems O, U and F, the ring $\mathcal{K}$ with Theorem 10.15, and Theorem 10.12. Each ingredient we did not invent is credited where it is used.

What is not ours. Bai determined the odd part and proved Reiner’s two counts [Bai03]. Gao, Marx-Kuo, McDonald and Yuen determined the $n - 1$ largest factors and posed the conjectures we prove; their Proposition 1.9 gives the odd part of every Cayley graph of $\mathbb{F}_2^n$, their Proposition 2.12 the $\mathbb{F}_2$ -rank of the Laplacian of the folded cube (§10.8), their Remark 4.4 contains Lemma 9.2(b), their Table 1 reproduces Bai’s table of $\text{Syl}_2 K(Q_n)$ for $n \leq 11$, and the monomials of their change of variables $u_i = x_i - 1$ (proof of their Proposition 2.5) are, up to the sign $(-1)^{|S|}$, the basis we use [GMMY24]. Chandler, Sin and Xiang computed the Smith group of the adjacency matrix with the same monomial basis and Bier’s vectors [CSX17], [Bie93], [Wil90]. Reiner and Tseng gave the exact sequence relating the critical groups of a double cover and of its base [RT14]; we use none of it, but the fold law is a statement about such a cover. The multiplicities of tilting modules of SL_2 in tensor powers of V in characteristic 2 are known: they follow from Donkin’s tensor product theorem [Don93] and the characters of [TW21], and Larsen gives the fusion rule of $V^{\otimes 2}$ at $p = 2$ [Lar24, §2], which reproduces our recursion (§12.2). The Weyl and dual Weyl filtrations, and the lifting of tilting decompositions over complete rings, are standard [Jan03, Don93]; we give the short elementary proofs needed for SL_2 over $\mathbb{Z}_2$ (§3.2), because the general references are written over fields or for other purposes. Kostant’s commutation formula in the $\mathbb{Z}$ -form of $U(\mathfrak{sl}_2)$ is [Kos66], [Hum72, §26]. Kummer’s theorem [Kum52], Legendre’s formula $v_2(m!) = m - s_2(m)$, the von Staudt–Clausen theorem [vSt40], [Cla40], Strassmann’s theorem [Str28], and the pool-adjacent-violators algorithm [ABERS55] are classical. Related representation-theoretic computations of Smith and critical groups are those of Sin’s school [CSX17], [DHS18] and the subset-intersection diagonal forms of [DEGJPP23].

Contemporary context. Vetluzhskikh and Zakharov compute the order of the kernel of the pushforward between critical groups of abelian covers [VZ24], and Len and Zakharov the order of the Prym group of a double cover [LZ22]; both give orders, not group structure. Akyar et al. recall that the exact sequence of Reiner and Tseng for a graph cover splits on primary components away from the primes dividing the covering degree, which makes the prime 2 «a natural place to look for additional structure under an involution» [Aky26, §4]. Their reflections of odd grids have fixed vertices, so their fold is not a regular double cover, and they ask: «Can an analogous integral description with fixed vertices explain both the folded all-ones class and the remaining 2-primary factors?» The antipodal fold of the cube has no fixed vertex: it is a regular double cover, and there the fold law describes the 2-part.

If a result here was obtained earlier, we would be grateful to be told, and we will correct the record.

2. The group ring as a module over the hyperalgebra of SL_2

2.1 The Laplacian in the group ring

Let $G = (\mathbb{Z}/2)^n$ with generators $x_1, \dots, x_n$, and $R := \mathbb{Z}[G]$. The Laplacian of the Cayley graph Q_n of G acts on $\mathbb{Z}^G = R$ as multiplication by

$$\sigma := \sum_{i=1}^n (1 - x_i).$$

Since Q_n is connected, $\text{coker}(\sigma \text{ on } R) \cong \mathbb{Z} \oplus K(Q_n)$ [Lor91], [Big99]. Put $s_i := 1 - x_i$ and, for $S \subseteq \{1, \dots, n\}$, $s_S := \prod_{i \in S} s_i$. The elements s_S form a $\mathbb{Z}$ -basis of R (the change of basis from $\{x_S\}$ is unitriangular for inclusion).

Lemma 2.1. In R ,

$$s_j s_S = s_{S \cup \{j\}} \quad \text{if } j \notin S, \quad s_j s_S = 2s_S \quad \text{if } j \in S.$$

Hence $\sigma s_S = \sum_{j \notin S} s_{S \cup \{j\}} + 2|S| \cdot s_S$, that is, $\sigma = U + 2D$, where $U s_S = \sum_{j \notin S} s_{S \cup \{j\}}$ and $D s_S = |S| s_S$.

Proof. $s_j^2 = (1 - x_j)^2 = 2 - 2x_j = 2s_j$. ■

Up to the sign $(-1)^{|S|}$, these are the monomials of the change of variables $u_i = x_i - 1$ used by Gao et al. [GMMY24, proof of Proposition 2.5], and the basis is the Laplacian analogue of the monomial basis in which Chandler, Sin and Xiang write the adjacency matrix [CSX17, §5, (5)]. Since $(\mathbb{Z} \oplus K) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \text{Syl}_2 K$ for a finite abelian group K , and $\otimes \mathbb{Z}_2$ is right exact, we have $\mathbb{Z}_2 \oplus \text{Syl}_2 K(Q_n) \cong \text{coker}(\sigma \text{ on } R \otimes \mathbb{Z}_2)$.

2.2 The hyperalgebra and the dictionary

Let $U_{\mathbb{Q}}$ be the universal enveloping algebra of $\mathfrak{sl}_2$ over $\mathbb{Q}$, with generators E, F, H , $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$. Its Kostant $\mathbb{Z}$ -form $\text{Dist}_{\mathbb{Z}}$ is the subring generated by the divided powers $E^{(r)} = E^r/r!$, $F^{(r)} = F^r/r!$ and the elements $C(H, r) = H(H-1) \cdots (H-r+1)/r!$ [Kos66], [Hum72, §26]; it is the distribution algebra of SL_2 over $\mathbb{Z}$. Kostant's formula

$$E^{(s)} F^{(r)} = \sum_{j=0}^{\min(r,s)} F^{(r-j)} C(H-r-s+2j, j) E^{(s-j)} \quad (\text{K})$$

holds in $U_{\mathbb{Q}}$. Put $A := \mathbb{Z}_2$ and $\text{Dist}_A := A \otimes \text{Dist}_{\mathbb{Z}}$, a Hopf algebra with $\Delta(F^{(r)}) = \sum_{a+b=r} F^{(a)} \otimes F^{(b)}$, the same for E , and $\Delta(H) = H \otimes 1 + 1 \otimes H$.

A **lattice** is a Dist_A -module that is free of finite rank over A , on which H acts diagonally with integer eigenvalues (the **weights**): the module is the direct sum of its weight spaces. A lattice X has the **floor** grading for any integer $c \geq$ every weight of X with $c \equiv$ the weights (mod 2): the floor of a weight vector of weight w is $(c - w)/2$. We write D for the floor operator (the floor relative to the top weight, unless said otherwise); F raises the floor by one.

Let $V := Ab_0 \oplus Ab_1$ with $Hb_0 = b_0$, $Hb_1 = -b_1$, $Fb_0 = b_1$, $Eb_1 = b_0$, $Fb_1 = Eb_0 = 0$, and $E^{(r)} = F^{(r)} = 0$ on V for $r \geq 2$: the natural module.

Lemma 2.2 (the dictionary). The A -linear isomorphism $R \otimes A \rightarrow V^{\otimes n}$ that sends s_S to the pure tensor with b_1 in the positions of S and b_0 elsewhere carries U to F , D to $(n - H)/2$, and therefore

$$\sigma = F + n - H.$$

Proof. On $V^{\otimes n}$, $F = \sum_j 1 \otimes \cdots \otimes F \otimes \cdots \otimes 1$ turns one b_0 into b_1 in every possible way, which is U ; and a pure tensor with $|S|$ letters b_1 has weight $n - 2|S|$. ■

Corollary 2.3. If $V^{\otimes n} \cong \bigoplus_j M_j$ as Dist_A -modules, then $\mathbb{Z}_2 \oplus \text{Syl}_2 K(Q_n) \cong \bigoplus_j \text{coker}(F + n - H \text{ on } M_j)$.

Proof. $\sigma = F + n - H$ is an element of Dist_A (it acts through F and H), so every isomorphism of Dist_A -modules carries σ to σ , and the cokernel of an endomorphism that preserves a direct sum decomposition is the sum of the cokernels. ■

2.3 Weyl lattices, duality, filtrations

For $\lambda \geq 0$ let $\Delta_{\mathbb{Q}}(\lambda)$ be the simple $U_{\mathbb{Q}}$ -module of highest weight λ with highest weight vector v , and $\Delta_A(\lambda) := \text{Dist}_A \cdot v$, the **Weyl lattice**. It has the basis $e_r := F^{(r)}v$ ($0 \leq r \leq \lambda$), with

$$Fe_r = (r+1)e_{r+1}, \quad Ee_r = (\lambda - r + 1)e_{r-1}, \quad He_r = (\lambda - 2r)e_r.$$

Let ι be the anti-involution of Dist_A with $\iota(E^{(r)}) = F^{(r)}$, $\iota(F^{(r)}) = E^{(r)}$, $\iota(H) = H$. For a lattice X the **contravariant dual** $X^\iota := \text{Hom}_A(X, A)$ with $(g \cdot \xi)(x) := \xi(\iota(g)x)$ is a lattice; ι is compatible with the coproduct, so $(X \otimes Y)^\iota \cong X^\iota \otimes Y^\iota$, and $V^\iota \cong V$ (the dual basis of b_0, b_1). The **dual Weyl lattice** is $\nabla_A(\lambda) := \Delta_A(\lambda)^\iota$; in the dual basis w_r of the e_r ,

$$Fw_r = (\lambda - r)w_{r+1}, \quad Ew_r = rw_{r-1}.$$

A **Δ -filtration** of a lattice X is a chain of sublattices $0 = X_0 \subset X_1 \subset \cdots \subset X_s = X$, each X/X_j free over A , with $X_j/X_{j-1} \cong \Delta_A(\lambda_j)$; a **∇ -filtration** is the same with ∇_A . The map ι exchanges them.

Lemma 2.4. For $m \geq 1$, $\Delta_A(m) \otimes V$ has a Δ -filtration with factors $\Delta_A(m+1)$ (a sublattice) and $\Delta_A(m-1)$ (the quotient), and $\nabla_A(m) \otimes V$ a ∇ -filtration with factors $\nabla_A(m-1)$, $\nabla_A(m+1)$. Moreover, $\Delta_A(0) \otimes V = V = \Delta_A(1) = \nabla_A(1)$.

Proof. Let v be the highest weight vector of $\Delta_A(m)$. The vectors $F^{(r)}(v \otimes b_0) = F^{(r)}v \otimes b_0 + F^{(r-1)}v \otimes b_1$ ($0 \leq r \leq m+1$) are primitive in the basis $\{F^{(r)}v \otimes b_e\}$, so they span a saturated sublattice X_1 ; it is $\text{Dist}_A \cdot (v \otimes b_0)$. Over $\mathbb{Q}$, $X_1 \otimes \mathbb{Q}$ is generated by a vector of weight $m+1$ killed by every $E^{(s)}$, $s \geq 1$, and has dimension $m+2$, so it is the simple module $\Delta_{\mathbb{Q}}(m+1)$, with $v \otimes b_0$ as highest weight vector; and Dist_A applied to the highest weight vector gives $\Delta_A(m+1)$ on one side and X_1 on the other. So $X_1 \cong \Delta_A(m+1)$. The quotient is free of rank m , spanned by the images of $F^{(s)}v \otimes b_1$ ($0 \leq s < m$): indeed $F^{(m)}v \otimes b_1 = F^{(m+1)}(v \otimes b_0) \in X_1$. The image v' of $v \otimes b_1$ is killed by E ($E(v \otimes b_1) = v \otimes b_0 \in X_1$) and by every $E^{(s)}$, $s \geq 2$, and $F^{(s)}(v \otimes b_1) = F^{(s)}v \otimes b_1$ (since $F^{(r)}b_1 = 0$ for $r \geq 1$); so the quotient is $\text{Dist}_A \cdot v' \cong \Delta_A(m-1)$. The statement for ∇ follows by the duality ι . ■

3. The dance lattices and the family decomposition

3.1 The doubling functor

Let N be a lattice. On the free A -module $\Phi(N) := b_0 \otimes N \oplus b_1 \otimes N$ (we write b_en for $b_e \otimes n$) define, for n of weight w ,

$$H(b_0n) = (2w+1)b_0n, \quad H(b_1n) = (2w-1)b_1n,$$

$$F(b_0n) = b_1n, \quad F(b_1n) = 2b_0(Fn),$$

$$E(b_0n) = 2b_1(En), \quad E(b_1n) = b_0(\Omega - H^2)n,$$

where $\Omega := 4FE + (H + 1)^2$ acts on N ; on a vector of weight w , $\Omega - H^2 = 4FE + 2w + 1$. For a morphism f of lattices put $\Phi(f) := 1 \otimes f$.

Proposition 3.1.

- (a) $\Phi(N)$ is a lattice, Φ is an exact functor, and it sends saturated sublattices to saturated sublattices.
- (b) On $\Phi(N)$, for $s \geq 0$ and $e = 0, 1$:

$$\begin{aligned} F^{(2s)}(b_en) &= b_e F^{(s)}n / (2s - 1)!!, \\ F^{(2s+1)}(b_0n) &= b_1 F^{(s)}n / (2s + 1)!!, \end{aligned}$$

$$F^{(2s+1)}(b_1n) = ((2s + 2)/(2s + 1)!!) \cdot b_0 F^{(s+1)}n.$$

- (c) $\Phi(N) \otimes \mathbb{F}_2 \cong V \otimes (N \otimes \mathbb{F}_2)^{[1]}$ as $\text{Dist}_{\mathbb{F}_2}$ -modules, where $M^{[1]}$ is the Frobenius twist ($F^{(2s)}$, $E^{(2s)}$ act as $F^{(s)}$, $E^{(s)}$, the odd divided powers act as 0, and H acts as $2H$).
- (d) $\Phi(\Delta_A(\mu)) \cong \Delta_A(2\mu + 1)$ and $\Phi(\nabla_A(\mu)) \cong \nabla_A(2\mu + 1)$; $\Phi(A) = V$.

Proof. (a) *The relations.* On b_0n (weight $2w + 1$): $EF(b_0n) = E(b_1n) = b_0(4FE + 2w + 1)n$ and $FE(b_0n) = F(2b_1En) = 4b_0FEn$, so $[E, F] = 2w + 1$. On b_1n (weight $2w - 1$): $EF(b_1n) = 4b_1EFn$ and $FE(b_1n) = b_1(4FE + 2w + 1)n$, so $[E, F] = 4(EF - FE) - 2w - 1 = 4w - 2w - 1 = 2w - 1$. The relations with H hold by weights. So $\Phi(N) \otimes \mathbb{Q}$ is a $U_{\mathbb{Q}}$ -module, and $\Phi(N)$ is stable under F and H . *Divided powers.* From the formulas, $F^{2s}(b_0n) = 2^s b_0 F^s n$, $F^{2s+1}(b_0n) = 2^s b_1 F^s n$, $F^{2s}(b_1n) = 2^s b_1 F^s n$, $F^{2s+1}(b_1n) = 2^{s+1} b_0 F^{s+1} n$; dividing by $(2s)! = 2^s s!(2s - 1)!!$ and $(2s + 1)! = 2^s s!(2s + 1)!!$ gives (b), whose denominators are odd. For E , Ω is central in $U_{\mathbb{Q}}$ and integral, and $Ef(H) = f(H - 2)E$; by induction, on n of weight w ,

$$\begin{aligned} E^{2s}(b_1n) &= 2^s \prod_{0 \leq j < s} (\Omega - (w + 2j)^2) \cdot b_1 E^s n, \\ E^{2s+1}(b_1n) &= 2^s \prod_{0 \leq j \leq s} (\Omega - (w + 2j)^2) \cdot b_0 E^s n, \\ E^{2s}(b_0n) &= 2^s \prod_{1 \leq j \leq s} (\Omega - (w + 2j)^2) \cdot b_0 E^s n, \\ E^{2s+1}(b_0n) &= 2^{s+1} \prod_{1 \leq j \leq s} (\Omega - (w + 2j)^2) \cdot b_1 E^{s+1} n; \end{aligned}$$

dividing by the factorials leaves $E^{(s)}$ or $E^{(s+1)}$ of N times odd denominators ($2(s + 1)/(2s + 1)!!$ for the last one). The $C(H, r)$ act by integers. So $\Phi(N)$ is stable under Dist_A , and it is a lattice. The map $\Phi(f) = 1 \otimes f$ commutes with all the formulas because they use only the action on N ; exactness and saturation are clear because $\Phi(N) = N^2$ as A -modules.

(c) On a vector of weight $w + 2s$, $\Omega \equiv (H + 1)^2 \pmod{4}$, so $\Omega - (w + 2j)^2 \equiv (w + 2s + 1)^2 - (w + 2j)^2 = (2s - 2j + 1)(2w + 2s + 2j + 1)$, an odd number, modulo 2. Hence modulo 2, $F^{(2s)}$ and $E^{(2s)}$ act as $1 \otimes F^{(s)}$, $1 \otimes E^{(s)}$; $F^{(2s+1)}(b_0n) = b_1 F^{(s)}n$, $F^{(2s+1)}(b_1n) = 0$, $E^{(2s+1)}(b_1n) = b_0 E^{(s)}n$, $E^{(2s+1)}(b_0n) = 0$. These are the formulas of $V \otimes M^{[1]}$ computed with the coproduct, since on

$M^{[1]}$ only the even divided powers act.

(d) Order the basis of $\Phi(\nabla_A(\mu))$ by weight: $b_0 w_s$ (index $2s$), $b_1 w_s$ (index $2s+1$). Then $F(b_0 w_s) = b_1 w_s$ and $F(b_1 w_s) = 2(\mu - s)b_0 w_{s+1}$, while in $\nabla_A(2\mu + 1)$, $Fw'_R = (2\mu + 1 - R)w'_{R+1}$, that is $2\mu + 1 - 2s$ for $R = 2s$ and $2(\mu - s)$ for $R = 2s + 1$. The diagonal map $w'_R \mapsto c_R \cdot (R - \text{th vector})$ with $c_{2s} = (2\mu + 1 - 2s)c_{2s+1}$, $c_{2s+1} = c_{2s+2}$, $c_0 = 1$, has odd ratios, so it is an isomorphism of A -modules; it commutes with F and H . Over $\mathbb{Q}$ both modules are simple: $\Phi(\nabla_A(\mu)) \otimes \mathbb{Q}$ has the character of $\Delta(2\mu + 1)$, and by Weyl's theorem on complete reducibility a finite-dimensional $U_{\mathbb{Q}}$ -module with the character of a simple module is simple; and on a simple module E is determined by F , H and $E(\text{top}) = 0$, through $EFx = FEx + Hx$; so the map commutes with E , hence with $\text{Dist}_A \otimes \mathbb{Q}$ and with Dist_A . The same argument with $Fe_r = (r + 1)e_{r+1}$ gives $\Phi(\Delta_A(\mu)) \cong \Delta_A(2\mu + 1)$ (ratios $(2s + 1)$ and 1). Finally, $\Phi(A) = V$ is (d) with $\mu = 0$. ■

3.2 Lifting from $\mathbb{F}_2$

Lemma 3.2 (Lemma W). Let X be a lattice with a sublattice N such that $X/N \cong \Delta_A(\lambda)$ and every weight of N lies in $[-\lambda, \lambda]$. Then the extension splits.

Proof. Lift the highest weight vector v of $\Delta_A(\lambda)$ to $x \in X$ of weight λ . Every weight of X lies in $[-\lambda, \lambda]$, so $E^{(s)}x = 0$ for $s \geq 1$ and $F^{(r)}x = 0$ for $r > \lambda$. Define the A -linear map $f : \Delta_A(\lambda) \rightarrow X$ by $f(e_r) := F^{(r)}x$. Then

$$F^{(t)}f(e_r) = C(r + t, t)F^{(r+t)}x = f(F^{(t)}e_r)$$

(both sides vanish when $r + t > \lambda$), and by (K)

$$E^{(s)}F^{(r)}x = C(\lambda - r + s, s)F^{(r-s)}x,$$

which is the same formula as on v ; so f is a Dist_A -map splitting the extension. ■

Lemma 3.3. For all $\lambda, \mu \geq 0$, $\text{Ext}^1(\Delta_A(\lambda), \nabla_A(\mu)) = 0$ (extensions of Dist_A -modules; an extension X of two lattices is a lattice: $X \otimes \mathbb{Q}$ is semisimple by Weyl's theorem, so H is diagonalizable on it with integer eigenvalues, and the projection onto the weight space of weight w is $P(H)$, where P is the polynomial of degree at most $w_1 - w_0$ that takes the value 1 at w and 0 at the other integers of $[w_0, w_1]$, w_0 and w_1 the least and the largest weight; by Newton's forward differences

$$P(H) = \sum_r (\Delta^r P)(w_0) \cdot C(H - w_0, r)$$

with integer coefficients $(\Delta^r P)(w_0)$, and the $C(H - w_0, r)$ lie in $\text{Dist}_{\mathbb{Z}}$; so X is the direct sum of its weight spaces). Consequently, if M has a Δ -filtration and N a ∇ -filtration, then $\text{Ext}^1(M, N) = 0$, and the reduction map $\text{Hom}_{\text{Dist}_A}(M, N) \rightarrow \text{Hom}_{\text{Dist}_{\mathbb{F}_2}}(M \otimes \mathbb{F}_2, N \otimes \mathbb{F}_2)$ is onto.

Proof. If $\mu \leq \lambda$, the weights of $\nabla_A(\mu)$ lie in $[-\mu, \mu] \subseteq [-\lambda, \lambda]$: Lemma 3.2. If $\mu > \lambda$, the duality ι identifies the group with $\text{Ext}^1(\Delta_A(\mu), \nabla_A(\lambda))$, which vanishes by the first case. The long exact sequences of Ext along the filtrations give $\text{Ext}^1(M, N) = 0$. Applying $\text{Hom}(M, -)$ to $0 \rightarrow N \rightarrow N \rightarrow N \otimes \mathbb{F}_2 \rightarrow 0$ (the first map is multiplication by 2) gives the exact sequence $\text{Hom}(M, N) \rightarrow \text{Hom}(M, N \otimes \mathbb{F}_2) \rightarrow \text{Ext}^1(M, N) = 0$, and

$$\text{Hom}_{\text{Dist}_A}(M, N \otimes \mathbb{F}_2) = \text{Hom}_{\text{Dist}_{\mathbb{F}_2}}(M \otimes \mathbb{F}_2, N \otimes \mathbb{F}_2). \quad \blacksquare$$

Corollary 3.4. Let X, Y be lattices that both have a Δ -filtration and a ∇ -filtration. If $X \otimes \mathbb{F}_2 \cong Y \otimes \mathbb{F}_2$, then $X \cong Y$.

Proof. Lift an isomorphism $\bar{f}$ and its inverse to $f : X \rightarrow Y$, $g : Y \rightarrow X$ (Lemma 3.3). Then $gf \equiv 1 \pmod{2}$ in the ring $\text{End}(X)$, which is a free A -module of finite rank, so $gf = 1 + 2h$ is invertible (A is complete); likewise fg . So f is an isomorphism. ■

3.3 The dance lattices

Define lattices $T(\lambda)$, $\lambda \geq 0$, by

$$T(0) := A, \quad T(2l+1) := \Phi(T(l)), \quad T(2l+2) := V \otimes \Phi(T(l)).$$

Reading $\lambda + 1$ in binary from the leading digit down, $T(\lambda)$ is obtained from A by k moves, one per binary digit below the leading one: a digit 0 is a Φ and a digit 1 is a $V \otimes \Phi$. If $\lambda + 1 = 2^k + \sum_{t \in I} 2^t$, the rank of $T(\lambda)$ is $2^{k+|I|}$; its weights are $\lambda, \lambda - 2, \dots, -\lambda$ (the top weight occurs once), and its character is

$$\text{ch } T(\lambda) = \prod_{t < k} (e^{2^t} + e^{-2^t}) \cdot \prod_{t \in I} (e^{2^t} + e^{-2^t}),$$

because

$$\text{ch } \Phi(N) = (e + e^{-1}) \cdot (\text{ch } N)(e^2) \quad \text{and} \quad \text{ch}(V \otimes M) = (e + e^{-1}) \text{ch } M.$$

By Lemma 2.4 and Proposition 3.1(d), every $T(\lambda)$ and every $V^{\otimes n} \otimes A$ has a Δ -filtration and a ∇ -filtration (both operations preserve each kind of filtration).

Lemma 3.5. Over $\mathbb{F}_2$, $V^{\otimes 3} \cong \Delta(3) \oplus V^2$ and $\Delta(3) \cong V \otimes V^{[1]}$. Consequently, for every lattice N ,

$$(V \otimes V \otimes \Phi(N)) \otimes \mathbb{F}_2 \cong (\Phi(N)^2 \oplus \Phi(V \otimes N)) \otimes \mathbb{F}_2.$$

Proof. The submodule of $V^{\otimes 3}$ generated by $b_0^{\otimes 3}$ is $\Delta(3)$; its contravariant form takes the values 1, 3, 3, 1 (that is, $C(3, r)$), all units, so $V^{\otimes 3} = \Delta(3) \oplus \Delta(3)^\perp$, and $\Delta(3)^\perp$ is stable because the form is contravariant. This complement has weights ± 1 only; on its weight-1 space $E = 0$, so $EF = H = 1$ there, F maps it isomorphically onto the weight-(-1) space, and each pair $\{p, Fp\}$ spans a copy of V : $\Delta(3)^\perp \cong V^2$. The construction in the proof of Lemma 3.2, which works verbatim over $\mathbb{F}_2$, gives a map $\Delta(3) \rightarrow V \otimes V^{[1]}$, $v \mapsto b_0 \otimes b_0$; its image contains $b_0 b_0$, $b_1 b_0$, $b_0 b_1 = F^{(2)}(b_0 b_0)$ and $b_1 b_1$, so it is onto, and both have dimension 4. Finally, with Proposition 3.1(c) and the monoidality of the twist,

$$(V \otimes V \otimes \Phi(N)) \otimes \mathbb{F}_2 \cong V^{\otimes 3} \otimes \bar{N}^{[1]} \cong (V \otimes V^{[1]} \otimes \bar{N}^{[1]}) \oplus (V \otimes \bar{N}^{[1]})^2 = (V \otimes (V \otimes \bar{N})^{[1]}) \oplus (V \otimes \bar{N}^{[1]})^2,$$

which is $(\Phi(V \otimes N) \oplus \Phi(N)^2) \otimes \mathbb{F}_2$ ($\bar{N} := N \otimes \mathbb{F}_2$). ■

Theorem 3.6 (the family decomposition). For every $n \geq 0$, as Dist_A -lattices,

$$V^{\otimes n} \otimes A \cong \bigoplus_{\lambda} T(\lambda)^{m_{\lambda}(n)},$$

with the multiplicities $m_{\lambda}(n)$ of §1.2.

Proof. Induction on n , applying $V \otimes -$ to the decomposition for $n - 1$; it suffices that $V \otimes T(\lambda)$ decomposes according to the rule $V \cdot [\lambda]$ of §1.2. Now $V \otimes T(0) = V = \Phi(A) = T(1)$, and $V \otimes T(2l + 1) = V \otimes \Phi(T(l)) = T(2l + 2)$ by definition. For $V \otimes T(2l + 2) = V \otimes V \otimes \Phi(T(l))$: both it and $\Phi(T(l))^2 \oplus \Phi(V \otimes T(l))$ have Δ - and ∇ -filtrations, and their reductions are isomorphic (Lemma 3.5), so they are isomorphic (Corollary 3.4). Here $\Phi(T(l))^2 = T(2l + 1)^2$, and $V \otimes T(l)$ decomposes by induction on l (Φ is additive and $\Phi(T(\nu)) = T(2\nu + 1)$). ■

Remarks. (1) $T(\lambda) \otimes \mathbb{F}_2$ is the indecomposable tilting module of SL_2 of highest weight λ in characteristic 2 (Donkin's tensor product theorem [Don93]), so $m_\lambda(n)$ is its multiplicity in $V^{\otimes n}$; this identification is not used. The multiplicities are known: they can be read off by peeling the characters above off $(e + e^{-1})^n$ [TW21], and Larsen gives the fusion rule of $V^{\otimes 2}$ in characteristic 2 [Lar24, §2]; iterating Larsen's rule reproduces the recursion of §1.2 for every even $n \leq 64$ (§12.2). What we add is the integral statement over $\mathbb{Z}_2$, with explicit lattices. (2) The proof uses no result of the general theory of tilting modules: only (K), Lemma 3.2, Weyl's theorem on complete reducibility over $\mathbb{Q}$, and the completeness of $\mathbb{Z}_2$.

Lemma 3.7. For $n \geq 1$: $m_0(n) = 0$; $m_n(n) = 1$; $m_1(n) = 2^{(n-1)/2}$ for odd n ; $m_2(n) = 2^{(n-2)/2}$ for even n .

Proof. In the rule $V \cdot [\lambda]$ the produced symbols are $[1]$, $[\lambda + 1]$ with λ odd (even and ≥ 2), $[2l + 1]$, and $\Phi(\cdot) = [2\nu + 1]$ (odd): $[0]$ is never produced. The symbol $[\lambda + 1]$ is the largest symbol of $V \cdot [\lambda]$, with coefficient 1, so $m_n(n) = 1$. The symbol $[2]$ is produced only by $V \cdot [1]$, so $m_2(n + 1) = m_1(n)$. The symbol $[1]$ is produced by $V \cdot [0]$ and by the term $2[2l + 1]$ of $V \cdot [2]$ ($l = 0$), and by $\Phi(V \cdot [l])$ only if $V \cdot [l]$ contains $[0]$, which never happens; so $m_1(n + 1) = m_0(n) + 2m_2(n) = 2m_2(n)$ for $n \geq 1$, with $m_1(1) = 1$. ■

3.4 The groups of the families

For a family λ , an integer $\alpha \geq 1$ and a shift $i \geq 0$, let D be the floor operator of $T(\lambda)$ (floor $(\lambda - w)/2$ on weight w) and put

$$\sigma_i^{(\alpha)} := F + 2^\alpha(D + i), \quad X^{(\alpha)}(\lambda, i) := \mathrm{coker}(\sigma_i^{(\alpha)} \text{ on } T(\lambda)).$$

For $\alpha = 1$, $\sigma_i^{(1)} = F + \lambda - H + 2i$.

Theorem 3.8. For every $n \geq 1$, $\mathbb{Z}_2 \oplus \mathrm{Syl}_2 K(Q_n) \cong \bigoplus_\lambda X^{(1)}(\lambda, (n - \lambda)/2)^{m_\lambda(n)}$.

Proof. Corollary 2.3 and Theorem 3.6. On a summand $T(\lambda)$ of $V^{\otimes n}$,

$$\sigma = F + n - H = F + (\lambda - H) + (n - \lambda) = \sigma_{(n-\lambda)/2}^{(1)}. \quad \blacksquare$$

The cube uses $\alpha = 1$; the fold law (§10) uses $\alpha = 2$. The rest of the paper computes $X^{(\alpha)}(\lambda, i)$.

4. The dance: Theorem D

4.1 Payment systems

Let N be a lattice graded by floors $0, 1, \dots, f_{\max}$, on which F maps floor f into floor $f + 1$. A **payment** is a function $\pi : \mathbb{Z}_{\geq 0} \rightarrow A$; $\pi(D)$ denotes the operator that multiplies floor f by $\pi(f)$. A **payment system of size r** on N is an operator on $N^{\oplus r}$, with slots $1, \dots, r$, of the form

$$G(x\iota_{s'}) = (F + \pi_{s'}(D))x\iota_{s'} + \sum_{s > s'} K_{ss'}(D)x\iota_s \quad (x \in N),$$

where ι_s places an element in slot s and the **couplings** $K_{ss'}$ are payments. It is **even** if every value of every π_s and $K_{ss'}$ lies in $2A$. The operator $\sigma_i^{(\alpha)}$ on $T(\lambda)$ (§3.4) is an even payment system of size 1, with $\pi(d) = 2^\alpha(i + d)$, because $\alpha \geq 1$.

4.2 The two moves

Proposition 4.1 (the twist move). Let $N = \Phi(N')$ with its floor grading (the floor of b_en' is $2f + e$ if n' has floor f). For an even payment system G of size r on N ,

$$\text{coker } G \cong \text{coker}(2G'),$$

where G' is the even payment system of size r on N' with

$$\pi'_s(f) = -\pi_s(2f)\pi_s(2f + 1)/2,$$

$$K'_{us'}(f) = -\frac{1}{2}[\pi_{s'}(2f + 1)K_{us'}(2f) + K_{us'}(2f + 1)\pi_u(2f) + \sum_{s' < s < u} K_{ss'}(2f + 1)K_{us}(2f)].$$

Proof. On $\Phi(N')$, $F(b_0n) = b_1n$ and $F(b_1n) = 2b_0(Fn)$. For $n \in N'$ of floor f , the relation $G(b_0n\iota_{s'}) = b_1n\iota_{s'} + \pi_{s'}(2f)b_0n\iota_{s'} + \sum_{s > s'} K_{ss'}(2f)b_0n\iota_s$ has coefficient 1 on $b_1n\iota_{s'}$, and every other term lies in $b_0 \otimes N'$. Eliminating these relations (one for each n and s'), we see that $\text{coker } G$ is the quotient of $(b_0 \otimes N')^{\oplus r}$ by the relations $G(b_1n\iota_{s'})$, where every $b_1x\iota_s$ is replaced by $-\pi_s(2f)b_0x\iota_s - \sum_{u > s} K_{us}(2f)b_0x\iota_u$. That relation is

$$\begin{aligned} & 2b_0(Fn)\iota_{s'} - \pi_{s'}(2f + 1)\pi_{s'}(2f)b_0n\iota_{s'} \\ & - \sum_{u > s'} [\pi_{s'}(2f + 1)K_{us'}(2f) + K_{us'}(2f + 1)\pi_u(2f) + \sum_{s' < s < u} K_{ss'}(2f + 1)K_{us}(2f)]b_0n\iota_u, \end{aligned}$$

which is $2G'(n\iota_{s'})$ under $b_0x \mapsto x$. The values of G' are halves of products of two elements of $2A$, so they lie in $2A$. ■

Proposition 4.2 (the split move). Let $N = V \otimes \Phi(N')$. For $n \in N'$ of floor f put

$$\begin{aligned} A_n &:= b_0 \otimes b_0n, & B_n &:= b_0 \otimes b_1n - b_1 \otimes b_0n, & C_n &:= b_1 \otimes b_0n, \\ & & \text{Dd}_n &:= b_1 \otimes b_1n, \end{aligned}$$

of floors $2f, 2f + 1, 2f + 1, 2f + 2$. They form a basis of N over the bases of N' , and

$$FA_n = B_n + 2C_n, \quad FB_n = 2A_{Fn}, \quad FC_n = \text{Dd}_n, \quad F\text{Dd}_n = 2C_{Fn}.$$

For an even payment system G of size r on N , $\text{coker } G \cong \text{coker}(2G'')$, where G'' is the even payment system of size $2r$ on N' with slots (s, A) , (s, C) , ordered lexicographically with A before C , and

- $\pi''_{(s,A)}(f) = -\pi_s(2f)\pi_s(2f+1)/2$, $\pi''_{(s,C)}(f) = -\pi_s(2f+1)\pi_s(2f+2)/2$;
- the couplings $(s', A) \rightarrow (u, A)$ are the formula of Proposition 4.1 at the floors $(2f, 2f+1)$, and the couplings $(s', C) \rightarrow (u, C)$ are the same formula at $(2f+1, 2f+2)$;
- $K''_{(s',C),(s',A)}(f) = -\pi_{s'}(2f+1)$, $K''_{(u,C),(s',A)}(f) = -K_{us'}(2f+1)$ ($u > s'$), and there is no coupling from a slot C to a slot A .

Proof. The formulas for F follow from $\Delta(F) = F \otimes 1 + 1 \otimes F$ and §3.1: for instance $FA_n = b_1 \otimes b_0 n + b_0 \otimes b_1 n = B_n + 2C_n$ and $FB_n = b_1 \otimes b_1 n + 2b_0 \otimes b_0 F n - b_1 \otimes b_1 n$. The relations $G(A_n \iota_{s'})$ and $G(C_n \iota_{s'})$ have coefficient 1 on $B_n \iota_{s'}$ and $Dd_n \iota_{s'}$ respectively; eliminate those vectors. Substituting into $G(B_n \iota_{s'})$ gives

$$2A_{Fn \iota_{s'}} - \pi_{s'}(2f+1)\pi_{s'}(2f)A_n \iota_{s'} - \sum_u [\text{coupling of Proposition 4.1 at } (2f, 2f+1)]A_n \iota_u \\ - 2\pi_{s'}(2f+1)C_n \iota_{s'} - 2 \sum_{u>s'} K_{us'}(2f+1)C_n \iota_u,$$

and substituting into $G(Dd_n \iota_{s'})$ gives

$$2C_{Fn \iota_{s'}} - \pi_{s'}(2f+2)\pi_{s'}(2f+1)C_n \iota_{s'} - \sum_u [\text{coupling of Proposition 4.1 at } (2f+1, 2f+2)]C_n \iota_u.$$

Both are twice the relations of G'' under $A_{x \iota_s} \mapsto x \iota_{(s,A)}$, $C_{x \iota_s} \mapsto x \iota_{(s,C)}$. Evenness is as in Proposition 4.1. ■

4.3 The dance

Proposition 4.3. Let $\lambda+1 = 2^k + \sum_{t \in I} 2^t$, $\alpha \geq 1$ and $i \geq 0$. Apply to $\sigma_i^{(\alpha)}$ on $T(\lambda)$ the moves of the binary digits $t = 0, 1, \dots, k-1$ of $\lambda+1$, in this order: a twist move if $t \notin I$, a split move if $t \in I$. After the k moves one obtains an even payment system on $T(0)^{\oplus 2^{|I|}} = A^{2^{|I|}}$, that is, a lower triangular matrix $M = M^{(\alpha)}(\lambda, i)$ of size $2^{|I|}$, and

$$X^{(\alpha)}(\lambda, i) \cong \bigoplus_{a=1}^{k-1} (\mathbb{Z}/2^a)^{2^{|I|+k-1-a}} \oplus \text{coker}(2^k M).$$

Proof. If $\lambda = 2l+1$, then $\lambda+1$ has last digit 0, $T(\lambda) = \Phi(T(l))$ and $l+1 = (\lambda+1)/2$; if $\lambda = 2l+2$, then $\lambda+1$ has last digit 1, $T(\lambda) = V \otimes \Phi(T(l))$ and $l+1 = \lambda/2 = \lfloor (\lambda+1)/2 \rfloor$. In both cases the floor grading of $T(\lambda)$ (floor $(\lambda-w)/2$ at weight w) is the grading of Propositions 4.1–4.2: for instance $b_0 n$ has weight $2w+1$ and floor $(2l+1-2w-1)/2 = l-w = 2f$. So the moves can be applied digit by digit, and $T(0) = A$ has the single floor 0, on which $F = 0$. Each move halves the rank, $\text{rank } T(\lambda) = 2^{k+|I|}$, and turns $\text{coker}(2^h G)$ into $\text{coker}(2^{h+1} G')$ plus cyclic factors $\mathbb{Z}/2^h$, as many as half the rank (the eliminated unit pivots, multiplied by 2^h): the Smith exponents of $2^h G$ are h , with multiplicity half the rank, and the exponents of $2G'$ raised by h . At move $h = 0, 1, \dots, k-1$ the rank is $2^{k+|I|-h}$; the factors of move 0 are trivial, and move $a \geq 1$ gives $(\mathbb{Z}/2^a)^{2^{k+|I|-a-1}}$. ■

We use the notation $\text{coker}(2^k M)$ for the module whose Smith exponents are those of M raised by k (a zero exponent of M gives $\mathbb{Z}/2^k$, an infinite one stays $\mathbb{Z}_2$). The **dancers** are the subsets $D \subseteq I$: a split move at the digit t splits each slot into a slot A and a slot C , and the final slot s is identified with $D(s) := \{t \in I : s_t = C\}$. We index M by dancers, in the o-order. The order in which the moves produce the slots is another linear extension of inclusion (for $I = \{0, 1\}$ it is $\emptyset, \{1\}, \{0\}, \{0, 1\}$); re-indexing does not change the Smith form, and M is lower triangular in either order, since $M_{DD'} \neq 0$ only if $D' \subseteq D$ (Theorem 4.4).

4.4 The final matrix in closed form

Put $\pi_0(d) := 2^\alpha(i + d)$, $K := 2^k$ and $o(\Delta) := \sum_{t \in \Delta} 2^t$.

Theorem 4.4 (windows). Let $q_0 := -1$ and $q_{t+1} := q_t^2 - [t \in I] \cdot y_t q_t$ in $\mathbb{Z}[y_t : t \in I]/(y_t^2 : t \in I)$, and let $a_k(\Delta)$ be the coefficient of $y^\Delta := \prod_{t \in \Delta} y_t$ in q_k . Then for dancers D, D' :

$$M_{DD'} = -2^{1-K+o(D \setminus D')} \cdot a_k(D \setminus D') \cdot \prod_{d=o_D}^{o_{D'}+K-1} \pi_0(d) \quad \text{if} \\ D' \subseteq D,$$

$$M_{DD'} = 0 \quad \text{otherwise.}$$

The window $[o_D, o_{D'} + K - 1]$ has $K - o(D \setminus D') \geq 1$ elements.

Proof. Let $I_{<h} := I \cap [0, h)$. We show by induction on $h = 0, \dots, k$ that after h moves the system has one slot for each $D \subseteq I_{<h}$, couplings only from D' to $D \supsetneq D'$, and

$$\pi_D^{(h)}(f) = c_\emptyset^{(h)} \cdot \prod_{d \in W} \pi_0(d), \\ W = [2^h f + o_D, 2^h f + o_D + 2^h - 1],$$

$$K_{DD'}^{(h)}(f) = c_{D \setminus D'}^{(h)} \cdot \prod_{d \in W} \pi_0(d), \\ W = [2^h f + o_D, 2^h f + o_{D'} + 2^h - 1],$$

where $c^{(h)} = \sum_{\Delta} c_{\Delta}^{(h)} x^{\Delta} \in \mathbb{Q}[x_t : t \in I_{<h}]/(x_t^2)$ satisfies $c^{(0)} = 1$ and $c^{(h+1)} = -(c^{(h)})^2/2 - [h \in I] \cdot x_h c^{(h)}$. For $h = 0$ there is one slot, $\pi(f) = \pi_0(f)$, $c = 1$. In the formula of Proposition 4.1, each of the three kinds of terms is a product of the values of π_0 over two **adjacent** windows of level h , whose union is the window of level $h + 1$: for instance $\pi_{D'}(2f + 1)K_{DD'}(2f)$ multiplies over $[2^{h+1}f + o_D, 2^{h+1}f + o_{D'} + 2^h - 1]$ and $[2^{h+1}f + 2^h + o_{D'}, 2^{h+1}f + o_{D'} + 2^{h+1} - 1]$. So every term is the same monomial, and the coefficient of the pair (D, D') is $-\frac{1}{2} \sum c_{\Delta_1} c_{\Delta_2}$ over the ordered decompositions $D \setminus D' = \Delta_1 \sqcup \Delta_2$ (the intermediate slot is $U = D' \sqcup \Delta_1$; the decompositions with $\Delta_1 = \emptyset$ or $\Delta_2 = \emptyset$ are the first two terms); this is $-\frac{1}{2}[(c^{(h)})^2]_{D \setminus D'}$, since $x_t^2 = 0$. In a split move at $h \in I$ the slot D gives the slots D (from A) and $D \cup \{h\}$ (from C), with $o_{D \cup \{h\}} = o_D + 2^h$. In the formulas of Proposition 4.2, the diagonal entries have the windows $[2^{h+1}f + o_D, \dots + 2^{h+1} - 1]$ and $[2^{h+1}f + o_D + 2^h, \dots + 2^{h+1} - 1]$; the coupling $-\pi_{D'}(2f + 1)$ from D' to $D' \cup \{h\}$ and the coupling $-K_{DD'}(2f + 1)$ from D' to $D \cup \{h\}$ have the windows $[2^{h+1}f + o_{D'} + 2^h, 2^{h+1}f + o_{D'} + 2^{h+1} - 1]$ and $[2^{h+1}f + o_D + 2^h, 2^{h+1}f + o_{D'} + 2^{h+1} - 1]$, which are the windows of the new pairs. Their coefficients $-c_\emptyset$ and $-c_{D \setminus D'}$ are the coefficients of x_h and $x_h x^{D \setminus D'}$ in $-x_h c^{(h)}$; the other couplings carry $-\frac{1}{2}[(c^{(h)})^2]_{\Delta}$ with $h \notin \Delta$, and $(c^{(h)})^2$ has no x_h . This proves the recursion. Put $y_t := 2^{2^t} x_t$ and $c^{(h)} = -2^{1-2^h} q_h$. Then $c^{(0)} = 1$ gives $q_0 = -1$, and

$$c^{(h+1)} = -2^{1-2^{h+1}} q_h^2 + [h \in I] 2^{1-2^h} 2^{-2^h} y_h q_h = -2^{1-2^{h+1}} (q_h^2 - [h \in I] y_h q_h),$$

so $c_{\Delta}^{(k)} = -2^{1-K} a_k(\Delta) 2^{o(\Delta)}$. At $h = k$ there is only the floor 0, and $M_{DD'} = K_{DD'}^{(k)}(0)$. ■

4.5 The coefficients: the deficit law

Lemma 4.5 (deficit law). $a_k(\emptyset) = 1$, and for $\emptyset \neq \Delta \subseteq I$,

$$v_2(a_k(\Delta)) = k - |\Delta| - \min \Delta =: \delta(\Delta); \quad \text{put} \quad \delta(\emptyset) := 0.$$

Proof. The constant term of q_t is 1 for $t \geq 1$. We prove $v_2(a_t(\Delta)) = t - |\Delta| - \min \Delta$ for every $t > \max \Delta$, by induction on $|\Delta|$ and then on t . Let $m := \max \Delta \in I$ and $\mu := \min \Delta$. Since q_m does not involve y_m , the coefficient of y^Δ in q_m^2 is 0 and $a_{m+1}(\Delta) = -a_m(\Delta \setminus \{m\})$. This is ± 1 if $\Delta = \{m\}$ (valuation $0 = (m+1) - 1 - m$); otherwise it has valuation $m - (|\Delta| - 1) - \mu$ by induction. For $t \geq m+1$, the term $y_t q_t$ does not contribute to y^Δ , and

$$a_{t+1}(\Delta) = 2a_t(\Delta) + 2 \sum a_t(\Delta_1)a_t(\Delta_2),$$

over the unordered decompositions $\Delta = \Delta_1 \sqcup \Delta_2$ into non-empty parts. One of $\min \Delta_1$, $\min \Delta_2$ is μ and the other is $\leq m$, so each cross term has valuation at least $1 + 2t - |\Delta| - \mu - m$, which exceeds $v_2(2a_t(\Delta)) = 1 + t - |\Delta| - \mu$ by $t - m \geq 1$. $\blacksquare$

So $v_2(a_k(\Delta))$ **depends only on k and Δ** , not on the other digits of $\lambda + 1$; and δ is **strictly subadditive**: $\delta(\Delta_1) + \delta(\Delta_2) - \delta(\Delta_1 \sqcup \Delta_2) = k - \max(\min \Delta_1, \min \Delta_2) \geq 1$ for non-empty disjoint Δ_1, Δ_2 .

Theorem D. For every family λ , every $\alpha \geq 1$ and every shift $i \geq 0$,

$$X^{(\alpha)}(\lambda, i) \cong \bigoplus_{a=1}^{k-1} (\mathbb{Z}/2^a)^{2^{|I|+k-1-a}} \oplus \text{coker}(2^k M^{(\alpha)}(\lambda, i)),$$

with $M^{(\alpha)}(\lambda, i)$ the explicit integer matrix of Theorem 4.4, whose entries have the valuations

$$v_2(M_{DD'}) = 1 - K + o(D \setminus D') + \delta(D \setminus D') + \sum_{d=o_D}^{o_{D'}+K-1} (\alpha + v_2(i+d)) \quad (D' \subseteq D).$$

Hence, by Theorem 3.8, $\mathbb{Z}_2 \oplus \text{Syl}_2 K(Q_n)$ is the direct sum, over the families λ , of $m_\lambda(n)$ copies of the small clocks and of $\text{coker}(2^k M^{(1)}(\lambda, (n-\lambda)/2))$.

Proof. Proposition 4.3, Theorem 4.4 and Lemma 4.5. $\blacksquare$

Example. $\lambda = 6$ ($k = 2$, $I = \{0, 1\}$): $q_1 = 1 + y_0$, $q_2 = 1 + 2y_0 - y_1 - y_0y_1$. With $p_d := \pi_0(d)$, $M = -(1/8) \cdot A$, where A is, in the o-order (row D , column D'):

$D \setminus D'$	$\emptyset$	$\{0\}$	$\{1\}$	$\{0, 1\}$
$\emptyset$	$p_0 p_1 p_2 p_3$	0	0	0
$\{0\}$	$4p_1 p_2 p_3$	$p_1 p_2 p_3 p_4$	0	0
$\{1\}$	$-4p_2 p_3$	0	$p_2 p_3 p_4 p_5$	0
$\{0, 1\}$	$-8p_3$	$-4p_3 p_4$	$4p_3 p_4 p_5$	$p_3 p_4 p_5 p_6$

The matrix M has $2^{|I|} \leq \lfloor \lambda/2 \rfloor + 1$ rows (with equality, for instance, when $\lambda = 2^{k+1} - 2$), against 2^n for the Laplacian and $2^{k+|I|} = \dim T(\lambda)$ for the family; its entries are closed forms. What remains is its Smith form.

5. The valuations: rulers, gold and clocks

Fix a family λ ($\lambda + 1 = 2^k + \sum_{t \in I} 2^t$, $K = 2^k$), $\alpha \geq 1$, and a shift $i \geq 1$; put $m := i - 1$. For $t \in I$ recall $h_t = \text{next}(t) - t \geq 1$. Let $t_1 := \min I$ ($t_1 := k$ if $I = \emptyset$). For dancers $a \supseteq b$ write $\Delta := a \setminus b$.

Definitions. For $\mu \geq 0$ and $E \subseteq I$, the **ruler**

$$R_\mu(E) := \sum_{t \in E} (h_t - \alpha 2^t) - \kappa(\mu, o_E);$$

the **gold** of $\Delta \subseteq I$,

$$\gamma(\Delta) := \sum_{u \in I \setminus \Delta, u > \min \Delta} h_u \quad (\gamma(\emptyset) := 0),$$

the gaps of I above $\min \Delta$ that Δ skips; and the **clocks of the cell**

$$x_D := R_m(D) - R_{m+K}(I \setminus D).$$

For $\alpha = 1$ this is the definition of §1.2.

Lemma 5.1. (a) $\delta(\Delta) = \sum_{t \in \Delta} (h_t - 1) + \gamma(\Delta)$. (b) $\gamma(\Delta) = 0$ if and only if $\Delta = \emptyset$ or $\Delta = I \cap [\min \Delta, k)$ (a **tail** of I).

Proof. (a) $k - \min \Delta = \sum_{t \in I, t \geq \min \Delta} h_t$ (the gaps telescope), and the digits $t \geq \min \Delta$ of I are those of Δ and those counted in $\gamma(\Delta)$. (b) Each $h_u \geq 1$. ■

Proposition 5.2 (the cost form). For dancers $b \subseteq a$,

$$v_2(M^{(\alpha)}(\lambda, i)_{ab}) = \alpha K + \kappa(m, K) + R_m(a) - R_{m+K}(b) + \gamma(a \setminus b).$$

Proof. By Theorem D, $v_2(M_{ab}) = 1 - K + o(\Delta) + \delta(\Delta) + \alpha(K + o_b - o_a) + \sum_{d=o_a}^{o_b+K-1} v_2(i + d)$. By Legendre's formula $v_2(x!) = x - s_2(x)$, $\sum_{d=x}^{y-1} v_2(m + 1 + d) = (y - x) - s_2(m + y) + s_2(m + x)$. With $x = o_a$, $y = o_b + K$ and $o(\Delta) = o_a - o_b$:

$$v_2(M_{ab}) = 1 + \delta(\Delta) + \alpha(K + o_b - o_a) - s_2(m + K + o_b) + s_2(m + o_a).$$

By Lemma 5.1(a), $\delta(\Delta) = G(a) - G(b) + \gamma(\Delta)$ with $G(E) := \sum_{t \in E} (h_t - 1)$. Since $\kappa(\mu, o_E) = s_2(\mu) + |E| - s_2(\mu + o_E)$, we have

$$R_\mu(E) = G(E) - \alpha o_E + s_2(\mu + o_E) - s_2(\mu).$$

Substituting,

$$v_2(M_{ab}) - [R_m(a) - R_{m+K}(b) + \gamma(\Delta)] = 1 + \alpha K + s_2(m) - s_2(m + K) = \alpha K + \kappa(m, K),$$

because $s_2(K) = 1$. ■

So every entry of M is a constant of the cell, plus a **row term** read from the carries of $m + o_a$, minus a **column term** read from the carries of $m + K + o_b$, plus the gold of the arrow. We write

$$\hat{c}(a, b) := R_m(a) - R_{m+K}(b) + \gamma(a \setminus b) \quad (b \subseteq a)$$

for the **normalized cost** of the arrow $a \rightarrow b$.

Proposition 5.3 (the clocks of §1.2). For $\alpha = 1$, every family λ , every $i \geq 1$ and every dancer D ,

$$u_i(D) = k + K + \kappa(m, K) + x_D.$$

Proof. Put $E := I \setminus D$ and $\xi := \xi_D = \lambda + 1 - 2o_D = K + o_E - o_D$, so $o_D + \xi = K + o_E$. *Step 1* ($i = 1$). We show

$$u_1(D) = \varphi(\lambda + 1) + 2\psi(D), \quad \text{with} \quad \psi(D) := \sum_{t \in D} (h_t - 2^t).$$

Let $\mu := \xi - 1 = \lambda - 2o_D$. Since $K - 1 = o([0, k] \setminus I) + o_I$, we have $\mu = o([0, k] \setminus I) + 2o_E$. Adding 2^{t+1} ($t \in E$) to $o([0, k] \setminus I)$ carries through the positions $t+1, \dots, \text{next}(t)-1$, which lie in $[0, k] \setminus I$, and stops at $\text{next}(t)$, a position that is set neither in $o([0, k] \setminus I)$ nor by any other t ; the stretches $[t+1, \text{next}(t)]$ of different t are disjoint. Hence $s_2(\mu) = (k - |I|) + |E| - G(E)$. Now $\varphi(\xi) = \xi + v_2(\xi)$ and $s_2(\xi) = s_2(\mu) + 1 - v_2(\xi)$, and $\kappa(o_D, \xi) = |D| + s_2(\xi) - s_2(K + o_E) = |D| + s_2(\xi) - 1 - |E|$. So

$$u_1(D) = \xi + v_2(\xi) + s_2(\xi) - 1 + |D| - |E| + h(D) = \xi + s_2(\mu) + |D| - |E| + h(D)$$

$$= K + o_I - 2o_D + k - |I| + 2|D| + G(D) - G(E),$$

using $\xi = K + o_I - 2o_D$ and $h(D) = G(D) + |D|$. On the other side

$$\varphi(\lambda + 1) = K + o_I + t_1, \quad 2\psi(D) = 2G(D) + 2|D| - 2o_D.$$

The difference of the two sides is $k - |I| - G(D) - G(E) - t_1 = k - t_1 - \sum_{t \in I} h_t = 0$. *Step 2* ($i \geq 1$). From $\kappa(a, b) = s_2(a) + s_2(b) - s_2(a + b)$, $\kappa(m + o_D, \xi) - \kappa(o_D, \xi) = \kappa(m, o_D + \xi) - \kappa(m, o_D)$, so

$$u_i(D) = u_1(D) + \kappa(m, K + o_E) - \kappa(m, o_D).$$

Step 3. $x_D = \psi(D) - \kappa(m, o_D) - \psi(E) + \kappa(m + K, o_E)$, and

$$\kappa(m, K + o_E) + \kappa(K, o_E) = \kappa(m, K) + \kappa(m + K, o_E)$$

(both sides are $s_2(m) + s_2(K) + s_2(o_E) - s_2(m + K + o_E)$), with $\kappa(K, o_E) = 0$. Hence $u_i(D) - k - K - \kappa(m, K) - x_D = \varphi(\lambda + 1) + \psi(D) + \psi(E) - k - K = \varphi(\lambda + 1) + \psi(I) - k - K$, and $\psi(I) = (k - t_1) - o_I$, so this is 0. $\blacksquare$

The shift $i = 0$. Here the payment $\pi_0(0) = 0$ makes the row $\emptyset$ of $M(\lambda, 0)$ zero (its window $[0, K - 1]$ contains $d = 0$), and no other window contains 0. The same computation with Legendre's formula at $i = 0$, $\sum_{d=x}^{y-1} v_2(d) = (y - x) - s_2(y - 1) + s_2(x - 1)$ ($x \geq 1$), is the case « $m = -1$ »: since $m + K + o_b = K - 1 + o_b$, both rulers are read at $K - 1$ (Proposition 5.4). In the language of §7.3 this is the house with $\ell = K - 1$.

Proposition 5.4. For $D \neq \emptyset$ and every $b \subseteq D$,

$$v_2(M(\lambda, 0)_{Db}) = \alpha K + R^\sharp(D) - R^\sharp(b) + \gamma(D \setminus b),$$

$$R^\sharp(E) := \sum_{t \in E} (h_t - \alpha 2^t) - \kappa(K - 1, o_E),$$

and, for $\alpha = 1$, $u_0(D) = k + K + x_D^\sharp$ with $x_D^\sharp := R^\sharp(D) - R^\sharp(I \setminus D)$. Explicitly $\kappa(K - 1, o_E) = k - \min E$ (with $\min \emptyset := k$).

Proof. Let $E \neq \emptyset$. Then $K - 1 + o_E = K + (o_E - 1)$ and $s_2(o_E - 1) = |E| - 1 + \min E$, so that $\kappa(K - 1, o_E) = k + |E| - 1 - s_2(o_E - 1) = k - \min E$; for $E = \emptyset$ both sides are 0. As in Proposition 5.2, $R^\sharp(E) = G(E) - \alpha o_E + s_2(K - 1 + o_E) - k$, and, for $D \neq \emptyset$, $s_2(K - 1 + o_D) = 1 + s_2(o_D - 1)$. Legendre's formula, applied at $i = 0$ with $x = o_D \geq 1$ and $y = o_b + K$, gives

$$v_2(M_{Db}) = 1 + \delta(\Delta) + \alpha(K + o_b - o_D) - s_2(K - 1 + o_b) + s_2(o_D - 1),$$

and with $\delta(\Delta) = G(D) - G(b) + \gamma(\Delta)$ the difference $v_2(M_{Db}) - [R^\sharp(D) - R^\sharp(b) + \gamma(\Delta)]$ is αK . For the clocks: in $u_i(D)$ the first argument of κ is $i + o_D - 1$, which is $o_D - 1$ at $i = 0$ against o_D at $i = 1$, while ξ is unchanged; and $\kappa(o_D - 1, \xi) - \kappa(o_D, \xi) = v_2(o_D) - v_2(o_D + \xi) = \min D - \min E$ ($v_2(K + o_E) = \min E$). So $u_0(D) = u_1(D) + \min D - \min E = k + K + \psi(D) - \psi(E) + \min D - \min E$, and $x_D^\sharp = \psi(D) - (k - \min D) - \psi(E) + (k - \min E)$ is the same. ■

6. The Smith form of M : the ceiling

6.1 Determinantal divisors and the tropical bound

For a matrix M over A with Smith exponents $e_1 \leq e_2 \leq \dots$, the r -th **determinantal divisor** $d_r(M)$, the least valuation of an $r \times r$ minor, is $e_1 + \dots + e_r$. So $r \mapsto d_r$ is convex with integer increments, and the increments are the exponents.

Fix a cell (λ, i, α) with $i \geq 1$, and number the dancers in o-order, $D_0 = \emptyset, D_1, \dots, D_{N-1} = I$ ($N = 2^{|I|}$). Since $o_{I \setminus D} = o_I - o_D$, the map $D \mapsto I \setminus D$ reverses the o-order. Put $L_r := \{D_{N-r}, \dots, D_{N-1}\}$ (the last r dancers) and $F_r := \{D_0, \dots, D_{r-1}\} = \{I \setminus D : D \in L_r\}$ (the first r). Write

- $\hat{d}_r := d_r(M) - r(\alpha K + \kappa(m, K))$ (the normalized divisors);
- $T(r) :=$ the least normalized cost $\sum \hat{c}(a, \beta(a))$ of an **r -matching**: a bijection β from r dancers onto r dancers with $\beta(a) \subseteq a$ for every a ;
- $U(r) := \sum_{D \in L_r} x_D$, the sum of the last r clocks (the **natural** partial sums);
- $Y(r) := \sum_{D \in L_r} \hat{y}_D$, the same for the integer antitonic fit $\hat{y}$ of x (the **pooled** partial sums).

Lemma 6.1 (tropical bound). $\hat{d}_r \geq T(r)$.

Proof. An $r \times r$ minor is a signed sum of products of r entries along a bijection from its rows to its columns. A product is non-zero only if every entry lies in the support $\{b \subseteq a\}$ of M , and then its normalized valuation is the cost of the matching (Proposition 5.2). ■

6.2 Theorem O: the unique golden matching

Call an arrow $a \rightarrow b$ ($b \subseteq a$) **golden** if $\gamma(a \setminus b) = 0$, that is (Lemma 5.1(b)), if $b = a$ or $b = a \setminus T$ for a tail $T = I \cap [t, k)$ of I contained in a .

Theorem 6.2 (Theorem O). For every $I \subseteq [0, k)$ and every $0 \leq r \leq N$, there is **exactly one** bijection $\beta_r : L_r \rightarrow F_r$ all of whose arrows $D \rightarrow \beta_r(D)$ are golden.

Proof. Let $t_1 < \dots < t_s$ be the digits of I and identify a dancer D with the integer $p(D) := \sum_j [t_j \in D] 2^{j-1} \in [0, 2^s)$; the o-order is the order of p . A tail of I contained in D is a set of top bits of $p(D)$ that are all equal to 1, so the golden images of p are p itself and the numbers obtained from p by clearing a non-empty block of its top bits, all equal to 1. We must show that there is exactly one bijection from $\{N - r, \dots, N - 1\}$ onto $\{0, \dots, r - 1\}$ that sends every p to a golden image. Induction on s ; $s = 0$ is trivial. Let $N = 2^s \geq 2$.

- $r \leq N/2$: every row has top bit 1 and every column has top bit 0, so every arrow clears the top bit, followed by a golden arrow of the remaining $s - 1$ bits. Removing the top bit of the rows gives exactly the problem $(s - 1, r)$, which has one solution.
- $r = N/2 + q$ with $0 < q \leq N/2$: the columns $N/2, \dots, N/2 + q - 1$ have top bit 1 and are reached only by loops, so they are matched with themselves, and they are rows. The rows $N/2 - q, \dots, N/2 - 1$ have top bit 0 and admit only loops, so they are matched with themselves, and they are columns. The remaining rows $N/2 + q, \dots, N - 1$ must go to the remaining columns $0, \dots, N/2 - q - 1$ by clearing the top bit: this is the problem $(s - 1, N/2 - q)$, which has one solution. ■

The golden matching is explicit: for $r \leq N/2$ it is «remove the top digit of I and solve the smaller problem», and for $r > N/2$ it fixes the $2r - N$ middle dancers and solves the smaller problem on the rest.

Corollary 6.3 (the natural minor). $v_2(\det M[L_r, F_r]) = r(\alpha K + \kappa(m, K)) + U(r)$; hence $\hat{d}_r \leq U(r)$ for every r .

Proof. For a bijection $\beta : L_r \rightarrow F_r$ inside the support, the normalized valuation of its term is

$$\sum_{D \in L_r} R_m(D) - \sum_{E \in F_r} R_{m+K}(E) + \sum_D \gamma(D \setminus \beta(D)).$$

The first two sums do not depend on β and add up to $\sum_{D \in L_r} (R_m(D) - R_{m+K}(I \setminus D)) = U(r)$. The gold is a non-negative integer, zero exactly for the golden matching β_r . So the minor has a unique term of least valuation, and its valuation is that of the term. ■

6.3 From the natural sums to the pooled sums

Lemma 6.4 (convexity). Let $d : \{0, \dots, N\} \rightarrow \mathbb{Z}$ be convex with integer increments, $d(0) = 0$, and $d(r) \leq U(r)$ for every r . Then $d(r) \leq Y(r)$ for every r .

Proof. Read the dancers from the end. On every level set of the real antitonic fit (§7.1) the sums of x and of $\hat{y}$ agree, so $Y = U$ at the two ends $r_0 < r_1 = r_0 + p$ of each level set, counted from the end. Inside a level set with sum $S = pq + \rho$ ($0 \leq \rho < p$), Y increases by q on the first $p - \rho$ steps and by $q + 1$ on the last ρ steps (the fit takes the larger values first in o-order). We argue by induction over the level sets from the end, so we may assume $d(r_0) \leq Y(r_0)$. Suppose $d(r^*) > Y(r^*)$ with $r_0 < r^* < r_1$.

- If $r^* \leq r_0 + p - \rho$, some increment of d in $(r_0, r^*]$ is at least $q + 1$, hence so is every later one, and $d(r_1) \geq d(r^*) + (r_1 - r^*)(q + 1) > Y(r^*) + (r_1 - r^*)(q + 1) \geq Y(r_1) = U(r_1)$, a contradiction.
- If $r^* > r_0 + p - \rho$, then $d(r_1) - d(r^*) < Y(r_1) - Y(r^*) = (r_1 - r^*)(q + 1)$, so some increment of d in $(r^*, r_1]$ is at most q (they are integers), hence so is every earlier one, and $d(r^*) \leq d(r_0) + (r^* - r_0)q \leq Y(r^*)$, a contradiction. ■

Theorem 6.5. Suppose $T(r) \geq Y(r)$ for every r and that the integer fit $\hat{y}$ is non-increasing. Then $\hat{d}_r = Y(r)$ for every r , and the Smith exponents of $M^{(\alpha)}(\lambda, i)$ are $\alpha K + \kappa(m, K) + \hat{y}_D$, $D \subseteq I$.

Proof. $Y(r) \leq T(r) \leq \hat{d}_r \leq Y(r)$, by the hypothesis, Lemma 6.1, and Corollary 6.3 with Lemma 6.4 (applied to $\hat{d}$). Since $\hat{y}$ is non-increasing, $Y(r)$ is the sum of the r smallest values of $\hat{y}$, and the exponents are the increments. ■

The two hypotheses are proved in §7 (Theorem 7.8 and Lemma 7.9).

7. The Smith form of M : the floor

7.1 Antitonic fits

For a real sequence $x = (x_0, \dots, x_{N-1})$, its **antitonic fit** $z = \text{fit}(x)$ is the non-increasing sequence nearest to x in the Euclidean norm; it exists and is unique (projection onto a closed convex cone). Its **level sets** are the maximal runs of positions with equal value. The **integer fit** replaces, on each level set of p positions with sum S (necessarily $S = p \cdot z$), the values by $S \bmod p$ copies of $\lceil S/p \rceil$ followed by $p - (S \bmod p)$ copies of $\lfloor S/p \rfloor$; this is the integer antitonic fit of §1.2, computed there by the pool-adjacent-violators algorithm [ABERS55], [BBBB72].

Lemma 7.1 (characterization). A non-increasing sequence z is $\text{fit}(x)$ if and only if, for every level set $[p, q]$ of z , $\sum_{j=p}^q (x_j - z_j) = 0$ and $\sum_{j=p}^s (x_j - z_j) \leq 0$ for $p \leq s \leq q$.

Proof. If a prefix of a level set had a positive sum, raising the values on that prefix by a small ε and lowering the rest of the level set would keep z non-increasing and decrease the distance; the same with a non-zero total. Conversely, let y be non-increasing. Then $\|x - y\|^2 = \|x - z\|^2 + \|z - y\|^2 + 2 \sum_j (x_j - z_j)(z_j - y_j)$, and on a level set $[p, q]$ with value c , Abel summation with $S_s := \sum_{j=p}^s (x_j - c) \leq 0$ ($S_q = 0$) gives

$$\sum_{j=p}^q (x_j - c)(c - y_j) = - \sum_{s=p}^{q-1} S_s (y_s - y_{s+1}) \geq 0,$$

since y is non-increasing. So $\|x - y\| \geq \|x - z\|$. ■

We say that x is **cut** at a position b ($0 < b < N$) if $\text{fit}(x)$ is the concatenation of $\text{fit}(x_{<b})$ and $\text{fit}(x_{\geq b})$. The cut is **strict** if moreover $\text{fit}(x)_{b-1} > \text{fit}(x)_b$. A position b with $\text{fit}(x)_{b-1} > \text{fit}(x)_b$ is always a cut, since Lemma 7.1 is a condition on each level set separately and so holds for the two pieces; thus b is a strict cut if and only if no level set of $\text{fit}(x)$ contains both $b - 1$ and b .

Lemma 7.2 (cuts and shifts).

- If $\text{fit}(x_{<b})$ followed by $\text{fit}(x_{\geq b})$ is non-increasing, then x is cut at b .
- Let x be cut at the positions $b_1 < \dots < b_s$, and let δ be a non-increasing sequence that is constant between consecutive cut positions. Then $\text{fit}(x + \delta) = \text{fit}(x) + \delta$, and $x + \delta$ is cut at the same positions. If moreover δ is integer and constant on every level set of $\text{fit}(x)$ — in particular, if every position at which δ changes is a strict cut of x — then $\text{fit}(x + \delta)$ has the level sets of

$\text{fit}(x)$, the strict cuts of x are strict cuts of $x + \delta$, and the integer fit of $x + \delta$ is the integer fit of x plus δ .

(c) If $\text{fit}(x)_{<b} = \text{fit}(x_{<b})$, or $\text{fit}(x)_{\geq b} = \text{fit}(x_{\geq b})$, then x is cut at b .

Proof. (a) Each level set of the concatenation is a level set of one piece, or the union of a final level set of the first piece and an initial one of the second with the same value; Lemma 7.1 for the pieces gives it for the union (a prefix that crosses the cut has the total 0 of the first part plus a prefix sum ≤ 0). (b) On each piece the fit of $x + \text{const}$ is the fit plus the constant; the concatenation is non-increasing, so (a) applies. Now let δ be integer and constant on every level set. Two adjacent level sets of $\text{fit}(x)$, with values $\mu_A > \mu_B$, receive constants $\delta_A \geq \delta_B$, so $\mu_A + \delta_A > \mu_B + \delta_B$: the level sets of $\text{fit}(x) + \delta$ are those of $\text{fit}(x)$, and a strict cut stays strict (a strict drop of the fit is a cut). On a level set of p positions with sum S , adding the integer δ_L to every position adds $p\delta_L$ to the sum, keeps $S \bmod p$, and adds δ_L to $\lceil S/p \rceil$ and to $\lfloor S/p \rfloor$; so the integer fit is shifted by δ . A level set never contains the two sides of a strict cut, so a δ that changes only at strict cuts is constant on every level set. Without this hypothesis the integer claim fails: $x = (3, 4, 3, 4)$ is cut at 2 but not strictly ($\text{fit}(x) = (7/2, 7/2, 7/2, 7/2)$), and for $\delta = (1, 1, 0, 0)$ the integer fit of $x + \delta$ is $(5, 4, 4, 3)$, while the integer fit of x plus δ is $(4, 4, 3, 3) + \delta = (5, 5, 3, 3)$. (c) Suppose $z := \text{fit}(x)$ agrees with $\text{fit}(x_{<b})$ before b . Let $L = L_1 \cup L_2$ be a level set of z , with value c , that crosses b (L_1 before b). The set L_1 is a final level set of $\text{fit}(x_{<b})$, so $\sum_{L_1} (x - c) = 0$; hence the sum of $x - c$ over a prefix of L_2 equals its sum over L_1 plus that prefix, which is ≤ 0 by Lemma 7.1 for z , and the total over L_2 is 0. The level sets of z after b are unchanged, so $z_{\geq b}$ satisfies Lemma 7.1 for $x_{\geq b}$. The second hypothesis is the first one for the reversed and negated sequence. ■

Antisymmetry. For dancers in o-order, the **mirror** is $D \mapsto I \setminus D$, the reversal of positions; x is **antisymmetric** if $x_{I \setminus D} = -x_D$. The fit of an antisymmetric sequence is antisymmetric, and so is its integer fit (on a level set of size p and sum $-S$ the rule gives $p - (S \bmod p)$ copies of $-\lfloor S/p \rfloor$ followed by $-\lceil S/p \rceil$); hence a position b is a cut if and only if its mirror $N - b$ is, and a strict cut if and only if its mirror is.

Lemma 7.3 (antisymmetric gluing). Let x be antisymmetric of length $2N'$, let $Z_1 = \text{fit}(x_{<N'})$ and let X_1 be the integer fit of $x_{<N'}$. Then

$$\text{fit}(x) = [\max(Z_1, 0), \min(Z_2, 0)] \quad \text{and the integer fit of } x \text{ is } [\max(X_1, 0), \min(X_2, 0)],$$

where Z_2, X_2 are the mirror-negatives of Z_1, X_1 (so $Z_2 = \text{fit}(x_{\geq N'})$).

Proof. The candidate z is non-increasing. Its level sets are the level sets of Z_1 with positive value, their mirrors, and one **central** level set $P \cup P^*$ with value 0, where $P = \{Z_1 \leq 0\}$ is a final segment of the first half, a union of level sets of Z_1 , and P^* is its mirror. The positive level sets satisfy Lemma 7.1 because they do for Z_1 ; their mirrors by antisymmetry. On $P \cup P^*$ the total of x is 0 by antisymmetry. A prefix contained in P has sum at most the corresponding sum of Z_1 , by Lemma 7.1 applied to the level sets of Z_1 that it covers, and that sum is ≤ 0 . A prefix that covers P and the first L positions of P^* has sum $\sum_P x - \sum_{\text{last } L \text{ positions of } P^*} x$, the sum over the first $|P| - L$ positions of P , which is ≤ 0 again. So $z = \text{fit}(x)$. The integer fit takes on the positive level sets the same values as X_1 , and 0 on the central one; and $X_1 \geq 0$ where $Z_1 > 0$, $X_1 \leq 0$ where $Z_1 \leq 0$. ■

7.2 Theorem U

Theorem 7.4 (Theorem U). Let a cell have costs $\hat{c}(a, b)$ on its arrows and pooled clocks $\hat{y}$. Suppose there are real functions v («rows») and w («columns») on the dancers such that

- (U1) v and w are non-increasing in o-order;
- (U2) $v(D) - w(I \setminus D) = \hat{y}_D$ for every dancer D ;
- (U3) $v(a) - w(b) \leq \hat{c}(a, b)$ for every arrow $b \subseteq a$ (loops included).

Then every r -matching costs at least $Y(r) = \sum_{D \in L_r} \hat{y}_D$; that is, $T(r) \geq Y(r)$.

Proof. For an r -matching with rows R and columns C , by (U3) its cost is at least $\sum_R v - \sum_C w$. By (U1), $\sum_R v \geq \sum_{L_r} v$ and $\sum_C w \leq \sum_{F_r} w$. Since $F_r = \{I \setminus D : D \in L_r\}$, (U2) gives $\sum_{L_r} v - \sum_{F_r} w = \sum_{L_r} \hat{y}$. ■

7.3 Houses, windows and penalties

A **house** is $H = (I, k, \ell, \alpha)$ with $I \subseteq [0, k]$, $0 \leq \ell < 2^k$ and $\alpha \geq 1$. Its **base ruler**, **carry-out set**, **base clocks** and **base costs** are

$$R_H(E) := \sum_{t \in E} (h_t - \alpha 2^t) - \kappa(\ell, o_E), \quad J_H(E) := [\ell + o_E \geq 2^k],$$

$$x_H(D) := R_H(D) - R_H(I \setminus D), \quad \hat{c}_H(a, b) := R_H(a) - R_H(b) + \gamma(a \setminus b).$$

The sequence x_H is antisymmetric, and J_H is an up-set of the o-order (it is monotone in o_E). For integers $P_r, P_c \geq 0$ the **penalty cell** $\text{Cell}(H; P_r, P_c)$ has the costs $\hat{c}_H(a, b) - P_r J_H(a) + P_c J_H(b)$ and the clocks $x_H(D) - P_r J_H(D) + P_c J_H(I \setminus D)$.

For $\mu \geq 0$ let $r_k(\mu)$ be the number of consecutive ones of μ starting at bit k , that is $r_k(\mu) = v_2(\lfloor \mu/2^k \rfloor + 1)$.

Lemma 7.5 (real cells). Let $i \geq 1$, $m = i - 1$ and $\ell := m \bmod K$, and let $H = (I, k, \ell, \alpha)$. Then $R_m = R_H - r_k(m)J_H$ and $R_{m+K} = R_H - r_k(m + K)J_H$; so the cell (λ, i, α) is $\text{Cell}(H; r_k(m), r_k(m + K))$.

Proof. In the sum $m + o_E$ the low part $\ell + o_E$ produces the carries of $\kappa(\ell, o_E)$, and a carry out of bit $k - 1$ exactly when $J_H(E) = 1$; that carry then runs through the $r_k(m)$ ones of m that start at bit k . The same holds for $m + K$, whose low part is also ℓ . (One of the two penalties is 0: the bit k is 1 in exactly one of m and $m + K$.) ■

Windows. For $t \in I$ let $W_t := [t, \text{next}(t))$, of length h_t . The **type** of t is the bit T_t of ℓ at position t . Let $r_u(\ell)$ be the number of consecutive ones of ℓ starting at bit u , and put

$$\kappa_t := \min(r_t(\ell), h_t) \quad \text{if } T_t = 1, \quad \kappa_t := 1 + \min(r_{t+1}(\ell), h_t - 1) \quad \text{if } T_t = 0;$$

$$\text{sat}_t := [\kappa_t = h_t] \quad (\text{the window is saturated if } \text{sat}_t = 1), \quad \tau_t := h_t - \alpha 2^t - [T_t = 1]\kappa_t.$$

Lemma 7.6 (the gates). For $E \subseteq I$, let $c_t(E) \in \{0, 1\}$ be the carry into bit t in the sum $\ell + o_E$, and $d = [t \in E]$. Then $c_{t_1}(E) = 0$ and:

- if $T_t = 1$, the window W_t makes κ_t carries if $d \vee c_t$, and none otherwise;
- if $T_t = 0$, it makes κ_t carries if $d \wedge c_t$, and none otherwise;
- a carry leaves W_t ($c_{\text{next}(t)} = 1$) if and only if the window makes carries and is saturated.

Hence $R_H(E) = \sum_{t \in E} \tau_t - \sum_{t \in I} \kappa_t c_t(E) \cdot [(T_t = 1 \wedge t \notin E) \vee (T_t = 0 \wedge t \in E)]$, and $J_H(E) = c_k(E)$.

Proof. Inside W_t the addend o_E has at most the bit t , and ℓ has the bit T_t at t . If $T_t = 1$ and $d + c_t \geq 1$, bit t carries, and the carry runs through the ones of ℓ at $t + 1, \dots$: there are $\min(r_t(\ell), h_t)$ carries inside the window (for $d + c_t = 2$ the carry out of bit t is the same and the run starts at $t + 1$, which gives the same count). If $T_t = 0$, bit t carries only when $d = c_t = 1$, and then the carry runs through the $r_{t+1}(\ell)$ ones above t . The carry leaves the window exactly when all its positions carry. Below t_1 the addend is zero, so no carry is born there. Finally, $\kappa(\ell, o_E) = \sum_t (\text{carries in } W_t)$ and $\sum_{t \in E} \kappa_t [T_t = 1]$ is absorbed into τ . ■

Lemma 7.7 (the top digit). Let $I \neq \emptyset$, $c := \max I$, $I' := I \setminus \{c\}$, $H' := (I', c, \ell \bmod 2^c, \alpha)$, and let $T, \kappa, \text{sat}, \tau$ be the data of the window $W_c = [c, k]$ of H , of length $h := k - c$. For $D_0 \subseteq I'$:

$$\begin{aligned} R_H(D_0) &= R_{H'}(D_0) - [T = 1] \kappa J_{H'}(D_0), \\ R_H(D_0 \cup \{c\}) &= \tau + R_{H'}(D_0) - [T = 0] \kappa J_{H'}(D_0), \end{aligned}$$

$$\begin{aligned} J_H(D_0) &= [T = 1] \cdot \text{sat} \cdot J_{H'}(D_0), \\ J_H(D_0 \cup \{c\}) &= \text{sat} \cdot ([T = 1] + [T = 0] J_{H'}(D_0)), \end{aligned}$$

and for $\Delta \subseteq I$: $\gamma_I(\Delta) = \gamma_{I'}(\Delta) + h$ if $c \notin \Delta \neq \emptyset$, and $\gamma_I(\Delta_0 \cup \{c\}) = \gamma_{I'}(\Delta_0)$. Consequently the first half of the base clocks of H (the dancers without c) is $x_{H'} + d_1$, with

$$d_1(D_0) := -\tau - [T = 1] \kappa J_{H'}(D_0) + [T = 0] \kappa J_{H'}(I' \setminus D_0),$$

and the second half is its mirror-negative.

Proof. The windows of the digits of I' are the same in H and in H' (the predecessor of c has next = c in both), with the same types and data; the carry into W_c is $c_c(E) = J_{H'}(E \cap I')$. Apply Lemma 7.6 to H and to H' . For γ : the digit c is skipped by every non-empty $\Delta \not\ni c$, and is never skipped otherwise. The clocks follow by subtraction, with $I \setminus D_0 = (I' \setminus D_0) \cup \{c\}$. ■

7.4 Theorem F

Theorem 7.8 (Theorem F, the floor). For every house $H = (I, k, \ell, \alpha)$, let $\hat{y}_H$ be the integer fit of x_H . There is a function $V : 2^I \rightarrow \mathbb{Z}$ with

- (E) $V(D) - V(I \setminus D) = \hat{y}_H(D)$ for every D ;
- (M) V non-increasing in o-order;
- (G) $V(a) - V(b) \leq \hat{c}_H(a, b)$ for every $b \subsetneq a$;

and moreover:

- (Cut) x_H is cut at the first dancer b with $J_H(b) = 1$ (if any), and hence at its mirror; and these cuts are **strict**: $\text{fit}(x_H)$ is larger at the dancer before b than at b , and likewise at the mirror. ($b \neq \emptyset$, since $J_H(\emptyset) = [\ell \geq 2^k] = 0$.)
- (Λ) $|\hat{y}_H| \leq \alpha(2^k - 1)$.

Proof. Induction on $|I|$. For $I = \emptyset$ there is one dancer, $x = \hat{y} = 0$, $V := 0$, and $J_H(\emptyset) = [\ell \geq 2^k] = 0$. Let $I \neq \emptyset$ and let H' , c , T , κ , sat , τ , h be as in Lemma 7.7, with V' , $\hat{y}'$, J' given by the induction hypothesis for H' . Write $X_1 := \hat{y}' + d_1$ on the dancers $D_0 \subseteq I'$ and X_2 for its mirror-negative.

1. **The walk.** The sequence d_1 is integer, non-increasing, and changes only at the first dancer with $J' = 1$ and at its mirror (J' is an up-set; $J' \circ \text{mirror}$ is a down-set). These are strict cuts of $x_{H'}$, by (Cut) for H' . So, by Lemma 7.2(b), the real fit of $x_{H'} + d_1$ is $\text{fit}(x_{H'}) + d_1$, with the level sets of $\text{fit}(x_{H'})$, and its integer fit is X_1 . By Lemma 7.3,

$$\hat{y}_H = [\max(X_1, 0), \min(X_2, 0)].$$

2. **(Λ).** $d_1 \leq -\tau + [T = 0]\kappa = \alpha 2^c - h + \kappa \leq \alpha 2^c$, since $\kappa \leq h$. With (Λ) for H' , $X_1 \leq \alpha(2^c - 1) + \alpha 2^c \leq \alpha(2^k - 1)$; so $0 \leq \max(X_1, 0) \leq \alpha(2^k - 1)$, and the second half follows by antisymmetry.

3. **(Cut).** By Lemma 7.7, $J_H = [T = 1] \cdot \text{sat} \cdot (J', 1)$, or $\text{sat} \cdot [T = 0] \cdot (0, J')$, or $J_H \equiv 0$ (we write a function on 2^I as the pair of its values on the two halves).

(a) $T = 1$, $\text{sat} = 1$, $J' \not\equiv 0$: the boundary b of J_H is the boundary of J' in the first half. Saturation of type 1 means $\kappa = h$, so $\tau = -\alpha 2^c$ and $d_1 = \alpha 2^c - hJ'$. Just before b , $J' = 0$ and $X_1 = \hat{y}' + \alpha 2^c \geq \alpha > 0$ by (Λ) for H' ; so the real fit Z_1 of the first half is positive before b , and by Lemma 7.3 $\text{fit}(x_H)_{<b} = (Z_1)_{<b}$. The first half is cut at b (step 1), so $(Z_1)_{<b} = \text{fit}((x_H)_{<b})$, and Lemma 7.2(c) gives the cut. It is strict: on the first half $\text{fit}(x_H) = \max(Z_1, 0)$ (Lemma 7.3), and $Z_1 = \text{fit}(x_{H'}) + d_1$ with $\text{fit}(x_{H'})$ non-increasing and d_1 dropping by $h \geq 1$ at b ; so, if b^- is the dancer before b , $Z_1(b^-) \geq Z_1(b) + h$, and $Z_1(b^-) > 0$. So $\max(Z_1, 0)$ drops strictly at b .

(b) $T = 1$, $\text{sat} = 1$, $J' \equiv 0$: the boundary is the middle of the sequence, and $X_1(I') = \hat{y}'(I') + \alpha 2^c \geq \alpha > 0$; so $Z_1 > 0$, $\text{fit}(x_H)$ restricted to the first half is Z_1 , and Lemma 7.2(c) gives the cut at the middle. It is strict: $\text{fit}(x_H)$ is $Z_1(I') > 0$ at I' and $\min(Z_2(\{c\}), 0) \leq 0$ at $\{c\}$.

(c) $T = 0$, $\text{sat} = 1$, $J' \not\equiv 0$: the boundary is $b' \cup \{c\}$, with b' the boundary of J' . Saturation of type 0 means $\kappa = h$ and $\tau = h - \alpha 2^c$. The second half of the clocks is $x_{H'} + d_2$ with $d_2(D_0) = \tau - [T = 0]\kappa J'(D_0) + [T = 1]\kappa J'(I' \setminus D_0)$ (Lemma 7.7), which is non-increasing and changes only at cuts of $x_{H'}$; so the second half is cut at $b' \cup \{c\}$, and its integer fit is $X_2 = \hat{y}' + d_2$. Here $X_2(b') = \hat{y}'(b') + \tau - \kappa = \hat{y}'(b') - \alpha 2^c \leq -\alpha < 0$, so the real fit Z_2 of the second half is negative from $b' \cup \{c\}$ on, $\text{fit}(x_H)_{\geq b' \cup \{c\}} = (Z_2)_{\geq b' \cup \{c\}} = \text{fit}((x_H)_{\geq b' \cup \{c\}})$, and Lemma 7.2(c) gives the cut. It is strict: $b' \neq \emptyset$ (as $J'(\emptyset) = 0$), on the second half $\text{fit}(x_H) = \min(Z_2, 0)$ (Lemma 7.3), and $Z_2 = \text{fit}(x_{H'}) + d_2$ (Lemma 7.2(b), b' being a strict cut of $x_{H'}$) drops by at least $\kappa = h \geq 1$ at $b' \cup \{c\}$, where it is negative. So $\min(Z_2, 0)$ drops strictly at $b' \cup \{c\}$.

(d) Otherwise $\text{sat} = 0$, or $T = 0$ and $J' \equiv 0$; then $J_H \equiv 0$ (Lemma 7.7), and there is nothing to prove. In each case the mirror cut is strict by antisymmetry (§7.1).

4. **The inherited potential.** On the first half put $v'(D_0) := V'(D_0) - [T = 1]\kappa J'(D_0)$, and on the second half $v'(D_0 \cup \{c\}) := \tau + V'(D_0) - [T = 0]\kappa J'(D_0)$. Then $v' = R_H + \omega$ with $\omega(D) := (V' - R_{H'})(D \cap I')$, and:

(G) for v' , with **slack h on same-half arrows**: $\hat{c}_H(a, b) \geq v'(a) - v'(b) + h$ if $c \in a, b$ or $c \notin a, b$, and $\hat{c}_H(a, b) \geq v'(a) - v'(b)$ if $c \in a \setminus b$. Indeed $v'(a) - v'(b) - R_H(a) + R_H(b) = \omega(a) - \omega(b) = (V' - R_{H'})(a \cap I') - (V' - R_{H'})(b \cap I')$, which is $\leq \gamma_{I'}((a \setminus b) \cap I')$ by (G) for H' (and 0 when $a \cap I' = b \cap I'$), and Lemma 7.7 compares γ_I with $\gamma_{I'}$.

(E) for v' with X_1 : $v'(D_0) - v'(I \setminus D_0) = (V'(D_0) - V'(I \setminus D_0)) + d_1(D_0) = \hat{y}'(D_0) + d_1(D_0) = X_1(D_0)$, by (E) for H' ; and on the second half with X_2 .

(M) for v' inside each half (V' is non-increasing and J' is an up-set).

5. **The junction.** $v'(\{c\}) - v'(I') = -X_1(I')$, and, by (Λ) for H' ,

$$-X_1(I') = -\hat{y}'(I') + \tau + [T = 1]\kappa J'(I') - [T = 0]\kappa J'(\emptyset) \leq \alpha(2^c - 1) + h - \alpha 2^c = h - \alpha.$$

6. **No central flat.** If $X_1(I') > 0$, then $\hat{y}_H = (X_1, X_2)$, and $V := v'$ satisfies (E), (M) (at the junction by step 5) and (G).

7. **The central flat.** If $X_1(I') \leq 0$, let $F_1 := \{X_1 \leq 0\}$, a non-empty final segment of the first half containing I' , and F_2 its mirror, an initial segment of the second half containing $\{c\}$. Put $F := F_1 \cup F_2$; $\hat{y}_H = 0$ exactly on F . Let f be the first dancer of F_1 , p the dancer just before f (if any) and $p^* = I \setminus p$ the dancer just after F (if any). Put

$$c_3 := v'(f) = \max_{F_1} v', \quad c_4 := v'(I \setminus f) = \min_{F_2} v'.$$

Then $c_3 - c_4 = X_1(f) \leq 0$ by (E) for v' , $c_3 \leq v'(p)$, $v'(p^*) \leq c_4$, and $v'(p^*) < v'(p)$ (because $v'(p) - v'(p^*) = X_1(p) > 0$). So **the interval** $[\max(c_3, v'(p^*)), \min(c_4, v'(p))]$ **is non-empty**. Choose an integer c_F in it, and put $V := c_F$ on F and $V := v'$ elsewhere.

8. **Checks.** (E): on F both sides are c_F and $\hat{y}_H = 0$; off F nothing changed. (M): V is constant on the segment F , and $v'(p) \geq c_F \geq v'(p^*)$. (G) for an arrow $a \rightarrow b$ ($b \subsetneq a$, so b comes before a in o-order) with at least one end in F . Since F_1 is a final segment of the first half and F_2 an initial segment of the second, the cases are:

- $a \in F_2$, b before F : $c_F \leq c_4 \leq v'(a)$, and $v'(a) - v'(b) \leq \hat{c}_H(a, b)$.
- $a \in F_1$, b before F (same half): $c_F \leq c_4 \leq v'(\{c\}) \leq v'(I') + h - \alpha \leq v'(a) + h - \alpha$, and $\hat{c}_H(a, b) \geq v'(a) - v'(b) + h$; so $c_F - v'(b) \leq \hat{c}_H(a, b) - \alpha$.
- $b \in F_1$, a after F : $c_F \geq c_3 \geq v'(b)$, and $v'(a) - v'(b) \leq \hat{c}_H(a, b)$.
- $b \in F_2$, a after F (same half): $c_F \geq c_3 \geq v'(I') \geq v'(\{c\}) - h + \alpha \geq v'(b) - h + \alpha$, and $\hat{c}_H(a, b) \geq v'(a) - v'(b) + h$; so $v'(a) - c_F \leq \hat{c}_H(a, b) - \alpha$.
- Both ends in F , same half: $\hat{c}_H(a, b) \geq v'(a) - v'(b) + h \geq v'(I') - v'(\{c\}) + h \geq \alpha > 0$. If both lie in F_1 , then $v'(a) \geq v'(I')$ and $v'(b) \leq c_3 \leq c_4 \leq v'(\{c\})$; if both lie in F_2 , then $v'(a) \geq c_4 \geq c_3 \geq v'(I')$ and $v'(b) \leq v'(\{c\})$. Both ends in F , $a \in F_2$ and $b \in F_1$: $\hat{c}_H(a, b) \geq v'(a) - v'(b) \geq c_4 - c_3 \geq 0$.

In every case $V(a) - V(b) \leq \hat{c}_H(a, b)$. ■

The construction has a plain reading: at each new digit **exactly one new flat appears, mirror-symmetric, and it pays one price, chosen between the dearest dancer of its lower half and the cheapest of its upper half; every other dancer keeps the price it had one digit below.**

7.5 Every cell

Lemma 7.9 (integrality). In every house the real fit of x_H takes integer values. So does the real fit of the clocks of every penalty cell $\text{Cell}(H; P_r, P_c)$, and that of x_H with its first dancer deleted, in the house $\ell = 2^k - 1$. Consequently, in each of these the integer fit is the real fit, and it is non-increasing.

Proof. Induction on $|I|$. For $I = \emptyset$ the sequence is (0). Let $I \neq \emptyset$, with H' and d_1 as in Lemma 7.7. The shift d_1 is integer, non-increasing, and changes only at cuts of $x_{H'}$ (step 1 of the proof

of Theorem 7.8); so, by Lemma 7.2(b), the real fit Z_1 of the first half is $\text{fit}(x_{H'}) + d_1$, integral by induction, and by Lemma 7.3 $\text{fit}(x_H) = [\max(Z_1, 0), \min(Z_2, 0)]$ is integral. The penalty shift of a cell is integer, non-increasing, and changes only at cuts of x_H (Theorem 7.8, (Cut)), so Lemma 7.2(b) again gives an integral fit. In the house $\ell = 2^k - 1$ with $I \neq \emptyset$, x_H is cut at its second dancer $\{t_1\}$ (Theorem 7.8, (Cut)), so the fit of the sequence without its first dancer is the restriction of $\text{fit}(x_H)$. Finally, on a level set of integral value z with p positions the sum is $S = pz$, so $S \bmod p = 0$ and the integer fit is z there. ■

Strictness is not used in this proof: a cut is enough. As written, strictness is also used in Theorem 7.8 (steps 1 and 3) and in Theorem 7.10, to identify integer fits; with Lemma 7.9 a cut suffices there (in Theorem 7.8 by a joint induction with Lemma 7.9, which uses the (Cut) of Theorem 7.8). It is indispensable only in Proposition 9.6 (H), twice: in the induction for the base cell, and for the penalty shift. Lemma 7.9 also shows that the integer part of Lemma 7.2(b) was never needed beyond integral fits; we keep the strict (Cut) of Theorem 7.8 because Proposition 9.6 needs it.

Theorem 7.10. For every family λ , every $\alpha \geq 1$ and every shift $i \geq 1$, $T(r) \geq Y(r)$ for every r , and the Smith exponents of $M^{(\alpha)}(\lambda, i)$ are $\alpha K + \kappa(m, K) + \hat{y}_D$ ($D \subseteq I$), where $\hat{y}$ is the integer fit of the clocks x_D of §5. For $i = 0$ the Smith exponents are ∞ (once) and $\alpha K + \hat{y}_D^\#$ ($D \neq \emptyset$), where $\hat{y}^\#$ is the integer fit of the sequence $x_D^\#$ ($D \neq \emptyset$) of Proposition 5.4.

Proof. The case $i \geq 1$. By Lemma 7.5 the cell is $\text{Cell}(H; P_r, P_c)$ with $H = (I, k, m \bmod K, \alpha)$. Let V be given by Theorem 7.8 and put $v := V - P_r J_H$, $w := V - P_c J_H$. (U1) holds because J_H is an up-set. (U3): $v(a) - w(b) - \hat{c}(a, b) = V(a) - V(b) - \hat{c}_H(a, b) \leq 0$ by (G), and it is 0 on loops. (U2): $v(D) - w(I \setminus D) = \hat{y}_H(D) - P_r J_H(D) + P_c J_H(I \setminus D)$, which is the integer fit of the clocks of the cell by Lemma 7.2(b): the penalty shift $-P_r J_H(D) + P_c J_H(I \setminus D)$ is integer, non-increasing, and changes only at the first dancer with $J_H = 1$ and at its mirror, which are strict cuts of x_H (Theorem 7.8, (Cut)). Theorem 7.4 gives $T(r) \geq Y(r)$; Lemma 7.9 gives monotonicity, and Theorem 6.5 concludes.

The case $i = 0$. The row $\emptyset$ of $M(\lambda, 0)$ is zero, so the Smith exponents are ∞ and those of the submatrix $M^\#$ of the rows $D \neq \emptyset$. By Proposition 5.4 its normalized costs are those of the house $H^\# = (I, k, K - 1, \alpha)$, for which $J_{H^\#}(D) = [D \neq \emptyset]$, and $x_D^\# = x_{H^\#}(D)$ for $D \neq \emptyset$. If $I \neq \emptyset$, the cut of $x_{H^\#}$ at the dancer $\{t_1\}$ is strict (Theorem 7.8, (Cut)), so $\emptyset$ is a level set by itself, and the fit and the integer fit of $x^\#$ (the sequence without $\emptyset$) are those of $x_{H^\#}$ restricted to $D \neq \emptyset$; if $I = \emptyset$, $M(\lambda, 0) = 0$. For $r \leq N - 1$ the rows L_r avoid $\emptyset$ and the columns F_r avoid I . So Lemma 6.1, Theorem O, Corollary 6.3 and Lemma 6.4 hold for $M^\#$ with the same proofs: they involve only the rows L_r and the columns F_r , and every entry of $M^\#$ in the support is non-zero, because no window of a row $D \neq \emptyset$ contains $d = 0$; and Theorem 7.4 holds with $v = w = V$ restricted to the dancers that occur. ■

8. Proof of the Main Theorem

Let $n \geq 1$. By Theorem 3.8,

$$\mathbb{Z}_2 \oplus \text{Syl}_2 K(Q_n) \cong \bigoplus_{\lambda} X^{(1)}(\lambda, (n - \lambda)/2)^{m_\lambda(n)},$$

the sum over $\lambda \equiv n \pmod{2}$ with $1 \leq \lambda \leq n$ (Lemma 3.7: $m_0(n) = 0$). Fix a family λ and $i = (n - \lambda)/2$. By Theorem D, $X^{(1)}(\lambda, i)$ is the small clocks plus $\text{coker}(2^k M)$, $M = M^{(1)}(\lambda, i)$.

- If $i \geq 1$, by Theorem 7.10 (with $\alpha = 1$) the Smith exponents of M are $K + \kappa(m, K) + \hat{y}_D$, where $\hat{y}$ is the integer fit of the clocks x_D . Those of $\text{coker}(2^k M)$ are $k + K + \kappa(m, K) + \hat{y}_D$.

By Proposition 5.3 the raw clocks are $u_i(D) = k + K + \kappa(m, K) + x_D$, and adding an integer constant commutes with the integer fit. So these exponents are the pooled clocks $\hat{y}_i(D)$ of §1.2.

- If $i = 0$ (so $\lambda = n$ and $m_n(n) = 1$), the Smith exponents of M are ∞ and $K + \hat{y}_D^\#$ ($D \neq \emptyset$), and by Proposition 5.4 $u_0(D) = k + K + x_D^\#$. So $\text{coker}(2^k M)$ is $\mathbb{Z}_2$ (the big clock $\hat{y}_0(\emptyset) = \infty$) plus the pooled clocks of the sequence $u_0(D)$, $D \neq \emptyset$. This is the convention of §1.2: the entry ∞ stands first and is not pooled.
- If $I = \emptyset$, M is 1×1 . For $i \geq 1$ its exponent is $K + \kappa(m, K)$ (Proposition 5.2 with $x_\emptyset = 0$), and $u_i(\emptyset) = \varphi(K) + \kappa(i - 1, K) = k + K + \kappa(m, K)$. For $i = 0$, $M = 0$.

The fits are integral, so the integer fits are the fits themselves, and they are non-increasing (Lemma 7.9). This proves the Main Theorem. $\blacksquare$

9. Consequences

9.1 Bai's counts

Lemma 9.1. Every raw clock $u_i(D)$ with $(i, D) \neq (0, \emptyset)$ is at least 2, and so is every pooled big clock.

Proof. $\varphi(\xi) \geq \xi \geq 1$, and $\kappa \geq 0$. If $\xi = K + o_{I \setminus D} - o_D = 1$, then, since $o_D \leq o_I < K$, we need $o_{I \setminus D} = 0$ and $o_D = K - 1$, that is $D = I = [0, k]$; then $h(D) = k \geq 1$ (the family has $\lambda \geq 1$). So $u_i(D) \geq 2$. A level set of numbers that are all ≥ 2 has mean ≥ 2 , and its integer fit takes values $\geq \lfloor \text{mean} \rfloor \geq 2$. $\blacksquare$

Proof of Corollary 1.5. A copy of a family with $k \geq 1$ contributes

$$\sum_{a=1}^{k-1} 2^{|I|+k-1-a} + 2^{|I|} = 2^{|I|+k-1} = \dim T(\lambda)/2$$

cyclic factors, all non-trivial (Lemma 9.1), one of which is $\mathbb{Z}_2$ exactly for $\lambda = n$. By Lemma 3.7 only families with $k \geq 1$ occur, and $\sum_\lambda m_\lambda(n) \dim T(\lambda) = 2^n$ (Theorem 3.6). So $\text{Syl}_2 K(Q_n)$ has $2^{n-1} - 1$ cyclic factors. A factor $\mathbb{Z}/2$ can only be a small clock with $a = 1$, and a family with $k \geq 2$ has $2^{|I|+k-2} = \dim T(\lambda)/4$ of them. The families with $k \leq 1$ and $\lambda \geq 1$ are $\lambda = 1$ ($\dim 2$) and $\lambda = 2$ ($\dim 4$). So the number of factors $\mathbb{Z}/2$ is $(2^n - 2m_1(n) - 4m_2(n))/4$, which by Lemma 3.7 is $2^{n-2} - 2^{(n-3)/2}$ for odd n and $2^{n-2} - 2^{(n-2)/2}$ for even $n \geq 2$; that is a_n . $\blacksquare$

9.2 The podium

In this subsection the family is λ , with digits $t_1 < \dots < t_s$ of I ($t_{s+1} := k$), and $P_l := \{t_1, \dots, t_l\}$. Put

$$Y_l := \lambda + 1 - o_{P_l} = K + o(I \setminus P_l), \quad \text{so} \quad v_2(Y_l) = t_{l+1},$$

and $\psi(E) = \sum_{t \in E} (h_t - 2^t)$ as in §5, $\psi_t := \psi(\{t\}) = h_t - 2^t$. Recall $g(N) = \max_{1 \leq y < N} \varphi(y)$.

Lemma 9.2. (a) $\varphi(\lambda + 1) + \psi(P_l) = \varphi(Y_l)$ for $0 \leq l \leq s$. (b) For $N \geq 1$, $\max_{1 \leq y \leq N} \varphi(y)$ is attained at a **clearing** of N : a number obtained from N by clearing some of its lowest set bits, not all of them. (c) $Y_0, \dots, Y_s$ are the clearings of $\lambda + 1$; hence $\max_l \varphi(Y_l) = g(\lambda + 2)$, and $\max_{l \geq 1} \varphi(Y_l) = \max_{1 \leq y \leq Y_1} \varphi(y)$.

Proof. (a) $\psi(P_l) = \sum_{j \leq l} (t_{j+1} - t_j) - o_{P_l} = t_{l+1} - t_1 - o_{P_l}$, and $\varphi(\lambda + 1) = \lambda + 1 + t_1$. (b) Given $y \leq N$ with $v_2(y) = q$, let N_q be N with its bits below q cleared. Then $y \leq N_q$, because y is a multiple of 2^q not above N ; and $v_2(N_q) \geq q$, so $\varphi(N_q) \geq \varphi(y)$; and N_q is a clearing. (c) is (b) for $N = \lambda + 1$ and $N = Y_1$. ■

For $N = n - 1$, part (b) is the observation of [GMMY24, Remark 4.4].

Proposition 9.3 (shift 1). Let $i = 1$ and write $e(D)$ for the pooled big clocks. Then

- (a) $e(\emptyset) = \max_{0 \leq l \leq s} \varphi(Y_l)$;
- (b) $e(D) \leq \max_{1 \leq l \leq s} \varphi(Y_l)$ for $D \neq \emptyset$;
- (c) if $0 \in I$, then $e(\{0\}) = \max_{1 \leq l \leq s} \varphi(Y_l)$.

Proof. At $i = 1$, $m = 0$ and $\ell = 0$. In the house $(I, k, 0, 1)$ every window is of type 0 with $\kappa_t = 1$, no carry is ever born (Lemma 7.6), and $J \equiv 0$. So $R_0 = R_K = \psi$ and the cell is its base cell. In the proof of Theorem 7.8 at the level j (digit $c = t_j$) the datum τ of the window (§7.3) is ψ_{t_j} , and $d_1 = -\psi_{t_j}$. With the antisymmetry of $\hat{y}_{j-1}$, step 1 reads

$$\begin{aligned} \hat{y}_j(D_0) &= \max(\hat{y}_{j-1}(D_0) - \psi_{t_j}, 0) \quad \text{and} \\ \hat{y}_j(D_0 \cup \{t_j\}) &= \min(\hat{y}_{j-1}(D_0) + \psi_{t_j}, 0). \end{aligned}$$

Moreover $e(D) = k + K + \hat{y}_s(D)$ and $k + K = \varphi(\lambda + 1) + \psi(I)$ (proof of Proposition 5.3).

- (a) The recursion $y_j = \max(y_{j-1} - \psi_{t_j}, 0)$ with $y_0 = 0$ gives

$$\hat{y}_s(\emptyset) = \max_l \left(- \sum_{j>l} \psi_{t_j} \right).$$

So $e(\emptyset) = \max_l [\varphi(\lambda + 1) + \psi(P_l)] = \max_l \varphi(Y_l)$, by Lemma 9.2(a).

- (b) Let b be the largest index with $t_b \in D$. Then $\hat{y}_b(D \cap P_b) \leq 0$, and from level b on the same recursion gives $\hat{y}_s(D) \leq \max_{l \geq b} (-\sum_{j>l} \psi_{t_j})$.

- (c) $t_1 = 0$ and $\psi_0 = h_0 - 1 \geq 0$, so $\hat{y}_1(\{0\}) = \min(\psi_0, 0) = 0$. From level 1 on, the recursion is that of (a) started at 0. ■

Proposition 9.4 (shift 0). Let $\lambda = n$ and $i = 0$, and let $e(D)$ ($D \neq \emptyset$) be the pooled big clocks. Then

- (a) $e(\{t_1\}) = \max(\varphi(Y_1) - 2^{t_1}, \max_{2 \leq l \leq s} \varphi(Y_l))$;
- (b) $e(D) \leq \max_{2 \leq l \leq s} \varphi(Y_l)$ for every other $D \neq \emptyset$. (A maximum over an empty set is $-\infty$.)

Proof. By Theorem 7.10 the clocks are those of the house $H^\sharp = (I, k, K - 1, 1)$, shifted by $k + K$, with the dancer $\emptyset$ deleted after pooling. In $H^\sharp$ every window is of type 1 and saturated ($\kappa_t = h_t$, $\tau_t = -2^t$ in the notation of §7.3), and at every level $J' = [D_0 \neq \emptyset]$. So, in step 1 of the proof of Theorem 7.8, $d_1(D_0) = 2^{t_j} - h_{t_j}[D_0 \neq \emptyset]$. At level 1 ($I' = \emptyset$), $X_1(\emptyset) = 2^{t_1}$ and $\hat{y}_1(\{t_1\}) = -2^{t_1}$. At every level $j \geq 2$ a dancer $D_0 \neq \emptyset$ of the first half follows $y_j = \max(y_{j-1} + 2^{t_j} - h_{t_j}, 0)$, where $2^{t_j} - h_{t_j} = -\psi_{t_j}$. Since $k + K - \sum_{j>l} \psi_{t_j} = \varphi(Y_l)$ (as in Proposition 9.3), (a) follows. For (b), let $b \geq 2$ be the largest index with $t_b \in D$: at level b the value is ≤ 0 , and the recursion takes over. ■

Lemma 9.5 (Lemma K). For $a \geq 0$, $B \geq 1$ and $q \geq 0$: $\varphi(2^q B) + \kappa(a, B) \leq \max_{1 \leq y \leq 2^q(a+B)} \varphi(y)$.

Proof. If $\kappa(a, B) = 0$, take $y = 2^q B$. Otherwise let the highest carry of $a + B$ leave the bit $p - 1$. Carries leave only bits $v_2(B), \dots, p - 1$, so $\kappa \leq p - v_2(B)$. A carry leaves bit $p - 1$, so $(a \bmod 2^p) + (B \bmod 2^p) \geq 2^p$ and $(a + B) \bmod 2^p < a \bmod 2^p \leq a$. Then

$$y := 2^q((a + B) - ((a + B) \bmod 2^p))$$

satisfies $y \leq 2^q(a+B)$, $y \geq 2^q(B+1)$ and $v_2(y) \geq q+p$, so $\varphi(y) > 2^q B + q + v_2(B) + \kappa = \varphi(2^q B) + \kappa$. ■

Proposition 9.6 (a ceiling for the big clocks). In the cube Q_n , every big clock of every family at a shift $i \geq 1$ is at most $g(n-i+1) = \max_{1 \leq y \leq n-i} \varphi(y)$.

Proof. (H) *The head is dyadic.* The first level set of the real fit of the clocks of the cell is $\{D : D \subseteq P_l\}$ for some l . In the base cell this follows by induction along step 1 of the proof of Theorem 7.8, which keeps the level sets of $\text{fit}(x_{H'})$ (the house H' of Lemma 7.7) because d_1 changes only at strict cuts (Lemma 7.2(b)). The first level set of the first half is the first level set of $\text{fit}(x_{H'})$ if Z_1 is positive there; otherwise $Z_1 \leq 0$ on the whole first half, the fit vanishes on the whole level, and the first level set is everything. The penalty shift of a real cell keeps the level sets, because it changes only at strict cuts (Theorem 7.8, (Cut), and Lemma 7.2(b)); this is where strictness is needed, since a jump of the shift at a cut that is not strict, inside the head, would split it. So the largest big clock is the mean μ of the raw clocks $u_i(D)$, $D \subseteq P_l$, which is an integer (Lemma 7.9). (M) *The mean over a dyadic head.* By steps 1–2 of the proof of Proposition 5.3, $u_i(D) = \varphi(\lambda+1) + 2\psi(D) + \kappa(m, K + o_{I \setminus D}) - \kappa(m, o_D)$. Over $D \subseteq P_l$ each digit of P_l lies in half the sets, so the mean of $2\psi(D)$ is $\psi(P_l)$, and $\varphi(\lambda+1) + \psi(P_l) = \varphi(Y_l)$. Also $K + o_{I \setminus D} = Y_l + o_{P_l \setminus D}$, and $D \mapsto P_l \setminus D$ permutes the subsets of P_l . So

$$\mu = \varphi(Y_l) + \text{mean}_{D \subseteq P_l} [\kappa(m, Y_l + o_D) - \kappa(m, o_D)].$$

Let $q := t_{l+1} = v_2(Y_l)$, write $m = 2^q m_h + m_\ell$ with $m_\ell < 2^q$, and put $c_D := [m_\ell + o_D \geq 2^q]$ (note $o_D < 2^q$). The low parts carry in the same way in both sums, and $s_2(Y_l + o_D) = s_2(Y_l/2^q) + s_2(o_D)$, so

$$\kappa(m, Y_l + o_D) - \kappa(m, o_D) = \kappa(m_h + c_D, Y_l/2^q).$$

(K) *The bound.* Apply Lemma 9.5 with $B = Y_l/2^q$ and $a = m_h + c_D$. If $c_D = 1$, then $2^q \leq m_\ell + o_D \leq m_\ell + o_{P_l}$. In both cases $Y_l + 2^q(m_h + c_D) \leq Y_l + m + o_{P_l} = \lambda + 1 + m = n - i$. So every term of the mean is $\leq g(n-i+1)$, and so is μ . ■

Lemma 9.7 (multiplicities). $m_n(n) = 1$; $m_{n-2}(n) = n - 1 - [n \text{ even}]$ for $n \geq 2$; and for odd $n \geq 5$,

$$m_{n-4}(n) = (n-1)(n-2)/2 - 1 - [n \equiv 1 \pmod{4}] \geq 4.$$

Proof. By Theorem 3.6 the weight multiplicities of $V^{\otimes n}$, $C(n, j)$ at $n - 2j$, are $\sum_\lambda m_\lambda(n) \text{mult}_{n-2j} T(\lambda)$, and $T(\lambda)$ has highest weight λ with multiplicity 1. From the character of §3.3, $\text{mult}_{\lambda-2} T(\lambda) = [k \geq 1] + [0 \in I]$ (one factor $e^{\pm 1}$ turned), and $\text{mult}_{\lambda-4} T(\lambda) = [k \geq 2] + [1 \in I] + [k \geq 1][0 \in I]$ (one factor $e^{\pm 2}$, or two factors $e^{\pm 1}$). For $\lambda = n$: $0 \in I \iff n \text{ even}$, and for odd n , $1 \in I \iff n \equiv 1 \pmod{4}$. Peeling at the weights $n-2$ and $n-4$ gives the formulas, with $\text{mult}_{n-4} T(n-2) = 1$ for odd n . For $n = 5$ the last value is 4, and it increases with n . ■

Proof of Corollary 1.6. Let $n \geq 4$ and $G = g(n-1) = \max_{1 \leq y \leq n-2} \varphi(y)$; $G \geq \varphi(n-2) \geq n-2$.

1. *Nothing else rises above G .* Small clocks are $\leq k-1 \leq \log_2(n+1) - 1 < G$. Shifts $i \geq 2$: every dancer is $\leq g(n-i+1) \leq G$ (Proposition 9.6). Shift 1 (the family $n-2$, where $\lambda+1 = n-1$): by Proposition 9.3 and Lemma 9.2, $e(\emptyset) = \max_{y \leq n-1} \varphi(y) = \max(G, \varphi(n-1))$, and every other dancer is $\leq \max_{y \leq Y_1} \varphi(y) \leq G$ because $Y_1 = n-1-2^{t_1} \leq n-2$. Shift 0 (the family n): by Proposition 9.4 every dancer other than $\{t_1\}$ is $\leq G$, since $Y_l \leq n+1-2^{t_1}-2^{t_2} \leq n-2$ for $l \geq 2$. The dancer $\{t_1\}$ is at $V_0 = \max(\varphi(n+1-2^{t_1}) - 2^{t_1}, A_0)$,

where $A_0 := \max_{2 \leq l \leq s} \varphi(Y_l) \leq G$ ($A_0 = -\infty$ if $s < 2$). So the factors above G are at most the $m_{n-2}(n)$ copies of $\varphi(n-1)$, when $\varphi(n-1) > G$, and the single V_0 , when $V_0 > G$.

2. *The value of V_0 .* If n is even, $t_1 = 0$ and $V_0 = \max(\varphi(n) - 1, A_0)$. If $n \equiv 1 \pmod{4}$, $t_1 = 1$ and $V_0 = \max(\varphi(n-1) - 2, A_0)$. If $n \equiv 3 \pmod{4}$ (or $I = \emptyset$), $V_0 \leq G$, because $n+1-2^{t_1} \leq n-3$. Also, if $n \equiv 3 \pmod{4}$, then $v_2(n-1) = 1$ and $\varphi(n-1) - 2 = n-2 \leq G$; and if n is even, $\varphi(n-1) = n-1 \leq \varphi(n-2) \leq G$.
3. *Enough factors at G .* If n is even, each of the $n-2$ copies of the family $n-2$ has two dancers at exactly G : $\emptyset$, at $\max(G, \varphi(n-1)) = G$, and $\{0\}$, at $\max_{y \leq n-2} \varphi(y) = G$ (Proposition 9.3(c); $0 \in I$ because $n-1$ is odd). That is $2(n-2)$ factors at G : at least $n+1$ for $n \geq 6$, which is enough even when $c_1 = G$; for $n = 4$, $c_1 = \varphi(4) - 1 = 5 > G = 3$, and $2(n-2) = 4 = n$ factors are enough. If n is odd, the family $n-2$ gives $n-1$ copies of $\emptyset$ at $\max(G, \varphi(n-1)) \geq G$. The family $n-4$, at shift 2, gives $m_{n-4}(n) \geq 4$ copies of a factor at exactly G : there $m = 1$, $0 \notin I$ (λ odd), and as in Proposition 9.3 no carry is born and $J \equiv 0$, so the same walk gives $e(\emptyset) = \max_{y \leq n-3} \varphi(y) = g(n-2)$, which is G because $\varphi(n-2) = n-2 \leq \varphi(n-3)$.
4. *Reading off.* For n even, $c_1 = \max(G, \varphi(n) - 1)$ and $c_2 = \dots = c_{n+1} = G$. For n odd with $\varphi(n-1) \leq G$, nothing lies above G , so $c_1 = \dots = c_{n+1} = G$. For n odd with $\varphi(n-1) > G$, the $n-1$ copies of $\varphi(n-1)$ come first. Then comes $V_0 = \varphi(n-1) - 2$ if this exceeds G (which forces $t_1 = 1$), and otherwise G ; so $c_n = \max(G, \varphi(n-1) - 2)$ and $c_{n+1} = G$.

The particular cases: $g(n) = \max(G, \varphi(n-1))$, and $v_2 n + n - 1 = \varphi(n) - 1$. For n odd, $\varphi(n) - 1 = n - 1 \leq \varphi(n-1) \leq g(n)$, which gives [GMMY24, Theorems 4.1, 4.2] for $n \geq 4$. In all cases $c_n = \max(G, \varphi(n-1) - 2)$ and $v_2(n-1) + n - 3 = \varphi(n-1) - 2$ (for n even, $n-3 < G$). For $n = 3$, $\text{Syl}_2 K(Q_3) = \{1 : 1, 3 : 2\}$ and the three statements hold. ■

9.3 Gao et al.'s Conjecture 5.4

Write $N := 2^e - 1$ and $n := 2^e$.

Lemma 9.8. For odd N' , the families of $N' + 1$ are the $\lambda + 1$, λ a family of N' , with $m_{\lambda+1}(N' + 1) = m_\lambda(N')$ and the same shift. If $\lambda + 1 = 2^k + \sum_{t \in I} 2^t$, the family $\lambda + 1$ has the same k and the set $I \cup \{0\}$, with $h'_0 = t_1 := \min(I \cup \{k\})$ and the other gaps unchanged; its small clocks are those of λ taken twice, and its dancers are D and $D \cup \{0\}$, $D \subseteq I$, adjacent in o-order.

Proof. Every family of an odd N' is odd, and $V \cdot [2l + 1] = [2l + 2]$. Since $\lambda + 1$ is even, $\lambda + 2 = 2^k + o_I + 1$ with $o_I + 1 < 2^k$. ■

Lemma 9.9. Let $J := i + o_D \geq 1$. In the cube N , $u(D) = 2^e - 2J + v_2(J) + s_2(J) + h(D)$. In the cube n , for a family with dancer D' and $J' := i + o_{D'} \geq 1$, $u(D') = 2^e + 1 - 2J' + s_2(J' - 1) + h'(D')$.

Proof. In the cube N , $\xi = \lambda + 1 - 2o_D = N + 1 - 2J = 2^e - 2J$, and $J \leq 2^{e-1} - 1$, so $\varphi(\xi) = 2^e - 2J + 1 + v_2(J)$. With $a = J - 1$, $a + \xi = (2^e - 1) - J$ and $\xi/2 = (2^{e-1} - 1) - a$, so $s_2(a + \xi) = e - s_2(J)$, $s_2(\xi) = e - 1 - s_2(a)$, and $\kappa(a, \xi) = s_2(J) - 1$. In the cube n , $\xi' = 2^e + 1 - 2J'$ is odd. With $a = J' - 1$, $a + \xi' = (2^e - 1) - a$ and $\xi' = (2^e - 1) - 2a$, so $\kappa(a, \xi') = s_2(a)$. ■

Theorem 9.10 (the mirror that doubles). For every family $\lambda < N$ of the cube N (shift $i \geq 1$) and every $D \subseteq I$, the dancers D and $D \cup \{0\}$ of the family $\lambda + 1$ of the cube n have the raw clock $u_N(D)$.

Proof. For $D' = D$: $J' = J$, $h'(D) = h(D)$, and $s_2(J - 1) = s_2(J) - 1 + v_2(J)$, so Lemma 9.9 gives $u_n(D) = u_N(D)$. For $D' = D \cup \{0\}$: $J' = J + 1$ and $h'(D \cup \{0\}) = h(D) + t_1$, so $u_n(D \cup \{0\}) = u_N(D) + t_1 - 1 - v_2(J)$. Finally, $2i = 2^e - (\lambda + 1) = 2^e - 2^k - o_I$ with $k < e$, so $v_2(i) = t_1 - 1$. Every digit of D is $\geq t_1$, so $v_2(J) = v_2(i + o_D) = t_1 - 1$. ■

Proof of Corollary 1.7. By Lemma 9.8 and Theorem 9.10, the raw sequence of $(\lambda + 1, i)$ in $\mathfrak{o}$ -order is that of (λ, i) with every entry written twice. The real fit of a doubled sequence is the doubled fit (Lemma 7.1). A doubled level set, with $2p$ positions and sum $2S$, gets $2(S \bmod p)$ copies of $\lceil S/p \rceil$ and the rest $\lfloor S/p \rfloor$, the same multiset as two copies of the original. So for $\lambda < N$ the family $\lambda + 1$ of n contributes twice what λ contributes to N , small clocks included. The family N of N ($I = \emptyset$) contributes its small clocks and $\mathbb{Z}_2$. The family $n = N + 1$ of n ($I = \{0\}$) contributes these small clocks twice, $\mathbb{Z}_2$, and the dancer $\{0\}$ at shift 0, with $\xi = 2^e - 1$, $\kappa = 0$ and $h_0 = e$: $u = 2^e + e - 1$. Cancelling $\mathbb{Z}_2$, $\text{Syl}_2 K(Q_n) \cong \text{Syl}_2 K(Q_N)^2 \oplus \mathbb{Z}/2^{2^e+e-1}$. ■

10. The fold law

10.1 The fold on the families

Let $a := x_1 x_2 \cdots x_n \in G$. The folded cube is the Cayley graph of $G/\langle a \rangle$ with the images of the generators, so its Laplacian acts on $\mathbb{Z}[G/\langle a \rangle] = R/(a - 1)R$ as multiplication by σ , and $\mathbb{Z} \oplus \bar{K}(n) \cong R/(\sigma R + (a - 1)R)$ (for $n \geq 2$ the graph is connected).

Lemma 10.1. Under the dictionary of Lemma 2.2, a acts on $V^{\otimes n}$ as

$$a = (-1)^{(n-H)/2} \exp(F), \quad \exp(F) := \sum_{r \geq 0} F^{(r)}.$$

On a summand $T(\lambda)$ of $V^{\otimes n}$ at shift $i = (n - \lambda)/2$ this is $a = (-1)^{D+i} \exp(F)$, D the floor operator of $T(\lambda)$.

Proof. On V , $x = 1 - s$ acts by $g : b_0 \mapsto b_0 - b_1, b_1 \mapsto -b_1$ (Lemma 2.1), and $g = (-1)^{D_V} \exp(F)$ ($\exp(F)b_0 = b_0 + b_1, \exp(F)b_1 = b_1$). Since $\Delta(F^{(r)}) = \sum_{a+b=r} F^{(a)} \otimes F^{(b)}$, $\exp(F)$ is group-like, and so is $(-1)^{\text{floor}}$; so $a = g^{\otimes n}$ is $(-1)^{\text{floor}} \exp(F)$ on $V^{\otimes n}$, where the floor of a pure tensor is its number of letters $b_1, (n - H)/2$. On $T(\lambda)$ the weight is $\lambda - 2D$, so $(n - H)/2 = D + i$. ■

Since a is expressed through H and the divided powers of F , it is carried by every isomorphism of Dist_A -lattices, and Theorem 3.6 gives, with

$$\begin{aligned} \tau_i &:= \sigma_i^{(1)} = F + 2(D + i) \quad \text{and} \\ \bar{X}(\lambda, i) &:= T(\lambda)/(\tau_i T(\lambda) + (a - 1)T(\lambda)) \quad (a = (-1)^{D+i} \exp(F)) : \end{aligned}$$

Corollary 10.2. For every $n \geq 2$, with the multiplicities of Theorem 3.6,

$$\mathbb{Z}_2 \oplus \text{Syl}_2 \bar{K}(n) \cong \bigoplus_{\lambda} \bar{X}(\lambda, (n - \lambda)/2)^{m_{\lambda}(n)}.$$

Proposition 10.3. If $\bar{X}(\lambda, i) \cong 2X(\lambda, i)$ for every $\lambda \geq 1$ and every $i \geq 0$ (where $X := X^{(1)}$), then Theorem DW holds.

Proof. By Theorem 3.8 and Corollary 10.2,

$$\mathbb{Z}_2 \oplus \text{Syl}_2 \bar{K}(n) \cong \bigoplus_{\lambda} (2X)^{m_{\lambda}(n)} = 2(\mathbb{Z}_2 \oplus \text{Syl}_2 K(Q_n)) \cong \mathbb{Z}_2 \oplus 2 \text{Syl}_2 K(Q_n);$$

cancel $\mathbb{Z}_2$ (the torsion submodule of a finitely generated $\mathbb{Z}_2$ -module is determined). Passing to $2 \cdot \text{Syl}_2 K(Q_n)$ removes exactly the factors $\mathbb{Z}/2$ of $\text{Syl}_2 K(Q_n)$, whose number is a_n (Corollary 1.5). For $n = 1$ both groups are trivial. $\blacksquare$

So the fold law is a statement family by family. We prove it on the half-size lattice $N := T(l)$, where $\lambda = 2l + 1$ or $\lambda = 2l + 2$.

10.2 The two families on the half-size lattice

On $\Phi(N)$ put $\text{Ch} := \sum_{s \geq 0} F^{(s)}/(2s - 1)!!$ and $\text{Sh} := \sum_{s \geq 0} F^{(s)}/(2s + 1)!!$, operators on N ($F = F_N$); Sh is invertible. With $u^2 = 2F$, $\text{Ch} = \cosh(u)$ and $\text{Sh} = \sinh(u)/u$ as power series, since $F^{(s)}/(2s - 1)!! = (2F)^s/(2s)!$. For $j \geq 0$ put

$$\Psi_j := 2(2D + j) + (\text{Ch} - (-1)^j) \cdot \text{Sh}^{-1} \quad \text{and} \quad \Sigma_c := F + 4(D + c),$$

operators on N . As power series, $(\text{Ch} - 1)\text{Sh}^{-1} = u \cdot \tanh(u/2)$ and $(\text{Ch} + 1)\text{Sh}^{-1} = u \cdot \coth(u/2)$, so

$$\Psi_{2c} = 4(D + c) + \sum_{m \geq 1} \eta_m F^{(m)}, \quad \eta_m = 2(2^{2m} - 1)B_{2m}/(2m - 1)!!,$$

$$\Psi_{2c-1} = 4(D + c) + \sum_{m \geq 1} \eta'_m F^{(m)}, \quad \eta'_m = 2B_{2m}/(2m - 1)!!,$$

with Bernoulli numbers B_{2m} ($u^{2m} = 2^m m! F^{(m)}$ and $(2m)! = 2^m m! (2m - 1)!!$). By the von Staudt–Clausen theorem [vSt40], [Cla40], $v_2(B_{2m}) = -1$, so every η_m and η'_m is a 2-adic unit; $\eta_1 = 1$, $\eta'_1 = 1/3$. In particular Ψ_j **has the payment** $4(D + \lceil j/2 \rceil)$, **an odd constant coefficient of F , and constant integral coefficients of $F^{(m)}$, $m \geq 2$.**

Lemma 10.4 (odd families). For $\lambda = 2l + 1$ and every $j \geq 0$, with $N = T(l)$ and $c := \lceil j/2 \rceil$: (a) $\bar{X}(\lambda, j) \cong \text{coker}(\Psi_j \text{ on } N)$; (b) $2X(\lambda, j) \cong \text{coker}(\Sigma_c \text{ on } N)$.

Proof. (a) On $T(\lambda) = \Phi(N)$, Proposition 3.1(b) gives $\exp(F)(b_0 n) = b_0 \text{Ch}(n) + b_1 \text{Sh}(n)$. The floor of $b_0 F^{(s)} n$ in $T(\lambda)$ is $2(f + s)$ and that of $b_1 F^{(s)} n$ is one more, so $a(b_0 n) = (-1)^j [b_0 \text{Ch}(n) - b_1 \text{Sh}(n)]$. Since Sh is invertible, $\{b_0 n, ab_0 n\}$ (n over a basis of N) is a basis of $T(\lambda)$, which is therefore free over $A[a]/(a^2 - 1)$. From $b_1 n = b_0 \text{ChSh}^{-1} n - (-1)^j ab_0 \text{Sh}^{-1} n$ and $\tau_j(b_0 n) = b_1 n + 2(2f + j)b_0 n$, τ_j has the matrix $[[P, Q], [Q, P]]$ in this basis, with $P = \text{ChSh}^{-1} + 2(2D + j)$ and $Q = (-1)^{j+1} \text{Sh}^{-1}$. On the coinvariants $T(\lambda)/(a - 1)T(\lambda) \cong N$, τ_j acts as $P + Q = \Psi_j$.

(b) By Proposition 4.1 with $\alpha = 1$, $X(\lambda, j) \cong \text{coker}(2G')$, with $G' = F + \pi'(D)$ on N and

$$\pi'(f) = -2(j + 2f)(j + 2f + 1) = -4(c + f)u_f,$$

where u_f is odd ($u_f = j + 2f + 1$ for even j , $j + 2f$ for odd j). So $2X(\lambda, j) \cong \text{coker } G'$. Right multiplication by the floor unit $(-u(D))^{-1}$ and conjugation by the floor units $s(D)$ with $s(f + 1) = -u_f^{-1} s(f)$ turn G' into Σ_c and do not change the cokernel. $\blacksquare$

For operators X, Y on N , put on $N \oplus N$

$$\text{glue}(X, Y) := [[X, 0], [(Y - X)/2, Y]],$$

so that $\text{glue}(X, Y) = L^{-1} \text{diag}(X, Y) L$ with $L = [[1, 0], [1/2, 1]]$, and $\text{glue}(X, Y) \cdot \text{glue}(X', Y') = \text{glue}(XX', YY')$.

Lemma 10.5 (even families). For $\lambda = 2l + 2$, $N = T(l)$ and every $i \geq 0$:

- (a) $\bar{X}(\lambda, i) \cong \text{coker}(\text{glue}(\Psi_{i+1}, \Psi_i) \text{ on } N \oplus N)$;
- (b) $2X(\lambda, i) \cong \text{coker}(\text{glue}(\Psi_{i+1}(\Psi_{i+1} - 2)/2, \Psi_i(\Psi_i + 2)/2) \text{ on } N \oplus N)$.

Proof. Let $\nu = 2l + 1$, so $T(\lambda) = V \otimes T(\nu)$ (§3.3) at the same shift i . By Lemma 10.1 and the coproduct, on $V \otimes M$ ($M = T(\nu)$) we have $a = g \otimes a_M$ and $\tau_i = (1 - g) \otimes 1 + 1 \otimes \tau_M$. Here $(1 - g)b_0 = b_1$ and $(1 - g)b_1 = 2b_1$, and $V \cong A[g]/(g^2 - 1)$ is the regular module with generator b_0 .

- Modulo $a - 1$, $g \equiv a_M$, so $\bar{X}(\lambda, i) \cong \text{coker}(f \text{ on } M)$ with $f := 1 + \tau_M - a_M$.
- Modulo τ_i , $g \equiv 1 + \tau_M$, so $X(\lambda, i) \cong M/((1 + \tau_M)^2 - 1)M = \text{coker}(\tau_M(\tau_M + 2))$. With $J = D + i$ and $JF = F(J + 1)$, $\tau_M(\tau_M + 2)/2 = F^{(2)} + 2F(J + 1) + 2J(J + 1)$ is integral, so $2X(\lambda, i) \cong \text{coker}(\theta \text{ on } M)$ with $\theta := \tau_M(\tau_M + 2)/2 = f \cdot h$, $h := (1 + \tau_M + a_M)/2$.

Now $M = \Phi(N)$, and by the proof of Lemma 10.4(a) every $w \in A[\tau_M, a_M]$ has the circulant matrix $[[X, Y], [Y, X]]$ in the basis $(b_0N, a \cdot b_0N)$. Adding column 2 to column 1, subtracting row 1 from row 2, and exchanging the two slots give

$$\text{coker}(w) \cong \text{coker}(\text{glue}(w_-, w_+)),$$

where $w_+ := X + Y$ and $w_- := X - Y$ are the actions of w on the coinvariants and the anti-coinvariants of a_M . They are ring maps, with $\tau_M \mapsto P + Q = \Psi_i$, $a_M \mapsto 1$, and $\tau_M \mapsto P - Q = \Psi_{i+1} - 2$, $a_M \mapsto -1$. So $f_+ = \Psi_i$, $f_- = \Psi_{i+1}$, $\theta_+ = \Psi_i(\Psi_i + 2)/2$ and $\theta_- = \Psi_{i+1}(\Psi_{i+1} - 2)/2$. ■

So the fold law asks: (odd) $\text{coker } \Psi_j \cong \text{coker } \Sigma_{[j/2]}$ on $T(l)$; (even) the two glued systems of Lemma 10.5 have the same cokernel. Note $\Psi_{i+1} - \Psi_i = 2(1 + (-1)^i \text{Sh}^{-1})$.

10.3 The formal dance

The operators of §10.2 are not payment systems: they carry the divided powers $F^{(m)}$, $m \geq 2$. We extend the dance of §4 to them.

Formal operators. Let $\mathfrak{B}$ be the set of formal sums $x = \sum_{m \geq 0} x_m(D)F^{(m)}$, where each x_m is a function $\mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}_2$ (the *coefficient at the target floor*), with the product

$$(xy)_m(f) = \sum_{a+b=m} C(m, a) \cdot x_a(f) \cdot y_b(f - a).$$

It is an associative $\mathbb{Q}_2$ -algebra (the rule is that of $F^{(a)}F^{(b)} = C(a + b, a)F^{(a+b)}$ and $F^{(a)}p(D) = p(D - a)F^{(a)}$ for a function p of the floor), and x is *integral* if every x_m takes values in A . An element of $\mathfrak{B}$ acts on every floor-graded lattice with divided powers of F — in particular on every $T(l)$ — by $x \cdot v = \sum_m x_m(f + m)F^{(m)}v$ for v of floor f (a finite sum), and this is a ring homomorphism. A value $x_m(g)$ with $g < m$ never acts, and we identify two formal sums that agree at every $g \geq m$; then the action on all the $T(l)$ together is faithful, because on the sublattice generated by the top vector of $T(l)$ the operator $F^{(m)}$ sends the vector of floor g to $C(g + m, m)$ times the vector of floor $g + m$, which is non-zero for $g + m \leq l$. A *system of size r* is an $r \times r$ matrix $G = (G[s, s'])$ over $\mathfrak{B}$, acting on $N^{\oplus r}$; $G[s, s']$ maps the slot s' (the **source*) to the slot s (the *coordinate*). The operators $\sigma_i^{(\alpha)}$, Ψ_j , Σ_c , their products, and the glued operators of Lemma 10.5 are systems of size 1 or 2.

The twist map. On $\Phi(N')$, Proposition 3.1(b) writes the action of $x \in \mathfrak{B}$ in the slots $b_0 \otimes N'$, $b_1 \otimes N'$ as the block matrix $[[x^{00}, x^{01}], [x^{10}, x^{11}]]$ (coordinate slot first, source slot second), with D, F those of N' :

$$x^{00} = \sum_q x_{2q}(2D)F^{(q)}/(2q-1)!!,$$

$$x^{11} = \sum_q x_{2q}(2D+1)F^{(q)}/(2q-1)!!,$$

$$x^{10} = \sum_q x_{2q+1}(2D+1)F^{(q)}/(2q+1)!!,$$

$$x^{01} = \sum_q ((2q+2)/(2q+1)!!) \cdot x_{2q+1}(2D)F^{(q+1)}.$$

(The floor of $b_e \otimes n'$ is $2f+e$; for instance $F^{(2q+1)}(b_1 n') = ((2q+2)/(2q+1)!!) \cdot b_0 F^{(q+1)} n'$ lands at the floor $2(f+q+1)$.) The denominators are odd. Since this is the action on $\Phi(N')$ for every N' , and the action is faithful, $x \mapsto [[x^{00}, x^{01}], [x^{10}, x^{11}]]$ is multiplicative; it is applied entrywise to systems.

The V-split. On $V \otimes M$, $F^{(m)} = 1 \otimes F^{(m)} + F \otimes F^{(m-1)}$ and the floor of $b_v \otimes y$ is v plus the floor of y . So the action of x in the slots $b_0 \otimes M$, $b_1 \otimes M$ is

$$[[x, 0], [x', x^+]], \quad x_m^+(g) := x_m(g+1), \quad x'_{m-1}(g) := x_m(g+1) \quad (m \geq 1),$$

operators on M : a system of size r on $V \otimes M$ becomes a system of size $2r$ on M .

The moves. Let G be a system of size r on N .

- If $N = \Phi(N')$, apply the twist map. Order the coordinates as (the b_1 -slots, the b_0 -slots) and the sources as (the b_0 -slots, the b_1 -slots); then $G = [[P, C], [B, \text{Dd}]]$ with **pivot** $P := G^{10}$, $C := G^{11}$, $B := G^{00}$, $\text{Dd} := G^{01}$, blocks of size r over $\mathfrak{B}'$.
- If $N = V \otimes \Phi(N')$, first V-split (size $2r$ on $\Phi(N')$), then proceed as in the first case.

The move is **clean** if P is invertible on $N'^{\oplus r'}$ and $S := \text{Dd} - BP^{-1}C \equiv 0 \pmod{2}$; then the **next system** is $G' := S/2$, of size $r' = r$ or $2r$ on N' . The **dance** of a system G on $T(l)$ along the word w of l (§4.3: the digits $t = 0, \dots, k-1$ of $l+1 = 2^k + \sum_{t \in I} 2^t$, a twist move if $t \notin I$, a split move if $t \in I$) is the sequence $G_0 = G, G_1, \dots, G_k$; it is **clean** if every move is clean, and then its **final matrix** is $M_w(G) := G_k$, a matrix over A of size $r \cdot 2^{|I|}$ (on $T(0) = A$, F acts as 0). The final slots are indexed by (s, D) , s a slot of G and $D \subseteq I$ a dancer, as in §4.3.

Lemma 10.6. (a) For an even payment system the twist move is the elimination of Proposition 4.1, and the split move (V-split followed by the twist move) has the same next system as Proposition 4.2. (b) Hence the dance of $\sigma_c^{(\alpha)}$ on $T(l)$ is clean and $M_w(\sigma_c^{(\alpha)}) = M^{(\alpha)}(l, c)$, the matrix of Theorem 4.4; and for a unit $u \in A$, $M_w(uG) = u \cdot M_w(G)$.

Proof. (a) For an even payment system $G = F + \pi(D) + K(D)$ (couplings of degree 0), $P = G^{10}$ is the identity, $B = (\pi + K)(2D)$, $C = (\pi + K)(2D+1)$ and $\text{Dd} = 2F$, so $S/2 = F - \frac{1}{2}(\pi + K)(2D) \cdot (\pi + K)(2D+1)$, which is G' of Proposition 4.1 (expand the product of the two triangular matrices). So the twist move is clean: P is invertible, and $S = 2G'$ with G' even (Proposition 4.1), so that the next move applies to G' . For the split move, Proposition 4.2 eliminates the coordinates B_n, Dd_n against the sources A_n, C_n , in the basis $(A_n, B_n, C_n, \text{Dd}_n)$; the V-split followed by the twist move eliminates the coordinates $b_0 \otimes b_1 n, \text{Dd}_n$ against the same sources A_n, C_n , in the basis $(A_n, b_0 \otimes b_1 n, C_n, \text{Dd}_n)$. The two bases differ by $B_n = b_0 \otimes b_1 n - C_n$. In the matrix of G , this replaces the remaining column of the source $b_0 \otimes b_1 n$ by itself minus

the pivot column of C_n , and the remaining row of the coordinate C by itself plus the pivot row of the coordinate $b_0 \otimes b_1$; the pivot block does not change. Neither operation changes the Schur complement: for a pivot row p , $M_{p,c} - M_{p,P}M_{P,P}^{-1}M_{P,c} = 0$, and symmetrically for a pivot column. So the two next systems coincide, slot (s, A) with the dancer $v = 0$ and slot (s, C) with $v = 1$; and the split move is clean, because its pivot block is the one eliminated in Proposition 4.2 (triangular, with the coefficient 1 on each eliminated vector) and $S = 2G''$ with G'' even (Proposition 4.2).

(b) By (a) and Proposition 4.3. For the last claim, $uDd - uB(uP)^{-1}uC = uS$. ■

Proposition 10.7. Let G be a system of size r on $T(l)$ whose dance is clean. Then

$$\text{coker}(G \text{ on } T(l)^{\oplus r}) \cong \bigoplus_{a=1}^{k-1} (\mathbb{Z}/2^a)^{r \cdot 2^{|I|+k-1-a}} \oplus \text{coker}(2^k M_w(G)).$$

In particular, two systems of the same size on $T(l)$ with clean dances and final matrices of equal Smith form have isomorphic cokernels.

Proof. If P is invertible, unimodular row and column operations turn $[[P, C], [B, Dd]]$ into $\text{diag}(P, S)$ and then into $\text{diag}(1, S)$; so $\text{coker}(2^h G_h) \cong (\mathbb{Z}/2^h)^{\text{rank } P} \oplus \text{coker}(2^h S) = (\mathbb{Z}/2^h)^{\text{rank } P} \oplus \text{coker}(2^{h+1} G_{h+1})$. Count the ranks as in Proposition 4.3. ■

10.4 Rigidity: the inverse block

In this section $G = \Xi$ is a system of size 1 on $T(l)$, $l + 1 = 2^k + \sum_{t \in I} 2^t$, $K = 2^k$. A dancer of level h is a subset D of $I \cap [0, h)$.

Lemma 10.8 (inverse block). Suppose Ξ is invertible on $T(l) \otimes \mathbb{Q}$ and every pivot met by its dance is invertible over $\mathbb{Q}_2$. Then for every level h ,

$$G_h^{-1} = 2^h \cdot \text{out}_h \circ \Xi^{-1} \circ \text{in}_h,$$

where in_h sends the coordinate (D, g) of level h (dancer D , level- h floor g) to a vector of $T(l)$ of floor $2^h g + o_D$, out_h reads the source (D, g) of level h off the vectors of $T(l)$ of floor $2^h g + o_D + 2^h - 1$, and both are fixed by the word w alone.

Proof. In the order of §10.3, $G_h = [[P, C], [B, Dd]]$ with P invertible; since G_h is invertible, so is S , and the block of G_h^{-1} on (the remaining sources) $\times$ (the remaining coordinates) is S^{-1} (the standard identity for the inverse of a block matrix). So $G_{h+1}^{-1} = 2S^{-1}$ is 2 times a block of G_h^{-1} ; compose over the levels. The remaining coordinates of a twist move are the b_0 -slots, of level- h floor $2g$, and the remaining sources the b_1 -slots, of level- h floor $2g + 1$; after a V -split, the dancer v adds v to the level- h floor. By induction, a coordinate of level h sits at the floor $2^h g + o_D$ of $T(l)$ and a source at $2^h g + o_D + (2^h - 1)$: a twist move, or a split move with $v = 0$, adds 0 to the coordinate offset and 2^h to the source offset; a split move with $v = 1$ adds 2^h and 2^{h+1} . ■

Corollary 10.9. Suppose $\Xi^{-1} = \sum_L y_L(D)F^{(L)}$ in $\mathfrak{B}$. Then for dancers $X, Y \subseteq I$,

$$(M_w(\Xi)^{-1})_{YX} = 2^k \cdot \gamma_{YX} \cdot y_L(o_Y + K - 1), \quad L := o_Y - o_X + K - 1,$$

where $\gamma_{YX} \in \mathbb{Q}_2$ depends only on the word w , and $\gamma_{YX} = 0$ if $L < 0$. The floors visited form the **window** $[o_X, o_Y + K - 1]$, which is the window of Theorem 4.4.

Proof. $\text{in}(e_X)$ has floor o_X and out reads the source Y at floor $o_Y + K - 1$; of

$$\Xi^{-1} \text{in}(e_X) = \sum_L y_L(o_X + L) F^{(L)} \text{in}(e_X)$$

only the term $L = o_Y - o_X + K - 1$ has that floor. Put $\gamma_{YX} := \text{out}_Y(F^{(L)} \text{in}(e_X))$. ■

From now on let $\alpha \geq 2$, $c \geq 1$, $\eta \in A$ a unit, and

$$\Sigma := 2^\alpha(D + c) + \eta F, \quad \Xi := \Sigma + \sum_{m \geq 2} \zeta_m(D) F^{(m)},$$

with integral functions ζ_m . Put $\pi(d) := 2^\alpha(d + c) \neq 0$ and $\zeta_1 := \eta$.

Lemma 10.10 (path sum). For every $L \geq 0$ and every floor f ,

$$y_L^\Xi(f) = y_L^\Sigma(f) \cdot (1 + e_L(f)) \quad \text{with} \quad v_2(e_L(f)) \geq \alpha - 1, \quad \text{and}$$

$$y_L^\Sigma(f) = (-1)^L L! \eta^L / \prod_{d=f-L}^f \pi(d).$$

Proof. Write $\Xi = \pi(D) + N$. Then $\Xi^{-1} = \sum_{s \geq 0} (-1)^s (\pi(D)^{-1} N)^s \pi(D)^{-1}$, and the coefficient of $F^{(L)}$ at the target floor f is the sum, over the compositions $L = m_1 + \dots + m_s$ (landing floors $f - L = f_0 < f_1 < \dots < f_s = f$), of

$$(-1)^s (L! / \prod m_i!) \cdot \prod_i \zeta_{m_i}(f_i) / \prod_{j=0}^s \pi(f_j).$$

For Σ only the composition $1 + \dots + 1$ occurs. Any other composition, divided by that term, is $\pm \prod_i \zeta_{m_i}(f_i) \cdot \eta^{-L} / \prod_i m_i!$ times the payments of the $\sum_i (m_i - 1)$ floors of $[f - L, f]$ that it skips; its valuation is at least $\sum_i [\alpha(m_i - 1) - v_2(m_i!)] \geq (\alpha - 1) \sum_i (m_i - 1) \geq \alpha - 1$, since $v_2(m!) \leq m - 1$. ■

The final matrix of Σ is known. By Lemma 10.6(b), $M_w(\Sigma) = \eta \cdot M_w(F + \eta^{-1} \pi(D))$, and the proof of Theorem 4.4 uses only the products of the payment over windows. So, with $\mathcal{A}$ the matrix of multiplication by q_k in the basis y^D ($\mathcal{A}_{DD'} = a_k(D \setminus D')$ if $D' \subseteq D$, else 0):

$$M_w(\Sigma) = \mathfrak{d}_1 \cdot \mathcal{A} \cdot \mathfrak{d}_2, \quad (\mathfrak{d}_1)_D = -2^{1-K} \eta^{1-K} \cdot (2\eta)^{o_D} / \prod_{d < o_D} \pi(d),$$

$$(\mathfrak{d}_2)_{D'} = (2\eta)^{-o_{D'}} \cdot \prod_{d \leq o_{D'} + K - 1} \pi(d),$$

diagonal and invertible, and $v_2(\mathcal{A}_{DD'}) = \delta(D \setminus D')$ exactly (Lemma 4.5).

For matrices B, E of the same size, $B \circ (1 + E)$ is the matrix with entries $B_{ij}(1 + E_{ij})$.

Lemma 10.11 (stability). If $v_2(E_{ij}) \geq 1$ for all i, j , then $(M_w(\Sigma)^{-1} \circ (1 + E))^{-1} = M_w(\Sigma) \circ (1 + E')$ with $v_2(E'_{ij}) \geq 1$ for all i, j .

Proof. Diagonal scalings commute with $\circ$, so it suffices to treat $\mathcal{A}$. Its inverse $\mathcal{A}^{-1}$ is multiplication by $q_k^{-1} = \sum_j (1 - q_k)^j$ (q_k has constant term 1 for $k \geq 1$; for $k = 0$ everything is 1×1); its (D, D') entry is a sum over the ordered partitions of $D \setminus D'$ into non-empty blocks $B_1, \dots, B_j$ of $\pm a_k(B_1) \dots a_k(B_j)$, so by the strict subadditivity of δ (§4.5) $v_2((\mathcal{A}^{-1})_{DD'}) \geq \delta(D \setminus D')$, and

it vanishes unless $D' \subseteq D$. Put $B := \mathcal{A}^{-1}$. Then $(B \circ (1 + E))^{-1} = (1 + \mathcal{A}(B \circ E))^{-1} \mathcal{A} = \sum_{j \geq 0} (-1)^j [\mathcal{A}(B \circ E)]^j \mathcal{A}$, which converges because $\mathcal{A}(B \circ E) \equiv 0 \pmod{2}$. The (D, D') entry of the j -th term is a sum over chains $D \supseteq E_1 \supseteq F_1 \supseteq \dots \supseteq E_j \supseteq F_j \supseteq D'$ of products whose valuation is at least $\delta(D \setminus E_1) + \delta(E_1 \setminus F_1) + 1 + \dots + \delta(F_j \setminus D')$, which is $\geq \delta(D \setminus D') + j$ by subadditivity. Since $v_2(\mathcal{A}_{DD'})$ is exactly $\delta(D \setminus D')$, the terms $j \geq 1$ change $\mathcal{A}_{DD'}$ by a factor $1 + E'_{DD'}$ with $v_2(E'_{DD'}) \geq 1$, and they vanish where $\mathcal{A}$ does. ■

Theorem 10.12 (rigidity). If the dance of Ξ along w is clean, then $M_w(\Xi) = M_w(\Sigma) \circ (1 + E')$ **with** $v_2(E') \geq 1$ **entrywise**. Consequently $M_w(\Xi)$ and $M_w(\Sigma)$ have the same support and the same entry valuations, the same Smith form, and $\text{coker}(\Xi \text{ on } T(l)) \cong \text{coker}(\Sigma \text{ on } T(l)) \cong X^{(\alpha)}(l, c)$.

Proof. The dance of Σ is clean (Lemma 10.6). By Corollary 10.9 for Ξ and for Σ , with the same γ , and Lemma 10.10, $M_w(\Xi)^{-1} = M_w(\Sigma)^{-1} \circ (1 + E)$ with $v_2(E) \geq \alpha - 1 \geq 1$; Lemma 10.11 inverts this. The Smith form of $M^{(\alpha)}(l, c)$ is computed in Theorems 6.5 and 7.10 from the valuations of its entries and from its support alone: the tropical bound (Lemma 6.1) and the unique golden matching (Theorem 6.2) see nothing else. Since $M_w(\Sigma) = \eta \cdot M_w(F + \eta^{-1}\pi(D))$ has the valuations and the support of $M^{(\alpha)}(l, c)$ (Theorem D, with the payment $\eta^{-1}\pi$), all three matrices have the same Smith form. Proposition 10.7 gives the cokernels; and $\text{coker } \Sigma \cong \text{coker}(F + \pi(D))$ because $\eta^{-D}(\eta F + \pi(D))\eta^D = F + \pi(D)$. ■

The case $c = 0$. Then $\pi(0) = 0$ and Ξ is not invertible. We treat it by continuity in §10.6.

10.5 Cleanliness: the ring $\mathcal{K}$

Theorem 10.12 needs the dance of Ξ to be clean. This is not automatic: the payments are functions of the floor, and the moves mix floors.

Positions. At level h the dancers are the subsets D of $P_h := I \cap [0, h)$. The **position** of the dancer D at the level- h floor f is

$$y := 2^h f + o_D + c,$$

where c is a fixed element of $\mathbb{Z}_2$ (the shift). A system G of level h is **covariant** if $G[D, D'] = 0$ unless $D' \subseteq D$, and there are functions $\Gamma_{\Delta, m}$ such that

$$G[D, D']_m(f) = \Gamma_{D \setminus D', m}(2^h f + o_D + c)$$

(the coefficient of $F^{(m)}$ at the target floor f depends only on $\Delta = D \setminus D'$, m and the position of the target).

The ring $\mathcal{K}$. Let $\mathcal{K}$ be the set of functions $\vartheta : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ of the form

$$\vartheta(y) = a + 2\chi(2y), \quad a \in \mathbb{Z}_2, \quad \chi \in X \cdot \mathbb{Z}_2[[X]]$$

(the series converges at $X = 2y$).

Lemma 10.13. (a) The representation (a, χ) of $\vartheta \in \mathcal{K}$ is unique. The set $\mathcal{K}$ is a ring, it is stable under every shift $y \mapsto y - e$ ($e \in \mathbb{Z}_2$), and $1/\vartheta \in \mathcal{K}$ when a is odd.

(b) Let $\mathcal{S} := \mathbb{F}_2 \oplus \varepsilon \cdot X\mathbb{F}_2[[X]]$ with $\varepsilon^2 = 0$, a commutative ring of characteristic 2, and $\Theta(\vartheta) := \bar{a} + \varepsilon\bar{\chi}$ (reductions modulo 2). Then $\Theta : \mathcal{K} \rightarrow \mathcal{S}$ is a **ring homomorphism**, $\Theta(\vartheta(\cdot - e)) = \Theta(\vartheta)$ **for every** $e \in \mathbb{Z}_2$, **and** $\ker \Theta = 2\mathcal{K}$.

(c) $\vartheta(y) \equiv a \pmod{4}$ for every y .

Proof. (a) If $a + 2\chi(2y) = a' + 2\chi'(2y)$ for all $y \in \mathbb{Z}_2$, the power series $(a - a') + 2(\chi - \chi')(2Y)$ in Y has coefficients tending to 0 and infinitely many zeros, so it vanishes by Strassmann's theorem [Str28]. *Products.* $(a + 2\chi)(b + 2\omega) = ab + 2(a\omega + b\chi + 2\chi\omega)$. *Shifts.* $\chi(X - 2e) = \chi(-2e) + \tilde{\chi}(X)$ with $\tilde{\chi} \in X\mathbb{Z}_2[[X]]$, and $\chi(-2e) \in 2\mathbb{Z}_2$. *Inverses.* $(a + 2\chi)^{-1} = a^{-1} \sum_{j \geq 0} (-2\chi/a)^j$.

(b) The product rule in (a) reduces to $(\bar{a} + \varepsilon\bar{\chi})(\bar{b} + \varepsilon\bar{\omega})$. Modulo 2, the coefficient $\sum_{n \geq j} c_n C(n, j)(-2e)^{n-j}$ of X^j in $\tilde{\chi}$ is $\equiv c_j$, and $2\chi(-2e) \in 4\mathbb{Z}_2$; so Θ is shift-invariant. Finally, $\Theta(\vartheta) = 0$ means a, χ even, $\vartheta = 2(a/2 + 2(\chi/2)(2y))$.

(c) $2\chi(2y) \in 4\mathbb{Z}_2$. ■

Let $\mathcal{T}_h$ be the set of covariant systems of level h whose functions $\Gamma_{\Delta, m}$ all lie in $\mathcal{K}$, and let $R_{\mathcal{S}} := \mathcal{S}[F^{(m)} : m \geq 1][E_t : t \in P_h]/(E_t^2)$, a commutative ring (divided-power multiplication in F).

Lemma 10.14. (a) $\mathcal{T}_h$ is a ring.

(b) $\Theta_h(G) := \sum_{\Delta, m} \Theta(\Gamma_{\Delta, m}) F^{(m)} E^\Delta$ ($E^\Delta := \prod_{t \in \Delta} E_t$) is a ring homomorphism $\mathcal{T}_h \rightarrow R_{\mathcal{S}}$.

(c) In $R_{\mathcal{S}}$, every element β satisfies $\beta^2 = \beta_0^2$, where β_0 is its constant term.

Proof. (a) For $G, H \in \mathcal{T}_h$,

$$(GH)[D, D']_m(f) = \sum_{D \supseteq D_1 \supseteq D'} \sum_{a+b=m} C(m, a) \cdot \Gamma_{D \setminus D_1, a}^G(y) \cdot \Gamma_{D_1 \setminus D', b}^H(y - a \cdot 2^h - o_{D \setminus D_1}),$$

because the position of $(D_1, f - a)$ is $y - a \cdot 2^h - o_{D \setminus D_1}$ (o is additive on disjoint sets). The sum over D_1 is a sum over the decompositions $\Delta = (D \setminus D_1) \sqcup (D_1 \setminus D')$, so it depends only on Δ , m and y , and lies in $\mathcal{K}$ by Lemma 10.13(a). (b) Apply Θ to this formula: by shift-invariance the shifts disappear, and the sum over decompositions of Δ is the product of the E^Δ . (c) $R_{\mathcal{S}}$ is commutative of characteristic 2, so $\beta^2 = \sum_{\mu} \beta_{\mu}^2 \mu^2$ over the monomials $\mu = F^{(q)} E^\Delta$, and $\mu^2 = C(2q, q) F^{(2q)} E^{2\Delta} = 0$ unless $\mu = 1$ ($C(2q, q)$ is even for $q \geq 1$, and $E_t^2 = 0$). ■

Theorem 10.15 (cleanliness). Let $G_h \in \mathcal{T}_h$ satisfy

(a) the payments are $\equiv 0 \pmod{4}$: $\Gamma_{\emptyset, 0}(y) \in 4\mathbb{Z}_2$ for every y ;

(b) the diagonal coefficients of F are odd: $\Gamma_{\emptyset, 1}(y)$ is a unit for every y . Then the move of level h is clean, and $G_{h+1} \in \mathcal{T}_{h+1}$ again satisfies (a) and (b). Consequently, **for every $\alpha \geq 2$, every $c \in \mathbb{Z}_2$, every unit η and all constants $\zeta_m \in \mathbb{Z}_2$, the dance of $2^\alpha(D+c) + \eta F + \sum_{m \geq 2} \zeta_m F^{(m)}$ on every $T(l)$ is clean.**

Proof. *The twist move.* The level does not change the dancers. The position of (D, f') at level $h+1$ is $y' = 2^{h+1}f' + o_D + c$; the level- h floors $2f'$ and $2f' + 1$ have the positions y' and $y' + 2^h$. By the formulas of §10.3 the blocks have the coefficient functions

$$B : \quad \Gamma_{\Delta, 2q}(y')/(2q-1)!!, \quad C : \quad \Gamma_{\Delta, 2q}(y' + 2^h)/(2q-1)!!,$$

$$P : \quad \Gamma_{\Delta, 2q+1}(y' + 2^h)/(2q+1)!!, \quad \text{Dd} : \quad ((2q+2)/(2q+1)!!) \cdot \Gamma_{\Delta, 2q+1}(y')$$

(the last is the coefficient of $F^{(q+1)}$). All four lie in $\mathcal{T}_{h+1}$ (Lemma 10.13(a); the denominators are odd). The diagonal constant term of P is $\Gamma_{\emptyset, 1}(y' + 2^h)$, a unit by (b), and the other terms of P raise m or Δ ; so P^{-1} exists, is integral and lies in $\mathcal{T}_{h+1}$ (a Neumann series in which each coefficient receives finitely many terms). So $S = \text{Dd} - BP^{-1}C \in \mathcal{T}_{h+1}$. Apply Θ_{h+1} : $\Theta(\text{Dd}) = 0$, because every coefficient of Dd is $(2q+2)$ times an element of $\mathcal{K}$; $\Theta(B) = \Theta(C) =: \beta$, because the shift by 2^h is invisible to Θ . In the commutative ring $R_{\mathcal{S}}$,

$$\Theta(S) = -\beta \cdot \Theta(P)^{-1} \cdot \beta = -\beta^2 \cdot \Theta(P)^{-1},$$

and $\beta^2 = \Theta(\Gamma_{\emptyset,0})^2$ by Lemma 10.14(c). By (a) and Lemma 10.13(c) the constant a of $\Gamma_{\emptyset,0}$ is even, so $\Theta(\Gamma_{\emptyset,0}) = \varepsilon\bar{\chi}$ and $\beta^2 = \varepsilon^2\bar{\chi}^2 = 0$. So every coefficient function of S lies in $\ker \Theta = 2\mathcal{K}$: the move is clean and $G_{h+1} = S/2 \in \mathcal{T}_{h+1}$. For (a): the new payment is $-\Gamma_{\emptyset,0}(y') \cdot \Gamma_{\emptyset,1}(y' + 2^h)^{-1} \cdot \Gamma_{\emptyset,0}(y' + 2^h)/2$ (only constant terms contribute to the constant term, and Dd has none), which is $\equiv 0 \pmod{8}$. For (b): the coefficient of $F^{(1)}$ on the diagonal of S is $2\Gamma_{\emptyset,1}(y')$ (from Dd) minus terms of $BP^{-1}C$ that each contain a constant term of B or of C , that is a payment $\equiv 0 \pmod{4}$; after division by 2 it is odd. *The split move.* First the V -split: by §10.3 the dancer D gives the dancers D ($v = 0$) and $D \cup \{h\}$ ($v = 1$); the position of $(D \cup \{h\}, g)$ is $2^h(g+1) + o_D + c = 2^hg + o_{D \cup \{h\}} + c$, so the diagonal blocks keep their functions $\Gamma_{\Delta,m}$, and the new coupling from D' to $D \cup \{h\}$ has $\Gamma_{\Delta \cup \{h\},m-1}^{\text{new}} := \Gamma_{\Delta,m}$ at the same position. The split system is covariant with functions in $\mathcal{K}$ and still satisfies (a), (b). It has couplings with an odd constant term ($\Gamma_{\{h\},0}^{\text{new}} = \Gamma_{\emptyset,1}$); they are harmless, because in R_S they carry E_h and $E_h^2 = 0$, so they do not enter β^2 . Then the twist move, as above, with the dancers of P_{h+1} . *The operators.* At level 0 the only dancer is $\emptyset$, the position is $y = f + c$, and $\Gamma_{\emptyset,0}(y) = 2^\alpha y = 2\chi(2y)$ with $\chi(X) = 2^{\alpha-2}X$, which is in $\mathcal{K}$ because $\alpha \geq 2$; $\Gamma_{\emptyset,1} = \eta$, $\Gamma_{\emptyset,m} = \zeta_m$. ■

Remarks. (1) At $\alpha = 1$ the payment $2y$ is not in $\mathcal{K}$ (χ would be $X/2$), and the cleanliness of the dance of $\sigma_i^{(1)}$ (Lemma 10.6) comes from the fact that it is an even payment system. (2) The class $\mathcal{K}$ is sufficient, not necessary: every even payment system dances clean, whatever its payments (proof of Lemma 10.6(a), with Propositions 4.1 and 4.2). With higher divided powers some restriction is needed. We measured the dances on $T(l)$, $l = 2, \dots, 63$, of $p(D) + u(D)F + \sum_{m \geq 2} \zeta_m F^{(m)}$ with constant random integers $\zeta_m \in [-5, 5]$, four for each l (248 dances in each case; this is used nowhere, and the code is `gate/remark2/`, §12.1):

- random payments $p(d) = 4q(d)$, $q(d) \in [-9, 9]$ non-zero, and $u = 1$: three dances are not clean, all at $l = 63$;
- the payments $4(d + c)$, $c = 1, \dots, 4$, and random odd $u(d) \in [-17, 19]$: 79 dances are not clean, the first at $l = 32$.

With the coefficients η_m of Ψ_0 in place of the random ζ_m , all 496 dances are clean.

10.6 The odd families

Theorem 10.16. Let $\alpha \geq 2$, $c \geq 0$, $\eta \in A$ a unit, $\zeta_m \in \mathbb{Z}_2$ constants and $\Xi := 2^\alpha(D + c) + \eta F + \sum_{m \geq 2} \zeta_m F^{(m)}$. Then for every $l \geq 0$, $\text{coker}(\Xi \text{ on } T(l)) \cong X^{(\alpha)}(l, c)$.

Proof. For $c \geq 1$: Theorems 10.15 and 10.12. For $c = 0$, let Ξ_c and $\Sigma'_c := 2^\alpha(D + c) + \eta F$ be the operators with the shift $c \in \mathbb{Z}_2$. By Theorem 10.15 their dances are clean for every $c \in \mathbb{Z}_2$, and the entries of their final matrices are continuous functions of c : each level is obtained from the previous one by ring operations, odd denominators, the inversion of a pivot whose diagonal constant term is a unit, and a division by 2 that stays integral. For $c = 2^n$, $n \geq 0$, Theorem 10.12 gives $M_w(\Xi_c) - M_w(\Sigma'_c) = M_w(\Sigma'_c) \circ E'_c$ with $v_2(E'_c) \geq 1$, that is: for every entry, $v_2(M_w(\Xi_c) - M_w(\Sigma'_c)) \geq v_2(M_w(\Sigma'_c)) + 1$ (both sides ∞ if the entry of $M_w(\Sigma'_c)$ is 0). Let $n \rightarrow \infty$. Where $M_w(\Sigma'_0)$ has a non-zero entry, its valuation is that of $M_w(\Sigma'_{2^n})$ for large n , and the inequality passes to the limit. Where it has a zero entry, $v_2(M_w(\Sigma'_{2^n})) \rightarrow \infty$, hence $v_2(M_w(\Xi_{2^n})) \rightarrow \infty$ and the entry of $M_w(\Xi_0)$ is 0. So $M_w(\Xi_0)$ has the support and the entry valuations of $M_w(\Sigma'_0)$, which are those of $M^{(\alpha)}(l, 0)$ (Lemma 10.6 and Theorem D). Theorem 7.10 at the shift 0 uses only these data, so the Smith forms agree, and the cokernels follow from Proposition 10.7. ■

Corollary 10.17 (the fold law on the odd families). For every $l \geq 0$ and every $j \geq 0$,

$$\bar{X}(2l+1, j) \cong 2X(2l+1, j).$$

Proof. Let $c = \lceil j/2 \rceil$. By §10.2, $\Psi_j = 4(D+c) + \eta F + \sum_{m \geq 2} \eta_m F^{(m)}$ with constant integral η_m and $\eta = \eta_1$ a unit (1 for even j , $1/3$ for odd j). By Theorem 10.16 ($\alpha = 2$), $\text{coker}(\Psi_j \text{ on } T(l)) \cong X^{(2)}(l, c) = \text{coker}(\Sigma_c \text{ on } T(l))$. Lemma 10.4 identifies the two sides with $\bar{X}(2l+1, j)$ and $2X(2l+1, j)$. ■

10.7 The even families

Lemma 10.5 presents both sides of an even family as glued systems of size 2 on $N = T(l)$. Their arms are odd-family operators, which differ by multiples of 2. Two facts make the glue harmless: the dance modulo 2 sees only the operator modulo 2 (Lemma 10.18), and gluing commutes with the dance (Lemma 10.19). Then the second side differs from the first by a unit at the bottom of the dance (Lemma 10.20).

Fix the shift $c \in \mathbb{Z}_2$ used for the positions, and let $\mathfrak{a} : \mathcal{K} \rightarrow \mathbb{Z}_2$, $a + 2\chi(2y) \mapsto a$. It is a ring homomorphism, and $\mathfrak{a}(\vartheta(\cdot - e)) = \mathfrak{a}(\vartheta) + 2\chi(-2e) \equiv \mathfrak{a}(\vartheta) \pmod{4}$. So $\mathfrak{a}$ modulo 4 is a shift-invariant ring homomorphism; exactly as in Lemma 10.14, it induces a ring homomorphism from $\mathcal{T}_h$ to the commutative ring $R_{\mathbb{Z}/4} := (\mathbb{Z}/4)[F^{(m)}][E_t]/(E_t^2)$. By Lemma 10.13(c), every value $\vartheta(y)$ is $\equiv \mathfrak{a}(\vartheta) \pmod{4}$.

Lemma 10.18 (modulo 2). Let $G, G' \in \mathcal{T}_0$ both satisfy (a) and (b) of Theorem 10.15, and suppose $G \equiv G' \pmod{2}$ (every coefficient, at every position). Then $G_h \equiv G'_h \pmod{2}$ **at every level h** .

Proof. By induction it suffices to treat one twist move (a V -split does not change the congruence). By Theorem 10.15, $S \in 2\mathcal{T}$, so $\mathfrak{a}(G_{h+1}) = \mathfrak{a}(S)/2$, and G_{h+1} modulo 2 is determined by $\mathfrak{a}(S)$ modulo 4. In $R_{\mathbb{Z}/4}$,

$$\mathfrak{a}(S) \equiv \mathfrak{a}(\text{Dd}) - \beta^2 \pi^{-1}, \quad \beta := \mathfrak{a}(B) \equiv \mathfrak{a}(C), \quad \pi := \mathfrak{a}(P),$$

because the shift 2^h between B and C is invisible modulo 4. Now pass from G_h to G'_h , whose $\mathfrak{a}$ -image differs by 2δ . Then $\mathfrak{a}(\text{Dd})$ changes by a multiple of 4 (the factor $2q+2$). Next, β changes to $\beta + 2\delta'$ and β^2 by $4\beta\delta' + 4\delta'^2 \equiv 0$ (commutativity). The inverse π^{-1} changes by $-2\pi^{-2}\delta'' \pmod{4}$, so $\beta^2\pi^{-1}$ changes by $-2\beta^2\pi^{-2}\delta''$, which is $\equiv 0 \pmod{4}$ because $\beta^2 \equiv 0 \pmod{2}$: modulo 2, β^2 is the square of the constant term of β , that is of a payment, which is even. So $\mathfrak{a}(S') \equiv \mathfrak{a}(S) \pmod{4}$, and $G'_{h+1} \equiv G_{h+1} \pmod{2}$. ■

Lemma 10.19 (the glue). Let X, Y be systems of size 1 on $T(l)$ whose dances along w are clean, with $X_h \equiv Y_h \pmod{2}$ at every level h . Then the dance of $\text{glue}(X, Y)$ is clean, $\text{glue}(X, Y)_h = \text{glue}(X_h, Y_h)$ **at every level**, and in particular $M_w(\text{glue}(X, Y)) = \text{glue}(M_w(X), M_w(Y))$.

Proof. $\text{glue}(X, Y) = L^{-1} \text{diag}(X, Y)L$ with the scalar matrix $L = [[1, 0], [1/2, 1]]$, which acts on the outer slot. Every move acts slot by slot on the inner structure, so the move of $L^{-1}ZL$ is L^{-1} times the move of Z times L : the pivot, the other blocks and the Schur complement are conjugated by L . For $Z = \text{diag}(X, Y)$ the dance is the two dances side by side. So $\text{glue}(X, Y)_h = \text{glue}(X_h, Y_h)$ as rational systems. A glued system $\text{glue}(U, U')$ is integral if and only if $U \equiv U' \pmod{2}$; its pivot $\text{glue}(P_U, P_{U'})$ is then invertible over $\mathbb{Z}_2$; and its Schur complement $\text{glue}(S_U, S_{U'}) \equiv 0 \pmod{2}$ if and only if $S_U \equiv S_{U'} \equiv 0 \pmod{2}$ and $S_U \equiv S_{U'} \pmod{4}$, that is $U_{h+1} \equiv U'_{h+1} \pmod{2}$. ■

Lemma 10.20 (the bend). Let $c \geq 1$, $\Psi := 4(D+c) + \eta F + \sum_{m \geq 2} \zeta_m F^{(m)}$ (η a unit, $\zeta_m \in \mathbb{Z}_2$ constants), $\Sigma := 4(D+c) + \eta F$, and

$$N_{\pm} := 2^k \cdot \text{out}(\Psi \pm 2)^{-1} \text{in},$$

with the maps $\text{out} = \text{out}_k$, $\text{in} = \text{in}_k$ of Lemma 10.8 ($\Psi \pm 2$ is invertible: its payments $4(d+c) \pm 2$ never vanish). Then $M_w(\Psi) \cdot N_{\pm} \in 2 \cdot \text{Mat}(\mathbb{Z}_2)$.

Proof. Put $w(d) := v_2(4(d+c)) - 1 = 1 + v_2(d+c) \geq 1$ and $W[a, b] := \sum_{d=a}^b w(d)$. As in Corollary 10.9, $(N_{\pm})_{YX} = (M_w(\Sigma)^{-1})_{YX} \cdot R_{YX}$, where $R_{YX} = y_L^{\Psi \pm 2}(f)/y_L^{\Sigma}(f)$ on the window $[o_X, o_Y + K - 1]$. Expand $y^{\Psi \pm 2}$ as in Lemma 10.10. A composition, divided by the main term of y^{Σ} , is

$$\pm \left(\prod_i \zeta_{m_i} \cdot \eta^{-L} / \prod_i m_i! \right) \cdot \prod_{\text{landed}} \pi(d)/(\pi(d) \pm 2) \cdot \prod_{\text{skipped}} \pi(d),$$

$$\pi(d) = 4(d+c).$$

Here $v_2(\pi(d)/(\pi(d) \pm 2)) = w(d)$, $v_2(\pi(d)) = w(d) + 1$, and $-\sum_i v_2(m_i!) \geq -s'$, where s' is the number of skipped floors. So $v_2(R_{YX}) \geq W[o_X, o_Y + K - 1]$. By Theorem 10.12, $M_w(\Psi) = M_w(\Sigma) \circ (1 + E')$, and $M_w(\Sigma) = \mathfrak{d}_1 \mathcal{A} \mathfrak{d}_2$ with $v_2((\mathfrak{d}_1)_D) = -W[0, o_D - 1] + c_0$, c_0 independent of D (as $v_2(\pi(d)) = w(d) + 1$). Each term of $(M_w(\Psi)N_{\pm})_{DX} = \sum_Y M_w(\Psi)_{DY} (N_{\pm})_{YX}$ is non-zero only for $X \subseteq Y \subseteq D$, and then has valuation at least

$$v_2((\mathfrak{d}_1)_D) - v_2((\mathfrak{d}_1)_X) + \delta(D \setminus Y) + \delta(Y \setminus X) + W[o_X, o_Y + K - 1]$$

$$= \delta(D \setminus Y) + \delta(Y \setminus X) + W[o_D, o_Y + K - 1] \geq 1,$$

because $o_D - o_Y = o_{D \setminus Y} \leq K - 1$, so the window $[o_D, o_Y + K - 1]$ is non-empty and $w \geq 1$ on it. $((\mathfrak{d}_2)_Y$ cancels, and $v_2((\mathcal{A}^{-1})_{YX}) \geq \delta(Y \setminus X)$, proof of Lemma 10.11.) $\blacksquare$

Corollary 10.21. In Lemma 10.20, the dances of $\Psi(\Psi \pm 2)/2$ are clean, and

$$M_w(\Psi(\Psi + 2)/2) = W_+ \cdot M_w(\Psi), \quad M_w(\Psi(\Psi - 2)/2) = W_- \cdot M_w(\Psi), \quad W_{\pm} \in \pm I + 2 \cdot \text{Mat}(\mathbb{Z}_2).$$

Proof. Cleanliness. $\Psi \in \mathcal{T}_0$ with (a), (b). By Lemma 10.14, $\Theta(\Psi^2) = \Theta(\Psi)^2 = \Theta(4y)^2 = 0$ and $\Theta(2\Psi) = 0$, so $\Psi(\Psi \pm 2) \in 2\mathcal{T}_0$ and $\Psi(\Psi \pm 2)/2 \in \mathcal{T}_0$. Its payment is $4y(4y \pm 2)/2 = 4y(2y \pm 1) = 2\chi(2y)$ with $\chi = X^2 \pm X$, which is $\equiv 0 \pmod{4}$. Its coefficient of F is $(\eta \cdot 4(y-1) + 4y \cdot \eta \pm 2\eta)/2 = \eta(4y - 2 \pm 1)$, which is odd. Theorem 10.15 applies. *The identity.* By partial fractions (all operators are series in F and D , and $\Psi, \Psi \pm 2$ commute), $(\Psi(\Psi + 2)/2)^{-1} = \Psi^{-1} - (\Psi + 2)^{-1}$ and $(\Psi(\Psi - 2)/2)^{-1} = (\Psi - 2)^{-1} - \Psi^{-1}$. By Lemma 10.8 applied to the clean dances of Ψ and $\Psi(\Psi \pm 2)/2$ (same out, in):

$$M_w(\Psi(\Psi + 2)/2)^{-1} = M_w(\Psi)^{-1} - N_+ = M_w(\Psi)^{-1}(I - M_w(\Psi)N_+),$$

so $W_+ = (I - M_w(\Psi)N_+)^{-1}$. Likewise $W_- = -(I - M_w(\Psi)N_-)^{-1}$. By Lemma 10.20, $W_{\pm} \equiv \pm I \pmod{2}$. $\blacksquare$

Theorem 10.22 (the fold law on the even families). For every $l \geq 0$ and every $i \geq 0$,

$$\bar{X}(2l + 2, i) \cong 2X(2l + 2, i).$$

Proof. Let $Y_0 := 1 + (-1)^i \text{Sh}^{-1}$, so that $\Psi_{i+1} = \Psi_i + 2Y_0$ (§10.2). Take the positions $y = f + [i/2]$ at level 0. In these positions both Ψ_i and Ψ_{i+1} lie in $\mathcal{T}_0$ with (a) and (b): their payments are $4y$ and $4(y + [i \text{ even}])$, their coefficients of F are the units 1, $1/3$, and their other coefficients are constants. *The arms are congruent.* $\Psi_{i+1} - \Psi_i = 2Y_0 \equiv 0 \pmod{2}$. For the second side put $U := \Psi_{i+1}(\Psi_{i+1} - 2)/2$ and $U' := \Psi_i(\Psi_i + 2)/2$, both in $\mathcal{T}_0$ with (a) and (b) (proof of Corollary 10.21). With $\Psi := \Psi_i$,

$$U - U' = \Psi Y_0 + Y_0 \Psi - 2\Psi + 2Y_0^2 - 2Y_0 \equiv [\Psi, Y_0] = 4[D, Y_0] \equiv 0 \pmod{2},$$

because Y_0 and the F -part of Ψ are power series in F with constant coefficients and commute, while Ψ contains $4D$. (Ψ_i and Y_0 do not commute, so the product $(\Psi_{i+1} + \Psi_i)(Y_0 - 1)$ is not the difference.)

The case $i \geq 1$. By Lemma 10.18 the arms dance congruently modulo 2, so by Lemma 10.19 the dances of $\text{glue}(\Psi_{i+1}, \Psi_i)$ and $\text{glue}(U, U')$ are clean, and by Corollary 10.21 (with Ψ_{i+1} and Ψ_i , whose shifts $[(i+1)/2]$, $[i/2]$ are ≥ 1)

$$M_w(\text{glue}(U, U')) = \text{glue}(W_- M_w(\Psi_{i+1}), W_+ M_w(\Psi_i)) = \text{glue}(W_-, W_+) \cdot M_w(\text{glue}(\Psi_{i+1}, \Psi_i)),$$

since $\text{glue}(X, Y) \cdot \text{glue}(X', Y') = \text{glue}(XX', YY')$. Now $\text{glue}(W_-, W_+) = [[W_-, 0], [(W_+ - W_-)/2, W_+]]$ is integral ($W_+ \equiv I$ and $W_- \equiv -I \pmod{2}$, so $W_+ - W_- \in 2 \cdot \text{Mat}(\mathbb{Z}_2)$) and has invertible diagonal blocks, so it lies in $\text{GL}(\mathbb{Z}_2)$. The two final matrices are associates and have the same Smith form; Proposition 10.7 and Lemma 10.5 give $\bar{X}(2l+2, i) \cong 2X(2l+2, i)$. *The case $i = 0$.* Then $\Psi_0 = 4D + u \cdot \tanh(u/2)$ is not invertible. Replace the shift by a parameter: $i = 2c$, $c \in \mathbb{Z}_2$, with payments $4(D + c)$ for Ψ_i and $4(D + c + 1)$ for Ψ_{i+1} . Every object above is continuous in c : the dances are clean for every c (Theorem 10.15 and Lemmas 10.18 and 10.19); N_+ uses $(\Psi_i + 2)^{-1}$, whose payments $4(d + c) + 2$ never vanish, and N_- uses $(\Psi_{i+1} - 2)^{-1}$, with payments $4(d + c + 1) - 2$. For $c = 2^n$, $n \geq 0$, we have $M_w(\Psi_i)N_+ \in 2 \cdot \text{Mat}$, $M_w(\Psi_{i+1})N_- \in 2 \cdot \text{Mat}$, and the identity

$$M_w(\text{glue}(U, U')) = \text{glue}(W_-, W_+) \cdot M_w(\text{glue}(\Psi_{i+1}, \Psi_i)),$$

with $W_{\pm} = \pm(I - M_w(\cdot)N_{\pm})^{-1}$. The set $2 \cdot \text{Mat}(\mathbb{Z}_2)$ is closed, so at $c = 0$ the products $M_w(\Psi_0)N_+$ and $M_w(\Psi_1)N_-$ still lie in $2 \cdot \text{Mat}$; then $W_{\pm}$ are defined and $\equiv \pm I \pmod{2}$ there, and the identity passes to the limit. So the final matrices are associates at $c = 0$ as well. $\blacksquare$

10.8 Proof of Theorem DW, and the folded cube

Proof of Theorem DW. By Proposition 10.3 it suffices that $\bar{X}(\lambda, i) \cong 2X(\lambda, i)$ for every $\lambda \geq 1$ and every $i \geq 0$. For $\lambda = 2l + 1$ this is Corollary 10.17, and for $\lambda = 2l + 2$ Theorem 10.22. $\blacksquare$

Proof of Corollary 1.4. The 2-part is Theorem DW: $\text{Syl}_2 \bar{K}(n) \cong 2 \cdot \text{Syl}_2 K(Q_n)$ is $\text{Syl}_2 K(Q_n)$ with every exponent lowered by one, the a_n factors $\mathbb{Z}/2$ disappearing (Corollary 1.5). The odd part is given by [GMMY24, Proposition 1.9]. $\blacksquare$

The mod-2 layer. Let $\bar{L}$ be the Laplacian of the folded cube, a square matrix of size 2^{n-1} ($n \geq 2$), with $\text{coker } \bar{L} \cong \mathbb{Z} \oplus \bar{K}(n)$. Its invariant factors divisible by 2 are the zero one and one for each cyclic factor of $\text{Syl}_2 \bar{K}(n)$. By Corollary 1.5, $\text{Syl}_2 K(Q_n)$ has $2^{n-1} - 1$ cyclic factors, of which a_n are $\mathbb{Z}/2$; by Theorem DW, $\text{Syl}_2 \bar{K}(n)$ has $2^{n-1} - 1 - a_n$. So $\text{rank}_{\mathbb{F}_2} \bar{L} = 2^{n-1} - (2^{n-1} - a_n) = a_n$. This is [GMMY24, Proposition 2.12] (with their $r = n - 1$). Their Remark 2.13 asks whether its coincidence with Bai's count of the factors $\mathbb{Z}/2$ of $K(Q_n)$ is a «special case of some deeper connection». Theorem DW is such a connection: the coincidence is its first layer.

Remark (what the fold sees). The proof uses the antipode $a = x_1 \cdots x_n$ through Lemma 10.1 only: a is $(-1)^{(n-H)/2} \exp(F)$, an operator built from the weight and the divided powers of F , so it is carried by the family decomposition of Theorem 3.6. An element such as $x_1 x_2$ acts on two tensor factors only; it is not of this kind, and the quotient by it does not halve the group (we checked $3 \leq n \leq 12$, §12).

11. Open questions

1. **Other quotients of the cube.** The Cayley graphs of $\mathbb{F}_2^r$ are the quotients of cubes by binary linear codes [IKKY23, Proposition 37], and those containing $\mathbb{1}$ — the «non-generic» ones in the sense of [GMMY24], see also [Yue24] — are built from the folded cube by successive quotients of codimension one [GMMY24, §2.2]. Does the method of §10 compute their 2-parts? The antipode was special because it is $\pm \exp(F)$ on every family (Lemma 10.1); a general code element acts on some tensor factors only.
2. **The turned lid.** Write $C_b := Q_b / \langle \mathbb{1} \rangle$ (so C_1 is a point). For $a + b = n$, the graph $Q_a \square C_b$ is the quotient of Q_n by a translation of weight b . We measured, for every $n \leq 12$ and every b , that $\text{Syl}_2 K(Q_a \square C_b) \cong \text{Syl}_2 K(Q_b \square C_a)$, although the graphs are not isomorphic and the odd parts differ; and that $2 \cdot \text{Syl}_2 K(Q_n) \cong \text{Syl}_2 K(Q_a \square C_b)$ holds only for $b = n$ (Theorem DW). Theorem DW is the degenerate end of this symmetry (the twin of $b = n$ would be the quotient by 0). We have no proof.
3. **A direct proof of the fold law.** Reiner and Tseng give an exact sequence $0 \rightarrow K(G^\pm) \rightarrow K(Q_n) \rightarrow \bar{K}(n) \rightarrow 0$ for the double cover $Q_n \rightarrow Q_n / \langle \mathbb{1} \rangle$ ($n \geq 3$), with $K(G^\pm)$ the critical group of a signed graph [RT14]. Theorem DW identifies the 2-part of the target, abstractly, with $2 \cdot \text{Syl}_2 K(Q_n)$. Is there a proof through the signed graph, without tilting modules?
4. **The meaning of pooling.** The integer antitonic fit appears because the Smith form of $M(\lambda, i)$ is governed by the cheapest matchings of a triangular cost matrix (§6–§7). Is there a module-theoretic reason, such as a filtration of $X(\lambda, i)$ whose layers are the pooled blocks?
5. **Other primes and other graphs.** For the Hamming graphs $H(n, p)$, p odd, the group ring $\mathbb{Z}[(\mathbb{Z}/p)^n]$ is the n -th tensor power of $\mathbb{Z}[\mathbb{Z}/p]$, and the Laplacian is again a sum over the tensor factors. Is there a decomposition of their p -parts over tilting modules of a suitable group in characteristic p ?
6. **The tilting modules in Lean.** Every theorem of this paper is checked in Lean (§12.3), with the dance lattices $T(\lambda)$ of §3.3 as explicit lattices. That they are the indecomposable tilting modules of SL_2 in characteristic 2 is not formalized, and the proof does not need it. Formalizing it would join the formal proof to the representation theory that guided it.

12. Computations

12.1 The checks published with this paper

All the checks below were run on a laptop, under a watchdog with limits on memory and time; a log counts only if it ends with the watchdog’s final line. The code is in the folder `gate/` that accompanies this paper. A **control** is a deliberately wrong variant that must fail; «none» means that the row has no control.

statement, and the code that checks it	range	result	control
Main Theorem against the direct 2-adic Smith form of the Laplacian of Q_n ; Theorem DW against that of $Q_n/\langle 1 \rangle$; code <code>gate_direct.py</code>	$n = 2, \dots, 11$	10/10 and 10/10	the rule without pooling fails at $n = 6, 10$; the rule without the term $h(D)$ at $n = 2$ and $4, \dots, 11$; $Q_n/\langle x_1 x_2 \rangle$ in place of the folded cube fails for every $n = 3, \dots, 11$
Theorem D: direct Smith form of $\sigma_i^{(\alpha)}$ on $T(\lambda)$ against small clocks $\oplus \text{coker}(2^k M)$; Theorem 4.4 against the dance; code <code>gate_dance.py</code>	$\lambda \leq 40$, $\alpha = 1, 2$, $i \leq 6$, $\dim T(\lambda) \leq 64$	432/432; 0 mismatches	the closed form without the factor $2^{o(\Delta)}$ differs from the dance in 420/420 cells, and predicts a wrong Smith form in 262
Propositions 5.2 · 5.3 · 5.4 (5.2 against the closed form of Theorem 4.4; of 5.4, the formula for the clocks $u_0(D)$, $D \neq \emptyset$); code <code>gate_sections5to7.py</code>	$\lambda < 64$; $i = 1, \dots, 40$ ($\alpha = 1, 2$ for 5.2, $\alpha = 1$ for 5.3); $i = 0$, $\alpha = 1$ for 5.4	109 200 matrix entries · 14 560 clocks · 301 clocks, 0 failures	none
Theorem 7.10: Smith form of M against the pooled clocks; code <code>gate_sections5to7.py</code>	$\lambda < 64$, $i \leq 12$, $\alpha = 1, 2$	1 638 cells, 0 failures	the raw clocks, without pooling, are rejected in all 322 cells where pooling changes them
Theorem 7.8: the construction of V and the claims of its proof, for three admissible prices c_F ; code <code>gate_sections5to7.py</code>	houses with $I \neq \emptyset$ and $k \leq 6$ ($\lambda \leq 126$), $\alpha \leq 3$, every ℓ	16 002 houses, 48 006 house-checks, 0 failures	without gold, (G) fails in 164 houses ($\lambda < 64$, $\alpha = 1$); with $c_F := v'(I')$, in 24
Theorem 7.8, strict (Cut); Lemma 7.2(b) where it is applied: the penalty shifts $P_r, P_c \leq 4$ and the deletion at $i = 0$; code <code>gate_strict_cut.py</code>	every house with $1 \leq k \leq 6$, $\alpha \leq 3$	16 380 houses, 8 001 boundaries, 192 024 non-zero shifts (24 on each house with a boundary), 360 deletions, 0 failures	strictness tested one dancer after the boundary fails in 1 911 of 7 038 houses; the shift moved one dancer late breaks the integer shift in 18 606 of 192 024
Lemma 7.9: the fit of x_H , of its shifts and of its deletions is integral; code <code>gate_strict_cut.py</code>	the same	0 non-integral values	x_H with 1 added at its first dancer (not a house) has a non-integral fit in 1 942 houses
Lemma 7.2(b) on random sequences; code <code>gate_lemma72.py</code>	20 000 sequences of length ≤ 9	0 failures	jumps of δ at cuts that are not strict break the integer part in 189 of 2 841 trials
Theorem 6.2: exactly one golden bijection; code <code>gate_theorem0.py</code>	$ I \leq 5$, every r	69 cases, 0 failures	«golden» replaced by «any subset» or by «any run of digits»: the count is not 1 in 42 and in 26 cases
Corollary 1.5 · Lemma 9.7 · Corollary 1.6; code <code>gate_section9.py</code>	$n \leq 200$	199 · 199 · 197 values, 0 failures	none (a counter shows that $c_n = G$ fails for 37 values of n)
Propositions 9.3 · 9.4 · 9.6; code <code>gate_section9.py</code>	$\lambda \leq 300$	300 · 292 · 22 500 cells, 0 failures	none (a counter shows that the bound $g(n - i)$ is exceeded in 298 cells)

statement, and the code that checks it	range	result	control
Corollary 1.7; code <code>gate_section9.py</code>	$e = 1, \dots, 8$	8/8	none
the weights η_m, η'_m of §10.2: the Bernoulli formulas, and $v_2 = 0$; code <code>gate_section10.py</code>	$m \leq 39$	0 failures	none
Lemma 10.1 on $V^{\otimes n}$; code <code>gate_section10.py</code>	$n \leq 6$	0 failures	none
Lemmas 10.4 and 10.5, both sides, and $\bar{X}(\lambda, i) \cong 2X(\lambda, i)$, on the lattices $T(\lambda)$ of §3.3; code <code>gate_section10.py</code>	$\lambda \leq 22,$ $i \leq 5$	$66 + 66 + 66 + 66$ and 132/132	the sign-free fold fails in 124/132; a wrong sign in Ψ_j fails 66/66
Remark (2) of §10.5 (a measurement); code <code>gate_remark2.py</code> , in <code>remark2/</code>	$l = 2, \dots, 63,$ seed 11	as stated there	none

The folder `remark2/` contains three engines of the project, copied unchanged, and the driver that sets the four experiments of the remark. The direct computations of the first row also give the example of §1.3: $\text{Syl}_2 K(Q_5) = \{1 : 6, 3 : 4, 4 : 1, 6 : 4\}$ and $\text{Syl}_2 \bar{K}(5) = \{2 : 4, 3 : 1, 5 : 4\}$.

12.2 Independent re-computations

Six readers of the project (Appendix A) recomputed what follows, each with code of its own, never the authors'. Their code and logs are part of the project's record. As in §12.1, «none» means that the row has no control; «fires» means that the reader's control failed, as it must.

statement	reader	range	result	control
Main Theorem against the Smith form of the Laplacian of Q_n	1	$n \leq 13$	12/12	fires
Main Theorem, from the rule as printed in §1; Theorem DW	3	$n \leq 12$	12/12 and 12/12	the rule without the lowest digit of I , $a_n \pm 1$, and $Q_n/\langle x_1 x_2 \rangle$ fail
Theorem DW, in the vertex basis	2	$n \leq 12$	11/11	$Q_n/\langle x_1 x_2 \rangle$ fails for $n = 3, \dots, 12$
Gao et al.'s Table 1	3	$n = 7, \dots, 11$	agrees entry by entry	none
the recursion of §1.2 against Larsen's fusion rule [Lar24, §2]	1, 3	even $n \leq 64$	32/32	fires
Theorem 7.8	1	houses with $\lambda \leq 255, \alpha \leq 4$	0 failures	none
Theorem 7.8, every claim of its proof, strict (Cut), the integer separation of adjacent level sets (Lemma 7.9 of version 2), the integer shifts	3	all houses with $k \leq 7$ ($k = 0$ included), $\alpha = 1, 2$, penalties ≤ 6	43 690 houses, 0 failures	two wrong prices of the central flat are detected in 2 550 and 2 808 of 3 979 houses

statement	reader	range	result	control
Theorems D and 7.10 on the lattices $T(\lambda)$	3	two runs, $\lambda \leq 31$, $i \leq 10$ and $\lambda \leq 63$, $i \leq 8$, each with $\alpha = 1, 2$; the 43 and 372 cells whose predicted exponent is at least 60 skipped	639 and 762 cells compared, 0 failures	the raw clocks are rejected in all 102 and 172 compared cells where pooling acts
Theorem 7.10 on the matrices M	3	$\lambda \leq 127$, $i \leq 30$, $\alpha = 1, 2$	7 874 cells, 0 failures	the raw clocks are rejected in all 1 876 cells where pooling acts
Theorem 7.8, strict (Cut), the integer separation of adjacent level sets (Lemma 7.9 of version 2), the penalty shifts $P_r, P_c \leq 4$ and the deletion at $i = 0$	4	every house with $1 \leq k \leq 7$, $\alpha \leq 3$	65 532 houses, 777 240 non-zero shifts, 741 deletions, 0 failures	a price of the central flat one above or one below the interval fails in 10 374 houses each; the shift one dancer late fails in 78 070 of 721 872; strictness one dancer after the boundary fails in 7 967 of 30 078
Theorem 7.10 on the matrices M	4	$\lambda \leq 127$, $i \leq 30$, $\alpha = 1, 2$	7 874 cells, 0 failures	the raw clocks are rejected in all 1 834 cells with $i \geq 1$ where pooling acts; the penalty one dancer late in all 1 904
Lemma 7.2(b), restated, on random sequences	4	50 000 sequences of length ≤ 10	0 failures	a jump at a cut that is not strict breaks the integer part in 736 of 9 939 trials
Lemma 7.9 (integrality)	4	every house with $1 \leq k \leq 7$, $\alpha \leq 3$, and the 7 634 cells of its row on the matrices M other than the 240 with $i = 0$ and $I \neq \emptyset$ (the 7 620 with $i \geq 1$, and the 14 with $i = 0$ and $I = \emptyset$, where $M = 0$)	no non-integral level set	none
Lemma 7.9; the strict (Cut); Lemma 7.2(b) at the penalty shifts; the deletion at $i = 0$	5	every house with $1 \leq k \leq 7$, $\alpha \leq 3$ (65 532); penalties $P_r, P_c \leq 4$ on the 8 001 houses with $k \leq 6$ that have a boundary (192 024 cells); 741 deletions	0 failures	x_H with 1 added at its first dancer: non-integral in 1 942 of 16 380 houses; strictness one dancer after the boundary fails in 7 967 of 30 078

statement	reader	range	result	control
§1.2 (the rounding never acts); Propositions 5.3, 5.4 and Lemma 7.5; Proposition 9.6 (H)	5	every family $\lambda \leq 255$, every shift $i \leq 40$ (10 455 cells; (H) on the 10 200 with $i \geq 1$)	0 failures	for the integrality: the raw clocks with 1 added at the first dancer have a non-integral block mean in 2 316 of the 9 880 cells with $I \neq \emptyset$ and $i \geq 1$; none for Propositions 5.3 and 5.4, Lemma 7.5 and (H)
Theorem 6.2	3	$ I \leq 5$	69/69	«any run» and «any subset» fail in 26 and 42 cases
Corollaries 1.5, 1.6 (from the rule) and 1.7	3	$n \leq 300$; $e \leq 8$	0 failures	a wrong exponent in Corollary 1.5 and a wrong constant in Corollary 1.6 fail for 149 and 65 values of n
odd part of Corollary 1.4	3	$p = 3, 5, 7$, $n \leq 11$	30/30	fires
$\bar{X}(\lambda, i) \cong 2X(\lambda, i)$ on explicit lattices	2	$\lambda \leq 20$, $i \leq 6$	140/140	$\lambda = 0$ and the sign-free fold fail
$\bar{X}(\lambda, i) \cong 2X(\lambda, i)$ from the definition $T(\lambda)/(\tau_i T(\lambda) + (a - 1)T(\lambda))$	3	$\lambda \leq 63$, $i \leq 8$	513/513	a replaced by $-a$ fails 513/513
the formal dance of Ψ_j : clean, with the support, the entry valuations and the Smith form of $M^{(2)}(l, \lceil j/2 \rceil)$ (Theorems 10.12, 10.15, 10.16)	3	$l \leq 31$, $j \leq 7$	256/256	$\alpha = 1$, and a coefficient of $F^{(2)}$ depending on the parity of the floor, fail
Theorem 10.15: one move on random covariant systems	2	levels 0–3, 1–4 dancers	800/800	none
Lemma 10.20	2	$l \leq 24$, $c = 1, \dots, 4$, three kinds of weights	96/96 for each kind and sign	$\alpha = 1$ and $\Psi \pm 4$ fail in 95/96 and 71/72
the turned lid (§11, question 2)	3	$n = 3, \dots, 12$, every b	0 failures (a measurement)	none
the cells of the rows of readers 4 and 5; Lemma 7.9 on houses and on the 10 455 cells of the fifth reader's second row	6	$\lambda \leq 127$, $i \leq 30$, $\alpha = 1, 2$; houses with $1 \leq k \leq 6$, $\alpha \leq 3$; $\lambda \leq 255$, $i \leq 40$	7 874 cells: 254 with $i = 0$ (240 with $I \neq \emptyset$), 7 620 with $i \geq 1$; 16 380 houses and 10 455 cells, no non-integral fit	the controls of reader 5 reproduced: 1 942 of 16 380 houses, 2 316 of 9 880 cells

12.3 The Lean certificate

Every theorem of this paper has been checked in the proof assistant Lean 4 [dMU21], toolchain v4.28.0, with the Mathlib library [mC20], tag v4.28.0. The certificate [LC26] prints the Lean statements and every definition they use, compares them with this paper line by line, and says how the proof was obtained and how to check it.

What Lean proves. For every $n \geq 1$:

- the Main Theorem (`MainTheorem.M1`), with the rule of §1.2 written out in Lean: the recursion for the multiplicities, the binary digits of $\lambda + 1$, the raw clocks, the integer antitonic fit and the groups $X(\lambda, i)$;
- Theorem DW (`FoldLaw.DW`) and its equivalent form (`SmallCases.DW'_all`; for $n \geq 2$ it is `FoldLaw.DW'`);
- Corollary 1.4: the odd part (`OddParts.C5`); the 2-part is `SmallCases.DW'_all` read together with the Main Theorem, which Lean does not state as one theorem;
- Corollaries 1.5 (`BaiCounts.B1`, `BaiCounts.B2`), 1.6 (`PodiumCor.P1`, `PodiumCor.P2`, `PodiumCor.P3`, `PodiumCor.G1`, `PodiumCor.G2`, `PodiumCor.G3`; Gao et al.'s Theorem 4.1 for every $n \geq 2$ is `SmallCases.G1_all`) and 1.7 (`Mirror.Y2`);
- Bai's odd part of $K(Q_n)$ (`OddParts.B1`), which this paper cites [Bai03] and Lean proves;
- Theorem D (`Windows.theorem_D_i`), and each of the 68 numbered statements of §2–§10, from Lemma 2.1 to Theorem 10.22.

Nothing is cited. The statements are about a finite graph. Lean defines the Laplacian of Q_n , its cokernel and the torsion of that cokernel, and proves the theorems themselves; Kirchhoff's theorem is not used. The proof contains no `sorry`, and every theorem depends only on the three standard axioms `propext`, `Classical.choice` and `Quot.sound`.

The one point of trust is that the Lean definitions say what this paper says. The certificate prints every definition the statements use and compares it with this paper (its §1.2–§1.3). That comparison is what a reader should check.

How. The text of §2–§10 was cut into 68 pieces, each a short self-contained file; a 69th piece added the cases that the first 68 left out (the equivalent form of Theorem DW, the count of §10.8 and the $\mathbb{F}_2$ -rank of the folded Laplacian at $n = 1$, and Gao et al.'s Theorem 4.1 at $n = 2$). Aristotle, an AI system made by Harmonic [Har26], wrote the Lean proofs. On the author's machine each returned project was compared with the previous one, searched for forbidden constructs, and compiled module by module; the main statements of each piece were restated by hand in a check file, and their faithfulness to the paper was audited. At the end of the 68 pieces all 416 modules were compiled again from their sources, in one clean rebuild, all with exit code 0; the four modules of the 69th piece were then compiled from their sources, with the sources of the other 416 unchanged. The project has 420 Lean files (419 in its 69 folders, and the main file), 61 285 lines and 3 470 theorem and lemma declarations.

Read cold. Before release, the certificate was read by a reader other than its writer, with the compiled project at hand. The reader re-read every definition in the statements against this paper, re-elaborated from source the files of the Main Theorem and of Theorem DW and five more of the chain, and compared the rule with Smith forms it computed from the graphs ($n \leq 11$). Verdict: «holds as far as I can check»; no Lean statement says something other than this paper. It found four errors in the record and none in a statement: one sentence, the description of the edges of the folded cube in §1.3 of version 8 of this paper, which the certificate repeated (corrected in §1.3 here and in [LC26]), and three counts of the certificate (corrected in

[LC26]). The changes of version 2 of the certificate, which adds the 69th piece, were read cold in their turn before release, with the Lean of that piece.

What Lean does not state ([LC26], §1.3 and §2):

- the pool-adjacent-violators algorithm (Lean has the fit it computes);
- that the dance lattices are the indecomposable tilting modules (not used);
- the numerical facts of §1.2, §1.3 and §12.

The Main Theorem is an isomorphism of groups in Lean and of $\mathbb{Z}_2$ -modules here. The two statements are equivalent, because an additive map between finitely generated $\mathbb{Z}_2$ -modules is $\mathbb{Z}_2$ -linear ([LC26], §1.3).

Where. The certificate [LC26] is published with the Lean project, the check files, the files of the 69 pieces, and the logs of every piece and of the clean rebuild.

13. Exact status

- **Proved, with the grade «pencil, audited, read cold»:** the Main Theorem; Theorem D; Theorems O and F; Theorem DW and Corollary 1.4; Corollaries 1.5, 1.6 and 1.7. Their proofs were read cold as text in the first version of this text, which had one gap (Lemma 7.2(b)); the repair was read cold in its turn, and so were the changes of every later version (Appendix A).
- **Added in version 3, audited, read cold:** Lemma 7.9 in its new form (integrality of the fit), found and proved by the fourth reader, re-proved by the auditor, and re-derived by the fifth reader.
- **Numbered statements changed after the first version:** version 2 restated Lemma 7.2(b) and added the strictness of the cuts to the (Cut) of Theorem 7.8; version 3 strengthened Lemma 7.9, added the hypothesis «even» to Lemma 10.6(a), and added the explicit quantifier «for every $L \geq 0$ and every floor f » to Lemma 10.10. No other numbered statement changed in what it states; numbers, notation and typesetting changed.
- **Measured, not proved:** the turned lid (§11, question 2), and the measurements of Remark (2) of §10.5.
- **Checked in Lean 4 with Mathlib:** every proved result above and each numbered statement of §2–§10, with nothing cited; the certificate was read cold (§12.3).
- **Not done:** a human referee.

No human expert has refereed any part of this work. The results are stated as theorems because every step has a written proof that was re-derived by readers other than its author, and because the whole proof has been checked in Lean. Neither is a substitute for refereeing.

Acknowledgements and use of AI

The mathematics of this paper was developed with extensive use of Claude, an AI system made by Anthropic, working under the author’s direction in separate roles:

- instances that proposed and wrote proofs, each from a written mission;

- instances that re-derived them step by step and ran their own checks;
- instances that read the chains, and then the whole text, cold, with no access to the audits (Appendix A).

The Lean proofs (§12.3) were written by Aristotle, an AI system made by Harmonic [Har26], from pieces written from this paper. Instances of Claude wrote the pieces, audited what came back, compiled it on the author’s machine, and read the certificate cold.

The author directed the project and takes responsibility for the content.

The work began on 2 October 2026 as a search, inside another project of the author, for open problems within reach of the same methods. The full record — the missions, the reports, the audits, the cold readings, the engines and their logs, including the routes that failed — is kept with the project and is available from the author.

Appendix A. How this work was done

Writing and checking. The proofs were found, family of statements by family of statements, by separate instances of an AI system working as **constructors**, each from a written mission and in a folder of its own («flights» 1 to 9, 3–5 October 2026). Each flight was then **audited** by a different instance, which re-derived every step in its own words, marked it CORRECT, GAP or ERROR, and ran its own engines (never the constructor’s) with controls built to fail. A step went into the chain only after the audit. The chains and then the text were read by **cold readers**: further instances that never saw the audits, and wrote their own verdicts and their own code. Every computation ran under a watchdog with limits on memory and time, and every log is checked for its final line.

Readings.

- **The Main Theorem** (flights 1–4: Theorems D, O, U, F and the integer fit). Audited flight by flight. Read cold on 4 October (cold reader 1), from the statements and the flights’ reports, with no access to the audits: «**holds**», every link re-derived, with three remarks on presentation, applied in version 1.
- **Corollaries 1.5–1.7** (flights 5–6). Audited, with the auditor’s own gates; read cold numerically by cold reader 1 ($n \leq 64$), and as text with version 1 (below).
- **Theorem DW** (flights 6–9). Audited flight by flight. Read cold on 5 October (cold reader 2), with no access to the audits: «**holds**», with six remarks on presentation and a priority search, applied in version 1.
- **Version 1 of this text** was written by the auditor from the flights, the audits and the readings. Two parts were new in the writing: the proof of Theorem 6.2 by top bits, and the organization of §10 around Theorems 10.12 and 10.15. It was read cold as a whole on 5 October 2026 (cold reader 3) by a reader with no access to the flights, the audits or the earlier readings, who re-derived every numbered statement and wrote its own code: «**holds with gaps**». It found one gap: the integer part of Lemma 7.2(b) is false as stated (the example in its proof), and it was used in Theorem 7.8 (step 1), Lemma 7.9, Theorem 7.10 and Proposition 9.6. No theorem is affected. Version 2 restates Lemma 7.2(b), adds strict (Cut) to Theorem 7.8 with its proof, and cites it at each use; this repair is the reader’s, re-derived by the auditor and checked by machine (§12.1). The reader also found that Remark (2) of §10.5 stated more than had been measured, that the quotation of [Aky26] had been trimmed so as to change its meaning, that four entries of the verification record said more than the code did (the range of Theorem 6.2, the houses of Theorem 7.8 counted

as house-checks, a control announced and not implemented in `gate_dance.py`, and rows resting on code not printed), and about thirty points of presentation; all are corrected in version 2.

- **Version 2** repaired the gap. Its changes were read cold on 5 October 2026 (cold reader 4) by a reader with no access to the flights, the audits or the earlier readings, who re-derived every changed step, checked the repair on every house with $1 \leq k \leq 7$, $\alpha \leq 3$ and against the Smith form of M in 7 874 cells, and wrote its own code: the repair **holds**. It found one false sentence (Remark (2) of §10.5 said «every payment system» where Proposition 4.1 needs an even one; the author then found the same word missing in Lemma 10.6(a), harmless because its only use is for an even system), two rows of §12.1 that said more than their code (zero shifts counted as shifts, and a range wider than the one run), a control that did not control the house check, and points of presentation, the main one being that the cases of the proof of Theorem 7.8 ran together in the pdf. It also found, with a two-line proof, that the fit of every house is integral, so the rounding of the integer fit never acts in this paper: the gap of version 1 was a gap of proof, never of conclusion.
- **Later versions.** Each later version made corrections asked for by the reading before it; version 4 also set the mathematics anew, and version 5 the page. No numbered statement changed in what it states after version 3 (§13). The changes of each version were read cold by a new reader (cold reader 5 and each later one, from 5 October 2026), with no access to the flights or the audits, who checked every changed passage against the version before it, re-counted or re-ran with code of its own the numbers and checks it concerned, and looked at the pdf: at every page, or at the changed pages after comparing all the pages by image. None of these readings found a gap or an error in the mathematics. Between them, they found false or inexact sentences in this record and in §12, points of presentation, and defects of typesetting. This summary replaces a version-by-version account; that account, the list of corrections of each version with its sources, every report and every grading are kept with the project.

Formal verification. From 6 to 10 October 2026 the proofs were checked in Lean 4 with Mathlib, in 68 pieces; a 69th piece, for the small cases, followed on 11 October 2026. The certificate was read cold on 11 October 2026, and so were the changes of its version 2, which adds the 69th piece (§12.3, [LC26]).

References

- [ABERS55] M. Ayer, H. D. Brunk, G. M. Ewing, W. T. Reid, E. Silverman, An empirical distribution function for sampling with incomplete information, *Ann. Math. Statist.* 26 (1955), 641–647.
- [Aky26] T. Akyar, A. Beliaikov, K. Delchev, N. Kalinin, E. Lupercio, H. Serrano, M. Shkolnikov, D. Tabares, N. Terekhov, The 2-adic valuation of the order of the all-ones class in the sandpile group of a square, arXiv:2609.10625 (2026).
- [Bai03] H. Bai, On the critical group of the n -cube, *Linear Algebra Appl.* 369 (2003), 251–261.
- [BBBB72] R. E. Barlow, D. J. Bartholomew, J. M. Bremner, H. D. Brunk, *Statistical Inference under Order Restrictions*, Wiley, 1972.
- [Bie93] T. Bier, Remarks on recent formulas of Wilson and Frankl, *European J. Combin.* 14 (1993), 1–8.
- [Big99] N. L. Biggs, Chip-firing and the critical group of a graph, *J. Algebraic Combin.* 9 (1999), 25–45.
- [Cla40] T. Clausen, Lehrsatz aus einer Abhandlung über die Bernoullischen Zahlen, *Astron. Nachr.* 17 (1840), 351–352.

- [CSX17] D. B. Chandler, P. Sin, Q. Xiang, The Smith group of the hypercube graph, *Des. Codes Cryptogr.* 84 (2017), 283–294; arXiv:1511.00272.
- [DEGJPP23] J. E. Ducey, L. Engelthaler, J. Gathje, B. Jones, I. Pfaff, J. Plute, Integer diagonal forms for subset intersection relations, arXiv:2310.09227 (2023).
- [DHS18] J. E. Ducey, I. Hill, P. Sin, The critical group of the Kneser graph on 2-subsets of an n -element set, *Linear Algebra Appl.* 546 (2018), 154–168; arXiv:1707.09115.
- [DJ14] J. E. Ducey, D. M. Jalil, Integer invariants of abelian Cayley graphs, *Linear Algebra Appl.* 445 (2014), 316–325; arXiv:1308.2335.
- [dMU21] L. de Moura, S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction — CADE 28*, Lecture Notes in Computer Science 12699, Springer, 2021, 625–635.
- [Don93] S. Donkin, On tilting modules for algebraic groups and finite dimensional algebras, *Math. Z.* 212 (1993), 39–60.
- [EL91] A. El-Amawy, S. Latifi, Properties and performance of folded hypercubes, *IEEE Trans. Parallel Distrib. Syst.* 2 (1991), 31–42.
- [GMMY24] J. Gao, J. Marx-Kuo, V. McDonald, C. H. Yuen, Sandpile groups of Cayley graphs of $\mathbb{F}_2^r$, *Comm. Algebra* 52 (2024), 4459–4479; arXiv:1912.06919v3 (cited by its numbering).
- [Har26] Harmonic, Aristotle, an AI system for mathematics, <https://aristotle.harmonic.fun> (2026).
- [Hum72] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9, Springer, 1972.
- [IKKY23] K. Iga, C. Klivans, J. Kostiuk, C. H. Yuen, Eigenvalues and critical groups of Adinkras, *Adv. in Appl. Math.* 143 (2023), Paper No. 102450; arXiv:2202.02821.
- [Jan03] J. C. Jantzen, *Representations of Algebraic Groups*, 2nd ed., Mathematical Surveys and Monographs 107, AMS, 2003.
- [Kli18] C. J. Klivans, *The Mathematics of Chip-Firing*, CRC Press, 2018.
- [Kos66] B. Kostant, Groups over $\mathbb{Z}$, in *Algebraic Groups and Discontinuous Subgroups*, Proc. Sympos. Pure Math. 9, AMS, 1966, 90–98.
- [Kum52] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, *J. Reine Angew. Math.* 44 (1852), 93–146.
- [Lar24] M. Larsen, Bounds for SL_2 -indecomposables in tensor powers of the natural representation in characteristic 2, arXiv:2405.16015 (2024).
- [LC26] R. Amichis Luengo, *The Dancing Sand Theorem — Lean certificate*, version 2, Zenodo (2026), DOI 10.5281/zenodo.23298436; all versions, concept DOI 10.5281/zenodo.23296707.
- [Lor91] D. J. Lorenzini, A finite group attached to the Laplacian of a graph, *Discrete Math.* 91 (1991), 277–282.
- [LZ22] Y. Len, D. Zakharov, Kirchhoff’s theorem for Prym varieties, with an appendix by S. Casalaina-Martin, *Forum Math. Sigma* 10 (2022), Paper No. e11; arXiv:2012.15235.
- [mC20] The mathlib Community, The Lean mathematical library, in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, 367–381.
- [MGM19] J. Marx-Kuo, J. Gao, V. McDonald, The sandpile group of Cayley graphs, poster, Joint Mathematics Meetings, Baltimore, January 2019.
- [OEIS] The On-Line Encyclopedia of Integer Sequences, sequence A193134, accessed 5 October 2026.
- [RT14] V. Reiner, D. Tseng, Critical groups of covering, voltage and signed graphs, *Discrete Math.* 318 (2014), 10–40; arXiv:1301.2977.
- [Str28] R. Strassmann, Über den Wertevorrat von Potenzreihen im Gebiet der p -adischen Zahlen,

- J. Reine Angew. Math.* 159 (1928), 13–28.
- [TW21] D. Tubbenhauer, P. Wedrich, Quivers for SL_2 tilting modules, *Represent. Theory* 25 (2021), 440–480.
- [vSt40] K. G. C. von Staudt, Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend, *J. Reine Angew. Math.* 21 (1840), 372–374.
- [VZ24] M. Vetluzhskikh, D. Zakharov, Critical groups in harmonic abelian quotients, arXiv:2409.04629 (2024).
- [Wil90] R. M. Wilson, A diagonal form for the incidence matrices of t -subsets vs. k -subsets, *European J. Combin.* 11 (1990), 609–615.
- [Yue24] C. H. Yuen, The critical groups of Adinkras up to 2-rank of Cayley graphs, *Electron. J. Combin.* 31 (2024), Paper No. P1.38; arXiv:2301.02517.