

Finite quantum theta functions for skew-symmetrizable seeds

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Abstract

For integral compatible skew-symmetrizable seeds, finite quantum theta functions form a bar-invariant, mutation-compatible basis of an algebra between the cluster and upper cluster algebras. We prove coefficient parity, leading-coefficient positivity, valuative independence, unit Newton vertex coefficients, and independence under coefficientwise base change. We obtain upper theta bases for large cluster complexes, reddening classes, and Markov matrices and their one-way acyclic extensions. In mutation-acyclic classes, these are also bases of the cluster algebra. Specialization to the classical upper algebra is injective over the integers and is an isomorphism in these classes.

AI use and scholarly credit

The results of this paper were developed through AI-assisted exploration, drawing on our earlier work and the research materials collected for this project. The arguments underwent multiple rounds of dialogue and review, and the manuscript was organized using the built-in mathematical writing skill.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Broken lines associate formal Laurent series to tropical lattice points. To obtain a basis of the upper cluster algebra, one must establish Laurent finiteness and finite theta expansions of regular functions. Quantum coefficients may contain negative terms, so classical specialization can shrink their support.

Let v be an indeterminate. Gross–Hacking–Keel–Kontsevich [5] construct classical theta functions. Mandel’s refined scattering framework [7] and the skew-symmetrizable initial diagrams discussed in [4, Remark 4.4] precede the construction used here. Davison–Mandel prove coefficientwise quantum positivity for skew-symmetric cluster data. For general skew-symmetrizable seeds, we use the ordering of $\mathbb{Q}(v)$ determined by the highest term. It permits coefficients such as $v^2 - 1 + v^{-2}$, whose Laurent coefficients are not all nonnegative. Nakanishi [16, Theorem 5.10 and Corollary 5.12] proves nonpositivity for certain nonskew-symmetric quantum cluster scattering diagrams.

Set $R = \mathbb{Z}[v^{\pm 1}]$ and $K = \mathbb{Q}(v)$. A seed with $n \geq 1$ mutable and $f \geq 0$ frozen variables has exponent lattice $M = \mathbb{Z}^{n+f}$; write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For $m \in M$, let X^m be its based

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quantum monomial and Θ_m its broken-line theta series. The multiplication and broken-line conventions are given in Section 2. Let $\Theta_{\text{fin}} \subseteq M$ consist of the labels whose theta series has finitely many nonzero monomial coefficients in a seed chart. Bar fixes each X^m , sends v to v^{-1} , and reverses products. Define

$$\mathcal{M}_\Theta = \bigoplus_{m \in \Theta_{\text{fin}}} R\Theta_m.$$

Here $\mathcal{A}$ is the R -algebra generated by all mutable cluster variables and the frozen variables and their inverses in the common skew field; $\mathcal{U}$ is the intersection of all seed quantum tori in that skew field.

Theorem 1.1. *For every integral compatible seed with positive diagonal skew-symmetrizer, the theta series have coefficients in R , are invariant under bar, and are transported by quantum seed mutation. Laurent finiteness is independent of the seed. The module $\mathcal{M}_\Theta$ is an algebra and*

$$\mathcal{A} \subseteq \mathcal{M}_\Theta \subseteq \mathcal{U}.$$

Its structure constants have positive highest coefficient whenever nonzero. The set Θ_{fin} is closed under addition and is saturated: $km \in \Theta_{\text{fin}}$ for a positive integer k if and only if $m \in \Theta_{\text{fin}}$.

For a nonzero finite Laurent polynomial $f = \sum_{b \in M} f_b X^b$ with $f_b \in K$, set $\text{supp } f = \{b : f_b \neq 0\}$ and $\text{Newt}(f) = \text{conv}(\text{supp } f)$, where conv denotes the real convex hull in $M_{\mathbb{R}}$. For a real linear functional $\ell : M_{\mathbb{R}} \rightarrow \mathbb{R}$, set $\text{val}_\ell(f) = \min_{b \in \text{supp } f} \ell(b)$.

Theorem 1.2. *Fix a seed. If $J \subset \Theta_{\text{fin}}$ is finite and nonempty and $c_m \in K \setminus \{0\}$, then*

$$\text{Newt} \left(\sum_{m \in J} c_m \Theta_m \right) = \text{conv} \bigcup_{m \in J} \text{Newt}(\Theta_m).$$

Every vertex of the polytope on the right belongs to the support of exactly one participating theta function. Its coefficient in that theta function is one. For every nonzero commutative unital ring S and homomorphism $R \rightarrow S$, the specialized finite theta functions remain linearly independent in the specialized based quantum torus. Each finite coefficient matrix has a unit triangular maximal minor.

Cheung–Magee–Mandel–Muller prove valuative independence and Newton vertex monicity in [3, Theorems 3.8, 3.10 and 3.15]. Their Remark 1.3 extends these results to the quantum setting covered by Davison–Mandel positivity. Theorem 1.2 treats finite theta functions for general compatible skew-symmetrizable seeds and gives coefficientwise base-change independence on the fixed exponent lattice. Its proof follows the rational broken-line and reverse-reconstruction approach, using ordered wall-action supports and finite local certificates.

For an exchange matrix $B = (b_{ij})$, its directed graph has an arrow $i \rightarrow j$ when $b_{ij} > 0$. It is acyclic when this graph has no directed cycle, and mutation-acyclic when some matrix in its mutation class is acyclic. To define reddening, start with the auxiliary extended matrix $\begin{pmatrix} B \\ I_n \end{pmatrix}$ and apply the same matrix mutations. The bottom block is its c -matrix, and its columns are the c -vectors. A sequence is reddening when all final c -vectors are entrywise nonpositive; it is green when every selected c -vector is entrywise nonnegative before its mutation. For index sets I, F , $B_{F,I}$ denotes the submatrix with rows in F and columns in I . Freezing F means retaining its variables but allowing mutation only in the complementary mutable set I .

- Theorem 1.3.** 1. A reddening sequence gives finite support boxes for every theta label and an R -basis of $\mathcal{U}$. In a mutation-acyclic class this is also a basis of $\mathcal{A}$.
2. The principal enlargement in Section 10 has a central frozen kernel, and quotienting by its augmentation ideal gives $\mathcal{U}$ and $\mathcal{M}_\Theta$, respectively.
3. Suppose the classical upper algebra has a basis of Laurent-polynomial theta functions satisfying valuative independence and each basis label has a finite quantum theta function. Then these quantum theta functions form an R -basis of $\mathcal{U}$, and each nonzero summand in the expansion of f has support in $\text{Newt}(f)$.
4. These hypotheses hold for the Markov matrix with any compatible integral coefficient realization. Thus $\mathcal{M}_\Theta = \mathcal{U}$ in that class, although it admits no reddening sequence.
5. Partition the mutable indices as $I \sqcup F$, with $B_{F,I} \geq 0$ entrywise and $B_{F,F}$ acyclic. All-label finiteness and the equality $\mathcal{M}_\Theta = \mathcal{U}$ each pass from the frozen core to the full seed. In particular the connected Markov extensions of Section 12.4 have upper theta bases.
6. If the cluster complex is large in the sense of Section 11.5, all theta labels are finite and $\mathcal{M}_\Theta = \mathcal{U}$.

Berenstein–Zelevinsky [2, Theorem 7.5] prove equality of the quantum cluster and upper cluster algebras in mutation-acyclic classes, including skew-symmetrizable data. Equality with the intermediate theta algebra then follows from Theorem 1.1. The reddening application uses Qin’s criterion [9, Theorem 6.2.1(1)], and the Markov application uses Zhou’s classical principal-coefficient basis [11, Theorem 5.1].

Gao [13, Chapter 3] develops skew-symmetrizable quantum theta functions and their relation to quantum greedy bases. In Section 3.3, the term “finite type” is defined by requiring the basis multipliers to take values from a pair of positive integers with product at most three; Theorems 3.17 and 3.19 are stated in this multiplier-based setting.

Qin’s freezing operators identify theta functions on charge-coordinate faces and transfer upper bases to frozen seeds under the hypotheses of [15, Theorems 5.2, 6.13 and 6.18]. In the classical quiver-with-potential setting, Fei–Weyman [17, Corollary 2.13 and Theorem 2.14] establish source–sink and optimized-vertex extensions of cluster models under full-rank and Noetherian hypotheses. Our unfreezing theorem transfers a finite quantum theta upper basis from a frozen core to a one-way extension over the integral Laurent coefficient ring. Together with the finite theta structure for general compatible skew-symmetrizable seeds, this yields upper bases for the nonskew-symmetric Markov extensions in Section 12.4, which have no reddening sequence.

The formal theta coefficients and finite structure constants satisfy parity formulas in the original quantum parameter, including unequal symmetrizer entries (Theorem 3.5 and Corollary 6.4). Integral specialization embeds the quantum upper algebra modulo $v - 1$ into its classical counterpart; independent classical unfreezing gives an isomorphism for the upper-basis classes above (Theorem 11.2 and Corollary 11.12).

2 Compatible lattices and scattering conventions

Let $n \geq 1$ and $f \geq 0$ be integers. Put $N = \mathbb{Z}^n$ and $M = \mathbb{Z}^{n+f}$, with standard bases denoted by e_i in the indicated lattice. Write $m = (m_{\text{mut}}, m_{\text{fr}}) \in M$, where $m_{\text{mut}} \in \mathbb{Z}^n$ and $m_{\text{fr}} \in \mathbb{Z}^f$. We write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. A compatible datum consists of

integer matrices

$$p = \begin{pmatrix} B \\ C \end{pmatrix} : N \longrightarrow M, \quad \Lambda = -\Lambda^T, \quad p^T \Lambda = (D \ 0), \quad D = \text{diag}(d_1, \dots, d_n), \quad d_i > 0.$$

Here B is $n \times n$, C is $f \times n$, and Λ is $(n+f) \times (n+f)$. If $pa = 0$, then $a^T p^T \Lambda = (a^T D, 0) = 0$, so $a = 0$. Thus p is injective. The identity $p^T \Lambda p = DB$ makes DB skew-symmetric. The sublattice pN may be nonsaturated. For an extended matrix $p = (p_{ij})$, mutation μ_k at $1 \leq k \leq n$ is defined by

$$(\mu_k(p))_{ij} = \begin{cases} -p_{ij}, & i = k \text{ or } j = k, \\ p_{ij} + [p_{ik}]_+ [p_{kj}]_+ - [-p_{ik}]_+ [-p_{kj}]_+, & i \neq k, j \neq k, \end{cases}$$

where $[a]_+ = \max(a, 0)$; on vectors and matrices this notation applies entrywise.

The based quantum torus is the free R -module

$$\mathcal{T}_R = \bigoplus_{m \in M} R X^m, \quad X^m X^{m'} = v^{\Lambda(m, m')} X^{m+m'}.$$

Here $\Lambda(m, m') = m^T \Lambda m'$ and $X_i = X^{e_i}$ for $1 \leq i \leq n+f$. For a commutative unital R -algebra S , put $\mathcal{T}_S = S \otimes_R \mathcal{T}_R$, with the same based monomials and the image of v in its multiplication law. For a scalar localization L of R , write $\mathcal{U}_L$ for the intersection of the seed tori $\mathcal{T}_{s,L}$ in the common skew field; thus $\mathcal{U}_R = \mathcal{U}$. Bar is the $\mathbb{Z}$ -linear anti-involution $\bar{v} = v^{-1}$, $\overline{X^m} = X^m$. For $F = \sum_{b \in M} c_b X^b$, define $[X^m]F = c_m$, with value zero when m is absent. Products are first expressed in the based monomials using the displayed multiplication rule.

Let $P = \mathbb{N}^n$, including zero, and $N^+ = P \setminus \{0\}$. For $a \in P$ set $\text{ht}(a) = \sum_i a_i$. For a series, its support is again the set of exponents with nonzero coefficients. Formal series below have support in a finite union of translates of pP . On a fixed translate $b + pP$, truncation at height H retains the terms X^{b+pa} with $\text{ht}(a) \leq H$. For a finite union, take these truncations in each translate; their retained exponent sets are finite. Convergence on a fixed support set means eventual agreement on every such truncation. Products are defined coefficientwise: injectivity of p makes each coefficient a finite sum, since a fixed nonnegative charge has finitely many decompositions in P .

Set $G = M/pN$, and give X^m degree $[m] \in G$ and every scalar degree zero. Thus $\mathcal{T}_R = \bigoplus_{\gamma \in G} \mathcal{T}_\gamma$, where $\mathcal{T}_\gamma$ is the span of the monomials of degree γ . A finite union of translated positive cones meets only finitely many G -classes.

Define, for $a, b \in N$ and $m \in M$,

$$\nu_a(m) = \Lambda(pa, m) = \sum_i d_i a_i m_i, \quad \Omega(a, b) = \Lambda(pa, pb) = a^T D B b.$$

The map $a \mapsto \nu_a$ is injective. In particular, at a general point of a wall, all wall degrees with the same supporting hyperplane lie on the same ray in N .

Extend each ν_a linearly to $M_{\mathbb{R}}$, and write $\nu_a^\perp = \{Q \in M_{\mathbb{R}} : \nu_a(Q) = 0\}$. A wall of charge ray $\mathbb{R}_{>0}a$ has rational polyhedral support of codimension one in $M_{\mathbb{R}}$, contained in $\nu_a^\perp$ and a group element in $1 + X^{pa}K[[X^{pa}]]$. For a wall element g , write $\text{Ad}_g(F) = gFg^{-1}$. Its positive action is Ad_g ; the inverse action is used for the opposite crossing. Both preserve the G -degree, since their exponent shifts lie in pP . On a monomial X^m , the favorable action is the positive action when $\nu_a(m) > 0$ and the inverse when $\nu_a(m) < 0$. A central wall is incoming when its support contains pa , and outgoing otherwise. For an affine support the corresponding condition uses its recession cone. Diagrams are considered up to equality of path-ordered products. They have finitely many nontrivial walls at every bounded positive

height. Boundaries of walls and intersections of distinct supporting hyperplanes are called seams. Their codimension-two cells are joints. A joint is parallel to a charge a if pa belongs to its tangent space, and perpendicular to that charge otherwise. We use the rational existence and uniqueness theorem for scattering diagrams [4, Theorem 2.13], or [7, Theorem 2.6], and endpoint transport [7, Theorem 2.14].

For an indeterminate w , write

$$\Psi_w(z) = \exp \left(\sum_{j \geq 1} \frac{(-1)^{j+1} z^j}{j(w^j - w^{-j})} \right).$$

The initial wall of direction e_i is the full hyperplane $\nu_{e_i}^\perp$ with label $\Psi_{v^{d_i}}(X^{pe_i})$. The cluster diagram is its consistent outgoing completion. A broken line ending at a point Q is a continuous piecewise affine path on $(-\infty, 0]$ with finitely many genuine bends, whose segment decorated by cX^m has velocity $-m$. It begins with X^m , avoids seams, and at a transverse wall crossing selects one nonzero monomial from the wall action on its current decoration. Taking the constant action term is allowed. Its final monomial contributes to $\Theta_{m,Q}$; define $\Theta_{0,Q} = 1$. At fixed height these sums are finite. The positive seed chamber is $Q_i > 0$ for $1 \leq i \leq n$. Each $\Theta_{m,Q}$ is homogeneous of degree $[m] \in G$, because every bend adds a charge in pP .

For an integer $r \geq 0$, the centered Gaussian polynomials are defined by

$$\prod_{j=0}^{r-1} (1 + w^{r-1-2j} z) = \sum_{s=0}^r \begin{bmatrix} r \\ s \end{bmatrix}_w z^s.$$

For integers s , set $\begin{bmatrix} r \\ s \end{bmatrix}_w = 0$ when $s < 0$ or $s > r$. For $r \in \mathbb{Z}$ and $s \geq 0$, define

$$C_r(s; w) = \begin{cases} \begin{bmatrix} r \\ s \end{bmatrix}_w, & r \geq 0, \\ (-1)^s \begin{bmatrix} -r+s-1 \\ s \end{bmatrix}_w, & r < 0. \end{cases}$$

Since $\Psi_w(w^2 z) = (1 + wz)\Psi_w(z)$, the initial wall action is

$$\text{Ad}_{\Psi_{v^{d_i}}(X^{pe_i})}(X^m) = \sum_{s \geq 0} C_{m_i}(s; v^{d_i}) X^{m+spe_i}, \quad (1)$$

These coefficients belong to R and are bar invariant. The inverse action uses C_{-m_i} .

3 Integral actions and mutation

3.1 Integral ray factorization

Let G_K be the group obtained by exponentiating the completed positive Lie algebra $\prod_{a \in N^+} KX^{pa}$, with commutator bracket $[X^{pa}, X^{pb}] = (v^{\Omega(a,b)} - v^{-\Omega(a,b)})X^{p(a+b)}$, acting by conjugation on the completed torus. The exponential and group products are taken degreewise in positive height. Let G_R consist of those elements whose action coefficients on every X^m lie in R . The identity term is required to be X^m ; every other term has positive charge. Composition preserves G_R . Solving the identity $gg^{-1} = 1$ one height at a time proves that inversion also preserves it.

Lemma 3.1. *Choose any total order on the positive rational rays of P . Every $g \in G_K$ has a unique degreewise ordered factorization $g = \prod_{\rho} g_{\rho}$ into ray subgroups. If $g \in G_R$, all g_{ρ} belong to G_R . If g commutes with bar, so do its ray factors.*

Proof. Modulo height greater than one, each homogeneous component belongs to a unique ray. Having determined factors below height h , compare their ordered product with g in height h . The quotient is central in that truncation and splits uniquely by ray. This determines the next coefficient and proves existence and uniqueness by induction. At a fixed shift pa of height h , the coefficient of $g(X^m)$ is the coefficient of the factor on the ray of a , plus products of coefficients of smaller heights from the other factors, with powers of v . No other factor can contribute a single term of shift pa . Subtracting these integral terms gives the height- h action coefficient of the ray factor in R . Applying bar and uniqueness proves the last assertion. $\square$

Proposition 3.2. *The cluster diagram has a representative with integral, bar-compatible wall actions. Its theta series have integral bar-invariant coefficients, leading term X^m , and classical specialization equal to the classical theta series.*

Proof. Use full ray factors in the usual joint induction. At a joint where the diagram is consistent below height h , the full local monodromy is in G_R . Factor it by Lemma 3.1, and attach the required full ray factors, with the orientation prescribed by the loop, on the outgoing supports $j - \mathbb{R}_{\geq 0}pa$. Only factors whose leading height is h are needed at this stage.

Walls are cylindrical in the frozen subspace. At a codimension-two joint the charges annihilating its tangent space form a saturated rank-two sublattice. The restriction of Ω is either zero, in which case all local actions commute, or nondegenerate, in which case pa is transverse to the joint for every nonzero incident charge. Brackets have the sum charge and vanish when Ω does. These are precisely the properties used in [5, Lemmas C.6–C.7].

A newly introduced wall passing through a new joint away from its boundary contributes twice with inverse signs in height h . A side boundary of $j - \mathbb{R}_{\geq 0}pa$ is tangent to pa and hence cannot be a perpendicular joint. Thus all perpendicular local products are trivial. Comparing with the rational consistent diagram leaves central walls of height h . Their labels glue across perpendicular facets; every remaining facet is parallel to their direction. Such a polyhedral support contains its direction and is incoming. There is no new incoming datum, so the residual diagram is empty. This completes the induction. Bar is preserved by Lemma 3.1.

Each genuine bend consumes positive height. At bounded height there are finitely many walls, shifts, and bend sequences; a transverse sequence determines at most one path ending at Q . A tangent segment selects only the constant term. Thus the theta coefficients are finite sums of products of integral bar-invariant action coefficients. The unique zero-height path gives coefficient one at X^m .

The integral actions specialize at $v = 1$. On a ray they fix the kernel of its normal and multiply a normal coordinate by a one-variable unit; hence their specialization lies in the classical tropical-vertex group. The initial actions specialize to the classical initial actions. Consistency and rational uniqueness identify the specialized diagram. Degreewise finite broken-line sums commute with this specialization. $\square$

3.2 Transverse completion and the slab

Fix k and write $z = X^{pe_k}$. The transverse degree of a charge a is $\sum_{i \neq k} a_i$. Invert all nonzero polynomials in z . Monomials normalize this Ore set by $z \mapsto v^{2\Lambda(m, pe_k)} z$. A choice of representatives for $M/\mathbb{Z}pe_k$ identifies the localized torus, as a vector space, with a direct sum of copies of $K(z)$.

Complete in positive transverse degree, allowing only finitely many integral e_k -coordinates at each fixed transverse degree in wall labels. Let $\bar{G}$ be the associated group; let $\tilde{G}$ be its

subgroup with all original charge coordinates nonnegative. The group $\tilde{G}$ embeds in both the original completion and the transverse completion. Denote the slab action by $S = \text{Ad}_{\Psi_{v, d_k}(z)}$.

Lemma 3.3. *The cluster diagram admits a representative in the transverse completion with the pure e_k wall retained as a separate slab. Its other labels belong to $\tilde{G}$, and there are finitely many nontrivial walls at bounded transverse degree. This representative is consistent on the Ore-localized torus.*

Proof. Let $a_i \geq 0$ and let a lie off the e_k axis. Formula (1) gives a finite polynomial for $S(X^{pa})$ when $(Ba)_k \geq 0$, and for $S^{-1}(X^{pa})$ when $(Ba)_k \leq 0$. All resulting charges remain nonnegative. These are the favorable conjugations used at a slab joint.

Induct on transverse degree. At a general slab joint, the two possible oriented local products can be written $g_2 S g_1 S^{-1}$ when $(Ba)_k > 0$, and $S^{-1} g_2 S g_1$ when $(Ba)_k < 0$. Insert the ray correction on the opposite half-wall on the unconjugated side. The favorable conjugation just established puts this correction in $\tilde{G}$ and makes its e_k support finite. When $(Ba)_k = 0$ it commutes with the slab. Away from the slab, use the ordinary central joint correction.

At a perpendicular slab joint, all incident non-slab positive charges project to one positive ray modulo $\mathbb{Z}e_k$. Writing them as $\alpha e_k + \beta n$ with $\beta > 0$ shows that $(Ba)_k = \beta(Bn)_k$ has one common sign. Hence the chosen favorable conjugation works for the whole local product. The new support lies on the negative side when $(Ba)_k > 0$ and on the positive side when $(Ba)_k < 0$, precisely the unconjugated side in the displayed loop words. Its intersection with the slab is only its source joint, so it creates no further transverse slab joint. Away from the slab the three-case perpendicular-joint argument of Proposition 3.2 applies.

Parallel joints require a separate comparison. Fix transverse height J and use the original positive-charge completion, further truncated at original height H . Induct on H against the rational consistent diagram. A first discrepancy is central at that original height; perpendicular products are trivial, so the facet-gluing argument of Proposition 3.2 makes any residual support incoming, a contradiction. Letting H tend to infinity proves agreement in this positive-charge completion. The transverse-positive subgroup embeds in it: at each transverse height its e_k support is finite. Thus the non-slab parallel products are trivial there as well. At parallel slab joints, cancel the commuting slab and its inverse first.

For uniqueness in the transverse group, where negative e_k charges are allowed, take the first transverse height J of a discrepancy and minimize its finite polyhedral support, separately for each charge a . If $(Ba)_k \geq 0$, a discrepancy in the positive half-space can be followed along $+pa$ until its last exit from the finite union of outgoing cones. The exit remains on that side and its central local loop would force the discrepancy to vanish, a contradiction. Hence its support is in the negative half-space. For $(Ba)_k \leq 0$ it is in the positive half-space. When $(Ba)_k = 0$ these statements place it in the slab, impossible for a non-slab normal with positive transverse charge. At a perpendicular slab joint, the remaining discrepancy pieces consequently all lie on the unconjugated side of the favorable loop. Consistency then makes their local discrepancy product trivial. The same last-exit argument now removes all remaining pieces, proving uniqueness through J .

Taking the transverse limit gives the asserted representative. Products involving the slab act on $K(z)$; equality of their expansions is equality of these rational coefficients, since the Laurent expansion of a rational function is injective. Thus consistency also holds on the stated localized torus. This is the geometric construction of [5, Appendix C.2] with finite Gaussian conjugations. $\square$

3.3 Mutation and Laurent finiteness

For mutation in direction k , let E be the identity on M except for column k , which is $-e_k + [-pe_k]_+$, and let F be the identity on N except for row k , which is $-e_k^T + [B_{k,*}]_+$.

Then

$$p' = EpF, \quad \Lambda' = E^T \Lambda E, \quad T_k(m) = m + [m_k]_+ pe_k.$$

A linear lattice map L acts on based series by $L(\sum_m c_m X^m) = \sum_m c_m X^{Lm}$, fixing the coefficients; for a piecewise linear map this convention applies to each linear piece. Both E and F are involutions; the quantum mutation pullback is $S^{-1}E$. The piecewise linear map on labels and endpoints is ET_k . On a half-space containing Q its linear part is denoted $ET_{k,Q}$. Since F is integral and invertible, $EpN = p'N$, so E induces an isomorphism $G \rightarrow M/p'N$. Moreover,

$$ET_k(m) - Em = [m_k]_+ Epe_k = [m_k]_+ p'Fe_k \in p'N.$$

Thus tropical label transport carries degree $[m]$ to degree $[Em]$.

Proposition 3.4. *Theta series satisfy*

$$\Theta'_{ET_k(m), ET_k(Q)} = ET_{k,Q}(\Theta_{m,Q}).$$

For fixed transverse charge $(a_i)_{i \neq k}$, only finitely many values of a_k occur in $\Theta_{m,Q}$. If a theta function is finite in one positive seed chart, it is finite in every seed chart and belongs to $\mathcal{U}$.

Proof. The linear shear $\tau(m) = m + m_k pe_k$ is symplectic. Writing $S_{\pm pe_k} = \text{Ad}_{\Psi_{v^{d_k}}(X^{\pm pe_k})}$, the dilogarithm functional equation gives $S_{-pe_k} = \tau S_{pe_k}^{-1}$. Apply this identity to the slab representative of Lemma 3.3. On the two half-spaces, T_k changes wall supports and labels by the corresponding linear maps; the identity identifies their slab products. The resulting diagram has the mutated incoming walls, and rational uniqueness identifies it with the mutated cluster diagram.

A broken line is transformed segment by segment. At the slab, a favorable action with pairing $r \geq 0$ has indices $0 \leq j \leq r$. The transformed bend index is $r - j$, and $[j]_{v^{d_k}} = [r-j]_{v^{d_k}}$. Straight crossings are included as $j = 0$ or r . Away from the slab the linear maps preserve the quantum pairing and the selected action coefficients. This is a bijection, also after every finite truncation, and proves the displayed identity.

For slice finiteness, bound the transverse height by J . There are at most J non-slab bends and at most $J + 1$ slab crossings. The k th mutable exponent on every segment is bounded in absolute value by

$$|m_k| + J \max(\{0\} \cup \{|b_{ki}| : i \neq k\}),$$

since slab bends do not change that coordinate. Each slab action is a finite Gaussian with that bound; at the other bends, the transverse construction supplies finite e_k support. Finitely many sequences and choices remain.

Suppose now that the original positive-chart theta is finite. Move the endpoint to the adjacent seed chamber using the rational slab action. This preserves the finite set of transverse charge positions, although it can a priori introduce infinitely many e_k terms. The series after transport is a theta series, so slice finiteness makes each of these finitely many slices finite. The mutation identity then gives a finite Laurent polynomial in the adjacent torus. Iterate along any finite mutation sequence. The intersection of these tori is $\mathcal{U}$. $\square$

3.4 Parity of theta coefficients

Define the quadratic function $q_{B,D} : N \rightarrow \mathbb{Z}/2\mathbb{Z}$ by

$$q_{B,D}(a) = \sum_i d_i a_i^2 + \sum_{i < j} d_i b_{ij} a_i a_j \pmod{2}.$$

Its polarization is $q_{B,D}(a+b) - q_{B,D}(a) - q_{B,D}(b) = \Omega(a,b)$ modulo two: the diagonal cross terms are even, and the remaining terms give $a^T DB b$ since DB is skew-symmetric. We use $(-1)^e$ for the sign determined by $e \in \mathbb{Z}/2\mathbb{Z}$. The following is the weighted version of the parity property in [4, Proposition 3.11].

Theorem 3.5. *For $m \in M$, $a \in P$ and a general endpoint Q , put $c_{m,a,Q}(v) = [X^{m+pa}] \Theta_{m,Q}$. Then*

$$c_{m,a,Q}(-v) = (-1)^{\nu_a(m) + q_{B,D}(a)} c_{m,a,Q}(v).$$

Proof. On the positive charge completion over K , define, for $f \in K$ and $a \in P$,

$$\mathcal{J}(f(v)X^{pa}) = f(-v)(-1)^{q_{B,D}(a)} X^{pa}.$$

The polarization identity compensates for the sign acquired by $v^{\Omega(a,b)}$ under $v \mapsto -v$. Thus $\mathcal{J}$ is a semilinear algebra involution preserving every charge and height truncation. In the degree- $j e_i$ term of the initial wall logarithm, parameter substitution gives the sign $(-1)^{d_i j}$, while $q_{B,D}(j e_i) = d_i j^2 \equiv d_i j$. Hence each initial wall element is fixed.

Apply $\mathcal{J}$ to every wall label, leaving its support unchanged. Path products are transformed by this algebra involution, so the new diagram is consistent and has the same incoming data and outgoing supports. Rational scattering uniqueness makes it equivalent to the original diagram.

On series supported in $m + pN$ with the prescribed positive-cone bounds, define

$$\mathcal{J}_m(f(v)X^{m+pa}) = f(-v)(-1)^{\nu_a(m) + q_{B,D}(a)} X^{m+pa}, \quad a \in N.$$

Injectivity of p makes the charge a unique. For charge series g, h and a series F in this exponent coset, direct multiplication gives

$$\mathcal{J}_m(gFh) = \mathcal{J}(g) \mathcal{J}_m(F) \mathcal{J}(h).$$

For example, left multiplication by X^{pb} has phase $\nu_b(m) + \Omega(b, a)$, and

$$\nu_b(m) + \Omega(b, a) + \nu_{a+b}(m) + q_{B,D}(a+b) \equiv q_{B,D}(b) + \nu_a(m) + q_{B,D}(a) \pmod{2}.$$

Right multiplication has phase $-\nu_b(m) + \Omega(a, b)$ and gives the same identity. Consequently $\mathcal{J}_m$ intertwines each wall action, including its inverse, with the action in the transformed diagram.

Since $\mathcal{J}_m(X^m) = X^m$, applying it to the decorations of a broken line gives the corresponding decorations in the transformed diagram. The geometry and nonzero term choices are unchanged. At total charge a , genuine bends consume at most $\text{ht}(a)$ steps and use only finitely many walls and shifts in that truncation. A transverse decorated bend sequence is uniquely reconstructed backwards from Q ; a tangent crossing contributes only the constant term. Thus the coefficient sum is finite, as in Proposition 3.2, and $\mathcal{J}_m$ carries the theta series to the theta series of the transformed diagram. Equivalence of diagrams identifies the latter with $\Theta_{m,Q}$. Comparing coefficients proves the formula; integrality follows from Proposition 3.2. $\square$

Proposition 3.6. *For mutation at k , use the matrices E, F of Section 3 and fix a general endpoint Q . Let $\sigma = 1$ when $Q_k > 0$ and $\sigma = 0$ when $Q_k < 0$, and fix $m \in M$ and $a, b \in N$. Write B' for the mutable block of p' and ν' for the pairing defined by the mutated compatible datum. Set*

$$m' = ET_k(m), \quad a' = Fa + ([m_k]_+ - \sigma m_k - \sigma(Ba)_k) e_k.$$

Then, modulo two,

$$q_{B',D}(b) = q_{B,D}(Fb), \quad \nu'_{a'}(m') + q_{B',D}(a') = \nu_a(m) + q_{B,D}(a),$$

Hence the coefficient parity is preserved by the term transport of Proposition 3.4.

Proof. The two quadratic functions in the first equality have polarization $F^T DBF = DB'$. At e_k both have value d_k . For $i \neq k$, $Fe_i = e_i + [b_{ki}]_+ e_k$, so

$$q_{B,D}(Fe_i) = d_i + d_k([b_{ki}]_+^2 - b_{ki}[b_{ki}]_+) = d_i \pmod{2}.$$

Their difference is linear and vanishes on the standard basis, proving the first equality.

Put $t = \sigma((Ba)_k + m_k) - [m_k]_+$, the change of the k th coordinate in $Fa' = a + te_k$. Compatibility and the first equality give

$$\begin{aligned} \nu'_{a'}(m') + q_{B',D}(a') - \nu_a(m) - q_{B,D}(a) \\ \equiv d_k(tm_k - [m_k]_+(Ba)_k + t^2 - t(Ba)_k) \pmod{2}. \end{aligned}$$

For each $\sigma \in \{0, 1\}$, substitution reduces the parenthesis modulo two to $[m_k]_+([m_k]_+ - m_k) = 0$. Finally, subtract m' from $ET_{k,Q}(m + pa)$ and use $p' = EpF$ to obtain the displayed charge a' . $\square$

For a frozen exponent f , $\nu_a(f) = 0$, so the parity formula is unchanged by $m \mapsto m + f$. Changing the compatible form with p, D fixed also leaves $\nu_a(m)$ and $q_{B,D}$ unchanged.

4 Two-wall factorization and the ordering at infinity

For a nonzero rational function $c(v) \in \mathbb{Q}(v)$, let $\deg_\infty c$ be its numerator degree minus its denominator degree, and let $\text{lc}_\infty c$ be the quotient of the leading coefficients. Set $\deg_\infty 0 = -\infty$. Write $c >_\infty 0$ when $\text{lc}_\infty c > 0$. Together with zero these elements form the positive cone of an ordered field. Finite sums of positive elements cannot vanish. Multiplication by any integral power of v preserves this order.

4.1 Framed Hall series

Let q be a prime power, $k = \mathbb{F}_q$, $t = \sqrt{q}$, and let $\mathcal{C}$ be an essentially small hereditary k -linear length category with finite-dimensional Hom and Ext spaces. Suppose it has a positive additive integer degree $j(E)$ and finitely many isomorphism classes of bounded degree. Write

$$\chi(E, F) = \dim_k \text{Hom}(E, F) - \dim_k \text{Ext}^1(E, F).$$

Assume for the moment that χ is symmetric. Write $[E]$ for the isomorphism class of an object E . Let $F_{M,N}^E$ count the subobjects of E isomorphic to N with quotient M , and use the Hall product $[M] * [N] = \sum_E F_{M,N}^E [E]$. For a formal variable z , define

$$I([E]) = \frac{t^{\chi(E,E)}}{|\text{Aut } E|} z^{j(E)}, \quad G(z) = \sum_{[E]} I([E]).$$

The extension-counting identity

$$\sum_E \frac{F_{M,N}^E}{|\text{Aut } E|} = \frac{q^{-\chi(M,N)}}{|\text{Aut } M| |\text{Aut } N|}$$

shows that I is an algebra homomorphism. Without symmetry the same identity gives the quantum phase $t^{\chi(N,M) - \chi(M,N)}$.

Lemma 4.1. *For each simple S , write $\text{End } S = \mathbb{F}_{q^{\varepsilon_S}}$. Suppose $\lambda \in \mathbb{Z}_{>0}$ and $\varepsilon_S \mid \lambda j(S)$ for every simple. There is an exact k -linear functor F_λ to finite-dimensional vector spaces with $\dim F_\lambda(E) = \lambda j(E)$. If $\text{Gen}_\lambda(E)$ denotes vectors lying in no $F_\lambda(U)$ for a proper subobject U , then*

$$\frac{G(t^\lambda z)}{G(t^{-\lambda} z)} = \sum_{[E]} t^{\chi(E,E) - \lambda j(E)} \frac{|\text{Gen}_\lambda(E)|}{|\text{Aut } E|} z^{j(E)}, \quad (2)$$

and each orbit count on the right is a nonnegative integer.

Proof. The Grothendieck category $\text{Ind}(\mathcal{C})$ has injective envelopes; see [6, Section 2, following Theorem 2.3]. Let J_S be the injective envelope of S . Essentiality gives $\text{Hom}(T, J_S) = 0$ for a different simple T , and $\text{Hom}(S, J_S) = \text{End } S$. Writing $[E : S]$ for the multiplicity of S in a composition series of E , exactness gives

$$\dim_k \text{Hom}(E, J_S) = \varepsilon_S [E : S].$$

Thus the direct sum of $\lambda j(S)/\varepsilon_S$ copies of $\text{Hom}(-, J_S)^*$, where $*$ denotes the k -linear dual, over simple isomorphism classes, is the required functor. At any fixed object only finitely many summands are nonzero.

Exactness preserves finite intersections of subobjects. Each object has finitely many subobjects, by the bounded-degree and finite-Hom assumptions. A vector $x \in F_\lambda(E)$ therefore has a unique minimal generating subobject. Consequently

$$q^{\lambda j(E)} = \sum_{U \subset E} |\text{Gen}_\lambda(U)|.$$

If $g \in \text{Aut } E$ fixes a generating vector, that vector belongs to $F_\lambda(\ker(g - 1))$, so $g = 1$. The action on generating vectors is free. In the completed Hall algebra the displayed sum is $\mathbf{1} * \mathbf{h} = \mathbf{f}$, where the coefficients of $\mathbf{h}$ and $\mathbf{f}$ are respectively $|\text{Gen}_\lambda(E)|$ and $q^{\lambda j(E)}$. Applying I gives $G(z)I(\mathbf{h})(z) = G(t^{2\lambda}z)$. Substitute $t^{-\lambda}z$ to obtain (2). $\square$

If a rational wall series has these Hall realizations at unbounded prime powers, every nonzero coefficient of its favorable action is $>_\infty 0$: a nonzero rational function has only finitely many zeros and poles, and its eventual sign on positive real arguments is its leading sign. Converting $X^m z^j$ to based monomials supplies $v^{-\lambda j}$, giving the centered ratio $G(v^\lambda z)/G(v^{-\lambda} z)$.

4.2 Two-wall ray factors

Fix positive integers d_1, d_2, e with $d_1 \mid e$ and $d_2 \mid e$, and use $\Omega(a, b) = e(a_1 b_2 - a_2 b_1)$. Factor

$$\Psi_{v^{d_1}}(X^{(1,0)})\Psi_{v^{d_2}}(X^{(0,1)}) = \prod_a^{\mu(a) \text{ increasing}} G_a(X^a), \quad \mu(a) = \frac{a_1}{a_1 + a_2}, \quad (3)$$

where a runs over primitive nonnegative rays. Thus $(0, 1)$ comes first. Products and factorization are interpreted at each total height. Put

$$q(a) = d_1 a_1^2 + d_2 a_2^2 - e a_1 a_2, \quad \delta(a) = \gcd(d_1 a_1, d_2 a_2),$$

with the usual convention $\gcd(r, 0) = r$ for $r > 0$.

Proposition 4.2. *For $\lambda \in \delta(a)\mathbb{Z}_{>0}$, every nonzero coefficient of $G_a(v^\lambda z)/G_a(v^{-\lambda} z)$ is positive at infinity. If $G_a = 1 + \sum_{j \geq 1} g_j z^j$ and $g_j \neq 0$, then*

$$\deg_\infty g_j = -j^2 q(a), \quad \text{lc}_\infty g_j = 1.$$

If $q(a) > 0$ and $G_a \neq 1$, then $q(a) = \delta(a)$ and $G_a = \Psi_{v^{\delta(a)}}$. If $q(a) \leq 0$, every nonzero coefficient of $\log G_a$ is positive at infinity.

Proof. Over $\mathbb{F}_q$, take $K_i = \mathbb{F}_{q^{d_i}}$ and the (K_2, K_1) -bimodule $A = \mathbb{F}_{q^l}^{\oplus e/l}$, where $l = \text{lcm}(d_1, d_2)$. Representations are maps $A \otimes_{K_1} V_1 \rightarrow V_2$. The vertex-arrow exact sequence for Hom and Ext gives heredity and Euler form

$$\chi(\alpha, \beta) = d_1 \alpha_1 \beta_1 + d_2 \alpha_2 \beta_2 - e \alpha_1 \beta_2.$$

The difference $\chi(\beta, \alpha) - \chi(\alpha, \beta)$ is $\Omega(\alpha, \beta)$. Vertex vector-space Hall series are exactly the two dilogarithms, because their degree- j coefficients are

$$\frac{t^{d_i j^2}}{|\text{GL}_j(\mathbb{F}_{q^{d_i}})|} = \frac{t^{-d_i j^2}}{\prod_{h=1}^j (1 - t^{-2d_i h})}.$$

Every representation has subobject $(0, V_2)$ and quotient $(V_1, 0)$, fixing the order on the left of (3).

Use slope μ , with semistability meaning that every nonzero subobject has slope at most that of the object. The unique Harder–Narasimhan filtration puts the highest slope subobject on the right in our Hall convention. Its integrated factorization is precisely (3). The category $\mathcal{C}_a$ of semistable objects of slope $\mu(a)$ is abelian and extension closed. It is hereditary: an ambient lift killing a Yoneda two-extension has middle term an extension of same-slope objects, hence lies in $\mathcal{C}_a$. It is a length category with positive degree j for dimension vector ja , and its Euler form is symmetric. A stable object S of dimension ja has division endomorphism ring of degree ε_S over $\mathbb{F}_q$. Its action on every nonzero vertex space is faithful, so $\varepsilon_S \mid jd_i a_i$ for both vertices, and hence $\varepsilon_S \mid j\delta(a)$. Lemma 4.1 proves the action assertion. Rational factorization uses only additions and multiplications of initial coefficients, so it has no poles at $v = \sqrt{q} > 1$.

The representation space of dimension ja is affine of dimension $ej^2 a_1 a_2$. Its geometric semistable locus is open: each destabilizing dimension vector gives a closed incidence image under a projective Grassmannian projection. To compare geometric and rational semistability, split the species over an algebraic closure and give each split vertex the original slope and rank weights divided by d_i . The geometric HN filtration is unique and Frobenius invariant, hence descends. Thus this open set counts exactly $\mathcal{C}_a$.

For a nonzero g_j , choose a prime power q_0 at which it is nonzero. Along extensions $q = q_0^h$ with $h \equiv 1 \pmod{l}$, the same species and nonempty semistable open set base change correctly. This open set has $q^{ej^2 a_1 a_2} (1 + O(q^{-1}))$ points, by the elementary polynomial zero bound. Dividing by the two general linear groups and multiplying by $q^{j^2 q(a)/2}$ gives $g_j(\sqrt{q}) = q^{-j^2 q(a)/2} (1 + O(q^{-1}))$. This proves its asserted rational degree and leading coefficient. The set of nonzero degrees is additive, since direct sums preserve semistability and two nonzero rational coefficients are simultaneously nonzero for all sufficiently large prime powers.

If $q(a) > 0$, then $\delta(a) \mid q(a)$. For a stable constituent of dimension ja , heredity gives

$$j^2 q(a) = \varepsilon_S - \dim \text{Ext}^1(S, S) \leq \varepsilon_S \leq j\delta(a).$$

Thus $j = 1$, $q(a) = \varepsilon_S = \delta(a)$, and S is rigid. Two different stable objects of this dimension would have zero Hom and positive Euler pairing, which is impossible. The category is therefore the direct-sum category of this unique simple, giving the dilogarithm. This holds at all sufficiently large prime powers where the ray is nonzero and consequently as a rational identity.

If $q(a) < 0$, all terms of a nontrivial partition $j = j_1 + \cdots + j_s$ in the logarithm have strictly smaller degree, since $\sum j_i^2 < j^2$. The leading term of g_j survives; when $g_j = 0$, additivity of its support excludes such partitions altogether.

Finally suppose $q(a) = 0$ and both coordinates are positive. Set $b = e/d_1$, $c = e/d_2$. Rationality of a zero direction requires $bc(bc - 4)$ to be a square. The identity $(bc - 2)^2 -$

$r^2 = 4$ forces $bc = 4$. The cases $(b, c) = (2, 2), (1, 4), (4, 1)$ have primitive zero directions $(1, 1), (1, 2), (2, 1)$. Stable objects exist: use a nonzero arrow in the first case, a surjection from the larger field to a two-dimensional space over the smaller field in the second, and the dual injection in the third. The subobject slope criterion verifies stability directly. Direct sums give every $j \geq 1$. All g_j have degree zero and leading coefficient one, so the degree-zero leading series is $(1 - z)^{-1}$. Its logarithm has coefficient $1/j > 0$ at z^j . $\square$

4.3 Dilogarithm and exponential factors

For a charge $u \in N^+$ define $\delta_D(u) = \gcd_i(d_i u_i)$. We shall use factors of the two forms

$$\Psi_{v^d}(v^r X^{pu}), \quad d \mid \delta_D(u), \quad r \in \mathbb{Z}, \quad \exp(c(v) X^{pu}), \quad c >_\infty 0.$$

The charge u need not be primitive. An atomic factorization means a product of these factors on a fixed ray, finite at every bounded height.

Lemma 4.3. *Two such initial factors have a two-wall completion whose ray labels admit atomic factorizations. Every new dilogarithm parameter divides the pairing of its charge with every exponent of the original lattice.*

Proof. When the pairing vanishes the factors commute. Otherwise order the two directions so that their pairing is $e > 0$. For two dilogarithms with parameters d_1, d_2 , Proposition 4.2 applies. Multiplying the two initial variables by v^{r_1} and v^{r_2} multiplies the coefficient of degree $j(a_1, a_2)$ by $v^{j(r_1 a_1 + r_2 a_2)}$, preserving all conclusions.

Write the two original charges as u, w . For each original coordinate i , apply to a species representation the exact vector-space functor consisting of $d_i u_i / d_1$ copies of its first vertex space and $d_i w_i / d_2$ copies of its second. These are nonnegative integers. The division endomorphism ring of the rigid simple acts on every nonzero resulting space. Its degree $\delta(a)$ therefore divides $d_i(a_1 u_i + a_2 w_i)$ for each i .

For the remaining cases consider series with all positive-degree coefficients nonzero, leading degree $s j^2 + \ell j$, positive leading coefficient, and $s \geq 0$. Exponentials belong to this class with $s = 0$. In the product of two such series the coefficient of (a, b) has leading degree

$$s_1 a^2 + s_2 b^2 + eab + \ell_1 a + \ell_2 b.$$

For increments (a_i, b_i) on distinct increasing-slope rays, its excess over their reconstructed product is

$$2 \sum_{i < j} (s_1 a_i a_j + s_2 b_i b_j + e b_i a_j) > 0.$$

Thus height induction in the ray factorization leaves the positive leading coefficient unchanged. Every mixed ray has quadratic coefficient $s_1 a^2 + s_2 b^2 + eab > 0$, so its logarithm is positive at infinity by the strict square inequality used above. This treats two exponentials.

For a dilogarithm of parameter d followed by such a series, move the dilogarithm to the right by conjugation. Write the second series as $\sum_{b \geq 0} h_b X^{bpw}$. Here $e/d \in \mathbb{Z}_{>0}$, and the coefficient of (a, b) is

$$h_b v^{ra} \left[\begin{matrix} eb/d \\ a \end{matrix} \right]_{v^d}, \quad b \geq 1, \quad 0 \leq a \leq eb/d,$$

of leading degree $sb^2 + eab - da^2 + ra + \ell b$. For distinct increasing-slope increments the excess degree is

$$2 \sum_{i < j} (a_j (e b_i - d a_i) + s b_i b_j) > 0,$$

since $a_i/b_i < a_j/b_j \leq e/d$. The finite band is closed under addition, so factorization stays in it. With an exponential input, the interior rays have positive quadratic coefficient; the upper boundary is exactly $\exp(c(v)v^{re/d}X^{p((e/d)u+w)})$, since its extreme Gaussian coefficient is one. The lower boundary is the original exponential. If the dilogarithm is second, move it to the left using the favorable inverse action. The band is $0 \leq b \leq ea/d$ and the excess is $2 \sum_{i < j} (b_i(ea_j - db_j) + sa_ia_j) > 0$. $\square$

5 Positivity in arbitrary rank

The monoid lift and perturbation construction of [4, Sections 5.1–5.2 and Lemma 2.11] reduces higher-valency joints to the two-wall factorizations of Lemma 4.3 by induction on degree.

Denote the original charge lattice, symmetrizer and skew form by N_0 , D_0 and Ω_0 . An auxiliary free positive lattice N carries a map $\rho : N \rightarrow N_0$ sending every positive generator to a nonzero positive charge, with form $(\rho^*\Omega_0)(a, b) = \Omega_0(\rho a, \rho b)$ for $a, b \in N$. Divisibility for a dilogarithm of working charge u always means $d \mid \delta_{D_0}(\rho u)$. Working height is the sum of coordinates in N . If distinct working charge rays have the same geometric normal, their group elements commute. We separate their logarithms by working degree before applying the argument and multiply them only after projection. The local two-dimensional model lies in the effective normal quotient.

Lemma 5.1. *A finite set of positive incoming factors may be replaced by independent positive generators, preserving the original quantum coefficients, wall actions after projection, and incoming and outgoing conditions. If their original heights are h_j , a lifted charge a has original height $\sum_j h_j a_j$.*

Proof. If the old Hamiltonian lattice is L with form ω , and the shifts are $U_j \in L$, set

$$\tilde{L} = L \oplus \bigoplus_j \mathbb{Z}\epsilon_j, \quad \pi(x, \sum b_j \epsilon_j) = x + \sum b_j U_j, \quad \tilde{\omega} = \pi^*\omega.$$

The positive working generators ϵ_j act by their inclusions in $\tilde{L}$. Take inverse images of the initial wall hyperplanes. Since $\ker \pi$ is in the radical, the rational scattering construction can be made cylindrical along it. Projection then preserves codimension one, path products, and recession membership: a support saturated along $\ker \pi$ contains a working direction in its recession cone exactly when its projected support contains the projected direction. This verifies the saturation and surjectivity hypotheses of [4, Lemma 2.11].

The rule $X^x \mapsto X^{\pi x}$ is a homomorphism because the form is pulled back. Positive external charges ensure that only finitely many lifted charges can contribute below a fixed old height. Projection therefore respects completed products. Rational uniqueness identifies the projected completion with the old one. Coincident rays are multiplied after projection. The height formula follows by additivity. $\square$

Theorem 5.2. *At every bounded height the cluster diagram has a representative whose merged ray labels admit atomic factorizations. Every nonzero coefficient of a favorable wall action and of a theta series is positive at infinity.*

Proof. We prove the factorization assertion simultaneously for all auxiliary charge maps and for central and generically translated initial diagrams. First apply Lemma 5.1 to make the initial factors independent. At any finite truncation, generic translations of the initial hyperplanes make each codimension-two joint lie on at most two initial walls. The associated

affine consistent diagram and its central asymptotic diagram have the same initial factors. This is the coefficient-independent perturbation construction in [4, Section 5.2].

Induct on working height H . Height one is the initial diagram. Suppose the assertion holds through $H - 1$ and take a finite subdivided representative through H . At a joint, positive decompositions of incoming labels have at most two factors of height one: they are the original independent initial factors. A ray whose leading height is at least two cannot contain a height-one factor in a positive decomposition, because the first coefficients of all factors have positive leading sign.

Consider a local completion with these positive incoming factors. Lift them to independent generators of weights h_j . A lifted charge of old height at most H using any $h_j \geq 2$ has lifted height at most $H - 1$. For each fixed lifted ray using such a generator, every charge on that ray of old height at most H has working height at most $H - 1$. The induction hypothesis controls this truncated ray. The remaining coordinate face has at most two generators and is controlled at all heights by Lemma 4.3. Passing to this face is legitimate: the span of charges using another coordinate is a Lie ideal, so setting those coordinates to zero is a group quotient. Pushing back proves that this local positive completion is positive through H .

Fix now a working charge w of height H . Pure height- H changes to incoming labels are central modulo greater height. A change on any other direction travels straight and cannot alter the coefficient of X^w . If an outgoing piece on the ray of w is not positively factorizable through H , its incoming piece on that same direction must also fail: otherwise keep its positive factorization through H , replace all other incoming labels by their positive factorizations through $H - 1$, and use the preceding local argument. The central changes on other rays do not affect the chosen ray. For zero local pairing the factors commute.

Suppose a wall piece fails to admit a positive factorization through height H . From a general point of that piece, move in the positive working shift direction of w . This ray avoids the codimension-two skeleton within the wall and crosses finitely many transverse facets. The preceding paragraph propagates the failure of factorization across each facet. Every noninitial outgoing support is a translated closed rational polyhedral cone whose recession cone does not contain this direction. Its intersection with the ray is bounded. After leaving their finitely many bounded intersections, only initial full walls and the identity remain, all positively factorizable. This is a contradiction. If the direction is radical in a local quotient, its labels are unchanged across the joint; if it is entirely cylindrical, no new outgoing support in that direction is possible. Taking the asymptotic diagram proves the central statement and completes induction.

For a dilogarithm factor with positive pairing λ , divisibility gives

$$\frac{\Psi_{v^d}(v^{r+\lambda}z)}{\Psi_{v^d}(v^{r-\lambda}z)} = \sum_{j=0}^{\lambda/d} v^{rj} \left[\begin{matrix} \lambda/d \\ j \end{matrix} \right]_{v^d} z^j.$$

For an exponential factor the ratio is $\exp(c(v)(v^\lambda - v^{-\lambda})z)$. Both have positive leading signs. Reverse favorable crossings give the same formula with the absolute pairing. Products and finite coefficient sums preserve those signs. A fixed theta coefficient only uses a bounded-height diagram, so this proves its positivity. Invariance under equivalent representatives identifies the coefficients obtained at different truncation heights. $\square$

6 The finite theta algebra

The leading-term order is $m \preceq m + pa$ for $a \in P$. A series is m -pointed if its support lies in $m + pP$ and its coefficient at X^m is one. Pointed series are compatibly pointed across

seeds when their leading labels are related by the tropical mutation maps. Finite collections of theta series are independent: choose a minimal label in this order, read its coefficient one, and remove it. For infinite expansions, the support condition gives the following recursion.

Lemma 6.1. *Fix a general endpoint Q and a finite union $S = \bigcup_{j=1}^r (m_j + pP)$, with $r \geq 1$ and $m_j \in M$. Every formal series f with coefficients in R and support in S has a unique coefficientwise expansion*

$$f = \sum_{m \in S} c_m \Theta_{m,Q}, \quad c_m \in R.$$

On this fixed support set, the coefficient transformation and its inverse are continuous for the height truncations.

Proof. Partition the translating vectors m_j by their classes in G . In one class choose a representative $r_0 \in M$ and write $m_j = r_0 + pb_j$, with $b_j \in \mathbb{Z}^n$. Put $b = r_0 + p(\min_j(b_j)_1, \dots, \min_j(b_j)_n)^T$. Then the part of S in this class is contained in $b + pP$. Injectivity of p identifies its exponent positions with $b + pa$, $a \in P$.

Define the coefficients in increasing order of $\text{ht}(a)$ by

$$c_{b+pa} = [X^{b+pa}]f - \sum_{\substack{d \in P, d \leq a \\ d \neq a}} c_{b+pd} [X^{b+pa}] \Theta_{b+pd,Q},$$

where $d \leq a$ is coordinatewise. The sum is finite, and every d in it has smaller height. The diagonal theta coefficient is one, so this recursion gives existence and uniqueness. If $b+pa \notin S$, then $[X^{b+pa}]f = 0$ and no predecessor $b+pd \in S$ can contribute, since $S + pP = S$. Induction therefore gives $c_{b+pa} = 0$ outside S . Apply the recursion in each of the finitely many classes.

Both coefficient transformations through height H use only the finitely many input positions through that height. The common lower translate used above gives the same truncation topology on S : each original translating vector differs from b by a fixed nonnegative charge, and there are finitely many such vectors. This proves continuity. $\square$

Lemma 6.2. *Let $J \subset M$ be finite and fix $m \in M$. Define*

$$H = \max(\{0\} \cup \{\text{ht}(a) : a \in P, m - pa \in J\}).$$

Choose a point Q outside all proper rational linear hyperplanes and satisfying $\nu_a(Q)\nu_a(m) > 0$ whenever $0 < \text{ht}(a) \leq H$ and $\nu_a(m) \neq 0$. Then

$$[X^m] \sum_{j \in J} c_j \Theta_{j,Q} = c_m,$$

where $c_j \in K$ for $j \in J$ and $c_m = 0$ if $m \notin J$. Such points exist in an open neighborhood of m .

Proof. Injectivity of p and finiteness of J make the set defining H finite. The strict conditions hold at m and persist nearby. If a path from j ends with exponent m , its total bend charge a satisfies $m - pa = j$, so every bend has height at most H . Its last genuine bend would lie on the reverse ray $Q + \mathbb{R}_{\geq 0}m$. For a possible wall charge u , the value $\nu_u(Q) + s\nu_u(m)$ is never zero: its two terms have the same strict sign when $\nu_u(m) \neq 0$, and the first is nonzero otherwise. Thus no last bend exists. Only the straight path from m remains, with coefficient one. When $m = 0$ use the constant path. $\square$

This scattering-atlas coefficient detection follows [5, Proposition 6.4(3)] and [8, Theorem 1]. Its chart depends on the finite label set and target and may lie outside the cluster atlas.

Proposition 6.3. *If Θ_m and Θ_n are finite, their product has a finite theta expansion with coefficients in R and all output theta functions finite. Each nonzero coefficient is positive at infinity. Moreover, every output support is contained in the support of the product in a fixed positive seed chart.*

Proof. Lemma 6.1, applied to the support set $m+n+pP$, gives a formal expansion with labels $m+n+pa$, coefficients in R , and leading coefficient $v^{\Lambda(m,n)}$. For a fixed output label, only finitely many preceding labels can affect it. Apply Lemma 6.2 to this finite truncation and use endpoint transport. Its expansion coefficient is a monomial coefficient of the product in the detecting chart. By Theorem 5.2, it is zero or positive at infinity; quantum multiplication adds only powers of v .

In the fixed seed chart, each nonzero output contributes its label monomial with its nonzero expansion coefficient. All other contributions to that monomial also have positive leading sign. Therefore the label lies in the finite support of the product. Only finitely many output labels occur. The same noncancellation argument at every exponent shows that each entire output support is contained in that finite support. Proposition 3.4 gives membership in $\mathcal{U}$. $\square$

Corollary 6.4. *For finite theta functions, write the finite expansion from Proposition 6.3 as*

$$\Theta_m \Theta_n = \sum_{a \in P} \alpha_a(v) \Theta_{m+n+pa}.$$

Then

$$\alpha_a(-v) = (-1)^{\Lambda(m,n) + \nu_a(m+n) + q_{B,D}(a)} \alpha_a(v).$$

Proof. Use the involutions $\mathcal{J}_m$ defined in the proof of Theorem 3.5. For input charges $b, c \in P$, polarization gives

$$\begin{aligned} \nu_b(m) + q_{B,D}(b) + \nu_c(n) + q_{B,D}(c) + \Lambda(m+pb, n+pc) \\ \equiv \Lambda(m, n) + \nu_{b+c}(m+n) + q_{B,D}(b+c) \pmod{2}. \end{aligned}$$

Thus $\mathcal{J}_{m+n}(\Theta_m \Theta_n) = (-1)^{\Lambda(m,n)} \Theta_m \Theta_n$. Theorem 3.5 and the same polarization identity give

$$\mathcal{J}_{m+n}(\Theta_{m+n+pa}) = (-1)^{\nu_a(m+n) + q_{B,D}(a)} \Theta_{m+n+pa}.$$

Apply $\mathcal{J}_{m+n}$ to the finite expansion and compare its independent theta coordinates. $\square$

Proposition 6.5. *For every positive integer k ,*

$$k \operatorname{supp} \Theta_m \subseteq \operatorname{supp} \Theta_{km} \subseteq \underbrace{\operatorname{supp} \Theta_m + \cdots + \operatorname{supp} \Theta_m}_{k \text{ times}}.$$

In particular $\operatorname{Newt}(\Theta_{km}) = k \operatorname{Newt}(\Theta_m)$ whenever Θ_m is finite, and Θ_{fin} is an additive saturated submonoid of M .

Proof. At a general wall point with primitive charge $u \in N$, let $H(z)$ be the merged ray label, $z = X^{pu}$, and $\delta = \gcd_i(d_i u_i)$. For $\lambda, \mu \in \delta \mathbb{Z}_{>0}$ put $A_\lambda(z) = H(v^\lambda z)/H(v^{-\lambda} z)$. Cancellation of the middle factors gives

$$A_{\lambda+\mu}(z) = A_\lambda(v^\mu z) A_\mu(v^{-\lambda} z). \quad (4)$$

For a series in z , let supp_z be the set of powers with nonzero coefficients. All nonzero coefficients have positive leading sign, hence $\operatorname{supp}_z A_{k\delta} = k \operatorname{supp}_z A_\delta$, where the right side is the k -fold Minkowski sum. Stretching the slopes of a broken line by k , keeping its geometric

image, multiplies each selected bend by k and is permitted by this support identity. The resulting contribution cannot cancel. This proves the first inclusion.

The formal expansion of Θ_m^k has coefficient one at Θ_{km} : the total leading charge is zero and $\Lambda(m, m) = 0$. Lemma 6.2 makes all its coefficients nonnegative in the ordering at infinity. Thus no support term of Θ_{km} can disappear from the product, proving the second inclusion even before finiteness. The two inclusions imply scaling of convex hulls and equivalence of finiteness for m and km . Additivity follows from Proposition 6.3, since its leading output is Θ_{m+n} . $\square$

Corollary 6.6. *For a seed s , let $\mathcal{C}_s$ consist of all finite nonnegative real linear combinations of labels in $\Theta_{\text{fin},s}$. Then*

$$\mathcal{C}_s \cap M_s = \Theta_{\text{fin},s}.$$

For seeds s, t , let $\phi_{ts} : M_{s,\mathbb{R}} \rightarrow M_{t,\mathbb{R}}$ be the tropical map along a mutation sequence from s to t . Then ϕ_{ts} satisfies $\phi_{ts}(\mathcal{C}_s) = \mathcal{C}_t$.

Proof. If an integral m lies in the cone generated by finitely many finite labels, the rational linear system expressing m as their nonnegative combination has a rational solution. Clearing denominators and using additivity gives $km \in \Theta_{\text{fin},s}$ for some positive integer k . Saturation gives $m \in \Theta_{\text{fin},s}$.

For $x \in \mathcal{C}_s$, intersect a finitely generated rational cone containing x and generated by finite labels with a closed rational linearity cone of ϕ_{ts} containing x . This intersection is a rational polyhedral cone. Scale its finitely many rational generators to lattice points; the first assertion makes them finite labels. Their images are finite by mutation compatibility. Linearity on this cone gives $\phi_{ts}(x) \in \mathcal{C}_t$. Apply the same argument to the inverse map. $\square$

Corollary 6.7. *Let $M_{\text{fr}} \subset M$ consist of the exponents with zero mutable coordinates. For $f \in M_{\text{fr}}$,*

$$\Theta_{m+f} = v^{-\Lambda(f,m)} X^f \Theta_m = v^{-\Lambda(m,f)} \Theta_m X^f.$$

In particular $\Theta_{\text{fin}} + M_{\text{fr}} = \Theta_{\text{fin}}$.

Proof. Compatibility gives $\Lambda(pa, f) = 0$ for every charge a . If $\gamma(t)$ is a broken line, replace it by $\gamma(t) - tf$ and add f to every segment exponent. Wall supports are cylindrical in frozen directions, so this preserves crossing times, endpoint, and action coefficients. The charge coefficients of the two theta series agree. Since $\Lambda(f, pa) = 0$, based-monomial multiplication supplies exactly the displayed phases. Multiplication by a monomial preserves finiteness. $\square$

Proof of Theorem 1.1. Integrality, bar invariance, and mutation transport were established in Section 3. Propositions 6.3 and 6.5 give the algebra and label assertions. To include $\mathcal{A}$, take $m_i \geq 0$ for all mutable indices. Suppose a path has successive genuine bend charges $a_1, \dots, a_q$. After the first bend it lies in $\nu_{a_1} \leq 0$. If its previous segment lies in $\nu_{a_1+\dots+a_{j-1}} \leq 0$, the next bend point also has $\nu_{a_j} = 0$. On the following segment the cumulative normal has derivative

$$\nu_{a_1+\dots+a_j}(-m - p(a_1 + \dots + a_j)) = -\nu_{a_1+\dots+a_j}(m) \leq 0.$$

Thus the last segment cannot reach the positive chamber. A tangent first segment cannot select a nonconstant action term. Only the straight path remains, so $\Theta_{m,Q} = X^m$. The same statement in every seed, using Proposition 3.4, identifies normalized cluster monomials and frozen units with finite theta functions. These generate $\mathcal{A}$. $\square$

7 Valuative independence

Fix a seed and a point Q in its positive chamber outside every proper rational linear hyperplane. Such points will be called lattice general. At a general wall point y , let u be the primitive charge in N , and let $\delta = \gcd_i(d_i u_i)$. In particular $\nu_u(M) = \delta\mathbb{Z}$ and $\nu_u(pu) = 0$. With the merged ray label define $A_{y,\lambda}$ as in (4), and set

$$S_y = \text{supp}_z A_{y,\delta} \subseteq \mathbb{N}, \quad C_y = \text{conv}_{\mathbb{Q}}(S_y/\delta).$$

Here $\text{conv}_{\mathbb{Q}}$ denotes finite rational convex combinations. The set C_y is $[0, L_y] \cap \mathbb{Q}$ with $L_y = \max S_y/\delta$ if bounded, and $\mathbb{Q}_{\geq 0}$ otherwise. It includes zero.

A rational broken line has rational segment exponents and, at a transverse crossing at y , the rule

$$x' = x + |\nu_u(x)|c pu, \quad c \in C_y. \quad (5)$$

It has finitely many genuine bends and avoids seams. A crossing with $c = 0$ may be inserted as a record. Its shear is

$$T = I + \text{sgn}(\nu_u(x))c pu \otimes \nu_u, \quad T^{-1} = I - \text{sgn}(\nu_u(x))c pu \otimes \nu_u.$$

Here $(pu \otimes \nu_u)(b) = \nu_u(b)pu$. Since $\nu_u(pu) = 0$, the shear fixes the supporting hyperplane pointwise and has determinant one.

Lemma 7.1. *For $m \in M$, the rational final exponents are exactly*

$$\bigcup_{k \geq 1} k^{-1} \text{supp } \Theta_{km, Q}.$$

If Θ_m is finite, their convex hull is $\text{Newt}(\Theta_m)$, and every real linear functional attains its minimum on this set.

Proof. At bend i , let x_i be the incoming exponent and u_i the primitive charge, with $\nu_i = \nu_{u_i}$ and $\delta_i = \gcd_j(d_j(u_i)_j)$. Choose a finite expression $\delta_i c_i = \sum_j a_{ij} s_{ij}$, with $a_{ij} \in \mathbb{Q}_{\geq 0}$, $s_{ij} \in S_{y_i}$ and $\sum_j a_{ij} = 1$. Choose a common $k > 0$ making all kx_i integral and all

$$N_i = \frac{k|\nu_i(x_i)|}{\delta_i}, \quad N_i a_{ij}$$

integers. The identity $k|\nu_i(x_i)|c_i = \sum_j (N_i a_{ij}) s_{ij}$ expresses this increment as a sum of N_i members of S_{y_i} . Equation (4) permits this actual bend. Reparameterizing the path by $t \mapsto kt$ gives an actual broken line from km . Ordered positivity prevents cancellation of its final term. Conversely, divide all exponents of an actual path from km by k . Each selected integer in $\text{supp } A_{N\delta}$ is a sum of N members of S_y , so the divided rule is (5). This proves the equality. Proposition 6.5 gives the convex-hull assertion. The finite support for $k = 1$ is included and attains every minimum. $\square$

Lemma 7.2. *For a general wall point y and a finite subset $F \subset S_y$, there is a relative neighborhood W in its hyperplane such that $F \subset S_{y'}$ for every general $y' \in W$. Finite rational broken-line records with fixed initial exponent and endpoint vary continuously with their rational bend parameters while transversality, time order, and membership in the chosen relative neighborhoods persist.*

Proof. Choose a height at least $s \text{ht}(u)$ for every $s \in F$. In that finite diagram the merged truncated action is constant on the relative open cell containing y . Its nonzero coefficients therefore persist in a smaller relative neighborhood. Higher walls cannot change them.

For the continuity assertion, let $\nu_i = \nu_{u_i}$ be the normal of the i th recorded wall, where u_i is its primitive charge. Let $T_1, \dots, T_r$ be the recorded rational shears, $P_i = T_i \cdots T_1$ and $P_0 = I$. Unfolding gives the line $P_r^{-1}Q - tm$. The crossing times and points are

$$t_i = \frac{\nu_i(P_{i-1}P_r^{-1}Q)}{\nu_i(P_{i-1}m)}, \quad q_i = P_{i-1}(P_r^{-1}Q - t_i m). \quad (6)$$

The nonzero denominators, strict inequalities $t_1 < \cdots < t_r < 0$, and the required locations are open conditions. Formula (6) is continuous wherever they hold.

These rational records avoid the seams of the full diagram for lattice-general Q . At a transverse hyperplane, the crossing-point map in (6) is a surjective rational map of rank $M - 1$ onto that hyperplane. The preimage of any rational codimension-two seam span is a proper rational subspace. The same projection argument applies to an unrecorded crossing on any segment. A segment contained in a wall would itself require a proper rational linear condition on Q . There are countably many such conditions. The chosen Q avoids all of them. Additional wall crossings select the constant term. $\square$

Fix a real functional ℓ . Transport it backwards along a recorded path: just after bend i it is $\ell_i = \ell \circ T_r \cdots T_{i+1}$. Call a crossing taut if its parameter minimizes $\ell_i(x + |\nu_u(x)|c pu)$ over $c \in C_y$.

Lemma 7.3. *A rational path attaining the minimum of ℓ among all rational paths with its initial exponent and endpoint is taut at every transverse crossing, including those where it does not bend.*

Proof. If a crossing fails, insert it as a zero-bend record if necessary, and choose a rational $c' \in C_y$ giving a smaller value. Choose finite support certificates for all existing parameters and for c' . Lemma 7.2 provides relative neighborhoods preserving them. Replace the selected parameter by $(1 - \tau)c + \tau c'$ for a sufficiently small positive rational τ , leaving all other parameters and crossing signs fixed. Reconstruct the path from (6), keeping m and Q fixed. All open conditions persist, and the combined finite certificates make the new parameters legal. The full-diagram seam condition also persists by lattice generality and rationality of the new record.

The earlier exponent is unchanged, as are all later shears. Hence the change of final value is exactly

$$\tau |\nu_i(x)| (c' - c) \ell(T_r \cdots T_{i+1} p u_i) < 0,$$

contradicting minimality. The full cocycle identity and preserved finite certificates justify the bends after clearing denominators. $\square$

Lemma 7.4. *Suppose $\ell(w) \neq 0$ for every $0 \neq w \in M_{\mathbb{Q}}$. Two minimal rational paths with endpoint Q and the same final exponent have the same initial exponent and the same genuine bend record.*

Proof. At a reverse crossing, the outgoing exponent b determines $|\nu_u(x)| = |\nu_u(b)|$. The transported functional remains nonzero on nonzero rational vectors because the shears are invertible and rational. Thus minimization has a unique parameter: zero if $\ell_i(pu) > 0$, and L_y if $\ell_i(pu) < 0$ and C_y is bounded. If it is unbounded in the latter case, no minimum exists, so that state cannot occur on a minimal path. The incoming exponent is uniquely $b - |\nu_u(b)|c pu$.

The two paths have the same final straight segment. Compare the finite union of their genuine bends, starting from Q . At the first possible difference they meet the same wall with the same outgoing exponent and future functional. The other path, even if straight

there, must be taut. The unique parameter forces equality. Repeating over the finite union proves the claim. A zero final exponent forces all exponents to be zero by invertibility of the shears. $\square$

Proof of the Newton assertion in Theorem 1.2. First take ℓ as in Lemma 7.4. Each finite theta support has a unique minimal exponent. A path contributing it is minimal among rational paths by Lemma 7.1. Two distinct labels cannot have equal minimum values: genericity would make their minimal exponents equal, and Lemma 7.4 would make the labels equal. A finite combination therefore has a unique theta summand supplying its smallest term, irrespective of the signs of its nonzero coefficients. This proves

$$\text{val}_\ell \left(\sum c_m \Theta_m \right) = \min_m \text{val}_\ell(\Theta_m).$$

Both sides are minima of finitely many linear functions and hence continuous in ℓ . Such generic functionals are dense, so the equality holds for every real ℓ . Convex separation gives the asserted Newton equality. Choosing a generic ℓ in the interior of a vertex normal cone also proves unique ownership of every joint vertex. $\square$

Corollary 7.5. *For every convex set $P_0 \subset M_{\mathbb{R}}$,*

$$(K \otimes_R \mathcal{M}_\Theta) \cap \text{span}_K \{X^a : a \in P_0 \cap M\} = \bigoplus_{\substack{m \in \Theta_{\text{fin}} \\ \text{supp } \Theta_m \subset P_0}} K \Theta_m.$$

The spans consist of finite sums. If P_0 is bounded, the right side has finitely many labels. Any collection of linear valuation lower bounds holds for a finite theta combination exactly when it holds for every nonzero summand.

Proof. The Newton equality puts every participating support inside the convex hull of the support of the sum. A bounded set has finitely many lattice points, and each theta contains its label. The valuation assertion follows from the minimum identity. $\square$

8 Vertex coefficients and base change

Lemma 8.1. *At a general wall point y , abbreviate $A_{y,\lambda}$ to A_λ within this lemma. Suppose the basic-pairing action $A_\delta(z)$ is a polynomial of degree L . It is a finite product of Gaussian dilogarithm actions, has coefficients in $\mathbb{N}[v^{\pm 1}]$, and has leading z coefficient one. The same leading-coefficient assertion holds for $A_{h\delta}$, $h \geq 1$.*

Proof. For $L = 0$ the action is one. For $L > 0$, choose the atomic representative of Theorem 5.2 through charge height $2L \text{ht}(u)$. On the ray, its factors are $\Psi_{v^d}(v^t z^k)$, with $d \mid k\delta$, or $\exp(cz^k)$ with $c >_\infty 0$. Their actions are respectively

$$\sum_{j=0}^{k\delta/d} v^{tj} \begin{bmatrix} k\delta/d \\ j \end{bmatrix}_{v^d} z^{kj}, \quad \exp(c(v)(v^{k\delta} - v^{-k\delta})z^k).$$

Discard factors with $k > 2L$. A remaining $k > L$ would give a nonzero coefficient above L and at most $2L$, impossible by noncancellation. Thus $k \leq L$. An exponential factor would give such a coefficient at the smallest multiple of k above L , which is at most $L + k \leq 2L$. It too is impossible. Any remaining Gaussian factor of degree greater than L gives the same contradiction. Finally, if their degrees sum to more than L , the first partial sum exceeding L is at most $2L$, and taking these top terms gives another forbidden nonzero coefficient. Hence

their entire product has degree at most L . It agrees with A_δ through $2L$, so the polynomial equality is exact.

Its leading coefficient is a monomial v^e with integer coefficient one. The wall action commutes with bar by Proposition 3.2. Since $\delta \in \nu_u(M)$, this applies to A_δ itself, and forces $v^e = v^{-e}$, hence $e = 0$. Iterating (4) gives

$$A_{h\delta}(z) = \prod_{j=0}^{h-1} A_\delta(v^{(h-1-2j)\delta} z).$$

Its leading phase is $v^{L\delta \sum_{j=0}^{h-1} (h-1-2j)} = 1$. $\square$

Proposition 8.2. *Every vertex coefficient of a finite theta function is one. If b is a joint Newton vertex of a finite combination $\sum c_m \Theta_m$, then $[X^b] \sum c_m \Theta_m = c_{m_b}$ for its unique owner m_b .*

Proof. Choose a generic functional in the interior of the vertex's minimum normal cone; this cone has nonempty interior even for a lower-dimensional polytope. Every path to the vertex is rationally minimal. Lemmas 7.3 and 7.4 show that there is exactly one merged-wall bend record. At each crossing its parameter is zero or the finite right endpoint of C_y . The former selects coefficient one. The latter means S_y is bounded, so Lemma 8.1 applies and its maximal action coefficient is also one. All centered quantum phases are already part of these action coefficients. Their product is one. Unique vertex ownership, proved in Section 7, gives the final assertion. $\square$

Proposition 8.3. *For a finite nonempty set $J \subset \Theta_{\text{fin}}$ let $E = \bigcup_{m \in J} \text{supp } \Theta_m$. The matrix $([X^b] \Theta_m)_{b \in E, m \in J}$ has a unit triangular square minor of size $|J|$. Consequently the map*

$$S \otimes_R \mathcal{M}_\Theta \longrightarrow \mathcal{T}_S$$

is injective for every nonzero commutative unital R -algebra S , where the based monomial lattice is unchanged. A finite nonzero combination over S has Newton polytope equal to the convex hull of the Newton polytopes of its nonzero summands after specialization.

Proof. Choose a vertex of the joint polytope. Its row has entry one in its unique owner column and zero in every other column. Remove that column and choose a vertex of the remaining joint polytope. Repeat. Chosen rows are distinct, since an earlier row vanishes in every remaining column. In this order the selected square matrix is triangular with diagonal ones. It remains invertible after every base change, proving finite and hence arbitrary direct-sum independence. Vertex coefficients of each theta remain one, so its polytope does not shrink. A joint vertex still has only its unique owner's nonzero scalar as coefficient, which remains nonzero even if S has zero divisors. Thus the combination's joint vertices survive. All other terms lie in their convex hull. $\square$

Propositions 8.2 and 8.3 complete Theorem 1.2. In particular, for a finite theta function the quantum and classical Newton polytopes agree.

8.1 The ordered scattering cone

Let $R_{\geq 0}^\infty = \{0\} \cup \{a \in R : \text{lc}_\infty a > 0\}$. In $\mathcal{M}_\Theta$, require every scattering-chart coefficient of an element to lie in this set. Lemma 6.2 and Theorem 5.2 give exactly

$$\bigoplus_{m \in \Theta_{\text{fin}}} R_{\geq 0}^\infty \Theta_m.$$

This cone has no nonzero additively indecomposable element: every nonzero f splits as $v^{-1}f + (1 - v^{-1})f$, with both summands nonzero in the cone.

Proposition 8.4. *In the bar-invariant part of the ordered scattering cone, the additively indecomposable nonzero elements are exactly the finite theta functions. They do not in general generate that cone additively.*

Proof. Bar invariance of a finite combination is equivalent to bar invariance of every coefficient, by theta independence. The scalar semiring is therefore $\{0\} \cup \{a \in \mathbb{Z}[v + v^{-1}] : \text{lc}_\infty a > 0\}$. Its only nonzero additive indecomposable is 1. A nonconstant member decomposes as $1 + (a - 1)$, since subtraction of the constant does not change the positive highest term. A constant is a positive integer and is indecomposable only when it is one. Conversely a sum of nonconstant positive-leading bar-invariant polynomials cannot be the constant one: their highest degrees cannot cancel. Independence now gives the asserted indecomposables. For example $(v + v^{-1})\Theta_m$ is in the cone but cannot be a finite sum of theta functions with nonnegative integral multiplicities. $\square$

9 Support certificates and reddening sequences

For a mutation at k , let $\sigma = 1$ if the current endpoint coordinate $Q_k > 0$, and $\sigma = 0$ if it is negative. Direct subtraction of the transformed label from the transformed final exponent in Proposition 3.4 gives

$$a' = J_{k,\sigma}(B)a + ([m_k]_+ - \sigma m_k)e_k, \quad (7)$$

where J is the identity off row k , $J_{kk} = -1$, and $J_{kj} = [b_{kj}]_+ - \sigma b_{kj}$ for $j \neq k$. The full frozen columns enter the exponent identity, while the charge recurrence depends only on the mutable matrix.

Proposition 9.1. *Choose finitely many mutation words along which all selected endpoint coordinates are nonzero. Iterate (7) to write $a_w = A_w a + b_w(m)$. If nonnegative rational weights satisfy*

$$-\sum_{w,i} \lambda_{w,i} e_i^T A_w = \eta^T, \quad \eta_i > 0,$$

then every support charge obeys

$$0 \leq a_j \leq \left\lfloor \frac{\sum_{w,i} \lambda_{w,i} (b_w(m))_i}{\eta_j} \right\rfloor.$$

For a fixed finite family W of words, such weights exist if and only if

$$\{a \in \mathbb{R}_{\geq 0}^n : A_w a \geq 0 \text{ for every } w \in W\} = \{0\}.$$

There exists some finite family W with such weights if and only if the same cone, with inequalities for all admissible finite words, is zero. All words use the same chosen endpoint and its successive transforms.

Proof. Actual transport preserves each nonzero coefficient and the target positive charge condition, so $a_w \geq 0$. Weighted addition gives the bound. For fixed W , a certificate forces its cone to vanish. Conversely, finite-dimensional separation, equivalently the alternative theorem for linear inequalities, gives a nonnegative combination of the rows of the A_w and the coordinate rows that is strictly negative. Dropping the nonnegative coordinate rows preserves strict negativity. Rational approximation of the positive weights, with zero weights fixed, gives the required rational certificate. For the all-word assertion, intersect all homogeneous conditions with the compact simplex $a \geq 0$, $\sum a_i = 1$. If the intersection is empty, compactness gives a finite subfamily with empty intersection. Apply the fixed-family result to it. $\square$

For one mutation from the positive chamber, (7) yields

$$a_k \leq [-m_k]_+ + \sum_{i \neq k} [-b_{ki}]_+ a_i. \quad (8)$$

If B is acyclic, $H = [-B]_+$ is nilpotent after ordering its vertices. Iterating $a \leq [-m_{\text{mut}}]_+ + Ha$ gives the finite bound

$$0 \leq a \leq (I + H + \cdots + H^{n-1})[-m_{\text{mut}}]_+.$$

For the sharper reddening bound, suppose a mutation word takes r to s with auxiliary principal c -matrix $C^r(s) = -P$, where P is a permutation matrix. This auxiliary matrix is distinct from the original coefficient block. Let $\phi : M_s \rightarrow M_r$ be the tropical label map along the inverse word, and L its integral linear piece on the positive chamber of s .

Lemma 9.2. *There is an integral vector $b(m)$, computable by (7), such that*

$$Lp_s = -p_r P, \quad Lm - \phi(m) = p_r b(m).$$

The mutable block of L is $-P$ and $DP = PD$.

Proof. On nonnegative mutable labels the theta functions are cluster monomials. Their degrees add under normalized multiplication, so ϕ is linear on the positive chamber with mutable columns the full g -vectors and frozen columns the identity columns. Tropical duality gives $G^T DC = D$; see [9, Theorem 2.2.6 and Proposition 2.3.3]. With $C = -P$ this gives $G = -D^{-1}PD$. Its nonzero entries have absolute values $d_j/d_{\sigma(j)}$. They are positive integers whose product along each permutation cycle is one. Hence all are one, proving $DP = PD$ and $G = -P$.

The exponent recurrence gives $Lp_s = p_r A$ for an integral product of J matrices, and L is symplectic. Compatibility then yields

$$(D \ 0) = p_s^T \Lambda_s = A^T (D \ 0) L = (-A^T DP \ 0),$$

so $A = -P$. Summing the integral affine offsets in (7) gives the second equality. $\square$

Theorem 9.3. *Put $d_s(m) = P^{-1}b(m)$. Then $d_s(m) \in \mathbb{N}^n$ and*

$$\Theta_m^s = \sum_{0 \leq a \leq d_s(m)} t_{m,a}(v) X^{m+psa}, \quad t_{m,0} = t_{m,d_s(m)} = 1.$$

All labels are finite, and these theta functions are an R -basis of $\mathcal{U}$. If the mutation class is acyclic at some seed, $\mathcal{A} = \mathcal{U}$.

Proof. A term of charge a transports to a term with charge $b(m) - Pa$ at endpoint $\phi(Q)$ in the r diagram. At this endpoint the theta series has nonnegative charges. Hence $0 \leq a \leq P^{-1}b(m)$. The initial term gives $b(m) \geq 0$. The target theta's leading term pulls back to the corner $a = P^{-1}b(m)$, with coefficient one. This proves finiteness and the two corner assertions.

Proposition 3.4 now gives upper-algebra membership and compatible pointedness at every seed. In Qin's notation take $q^{1/2} = v$, $p^* = p$, and the dominance order $g' = g + pa$. Full rank follows from compatibility, and the reddening word gives injective reachability. Thus [9, Theorem 6.2.1(1), referring to Theorem 1.2.1(1)] applies. In the mutation-acyclic case, [2, Theorem 7.5] gives $\mathcal{A} = \mathcal{U}$. $\square$

The corner $m + p_s d_s(m) = L^{-1} \phi(m)$ is a bijective function of m . The support box and its occupied far corner show that d_s is independent of the chosen opposing word. It is integer piecewise linear, homogeneous, and unchanged by frozen translation.

Proposition 9.4. *In the situation of Theorem 9.3, every nonzero output of $\Theta_m \Theta_n$ has label $g = m + n + p_s a$ satisfying*

$$a \geq 0, \quad a + d_s(g) \leq d_s(m) + d_s(n).$$

The function d_s is componentwise subadditive and convex.

Proof. The product is supported in the rectangle of charges from zero to $d_s(m) + d_s(n)$. Proposition 6.3 puts the entire support of each output, including both its corners, inside that rectangle. Its leading label gives $a \geq 0$, and its far corner gives the second inequality. The output $g = m + n$ occurs, so $d_s(m + n) \leq d_s(m) + d_s(n)$. Positive homogeneity and continuous piecewise linearity extend this inequality to real arguments and imply convexity. $\square$

9.1 Finiteness under unfreezing

Partition the mutable indices as $I \sqcup F$, where F is nonempty, and let s^{fr} denote the seed obtained by freezing F . Its exponent lattice remains M ; its charge lattice is $\mathbb{Z}^I$, with charge map p_I given by the columns indexed by I . Write p_F for the remaining columns and $D_I = \text{diag}(d_i)_{i \in I}$. Subscripts I and F on vectors denote their corresponding coordinate subvectors. Assume

$$B_{F,I} \geq 0 \quad \text{entrywise}, \quad B_{F,F} \text{ is acyclic.} \quad (9)$$

Put $H = [-B_{F,F}]_+$ and define, for each label m ,

$$t(m) = \sum_{j=0}^{|F|-1} H^j [-m_F]_+.$$

The matrix H becomes strictly triangular after ordering F , so $H^{|F|} = 0$ and $(\text{id} - H)t(m) = [-m_F]_+$.

Lemma 9.5. *Every support charge of Θ_m^s satisfies $0 \leq a_F \leq t(m)$. If $m_F \geq 0$, then*

$$\Theta_m^s = \Theta_m^{s^{\text{fr}}}$$

as formal series in the same based monomials.

Proof. For $k \in F$, equation (8) and $B_{F,I} \geq 0$ give

$$a_k \leq [-m_k]_+ + \sum_{j \in F} [-b_{kj}]_+ a_j.$$

Iteration of $a_F \leq [-m_F]_+ + H a_F$ through $|F|$ steps gives the bound.

The sum of Lie components with a positive F -charge is an ideal: adding a nonnegative charge preserves that property. The quotient removes the initial F -walls and retains the initial I -walls. Scattering uniqueness identifies the quotient diagram with that of s^{fr} . This is the charge-face construction underlying [15, Lemma 5.1 and Theorem 5.2].

At finite height, a broken line of total F -charge zero uses only I -charge bends. Conversely a frozen broken line crosses the removed walls with their constant terms. Combined wall actions identify the coefficients at coincident supports. Hence the zero- F -charge part of Θ_m^s equals $\Theta_m^{s^{\text{fr}}}$, coefficientwise through every height. When $m_F \geq 0$, the bound gives $a_F = 0$ for the entire series, proving the equality. $\square$

Theorem 9.6. Assume (9). For a label m , define

$$g^{(k)} = -e_k + [-pe_k]_+ \quad (k \in F), \quad b = m - \sum_{k \in F} t_k(m)g^{(k)}.$$

If Θ_b^{fr} is finite, then Θ_m^s is finite. Writing

$$A_b = \{a_I \in \mathbb{N}^I : [X^{b+p_I a_I}] \Theta_b^{\text{fr}} \neq 0\},$$

one has

$$\text{supp } \Theta_m^s \subseteq \{m + p_I a_I + p_F a_F : a_I \in A_b, 0 \leq a_F \leq t(m)\}. \quad (10)$$

Consequently, if every frozen theta label is finite, every theta label of s is finite.

Proof. The F -coordinate columns of $g^{(k)}$ form $H - \text{id}$. Thus

$$b_F = m_F + (\text{id} - H)t(m) = [m_F]_+.$$

Lemma 9.5 identifies Θ_b^s with the finite frozen function. For each $k \in F$, the one-step cluster variable is

$$\Theta_{g^{(k)}}^s = X^{g^{(k)}} + X^{g^{(k)} + pe_k}.$$

Fix an order on F and multiply Θ_b^s by $(\Theta_{g^{(k)}}^s)^{t_k(m)}$ in that order. Its leading exponent is m , with coefficient a power of v . Iterating Proposition 6.3 gives a finite theta expansion whose leading output is Θ_m^s . Ordered noncancellation puts its entire support in the support of this product. The first factor contributes a charge in A_b , and the k th power contributes between zero and $t_k(m)$ copies of e_k . This proves (10) and finiteness. $\square$

10 Central coefficient quotients and pairing twists

Define the principal enlargement

$$M^+ = \mathbb{Z}_g^n \oplus \mathbb{Z}_h^n \oplus \mathbb{Z}_z^f, \quad p^+ a = (Ba, a, 0), \quad \varphi(g, h, z) = (g, Ch + z), \quad \Lambda^+ = \varphi^* \Lambda.$$

The subscripts name the coordinate blocks. Then $\varphi p^+ = p$ and $(p^+)^T \Lambda^+ = (D, 0, 0)$. The kernel

$$K_0 = \{(0, h, -Ch) : h \in \mathbb{Z}^n\}$$

consists of central frozen exponents. Superscript $+$ on a torus, algebra, or theta function refers to this enlarged seed. The based-monomial map $\Phi(X^{m^+}) = X^{\varphi m^+}$ is a torus algebra surjection. Along a fixed translated positive cone it is injective on exponent positions, because p is injective. Pullback of wall supports and their actions therefore identifies every fixed-label theta coefficient and its finiteness on the two sides. This follows either directly by lifting broken lines modulo the cylindrical kernel or by the saturated projection in Lemma 5.1.

We shall use the finite adjacent description of Berenstein–Zelevinsky [2, Theorem 5.1 and Lemmas 5.4–5.5]. For mutation at k , let $\mathcal{T}_k$ be the adjacent torus and set $X'_k = X^{-e_k + [-pe_k]_+} + X^{-e_k + [pe_k]_+}$. Then

$$S_k = \mathcal{T}_R \cap \mathcal{T}_{k,R} = R\langle X_i^{\pm 1} \ (i \neq k), X_k, X'_k \rangle, \quad \mathcal{U}_R = \bigcap_{k=1}^n S_k. \quad (11)$$

Here $R\langle \rangle$ denotes the generated subalgebra of the common skew field. Each S_k has the unique finite expression

$$c_0 + \sum_{r>0} (c_r X_k^r + c'_r (X'_k)^r), \quad c_0, c_r, c'_r \in \mathcal{T}_{k,R}, \quad (12)$$

where $\mathcal{T}_{\widehat{k},R}$ is the torus generated by all variables other than X_k . Coefficients are placed on the left. These statements remain valid after scalar localization. Each adjacent algebra S_k is G -graded: the generators $X_i^{\pm 1}$ are homogeneous, and the two exponents of X'_k differ by pe_k . Taking homogeneous components in the finite intersection therefore makes $\mathcal{U}_R$ a G -graded subalgebra. The finite theta algebra is graded by $\deg \Theta_m = [m]$.

Theorem 10.1. *The map Φ induces*

$$\mathcal{U}^+/(X^\kappa - 1 : \kappa \in K_0) \simeq \mathcal{U}, \quad \mathcal{M}_\Theta^+/(X^\kappa - 1 : \kappa \in K_0) \simeq \mathcal{M}_\Theta.$$

Moreover $\mathcal{M}_\Theta^+ = \mathcal{U}^+$ if and only if $\mathcal{M}_\Theta = \mathcal{U}$.

Proof. Put $G^+ = M^+/p^+N$. Injectivity of p gives the exact sequence

$$0 \longrightarrow K_0 \longrightarrow G^+ \longrightarrow G \longrightarrow 0.$$

Define $\mathcal{T}_\alpha^+$ for $\alpha \in G^+$ by the same degree convention as $\mathcal{T}_\gamma$. For any $\alpha \in G^+$ mapping to $\gamma \in G$, the map $\Phi : \mathcal{T}_\alpha^+ \rightarrow \mathcal{T}_\gamma$ is an R -module isomorphism: choosing a representative a^+ identifies the exponents on the two sides as $a^+ + p^+b$ and $\varphi a^+ + pb$, respectively, with $b \in \mathbb{Z}^n$.

The exchange variable requires a frozen correction. Let $a_k = -e_k + \sum_{i \leq n} [-b_{ik}]_+ e_i$, with zero frozen component, and $t_k = (0, [-C_{\bullet k}]_+)$. In the target it is $X^{a_k + t_k} + X^{a_k + t_k + pe_k}$. In the source the exchange variable has exponents $a_k^+ = (a_{k,\text{mut}}, 0, 0)$ and $a_k^+ + p^+e_k$. If $\tilde{t}_k = (0, 0, [-C_{\bullet k}]_+)$, its corrected lift is

$$v^{-\Lambda^+(\tilde{t}_k, a_k^+)} X^{\tilde{t}_k} X_k'^+.$$

Its two terms acquire the same phase because $\Lambda(t_k, pe_k) = 0$. Its image is exactly X_k' . All other generators also lift, so $\Phi(S_k^+) = S_k$.

For $f \in S_k \cap \mathcal{T}_\gamma$, express it as a finite sum of homogeneous generator words and lift these words. Multiply each lift by its unique central kernel monomial to bring its degree to α . The result lies in $S_k^+ \cap \mathcal{T}_\alpha^+$ and maps to f . By the torus isomorphism in that degree this lift is unique, independently of the expression and of k . It therefore belongs to every S_k^+ when $f \in \mathcal{U}$. Equation (11) proves surjectivity on upper algebras.

Choose one degree lift $\alpha(\gamma)$ for each γ . A finite kernel element can be written $F = \sum_{\gamma, \kappa} X^\kappa F_{\gamma, \kappa}$, with $F_{\gamma, \kappa} \in \mathcal{U}^+ \cap \mathcal{T}_{\alpha(\gamma)}^+$. Its zero image implies $\sum_{\kappa} F_{\gamma, \kappa} = 0$ by degreewise injectivity. Thus $F = \sum_{\gamma, \kappa} (X^\kappa - 1) F_{\gamma, \kappa}$, proving the stated kernel.

Fixed-label coefficient correspondence gives $\Phi(\Theta_{m^+}^+) = \Theta_{\varphi m^+}$ and equivalence of finiteness. Central frozen translation gives $\Theta_{m^+ + \kappa}^+ = X^\kappa \Theta_{m^+}^+$. The same degreewise argument proves the quotient for $\mathcal{M}_\Theta$. If $\mathcal{M}_\Theta = \mathcal{U}$, lift each finite homogeneous theta expansion uniquely to a chosen degree α to obtain $\mathcal{M}_\Theta^+ = \mathcal{U}^+$. The opposite implication follows by quotienting. $\square$

Proposition 10.2. *For fixed p, D , changing the compatible form preserves all theta coefficients and upper-algebra generation, under the based-monomial identification. More precisely, the change is a G -graded cocycle twist. Consequently*

$$\mathcal{M}_\Theta(p, \Lambda) = \mathcal{U}(p, \Lambda) \iff \mathcal{M}_\Theta^{\text{prin}}(B, D) = \mathcal{U}^{\text{prin}}(B, D),$$

where the right side uses $p^{\text{prin}} = \begin{pmatrix} B \\ I \end{pmatrix}$ and $\Lambda^{\text{prin}} = \begin{pmatrix} 0 & -D \\ D & -DB \end{pmatrix}$. All-label finiteness is also equivalent on the two sides.

Proof. Let I denote the coefficientwise identification of their based monomials. For two forms write $\Delta = \Lambda' - \Lambda$. Then $p^T \Delta = \Delta p = 0$, so it descends to an integral skew form on G , automatically zero on torsion. If F, H have degrees γ, η , their based-monomial images satisfy

$$I(F)I(H) = v^{\Delta(\gamma, \eta)} I(FH).$$

The exchange generators are homogeneous, so (11) identifies the upper algebras as modules with this twisted product. Wall brackets use only $\Lambda(pa, pb)$, and wall actions only $\Lambda(pa, m)$; neither changes. Thus all theta coefficients agree.

On the principal enlargement compare Λ^+ with

$$\Lambda^{\text{st}} = \begin{pmatrix} 0 & -D & 0 \\ D & -DB & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Their difference is such a twist. For Λ^{st} the z variables are central and absent from the walls. Taking the common direct-sum decomposition by z exponent in every seed shows that both $\mathcal{U}$ and $\mathcal{M}_\Theta$ are obtained from their standard principal versions by adjoining these Laurent variables. Combine this with Theorem 10.1. $\square$

Corollary 10.3. *For fixed B, D , the coefficient of charge a in $\Theta_{(g, h)}$ depends only on $g \in \mathbb{Z}^n$ and $a \in \mathbb{N}^n$, independently of the frozen label h and compatible realization (C, Λ) . In particular there is a subset $F(B, D) \subseteq \mathbb{Z}^n$ such that*

$$\Theta_{\text{fin}}(B, C, \Lambda) = F(B, D) \times \mathbb{Z}^f.$$

Replacing D by a positive integer multiple preserves this set.

Proof. Corollary 6.7 preserves charge coefficients under frozen translation. The coefficient correspondence in Theorem 10.1 is injective on the charge positions of each fixed label. Proposition 10.2 identifies those coefficients with the standard principal ones. These two steps give the first assertions. In the standard principal model, replacing D by rD substitutes v^r for v in every coefficient. This substitution is injective on R , so it preserves the nonzero charge positions. $\square$

11 Theta bases of upper cluster algebras

Lemma 11.1. *If a finite combination $\sum_{m \in J} c_m \Theta_m$, with $c_m \in K$, has all monomial coefficients in R , then all $c_m \in R$.*

Proof. Choose a minimal label of the finite set J in the positive-charge order. Its coefficient is precisely its scalar c_m , since its leading coefficient is one. Subtract this integral multiple and repeat. The process has $|J|$ steps. $\square$

11.1 Integral specialization

Write $\mathcal{T}_{s, \text{cl}, \mathbb{Z}}$ for the classical Laurent ring over $\mathbb{Z}$ in seed s , with variables x_i obtained from the quantum seed variables at $v = 1$ and monomials $x^m = \prod_{i=1}^{n+f} x_i^{m_i}$ for $m \in M$. Set $\mathcal{U}_{\text{cl}, \mathbb{Z}} = \bigcap_s \mathcal{T}_{s, \text{cl}, \mathbb{Z}}$ in the common classical fraction field. We write θ_m for the classical theta function obtained by coefficientwise specialization $v = 1$. Geiss–Leclerc–Schröer prove the upper-algebra specialization injection over a characteristic-zero field in [18, Proposition 3.5 and Corollary 3.6].

Theorem 11.2. *For every seed s , coefficientwise specialization gives*

$$\mathcal{U}_R \cap (v-1)\mathcal{T}_{s,R} = (v-1)\mathcal{U}_R, \quad \mathcal{U}_R/(v-1)\mathcal{U}_R \hookrightarrow \mathcal{U}_{\text{cl},\mathbb{Z}}.$$

Proof. For two adjacent seed tori, use the finite expression (12). Its existence can be read by the degree of X_k in the skew Laurent ring over the fraction field of $\widehat{\mathcal{T}}_{k,R}$: nonnegative degrees come from the original torus and negative degrees from powers of X'_k . The latter have degree -1 , and both exchange variables belong to both tori. Different degrees give uniqueness.

Modulo $v-1$, the ring $\widehat{\mathcal{T}}_{k,R}$ becomes an integer Laurent domain. The term $(X'_k)^r$ has x_k -degree $-r$, with a nonzero classical exchange-polynomial factor. Hence a PBW expression whose initial monomial coefficients vanish modulo $v-1$ has every PBW coefficient divisible by $v-1$. Dividing those coefficients gives

$$(\mathcal{T}_{s,R} \cap \mathcal{T}_{s',R}) \cap (v-1)\mathcal{T}_{s,R} = (v-1)(\mathcal{T}_{s,R} \cap \mathcal{T}_{s',R})$$

for adjacent s, s' .

If $F = (v-1)H \in \mathcal{U}_R$ with $H \in \mathcal{T}_{s,R}$, this equality puts H in each successive torus along any finite mutation path. Thus $H \in \mathcal{U}_R$. The same PBW expressions identify the reductions of F in adjacent charts as one classical rational function, since the exchange polynomials remain nonzero. Following paths shows that this reduction belongs to every $\mathcal{T}_{s,\text{cl},\mathbb{Z}}$. The kernel equality proves injectivity. $\square$

Lemma 11.3. *Let $J \subseteq M$. Suppose $\{\theta_j : j \in J\}$ is a basis of the classical upper algebra over $\mathbb{Q}$, and every θ_j is an integer Laurent polynomial in every seed with unit leading term x^j and other exponents in $j + p(P \setminus \{0\})$. Then it is a $\mathbb{Z}$ -basis of $\mathcal{U}_{\text{cl},\mathbb{Z}}$. If $\{\Theta_j : j \in J\}$ is also an R -basis of $\mathcal{U}_R$ and specializes to this basis, the map in Theorem 11.2 is an isomorphism.*

Proof. Expand an integer upper-algebra element in the given rational basis. In the finite set of participating labels, a minimal label in the positive-charge order receives no contribution from another label. Its monomial coefficient is its basis coordinate and is integral. Subtract that integer multiple and repeat. This gives a finite integer expansion, and independence is inherited from the rational basis. For the final assertion, lift every integer basis expansion using the same coefficients; injectivity is Theorem 11.2. $\square$

11.2 Finite support and specialization

Let $\mathcal{O} = \mathbb{Q}[v^{\pm 1}]_{(v-1)}$, a discrete valuation ring with uniformizer $\pi = v-1$, fraction field K , and residue field $\mathbb{Q}$. For finite $E \subset M$ write $W_E(L) = \bigoplus_{a \in E} LX^a$ when $L = K, \mathcal{O}, \mathbb{Q}$. When $L = \mathbb{Q}$, the map $R \rightarrow \mathbb{Q}$ sends v to one. Let $\mathcal{U}_{\text{cl}}$ be the classical upper algebra over $\mathbb{Q}$ for the same extended matrix. Write $S_k(L) = \mathcal{T}_L \cap \mathcal{T}_{k,L}$ for scalar localizations L of R .

Theorem 11.4. *For every finite $E \subset M$,*

$$\dim_K(\mathcal{U}_K \cap W_E(K)) \leq \dim_{\mathbb{Q}}(\mathcal{U}_{\text{cl}} \cap W_E(\mathbb{Q})).$$

Proof. Reduce the PBW expression (12) modulo π . Terms X_k^r have positive X_k degree r ; terms $(X'_k)^r$ have degree $-r$, with a nonzero exchange-polynomial factor in the other variables. The classical Laurent ring is a domain. Hence this expression is still linearly independent over $\widehat{\mathcal{T}}_{k,\mathbb{Q}}$ after reduction. If an element of $S_k(\mathcal{O})$ has every initial monomial coefficient divisible by π , all its PBW coefficients are divisible by π . Thus

$$S_k(\mathcal{O}) \cap \pi\mathcal{T}_{\mathcal{O}} = \pi S_k(\mathcal{O}), \quad \mathcal{U}_{\mathcal{O}} \cap \pi\mathcal{T}_{\mathcal{O}} = \pi\mathcal{U}_{\mathcal{O}}.$$

The second equality uses the finite intersection (11). Scalar localization commutes with that finite intersection by clearing finitely many scalar denominators.

Put $V_E = \mathcal{U}_K \cap W_E(K)$ and $L_E = V_E \cap W_E(\mathcal{O})$. If $f \in L_E$, finitely many adjacent expressions let us multiply by π^a to obtain an element of $\mathcal{U}_{\mathcal{O}}$. Repeated saturation then gives $f \in \mathcal{U}_{\mathcal{O}}$. The module L_E is a full finite free lattice in V_E and $L_E \cap \pi W_E(\mathcal{O}) = \pi L_E$. Its reduction therefore injects into $W_E(\mathbb{Q})$ with dimension $\dim_K V_E$. Since its elements belong to every quantum torus over $\mathcal{O}$, reduction gives the same classical rational function in every classical torus; exchange polynomials remain nonzero. Its image lies in $\mathcal{U}_{\text{cl}}$, proving the inequality. $\square$

Theorem 11.5. *Suppose $\mathcal{U}_{\text{cl}}$ has a basis $\{\theta_j : j \in J\}$ of Laurent-polynomial theta functions satisfying valutive independence for finite combinations. Suppose the quantum theta function Θ_j is finite for each $j \in J$. Then these quantum theta functions form an R -basis of $\mathcal{U}_R$, and $\mathcal{M}_{\Theta} = \mathcal{U}_R$. Every nonzero summand in the theta expansion of $f \in \mathcal{U}_R$ has support in $\text{Newt}(f)$.*

Proof. Proposition 3.2 identifies the specialization of Θ_j with θ_j . Every vertex coefficient of Θ_j is one by Proposition 8.2, so all its vertices survive specialization. Consequently

$$\text{Newt}(\Theta_j) = \text{Newt}(\theta_j).$$

For a bounded rational polytope P_0 , set $E = P_0 \cap M$ and

$$J(P_0) = \{j \in J : \text{supp } \theta_j \subseteq P_0\}.$$

This set is finite because every theta contains its label. The classical basis and valutive independence give

$$\mathcal{U}_{\text{cl}} \cap W_E(\mathbb{Q}) = \bigoplus_{j \in J(P_0)} \mathbb{Q}\theta_j.$$

The Newton equality places the corresponding quantum functions in $W_E(K)$. They belong to $\mathcal{U}_K$ by Proposition 3.4 and are independent by their leading terms. Theorem 11.4 gives

$$|J(P_0)| \leq \dim_K(\mathcal{U}_K \cap W_E(K)) \leq \dim_{\mathbb{Q}}(\mathcal{U}_{\text{cl}} \cap W_E(\mathbb{Q})) = |J(P_0)|.$$

Thus they form a basis of the quantum window. For nonzero $f \in \mathcal{U}_R$, take $P_0 = \text{Newt}(f)$ and apply Lemma 11.1 to its finite expansion. This gives coefficients in R and the required support inclusion. The zero element has the empty expansion. Since $\mathcal{M}_{\Theta} \subseteq \mathcal{U}_R$ and the displayed basis lies in $\mathcal{M}_{\Theta}$, the two algebras coincide. $\square$

For skew-symmetric cluster data, the positive quantum coefficients and classical specialization of [4, Theorems 1.1–1.2 and Appendix A] give finite quantum lifts of the finite classical theta functions. Classical valutive independence is supplied by [3, Theorems 3.8 and 3.10]; for principal coefficients the finite-support situation uses its unspecialized version, and finite frozen translations retain the full lattice. Theorem 11.5 therefore transfers a classical theta basis to the quantum upper algebra in this setting.

Specialization can remove an infinite tail from a formal wall action. For example, the wall series

$$H(z) = \frac{1 - v^{-1}z}{1 - vz}, \quad \frac{H(vz)}{H(v^{-1}z)} = \frac{(1 - z)^2}{(1 - v^2z)(1 - v^{-2}z)}$$

gives an action with infinitely many nonconstant coefficients, each bar invariant and with positive leading coefficient, whose specialization is one. Parity imposes an additional divisibility condition on coefficients that vanish at $v = 1$.

Lemma 11.6. *If $c \in R$ satisfies $c(v^{-1}) = c(v)$, $c(-v) = \pm c(v)$ and $c(1) = 0$, then $(v - v^{-1})^2$ divides c in R .*

Proof. An even bar-invariant Laurent polynomial has the form $P(v^2 + v^{-2})$, with $P \in \mathbb{Z}[T]$. An odd one has the form $(v + v^{-1})P(v^2 + v^{-2})$: this follows by induction from

$$(v^2 + v^{-2})(v^r + v^{-r}) = (v^{r+2} + v^{-r-2}) + (v^{r-2} + v^{2-r})$$

for odd $r \geq 1$. In both cases $c(1) = 0$ gives $P(2) = 0$. The factor theorem and $v^2 + v^{-2} - 2 = (v - v^{-1})^2$ prove the claim. $\square$

11.3 Computing theta expansions

Under Theorem 11.5, a single finite window determines an exact theta expansion algorithm. For $f \neq 0$, put $P_0 = \text{Newt}(f)$ and choose a rational linear functional h on $M_{\mathbb{R}}$ with $h(pe_i) = 1$ for every i ; this exists because p is injective. Set

$$N_0 = \left\lceil \max_{x \in P_0} h(x) - \min_{x \in P_0} h(x) \right\rceil.$$

For every candidate label $m \in P_0 \cap M$, compute the theta series to charge height N_0 , retain its terms in P_0 , and test the resulting finite polynomial in all adjacent algebras using (12). Precisely the labels whose entire theta is supported in P_0 pass this test. Indeed, the truncation contains every possible support term in that window. If it is in $\mathcal{U}$, the window basis from Theorem 11.5 expresses it using full theta functions supported there. Triangular independence of the truncated coefficient columns, each with its distinct unit leading term in P_0 , then identifies it with the theta of the candidate label. Finally solve the finite unit triangular system.

11.4 Upper bases under unfreezing

Work first in the standard principal realization

$$M = \mathbb{Z}^{2n}, \quad p = \begin{pmatrix} B \\ I_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -D \\ D & -DB \end{pmatrix}.$$

Write $x_i = X^{e_i}$ and $y_i = X^{e_{n+i}}$. For $J \subseteq \{1, \dots, n\}$, let $\mathcal{U}_R(J)$ and $\mathcal{M}_{\Theta}(J)$ be the upper and finite theta algebras obtained by allowing mutation only in J . All y_i and all x_i with $i \notin J$ are invertible frozen variables. These algebras are compared inside the original skew field.

Theorem 11.7. *Let $k \notin J$ and suppose $B_{k,J} \geq 0$ entrywise. If $\mathcal{M}_{\Theta}(J) = \mathcal{U}_R(J)$, then*

$$\mathcal{M}_{\Theta}(J \cup \{k\}) = \mathcal{U}_R(J \cup \{k\}).$$

For a nonzero Laurent polynomial f , let $\nu_k(f)$ be its minimum initial x_k exponent, and set $\nu_k(0) = +\infty$. Each element f of the latter upper algebra has a finite integral theta expansion obtained in at most $\max(0, -\nu_k(f))$ steps of strict pole-order descent.

Proof. The initial x_i commute. Define the exchange monomials

$$A = \prod_{i \neq k} x_i^{[-b_{ik}]_+}, \quad C = \prod_{i \neq k} x_i^{[b_{ik}]_+}.$$

Skew-symmetrizability and $B_{k,J} \geq 0$ imply $B_{J,k} \leq 0$. Thus C involves only frozen x_i and is a unit in $\mathcal{U}_R(J)$; A is a product of initial cluster variables and also belongs to $\mathcal{U}_R(J)$. Both commute with x_k and y_k . In the based normalization,

$$x_k y_k = v^{-2d_k} y_k x_k, \quad x'_k = x_k^{-1} (A + v^{-d_k} C y_k), \quad x'_k x_k = A + v^{d_k} C y_k.$$

For a positive integer t , put

$$P_t(y_k) = \prod_{j=1}^t (y_k + v^{-(2j-1)d_k} AC^{-1}) = v^{-d_k t^2} C^{-t} (x'_k)^t x_k^t. \quad (13)$$

The second equality follows by commuting x_k through the exchange factors. This polynomial is monic of degree t in y_k .

Assign degree one to y_k and degree zero to every other initial variable. The row corresponding to y_k in the principal coefficient block vanishes in columns J , and remains zero under mutations in J . Every resulting cluster variable therefore has y_k degree zero. If $\mathcal{T}_s^0$ is the degree-zero subring of such a seed torus, that torus is

$$\mathcal{T}_s^0[y_k^{\pm 1}; \tau], \quad \tau(a) = y_k a y_k^{-1}.$$

Here the skew Laurent notation means $y_k a = \tau(a) y_k$, with coefficients on the left. Conjugation by the frozen unit y_k preserves every $\mathcal{T}_s^0$. The common degree decomposition gives

$$E = \bigcap_s \mathcal{T}_s^0, \quad \mathcal{U}_R(J) = E[y_k^{\pm 1}; \tau]. \quad (14)$$

Indeed, membership of a finite Laurent sum in each seed torus is equivalent to membership of each of its degree components. We have $A, C, C^{-1} \in E$.

First suppose $h \in \mathcal{U}_R(J)$ and $\nu_k(h) \geq 0$. For a finite theta function of the frozen seed, its support has exponents $b + p_J a$, $a \geq 0$. Thus $B_{k,J} \geq 0$ and its leading coefficient one give

$$\nu_k(\Theta_b^J) = b_k.$$

Expand h in its finite R -basis of frozen theta functions. Valiative independence shows that every participating b has $b_k \geq 0$. Lemma 9.5, applied to the single added vertex, identifies these functions with theta functions for $J \cup \{k\}$. Consequently

$$h \in \mathcal{U}_R(J), \quad \nu_k(h) \geq 0 \quad \implies \quad h \in \mathcal{M}_\Theta(J \cup \{k\}). \quad (15)$$

Take $f \in \mathcal{U}_R(J \cup \{k\}) \subseteq \mathcal{U}_R(J)$. Right multiplication by a sufficiently large power of y_k makes f polynomial in y_k and preserves $\nu_k(f)$. Suppose $\nu_k(f) = -t < 0$. Monic left division in $E[y_k; \tau]$ gives

$$f = P_t Q + R_t, \quad \deg_{y_k} R_t < t.$$

Explicitly, a highest term $a_s y_k^s$ is removed by subtracting $P_t \tau^{-t}(a_s) y_k^{s-t}$. These operations take place in E and require only addition and multiplication. The coefficients of P_t have initial x_k degree zero, and τ scales initial based monomials by powers of v . Hence

$$\nu_k(Q), \nu_k(R_t) \geq -t.$$

Perform the same monic division in the initial skew Laurent ring in the generators other than y_k , with y_k adjoined polynomially. Decompose its coefficients by their initial x_k degree. Each coefficient of P_t has x_k degree zero, and τ preserves x_k degree. Thus every subtraction in the division preserves the homogeneous degree, and taking an x_k component commutes with the remainder map.

The adjacent k -mutated Laurent expansion has the form

$$f = \sum_j (x'_k)^j a_j,$$

where each a_j is Laurent in the other initial generators and has x_k degree zero. The nonzero summands have distinct x_k degrees $-j$. Therefore the largest occurring j is t , and the lowest component is $(x'_k)^t a_t$. By (13) it equals

$$P_t(v^{d_k t^2} x_k^{-t} C^t a_t).$$

The parenthesized expression is initially Laurent in y_k . Its lowest y_k degree is nonnegative: otherwise multiplication by the nonzero constant coefficient of P_t would leave a nonzero negative-degree term, contradicting polynomiality of the component of f . Thus its monic-division remainder is zero. Commutation of the remainder map with the ambient homogeneous projection gives

$$\nu_k(R_t) > -t.$$

Now set $h = v^{-d_k t^2} C^{-t} x_k^t Q$. This element belongs to $\mathcal{U}_R(J)$, since C and x_k are frozen units there, and has $\nu_k(h) \geq 0$. Equation (15) gives $h \in \mathcal{M}_\Theta(J \cup \{k\})$. Since C commutes with x'_k , equation (13) yields

$$P_t Q = (x'_k)^t h \in \mathcal{M}_\Theta(J \cup \{k\}).$$

It follows that $R_t = f - (x'_k)^t h$ remains in $\mathcal{U}_R(J \cup \{k\})$ and has smaller pole order. Induction terminates after at most t steps, with (15) handling the final remainder. Every division, frozen-basis expansion and theta product is finite and integral. Right multiplication by the original compensating power of y_k^{-1} finishes the expansion of the original f . $\square$

Corollary 11.8. *Suppose $B_{F,I} \geq 0$ and $B_{F,F}$ is acyclic. If the frozen core has $\mathcal{M}_\Theta = \mathcal{U}$ for one compatible realization with symmetrizer D_I , then the full matrix has $\mathcal{M}_\Theta = \mathcal{U}$ for every compatible realization with symmetrizer D .*

Proof. First use the standard principal realization of the full matrix, freezing F . Proposition 10.2 and Theorem 10.1, applied to the core matrix and D_I , give its theta upper basis in this induced compatible realization. Order $F = \{k_1, \dots, k_s\}$ so that $b_{k_j k_i} \geq 0$ for $i < j$. Theorem 11.7 applies successively with mutable sets I , $I \cup \{k_1\}$, and so on. This proves the equality for the full standard principal seed. Apply Proposition 10.2 and Theorem 10.1 again, now to the full matrix and D , to obtain every compatible realization. $\square$

11.5 Large cluster complexes

In a fixed seed chart of the standard principal realization, project the labels of normalized cluster monomials to their mutable coordinates $\mathbb{Z}^n$. We say that the cluster complex is large when these projected labels are contained in no closed half-space $\{g : \ell(g) \geq 0\}$ for a nonzero real linear functional ℓ . This is the condition of [5, Definition 8.27], expressed in the seed basis for the label lattice. The projection sends a principal cluster-monomial label to its usual g -vector: both are obtained from the initial exponent by the same tropical mutation maps. The full principal labels include every frozen translate of each such label.

Corollary 11.9. *If the cluster complex is large, every quantum theta label is finite and the theta functions form an R -basis of $\mathcal{U}$ for every compatible realization of the same B, D .*

Proof. Consider first the full principal label lattice $M = \mathbb{Z}^{2n}$. A functional nonnegative on all cluster-monomial labels and their frozen translates must vanish on the frozen subspace, because both signs of each frozen unit occur. It then descends to a functional on the mutable label space. Largeness implies that this functional is zero.

For each functional on the unit sphere in $M_{\mathbb{R}}^*$, choose a cluster-monomial label on which it is negative. These strict inequalities give an open cover of the sphere. Compactness provides

finitely many labels $g_1, \dots, g_q$ with the same property. Their finitely generated cone is closed and has zero dual cone, hence equals $M_{\mathbb{R}}$. For any $m \in M$, the rational system $\sum_i \lambda_i g_i = m$, $\lambda_i \geq 0$, has a nonnegative rational solution. Clearing denominators gives $km = \sum_i n_i g_i$ with $k > 0$ and $n_i \geq 0$ integral. Cluster monomials are finite theta functions, so additivity and saturation of finite labels imply $m \in \Theta_{\text{fin}}$.

Under the projection from the principal label lattice to the mutable label lattice, the classical cluster complex is the inverse image of the original one, as in the proof of [5, Proposition 8.28]. That proposition supplies the classical full theta basis of $\mathcal{U}_{\text{cl}}$ for this principal $\mathcal{A}$ -type object. The functions are defined over $\mathbb{Q}$; their basis property descends to $\mathbb{Q}$ by finite coefficient comparison in each unique expansion. The finite valuative independence used in Theorem 11.5 is given by [3, Theorems 3.8 and 3.10]. The finiteness just proved verifies the quantum-lift hypothesis, so Theorem 11.5 gives the principal quantum upper basis. Proposition 10.2 and Theorem 10.1 transfer the result to the other compatible realizations. $\square$

11.6 Classical integral bases under unfreezing

In the standard principal model, write x_i, y_i for the classical initial variables. For a mutable set $J \subseteq \{1, \dots, n\}$, let $\mathcal{U}_{\text{cl}, \mathbb{Z}}(J)$ be the integer classical upper algebra with mutation restricted to J , and let $\mathcal{M}_{\text{cl}, \mathbb{Z}}(J)$ be the $\mathbb{Z}$ -span of its finite classical theta functions. Classical finite theta multiplication [5] makes the latter an algebra. We use finite classical valuative independence over $\mathbb{Q}$ from [3, Theorems 3.8 and 3.10]. Define $\nu_k(f)$ as the least initial x_k exponent of a nonzero Laurent polynomial and set $\nu_k(0) = +\infty$.

Theorem 11.10. *If $k \notin J$, $B_{k,J} \geq 0$ entrywise and $\mathcal{M}_{\text{cl}, \mathbb{Z}}(J) = \mathcal{U}_{\text{cl}, \mathbb{Z}}(J)$, then*

$$\mathcal{M}_{\text{cl}, \mathbb{Z}}(J \cup \{k\}) = \mathcal{U}_{\text{cl}, \mathbb{Z}}(J \cup \{k\}).$$

Every nonzero $f \in \mathcal{U}_{\text{cl}, \mathbb{Z}}(J \cup \{k\})$ has an integral theta expansion obtained in at most $\max(0, -\nu_k(f))$ strict pole-order reductions.

Proof. Define

$$A = \prod_{i \neq k} x_i^{[-b_{ik}]_+}, \quad C = \prod_{i \neq k} x_i^{[b_{ik}]_+}, \quad x'_k = x_k^{-1}(A + Cy_k).$$

Skew-symmetrizability gives $B_{J,k} \leq 0$, so C uses only frozen variables of the core and is a unit there. The principal row of y_k has zero entries in columns J and remains zero under mutations in J . Consequently all resulting cluster variables have y_k -degree zero. If $\mathcal{T}_{s, \text{cl}, \mathbb{Z}}^0$ is the degree-zero subring of the corresponding Laurent ring, its common degree decomposition gives

$$E = \bigcap_s \mathcal{T}_{s, \text{cl}, \mathbb{Z}}^0, \quad \mathcal{U}_{\text{cl}, \mathbb{Z}}(J) = E[y_k^{\pm 1}], \quad A, C, C^{-1} \in E.$$

The intersections here range over mutations in J .

For a core theta function θ_b^J , its exponents lie in $b + p_J \mathbb{N}^J$. The sign hypothesis and unit leading term give $\nu_k(\theta_b^J) = b_k$. If $h \in \mathcal{U}_{\text{cl}, \mathbb{Z}}(J)$ has $\nu_k(h) \geq 0$, its finite integer theta expansion therefore uses only labels $b_k \geq 0$, by valuative independence applied over $\mathbb{Q}$. For such labels the full classical theta function equals the core function. Indeed, the one-step charge inequality (8) gives $a_k \leq [-b_k]_+ = 0$, and the zero- k -charge quotient removes precisely the k th initial wall. These are the coefficientwise classical specializations of the support bound and frozen-face identity in Lemma 9.5; the identities hold through every fixed height. Thus $h \in \mathcal{M}_{\text{cl}, \mathbb{Z}}(J \cup \{k\})$.

Now take $f \in \mathcal{U}_{\text{cl}, \mathbb{Z}}(J \cup \{k\})$. Multiply by a power of y_k to make f polynomial in y_k ; this leaves $\nu_k(f)$ unchanged. If $\nu_k(f) = -t < 0$, divide in $E[y_k]$ by the monic polynomial

$$P_t(y_k) = (y_k + AC^{-1})^t = C^{-t}(x'_k)^t x_k^t, \quad f = P_t Q + R_t, \quad \deg_{y_k} R_t < t.$$

Every coefficient of P_t has initial x_k -degree zero. Each subtraction in monic division therefore preserves the lower bound $-t$, so $\nu_k(Q), \nu_k(R_t) \geq -t$.

In the adjacent torus write $f = \sum_j (x'_k)^j a_j$, where a_j is Laurent in the other initial variables. The nonzero summands have distinct x_k -degrees $-j$. The lowest component is consequently $(x'_k)^t a_t = P_t x_k^{-t} C^t a_t$. The quotient on the right is polynomial in y_k : if it had a negative lowest power, multiplication by the nonzero constant term $(AC^{-1})^t$ of P_t would preserve that power in the integer Laurent domain, contradicting polynomiality of this component of f .

Performing the same monic division in the initial Laurent coefficient ring commutes with taking an x_k -degree component, because the coefficients of P_t have degree zero. The degree $-t$ remainder is therefore zero, and $\nu_k(R_t) > -t$. Set $h = C^{-t} x_k^t Q$. Both C and x_k are core units, so $h \in \mathcal{U}_{\text{cl}, \mathbb{Z}}(J)$, and $\nu_k(h) \geq 0$. The first part of the proof gives

$$P_t Q = (x'_k)^t h \in \mathcal{M}_{\text{cl}, \mathbb{Z}}(J \cup \{k\}).$$

Hence R_t remains in the full upper algebra with strictly smaller pole order. Induction, followed by multiplication by the compensating power of y_k^{-1} , gives a finite integer theta expansion. Classical theta independence makes these functions a basis. $\square$

Lemma 11.11. *For the principal enlargement and map φ of Section 10, the monomial map $\Phi(x^{m^+}) = x^{\varphi m^+}$ induces*

$$\mathcal{U}_{\text{cl}, \mathbb{Z}}^+ / (x^\kappa - 1 : \kappa \in K_0) \simeq \mathcal{U}_{\text{cl}, \mathbb{Z}}.$$

The property that the finite classical theta functions form an integer upper-algebra basis passes in both directions between these two realizations.

Proof. Full column rank holds in every seed. By [1, Proposition 1.8, Corollary 1.9 and Lemma 4.1], the classical upper algebra is the finite intersection of the adjacent algebras over $\mathbb{Z}$. In that argument the exchange binomials are coprime in the integer Laurent ring: full rank excludes proportional columns, and their unit coefficients have no common integer prime factor.

Use the exact sequence $0 \rightarrow K_0 \rightarrow G^+ \rightarrow G \rightarrow 0$ from Theorem 10.1. Choose a degree $\alpha \in G^+$ over $\gamma \in G$ and a representative $a^+ \in M^+$ of α . The monomial map is an integer-module isomorphism of the two torus components, since their exponents are parametrized by the same $b \in N$ as $a^+ + p^+ b$ and $\varphi(a^+) + p b$. Each adjacent generator lifts: the exchange variable uses the frozen correction $x^{(0,0,[-C_{\bullet k}]_+)}$ from that theorem. Lift a finite sum of homogeneous generator words and move each word to degree α by its unique kernel monomial. Uniqueness in the torus component makes this lift independent of the adjacent chart. It therefore lies in their intersection.

For the kernel, group a finite zero-image sum by target degree and move its terms to one chosen source degree. Their images sum to zero, so their lifts sum to zero by the component isomorphism. Subtracting these zero sums writes the original element as a sum of $(x^\kappa - 1)F_\kappa$, with $\kappa \in K_0$ and $F_\kappa \in \mathcal{U}_{\text{cl}, \mathbb{Z}}^+$. This proves the quotient statement.

Fixed-label theta correspondence and frozen translation are the coefficientwise classical versions of Theorem 10.1. Each label in a target degree has a unique lift to a prescribed source degree, because $\varphi p^+ = p$ and p is injective. Thus finite homogeneous theta expansions transfer in both directions through the component isomorphism. Their coefficients remain integers, and independence follows from their unit leading terms. $\square$

Corollary 11.12. *For reddening classes, large cluster complexes, the Markov matrix, and their successive one-way acyclic extensions satisfying (9), every compatible coefficient realization has*

$$\mathcal{U}_R/(v-1)\mathcal{U}_R \simeq \mathcal{U}_{\text{cl},\mathbb{Z}}.$$

Proof. In a reddening class, Theorem 9.3 gives finite quantum theta functions for all labels, with integral classical specializations in every seed. Their leading coefficients remain one, and Proposition 3.4 supplies compatible pointedness. Compatibility makes p full rank, and the reddening word gives injective reachability. The classical criterion [9, Theorem 1.2.1(1)], with charge map p and order $g' = g + pa$, therefore gives an integer classical basis indexed by all labels.

For a large cluster complex, [5, Proposition 8.28] gives the principal classical basis used in Corollary 11.9. For the Markov matrix, use [11, Theorem 5.1]. These functions are integral by Proposition 3.2; Lemma 11.3 descends the rational bases to $\mathbb{Z}$. Lemma 11.11 transfers them to each compatible realization. Extra frozen variables absent from the exchange columns are handled by their independent Laurent degrees.

For an extension $I \sqcup F$, order $F = \{k_1, \dots, k_s\}$ so that $b_{k_j k_i} \geq 0$ for $i < j$. In the standard principal realization, apply Theorem 11.10 successively, starting with the frozen core; its basis is supplied by Lemma 11.11 applied to the core matrix. All classical labels remain finite: the construction in Theorem 9.6 uses a finite core theta function and powers of one-step cluster variables with leading product label m . Classical positive theta multiplication puts the support of its leading output θ_m inside that finite product. This verifies all-label finiteness at each step. Lemma 11.11 then gives every compatible realization.

On the quantum side the corresponding bases, also indexed by all labels, are given by Theorem 9.3, Corollary 11.9, the Markov argument in Section 12.3, and Theorem 9.6 with Corollary 11.8. Their classical specializations have the same labels. Lemma 11.3 proves the isomorphism. $\square$

12 Low-rank applications

12.1 Non-cluster families in rank three

Consider principal coefficients and the standard compatible form for

$$B = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}, \quad D = \text{diag}(1, 2, 1).$$

The matrix is connected, skew-symmetrizable, singular, and acyclic. For

$$m_r = (r, 0, -r; 0, 0, 0), \quad r \in \mathbb{Z}_{\geq 1},$$

the acyclic bound gives

$$\text{supp } \Theta_{m_r} \subseteq \{m_r + pa : 0 \leq a \leq (2r, r, r)\}.$$

These are not cluster-monomial labels. To verify this, use the acyclic valued representation description [10, Theorem 7.3 and Corollaries 7.4–7.5]. With $H = [-B]_+$, a cluster monomial has mutable leading degree $(H - I)d + s$, where d is a rigid representation dimension vector, $s \geq 0$ records initial-variable factors, and $s_i d_i = 0$. In the quotient-dimension convention for the cluster character, the leading degree is $(H - I)d$. The counting-polynomial result for rigid representations in the same reference gives this formula over the formal parameter v .

Here $(H - I)(1, 1, 1)^T = (1, 0, -1)^T$. The assumed label would force

$$d = r(1, 1, 1)^T + (I - H)^{-1}s.$$

The inverse is $I + H + H^2 \geq 0$ with diagonal ones. If any $s_i > 0$ then $d_i > 0$, a contradiction. Thus $s = 0$, but the Euler quadratic form at $r(1, 1, 1)$ is zero, whereas a nonzero rigid object has Euler value $\dim \text{End} > 0$.

Likewise, for

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 3 \\ 0 & -1 & 0 \end{pmatrix}, \quad D = \text{diag}(2, 1, 3), \quad m_r = (-r, 0, r; 0, 0, 0),$$

the support box is $0 \leq a \leq (r, 2r, 2r)$. The same calculation forces rigid dimension $r(1, 2, 1)$, whose Euler value is $-r^2$, again impossible. After mutation at vertex two the matrix is

$$B_c = \begin{pmatrix} 0 & -1 & 3 \\ 2 & 0 & -3 \\ -2 & 1 & 0 \end{pmatrix}.$$

For endpoint anchor $(11, 17, 23)$ the words 21, 23, 31, executed left to right, have rows

$$e_1^T A_{21} = (-1, 0, 0), \quad e_3^T A_{23} = (0, -1, 2), \quad e_1^T A_{31} = (5, 0, -3).$$

Their weighted sum with weights 6, 1, 1 is $(-1, -1, -1)$. All selected second-step endpoint coordinates are positive: after the first step they are read from $(11, -17, 40)$ or $(80, 17, -23)$. Formula (7) gives offsets $r, 0, 0$ on these rows for m_r . Thus the cyclic-chart support satisfies $a_1 + a_2 + a_3 \leq 6r$. The mutated original principal coefficient block is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$; the charge bound also applies to a new principal realization at this seed.

12.2 A rank-four class with no acyclic seed

Take

$$B^{(0)} = \begin{pmatrix} 0 & 2 & -1 & -1 \\ -2 & 0 & 1 & 1 \\ 2 & -2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix}, \quad D = \text{diag}(2, 2, 1, 1),$$

with initial principal coefficients at seed r . The mutation class, up to relabeling, is closed on the following additional representatives:

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 2 & 2 & 0 & -2 \\ -2 & -2 & 2 & 0 \end{pmatrix}.$$

In the table, (j, π) means $\mu_k(B^{(j)})_{ab} = B_{\pi(a), \pi(b)}^{(j)}$; permutation strings list the four images.

$i \setminus k$	1	2	3	4
0	(0, 2134)	(0, 2134)	(1, 1234)	(1, 1243)
1	(2, 1234)	(2, 1243)	(0, 1234)	(0, 2134)
2	(1, 1234)	(1, 1243)	(2, 1243)	(2, 1243)

Each identity follows by the matrix mutation formula, and all permutations preserve D . Each representative contains a directed cycle: respectively 1231, 13241, and 1431. Mutation commutes with relabeling, so induction on arbitrary word length proves that no seed is acyclic.

Nevertheless the word $(3, 2, 1, 3, 4, 2, 1, 4)$ is green and ends at $B^{(0)}$ with principal c -matrix $-I$. Its successive selected c -columns are

$$e_3, e_2, e_1 + 2e_3, e_1 + e_3, e_2 + e_4, e_2 + 2e_4, e_1, e_4.$$

They are all positive. At the final seed s , $p_s = \begin{pmatrix} B^{(0)} \\ -I \end{pmatrix}$ and the inverse positive-chamber map is $L = \text{diag}(-I, I)$. Thus Theorem 9.3 gives an upper theta basis in this non-mutation-acyclic class.

For a label with mutable coordinates $x = (x_1, x_2, x_3, x_4)$, write $d_i(x)$ for the i th component of $d_s(m)$; the frozen coordinates do not affect it. The exact support dimensions are

$$\begin{aligned} d_1(x) &= \max\{0, -x_1, -x_1 - x_3, -x_1 - x_4, -x_1 - x_3 - x_4, \\ &\quad -x_1 - 2x_2 - x_3 - x_4, -2x_1 - 2x_2 - x_3 - x_4\}, \\ d_2(x) &= \max\{0, -x_2, -2x_1 - x_2, -2x_1 - x_2 - x_3, -2x_1 - x_2 - x_4, \\ &\quad -2x_1 - x_2 - x_3 - x_4, -2x_1 - 2x_2 - x_3 - x_4\}, \\ d_3(x) &= \max\{0, -x_3, -2x_2 - x_3, -2x_1 - 2x_2 - x_3, -2x_1 - 2x_2 - 2x_3\}, \\ d_4(x) &= \max\{0, -x_4, -2x_2 - x_4, -2x_1 - 2x_2 - x_4, -2x_1 - 2x_2 - 2x_4\}. \end{aligned}$$

Apply (7) along $(4, 1, 2, 4, 3, 1, 2, 3)$ on all 2^8 sign branches. Seventy-two branches have strict rational witnesses. On each of the other 184 branches a nonzero nonnegative combination of the strict sign rows is zero, proving infeasibility. On each feasible branch, the active gradient minus every displayed competing gradient is a nonnegative rational combination of its sign rows. The 1440 resulting equalities prove the maxima on the closures of the branches; continuity covers their boundaries. The companion certificate lists these rational vectors and verifies the equalities after reconstructing each mutation branch. For example $x = (-r, 0, 0, 0)$ gives $d_s = 2r(1, 1, 1, 1)$ for every $r > 0$.

12.3 The Markov matrix

Let

$$B = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}.$$

Every positive diagonal skew-symmetrizer is $D = dI$. For every compatible integral $p = \begin{pmatrix} B \\ C \end{pmatrix}$ and Λ , all theta labels are finite and their theta functions form an R -basis of $\mathcal{U}$.

Indeed, Zhou [11, Theorem 5.1] proves the full classical theta-basis statement for A_{prin} of the once-punctured torus. His row pairing matrix is B^T in our column convention, so its principal model is the one used here. Davison–Mandel positivity identifies the full quantum and classical supports in the standard principal model for $d = 1$; replacing the formal parameter by v^d gives every positive d . Theorem 11.5, with classical valuative independence from [3], gives its quantum upper basis. Proposition 10.2 and Theorem 10.1 transfer it to every compatible realization.

There is no reddening sequence. Every principal matrix in the mutation history is B or $-B$. By sign coherence, if the selected c -column has sign ε , the column-sum update is

$$\sum_j c'_j = \sum_j c_j + \left(-2 + \sum_{j \neq k} [\varepsilon b_{kj}]_+ \right) c_k = \sum_j c_j.$$

It remains $(1, 1, 1)^T$ from the initial identity matrix. A terminal matrix with all columns nonpositive is therefore impossible. In the standard principal model the degree assigning one to mutable variables and zero to frozen variables is also preserved by mutation. Every global monomial has nonnegative mutable exponents in its monomial seed: a negative exponent would require a Laurent unit to be divisible by a nonunit adjacent exchange polynomial. Its degree is consequently nonnegative. Evaluation at the corresponding nonzero tropical degree vector shows that the A_{prin} enough-global-monomials condition of [4, Definition 4.17] fails.

A compatible nonsaturated example has one frozen variable:

$$p = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -1 & 1 & -2 \\ 1 & 0 & -1 & -2 \\ -1 & 1 & 0 & -2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

Here $p^T \Lambda = (6I, 0)$ and the Smith invariants are $(1, 2, 6)$, so $M/pN \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6$.

For arbitrary compatible Markov data, take e_1 with zero frozen part and list charges

$$a_0 = 000, \ a_1 = 100, \ a_2 = 101, \ a_3 = 102, \ a_4 = 112, \ a_5 = 122, \ a_6 = 222,$$

where each three-digit string denotes its coordinate triple. Then

$$\Theta_{-e_1} = \sum_{j \in \{0,1,3,5,6\}} X^{-e_1 + pa_j} + (v^d + v^{-d})(X^{-e_1 + pa_2} + X^{-e_1 + pa_4}). \quad (16)$$

Zhou's negative-basic theta formula gives these seven classical support positions, and the quantum support is the same. On a negative k th-exponent slice of degree $-r$, membership in S_k is divisibility along the pe_k orbit by

$$\prod_{s=1}^r (1 + v^{(r+1-2s)d} z).$$

This follows by expanding $(X'_k)^r$ in based monomials; multiplying by an exponent with k th coordinate zero gives the same phase in all orbit positions. For the seven positions, the negative slices are $\{a_0, a_1\}$ and $\{a_5, a_6\}$ at $k = 1, r = 1$, $\{a_3, a_4, a_5\}$ at $k = 2, r = 2$, and $\{a_1, a_2, a_3\}$ at $k = 3, r = 2$. Writing $c_j = [X^{-e_1 + pa_j}] \Theta_{-e_1}$, the coefficients must satisfy

$$c_0 = c_1 = c_3 = c_5 = c_6, \quad c_2 = (v^d + v^{-d})c_1, \quad c_4 = (v^d + v^{-d})c_3.$$

Conversely these conditions put the polynomial in every S_k . There is exactly one with $c_0 = 1$, proving (16). In the four-variable example it reads

$$\begin{aligned} \Theta_{-e_1} = & X^{(-1,0,0,0)} + X^{(-1,2,-2,1)} + (v^6 + v^{-6})X^{(1,0,-2,2)} + X^{(3,-2,-2,3)} \\ & + (v^6 + v^{-6})X^{(1,-2,0,4)} + X^{(-1,-2,2,5)} + X^{(-1,0,0,6)}. \end{aligned}$$

Corollary 11.12 also gives $\mathcal{U}_R/(v-1)\mathcal{U}_R \simeq \mathcal{U}_{\text{cl}, \mathbb{Z}}$ in this Markov class.

12.4 One-way Markov extensions

Write B_M for the preceding Markov matrix. For an integer $r \geq 2$ and a nonzero column $c \in \mathbb{N}^3$, define

$$B_{r,c} = \begin{pmatrix} B_M & -rc \\ c^T & 0 \end{pmatrix}, \quad D_{r,c} = \text{diag}(1, 1, 1, r).$$

The matrices $B_{r,c}$ are connected, skew-symmetrizable, and not skew-symmetric. A compatible principal realization is

$$p = \begin{pmatrix} B_{r,c} \\ I_4 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -D_{r,c} \\ D_{r,c} & -D_{r,c}B_{r,c} \end{pmatrix}.$$

Corollary 12.1. *For every compatible realization with symmetrizer $D_{r,c}$, all theta labels for $B_{r,c}$ are finite and their theta functions form an R -basis of $\mathcal{U}$. For each label m , every support charge satisfies*

$$0 \leq a_4 \leq [-m_4]_+, \quad 0 \leq a_i \leq 2 \sum_{j=1}^3 [rc_j[-m_4]_+ - m_j]_+ \quad (1 \leq i \leq 3). \quad (17)$$

The matrix $B_{r,c}$ admits no reddening sequence and is not mutation-acyclic.

Proof. Direct multiplication gives $p^T \Lambda = (D_{r,c}, 0)$. Freeze vertex four. The core is a compatible Markov seed with symmetrizer I_3 and additional frozen coefficients, so all its theta labels are finite by the Markov result above. Equation (9) holds with $F = \{4\}$ and $H = 0$. Theorem 9.6 gives finiteness, and Corollary 11.8 gives the upper basis. For the support bound, $t_4(m) = [-m_4]_+$ and the core coordinates are

$$b_i = m_i - rc_i[-m_4]_+ \quad (1 \leq i \leq 3).$$

Each negative basic Markov theta function has charges with every coordinate at most two, by (16) and cyclic permutation. Positive basic theta functions and frozen monomials have charge zero. A product of these functions with leading label b therefore has core charge coordinates at most $2 \sum_j [-b_j]_+$. Proposition 6.3 places the entire support of Θ_b in this product. Equation (10) gives the stated box.

Cao–Li [12, Lemma 4.9] prove that existence of a green-to-red sequence for a skew-symmetrizable matrix passes to its principal submatrices. This is the valued version of the scattering deletion and projection argument in [14, Section 5.1]. The Markov principal submatrix has no such sequence, so $B_{r,c}$ has no reddening sequence. Since mutation-acyclic matrices admit reddening sequences, it is not mutation-acyclic. $\square$

The bound gives charge height at most

$$6 \sum_{j=1}^3 [rc_j[-m_4]_+ - m_j]_+ + [-m_4]_+.$$

For $r = 2$, $c = e_1$, and $m = -e_4$, this is 13. For $c = e_1$ and $h \in \mathbb{Z}_{\geq 0}$, Lemma 9.5 gives the explicit family

$$\begin{aligned} \Theta_{-e_1+he_4} = & \sum_{a \in \{000,100,102,122,222\}} X^{-e_1+he_4+p_I a} \\ & + (v + v^{-1})(X^{-e_1+he_4+p_I(101)} + X^{-e_1+he_4+p_I(112)}). \end{aligned}$$

Here the three-digit strings denote charges in the Markov core and p_I consists of its columns in the full extended matrix. The core symmetrizer is I_3 , which gives the parameter v in these coefficients.

Proposition 12.2. *For B_{2,e_1} with standard principal coefficients, let $e_1, \dots, e_8$ be the standard basis of the exponent lattice, so $y_4 = X^{e_8}$. Let $\Theta_{-e_1}^I$ be the seven-term Markov theta function for the seed obtained by freezing vertex four, embedded in the same based quantum torus. Then*

$$\Theta_{-e_4} = X^{-e_4} + X^{-e_4+e_8} (\Theta_{-e_1}^I)^2.$$

The right side has 26 nonzero based-monomial terms.

Proof. Equation (17) places the support in the 250 charge positions $0 \leq a_1, a_2, a_3 \leq 4$, $0 \leq a_4 \leq 1$. The charge-face quotient in Lemma 9.5 makes its $a_4 = 0$ part equal to X^{-e_4} , because the frozen label has zero mutable core coordinates. Membership in each adjacent torus imposes the centered Gaussian divisibility conditions used in (16).

These conditions form a linear system over R in the 250 charge coefficients. The accompanying exact certificate constructs this system directly from B_{2,e_1} and $D = \text{diag}(1, 1, 1, 2)$. It verifies all 90 nontrivial Laurent-polynomial remainder tests for the displayed candidate. After specialization $v = 2$ over $\mathbb{F}_{1000003}$, the homogeneous system, including the fixed zero-charge face, has column rank 250. Since its entries are Laurent polynomials and 2 is nonzero in that field, a nonzero specialized maximal minor proves full column rank over $\mathbb{Q}(v)$. Thus there is at most one candidate in the stated box with the required face and adjacent divisibilities. The theta function satisfies these conditions, which identifies the candidate.

To obtain the compact expression, expand the seven-term core formula by based-monomial multiplication using the original eight-dimensional Λ . The square has 25 terms. The vector $-e_4 + e_8$ pairs to zero with $-2e_1$ and every $p_I a$, and $p e_4 = -2e_1 + e_8$. Multiplying this square on the left by $X^{-e_4 + e_8}$ therefore puts these terms on the charge face $a_4 = 1$, with their unchanged coefficients. Adding X^{-e_4} gives exactly the verified 26-term candidate. The companion data contain the coefficient table and exact divisibility and multiplication checks. $\square$

Supplementary data. The companion archive `quantum_theta_supplement_20261012.zip`, supplied with this manuscript, contains the 26-term coefficient table at `spanning/incoming/theta_minus_e4.json`, the divisibility and uniqueness check `spanning/incoming/verify_explicit.py`, and the multiplication check `spanning/check_compact_formula.py`. Its `README.md` gives the commands and explains the certificate's dependence on the support bound, frozen face and mutation results used in Proposition 12.2. The archive also contains the rational rank-four support certificate and its checker.

For comparison with Proposition 9.1, every elementary Markov charge map satisfies

$$J_{k,\sigma}(\pm B_M)\mathbf{1} = \mathbf{1}, \quad \mathbf{1} = (1, 1, 1)^T,$$

because the changed coordinate is $-1 + 2 = 1$. Since every mutated matrix is $\pm B_M$, one has $A_w \mathbf{1} = \mathbf{1}$ for every word. The transported leading term has charge $b_w(m) \geq 0$. Hence all vectors $t\mathbf{1}$, $t \in \mathbb{N}$, satisfy every word constraint $A_w a + b_w(m) \geq 0$. Even the full collection of these constraints is unbounded, although the theta functions are finite.

Proof of Theorem 1.3. The first assertion is Theorem 9.3, the second is Theorem 10.1, and the third is Theorem 11.5. The Markov calculation and transfer in Section 12 prove the fourth assertion, including the absence of a reddening sequence. Theorem 9.6 and Corollaries 11.8 and 12.1 give the fifth assertion. Corollary 11.9 gives the sixth. $\square$

13 Further questions

Does $\mathcal{M}_\Theta = \mathcal{U}$ hold for every integral compatible skew-symmetrizable seed? Proposition 10.2 reduces the question to standard principal coefficients. Theorem 11.5 applies when a classical theta basis has finite quantum lifts. Theorem 11.7 applies to one-way extensions of a quantum theta upper basis. For mixed-sign couplings, its exchange monomial C can contain mutable variables and need not be invertible in the frozen-core upper algebra, so the monic division requires a different argument.

A second problem is to characterize Θ_{fin} from the mutation data. It is the set of lattice points in its conical hull by Corollary 6.6. The Markov example shows that the finite-word

inequalities of Proposition 9.1 need not bound the support even when every theta label is finite.

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