

REPORT: A QUASI-RIEMANN HYPOTHESIS FOR DEDEKIND ZETA FUNCTIONS

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ABSTRACT. Recent work on Artin's primitive root conjecture establishes a zero-free half-plane $\operatorname{Re} s > 1 - 10^{-6}$ for finite-order Hecke L -functions over cyclotomic fields containing $e^{2\pi i/12}$. Following their strategy, we prove that if K is any number field, then the Dedekind zeta function $\zeta_K(s)$ does not vanish in the half-plane $\operatorname{Re} s > 31/32$. As a corollary, the L -function of any finite-order Hecke character defined over K also does not vanish in the half-plane $\operatorname{Re} s > 31/32$, wherever it is holomorphic. These results were proved and formalized in Lean using AxiomProver. Compared to the work on Artin's primitive root conjecture, the hardest part was the formalization of the work of Kazhdan and Patterson on metaplectic forms in order to accommodate all number fields.

1. INTRODUCTION

1.1. Recent work on Dirichlet L -functions. The Riemann zeta function is defined by an absolutely convergent Dirichlet series that factors as an Euler product over primes p :

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}} \quad (\operatorname{Re} s > 1).$$

Riemann proved that $\xi(s) = s(1-s)\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$. We may therefore identify $\zeta(s)$ with its analytic continuation when $\operatorname{Re} s \leq 1$. The zeros of $\zeta(s)$ govern the distribution of primes through the (conditionally convergent) explicit formula: if $x \geq 2$ is not a prime power, then

$$\sum_{k=1}^{\infty} \sum_{p^k \leq x} \log p = x - \sum_{\zeta(\rho)=0} \frac{x^{\rho}}{\rho} - \frac{\zeta'}{\zeta}(0).$$

Zeros are counted with multiplicity, with the nontrivial zeros summed by symmetric truncation in height; the trivial-zero sum converges absolutely. See [6, Chapters 17–18].

In 1896, Hadamard and de la Vallée Poussin independently proved that $\zeta(s) \neq 0$ in the half-plane $\operatorname{Re} s \geq 1$, from which one deduces the prime number theorem

$$\#\{p \leq x\} \sim \int_2^x \frac{dt}{\log t}.$$

If $\theta < 1$ is a fixed constant such that $\zeta(s) \neq 0$ for $\operatorname{Re} s > \theta$, then

$$\#\{p \leq x\} = \int_2^x \frac{dt}{\log t} + O(x^{\theta} \log x).$$

The Riemann hypothesis asserts that $\theta = \frac{1}{2}$ is permissible. Platt and Trudgian rigorously verified that $\zeta(s) \neq 0$ when $\operatorname{Re} s > \frac{1}{2}$ and $|\operatorname{Im} s| \leq 3 \cdot 10^{12}$ [32].

In 1957, I. M. Vinogradov [36] and N. M. Korobov [22] independently proved that there exists an absolute constant $c > 0$ such that

$$\zeta(\sigma + it) \neq 0, \quad \sigma \geq 1 - \frac{1}{c(\log |t|)^{2/3}(\log \log |t|)^{1/3}}, \quad |t| \geq 3.$$

Ford proved that $c = 49.13$ is permissible when $|t|$ is suitably large, and $c = 57.54$ for all t [9, Theorems 1 and 5], from which one can prove

$$\#\{p \leq x\} = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-0.2098 \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right).$$

See also Khale's explicit Vinogradov–Korobov zero-free region for Dirichlet L -functions [19].

OpenAI has released two proofs of nonvanishing in a fixed half-plane: $\operatorname{Re} s > 7/8$ in [28] and $\operatorname{Re} s > 11/12$ in the shorter argument [29]. Both also treat Dirichlet L -functions and finite-order Hecke L -functions over $\mathbb{Q}(\sqrt{-3})$. Their conclusion replaces a boundary tending to $\operatorname{Re} s = 1$ with a fixed boundary of the form $\operatorname{Re} s = 1 - \theta$, a monumental advance.

1.2. The main result. Let K be a number field and $\mathcal{O}_K$ its ring of integers, a Dedekind domain. Let $\mathfrak{a}$ (resp. $\mathfrak{p}$) denote a nonzero (resp. nonzero prime) ideal of $\mathcal{O}_K$. The norm of $\mathfrak{a}$ is $N\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$. Unique factorization of nonzero ideals gives

$$(1.1) \quad \zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}, \quad \operatorname{Re} s > 1.$$

For $K = \mathbb{Q}$ this is the Riemann zeta function. More generally, if $K = \mathbb{Q}(e^{2\pi i/m})$, abelian class-field theory gives

$$\zeta_{\mathbb{Q}(e^{2\pi i/m})}(s) = \zeta(s) \prod_{\substack{\chi \pmod{m} \\ \chi \text{ nontrivial}}} L(s, \chi^*),$$

where χ^* is the primitive Dirichlet character inducing χ . In particular, the common zero-free half-plane proved below for Dedekind zeta functions also holds for the Riemann zeta function and Dirichlet L -functions.

Let D_K be the absolute discriminant of K , and let r_1 and r_2 count real embeddings and pairs of complex embeddings $K \hookrightarrow \mathbb{C}$, so that $[K : \mathbb{Q}] = r_1 + 2r_2$. The function ζ_K continues meromorphically to $\mathbb{C}$, with a simple pole at one. Moreover, the function

$$(1.2) \quad \xi_K(s) = s(1-s) D_K^{s/2} [\pi^{-s/2} \Gamma(s/2)]^{r_1} [2(2\pi)^{-s} \Gamma(s)]^{r_2} \zeta_K(s)$$

is entire and satisfies the functional equation

$$(1.3) \quad \xi_K(s) = \xi_K(1-s).$$

We may therefore identify ζ_K with its analytic continuation via ξ_K when $\operatorname{Re} s \leq 1$.

The zeros of ζ_K govern the distribution of prime ideals. Define $c_K^{(0)}$ by the Laurent expansion

$$\frac{\zeta'_K(s)}{\zeta_K(s)} = \frac{\operatorname{ord}_{s=0} \zeta_K(s)}{s} + c_K^{(0)} + O(s) \quad (s \rightarrow 0).$$

If $x \geq 2$ is not the norm of a power of a prime ideal, then we have the following (conditionally convergent) explicit formula:

$$\sum_{k \geq 1} \sum_{N\mathfrak{p}^k \leq x} \log N\mathfrak{p} = x - \sum_{\substack{\zeta_K(\rho)=0 \\ \rho \neq 0}} \frac{x^\rho}{\rho} - c_K^{(0)} - (\log x) \operatorname{ord}_{s=0} \zeta_K(s).$$

As before, zeros are counted with multiplicity, with the nontrivial zeros summed by symmetric truncation in height; the trivial-zero sum converges absolutely. The constant Laurent coefficient is needed because ζ_K may vanish at zero.

Landau [24] proved that the ideas of Hadamard and de la Vallée Poussin generalize to Dedekind zeta functions, so $\zeta_K(s) \neq 0$ for $\operatorname{Re} s \geq 1$. This leads to

$$\#\{\mathbf{Np} \leq x\} \sim \int_2^x \frac{dt}{\log t}.$$

If there exists a constant $\theta < 1$ such that $\zeta_K(s) \neq 0$ for $\operatorname{Re} s > \theta$, then

$$\#\{\mathbf{Np} \leq x\} = \int_2^x \frac{dt}{\log t} + O(x^\theta \log(D_K x^{[K:\mathbb{Q}]})).$$

The Vinogradov–Korobov method also gives zero-free regions for Dedekind zeta functions. In 1968, A. V. Sokolovskii [33] proved that for all number fields K , there exists a constant $c_K > 0$ such that

$$\zeta_K(\sigma + it) \neq 0 \quad \sigma \geq 1 - \frac{c_K}{(\log(|t| + 3))^{2/3} (\log \log(|t| + 3))^{1/3}},$$

from which one can deduce that there exists another constant $c'_K > 0$ such that

$$\#\{\mathbf{Np} \leq x\} = \int_2^x \frac{dt}{\log t} + O_K\left(x \exp\left(-c'_K \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right).$$

See also the related work of K. M. Bartz [1, 2], Hinz [16], and Coleman [5].

OpenAI proved that the L -function of any finite-order Hecke character defined over cyclotomic fields containing the group μ_{12} of twelfth roots of unity does not vanish when $\operatorname{Re} s > 1 - 10^{-6}$ [30, Theorem 1.2]. By abelian factorization and descent, they obtain the same zero-free half-plane for the Kummer fields considered in [30, Section 9]. Building on [30], we prove the following result.

Theorem 1.1. *If K is any number field, then $\zeta_K(s) \neq 0$ when $\operatorname{Re} s > 31/32$ and $s \neq 1$.*

By abelian class field theory, Theorem 1.1 immediately implies the following result.

Corollary 1.2. *If χ is a finite-order Hecke character defined over a number field K , then its Hecke L -function $L(s, \chi, K)$ does not vanish when $\operatorname{Re} s > 31/32$, wherever it is holomorphic.*

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2. SUMMARY OF THE STRATEGY

2.1. The reciprocal and the required saving. Fix a number field F containing μ_{12} . Following [30, Sections 4–8], we reduce nonvanishing of ζ_F to a bound for a smoothed reciprocal.

Let S be a fixed finite set of places containing all infinite places, and write ζ_F^S for ζ_F with the finite Euler factors at S omitted. For the moment, let H be a holomorphic, nowhere-zero function on $\operatorname{Re} s > 2/3$, with H and its reciprocal bounded on every closed half-plane strictly inside this region. The Poisson calculation will specify it as the product in (11.3).

We write $\int_{(c)}$ for an integral upward along $\operatorname{Re} s = c$, and $\hat{g}(s) = \int_0^\infty g(x)x^{s-1} dx$ for the Mellin transform of a function on $(0, \infty)$. Additive Fourier transforms on products of local fields use (7.1). Therefore, a hat denotes a Mellin transform for functions of a positive real variable and an additive transform for functions on those products.

For $Z > 0$, define the absolutely convergent integral

$$(2.1) \quad f(Z) = \frac{1}{2\pi i} \int_{(2)} Z^{s-61/100} e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)} ds.$$

Its Mellin transform satisfies

$$(2.2) \quad \int_0^\infty f(Z) Z^{-s+61/100} \frac{dZ}{Z} = e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}.$$

Initially, (2.2) holds on the line $\operatorname{Re} s = 2$; Proposition 6.1 gives the convergence and continuation argument. The numerator has no zeros. Consequently, holomorphic continuation of this transform excludes zeros of the denominator.

The goal is to prove that if $\varepsilon > 0$, then

$$(2.3) \quad |f(Z)| \ll_\varepsilon Z^{287/800+\varepsilon} \quad (Z \geq 2).$$

Here F, S, H , and hence f , are fixed before ε is chosen. The shift $61/100$ in (2.1) means that this bound continues the transform to $\operatorname{Re} s > 61/100 + 287/800 = 31/32$. Equivalently, the saving from the power $Z^{1-61/100}$ corresponding to $s = 1$ is

$$(2.4) \quad \frac{39}{100} - \frac{287}{800} = \frac{1}{32}.$$

Both (2.3) and the nonvanishing of H are needed; a zero of the numerator could cancel the pole produced by a zero of $\zeta_F^S(s)$.

The weighted average constructed in Section 5.6 gives (2.3) through two evaluations. The cubic transformation supplies an upper bound, while Poisson summation identifies a principal term proportional to f . Both evaluations use the same arithmetic data and smoothing functions, fixed independently of ε .

2.2. Nonvanishing in $\operatorname{Re} s > 11/12$. We describe the shorter argument of [29], emphasizing the cubic transformation on which it depends. Throughout this subsection, $F = \mathbb{Q}(\sqrt{-3})$, and η is a fixed finite-order Hecke character of F . For the conclusion concerning the Riemann zeta function, take $\eta = \mathbf{1}$ and use

$$L(s, \mathbf{1}, F) = \zeta_F(s) = \zeta(s) L(s, \chi_{-3}),$$

where χ_{-3} is the primitive quadratic Dirichlet character modulo 3. Thus nonvanishing of ζ_F implies nonvanishing of ζ ; the same argument with characters induced by Dirichlet characters gives the Dirichlet L -function conclusion [29, Section 3].

The proof estimates the Möbius coefficients of the reciprocal by averaging auxiliary sextic twists. Every ideal $\mathfrak{a}$ prime to 3 has a unique primary generator $a \equiv 1 \pmod{3}$. For $\mathfrak{a}$ prime to 6, put $\chi_{\mathfrak{a}}(u) = (u/a)_6$, the sextic residue symbol, extended by zero when $(u, a) \neq 1$, and fix a finite set S containing the primes above 6 and those dividing the conductor of η . For the ideal Möbius function μ_F , a fixed $W \in C_c^\infty(0, \infty)$, and $D \geq 2$, define, for $u \in \mathcal{O}_F$,

$$(2.5) \quad A_u(D) = \sum_{(\mathfrak{a}, S) = \mathcal{O}_F} \mu_F(\mathfrak{a}) \eta(\mathfrak{a}) \chi_{\mathfrak{a}}(u) W\left(\frac{N\mathfrak{a}}{D}\right).$$

The central estimate is

$$(2.6) \quad \sum_{\substack{u \in \mathcal{O}_F \\ 0 < N(u) \leq H}} |A_u(D)|^2 \ll_{\eta, S, W, \kappa, \varepsilon} D^{1+\varepsilon} H, \quad H = D^{1+\kappa}, \quad 0 < \kappa \leq \frac{1}{10},$$

where $N(u) = N(u\mathcal{O}_F)$ [29, Proposition 3.1]. There are $\asymp H$ values of u in this range, so the estimate gives square-root cancellation in sums of length D , on average over u . The argument has the following ingredients.

- (1) **Poisson summation and Gauss–Jacobi identities.** Poisson summation in the averaging variable u replaces the sextic characters in (2.6) by their finite Fourier transforms, which are sextic Gauss sums. Classical Gauss–Jacobi identities, in the form used by

Heath-Brown [14, equation (2)], combine these Gauss sums with the Möbius coefficients to give cubic Gauss sums, multiplied by a fixed angular character and fixed ray-class factors [29, Section 4 and Appendix A.1]. After the diagonal terms and common divisors have been separated, the required estimate is therefore a mean square of weighted cubic Gauss sums; this is the reason for introducing sextic twists in (2.5).

- (2) **Cubic theta coefficients and completion by cubes.** Kubota's theory and Patterson's construction provide a cubic theta function whose squarefree Fourier coefficients are given explicitly in terms of normalized cubic Gauss sums [23], [31, Theorem 8.1]. The full Fourier expansion also contains indices of the form $(\sqrt{-3})^{-3}cb^3$, where c, b are primary and c is squarefree; the ideals generated by c and b may share primes. Following Dunn–Radziwiłł's completion by cube factors, OpenAI adjoins these terms so that the sum obtained in item (1) becomes part of a weighted sum of Fourier coefficients to which the theta transformation applies [8, Lemma 5.4 and Proposition 5.3], [29, Sections 5.1 and 6.1]. Heath-Brown and Patterson had used the connection between cubic Gauss sums and theta coefficients in their study of Kummer sums at prime arguments [15].
- (3) **Theta transformation and the quadratic large sieve.** Dunn–Radziwiłł give explicit Fourier expansions of the same cubic theta function at the cusps reached by the transformation [8, Section 5 and Appendix A]; OpenAI incorporates the fixed ray-class twists in [29, Appendix A.2]. These formulas show how the character dependence changes: if the twisting parameter h is squarefree, then at a prime $\mathfrak{p} \mid (h)$ the finite Fourier factor and the theta factor combine as

$$\chi_{\mathfrak{p}}^{-1}\chi_{\mathfrak{p}}^{-2} = \chi_{\mathfrak{p}}^{-3} = \chi_{\mathfrak{p}}^3.$$

The transformed coefficients are therefore twisted by quadratic characters [29, Proposition 6.2]. A sum originally supported at norm scale X has transformed scale $N(h)^2/X$, and, after fixing the ray class of h , its coefficient sequence and smooth weight are independent of h [29, Lemma 6.3]. Goldmakher–Louvel's quadratic large sieve, extending Heath-Brown's real-character estimate, can now average these sums over h [13], [11, Theorem 1.1 and Corollary 1.2]. For two squarefree ideal ranges of lengths M and N , this sieve costs $(M+N)(MN)^\varepsilon$; combining it with Cauchy–Schwarz in the cube variable gives the completed mean-square estimate in [29, Proposition 5.2], including the extra terms needed when h is not squarefree.

- (4) **Removing the cubes and closing the mean-square argument.** The estimate in item (3) concerns the completed theta expansion, whereas (2.6) requires the original squarefree coefficients. One cannot discard the added terms, since different cube contributions may cancel. Möbius inversion removes them, following Dunn–Radziwiłł [8, Proposition 5.3 and equation (8.2)], with earlier related completions in Patterson [31, Theorem 6.1] and Heath-Brown [14, Section 3]. The completed estimate suffices for small cube divisors; for larger divisors, applying it directly loses too much. OpenAI instead uses two further Poisson transformations to express the remaining mean squares in terms of the same family at smaller norm scales, and induction completes the proof of (2.6) [29, Sections 5.3–5.6 and 7].
- (5) **Sixth powers recover the untwisted Möbius sum.** Let p be the primary generator of a prime ideal $\mathfrak{p} \notin S$. Since a sextic character is trivial on sixth powers away from its modulus,

$$(2.7) \quad \chi_{\mathfrak{a}}(p^6) = \mathbf{1}_{\mathfrak{p} \nmid \mathfrak{a}}, \quad A_{p^6}(D) = A_1(D) + O_W(D/N\mathfrak{p}).$$

Landau's prime ideal theorem supplies $\asymp H^{1/6}/\log H$ such primes with $\frac{1}{2}H^{1/6} < N\mathfrak{p} \leq H^{1/6}$ [24]. Their sixth powers are distinct terms of (2.6); averaging (2.7) over them gives

$$|A_1(D)|^2 \ll_{\eta, S, W, \kappa, \varepsilon} D^{1+\varepsilon} H^{5/6} + D^2 H^{-1/3}.$$

With $H = D^{1+\kappa}$, the first term gives exponent $11/12 + 5\kappa/12 + \varepsilon$ for $A_1(D)$. Taking κ and the mean-square loss sufficiently small for each requested $\varepsilon > 0$ yields

$$(2.8) \quad A_1(D) \ll_{\eta, S, W, \varepsilon} D^{11/12+\varepsilon}$$

as in [29, Section 3].

- (6) **Mellin continuation excludes zeros.** Write $L^S(s, \eta, F)$ for the Euler product with the factors at S omitted, and put $\widehat{W}(s) = \int_0^\infty W(y)y^s dy/y$. Termwise integration in $\operatorname{Re} s > 1$ gives

$$(2.9) \quad \int_0^\infty A_1(D)D^{-s} \frac{dD}{D} = \frac{\widehat{W}(s)}{L^S(s, \eta, F)}.$$

The bound (2.8) makes the left side holomorphic for $\operatorname{Re} s > 11/12$, since $A_1(D)$ vanishes for sufficiently small D . At any hypothetical zero ρ in this half-plane, choose $W(y) = y^{-\rho}\phi(y)$, where ϕ is nonnegative, smooth, compactly supported, and not identically zero; then $\widehat{W}(\rho) > 0$, contradicting (2.9) [29, Section 3]. The omitted Euler factors are nonzero for $\operatorname{Re} s > 0$, so restoring them preserves the conclusion.

We emphasize that Dunn–Radziwiłł's unconditional coefficient and cusp formulas are essential inputs in items (2)–(4), while their GRH-conditional asymptotic for prime Gauss sums is not used.

The improvement from $11/12$ to $7/8$ refines the cubic method by introducing additional prime averages and subtracting rescaled averages to cancel the parts of local Euler factors independent of the varying sextic character [28, Sections 12 and 16]. The sharper exponent also requires stronger second and fourth moments and a more precise count of characters whose L -functions have zeros near one [28, Lemmas 17.1 and 18.1 and Proposition 19.2]. Extending these character-dependent local calculations and moment estimates would impose further requirements beyond the cubic transformation and quadratic sieve, making this refinement less suited to the present work on Dedekind zeta functions.

2.3. The passage to general Dedekind zeta functions. The preceding argument identifies the central arithmetic feature: cubic theta summation turns sextic twists into a family controlled by a quadratic large sieve. OpenAI's adelic construction in [30, Sections 3–8] uses this feature in a different comparison. It estimates one weighted average of completed theta sums twice: theta summation gives an upper bound, while Poisson summation produces a principal term containing a reciprocal Hecke L -function. The cube factors are retained, and zero detection controls the other Poisson terms. This argument therefore does not require a general-field version of the Möbius mean square (2.6).

Although [30, Theorem 1.2] is stated for cyclotomic fields, the constructions used in its proof are already formulated in terms of ideals, local fields, and adeles. We take $F = K(\zeta_{12})$, with K arbitrary, and explain which field hypotheses those constructions use. We use the principal-character case of the cited argument to study ζ_F . The conclusion for all finite-order Hecke characters in Corollary 1.2 will then follow from Theorem 1.1 by class-field factorization. The six ingredients below are supplied by the cited argument and its classical sources; the sharper detector and enlarged Euler domains are proved in Sections 11 and 12. The cyclotomic theorem alone does not imply the general-field statement: each use of its proofs must be justified under the

weaker field hypothesis. We make those checks below and alongside the corresponding statements in Sections 7 to 13, and then improve the numerical choices in the comparison.

- (1) **Ideal generators, units, and reciprocity.** Over $\mathbb{Q}(\sqrt{-3})$, primary generators identify ideals with a discrete set of elements. Over F , ideals need not be principal and the unit group may be infinite. OpenAI handles both points by choosing a finite set S containing the infinite places, the primes above 6, the ramified data, and enough primes to generate the ideal class group [30, Lemmas 3.1–3.2]. Every ideal of $R = \mathcal{O}_{F,S}$ is then principal, and generators are considered modulo $\Gamma = (R^\times)^6$. Dirichlet’s S -unit theorem makes the norm-one quotient compact, so generators can be chosen with bounded shape. Kummer theory and Hilbert reciprocity give a complementary subgroup of local sixth-power classes on which the Hilbert pairing is trivial; this makes reciprocity symmetric for the chosen generators [26, Chapters I, V and VI], [37]. These facts hold for any number field containing μ_{12} : that inclusion also makes -1 a sixth power, as required by the alternating pairing. No assumption on the class number or on the rank of the unit group remains.
- (2) **Gauss-sum phases and the desired Euler coefficient.** The Poisson calculation must produce -1 , the first coefficient of a reciprocal Euler factor of ζ_F . Let $\gamma_j(\mathfrak{p})$ denote the normalized Gauss sum of the j -th power of the sextic residue character, using the trace additive character. OpenAI constructs a unitary Hecke character ϑ with

$$(2.10) \quad \gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p})\vartheta(\mathfrak{p}) = -1 \quad (\mathfrak{p} \notin S)$$

[30, Lemma 3.4]. Its cubic part is the pullback of the Jacobi-sum character over $\mathbb{Q}(\zeta_3)$ by $N_{F/\mathbb{Q}(\zeta_3)}$. Davenport–Hasse lifting identifies its prime values through residue-field norms [7]; Weil’s quadratic indices and their product formula supply the remaining sign [38]. Both operations are valid for a nonnormal extension. Thus the phase correction needed to produce $1/\zeta_F$ is available over our F .

- (3) **A theta function with the required Fourier coefficients.** The adelic argument uses Kazhdan–Patterson’s cubic exceptional representation, constructed as a residue of an Eisenstein series [20, Section II.1 and Theorem II.2.1]. Their Whittaker results, with the published correction, allow Fourier coefficients to be factored into local coefficients [20, Corollary I.3.6 and Theorem II.2.5], [21]. A Whittaker functional extracts a Fourier coefficient associated with a fixed additive character; its uniqueness fixes the factorization up to scalars. For the cubic cover used here, with determinant parameter $c = 2$, Cai–Fan–She’s formulation gives dimension one also at places above three [4, Proposition 2.2]. OpenAI fixes the scalars, calculates the exterior coefficients, and constructs the fixed local weights in [30, Proposition 5.1, Corollary 5.2, Lemmas 5.3–5.5 and Appendix A]. These are general local and adelic constructions. They need the roots of unity and the arithmetic choices in item (1), rather than a cyclotomic description of F . In particular, exact Gaussian smoothing and the terms whose squarefree and cube factors share primes are part of the cited construction.
- (4) **The complete transformed sum and its quadratic sieve.** OpenAI’s local reflection formula changes the interaction between two varying squarefree ideals into a quadratic residue symbol [30, Lemmas 6.2–6.3]. Its proof also treats higher powers in the twisting parameter and retains the zero values of characters at common primes. Goldmakher–Louvel’s quadratic large sieve is already a number-field theorem, extending Heath-Brown’s real-character estimate [13], [11, Theorem 1.1]. To apply it, one must check reciprocity with finitely many class parameters and the primitive conductor of products of characters. These checks are supplied in [30, Lemma 6.4]: two coprime squarefree indices in the same

local class give a product character whose exterior conductor is exactly their product. This uses quadratic Kummer theory over F and the fixed local classes at S . Summing the higher-power contributions then gives the complete reflected moment in [30, Proposition 4.1]; no new large-sieve inequality is needed.

- (5) **Tate’s Fourier analysis and the Poisson evaluation.** Tate’s thesis treats the finite congruence sums and infinite-place integrals within one Fourier theory, with self-dual measures fixing their normalizations [35]. Its local zeta integrals identify Euler and gamma factors, and global Poisson summation yields the functional equation of Hecke L -functions. This theory applies to every number field. We recall its relevant consequences in Section 4, keeping the introductory discussion here to its role in the comparison.

OpenAI’s Poisson evaluation produces a principal term containing

$$\frac{H(s)}{\zeta_F^S(s)},$$

where S indicates omitted finite Euler factors. The product H consists of the remaining local corrections [30, Proposition 7.3]. Identity (2.10) identifies the desired first coefficient, while the full calculation retains common-prime terms and determines H . The explicit factors of H are holomorphic and nowhere zero for $\operatorname{Re} s > 2/3$; see (11.3). Their estimates depend on the residue-field norm and unitary character values, so the same noncancellation holds over our F .

- (6) **Zero detection and dependence on the field.** Other Poisson frequencies introduce varying Hecke L -functions in the denominator. OpenAI combines the conductor large sieve with a Möbius-polynomial zero detector to bound the number of characters having a zero in a rectangle near one [30, Lemma 8.1 and Proposition 8.2]. The classical inputs are Huxley’s additive sieve and Hecke’s analytic theory, in Tate’s formulation [17, 12, 35]. The exceptional characters retain contours to the right of one; for the others, a reciprocal bound permits a finite contour segment to move left. The degree of F enters the polynomial growth in height. The cited proof chooses the height cutoff after that exponent is fixed, then uses enough decay of the fixed smooth weights to bound the horizontal integrals [30, Section 8.2]. This order of choices allows field-dependent constants and thresholds without changing the saving in the norm variable.

These observations identify the broader scope of the existing argument. Its weighted average and Gaussian smoothing are those of [30]. We change the unequal norm scales to (5.4), retain the conductor dependence in convexity, and use the sharper detector and enlarged Euler domains below. The resulting upper-bound exponent and principal-term exponent differ by the saving (2.4). The refinement retains an arbitrary $\varepsilon > 0$ in the upper bound, with all functions and weights fixed independently of ε . All Poisson remainders have a larger saving. The argument in Proposition 6.1 therefore yields the strict open half-plane of width $1/32$; it makes no assertion on the boundary.

Finally, F/K is abelian of degree at most four. By class field theory, $\zeta_F(s)/\zeta_K(s)$ factors as a product of entire Hecke L -functions of the nontrivial characters of $\operatorname{Gal}(F/K)$ [26, Chapter VII]. A zero of ζ_K away from one would also be a zero of ζ_F . We give this short descent explicitly in Section 5.1; the same principle is used for noncyclotomic Kummer fields in [30, Section 9].

3. DEDEKIND ZETA FUNCTIONS

The proof of Theorem 1.1 uses three classical properties of Dedekind zeta functions: the Möbius series for the reciprocal, meromorphic continuation and the functional equation, and

factorization under abelian extension. The first identifies the coefficients to estimate, the second supplies analytic bounds, and the third passes from a field containing the required roots of unity to an arbitrary field. We record the classical facts needed for these steps. Standard sources are Neukirch, Weil, and Iwaniec–Kowalski [26, 37, 18]. In this section K is arbitrary. Write $n = [K : \mathbb{Q}] = r_1 + 2r_2$, where r_1 counts real embeddings and r_2 counts pairs of complex embeddings, and let D_K be the absolute discriminant.

3.1. Ideals, Euler factors, and convergence. The Dirichlet series and Euler product in (1.1) converge absolutely and locally uniformly for $\operatorname{Re} s > 1$. The ideal norm is multiplicative. If $p\mathcal{O}_K = \prod_{j=1}^g \mathfrak{p}_j^{e_j}$, with $N\mathfrak{p}_j = p^{f_j}$, the factor above p is $\prod_j (1 - p^{-f_j s})^{-1}$, and $\sum_j e_j f_j = n$. This includes ramified primes. For real $\sigma > 1$, $\zeta_K(\sigma) \leq \zeta(\sigma)^n$.

The ideal Möbius function μ_K vanishes on ideals divisible by a square of a prime ideal and equals $(-1)^j$ on products of j distinct prime ideals. For $\operatorname{Re} s > 1$, taking the reciprocal and logarithmic derivative of (1.1) gives

$$\frac{1}{\zeta_K(s)} = \sum_{\mathfrak{a}} \frac{\mu_K(\mathfrak{a})}{N\mathfrak{a}^s}, \quad -\frac{\zeta'_K(s)}{\zeta_K(s)} = \sum_{\mathfrak{a}} \frac{\Lambda_K(\mathfrak{a})}{N\mathfrak{a}^s},$$

where $\Lambda_K(\mathfrak{p}^j) = \log N\mathfrak{p}$, with zero values elsewhere. The fact that ζ_K is holomorphic away from a simple pole at $s = 1$ gives $\#\{\mathfrak{a} : N\mathfrak{a} \leq x\} \ll_K x$, and partial summation gives $\sum_{N\mathfrak{a} \leq x} N\mathfrak{a}^{-1} \ll_K \log(2x)$. All constants in these elementary estimates may depend on K .

3.2. Continuation, the residue, and the functional equation. Hecke’s analytic theory continues ζ_K meromorphically to $\mathbb{C}$, with a single, simple pole at one. The analytic class number formula gives its positive residue:

$$\kappa_K := \operatorname{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

Here h_K is the class number, R_K the regulator, and w_K the number of roots of unity in K . We use $R_K = 1$ when the unit rank is zero.

The completed function ξ_K in (1.2) is of order one and satisfies the functional equation (1.3). For its gamma factors, we use the standard notation

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s).$$

The factor $s(1-s)$ in (1.2) cancels the simple poles at zero and one of the product $D_K^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(s)$. The gamma factors record the real and complex embeddings, and the discriminant records the arithmetic lattice’s covolume. One classical proof applies Poisson summation to lattices in ideal classes; Tate’s thesis gives the adelic formulation [35].

3.3. Finite Euler factors and abelian factorization. For a fixed finite set S of places, write S_f for its finite places. Superscript S means that the finite Euler factors in S are omitted. Thus

$$\zeta_K^S(s) = \zeta_K(s) \prod_{\mathfrak{p} \in S_f} (1 - N\mathfrak{p}^{-s}), \quad R_S := \operatorname{Res}_{s=1} \zeta_K^S(s) = \kappa_K \prod_{\mathfrak{p} \in S_f} (1 - N\mathfrak{p}^{-1}) > 0.$$

The removed factors have zeros only on $\operatorname{Re} s = 0$, so this operation preserves zero-free statements in positive half-planes.

For an abelian extension L/K , class-field theory gives

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\chi \in \widehat{\operatorname{Gal}(L/K)} \\ \chi \text{ nontrivial}}} L(s, \chi, K).$$

The factors are the Hecke functions defined in the next section. All factors in the product over χ are entire. A zero of ζ_K , away from one, therefore persists in ζ_L . This is the descent principle in (5.1). No general Artin holomorphy conjecture is needed for our abelian descent.

4. HECKE CHARACTERS, THEIR L -FUNCTIONS, AND LARGE SIEVES

Hecke characters are the automorphic forms on $\mathrm{GL}(1)$, the multiplicative group of a number field. We recall their ideal and idelic descriptions, the analytic properties of their L -functions, and the three large-sieve estimates used below. The application of the quadratic sieve requires a particular family of characters; its hypotheses are checked in [Section 9.1](#).

4.1. Hecke characters in ideal and idelic language. A ray-class character modulo a modulus $\mathfrak{m}$ is a character of the group of fractional ideals prime to its finite part, trivial on principal ideals (a) with $a \equiv 1$ modulo that part and positive at the real places included in $\mathfrak{m}$. It has finite order. Its conductor is the smallest modulus through which it factors; the inducing character of that conductor is called primitive. Allowing a prescribed character on principal ideals at infinity gives the more general Hecke characters, or Grössencharaktere, introduced by Hecke [12].

Let K_v be the completion at a place v . The adèle ring $\mathbb{A}_K$ is the restricted product of the K_v , with respect to $\mathcal{O}_v$ at finite places; its idele group $\mathbb{A}_K^\times$ is the restricted product of $K_v^\times$ with respect to $\mathcal{O}_v^\times$. Thus an idele is a local unit at almost every finite place. The quotient by the diagonal group $K^\times$ imposes the global compatibility of the local characters.

Equivalently, a unitary Hecke character is a continuous homomorphism

$$\eta : K^\times \backslash \mathbb{A}_K^\times \rightarrow \{z : |z| = 1\}.$$

Its local characters are trivial on $\mathcal{O}_v^\times$ at almost all finite places. At an unramified prime set $\eta(\mathfrak{p}) = \eta_v(\varpi_v)$, where ϖ_v is a uniformizer; this extends multiplicatively to ideals outside the conductor. Triviality on $K^\times$ relates ideal values to the local components at the omitted places. Let $\mathcal{O}_{K,S}$ be the ring of elements integral outside S . When S contains the infinite places and all ramification of η , for $a \in K^\times$ we have

$$\eta(a\mathcal{O}_{K,S}) = \prod_{v \in S} \eta_v(a)^{-1}.$$

This identity is why ideal sums can be combined with Fourier analysis on $K_S = \prod_{v \in S} K_v$.

At complex places we use $|z|_v = |z|^2$. At a real place a unitary character is $\mathrm{sgn}(x)^{a_v} |x|^{it_v}$; at a complex place it is $(z/|z|)^{m_v} |z|^{it_v}$, with $a_v \in \{0, 1\}$, $m_v \in \mathbb{Z}$, and real t_v . Finite-order characters have $t_v = 0$ everywhere and $m_v = 0$ at complex places. The character η in $L(s, \eta, K)$ has finite order. The auxiliary character used to correct the cubic Gauss-sum phases has nontrivial angular type, so the proof also needs this broader class of unitary characters.

For a prime-power modulus, a character is extended by zero on nonunits. Even a power that becomes the principal character keeps this zero extension. This convention preserves coprimality conditions and is essential in both the Poisson and theta expansions.

4.2. Hecke L -functions and analytic bounds. For a primitive finite-order η of finite conductor $\mathfrak{f}$, put

$$L(s, \eta, K) = \sum_{\mathfrak{a}} \frac{\eta(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p} \nmid \mathfrak{f}} (1 - \eta(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}.$$

These expressions converge absolutely for $\mathrm{Re} s > 1$. Its reciprocal has coefficients $\mu_K(\mathfrak{a})\eta(\mathfrak{a})$. Hecke's continuation theorem makes it entire unless η is principal, when it is ζ_K . With the real

parities $a_v \in \{0, 1\}$ of η ,

$$\Lambda(s, \eta, K) = (D_K \mathfrak{N}f)^{s/2} \prod_{v \text{ real}} \Gamma_{\mathbb{R}}(s + a_v) \prod_{v \text{ complex}} \Gamma_{\mathbb{C}}(s) L(s, \eta, K)$$

satisfies $\Lambda(s, \eta, K) = \varepsilon(\eta) \Lambda(1 - s, \bar{\eta}, K)$, where $|\varepsilon(\eta)| = 1$. Hecke's theory and Tate's local zeta integrals are the sources of this statement [12, 35].

For fixed archimedean type the analytic conductor has size $D_K \mathfrak{N}f(2 + |t|)^n$, up to fixed factors. Convexity, obtained from the functional equation and the Phragmén–Lindelöf principle, gives the conductor-dependent bound

$$|L(\sigma + it, \eta, K)| \ll_{K, \varepsilon} \mathfrak{N}f^{(1-\sigma)/2+\varepsilon} (2 + |t|)^{A_K} \quad (0 \leq \sigma \leq 1)$$

for nonprincipal finite-order characters, with some fixed A_K independent of ε ; see the deduction in Section 12. On the remaining fixed strips we use polynomial height bounds and absolute convergence to the right of one. For the principal character, (1.3) gives a polynomial height bound for $(s - 1)\zeta_K(s)$, with the pole removed. The contour argument needs a bound for $1/L$ as well. Lemma 12.1 obtains it on a smaller rectangle after excluding characters with a zero in the larger rectangle; the growth bound for L alone would not give this conclusion.

When S contains the conductor, $L^S(s, \eta, K)$ omits precisely the finite Euler factors indexed by S_f . Extra omissions at primes dividing an element u can enlarge bounds, or reciprocal bounds, by at most $\mathfrak{N}(u)^\varepsilon$ on a fixed positive-real-part strip. Indeed, separating finitely many small primes bounds the products of $1 + Q^{-\sigma_0}$ and $(1 - Q^{-\sigma_0})^{-1}$ by that amount. This observation does not discard the zero-extension convention.

4.3. Reciprocity and the three large sieves. When K contains the relevant roots of unity, Kummer theory associates characters to extensions $K(\sqrt[k]{u})/K$. Hilbert reciprocity relates the local power-residue symbols; their conductor outside primes above k is tame and is supported on primes where the valuation of u is not divisible by k . These are classical consequences of local and global class-field theory [26, 37]. Our choices making the symbols symmetric on ideal generators are given in Section 7.

Three different sieve estimates occur. In each, constants may depend on the fixed field, local data, and box shape. The summation parameters and coefficients are the varying data. An additive large sieve bounds sums against separated additive phases. For S -integers in a balanced box, the form needed in Section 10 is

$$\sum_{m \in \mathcal{B}_N} \left| \sum_{\alpha} b_{\alpha} e(m\alpha) \right|^2 \ll (N + V^2) \sum_{\alpha} |b_{\alpha}|^2,$$

where e is a fixed additive character, $\mathcal{B}_N$ is a fixed-shape box of volume comparable to N , and the reduced fractions α have denominator norm at most V . Huxley established the number-field additive large-sieve framework [17]. We use the form in [30, Lemma 7.1]; Section 10 records the normalization and explains why the cited Fourier-majorant argument applies over our field.

A conductor large sieve, obtained by expanding primitive multiplicative characters into Gauss sums and applying the additive sieve, gives

$$\sum_{\omega} \left| \sum_{\mathfrak{N}\mathfrak{n} \leq N} b_{\mathfrak{n}} \omega(\mathfrak{n}) \right|^2 \ll_{\varepsilon} (N + U^2) (NU)^{\varepsilon} \sum_{\mathfrak{n}} |b_{\mathfrak{n}}|^2.$$

Here the characters are distinct and primitive, have conductor norm $O(U)$, and their components at the fixed places lie in a fixed finite list. The dependence on U^2 is sufficient for the zero-density estimate in Lemma 12.1. We use [30, Lemma 8.1]; Section 12 explains its reduction to the additive sieve and checks the dependence on the field.

The stronger quadratic large sieve exploits the special reciprocity of quadratic characters. Goldmakher and Louvel's Theorem 1.1 [11], extending Heath-Brown's quadratic method [13], gives

$$\sum_{\substack{\mathfrak{d} \text{ squarefree} \\ N\mathfrak{d} \leq D}} \left| \sum_{\substack{\mathfrak{c} \text{ squarefree} \\ N\mathfrak{c} \leq C}} b_{\mathfrak{c}} \Xi_{\mathfrak{d}}(\mathfrak{c}) \right|^2 \ll_{\varepsilon} (C + D)(CD)^{\varepsilon} \sum_{\mathfrak{c}} |b_{\mathfrak{c}}|^2.$$

Here $\Xi_{\mathfrak{d}}$ denotes the quadratic Hecke character indexed by the squarefree ideal $\mathfrak{d}$. This estimate requires their Hecke-family axioms, including finite class data, reciprocity, and a primitive-conductor condition for products of characters. In Section 9.1 we verify these conditions for our Kummer family before using the theorem. The difference between $C + D$ here and the conductor sieve's $N + U^2$ is the analytic reason to transform the original sextic interaction into a quadratic one. The transformation provides the quadratic family to which the cited theorem applies.

5. THE CUBIC INGREDIENTS AND THE PROOF STRATEGY

The function f and the required bound (2.3) were introduced in Section 2.1. Following [30, Sections 4–8], we now define the character sum and the fixed weighted average that produce this bound. The cubic transformation and Poisson summation estimate the same average in two ways.

5.1. The zero-free statement and abelian descent. Write μ_m for the group of m -th roots of unity. We apply the argument of [30] with the principal Hecke character. With the field hypotheses checked in Section 2.3 and the numerical choices refined below, it gives the following statement. Abelian descent then gives Theorem 1.1; the general Hecke conclusion is deduced separately in the proof of Corollary 1.2.

Theorem 5.1. *Let F be a number field containing μ_{12} . Then $\zeta_F(s) \neq 0$ for $\operatorname{Re} s > 31/32$ and $s \neq 1$.*

Proposition 6.1 derives Theorem 5.1 from two estimates for the fixed average defined in Section 5.6. Proposition 7.1, whose proof is completed in Section 13, establishes those estimates over every $F \supset \mu_{12}$. We first give the descent from Theorem 5.1 to Theorem 1.1.

Proof of Theorem 1.1. Set $F = K(\zeta_{12})$. The extension F/K is abelian of degree at most four: its Galois group embeds in $\operatorname{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$. Abelian class-field factorization gives

$$(5.1) \quad \zeta_F(s) = \zeta_K(s) \prod_{\substack{\chi \in \widehat{\operatorname{Gal}(F/K)} \\ \chi \neq 1}} L(s, \chi, K).$$

Every nonprincipal factor on the right is entire. A zero of ζ_K away from one would therefore be a zero of ζ_F , contrary to Theorem 5.1. No assumption on the normal closure of K , and no higher-dimensional Artin holomorphy theorem, is used. $\square$

5.2. Cubic Gauss sums and coefficient normalizations. Let k be a finite field of cardinality $Q \equiv 1 \pmod{3}$, let $\rho : k^{\times} \rightarrow \mu_3$ have order three, and let ψ be a nontrivial additive character. Extend ρ by zero at zero. Its Gauss and Jacobi sums are

$$G(\rho) = \sum_{x \in k} \rho(x) \psi(x), \quad J(\rho, \rho) = \sum_{x \in k} \rho(x) \rho(1 - x).$$

Finite Fourier orthogonality gives $|G(\rho)| = Q^{1/2}$, and multiplying out gives

$$G(\rho)^2 = J(\rho, \rho) G(\bar{\rho}), \quad (Q^{-1/2} G(\rho))^3 = Q^{-1/2} J(\rho, \rho).$$

These are classical Gauss–Jacobi identities; Davenport and Hasse’s lifting relation transfers them compatibly through finite-field extensions [7]. Their phases, rather than just their absolute values, determine the reciprocal Euler factor in this proof.

The completion by cubes described in Section 2.2, item (2), uses Patterson’s Fourier-coefficient formula [31, Theorem 8.1]. To compare the classical series over $\mathbb{Q}(\zeta_3)$ with the adelic normalization, we absorb the fixed factor $(\sqrt{-3})^{-3}$ into the local data at the prime above 3. Away from that prime, the Fourier indices have the form cn^3 , with c squarefree; the ideals generated by c and n may share prime factors.

For the use of these coefficients in estimates for cubic Gauss sums, see Heath-Brown–Patterson [15], Heath-Brown [14], and Dunn–Radziwiłł [8].

5.3. The completed character sum. The cubic summation formula will be applied to the sums $B_m(Z)$ defined below. Their definition requires compatible ideal generators and a correction to the Gauss-sum phase; these are constructed in Lemmas 7.2 and 7.3. Enlarge the fixed set S as specified in Section 7, and put

$$R = \mathcal{O}_{F,S}, \quad U_S = R^\times, \quad \Gamma = U_S^6, \quad F_S = \prod_{v \in S} F_v.$$

An exterior ideal is an ideal of R , identified with an ideal of F supported outside S . Unless fractional ideals are specified, our ideal sums run over nonzero integral exterior ideals. Write $q_{\mathfrak{a}} = N\mathfrak{a}$. A Roman letter such as c denotes an allowed generator of the corresponding ideal $\mathfrak{c}$: these generators are chosen multiplicatively modulo Γ , with the reciprocity convention of (7.2).

At an exterior prime $\mathfrak{p}$, let $\chi_{\mathfrak{p}}$ be the sextic residue character and put $\chi_{\mathfrak{a}}(m) = \prod_{\mathfrak{p}|\mathfrak{a}} \chi_{\mathfrak{p}}(m)^{v_{\mathfrak{p}}(\mathfrak{a})}$. Every character power is extended by zero on nonunits, even when its exponent is divisible by six. Thus the twist excludes precisely the primes shared by mR and $\mathfrak{a}$. Let e be the trace additive character on F_S , fixed in Section 7.2. For squarefree $\mathfrak{c}$, the normalized sums $\gamma_j(\mathfrak{c})$ are defined in (7.3); explicitly,

$$\gamma_j(\mathfrak{c}) = q_{\mathfrak{c}}^{-1/2} \sum_{x \bmod \mathfrak{c}} \chi_{\mathfrak{c}}(x)^j e(x/c).$$

The fixed unitary Hecke character ϑ is chosen in Lemma 7.3 so that $\gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p})\vartheta(\mathfrak{p}) = -1$. It makes the first local coefficient produced by Poisson summation equal to that of $1/\zeta_F^S(s)$.

We use a Gaussian weight in the logarithm of the ideal norm. Let

$$(5.2) \quad P_G(x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{(\log x)^2}{4}\right), \quad \hat{P}_G(s) = e^{s^2}.$$

For $m \in R \setminus \{0\}$, put

$$(5.3) \quad B_m(Z) = \sum_{\substack{\mathfrak{c} \text{ squarefree} \\ \mathfrak{n}}} \frac{\gamma_2(\mathfrak{c})}{q_{\mathfrak{c}}^{1/2} q_{\mathfrak{n}}} \vartheta(\mathfrak{c}\mathfrak{n}^3) \chi_{\mathfrak{c}\mathfrak{n}^3}(m) P_G\left(\frac{q_{\mathfrak{c}} q_{\mathfrak{n}}^3}{Z}\right).$$

All ideals are exterior. The ideals $\mathfrak{c}$ and $\mathfrak{n}$ may share primes. The Gaussian gives absolute convergence for every fixed $Z > 0$. Moreover $B_{\gamma m} = B_m$ for $\gamma \in \Gamma$.

The coefficient $\gamma_2(\mathfrak{c})/(q_{\mathfrak{c}}^{1/2} q_{\mathfrak{n}})$ is the normalized exterior theta coefficient in (8.7).

5.4. Kubota’s cover and Patterson’s theta series. Cubic theta functions transform with an additional cube-root-of-unity multiplier. Kubota’s metaplectic construction incorporates this multiplier in a threefold central extension of the matrix group [23]. Patterson constructs the corresponding cubic theta series as a residue of an Eisenstein series [31].

The Weyl matrix $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ interchanges the two cusps. Expressing the same automorphic function before and after multiplication by w gives a transformation formula. Finite additive

Fourier inversion first inserts the desired multiplicative character twist; the cusp transformation then changes its arithmetic form. Constant terms must be treated along with nonzero Fourier coefficients. In the general-field construction, averaging against a nontrivial angular character at a complex place makes the constant term vanish. This cancellation is part of [30, Lemma 6.1].

In the 11/12 argument, the transformation uses Dunn–Radziwiłł’s coefficient and cusp formulas [8, Section 5 and Appendix A], with the fixed ray-class twists treated in [29, Appendix A.2]. The completion and subsequent removal of cubes are described in Section 2.2, items (2)–(4).

5.5. Kazhdan–Patterson over a general number field. Over a general field containing μ_3 , the appropriate object is the cubic exceptional representation constructed by Kazhdan and Patterson [20]. Their global Eisenstein-residue construction and local representation theory replace a direct appeal to the explicit Eisenstein-integer theta series. We use their work together with its correction [21].

Write $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and let π_v be a local representation. A Whittaker functional λ_v obeys $\lambda_v(\pi_v(n(x))\varphi) = \psi_v(x)\lambda_v(\varphi)$. It extracts a Fourier coefficient. Uniqueness means that all such functionals are scalar multiples of one another; it is what allows a global coefficient to factor into local coefficients after one consistent normalization. Such uniqueness is not automatic for metaplectic covers. For the cubic rank-two setting here we choose Kazhdan–Patterson’s determinant parameter $c = 2$. In the notation of Cai, Fan and She [4, Proposition 2.2],

$$d = \frac{3}{\gcd(1 + 4c, 3)} = 1,$$

so the relevant Whittaker space has dimension one, including at places above three. Their modern formulation relies on Kazhdan–Patterson; it is not a second independent proof of that theory.

At unramified finite places the spherical Whittaker coefficients vanish unless the index has valuation 0 or 1 (mod 3). At complex places the Whittaker functions are expressed in terms of Bessel functions. At the finitely many remaining places one chooses fixed smooth vectors. OpenAI uses these existence and uniqueness results to fix the coefficient phases and prove the reflection formula [30, Sections 5–6 and Appendix A]. We record the resulting statements in Lemma 8.2 and Proposition 8.3, with the Hilbert-symbol convention (8.6). This distinction matters because a scalar or conjugation error would change the Euler product used to detect zeros.

5.6. The weighted average. To obtain a principal term containing $1/\zeta_F^S$ by Poisson summation, we average $B_m(Z)$ against additive Gauss sums at a second scale. The squared norm of this additive factor determines the bound obtained from Cauchy–Schwarz. For $Z \geq 2$, fix

$$(5.4) \quad \begin{aligned} \alpha_0 &= \frac{67}{100}, & \beta_0 &= \frac{151}{400}, & X &= Z^{\alpha_0}, & Y &= Z^{\beta_0}, \\ M_0 &= \alpha_0 + \beta_0 = \frac{419}{400}, & h_0 &= 1 - \alpha_0 = \frac{33}{100}. \end{aligned}$$

At finite places use $|\varpi_v|_v = Nv^{-1}$, and at complex places use the square of the ordinary modulus. For $y \in F_S^\times$, put $|y|_S = \prod_{v \in S} |y_v|_v$ and $F_S^1 = \{y : |y|_S = 1\}$. The element ℓ_T has finite components one and complex components $T^{1/[F:\mathbb{Q}]}$, so $|\ell_T|_S = T$. The quotient F_S^1/Γ is compact. We use the self-dual additive measures for $\mathbf{e}$, the Fourier convention $\widehat{W}_0(y) = \int_{F_S} W_0(x)\mathbf{e}(-xy) dx$, and the multiplicative disintegration $d^\times y = dt dT/T$ for $y = t\ell_T$. These conventions are recorded in Section 7.2.

Choose a nonzero, nonnegative $W_1 \in C_c^\infty(0, \infty)$. Choose a tensor Bruhat–Schwartz function W_0 , compactly supported in $F_S^\times$, for which

$$(5.5) \quad \widehat{W}_0(0) = 0, \quad M(1/6) \neq 0, \quad M(z) = \int_{F_S^\times} \widehat{W}_0(y) |y|_S^z d^\times y.$$

Here a Bruhat–Schwartz function is smooth and rapidly decreasing at infinite places, and locally constant with compact support at finite places. The condition $\widehat{W}_0(0) = 0$ removes the zero Poisson frequency; $M(1/6) \neq 0$ keeps the principal residue. The construction in [Section 10.2](#) satisfies both conditions with fixed weights, independently of ε .

For an exterior ideal $\mathfrak{s} = sR$ and $k \in R$, set

$$g_{\mathfrak{s}}(k) = \sum_{d \bmod \mathfrak{s}} \chi_{\mathfrak{s}}(d) e(kd/s).$$

For $X, Y, Z > 0$, define

$$(5.6) \quad I(X, Y, Z) = \int_{F_S^1/\Gamma} \frac{1}{Y} \sum_{\mathfrak{s}} W_1(q_{\mathfrak{s}}/Y) \frac{1}{\sqrt{q_{\mathfrak{s}}X}} \\ \times \sum_{m \in R \setminus \{0\}} q_{\mathfrak{s}}^{-1/2} g_{\mathfrak{s}}(-m) W_0\left(\frac{m}{t\ell_{q_{\mathfrak{s}}X}}\right) B_m(Z) dt.$$

We sum over nonzero m explicitly because B_m is defined there; the support of W_0 excludes zero in the original average. The change $m \mapsto \gamma m$, $\gamma \in \Gamma$, shows that the integrand descends to the quotient. Set $I_Z = I(Z^{\alpha_0}, Z^{\beta_0}, Z)$, with α_0, β_0 as in (5.4). This is the average used in both estimates that follow.

5.7. How cubic reflection exposes a quadratic sieve. The summand in (5.3) contains both a cubic Gauss coefficient and a sextic twist. The theta transformation changes the latter to a quadratic interaction between the varying squarefree ideals. At a prime dividing mR exactly once, it leaves the first power of the sextic character and forces the transformed frequency to have valuation -2 . The spherical coefficient at a prime dividing the squarefree coefficient index contributes exponent -4 . Reciprocity combines these exponents:

$$1 - 4 \equiv 3 \pmod{6}.$$

The resulting quadratic symbol is displayed in (8.14) and proved in [30, Lemma 6.2]. Higher prime powers in mR require additional terms. Shared primes must also retain the zero values imposed by the characters. [Proposition 9.1](#) accounts for these terms when applying Goldmakher–Louvel’s inequality.

A balanced annular set at scale T is a set $R \cap \ell_T \mathcal{B}$, where $\mathcal{B} \subset F_S^\times$ is fixed and compact. Its elements have norm comparable to T , and their normalized shapes stay in $\mathcal{B}$. At the fixed scale $M_0 = 419/400$, the required estimate, obtained from the proof of [30, Proposition 4.1] as explained in [Section 9](#), is

$$\sum_{m \in R \cap \ell_{Z^{M_0}} \mathcal{B}} |B_m(Z)|^2 \ll_{\varepsilon} Z^{2M_0-1+\varepsilon}.$$

In (5.6), the elements m have norm comparable to $XY = Z^{M_0}$. After the common normalization $(XY)^{-1/2}$, the additive factor has squared norm $O(Y^{-1})$, as shown in (10.3) by the additive large sieve. Cauchy–Schwarz then gives

$$|I_Z| \ll_{\varepsilon} XY^{1/2} Z^{-1/2+\varepsilon}.$$

Inserting (5.4) yields

$$(5.7) \quad |I_Z| \ll_{\varepsilon} Z^{287/800+\varepsilon}.$$

The quadratic inequality is Goldmakher–Louvel’s; OpenAI verifies its application to the complete reflected family in [30, Lemmas 6.3–6.4]. We recall the character structure and the treatment of higher prime powers in Sections 8.3 and 9, explaining why those verifications apply over F .

5.8. The reciprocal in the Poisson evaluation. Poisson summation of (5.6) produces nonzero frequencies, which Lemma 7.4 writes uniquely as ur^6 , modulo sixth powers of units. Here the ideal generated by u has prime exponents between zero and five. For $u = 1$, two zeta factors occur in the numerator of (11.1); taking their residues leaves the reciprocal $1/\zeta_F^S(s)$. The identity (7.5) supplies the first Euler coefficient -1 ; the remaining local terms form H . The local formula in Lemma 11.1 shows that this correction cannot cancel a pole.

For $u \neq 1$, Lemma 12.1 counts the denominator characters with a zero in a rectangle near one. The corresponding contours remain to the right of one. For every other character, the reciprocal bound in (12.3) permits a finite segment of the contour to move left. Direct estimates control small frequencies, and the fixed smooth weights control large frequencies. With the ranges in Section 13, the result is

$$(5.8) \quad I_Z = c_0 f(Z) + O\left(Z^{717/2000}\right), \quad c_0 \neq 0,$$

The nonzero scalar is given by (13.2). Since $717/2000 < 287/800$, combining (5.8) with (5.7) gives (2.3). The Mellin argument then excludes zeros in the stated half-plane.

Both estimates follow the comparison of [30, Proposition 7.2 and Section 8.3], with the new scales, detector, and contours justified below. The field checks in Section 2.3 explain the retained inputs over every $F \supset \mu_{12}$. The next section passes the new saving $1/32$ through Mellin continuation. The shorter $11/12$ proof uses the different comparison described in Section 2.2.

6. MELLIN CONTINUATION

We retain the Mellin argument of [30, Section 8.3], allowing the upper-bound loss to be arbitrarily small. This proves the analytic implication described in Section 2.1: the two estimates for I_Z , together with a nonzero correction factor, exclude zeros of $\zeta_F^S(s)$.

Proposition 6.1. *Suppose Equations (2.1), (5.7) and (5.8) hold for every $\varepsilon > 0$, with I_Z, f, H, c_0 and all defining weights fixed independently of ε . Suppose also that H and its reciprocal are bounded on every closed half-plane $\operatorname{Re} s \geq \sigma_0 > 2/3$. Then Theorem 5.1 follows.*

Proof. Since $717/2000 < 287/800$, the two estimates imply

$$(6.1) \quad f(Z) \ll_{\varepsilon} Z^{287/800+\varepsilon} \quad (Z \geq 2).$$

For $0 < Z \leq 1$, move the line in (2.1) arbitrarily far to the right. The reciprocal Euler product is absolutely convergent there; H is bounded, and the Gaussian decays in height. Thus

$$(6.2) \quad f(Z) = O_A(Z^A) \quad \text{for every } A > 0.$$

Define

$$(6.3) \quad T(s) = \int_0^\infty f(Z) Z^{-s+61/100} \frac{dZ}{Z}.$$

This integral is holomorphic for $\operatorname{Re} s > 31/32$, since $61/100 + 287/800 = 31/32$. Indeed, on a compact subset of that half-plane let $d > 0$ be the minimum distance to its boundary. Choose $\varepsilon = d/2$ in (6.1). The large- Z integrand is then $O(Z^{-d/2})$, uniformly on the compact set. At zero use (6.2). Both bounds remain integrable after multiplication by any fixed power of $\log Z$. They justify locally uniform differentiation in (6.3).

To identify the transform without an interchange of two unbounded contour integrals, put $Z = e^t$. On the original line (2.1) becomes

$$e^{-(2-61/100)t} f(e^t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau t} e^{(2+i\tau-5/6)^2} \frac{H(2+i\tau)}{\zeta_F^S(2+i\tau)} d\tau.$$

The right integrand is rapidly decreasing; the left side is integrable by the endpoint bounds. Fourier inversion gives

$$T(s) = e^{(s-5/6)^2} \frac{H(s)}{\zeta_F^S(s)}.$$

First this holds on $\operatorname{Re} s = 2$, then throughout $\operatorname{Re} s > 1$, and finally meromorphically on $\operatorname{Re} s > 31/32$. The numerator is nowhere zero there. A zero of the denominator would give a pole of T , contradicting its holomorphy. The pole of ζ_F^S at one gives only a zero of its reciprocal. The finitely removed Euler factors have zeros on $\operatorname{Re} s = 0$, so restoring them does not change the conclusion. $\square$

Here ε is chosen after the compact set. Thus the bound for every $\varepsilon > 0$ in (6.1) gives the open half-plane, without any assertion of uniformity at its boundary.

7. ARITHMETIC CHOICES OVER A GENERAL FIELD

Throughout the rest of the proof, $F \supset \mu_{12}$ is fixed and $[F : \mathbb{Q}] = 2r$. Such an F is totally imaginary. It need not be Galois over $\mathbb{Q}$ or a CM field.

7.1. The additional field checks. The theorem in [30] assumes that F is cyclotomic. We use its proofs under the weaker assumption $\mu_{12} \subset F$. To justify that use, we must check the constructions as well as the inequalities: a cited estimate for a family over a cyclotomic field does not, by itself, estimate the corresponding family over F .

Proposition 7.1. *Let F be any number field containing μ_{12} . The arithmetic data, local vectors, and weights defining B_m and I_Z can be chosen, independently of ε , so that the estimates (5.7) and (5.8) hold. The principal correction H and its reciprocal are bounded on every closed half-plane $\operatorname{Re} s \geq \sigma_0 > 2/3$. Constants and thresholds may depend on F and the fixed data.*

The additional issues, and the checks that resolve them, are the following.

- (1) The class group, infinite unit group, and ramified places must permit the same generator and measure conventions. We check these in [Section 7.2](#); the phase corrections in [Lemma 7.3](#) must also exist without a normality or CM assumption.
- (2) The prescribed coefficients must belong to one global theta representation, with compatible local normalizations. The rational compatibility and the treatment of all complex and ramified places are checked in [Section 8](#); the reflected functions must retain their fixed weights and finite class dependence.
- (3) The transformed characters must satisfy the number-field quadratic-sieve hypotheses, including the exact primitive conductor of a product. [Section 9.1](#) checks this, and [Section 9](#) checks that higher prime powers and ideal counting preserve the full moment estimate.
- (4) Poisson summation must give the same reciprocal Euler factor, with a nonzero principal correction. The normalization in [Section 10](#) and the local factors in [Section 11](#) provide these checks, including primes shared by the summation indices.
- (5) The conductor and height estimates must preserve the numerical exponents when $[F : \mathbb{Q}]$ varies. [Sections 12](#) and [13](#) track the field dependence, the order of the height choices, and the independence of the fixed weights from ε .

We carry out these checks alongside the cited statements, then assemble the proof of [Proposition 7.1](#) in [Section 13](#). No uniformity in the field is required for this deduction.

7.2. Norms, measures and allowed generators. At a finite place use $|\varpi_v|_v = Nv^{-1}$; at a complex place use the square of the ordinary modulus. Fix the standard trace additive character $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ and its self-dual additive measures. On F_S this gives $e(x) = \prod_{v \in S} \psi_v(x_v)$ and

$$(7.1) \quad \widehat{f}(y) = \int_{F_S} f(x) e(-xy) dx.$$

This is the additive Fourier convention in (5.5) and in the Poisson formula (10.1). Choose S to contain the infinite places, the places over six, the conductors of the additive data, the exceptional places of the theta tensor chosen in [Section 8](#), the fixed small primes excluded in [30, Proposition 7.3], and enough primes to kill the ideal class group. The preliminary theta tensor is chosen before this enlargement; none of the following choices is used to obtain its existence.

Write

$$R = \mathcal{O}_{F,S}, \quad U_S = R^\times, \quad \Gamma = U_S^6, \quad F_S = \prod_{v \in S} F_v, \quad F_S^1 = \{y \in F_S^\times : |y|_S = 1\}.$$

As in [Section 5.3](#), an *exterior ideal* is supported outside S . We write $q_a = N\mathfrak{a}$. Lowercase Roman letters denote generators of the corresponding fraktur ideals. For $a \in F^\times$, the product formula gives $q_a := |a|_S = N(aR)$. Let ℓ_T have finite components one and component $T^{1/(2r)}$ at each complex place; then $|\ell_T|_S = T$.

Use arithmetic local reciprocity and the convention

$$(a, b)_{n,v} = \frac{\text{Art}_v(b)(a^{1/n})}{a^{1/n}}, \quad H_S(x, y) = \prod_{v \in S} (x_v, y_v)_{6,v}.$$

For $y \in F_S^\times$, write $[y]$ for its class in $\mathcal{V} = F_S^\times / F_S^{\times 6}$. For a unit u at a tame place, $(u, \varpi)_{n,v} \equiv u^{(q_v-1)/n} \pmod{v}$.

Lemma 7.2. *In $\mathcal{V} = F_S^\times / F_S^{\times 6}$, the image E of U_S / U_S^6 is a maximal isotropic subgroup for H_S . It admits an isotropic complement B , so $\mathcal{V} = E \oplus B$. Every exterior fractional ideal has a generator whose class lies in B , unique modulo Γ . Such generators can be chosen with a/ℓ_{q_a} in one fixed compact subset of F_S^1 .*

This is the generator construction in [30, Lemmas 3.1–3.2]. The proof uses the trivial S -class group, the S -unit theorem, local Kummer duality, and Hilbert reciprocity. For $\ell = 2, 3$, the dimension calculation is

$$\dim_{\mathbb{F}_\ell}(F_S^\times / F_S^{\times \ell}) = 2|S_f| + 2r, \quad \dim_{\mathbb{F}_\ell}(U_S / U_S^\ell) = |S_f| + r.$$

The first identity uses $\sum_{v|\ell} [F_v : \mathbb{Q}_\ell] = 2r$; the second is the S -unit theorem. A unit locally an ℓ -th power at S would give an everywhere unramified Kummer extension split at S , hence a quotient of the trivial S -class group. This proves that the unit classes inject. Hilbert reciprocity makes their image isotropic, and the displayed dimensions make it maximal. Since $-1 \in F^{\times 6}$, the pairing is alternating and admits the required complement. Compactness of F_S^1/Γ gives the bounded representatives. Each input is valid for the present F ; cyclotomicity is not used.

We call the generators in [Lemma 7.2](#) *allowed*, and *balanced* when the compactness condition is imposed. Allowed generators multiply modulo Γ . These choices let us apply local Fourier analysis to ideal sums while keeping the multiplicity of generators bounded.

7.3. Residue symbols and Gauss corrections. At an exterior prime $\mathfrak{p}$, define $\chi_{\mathfrak{p}}(x) \equiv x^{(q_{\mathfrak{p}}-1)/6} \pmod{\mathfrak{p}}$ on units, and extend by zero to nonunits. Every power of this character, *including exponent zero*, has that zero extension. Put $\chi_{\mathfrak{a}}(x) = \prod_{\mathfrak{p}|\mathfrak{a}} \chi_{\mathfrak{p}}(x)^{v_{\mathfrak{p}}(\mathfrak{a})}$, with $\chi_R = 1$. Exact reciprocity for allowed generators reads

$$(7.2) \quad \chi_{\mathfrak{a}}(b) = \chi_{\mathfrak{b}}(a), \quad \chi_{\mathfrak{a}}(\epsilon) = H_S(\epsilon, a)^{-1} \quad (\epsilon \in U_S).$$

The first equality includes the common-prime case, when both sides are zero. For coprime arguments it follows from the product formula for the Hilbert symbol and isotropy of B .

With the additive character e fixed in Section 7.2, for squarefree $\mathfrak{c}$, define

$$(7.3) \quad \gamma_j(\mathfrak{c}) = q_{\mathfrak{c}}^{-1/2} \sum_{x \bmod \mathfrak{c}} \chi_{\mathfrak{c}}(x)^j e(x/c).$$

Changing an allowed generator by a sixth-power unit leaves this unchanged. At a prime its additive character is primitive, and $\chi_{\mathfrak{p}}(-1) = 1$, since $q_{\mathfrak{p}} \equiv 1 \pmod{12}$.

The principal term of the Poisson calculation contains $\gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p})$. Its phase must be corrected to give -1 , the first coefficient of a reciprocal Euler factor of ζ_F . The following lemma supplies a Hecke character with exactly this correction.

Lemma 7.3. *There are a unitary Hecke character Λ and a finite-order Hecke character G , unramified outside S , such that*

$$(7.4) \quad \gamma_2(\mathfrak{p})^3 = -\Lambda(\mathfrak{p}), \quad G(\mathfrak{p}) = \overline{\chi_{\mathfrak{p}}(4)}\gamma_3(\mathfrak{p}), \quad \gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p}) = -\Lambda(\mathfrak{p})G(\mathfrak{p}).$$

At every complex place Λ has purely angular exponent 1 or -1 . In particular, for

$$(7.5) \quad \vartheta = \overline{\Lambda} \overline{G}, \quad \gamma_1(\mathfrak{p})\gamma_2(\mathfrak{p})\vartheta(\mathfrak{p}) = -1.$$

The construction and identities are [30, Lemma 3.4]. To see their field dependence, put $E_0 = \mathbb{Q}(\zeta_3)$. The cubic character is

$$\Lambda = \Lambda_{E_0} \circ N_{F/E_0},$$

where Λ_{E_0} is the normalized primary-generator character. Davenport–Hasse lifting proves the first identity in (7.4) prime by prime, for arbitrary residue degrees [7]. Weil’s quadratic index has multiplicative defect given by the quadratic Hilbert symbol; on the allowed generator classes that defect is trivial by isotropy, giving the finite-order correction G [38]. The Gauss-sum duplication identity then gives the last identity. Neither the norm pullback nor the index product formula requires F/E_0 to be normal.

Lemma 7.4. *Choose a representative set $\mathcal{U}$, modulo Γ , of nonzero S -integers with exterior valuations in $\{0, \dots, 5\}$, including one. Every nonzero frequency modulo Γ has a unique expression ur^6 , with $u \in \mathcal{U}$ and r an allowed generator of an exterior ideal. The inducing primitive Hecke characters $\chi_{\bullet}(u)$ are distinct, only $u = 1$ induces the principal character, and*

$$N \text{ cond}(\chi_{\bullet}(u)) \ll_{F,S} q_u.$$

See [30, Lemma 3.3]. Dividing exterior valuations by six gives the frequency decomposition, and Kummer theory identifies equality of characters with equality modulo $F^{\times 6}$. The fixed local power classes at S bound the ramified conductor there. In particular, the denominator characters for $u \neq 1$ in (11.1) are nonprincipal.

8. GLOBAL REALIZATION AND REFLECTION OF THE CUBIC FAMILY

OpenAI realizes the coefficients of $B_m(Z)$ in a global theta vector and transforms the resulting sum [30, Sections 5–6]. Finite Fourier inversion inserts the sextic twist, and invariance under the rational Weyl element gives the transformation. The construction keeps the coefficient phases and Gaussian smoothing throughout, since the Poisson calculation must later evaluate the same sum. The resulting interaction between the squarefree ideals is the quadratic symbol in (8.14).

8.1. The cited representation and the fixed vector. We use the cubic exceptional representation with determinant parameter $c = 2$ in Kazhdan–Patterson’s notation. The classical inputs are their local exceptional quotient and its spherical vector [20, Theorem I.2.9], tame Whittaker uniqueness [20, Corollary I.3.6], the global automorphic residue and restricted tensor product [20, Section II.1 and Theorem II.2.1], and the nonzero global Whittaker functional and uniqueness at every place [20, Theorem II.2.5]. The representations at complex places are described in [20, Theorems I.6.4–I.6.5]. The correction [21] is part of these inputs: we do not use its withdrawn auxiliary spherical-functional vanishing assertion. In the modern formulation of [4, Proposition 2.2], the dimension parameter is $d = 3/\gcd(1 + 4c, 3) = 1$, including at the wild places.

The uniqueness statement at primes above three is supplied by the cited KP/CFS results. The formal proof treats these primes together. For each such place, take the twisted Jacquet quotient of the chosen local representation, namely the quotient on which the upper unipotent subgroup acts by the prescribed additive character. A nonzero global Fourier coefficient supplies a pure tensor on which the induced linear functional is nonzero.

To prove that this functional is injective, insert a vector in its kernel using compactly supported Kirillov sections. Local uniqueness at the remaining places makes every nonconstant Fourier coefficient of the resulting automorphic function vanish on upper-triangular translates. The constant term requires a separate argument. Each inserted section is a finite sum of differences between a section and one of its unipotent translates. To see this, decompose it into finitely many pieces supported where a suitable additive-character multiplier differs from one, and divide each piece by that multiplier minus one. For every upper-triangular translate, the constant-term integral kills these differences: the upper-triangular group normalizes the upper unipotent subgroup, and Haar measure on $F \backslash \mathbb{A}_F$ is translation invariant. Fourier uniqueness and injectivity of the chosen global realization therefore make the tensor-product functional injective. Fixing all but one of the nonzero tensor factors embeds each Jacquet quotient in $\mathbb{C}$, proving that it has dimension one. Its dual gives the required local Fourier functionals. The constant-term vanishing here concerns the inserted subspace.

Local Whittaker uniqueness gives a product formula on the restricted tensor product, using vectors finite under maximal compact subgroups at infinity. To extend the formula to smooth archimedean translates, we need continuity of the automorphic realization. The following argument proves the required estimate from the local intertwiner and Kazhdan–Patterson’s residual inner-product formula.

Fix all local vectors except at one complex place. Let V^∞ be its smooth local model and V^{fin} the subspace finite under a maximal compact subgroup. In the compact realization used here, identify these with smooth functions on the sphere $\mathbb{S}^2$ and finite sums of spherical harmonics, respectively. For $u = \sum_{\ell \geq 0} u_\ell$, where u_ℓ has harmonic degree ℓ , use the Sobolev seminorms

$$p_m(u)^2 = \sum_{\ell \geq 0} (1 + \ell)^{2m} \|u_\ell\|_{L^2(\mathbb{S}^2)}^2 \quad (m = 0, 1, \dots).$$

These seminorms define the smooth topology. For $u \in V^{\text{fin}}$, let $\Theta(u)$ be the corresponding automorphic residue. All such residues have one fixed finite level, since only an archimedean vector varies.

Lemma 8.1. *For every compact subset C of the adelic group and every product D of right-invariant archimedean derivatives, there are an integer $m \geq 0$ and a constant $A > 0$ such that*

$$(8.1) \quad \sup_{g \in C} |(D\Theta(u))(g)| \leq A p_m(u) \quad (u \in V^{\text{fin}}).$$

The constants and the order m may depend on C, D , the field, and the fixed local vectors.

Proof. We first bound the inducing section used to form $\Theta(u)$. Let $\mathcal{J}$ be the complex standard intertwiner from the inducing model to the chosen local model [20, Section I.6]. In their common compact realization it acts on degree- ℓ spherical harmonics by a positive scalar a_ℓ . If P_ℓ is the Legendre polynomial normalized by $P_\ell(1) = 1$, the Funk–Hecke formula and Rodrigues’ formula give

$$(8.2) \quad \frac{a_\ell}{a_0} = \frac{\int_{-1}^1 (1-t)^{-2/3} P_\ell(t) dt}{\int_{-1}^1 (1-t)^{-2/3} dt} = \prod_{j=1}^{\ell} \frac{j-1/3}{j+1/3}, \quad a_0 > 0.$$

The product is empty when $\ell = 0$. Since

$$\frac{j+1/3}{j-1/3} \leq \frac{j+1}{j} \quad (j \geq 1),$$

(8.2) gives $a_\ell^{-1} \leq a_0^{-1}(1+\ell)$. Thus the inducing lift $\mathcal{J}^{-1}u$ satisfies

$$(8.3) \quad p_m(\mathcal{J}^{-1}u) \leq a_0^{-1} p_{m+1}(u).$$

This establishes continuity of the lift directly in the smooth topology.

Let φ_u be the global inducing section with varying local factor $\mathcal{J}^{-1}u$. The residue constant term is obtained by applying the product of local intertwiners [20, Proposition II.1.2]; its varying factor is therefore u . On compact sets the other local factors are bounded and fixed. Sobolev embedding on $\mathbb{S}^2$, together with (8.3), bounds both φ_u and this constant term by a fixed multiple of $p_m(u)$, for some m .

Kazhdan–Patterson’s residual inner-product formula [20, Theorem II.1.4] expresses $\|\Theta(u)\|_{L^2}^2$ as a pairing of φ_u with the residual constant term. Here the automorphic norm is taken modulo the center with the fixed unitary central character, as required by that theorem. At the fixed finite level, the boundary pairing is a finite sum of integrals over compact sets of cusp parameters; finiteness of the cusp classes and compactness of the norm-one torus quotient make its total measure finite. Bounding the two factors by the preceding seminorm estimates gives

$$(8.4) \quad \|\Theta(u)\|_{L^2} \ll p_m(u).$$

Right differentiation commutes with taking the residue. A derivative at the varying complex place applies a Lie algebra operator to u ; a derivative at another archimedean place replaces its fixed vector by a derivative of that vector. These local differential operators are continuous for the seminorms p_m . Applying (8.4) to the finitely many resulting tensors shows that, for each archimedean differential operator D' ,

$$(8.5) \quad \|D'\Theta(u)\|_{L^2} \ll_{D'} p_{m(D')}(u).$$

The finite level remains fixed.

Finally, cover the image of C in the fixed-level automorphic quotient by finitely many local charts. The Euclidean Sobolev inequality on these charts, with order exceeding half their real dimension, gives a finite set $\mathcal{D}$ of archimedean differential operators such that

$$\sup_{g \in C} |(D\Theta(u))(g)| \ll_{C,D} \sum_{D' \in \mathcal{D}} \|D'\Theta(u)\|_{L^2}.$$

The chart measures and quotient Haar measure are comparable on these compact sets. Using (8.5) and taking the largest of the finitely many seminorm orders proves (8.1). $\square$

By Lemma 8.1 and density of V^{fin} , the residue map extends continuously to V^∞ , with convergence of every derivative on compact sets. When all complex factors vary, the same argument bounds the supremum by $A \prod_{v|\infty} p_{m_v}(u_v)$, so their smooth extensions can be taken simultaneously. The Lie algebra identities extend by continuity; integrating them along one-parameter subgroups gives equivariance for smooth translates. Integration against the fixed additive character over the compact quotient $F \backslash \mathbb{A}_F$ then gives a continuous Whittaker functional agreeing with the original Fourier integral on V^{fin} . Local uniqueness supplies the same product formula on smooth vectors. This gives the factorization used to identify the coefficients of $B_m(Z)$, whose mean square is estimated in Proposition 9.1.

Write

$$a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \bar{n}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The scalar section is central and split, since its commutator exponent is $1 + 4c = 9$. We work on the scalar quotient. Fix the genuine embedding so that the effective diagonal multiplier is

$$(8.6) \quad \mathcal{C}_v(x, y) = (x, y)_{6,v}^2, \quad \mathcal{C}_{\mathfrak{p}}(p, u) = \chi_{\mathfrak{p}}(u)^{-2}.$$

This is one global convention, not a prime-by-prime choice. On the center of the covering torus prescribe the exceptional datum below. Here $s(g)$ denotes the fixed section into the cover and ι its genuine central character:

$$\omega_v(\zeta s(\text{diag}(x^3, y^3))s(zI)) = \iota(\zeta) |x^3/y^3|_v^{1/6}.$$

It is trivial on split scalars and has the exceptional value $|t|_v$ on $\text{diag}(t^3, t^{-3})$. The product formula and the compatible inducing construction in [20, Section II.1] give the rational automorphic realization.

The global compatibility check in [30, Appendix A.1] also applies to a nonnormal F . A rational ratio that is a cube in every completion is a cube in F : otherwise its Kummer extension would have a nonsplit prime by Chebotarev. The product formula then makes the displayed norm datum trivial on the rational subgroup. Thus the passage from local exceptional representations to one global automorphic representation uses no automorphisms of $F/\mathbb{Q}$. At infinity, every place is complex, and the pullback character Λ has angular exponent 1 or -1 at each place. The local constructions treat both signs, so a CM hypothesis is unnecessary.

Lemma 8.2. *The automorphic vector can be normalized so that its exterior local Whittaker functions satisfy, for a unit u ,*

$$W_{\mathfrak{p}}(a(p^e u)) = \chi_{\mathfrak{p}}(u)^{2e} \begin{cases} 0, & e < 0, \\ Q^{-k}, & e = 3k, \\ Q^{-k-1/2} g_{2,\mathfrak{p}}, & e = 3k+1, \\ 0, & e = 3k+2, \end{cases} \quad Q = q_{\mathfrak{p}},$$

where $g_{2,\mathfrak{p}} = \gamma_2(\mathfrak{p})$. Consequently, for allowed $A = cn^3$,

$$(8.7) \quad \prod_{\mathfrak{p} \notin S} W_{\mathfrak{p}}(a(A)) = \frac{\gamma_2(\mathfrak{c})}{q_{\mathfrak{c}}^{1/2} q_{\mathfrak{n}}}.$$

The product vanishes on all other exterior ideal shapes.

These are the normalized coefficients of [30, Proposition 5.1, Corollary 5.2 and Appendix A]. Kazhdan–Patterson supplies the global automorphic representation and local Whittaker uniqueness; OpenAI fixes the remaining scalar by a nonzero global Fourier coefficient and performs the local calculation in the convention (8.6). The local expressions depend on Q and the residue character. The Chinese-remainder factors give (8.7), including primes shared by $\mathfrak{c}$ and $\mathfrak{n}$. The same calculation applies at every exterior place of our F .

Let $\mathcal{L}_S(y) = \prod_{v \in S} \vartheta_v(y_v)^{-1}$. It is trivial on U_S and satisfies $\mathcal{L}_S(A) = \vartheta(AR)$ for allowed A . Write $m = \epsilon_m m_0$, with m_0 allowed, and let $v = \epsilon_m \Gamma \in U_S/\Gamma$. Set

$$\mathcal{L}_v(y) = \mathcal{L}_S(y) H_S(\epsilon_m, y)^{-1}.$$

It is independent of the representative of v , trivial on U_S , and has angular exponent $m_v = 1$ or -1 at each complex place. There is a fixed S -vector with Whittaker function

$$f_v(y) = \mathbf{1}_{\prod_{v \in S_f} U_v}(y_f) \prod_{v \mid \infty} |y_v|_v \omega_{m_v}(y_v) K_{1/3}(c_v |y_v|_v^{1/2}),$$

where $\omega_j(z) = (z/|z|)^j$, $c_v > 0$ is fixed by the local additive character, and U_v is a sufficiently small open subgroup of local sixth powers. At finite places one obtains the indicator by upper-unipotent Fourier projection. At a complex place one takes a normalized complex linear combination of the two real upper-unipotent Lie derivatives of the spherical Whittaker function. The finite cutoff, complex-place functions, and bounds after Weyl translation are constructed in [30, Lemmas 5.3–5.5], using Kazhdan–Patterson’s complex Whittaker theory. Let θ_v be the automorphic form with this S -vector and the exterior spherical vectors of Lemma 8.2.

For $x > 0$, let the set of elements of norm x be $X_x = \{y \in F_S^\times : |y|_S = x\}$. Disintegrate the fixed multiplicative measure as $d^\times y = d_x^\times y dx/x$, and define

$$V(x) = \int_{X_x} \mathcal{L}_v(y)^{-1} f_v(y) d_x^\times y.$$

The angular factors cancel, so this is the same positive radial function for every v . For $\operatorname{Re} s > -5/6$, the defining Mellin integral converges and gives

$$(8.8) \quad \widehat{V}(s) = C_0 A_0^s [\Gamma(s + 5/6) \Gamma(s + 7/6)]^r, \quad C_0, A_0 > 0.$$

The reciprocal of the gamma-factor expression in (8.8) extends to an entire function. In the following definition, $1/\widehat{V}(s)$ denotes this continuation. The fixed kernel

$$(8.9) \quad \mathcal{K}(T) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{e^{s^2}}{\widehat{V}(s)} T^s ds \quad \text{satisfies} \quad \int_0^\infty \mathcal{K}(T) V(xT) \frac{dT}{T} = P_G(x).$$

The line parameter σ in (8.9) may be any real number: the continued integrand has no poles, and Gaussian decay permits shifting the contour. The kernel has rapid decay at both endpoints and every absolute Mellin moment is finite. To verify the convolution identity, choose $\sigma > -5/6$, interchange the absolutely convergent integrals, and use $\int_0^\infty V(xT) T^s dT/T = x^{-s} \widehat{V}(s)$. The factors $\widehat{V}(s)$ cancel, leaving Mellin inversion for the Gaussian (5.2). Thus the transformed average retains the Gaussian in (5.3).

8.2. An exact reflected expression. For each $\mathfrak{p} \mid mR$, let $j_{\mathfrak{p}} \in \{0, \dots, 5\}$ represent $v_{\mathfrak{p}}(mR) \bmod 6$, and put $Q = q_{\mathfrak{p}}$. In the finite sums that follow, v runs over $R/\mathfrak{p}$. For $j \neq 0$, define

$$\tau_{\mathfrak{p},j} = \sum_{v \neq 0} \chi_{\mathfrak{p}}(v)^{-j} \psi_{\mathfrak{p}}(v/p), \quad \alpha_{\mathfrak{p},j}(v) = \tau_{\mathfrak{p},j}^{-1} \chi_{\mathfrak{p}}(v)^{-j} \quad (v \neq 0), \quad \alpha_{\mathfrak{p},j}(0) = 0.$$

For $j = 0$, use $\alpha_{\mathfrak{p},0}(0) = 1 - Q^{-1}$ and $\alpha_{\mathfrak{p},0}(v) = -Q^{-1}$ for $v \neq 0$. Finite Fourier inversion says

$$\sum_{v \bmod \mathfrak{p}} \alpha_{\mathfrak{p},j}(v) \psi_{\mathfrak{p}}(tv/p) = \chi_{\mathfrak{p}}(t)^j.$$

In particular, the character still vanishes on nonunits when $v_{\mathfrak{p}}(mR)$ is divisible by six.

For $\nu \in F^\times$, define the product of the transformed exterior coefficients by

$$(8.10) \quad \mathcal{E}_m(\nu) = \prod_{\mathfrak{p} | mR} \left[\sum_{v \bmod \mathfrak{p}} \alpha_{\mathfrak{p},j_{\mathfrak{p}}}(v) W_{\mathfrak{p}}(a(\nu)wn(v/p)) \right] \prod_{\substack{\mathfrak{q} \notin S \\ \mathfrak{q} \nmid mR}} W_{\mathfrak{q}}(a(\nu)).$$

The next proposition transforms the sums $B_m(Z)$ with their specified coefficients and Gaussian weight. The transformed weights, denoted by $\Phi_{v,\kappa}$, belong to a fixed finite family. Their uniformity in m and Z is needed when [Proposition 9.1](#) averages over m .

Proposition 8.3. *There is a fixed finite family of functions $\Phi_{v,\kappa} \in C^2((0, \infty), \mathbb{C})$, $\kappa \in \mathcal{V}$, such that*

$$(8.11) \quad B_m(Z) = \sum_{\nu \in F^\times / \Gamma} \mathcal{E}_m(\nu) \mathcal{L}_v(\nu)^{-1} \Phi_{v, [\nu]}(Z|\nu|_S).$$

The sum converges absolutely. For every $A > 0$, $j \in \{0, 1, 2\}$, and $x > 0$,

$$(8.12) \quad |(x\partial_x)^j \Phi_{v,\kappa}(x)| \ll_{A,j} \min(x^{1/8}, x^{-A}).$$

These functions and the implied constants depend only on the fixed field and data.

This is the form of [\[30, Lemmas 6.1–6.3\]](#) needed here. Derivative bounds through order two suffice for the Mellin separation in [Section 9](#). Finite Fourier inversion inserts the sextic twist, rational automorphy identifies the two cusp expansions, and the fixed angular character kills their constant terms. The exact smoothing [\(8.9\)](#) recovers the Gaussian defining B_m . The local calculation and the bounds for the finitely many S -profiles then give [\(8.11\)](#) and [\(8.12\)](#). Rational automorphy is over F . At an exterior prime, $\mu_{12} \subset F$ gives $Q \equiv 1 \pmod{12}$, exactly the residue-field condition in the local calculation. The remaining places are in the fixed finite set S . Their local sixth-power class groups are finite, and F_S^1/Γ is compact by [Lemma 7.2](#). Consequently unfolding removes the continuous shape of a generator and leaves only finitely many class-dependent profiles. The complex-place estimates apply at each of the r complex places; their product changes constants and gamma-factor multiplicities, while exact Gaussian smoothing remains available by [\(8.9\)](#). These checks give the same fixed profile bounds over F , without a Galois or CM hypothesis.

8.3. Why the reflected interaction is quadratic. Separate the primes dividing mR exactly once from those dividing it to higher powers:

$$(8.13) \quad mR = \mathfrak{h}\mathfrak{d}, \quad \mathfrak{h} \text{ powerful, } \mathfrak{d} \text{ squarefree, } (\mathfrak{h}, \mathfrak{d}) = 1.$$

An ideal is powerful if each nonzero prime exponent is at least two. At a prime dividing $\mathfrak{d}$, the local formula in [\[30, Lemma 6.2 and Appendix A\]](#) forces the transformed frequency to have valuation -2 and leaves the first power of the sextic character. At each prime dividing $\mathfrak{h}$, separate the terms with $v = 0$ and $v \neq 0$ in [\(8.10\)](#). Let $\mathfrak{f} \mid \text{rad}(\mathfrak{h})$ be the product of primes at which the term with $v \neq 0$ is chosen; these include every prime with $j_{\mathfrak{p}} \neq 0$.

For each such choice, the transformed frequency has the unique form

$$\nu = \epsilon \frac{a_{\mathfrak{f}} c n^3}{(df)^2}, \quad \epsilon \in U_S/\Gamma, \quad \text{supp}(\mathfrak{a}_{\mathfrak{f}}) \subseteq \text{supp}(\mathfrak{f}), \quad (\mathfrak{c}n, \mathfrak{d}\mathfrak{f}) = 1,$$

with $\mathfrak{c}$ squarefree and the exponents of $\mathfrak{a}_{\mathfrak{f}}$ congruent to zero or one modulo three. Different choices can give the same frequency, but their contributions remain separate in the finite Fourier sum.

The only arithmetic factor depending jointly on the varying squarefree ideals $\mathfrak{d}$ and $\mathfrak{c}$ is

$$(8.14) \quad \prod_{\substack{\mathfrak{p}|\mathfrak{d} \\ \mathfrak{q}|\mathfrak{c}}} \chi_{\mathfrak{p}}(q) \chi_{\mathfrak{q}}(p)^{-4} = \chi_{\mathfrak{d}}(c)^3.$$

The exponent 1 comes from reflection at a prime dividing $\mathfrak{d}$. The exponent -4 comes from $(df)^{-2}$ in the spherical unit factor at a prime dividing $\mathfrak{c}$. Reciprocity combines them into exponent 3 (mod 6), allowing the quadratic large sieve to be applied.

After fixing $\mathfrak{h}, \mathfrak{f}, \mathfrak{a}_{\mathfrak{f}}, \mathfrak{n}$ and the finite power classes, each term has the form

$$(8.15) \quad \frac{\omega \rho(\mathfrak{d}) \lambda(\mathfrak{c})}{q_{\mathfrak{c}}^{1/2} q_{\mathfrak{n}}} \chi_{\mathfrak{d}}(c)^3 \Phi \left(\frac{Z q_{\mathfrak{a}_{\mathfrak{f}}} q_{\mathfrak{c}} q_{\mathfrak{n}}^3}{q_{\mathfrak{d}}^2 q_{\mathfrak{f}}^2} \right),$$

where ρ depends only on $\mathfrak{d}$, λ only on $\mathfrak{c}$, and ω only on the fixed data. They satisfy $|\rho|, |\lambda| \leq 1$ and $|\omega| \leq \beta_{\mathfrak{f}}(\mathfrak{a}_{\mathfrak{f}})$, where the multiplicative majorant β is given by

$$(8.16) \quad \beta_{\mathfrak{p}}(0) = 1, \quad \beta_{\mathfrak{p}}(1 + 3k) = Q^{-k}, \quad \beta_{\mathfrak{p}}(3k) = Q^{1/2-k} \ (k \geq 1), \quad \beta_{\mathfrak{p}}(2 + 3k) = 0.$$

The restrictions $(\mathfrak{d}, \mathfrak{h}\mathfrak{n}) = 1$ and $(\mathfrak{c}\mathfrak{n}, \mathfrak{f}) = 1$ remain in force. The quadratic symbol vanishes when $\mathfrak{d}$ and $\mathfrak{c}$ share a prime. The condition $(\mathfrak{d}, \mathfrak{n}) = 1$ must be imposed separately, while $\mathfrak{c}$ and $\mathfrak{n}$ may share primes.

9. THE REFLECTED MEAN SQUARE

Fix a compact subset $\mathcal{B} \subset F_S^\times$, and put

$$\mathcal{M}_Z = R \cap \ell_{Z^{M_0}} \mathcal{B}, \quad M_0 = 419/400.$$

For $m \in \mathcal{M}_Z$, let $mR = \mathfrak{h}(m)\mathfrak{d}(m)$ be the powerful-squarefree decomposition (8.13).

Proposition 9.1. *For every $\varepsilon > 0$, $Z \geq 2$, and dyadic lengths $P, D \geq 1$,*

$$(9.1) \quad \sum_{\substack{m \in \mathcal{M}_Z \\ P \leq q_{\mathfrak{h}(m)} < 2P \\ D \leq q_{\mathfrak{d}(m)} < 2D}} |B_m(Z)|^2 \ll_{\varepsilon} Z^{\varepsilon} P^{1/2} \left(D + \frac{D^2 P}{Z} \right).$$

Consequently,

$$\sum_{m \in \mathcal{M}_Z} |B_m(Z)|^2 \ll_{\varepsilon} Z^{2M_0-1+\varepsilon}.$$

The constants may depend on F, S and $\mathcal{B}$. The estimates are uniform when $\mathcal{B}$ is multiplied by elements of a fixed compact subset of F_S^1 .

9.1. Checking the quadratic family. For squarefree $\mathfrak{c}$, let $\Xi_{\mathfrak{c}}$ be the primitive quadratic Hecke character of $F(\sqrt{\mathfrak{c}})/F$, with c allowed. This is well-defined since Γ consists of squares. Its conductor is $\mathfrak{c}\mathfrak{s}_{\mathfrak{c}}$, with $\mathfrak{s}_{\mathfrak{c}}$ dividing a fixed S -ideal. At exterior primes its values are $\chi_{\bullet}(c)^3$. Classify $\mathfrak{c}$ by its allowed generator's S -square class. The class map is multiplicative, finite, and trivial on a sufficiently deep principal ray subgroup. If two coprime squarefree ideals have the same class, their product character is trivial at S and has primitive conductor exactly their product. Exact reciprocity gives the required symmetric pairing outside S .

These are the checks in [30, Lemma 6.4] of Definition 1 of Goldmakher–Louvel, including the primitive-conductor product condition. Their Theorem 1.1 gives

$$(9.2) \quad \sum_{\substack{\mathfrak{d} \text{ squarefree} \\ q_{\mathfrak{d}} \leq D}} \left| \sum_{\substack{\mathfrak{c} \text{ squarefree} \\ q_{\mathfrak{c}} \leq C}} z_{\mathfrak{c}} \chi_{\mathfrak{d}}(c)^3 \right|^2 \ll_b (CD)^b (C+D) \sum_{\mathfrak{c}} |z_{\mathfrak{c}}|^2 \quad (b > 0).$$

Apply the theorem first to the full family. Restrictions on $\mathfrak{d}$ can then be imposed by discarding nonnegative terms in the outer sum, and restrictions on $\mathfrak{c}$ by setting the corresponding coefficients to zero.

9.2. The full moment and its field hypotheses. Proposition 9.1 has the form of [30, Proposition 4.1], with a different fixed support exponent. That proposition states the cyclotomic case at the old scale; the applicability of its proof to our F and scale follows from the checks below. Its proof in [30, Section 6] combines (8.15) with (9.2); we retain that proof.

Two details explain why the complete sum is controlled. First, the transformed weight restricts the effective norm of the coefficient index to $D^2 P/Z$, up to an arbitrarily small power of Z . The separation of the two ideal variables must preserve this decay uniformly. Fix the auxiliary ideals and power classes as in (8.15), and restrict to $C \leq q_{\mathfrak{c}} < 2C$ and $D \leq q_{\mathfrak{d}} < 2D$. Then

$$\log(q_{\mathfrak{c}}/C) - 2\log(q_{\mathfrak{d}}/D) \in [-2\log 2, \log 2].$$

Choose a fixed $b_0 \in C_c^\infty(\mathbb{R})$ equal to one on this interval. For any profile Φ in Proposition 8.3 and $v > 0$, put

$$G_v(x) = b_0(x) \Phi(v e^x), \quad \tilde{G}_v(t) = \int_{\mathbb{R}} G_v(x) e^{-itx} dx.$$

The product rule and (8.12) give $\|G_v\|_1 + \|G_v''\|_1 \ll_A \min(v^{1/8}, v^{-A})$. Two integrations by parts therefore yield

$$(9.3) \quad \int_{\mathbb{R}} |\tilde{G}_v(t)| dt \ll_A \min(v^{1/8}, v^{-A}) \quad (A > 0),$$

uniformly in v and in the finite profile index. In (8.15), take $v = Z q_{\mathfrak{a}_f} C q_{\mathfrak{n}}^3 / (D^2 q_f^2)$. Fourier inversion expresses its profile factor as the integral of $\tilde{G}_v(t) (q_{\mathfrak{c}}/C)^{it} (q_{\mathfrak{d}}/D)^{-2it}$, with measure $dt/(2\pi)$. Both powers have modulus one. Minkowski's inequality and (9.2) therefore bound the separated sums without changing their coefficient norms, while (9.3) retains the decay in v needed to sum over the auxiliary ideals.

Second, the local majorants (8.16) are summable over the higher prime powers. The number of powerful ideals of norm at most P is $O_F(P^{1/2})$, since each has a unique form $\mathfrak{r}^2 \mathfrak{s}^3$ with $\mathfrak{s}$ squarefree. These facts account for $P^{1/2}(D + D^2 P/Z)$ in (9.1). The coprimality restrictions in (8.15) remain in force throughout.

The support exponent enters the proof in [30, Section 6.4, equations (6.42)–(6.45)] only through bounds of the form $P \ll Z^{M_0}$, $L \ll Z^{2M_0-1}$, and $DR_0 \ll Z^{3M_0-1+\delta}$. For any fixed $M_0 > 1$, increase the decay order of the same profiles and choose the auxiliary losses so that

$$\delta + 2M_0\delta_1 + (3M_0 - 1 + \delta)\delta_2 + \delta_3 < \varepsilon.$$

Thus (9.1) holds at $M_0 = 419/400$, without changing any profile. For each nonempty dyad, $PD \ll Z^{M_0}$, and

$$P^{1/2} D \ll Z^{M_0} \leq Z^{2M_0-1}, \quad P^{1/2} \frac{D^2 P}{Z} = \frac{(PD)^2}{Z P^{1/2}} \ll Z^{2M_0-1}.$$

Summing $O((\log Z)^2)$ dyads proves the second estimate of Proposition 9.1. The family verification immediately above is the one in [30, Lemma 6.4]. It uses quadratic extensions, the finite S -square

classes, and symmetric reciprocity for allowed generators. Ideal counting and bounded multiplicity of balanced generators hold over any fixed number field. Consequently both estimates hold for our F , with field-dependent constants and arbitrary $\varepsilon > 0$.

10. THE ADDITIVE LARGE SIEVE AND THE THETA BOUND

The average I_Z and all of its weights were fixed in [Section 5.6](#). We recall the cited weight construction and additive estimate, then combine them with [Proposition 9.1](#). Throughout this section X, Y, M_0 have the values in [\(5.4\)](#).

10.1. Additive normalization and the cited sieve. Use the character and self-dual measures of [\(7.1\)](#). The lattice $R \subset F_S$ is self-dual of covolume one. Indeed, apply adelic duality to $F \subset \mathbb{A}_F$, with the self-dual indicators of $\mathcal{O}_v$ outside S ; strong approximation identifies the remaining quotient with F_S/R . Consequently

$$(10.1) \quad \sum_{m \in aR} f(m) = q_{aR}^{-1} \sum_{h \in a^{-1}R} \widehat{f}(h), \quad |a|_S = q_{aR}.$$

These are the standard additive-duality conventions of Tate and Weil [\[35, 37\]](#).

For distinct reduced fractions $\alpha = d/s$ modulo R , with $q_{sR} \leq V$, the additive estimate is

$$(10.2) \quad \sum_{m \in R \cap \ell_N \mathcal{B}} \left| \sum_{\alpha} b_{\alpha} e(m\alpha) \right|^2 \ll_{F,S,\mathcal{B}} (N + V^2) \sum_{\alpha} |b_{\alpha}|^2.$$

Here $\mathcal{B} \subset F_S^{\times}$ is a fixed compact balanced set. This is [\[30, Lemma 7.1\]](#), using Huxley's number-field additive-sieve method [\[17\]](#). The product formula separates two reduced fractions by at least V^{-2} in product norm. A Fourier majorant on the balanced set then gives $N + V^2$, with the degree of F absorbed in the constant. The proof therefore supplies the same norm exponents for our F .

10.2. Construction and Mellin bounds for the fixed weights. The choice in [\(5.5\)](#) and the contour bounds are those used in [\[30, Section 7\]](#). We recall why the two required conditions on W_0 are compatible. Choose any nonzero, nonnegative $W_1 \in C_c^{\infty}(0, \infty)$. It remains to construct W_0 . The first condition in [\(5.5\)](#) removes the zero Poisson frequency; the second makes the principal residue nonzero. For $0 < s < 1$, Tate's local zeta identity identifies the Fourier-side Mellin functional with a nonzero multiple of pairing the original function against $|y|_v^{1-s} d^{\times} y$. At one complex place, subtract two radial annular bumps to make the additive integral zero while keeping the pairing at $s = 1/6$ nonzero. The two functionals are not proportional. At other places choose nonnegative annular bumps. This proves the assertion without an assumed nonzero residue.

The function M is holomorphic for $\operatorname{Re} z > 0$ and rapidly decreasing vertically on compact positive real strips. Boundedness of $\widehat{W}_0$ handles zero, Schwartz decay handles infinity, and logarithmic integration by parts at a complex place handles height. The Mellin transform $\widehat{W}_1(w)$ has rapid vertical decay on every fixed real strip. Normalize $d^{\times} y = (dT/T) dt$ for $y = t\ell_T$, $t \in F_S^1$, and use the induced measure on F_S^1/Γ .

For [\(5.6\)](#), compactness of F_S^1/Γ permits bounded representatives. On those representatives the element sum has finite support, and the ideal series defining B_m is absolutely convergent. Thus the fixed average is well-defined before either transformation is applied.

10.3. Combining the additive and theta estimates. For a bounded representative t of F_S^1/Γ , write the additive factor in (5.6) as

$$A_t(m; X, Y) = \frac{1}{Y} \sum_{\mathfrak{s}} W_1(q_{\mathfrak{s}}/Y) \frac{g_{\mathfrak{s}}(-m)}{q_{\mathfrak{s}} \sqrt{X}} W_0\left(\frac{m}{t\ell_{q_{\mathfrak{s}}X}}\right).$$

Thus $I(X, Y, Z)$ is the integral of $\sum_{m \neq 0} A_t(m; X, Y) B_m(Z)$ over the quotient. We estimate the two factors in this sum by Cauchy–Schwarz.

Proposition 10.1. *With the field, characters, local data, and weights in (5.5) fixed as just constructed, for every $\varepsilon > 0$,*

$$|I_Z| := |I(Z^{\alpha_0}, Z^{\beta_0}, Z)| \ll_{\varepsilon} Z^{287/800+\varepsilon}.$$

Proof. This follows the deduction in [30, Proposition 7.2] at the new scales. Expand the smooth weight in $\log(q_{\mathfrak{s}}/Y)$ by Fourier inversion. Two integrations by parts make the resulting suprema summable. Before the common normalization $(XY)^{-1/2}$, the coefficient of each reduced fraction d/s is of size $Y^{-1}q_{\mathfrak{s}}^{-1/2}$. The sum of coefficient squares is $O(Y^{-1})$: there are $O_F(Y)$ denominator ideals and at most $q_{\mathfrak{s}}$ reduced residues for each. Apply (10.2) with $N \asymp XY$, $V \asymp Y$. The square norm of the normalized additive factor is

$$(10.3) \quad \sum_{m \in R \setminus \{0\}} |A_t(m; X, Y)|^2 \ll (XY)^{-1} (XY + Y^2) Y^{-1} \ll Y^{-1},$$

since $\beta_0 < \alpha_0$. By Proposition 9.1, the sum of $|B_m(Z)|^2$ over the support of the additive weight is $O_{\varepsilon}(Z^{2M_0-1+\varepsilon})$. Cauchy–Schwarz, followed by integration over the compact quotient, gives

$$|I_Z| \ll_{\varepsilon} Z^{M_0-1/2-\beta_0/2+\varepsilon} = Z^{287/800+\varepsilon}. \quad \square$$

11. POISSON SUMMATION AND THE EULER PRODUCT

The second evaluation of I_Z is [30, Proposition 7.3 and Section 8]. We record the resulting frequency series and Euler product, since they identify the reciprocal whose poles are excluded. The finite Fourier calculation, including the common-prime terms, is used by citation.

11.1. The frequency series and its factorization. For an exterior modulus $\mathfrak{s} = sR$, a theta index $A = cn^3$, and a frequency $h \in R$, define

$$\mathcal{P}(s, A, h) = \sum_{\substack{d \bmod \mathfrak{s} \\ h \equiv Ad \pmod{\mathfrak{s}}}} \chi_{\mathfrak{s}}(d) \sum_{v \bmod AR} \chi_{AR}(v) e\left(\frac{(h - Ad)v}{sA}\right).$$

The unit ideal contributes one term equal to one, and all characters retain their zero values on nonunits. This finite sum is the one in [30, Section 7]; it is defined without a coprimality assumption on s and A .

With $V_0(T) = \int_{F_S^1} \widehat{W}_0(t\ell_T) dt$, invert the Gaussian, W_1 , and V_0 , and write the first Mellin variable as $\xi + z - 1$. With $h = ur^6$ as in Lemma 7.4, the result is

$$I(X, Y, Z) = \frac{1}{2\pi i} \int_{\operatorname{Re} \xi = 4} \frac{1}{2\pi i} \int_{\operatorname{Re} w = 4} \frac{1}{2\pi i} \int_{\operatorname{Re} z = 2} X^{1/2-z} Z^{\xi+z-1} Y^{w-1} \\ \times e^{(\xi+z-1)^2} M(z) \widehat{W}_1(w) \sum_{u \in \mathcal{U}} q_u^{-z} \mathcal{F}_u(\xi, w, z) dz dw d\xi,$$

where $q_u = |u|_S$ and

$$\mathcal{F}_u(\xi, w, z) = \sum_{\substack{\mathfrak{c} \text{ squarefree} \\ \mathfrak{n}, \mathfrak{s}, \mathfrak{r}}} \gamma_2(\mathfrak{c}) \vartheta(\mathfrak{c}n^3) \mathcal{P}(s, cn^3, ur^6) q_{\mathfrak{c}}^{-1/2-\xi} q_{\mathfrak{n}}^{-1-3\xi} q_{\mathfrak{s}}^{-w} q_{\mathfrak{r}}^{-6z}.$$

The zero frequency vanishes because $\widehat{W}_0(0) = 0$. Unfolding sixth-power unit orbits against the norm-one quotient gives V_0 , whose Mellin transform is M . These operations, and absolute convergence on the initial lines, are justified in [30, Section 7] using Tate's Poisson formula and the fixed local measures. They hold over any F with the arithmetic choices of Section 7.

The local calculation factors the series as follows:

$$(11.1) \quad \mathcal{F}_u(\xi, w, z) = \frac{\zeta_F^S(6z) L^S(w, \chi_\bullet(u), F)}{L^S(\xi, \chi_\bullet(u), F)} H_u(\xi, w, z).$$

Write $x = \operatorname{Re} \xi$, $y = \operatorname{Re} w$, and $z_0 = \operatorname{Re} z$. The product $H_u = \prod_{p \notin S} H_{u,p}$ is holomorphic in the domain

$$(11.2) \quad \begin{aligned} x, y, z_0 &> 0, & 6z_0 + y &> 1, & x + 6z_0 &> 1, & x + y &> 1, \\ x + y + 6z_0 &> 2, & 6x + 6z_0 &> 5, \\ x &> \frac{1}{2}, & 3x &> 2 + \max(0, \frac{1}{2} - y). \end{aligned}$$

On any closed subregion with fixed positive lower bounds for these coordinates and margins, $H_u \ll_\varepsilon q_u^\varepsilon$, uniformly in height.

Here H_u is the product of the local correction factors in [30, Proposition 7.3]; the larger domain follows directly from its local formulas (7.19)–(7.22). At a prime not dividing u , put $Q = \mathbf{N}p$. The local variables satisfy

$$|V| = Q^{-6z_0}, \quad |W| = Q^{-y}, \quad |D_1| = Q^{-x}, \quad |E_1| = Q^{4-6x-6z_0}.$$

The coefficient of $Q^{-\xi}$ is $-\overline{\chi_p(u)}$ by (7.5). After extracting the three L -factors, the local formula gives

$$H_{u,p} - 1 \ll |VW| + |D_1V| + |D_1W| + Q|D_1WV| + |E_1|.$$

The five inequalities on the first two lines of (11.2) make this $O(Q^{-1-\delta_H})$ for some $\delta_H > 0$. All local denominators are bounded away from zero on a region with fixed margins.

At a prime dividing u , the exterior valuation of u is between one and five. The leading nonconstant term is $O(Q^{1-x-y})$; the remaining shell exponents in the same local calculation are

$$(\frac{1}{2} - x)e_0 + (2 - 3x)\ell + (\frac{1}{2} - y)k - 6z_0m, \quad e_0, k \in \{0, 1\}, \quad \ell, m \geq 0, \quad e_0 + 3\ell > 0.$$

The positive-shell coefficients vanish for $k \geq 2$. If $\ell = 0$, the largest exponent is $\max(\frac{1}{2} - x, 1 - x - y) < 0$. If $\ell \geq 1$, it is at most $2 - 3x + \max(0, \frac{1}{2} - y) < 0$, and the ℓ, m sums are geometric. Thus $H_{u,p} = 1 + O(Q^{-\delta_H})$, after possibly decreasing δ_H . Normal convergence away from u and

$$\prod_{p|u} (1 + CQ^{-\delta_H}) \ll_\varepsilon q_u^\varepsilon$$

prove the assertions. For the last bound, absorb each sufficiently large prime factor into Q^ε , and the finitely many smaller primes into the constant. These estimates use only Q and unitary character values, so they apply over our F without further enlargement of S .

The lowest nonprincipal corner used below is $(x, y, z_0) = (0.8801, 0.12, 0.4)$. The five convergence margins are respectively 1.52, 2.2801, 0.0001, 1.4001, 2.6806, and the last two margins are 0.3801, 0.2603. The principal remainder corners $(1 + 10^{-7}, 0.9, 1/6 + 10^{-7})$ and $(1 + 10^{-7}, 1, 0.05)$ also satisfy (11.2), as do all $(4, 4, z_0)$ with $z_0 \geq 2$. Decreasing each coordinate of any of these corners by 10^{-6} preserves all inequalities, so the contours have open neighborhoods with fixed margins. The domain (11.2) is convex: its last inequality is equivalent to $3x > 2$ and $3x + y > 5/2$. Therefore (11.1) continues meromorphically from the initial region throughout this domain; the original frequency series need not converge there. Nonvanishing of the full three-variable correction is unnecessary; it is the principal specialization that must not cancel a pole.

11.2. Noncancellation of the principal term. At $(u, w, z) = (1, 1, 1/6)$, set $H(s) = H_1(s, 1, 1/6)$. The coefficient -1 in (7.5) identifies $1/\zeta_F^S(s)$. To ensure that H cannot cancel one of its poles, we need the full local factor. OpenAI's local calculation [30, Proposition 7.3 and Section 8.3] gives

$$(11.3) \quad H_p(s) = (1 - Q^{-2}) \frac{1 - (1 - Q^{-1})Q^{-s} - \vartheta(p)^6 Q^{2-6s}}{(1 - \vartheta(p)^6 Q^{3-6s})(1 - Q^{-s})}.$$

Lemma 11.1. *The product H is holomorphic and nowhere zero for $\operatorname{Re} s > 2/3$. It and its reciprocal are bounded on every $\operatorname{Re} s \geq \sigma_0 > 2/3$.*

Proof. At a prime of norm Q , put $r = Q^{-1}$, $D = Q^{-s}$, and $E = \vartheta(p)^6 Q^{3-6s}$. Each local factor is holomorphic and nonzero already for $\sigma = \operatorname{Re} s > 1/2$. Its denominators have $|D|, |E| < 1$, while the nonconstant part of its numerator has modulus at most

$$(1 - r)|D| + r|E| \leq (1 - r)Q^{-1/2} + r < 1.$$

Uniformly for $\sigma \geq \sigma_0 > 2/3$, subtraction of one gives

$$H_p(s) - 1 \ll_{\sigma_0} Q^{-2} + Q^{-1-\sigma_0} + Q^{3-6\sigma_0}.$$

Each exponent is less than -1 . Summing over prime ideals proves normal convergence. A convergent logarithm for the tail and the nonzero finitely many initial factors prove nonvanishing and the two uniform bounds. No extra removal of small primes is needed. $\square$

12. THE CONDUCTOR SIEVE AND ZERO DETECTION

For nonprincipal frequencies the denominator in (11.1) varies with u . We first show that few of these denominators have a zero in a small rectangle near one. Only after excluding those characters will we move a contour left of one. The resulting reciprocal bound is therefore conditional only on the exclusion of these exceptional characters.

12.1. A conductor sieve from the additive sieve. For $u \neq 1$, let ω_u be the primitive character inducing $\chi_{\bullet}(u)$. By Lemma 7.4 these characters are nonprincipal and distinct. On a dyadic interval $U \leq q_u < 2U$, they have conductor norm $O_{F,S}(U)$ and components at S in a fixed finite list. Ideal counting and the finite choice of unit classes give $O_{F,S}(U)$ such frequencies u .

For any family with these properties,

$$(12.1) \quad \sum_{\omega} \left| \sum_{N\mathfrak{n} \leq N} b_{\mathfrak{n}} \omega(\mathfrak{n}) \right|^2 \ll_{\varepsilon} (N + U^2)(NU)^{\varepsilon} \sum_{\mathfrak{n}} |b_{\mathfrak{n}}|^2.$$

This is [30, Lemma 8.1]. Fixing the components at S , expanding the primitive exterior character by its Gauss sum, and using finite character orthogonality reduces it to the dual of (10.2). Balanced generators and the fixed S -data contribute only field-dependent constants and arbitrarily small power losses. Thus the proof is unchanged for our F .

Hecke's functional equation and conductor-aspect convexity give, for some fixed A_F ,

$$(12.2) \quad |L(\sigma + it, \omega_u, F)| \ll_{\varepsilon} U^{(1-\sigma)/2+\varepsilon} (2 + |t|)^{A_F} \quad (0 \leq \sigma \leq 1)$$

for nonprincipal ω_u , with A_F independent of the loss. To retain the σ -dependence, let $Q_c = 1 + N \operatorname{cond}(\omega_u)$ and $0 < d \leq 1/10$. The functional equation and absolute convergence give uniform boundary bounds $Q_c^{1/2+d} (2 + |t|)^{A_F}$ at $\sigma = -d$ and $O_{F,d}(1)$ at $\sigma = 1 + d$. Apply Phragmén–Lindelöf on this strip to $Q_c^{(s-1-d)/2} L(s, \omega_u, F)$. The conductor powers cancel on both boundaries, giving

$$|L(\sigma + it, \omega_u, F)| \ll_{F,d} Q_c^{(1+d-\sigma)/2} (2 + |t|)^{A_F}.$$

Choose $d/2 \leq \varepsilon$ and use $Q_c \ll_{F,S} U$ to obtain (12.2). On other fixed strips, polynomial height bounds suffice; the principal character is handled by $(s-1)\zeta_F(s)$. Omitting the extra factors at primes dividing u costs q_u^ε , as explained in Section 4. The field's degree affects the height exponent, but not the conductor exponent $(1-\sigma)/2$.

12.2. Detecting a hypothetical zero.

Lemma 12.1. *Choose $0 < \rho < \min(10^{-4}, 10^{-3}/(A_F + 1))$. There is $U_0 \geq 2$ such that, on each dyadic interval $U \leq q_u < 2U$ with $U \geq U_0$, the set $\mathcal{E}(U)$ of frequencies $u \neq 1$ whose primitive denominator has a zero in*

$$\operatorname{Re} s \geq 0.88, \quad |\operatorname{Im} s| \leq 4U^\rho$$

has size $O_{F,S}(U^{317/500})$. The sets $\mathcal{E}(U)$ are defined independently of ε . For every $\varepsilon > 0$ there is $U_\varepsilon \geq U_0$ such that, whenever $U \geq U_\varepsilon$, all frequencies $u \neq 1$ in that interval outside $\mathcal{E}(U)$ satisfy

$$(12.3) \quad \left| L^S(\xi, \overline{\chi_\bullet(u)}, F)^{-1} \right| \ll_\varepsilon U^\varepsilon \quad (\operatorname{Re} \xi \geq 0.8801, |\operatorname{Im} \xi| \leq 2U^\rho).$$

Proof. We modify the detector in [30, Proposition 8.2]. Fix

$$\sigma_0 = \frac{22}{25}, \quad q = \frac{9}{5}, \quad c = \frac{1}{100}, \quad r = \frac{131}{50}, \quad r' = \frac{21}{8}.$$

For all integral ideals, including those meeting S , put

$$D_U(s, \omega) = \sum_{N\mathfrak{d} \leq U^q} \mu_F(\mathfrak{d}) \omega(\mathfrak{d}) (N\mathfrak{d})^{-s}, \quad a_U(\mathfrak{n}) = \sum_{\substack{\mathfrak{d}|\mathfrak{n} \\ N\mathfrak{d} \leq U^q}} \mu_F(\mathfrak{d}).$$

Then $a_U(1) = 1$, $a_U(\mathfrak{n}) = 0$ for $1 < N\mathfrak{n} \leq U^q$, and $|a_U(\mathfrak{n})| \ll_\eta (N\mathfrak{n})^\eta$ for every $\eta > 0$. Complete multiplicativity, with zero values at conductor primes, gives $LD_U = \sum_{\mathfrak{n}} a_U(\mathfrak{n}) \omega(\mathfrak{n}) (N\mathfrak{n})^{-s}$ in absolute convergence. The coefficients a_U are independent of ω .

At a zero $\beta + i\gamma$ with $\sigma_0 \leq \beta \leq 1$ and $|\gamma| \leq 4U^\rho$, exponential Mellin inversion gives

$$\sum_{\mathfrak{n}} a_U(\mathfrak{n}) \omega(\mathfrak{n}) (N\mathfrak{n})^{-\beta-i\gamma} e^{-N\mathfrak{n}/U^r} = \frac{1}{2\pi i} \int_{(2)} L(\beta + i\gamma + v, \omega, F) D_U(\beta + i\gamma + v, \omega) \Gamma(v) U^{rv} dv.$$

There are no zeros with $\beta > 1$. Move the line to $\operatorname{Re} v = c - \beta \in [-0.99, -0.87]$. Only the gamma pole at zero is crossed, and its residue vanishes because $L(\beta + i\gamma, \omega, F) = 0$; the pole at -1 is not crossed. The L -function is entire and the gamma factor controls the horizontal limits. Ideal counting bounds $D_U(c + it, \omega) \ll U^{q(1-c)}$, and (12.2) bounds the shifted integral by

$$U^{q(1-c)+(1-c)/2+r(c-\sigma_0)+\eta}(2+|\gamma|)^{A_F} = U^{-3/1250+\eta}(2+|\gamma|)^{A_F} = o(1).$$

Here one may take $\eta = 1/10000$, since $A_F \rho < 1/1000$.

The unit coefficient on the left is $1 + o(1)$. The terms with norm above $U^{r'}$ are bounded by a fixed power of U times $\exp(-U^{r'-r}/2) = \exp(-U^{1/200}/2)$. Thus the sum over $U^q < N\mathfrak{n} \leq U^{r'}$ has modulus bounded below. Splitting into $O(\log U)$ dyads and applying partial summation to the decreasing weight $t^{\sigma_0-\beta} e^{-t/U^r}$ forces a dyad $U^q/2 < N \leq U^{r'}$ on which

$$\max_{N \leq y \leq 2N} \left| \sum_{N < N\mathfrak{n} \leq y} a_U(\mathfrak{n}) \omega(\mathfrak{n}) (N\mathfrak{n})^{-\sigma_0-i\gamma} \right| \gg (\log U)^{-1}.$$

The squared coefficient mass is $O_\eta(N^{1-2\sigma_0+\eta})$. To make (12.1) maximal in y , order the ideals by norm and decompose prefixes into binary intervals; this costs only logarithmic factors. To allow varying γ , use, for each resulting polynomial P_ω ,

$$\sup_{t \in [j, j+1]} |P_\omega(t)|^2 \ll \int_{j-1}^{j+2} (|P_\omega(t)|^2 + |P'_\omega(t)|^2) dt.$$

The derivative inserts only $\log Nn$. The height range therefore costs $O(U^\rho)$ and logarithms. Both terms of the conductor sieve are needed, since $q < 2$:

$$(N + U^2)N^{1-2\sigma_0} = N^{2-2\sigma_0} + U^2N^{1-2\sigma_0} \ll U^{63/100} + U^{79/125}.$$

Use loss $1/100000$ in the sieve and in the coefficient-mass estimate. Since $N \ll U^{r'}$, their combined U -power cost is at most $(1 + 2r')/100000 = 1/16000$. Absorb all logarithms into $U^{1/10000}$. Chebyshev's inequality then bounds the exceptional count by a constant times

$$U^{79/125+\rho+1/16000+1/10000} \ll U^{317/500}.$$

Distinctness of the primitive characters transfers this to the count of frequencies.

For the reciprocal bound, put $T = U^\rho$ and take a nonexceptional character. For $|t| \leq 2T$, the disk centered at $2 + it$ with radius 1.11995 lies inside the zero-free rectangle for sufficiently large U . Choose the analytic logarithm of L agreeing with the Euler logarithm at the center. Convexity bounds its real part above by $O_F(\log U)$; Borel–Carathéodory bounds its modulus by $O_F(\log U)$ on radius 1.119925. On radius 0.999, absolute convergence bounds the Euler logarithm by $O_F(1)$. The three-circles theorem gives, on radius 1.1199,

$$|\log L(s, \omega, F)| \ll_F (\log U)^\theta, \quad \theta = \frac{\log(1.1199/0.999)}{\log(1.119925/0.999)} < 1.$$

These disks reach $\operatorname{Re} s = 0.8801$; absolute convergence covers the remaining right half-plane. Hence $1/L \ll_\varepsilon U^\varepsilon$. The extra omitted Euler factors cost another arbitrarily small power, proving (12.3). The exceptional sets are unchanged when these losses vary. $\square$

This is a zero-density argument for the auxiliary family, not an assumption of zero-freeness for the target function. The principal frequency $u = 1$ is treated separately in Section 13.

13. CONTOUR COMPARISON AND THE NUMERICAL REFINEMENT

We follow the contour comparison of [30, Sections 8.2–8.3], with the new scales, detector, and Euler domains established above. We record the altered contours and their margins. Together with the arbitrary loss in the theta bound, these changes improve the width from $1/6000$ to $1/32$.

13.1. The principal term and the other frequencies. For a contour with real coordinates x, y, z_0 , the external power of Z in the triple Mellin integral is

$$(13.1) \quad E(x, y, z_0) = \frac{\alpha_0}{2} + x - 1 + h_0 z_0 + \beta_0(y - 1).$$

Put $\delta = 10^{-7}$ and $x_1 = 1 + \delta$. For $u = 1$, first move to $(x, y, z_0) = (x_1, x_1, 1/6 + \delta)$. Move w to 0.9, taking the residue at one; in that residue move z to 0.05, taking the residue at $1/6$. The two remainder contours are

$$(x_1, 0.9, 1/6 + \delta), \quad (x_1, 1, 0.05).$$

The denominator remains in absolute convergence. The double residue has exponent $s - 61/100$ and gives $c_0 f(Z)$, where

$$(13.2) \quad c_0 = \frac{R_S^2}{6} M(1/6) \widehat{W}_1(1) \neq 0.$$

Here $R_S = \operatorname{Res}_{s=1} \zeta_F^S(s) > 0$. The nonvanishing of $M(1/6)$ was part of the fixed choice of W_0 , and $\widehat{W}_1(1) > 0$. The residue therefore gives exactly the function in (2.1), with the correction factor (11.3).

For the other frequencies, split into dyads $U \leq q_u < 2U$ on the initial absolutely convergent lines. Deform each finite dyadic sum before resumming. Use direct estimates for $U \leq Z^{1/5}$. In

$$Z^{1/5} < U \leq Z^{3301/10000},$$

the exceptional set from Lemma 12.1 keeps its denominator contour at x_1 ; the remaining characters permit a finite segment to move to $\operatorname{Re} \xi = 0.8801$. For all these rows use $\operatorname{Re} w = 3/25$, $\operatorname{Re} z = 2/5$. Move ξ , then z , then w ; no numerator pole is crossed for nonprincipal rows. All moves stay in (11.2).

On the w -line, (12.2) costs $U^{11/25+\varepsilon}$, while q_u^{-z} supplies $U^{-2/5}$. Thus a subset of $O(U^\nu)$ rows contributes

$$\ll_{\varepsilon} Z^{E(x_1, 3/25, 2/5)} U^{\nu+1/25+\varepsilon}.$$

Relative to the principal reference exponent $C(1) = 39/100$,

$$E(x_1, 3/25, 2/5) - C(1) = -0.2552 + \delta.$$

Small rows use $\nu = 1$, and exceptional intermediate rows use $\nu = 317/500$. Good intermediate rows move only $|\operatorname{Im} \xi| \leq U^\rho$ to $x_g = 0.8801$, where (12.3) gives

$$\ll_{\varepsilon} Z^{E(x_g, 3/25, 2/5)} U^{26/25+\varepsilon}, \quad E(x_g, 3/25, 2/5) - C(1) = -0.3751.$$

The endpoint powers of U give the savings in Table 1.

The only degree-dependent issue is growth in height. First fix A_F , then choose ρ as in Lemma 12.1. The fixed weights satisfy, on the real strips in use,

$$|e^{(\xi+z-1)^2} M(z) \widehat{W}_1(w)| \ll_A e^{-(\operatorname{Im} \xi + \operatorname{Im} z)^2} (1 + |\operatorname{Im} z|)^{-A} (1 + |\operatorname{Im} w|)^{-A}.$$

Increasing A absorbs the polynomial growth of the numerator factors. Convolution with the Gaussian then gives arbitrary decay in the remaining ξ -height. The horizontal segments of the finite move and the discarded original tails contribute, for any chosen A_1 ,

$$\ll_{A_1, \varepsilon} Z^{E(x_1, 3/25, 2/5)} U^{26/25+\varepsilon-\rho A_1}.$$

On the horizontal segments the reciprocal is bounded by (12.3); on the original tails it is bounded by absolute convergence at $x_1 > 1$. Since $U > Z^{1/5}$, choosing $A_1 > 10/\rho$ makes these negligible. This chooses a decay estimate, not different weights.

For $U > Z^{3301/10000}$, return to the initial dyadic integral, keep $x = y = 4$, and move z right to the fixed line 50000. No pole is crossed. Absolute convergence of the L -factors and (11.2) give a dyadic bound

$$\ll_{\varepsilon} Z^{4.4675+0.33 \cdot 50000} U^{1-50000+\varepsilon}.$$

The dyads sum geometrically; at their lower endpoint the exponent of Z , before small losses, is

$$4.4675 + 0.33 \cdot 50000 + \frac{3301}{10000} (1 - 50000) = -0.2024.$$

The slight gap $3301/10000 - h_0 = 10^{-4}$ is what makes this rightward shift effective.

13.2. Keeping the full margin. The exact savings before arbitrarily small power losses are recorded in Table 1. Choose the losses in all Poisson bounds, including the conversions of U^ε into powers of Z and the dyadic logarithms, so that their combined cost is less than $1/4000$. The least Poisson saving is $7949/250000$, and

$$\frac{7949}{250000} - \frac{1}{4000} > \frac{63}{2000} > \frac{1}{32}.$$

Enlarge the threshold for Z so that every intermediate dyad meets the reciprocal thresholds for these losses. Bounded Z is absorbed into the constants. Hence the comparison gives

$$(13.3) \quad I_Z = c_0 f(Z) + O\left(Z^{717/2000}\right).$$

The constants and the threshold in Z may depend on F and the fixed data. All defining weights are fixed independently of the loss in the theta estimate.

TABLE 1. Savings with the revised scales and contours, measured from 39/100.

Contribution	Saving
Theta-side upper bound	1/32
Principal w -remainder	37749867/1000000000
Principal z -remainder	384999/10000000
Small frequencies	471999/10000000
Exceptional intermediate frequencies	2617/80000
Nonexceptional central integrals	7949/250000
Large frequencies	1481/2500

Use [Proposition 10.1](#) for each $\varepsilon > 0$. Since $c_0 \neq 0$ and $717/2000 < 287/800$, [\(13.3\)](#) gives $f(Z) \ll_\varepsilon Z^{287/800+\varepsilon}$. For any compact subset of $\operatorname{Re} s > 31/32$, one can choose ε smaller than its distance to the boundary. This is precisely the hypothesis used in [Proposition 6.1](#); no estimate on the boundary is required.

Proof of Proposition 7.1. The construction in [Section 7.2](#) chooses one fixed S and compatible generators over F . [Lemma 7.3](#) gives the norm-pullback and Weil-index corrections, so the target identity [\(7.5\)](#) holds at every exterior prime. The rational compatibility check in [Section 8](#) supplies the global theta representation, and the local and complex-place checks following [Proposition 8.3](#) give the same completed family and fixed transformed weights as in the cited proof.

The primitive-conductor verification in [Section 9.1](#) permits Goldmakher–Louvel’s sieve. The higher-power bounds, support-exponent check, and ideal counting in [Section 9](#) then give [Proposition 9.1](#) over F at the new scale. With the additive normalization of [Section 10](#), this proves [\(5.7\)](#). The Poisson and local-factor checks in [Section 11](#) identify the second evaluation and give the enlarged correction domains and the required bounds for H and its reciprocal. Finally, the conductor-dependent convexity estimate and replacement detector in [Section 12](#) permit the contour comparison proved in this section, giving [\(13.3\)](#), which is [\(5.8\)](#).

All arithmetic data, local vectors, and smoothing functions were fixed before choosing the losses in these estimates. Shrinking those losses changes implied constants and thresholds, not B_m , I_Z , or f . Thus the two bounds concern the same fixed functions for every $\varepsilon > 0$, as required. $\square$

Completion of the proof of Theorem 5.1. [Proposition 7.1](#) establishes the two estimates for the same fixed average and weights over F . The local factor in [Lemma 11.1](#) is holomorphic and nowhere zero in the required half-plane. Thus [Proposition 6.1](#) gives $\zeta_F(s) \neq 0$ for $\operatorname{Re} s > 31/32$ and $s \neq 1$. Restoring the finite Euler factors preserves this assertion, and the pole at one causes no obstruction. The descent in [\(5.1\)](#) gives [Theorem 1.1](#). $\square$

14. PROOF OF COROLLARY 1.2

Let χ be a finite-order Hecke character over K . For the trivial character the claim is [Theorem 1.1](#). Otherwise, let $r > 1$ be its order. Global class-field theory associates to $\ker \chi$ a cyclic extension L/K of degree r , with

$$\zeta_L(s) = \zeta_K(s) \prod_{j=1}^{r-1} L(s, \chi^j, K).$$

Every factor in the product is entire. Away from $s = 1$, a zero of $L(s, \chi, K)$ would therefore be a zero of $\zeta_L(s)$, contrary to [Theorem 1.1](#). At $s = 1$, take the ratio of the simple poles of the two

zeta functions to obtain

$$\prod_{j=1}^{r-1} L(1, \chi^j, K) = \frac{\text{Res}_{s=1} \zeta_L(s)}{\text{Res}_{s=1} \zeta_K(s)} \neq 0.$$

Thus $L(1, \chi, K) \neq 0$ as well. If the character is represented with a modulus larger than its conductor, the extra omitted Euler factors have zeros only on $\text{Re } s = 0$, so they do not affect the assertion.

15. AXIOMPROVER LEAN CERTIFICATE

This paper was written using human-AI collaboration with AxiomProver. The investigators set the direction and made the mathematical judgments, and they take full responsibility for the correctness of the results. AxiomProver is an AI system for mathematical research through formal proof, currently under development at Axiom Math. Theorem 1.1 was formalized with AxiomProver in Lean. The repository contains the formal statements, proofs, and instructions for checking the certificate:

<https://github.com/AxiomMath/DedekindZetaZeroFree>

The challenge file gives the definitions. The solution proves the stated targets. The repository reports successful verification with the Lean comparator.

The formal proof includes the cubic rank-two metaplectic constructions needed from Kazhdan and Patterson [20, 21] over number fields containing μ_{12} . Abelian descent then gives the result for arbitrary number fields. In contrast, OpenAI [30, Section 9] uses abelian factorization and descent to pass from its result on zero-free regions for Hecke L -functions over cyclotomic fields to a zero-free region for the Dedekind-zeta functions of Kummer fields $\mathbb{Q}(e^{2\pi i/q}, a^{1/q})$.

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