
Finite Field Riesz-Frechet Representation

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Abstract: We derive a Riesz-Frechet representation for bounded linear functionals defined on the inner product spaces over sub-valued finite fields.

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1. INTRODUCTION

Derived over a century ago, the Riesz-Frechet representation (theorem) states that every bounded linear functional on a Hilbert space is uniquely determined by a single element in that same space.

Theorem 1.1. *[1, 2, 4, 6–12, 16–19, 21, 23, 25, 28–33, 36, 38, 39, 41] (**Riesz-Frechet Representation**) Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $f : \mathcal{H} \rightarrow \mathbb{K}$ be a bounded linear functional. Then there is a unique element $\tau \in \mathcal{H}$ satisfying the following conditions.*

- (i) $f(h) = \langle h, \tau \rangle$ for all $h \in \mathcal{H}$.
- (ii) $\|f\| = \|\tau\|$.

Over time, the Riesz-Frechet representation has been broadened to encompass Lebesgue spaces [27, 34, 37], as well as continuous functionals on compact Hausdorff spaces (known as Riesz-Markov-Kakutani representation) [2, 3, 5, 6, 13, 22, 26, 32, 35, 40]. It is known that the Riesz-Frechet representation extends naturally to quaternionic Hilbert spaces [14]. Giles [15] and James [20] derived Riesz-Frechet representation for certain classes of Banach spaces. The traditional proof of the Riesz-Frechet representation relies on the orthogonal decomposition theorem, which fundamentally depends on the parallelogram law [17, 29]. If the Hilbert space is separable, one can also prove the Riesz-Frechet representation using orthonormal bases, a method that relies on the Gram-Schmidt orthonormalization process [18]. We are fundamentally motivated by the following question.

Question 1.2. *What is the finite field version of Riesz-Frechet representation?*

We begin by the notion of sub-modulus field.

Definition 1.3. *Let $\mathbb{F}$ be a (finite) field. A map $|\cdot| : \mathbb{F} \rightarrow [0, \infty)$ is said to be a **sub-modulus** or **sub-valued** if the following conditions hold.*

- (i) *If $\lambda \in \mathbb{F}$ is such that $|\lambda| = 0$, then $\lambda = 0$.*
- (ii) *$|\lambda\mu| \leq |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*
- (iii) *$|\lambda + \mu| \leq |\lambda| + |\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*

In this case, we say $\mathbb{F}$ as sub-modulus or sub-valued field.

Example 1.4. Let p be a prime, let $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$ be the standard field of integers modulo p . Given $n \in \mathbb{Z}_p$, we define

$$|n| := \text{unique } a \text{ such that } 0 \leq a \leq p-1 \text{ and } n \equiv a \pmod{p}.$$

We next introduce the notion of sub-normed linear spaces and inner product spaces.

Definition 1.5. Let $\mathcal{X}$ be a vector space over a sub-modulus field $\mathbb{F}$. A map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be a **sub-norm** if the following conditions hold.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| \leq |\lambda| \|x\|$ for all $\lambda \in \mathbb{F}$, for all $x \in \mathcal{X}$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.

In this case, we say $\mathcal{X}$ is a **sub-normed linear space**.

Definition 1.6. Let $\mathbb{K}$ be a sub-modulus field (with sub-modulus $|\cdot|$) and $\mathcal{X}$ be a sub-normed space (with sub-norm $\|\cdot\|$) over $\mathbb{F}$. We say that $\mathcal{X}$ is an **inner product space** if there is a map (called as **inner product**) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{F}$ satisfying the following conditions:

- (i) If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.
- (ii) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{X}$.
- (iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $\alpha \in \mathbb{F}$, for all $x, y \in \mathcal{X}$.
- (iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathcal{X}$.
- (v) $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{X}$.

Example 1.7. For $d \in \mathbb{N}$, let $\mathbb{Z}_p^d$ be the standard vector space over $\mathbb{Z}_p$. We define the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j \in \mathbb{Z}_p, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Z}_p^d$$

and the sub-norm

$$\|(c_j)_{j=1}^d\|^2 := \sum_{j=1}^d |c_j|^2 \in \{0, 1, \dots, d(p-1)^2\}, \quad \forall (c_j)_{j=1}^d \in \mathbb{Z}_p^d.$$

Example 1.8. Let $\mathbb{F}$ be a sub-modulus field. Define

$$c_{00}(\mathbb{N}, \mathbb{Z}_p) := \{ \{a_n\}_{n=1}^\infty : a_n \in \mathbb{Z}_p, \forall n \in \mathbb{N}, a_n \neq 0 \text{ only for finitely many } n \}.$$

We define the inner product

$$\langle \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \rangle := \sum_{n=1}^\infty a_n b_n, \quad \forall \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in c_{00}(\mathbb{N}, \mathbb{Z}_p)$$

and the sub-norm

$$\|\{a_n\}_{n=1}^\infty\| := \left(\sum_{n=1}^\infty |a_n|^2 \right)^{\frac{1}{2}}, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}(\mathbb{N}, \mathbb{Z}_p).$$

Then $\|\cdot\|_{00}$ is a sub-norm on $c_{00}(\mathbb{N}, \mathbb{F})$.

We establish two distinct formulations of the Riesz-Frechet representation within the context of finite fields, each employing unique assumptions and yielding different conclusions.

2. FINITE FIELD RIESZ-FRECHET REPRESENTATION

We begin by introducing following notion of orthonormal basis.

Definition 2.1. Let $\mathcal{X}$ be an inner product space over sub-modulus field $\mathbb{F}$. A (Hamel) basis $\{\tau_n\}_n$ for $\mathcal{X}$ is said to be an **orthonormal basis** for $\mathcal{X}$ if the following conditions hold:

- (i) $\|\tau_n\| \leq 1$ for all $n \in \mathbb{N}$.
- (ii) $\langle \tau_n, \tau_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.

Note that given an orthonormal basis $\{\tau_n\}_n$ for $\mathcal{X}$, condition (ii) in Definition 2.1 says that

$$1 = |\langle \tau_n, \tau_n \rangle| \leq \|\tau_n\|^2, \quad \forall n \in \mathbb{N}.$$

By using condition (i) in Definition 2.1 we then get $\|\tau_n\| = 1$ for all $n \in \mathbb{N}$. Thus all elements of orthonormal basis lie on the unit sphere.

Gram-Schmidt orthonormalization says that every separable Hilbert space has an orthonormal basis [24]. It is clear that an inner product space over a sub-modulus field does not necessarily have an orthonormal basis. Let $\mathcal{X}$ be an inner product space over a sub-modulus field $\mathbb{F}$. A linear map $f : \mathcal{X} \rightarrow \mathbb{F}$ is said to be bounded if

$$\sup \left\{ \frac{|f(x)|}{\|x\|} : x \in \mathcal{X} \setminus \{0\} \right\} < \infty.$$

In this case, we define the norm of f as

$$\|f\| := \sup \left\{ \frac{|f(x)|}{\|x\|} : x \in \mathcal{X} \setminus \{0\} \right\}.$$

Following proposition is obvious.

Proposition 2.2. Let $\mathcal{X}$ be an inner product space over a sub-modulus field $\mathbb{F}$. Let $f, g : \mathcal{X} \rightarrow \mathbb{F}$ be bounded linear maps. Then

- (i) If $\|f\| = 0$, then $f = 0$.
- (ii) $\|\lambda f\| \leq |\lambda| \|f\|$ for all $\lambda \in \mathbb{F}$.
- (iii) $\|f + g\| \leq \|f\| + \|g\|$.

In particular, the space $\mathcal{X}^* := \{f : \mathcal{X} \rightarrow \mathbb{F} \text{ is bounded and linear}\}$ is a sub-normed space over $\mathbb{F}$ w.r.t. sub-norm $\|\cdot\|$.

It is clear that every bounded linear functional on an inner product space over a sub-modulus field cannot be represented using a vector in the space.

Following is our first version of Riesz-Frechet representation.

Theorem 2.3. (Finite Field Riesz-Frechet Representation) Let $\mathcal{X}$ be an inner product space over a sub-valued field $\mathbb{K}$. Let $f : \mathcal{X} \rightarrow \mathbb{F}$ be a bounded linear functional. Assume that there is an orthonormal basis $\{\omega_n\}_n$ for $\mathcal{X}$ such that

$$f(\omega_n) \neq 0 \quad \text{only for finitely many } n.$$

Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:

- (i) $f(x) = \langle x, \tau \rangle$ for all $x \in \mathcal{X}$.

(ii)

$$\frac{|\sum_{n=1}^{\infty} f(\omega_n)^2|}{\|\sum_{n=1}^{\infty} f(\omega_n)\omega_n\|} = \frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\| = \left\| \sum_{n=1}^{\infty} f(\omega_n)\omega_n \right\|.$$

Proof. Define

$$\tau := \sum_{n=1}^{\infty} f(\omega_n)\omega_n.$$

We then have

$$\begin{aligned} f(x) &= f\left(\sum_{n=1}^{\infty} \langle x, \omega_n \rangle \omega_n\right) = \sum_{n=1}^{\infty} \langle x, \omega_n \rangle f(\omega_n) \\ &= \left\langle x, \sum_{n=1}^{\infty} f(\omega_n)\omega_n \right\rangle = \langle x, \tau \rangle, \quad \forall x \in \mathcal{X}. \end{aligned}$$

Now

$$\|f\| = \sup \left\{ \frac{|f(x)|}{\|x\|} : x \in \mathcal{X}, x \neq 0 \right\} = \sup \left\{ \frac{|\langle x, \tau \rangle|}{\|x\|} : x \in \mathcal{X}, x \neq 0 \right\} \leq \|\tau\|.$$

We now observe that

$$\begin{aligned} \|\tau\| &= \left\| \sum_{n=1}^{\infty} f(\omega_n)\omega_n \right\|, \\ \langle \tau, \tau \rangle &= \left\langle \sum_{n=1}^{\infty} f(\omega_n)\omega_n, \sum_{m=1}^{\infty} f(\omega_m)\omega_m \right\rangle = \sum_{n=1}^{\infty} f(\omega_n)^2 \end{aligned}$$

and

$$|\langle \tau, \tau \rangle| = |f(\tau)| \leq \|f\| \|\tau\|.$$

Uniqueness easily follows using the definition of inner product. $\square$

Let $\mathcal{X}$ be an inner product space over sub-modulus field $\mathbb{F}$. A subspace $\mathcal{Y}$ of $\mathcal{X}$ is said to be orthogonally complementable if there exists a unique subspace $\mathcal{Z}$ of $\mathcal{X}$ satisfying the following: for every $x \in \mathcal{X}$, there exist unique $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ such that

- (i) $x = y + z$.
- (ii) $\langle y, z \rangle = 0$.
- (iii) $\|x\|^2 = \|y\|^2 + \|z\|^2$.

In this case, we write $x = y \oplus z$, $\mathcal{Z} = \mathcal{Y}^\perp$ and $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Y}^\perp$.

Following is our second version of Riesz-Frechet representation.

Theorem 2.4. (Finite Field Riesz-Frechet Representation) *Let $\mathcal{X}$ be an inner product space over a sub-valued field $\mathbb{F}$. Let $f : \mathcal{X} \rightarrow \mathbb{F}$ be a non-zero bounded linear functional. Assume that $\text{Ker}(f)$ is orthogonally complementable in $\mathcal{X}$ and there exists a (nonzero) vector $\omega \in \text{Ker}(f)^\perp$ such that $\langle \omega, \omega \rangle \neq 0$. Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:*

- (i) $f(x) = \langle x, \tau \rangle$ for all $x \in \mathcal{X}$.
- (ii)

$$\frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\|.$$

Moreover,

$$\|\tau\| = \frac{|f(\omega)|}{|\langle \omega, \omega \rangle|} \|\omega\| \leq \frac{\|f\| \|\omega\|^2}{|\langle \omega, \omega \rangle|}.$$

Proof. Define

$$\tau := \frac{f(\omega)}{\langle \omega, \omega \rangle} \omega.$$

Let $x \in \mathcal{X}$. Then

$$f\left(x - \frac{f(x)}{f(\omega)} \omega\right) = f(x) - f(x) = 0.$$

Hence

$$x - \frac{f(x)}{f(\omega)} \omega \in \text{Ker}(f).$$

We then have

$$\begin{aligned} 0 &= \left\langle x - \frac{f(x)}{f(\omega)} \omega, \tau \right\rangle = \langle x, \tau \rangle - \left\langle \frac{f(x)}{f(\omega)} \omega, \tau \right\rangle \\ &= \langle x, \tau \rangle - \frac{f(x)}{f(\omega)} \langle \omega, \tau \rangle = \langle x, \tau \rangle - \frac{f(x)}{f(\omega)} \left\langle \omega, \frac{f(\omega)}{\langle \omega, \omega \rangle} \omega \right\rangle \\ &= \langle x, \tau \rangle - f(x). \end{aligned}$$

Note that

$$|\langle \tau, \tau \rangle| = |f(\tau)| \leq \|f\| \|\tau\|.$$

□

We next extend the finite field Riesz-Frechet representation to finite fields admitting conjugation (involution). We use the following definition of conjugation.

Definition 2.5. Let $\mathbb{F}$ be a finite sub-valued field (with valuation $|\cdot|$). A map $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ is said to be a **conjugation** or **involution** if the following conditions hold:

- (i) $\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$ for all $\alpha, \beta \in \mathbb{F}$.
- (ii) $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ for all $\alpha, \beta \in \mathbb{F}$.
- (iii) $\sigma(\sigma(\alpha)) = \alpha$ for all $\alpha \in \mathbb{F}$.
- (iv) $|\sigma(\alpha)| = |\alpha|$ for all $\alpha \in \mathbb{F}$.

In this case, we say that $\mathbb{F}$ is σ -conjugated.

Definition 2.6. Let $\mathbb{F}$ be a σ -conjugated sub-valued field and $\mathcal{X}$ be a sub-normed space over $\mathbb{F}$. We say that $\mathcal{X}$ is a **σ -inner product space** if there is a map (called as **σ -inner product**) $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{K}$ satisfying the following:

- (i) If $x \in \mathcal{X}$ is such that $\langle x, y \rangle = 0$ for all $y \in \mathcal{X}$, then $x = 0$.
- (ii) $\langle x, y \rangle = \sigma(\langle y, x \rangle)$ for all $x, y \in \mathcal{X}$.
- (iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $\alpha \in \mathbb{F}$, for all $x, y \in \mathcal{X}$.
- (iv) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in \mathcal{X}$.
- (v) $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{X}$.

We now have generalized versions of Theorem 2.3 and Theorem 2.4.

Theorem 2.7. (Finite Field Riesz-Frechet Representation) Let $\mathcal{X}$ be a σ -inner product space over a σ -conjugated sub-valued field $\mathbb{F}$. Let $f : \mathcal{X} \rightarrow \mathbb{F}$ be a bounded linear functional. Assume that there is an orthonormal basis $\{\omega_n\}_n$ for $\mathcal{X}$ such that

$$f(\omega_n) \neq 0 \text{ only for finitely many } n.$$

Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:

- (i) $f(x) = \langle x, \tau \rangle$ for all $x \in \mathcal{X}$.
- (ii)

$$\frac{|\sum_{n=1}^{\infty} f(\omega_n)^2|}{\|\sum_{n=1}^{\infty} f(\omega_n)\omega_n\|} = \frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\| = \left\| \sum_{n=1}^{\infty} f(\omega_n)\omega_n \right\|.$$

Proof. Define

$$\tau := \sum_{n=1}^{\infty} \sigma(f(\omega_n))\omega_n.$$

The other arguments are similar to those used in the proof of Theorem 2.3. □

Theorem 2.8. (Finite Field Riesz-Frechet Representation) Let $\mathcal{X}$ be a p -adic σ -inner product space over a σ -conjugated sub-valued field $\mathbb{F}$. Let $f : \mathcal{X} \rightarrow \mathbb{K}$ be a non-zero bounded linear functional. Assume that $\text{Ker}(f)$ is orthogonally complementable in $\mathcal{X}$ and there exists a (nonzero) vector $\omega \in \text{Ker}(f)^\perp$ such that $\langle \omega, \omega \rangle \neq 0$. Then there is a unique element $\tau \in \mathcal{X}$ satisfying the following conditions:

- (i) $f(x) = \langle x, \tau \rangle$ for all $x \in \mathcal{X}$.
- (ii)

$$\frac{|\langle \tau, \tau \rangle|}{\|\tau\|} \leq \|f\| \leq \|\tau\|.$$

Moreover,

$$\|\tau\| = \frac{|f(\omega)|}{|\langle \omega, \omega \rangle|} \|\omega\| \leq \frac{\|f\| \|\omega\|^2}{|\langle \omega, \omega \rangle|}.$$

Proof. Define

$$\tau := \frac{\sigma(f(\omega))}{\langle \omega, \omega \rangle} \omega.$$

The other arguments are similar to those used in the proof of Theorem 2.4. □

3. CONCLUSIONS

- (1) In 1907, Riesz [31] and Frechet [11] independently showed that every continuous linear functional on a Hilbert space is completely determined by a single element in that Hilbert space [4].
- (2) In 1938, Markov [26] and in 1941, Kakutani [22] extended the Riesz-Frechet representation to function spaces.
- (3) In 1947, James [20] and 1967, Giles [15] extended the Riesz-Frechet representation to certain classes of Banach spaces.
- (4) In this article, we derived two finite field Riesz representation theorems under certain conditions for inner product spaces over sub-modulus fields.

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