

FRÖBERG'S CONJECTURE THROUGH DEGREE $2d$ IN SUFFICIENTLY MANY VARIABLES

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ABSTRACT. Let k be any infinite field. We prove that, for every fixed $d \geq 1$ and all sufficiently large n , the quotient of $k[x_1, \dots, x_n]$ by any number of general forms of degree d has the Hilbert function predicted by Fröberg's conjecture through degree $2d$. The threshold for n is independent of the field and the number of generators. For generating degrees at least three, a direct construction with two groups of variables gives a nonincreasing recurrence for the two defects at the critical generator counts. Divisibility of these bounded defects and polynomial equidistribution then force their eventual vanishing.

1. INTRODUCTION

Let k be an infinite field, let $S = k[x_1, \dots, x_n]$ with its standard grading, and let $I = (f_1, \dots, f_r)$, where the f_i are general forms of the same degree d . Fröberg's conjecture predicts

$$(1.1) \quad \text{HS}_{S/I}(t) = \left[\frac{(1-t^d)^r}{(1-t)^n} \right]_+.$$

Here $[\sum_{j \geq 0} a_j t^j]_+$ retains the coefficients preceding the first nonpositive coefficient and replaces that coefficient and all subsequent coefficients by zero. The conjecture originates in [2]. We write $[t^j]F(t)$ for the coefficient of t^j in a power series $F(t)$. A statement about general forms means that it holds on a nonempty Zariski open subset of their coefficient space.

Hochster and Laksov proved the expected behavior in degree $d+1$ [3]. More recently, Boij, Dannetun and Lundqvist proved that, for each fixed $d > 1$ and sufficiently large n , the prediction holds through degree $2d-1$, uniformly in the number of generators [1, Theorem 2]. At the next degree, the products $f_i f_j = f_j f_i$ give the first Koszul relations. The expected coefficient is therefore

$$\binom{n+2d-1}{2d} - r \binom{n+d-1}{d} + \binom{r}{2},$$

provided that positive truncation has not already occurred. The purpose of this paper is to treat this endpoint by a construction that retains these relations throughout the proof.

Theorem 1.1. *For every integer $d \geq 1$ there is an integer $n_0(d)$ such that, for every infinite field k , every $n \geq n_0(d)$ and every $r \geq 0$, the quotient by r general degree- d forms in $k[x_1, \dots, x_n]$ has Hilbert function*

$$\dim_k(S/I)_j = [t^j] \left[\frac{(1-t^d)^r}{(1-t)^n} \right]_+ \quad (0 \leq j \leq 2d).$$

Outline of the proof. The proof has three parts: a reduction to two generator counts, a comparison between different numbers of variables, and an arithmetic argument that turns this comparison into vanishing. The quadratic case is treated separately in Appendix D; the same comparison construction then works for every $d \geq 3$.

The main step compares the two critical defects in m variables with those in $m+h$ variables, for a fixed $h = h(d)$. Write $H_n(r)$ for the dimension of the degree- $2d$ first Koszul homology and $C_n(r)$

for the degree- $2d$ quotient dimension. If r_n^-, r_n^+ are the integers on either side of the smaller zero of the expected coefficient, put

$$E_d(n) = \max\{H_n(r_n^-), C_n(r_n^+)\}.$$

Monotonicity reduces the endpoint assertion for every number of generators to $E_d(n) = 0$.

For $d \geq 3$ we prove the recurrence

$$(1.2) \quad E_d(m+h) \leq E_d(m)$$

by splitting the variables into groups of sizes h and m . Monomial matchings supply spaces with independent symmetric products. A hyperplane replacement reduces the homology from the smaller ring exactly, and a final scalar perturbation gives the comparison of defects. No vanishing of the smaller-ring defects is assumed in this step. The recurrence bounds the defects. If g is the largest divisor of $\gcd(n, dr)$ prime to the characteristic, with $g = \gcd(n, dr)$ in characteristic zero, then

$$g \mid 2dH_n(r), \quad g \mid 2dC_n(r).$$

These divisibilities and polynomial equidistribution force eventual vanishing, uniformly in the characteristic.

The main results that carry the proof are Proposition 2.2 (the two-count reduction), Theorem 6.2 (the comparison inequalities), and Theorem 7.4 (eventual vanishing). Section 8 combines them with the quadratic endpoint, Corollary D.14, to prove the theorem. The comparison is presented first using four explicit properties of the constructed family. Appendices B and C establish these properties; the product spaces they use are constructed in Appendix A. Appendix E supplies the numerical estimates for degrees three through eight.

Notation and conventions. Throughout, k is an infinite field and all vector spaces are over k . Rank conditions may be checked over an algebraic closure: their nonvanishing minors define nonempty opens over k , which have k -rational points because k is infinite. The theorem of Boij, Dannetun and Lundqvist supplies the required multiplication statements below twice the relevant generating degree. The quadratic endpoint is proved in Appendix D.

We write $S(V) = \text{Sym}(V)$ and $S_a V = \text{Sym}^a V$, with $S_a V = 0$ for $a < 0$. For subspaces of forms, AL denotes the span of the products $a\ell$, $a \in A$, $\ell \in L$; this multiplication image is also called the shadow of L under A .

2. THE ENDPOINT AND THE TWO CRITICAL COUNTS

Fix an infinite field k and an integer $d \geq 2$. Write $S = \text{Sym}(k^n)$ and

$$N_t(n) = \dim_k S_t = \binom{n+t-1}{t}.$$

For $0 \leq r \leq N_d(n)$ let $W \subset S_d$ be a general r -dimensional subspace. The two dimensions relevant to the endpoint are

$$(2.1) \quad H_n(r) = \dim_k H_1(W; S)_{2d}, \quad C_n(r) = \dim_k (S/(W))_{2d}.$$

Here “general” means that the multiplication matrix has its largest rank; this is a nonempty Zariski open condition on the Grassmannian. The notation records these generic dimensions, rather than the dimensions at an arbitrary special choice of W .

In degree $2d$, the Koszul complex on W has the form

$$(2.2) \quad 0 \longrightarrow \bigwedge^2 W \xrightarrow{\delta_2} W \otimes_k S_d \xrightarrow{\mu_W} S_{2d} \longrightarrow 0, \quad \delta_2(f \wedge g) = f \otimes g - g \otimes f.$$

The first map is injective, and there is no preceding term because its coefficient degree would be $-d$. Consequently, with

$$(2.3) \quad \chi_n(r) = N_{2d}(n) - rN_d(n) + \binom{r}{2},$$

the Euler characteristic gives

$$(2.4) \quad C_n(r) - H_n(r) = \chi_n(r).$$

Thus maximal rank after removing the constant Koszul relations is equivalent to

$$C_n(r) = \max\{\chi_n(r), 0\}, \quad H_n(r) = \max\{-\chi_n(r), 0\}.$$

The monotonicity needed below concerns formal symmetric products. We distinguish them from their polynomial images: if $V = S_d$, set

$$W \odot V = \text{im}(W \otimes V \longrightarrow \text{Sym}^2 V).$$

The map $\text{Sym}^2 V \rightarrow S_{2d}$ in the following lemma is ordinary polynomial multiplication.

Lemma 2.1. *For an inclusion $W \subseteq W' \subseteq S_d$, there is a natural injection*

$$H_1(W; S)_{2d} \longrightarrow H_1(W'; S)_{2d}.$$

In particular, $H_n(r)$ is nondecreasing in r , and $C_n(r)$ is nonincreasing in r .

Proof. Choose a vector-space complement $V = W \oplus U$. Under this decomposition, the kernel of $W \otimes V \rightarrow \text{Sym}^2 V$ is exactly $\wedge^2 W$: its restriction to $W \otimes W$ has that kernel, and its restriction to $W \otimes U$ is injective into the separate summand $W \otimes U$ of $\text{Sym}^2 V$. Hence

$$(2.5) \quad H_1(W; S)_{2d} \simeq (W \odot V) \cap \ker(\text{Sym}^2 V \longrightarrow S_{2d}).$$

The asserted injection follows from $W \odot V \subseteq W' \odot V$. The quotient map $S/(W) \twoheadrightarrow S/(W')$ gives the opposite monotonicity for the cokernel.

To pass from particular subspaces to generic dimensions, use the irreducible flag variety of pairs of the prescribed dimensions. Both of its projections to the corresponding Grassmannians are dominant. The inverse images of the two nonempty rank opens therefore meet, so the same inequalities hold for their generic values. $\square$

The smaller real zero of χ_n is

$$(2.6) \quad \kappa_d(n) = N_d(n) + \frac{1}{2} - \sqrt{\left(N_d(n) + \frac{1}{2}\right)^2 - 2N_{2d}(n)}.$$

It lies in $(0, N_d(n)]$. Indeed, polynomial multiplication $\text{Sym}^2 S_d \rightarrow S_{2d}$ is surjective, since every monomial of degree $2d$ is a product of two degree- d monomials. Thus $\chi_n(N_d(n)) \leq 0$, whereas $\chi_n(0) > 0$; moreover, χ_n is strictly decreasing on $[0, N_d(n)]$. Define

$$(2.7) \quad r_n^- = \lfloor \kappa_d(n) \rfloor, \quad r_n^+ = \lceil \kappa_d(n) \rceil, \quad E_d(n) = \max\{H_n(r_n^-), C_n(r_n^+)\}.$$

These definitions also cover an integral root, in which case the two counts coincide.

Proposition 2.2. *If $E_d(n) = 0$, then*

$$C_n(r) = \max\{\chi_n(r), 0\} \quad (0 \leq r \leq N_d(n)).$$

For fixed d and sufficiently large n , this is exactly the degree $2d$ coefficient prescribed by the positive truncation of $(1 - t^d)^r / (1 - t)^n$. For $r \geq N_d(n)$, the quotient by general degree- d forms is already zero in every degree at least d .

Proof. If $r \leq r_n^-$, Lemma 2.1 gives $H_n(r) = 0$, so (2.4) gives $C_n(r) = \chi_n(r) \geq 0$. If $r \geq r_n^+$, it gives $C_n(r) = 0$. There are no integer counts strictly between r_n^- and r_n^+ . At an integral root, (2.4) shows that the two endpoint defects are equal, so the same argument applies.

For clarity, we also check the effect of positive truncation. Put

$$H = \binom{2d-1}{d-1}, \quad \rho_d = 1 - \sqrt{1 - \frac{1}{H}}.$$

The leading terms in (2.6) give

$$(2.8) \quad \kappa_d(n) = \frac{\rho_d}{d!} n^d + O(n^{d-1}), \quad \frac{N_{2d-1}(n)}{N_{d-1}(n)} = \frac{1}{Hd!} n^d + O(n^{d-1}).$$

Since $0 < \rho_d < 1/H$, the first quantity is strictly smaller than the second for all sufficiently large n . For $n > 1$, the ratio

$$\frac{N_t(n)}{N_{t-d}(n)} = \prod_{i=0}^{d-1} \frac{n+t-d+i}{t-d+1+i}$$

decreases as t runs from d to $2d-1$. Thus if $r \leq r_n^-$, every coefficient $N_t(n) - rN_{t-d}(n)$ with $d \leq t < 2d$ is positive. No earlier truncation occurs, and the coefficient in degree $2d$ is $\chi_n(r)$. If $r \geq r_n^+$, either truncation has occurred earlier, or the nonpositive coefficient $\chi_n(r)$ causes it in degree $2d$. In both cases the prescribed coefficient is zero. Finally, $N_d(n)$ general forms span S_d , which proves the last assertion, including all larger counts. $\square$

It therefore remains to make the single nonnegative integer $E_d(n)$ vanish eventually. The geometric construction will give a fixed step h for which $E_d(n+h) \leq E_d(n)$ for all sufficiently large n . The arithmetic argument in Section 7 shows why that recurrence suffices, without any separately prescribed initial value of zero.

3. AN EXACT HYPERPLANE REPLACEMENT

The formal-product description of homology permits an exact replacement of the homology contributed by a subspace of generators. The situation to keep in mind is that A consists of forms in a smaller polynomial ring and $Q \subset A$ is the family already chosen there. We replace one generator f_* by $f_* + \epsilon M$, with M outside $W + A$. The lemma shows that this removes precisely the relations involving f_* : the remaining homology is the homology contributed by Q_- .

Lemma 3.1 (Hyperplane replacement). *Let T be a commutative graded algebra with $T_0 = k$, put $V_T = T_d$ and $\mathcal{K}_T = \ker(S_2(V_T) \rightarrow T_{2d})$, and suppose $W \cap A = Q = Q_- \oplus kf_*$ for subspaces $W, A \subset V_T$. Products denoted by $\odot$ are formal products in $S_2(V_T)$. Suppose $M \notin W + A$ and, under the natural formal-product identification,*

$$(3.1) \quad H_1(W + kM; T)_{2d} = \mathcal{K}_T \cap (Q \odot A).$$

Choose $W = W_0 \oplus kf_$ with $Q_- \subset W_0$. For every $\epsilon \neq 0$, the space $W_\epsilon = W_0 + k(f_* + \epsilon M)$ has $\dim W_\epsilon = \dim W$ and*

$$(3.2) \quad H_1(W_\epsilon; T)_{2d} = \mathcal{K}_T \cap (Q_- \odot A).$$

Proof. The complement argument in Lemma 2.1 applies unchanged to T and gives the natural identification

$$(3.3) \quad H_1(L; T)_{2d} = \mathcal{K}_T \cap (L \odot V_T).$$

For any $L \subset V_T$, one also has

$$(3.4) \quad (L \odot V_T) \cap S_2(A) = (L \cap A) \odot A.$$

Indeed, $L \odot V_T$ is the kernel of $S_2(V_T) \rightarrow S_2(V_T/L)$, and $A/(A \cap L) \hookrightarrow V_T/L$ stays injective on symmetric squares, as is seen by extending a basis. The condition on M gives $W_\epsilon \cap A = Q_-$ and preserves the dimension of W : a nonzero coefficient of $f_* + \epsilon M$ in an element of A would put M in $W + A$. Inclusion in $W + kM$ places $H_1(W_\epsilon; T)_{2d}$ in $\mathcal{K}_T \cap S_2(A)$ by (3.1). Intersecting with $W_\epsilon \odot V_T$ and using (3.4) proves one containment in (3.2); $Q_- \subset W_\epsilon$ proves the other. $\square$

4. SPACES WITH INDEPENDENT SYMMETRIC PRODUCTS

We first construct a large space of forms whose symmetric products are independent and have a single bidegree. The argument uses only matchings between products and monomials.

Lemma 4.1. *Let $s \geq 1$, $w \geq s$, and $a = \lfloor s/2 \rfloor$. In the paired variables $x_1, \dots, x_w, y_1, \dots, y_w$, there is a space*

$$C \subset S_a \langle x_1, \dots, x_w \rangle \otimes S_{s-a} \langle y_1, \dots, y_w \rangle, \quad \dim C = \binom{w}{s},$$

whose symmetric multiplication $S_2 C \rightarrow S_{2s}(k^{2w})$ is injective. In particular, $\dim C = N_s(w) - O_s(w^{s-1})$.

Proof. For each s -subset $S \subset [w]$, choose a form

$$(4.1) \quad f_S = \sum_{\substack{A \subset S \\ |A|=a}} c_{S,A} x_A y_{S \setminus A}, \quad x_A = \prod_{i \in A} x_i,$$

with independent coefficient parameters. Give both x_i and y_i index degree $e_i \in \mathbb{Z}^w$. Products with different index degrees are independent. Fix one product degree, specified by a set I of doubled indices and a disjoint set R of single indices. Write $i = |I|$ and $t = s - i$, so $|R| = 2t$. For $t > 0$, its source columns are the unordered partitions $R = P \sqcup Q$ with $|P| = |Q| = t$, representing $f_{I \cup P} f_{I \cup Q}$. Each form in this fiber has a unique partner. Thus distinct columns use disjoint sets of coefficient parameters.

Suppose first that s is even. Use target monomials with $x_j y_j$ on I , with x_j on a t -subset $D \subset R$, and with y_j on $R \setminus D$. Such a target factors into an allowed term of each source form exactly when

$$a - i \leq |D \cap P| \leq a.$$

Indeed, the first factor must receive $a - |D \cap P|$ of the x variables on I , and the second receives the rest. If s is odd and $i \geq 1$, choose one fixed index of I to carry y_j^2 , let the others carry $x_j y_j$, and use the same t -subsets D . The condition is now

$$a - i + 1 \leq |D \cap P| \leq a.$$

In either case the interval contains an integer nearest $t/2$. When its endpoints coincide, t is even. The graph joining source partitions to the allowed target subsets is therefore nonempty. Permutations of R act transitively on each vertex class and preserve adjacency. The graph is biregular, and its target class has twice as many vertices as its source class. Counting edges from any set of source vertices proves Hall's inequality. There is consequently a matching covering every source column.

If s is odd and $i = 0$, use instead the $(s-1)$ -subsets $D \subset R$, with x on D and y elsewhere. Adjacency means $|D \cap P| = |D \cap Q| = a$. The same transitivity and edge-counting argument applies, with target-to-source ratio

$$\frac{\binom{2s}{s-1}}{\frac{1}{2} \binom{2s}{s}} = \frac{2s}{s+1} \geq 1.$$

If $t = 0$, the fiber has the single column f_I^2 , and any one allowed monomial gives a nonzero entry.

For a matching in a fiber with $t > 0$, specialize the two forms in each column to the factors of its matched monomial. Unique partners make these choices consistent. The selected square submatrix

is a permutation matrix, so its determinant is a nonzero polynomial in the parameters. For $t = 0$, specialize the one form to one monomial. There are finitely many fibers, and the product of their nonzero determinants is nonzero. Choose parameters in this common open set. The forms f_S themselves have distinct index degrees, so they are independent and span the required space. $\square$

The degrees three through eight use two larger spaces in the fixed group of variables.

Proposition 4.2. *Let k be any infinite field. For $s \in \{2, 4\}$ and $w \geq s$, there is a space $C_s \subset S_s(k^{2w})$ of dimension $2\binom{w}{s}$ whose symmetric multiplication map is injective. In particular, there are spaces*

$$H_2 \subset S_2(k^h), \quad \dim H_2 = \lfloor 5h^2/32 \rfloor \quad (h \geq 65),$$

and

$$H_4 \subset S_4(k^h), \quad \dim H_4 = \lfloor h^4/256 \rfloor \quad (h \geq 144, 4 \mid h),$$

with injective symmetric multiplication.

Proof. Use two general multilinear forms on every s -element set of variable pairs. Lemmas A.1 and A.2 prove product independence by three quadratic and five quartic intersection patterns, and verify the stated dimension bounds. The full spaces have dimensions $h^2/4 + O(h)$ and $h^4/192 + O(h^3)$, respectively, for $w = \lfloor h/2 \rfloor$. $\square$

The space in Lemma 4.1 has one product bidegree and dimension $N_s(w) - O_s(w^{s-1})$. These are the two properties used in the multiplication estimates. All the capacity inequalities there are strict, so their lower-order terms are absorbed by first choosing the fixed block large and then increasing m .

5. THE CONSTRUCTION AND ITS GENERATOR COUNTS

We parametrize each family by an ordered list of independent forms; all assertions depend only on its span. The choices below are made on nonempty open subsets of the corresponding parameter spaces.

The construction takes place in $S(X \oplus Y)$, where $\dim X = h$ is fixed before $\dim Y = m$ tends to infinity. We use bidegrees with respect to (X, Y) . A row of X -degree b is the component $S_b X \otimes S_{2d-b} Y$ of the degree- $2d$ target. The following families will be chosen so that successive bidegrees of the multiplication map can be treated separately:

family	bidegree	role
Q	$(0, d)$	forms in the smaller polynomial ring
F	$(1, d-1)$	comparison of the two defects
E_j	$(j, d-j)$	intermediate multiplication maps
U	$(d, 0)$	the highest target bidegrees.

We call F the outer family and the E_j the intermediate families; only even j occur. First one chooses Q, E_j, U , with suitable lower-bidegree terms, so that their first homology is exactly the homology contributed by Q . Replacing one member of Q then reduces this to the homology at the smaller generator count. The forms F are adjoined and varied by adding forms in $S_d Y$. The induced map from homology to cokernel is the normal map studied in Appendix C; its rank gives the comparison of defects. All auxiliary forms are chosen within the exact count $r = q + f + \sum_j e_j + u$; the count is proved below.

Fix $d \geq 3$. The following single count construction is used throughout the comparison. We first specify fixed constants depending on d , then choose the size h of the first block, and finally choose the integer counts as the second block size m grows. This order of choices ensures that rounding does not affect any of the strict dimension inequalities. Write

$$H = \binom{2d-1}{d-1}, \quad \rho = \rho_d = 1 - \sqrt{1 - H^{-1}}, \quad \rho_2 = 1 - \sqrt{2/3}.$$

$$\mathcal{J} = \begin{cases} \{2, 4\} \cap \{1, \dots, d-1\}, & 3 \leq d \leq 8, \\ \{2, 4, \dots, 2\lceil d/4 \rceil\}, & d \geq 9. \end{cases}$$

Put

$$\tau_2 = \frac{9}{32(d-2)!\binom{2d-2}{d-2}}, \quad \tau_3 = \frac{1}{6(d-2)!\binom{2d-3}{d-2}}, \quad \tau_4 = \frac{\rho_2}{2(d-2)!\binom{2d-4}{d-2}},$$

and define

$$(5.1) \quad \gamma_2 = \begin{cases} \max\{\tau_2, \tau_3, \tau_4\}, & 3 \leq d \leq 8, \\ \tau_4, & d \geq 9. \end{cases}$$

For the other active indices set

$$\begin{aligned} \gamma_4 &= \frac{(d-1)!}{120(2d-5)!}, \\ \gamma_j &= \frac{(j-4)!(d-j+4)!}{(2j-4)!(2d-2j+4)!} \quad (j \in \mathcal{J}, j \geq 6). \end{aligned}$$

Set

$$(5.2) \quad \alpha = (101/100)\gamma_2, \quad \beta = (1011/1000)\gamma_2.$$

Choose a sufficiently large integer h , divisible by 180180 if $d \leq 8$ and by $4H$ otherwise, put $X = k^h$, and let $m = \dim Y$ tend to infinity with h fixed. In particular, constants implicit in $O_h(\cdot)$ may depend on this already fixed h . Put

$$K = h/H, \quad L = \left\lceil \frac{11}{10} 2^d \rho h \right\rceil, \quad c = \binom{L+1}{2}.$$

The choice of h can and will satisfy

$$(5.3) \quad L \leq h, \quad c \geq 2h-1, \quad c < h^2/10000 \quad \text{if } d \geq 9.$$

Let $D \subset S_2X$ be a general codimension- c subspace. Equivalently, D is the kernel of a general surjection $S_2X \rightarrow k^c$. This quotient separates the products of F in Proposition C.2. The multiplication estimates are verified for $d \geq 9$ in Section B.3, and for $3 \leq d \leq 8$ in Appendix E.

Proposition 5.1 (The generator counts). *For either $r = r_{m+h}^-$ or $r = r_{m+h}^+$, take $q = r_m^+$ and prescribe*

$$e_j = \left\lceil \frac{101}{100} \gamma_j h^j m^{d-j} \right\rceil \quad (j \in \mathcal{J}, j \geq 4), \quad u = \begin{cases} \lceil 2h^d/(d+1)! \rceil, & d \text{ odd}, \\ N_d(h), & d \text{ even}. \end{cases}$$

There are integers a, f, e_2 with

$$(5.4) \quad q + f + \sum_{j \in \mathcal{J}} e_j + u = r, \quad f = KN_{d-1}(a), \quad \alpha \leq \frac{e_2}{h^2 m^{d-2}} < \beta,$$

for every sufficiently large m , and $a = \sigma m + O_h(1)$, where $\sigma^{d-1} = H\rho$. Moreover, if

$$A = hN_{d-1}(m) - f, \quad T = hN_{2d-1}(m) - fN_d(m),$$

then $T - qA$ is bounded below by a positive constant times m^{2d-2} . This remains positive after adjoining $O_h(m^{d-2})$ scalar forms and any fixed number of additional outer columns.

Proof. The count e_2 will absorb the rounding in the other families. We first choose the largest allowed outer family that leaves the prescribed lower bound for e_2 , and then check that the unused count stays below its upper bound.

Put $g = u + \sum_{j \geq 4} e_j$, $R = r - q - g$, and $e_{\min} = \lceil \alpha h^2 m^{d-2} \rceil$. Define

$$a = \max\{v \geq 0 : KN_{d-1}(v) \leq R - e_{\min}\}, \quad f = KN_{d-1}(a), \quad e_2 = R - f.$$

The critical-root expansion gives $R = (h\rho/(d-1)!)m^{d-1} + O_h(m^{d-2})$, so a exists and has the asserted asymptotics. Since $1/2 < H\rho < 1$, one has $0 < \sigma < 1$. Maximality gives

$$0 \leq e_2 - e_{\min} < KN_{d-2}(a+1).$$

After division by h^2m^{d-2} , the limiting width is at most $1/(Hh(d-2)!)$. Taking h large makes this smaller than $\beta - \alpha$, proving every assertion about the exact count.

It remains to verify the dimension reserve $T - qA$. The leading terms cancel at the critical density, so the next coefficient determines its sign. Write

$$\kappa_d(m) = \frac{\rho}{d!}m^d + b_0m^{d-1} + O(m^{d-2}), \quad b_0 = \frac{d!d(2d-1)/(2d)! - \rho d(d-1)/(2d)!}{1 - \rho}.$$

Substitution in $T - qA$, using $2\rho - \rho^2 = H^{-1}$, cancels its coefficient of m^{2d-1} . If $e_2 = e_0(m)m^{d-2}$, the next coefficient is

$$(5.5) \quad \frac{(1-\rho)e_0(m)}{d!} + \frac{d(d-1)}{2(d!)^2}((h^2+h)\rho - h^2/H) - \frac{hd(2d-1)}{(2d)!}.$$

Its quadratic coefficient in h is positive as soon as $\alpha > d(d-1)\rho/(2d!)$. This follows from $\rho \binom{2d-4}{d-2} < \rho_2$; indeed $H\rho < 13/25$ and $\binom{2d-4}{d-2}/H < 1/4$, whereas $\rho_2 > 2/11$. Consequently (5.5) is uniformly positive for our whole interval once h is large. Additional scalar columns cost $O_h(m^{2d-3})$. Adding a fixed number v of outer columns changes $T - qA$ by $-v(N_d(m) - q) = O_h(m^d)$, also lower order. $\square$

6. THE LOCAL COMPARISON OF DEFECTS

Fix $d \geq 3$, using the quadratic endpoint of Corollary D.14, and take the parameters of Section 5. The purpose of this section is to deduce the comparison inequalities from four properties of a family of forms. The appendices establish those properties for the counts already chosen, so the argument here can be read independently of their dimension estimates. The construction uses forms

$$Q \subset S_d Y, \quad F \subset X \otimes S_{d-1} Y, \quad E_j \subset S_j X \otimes S_{d-j} Y,$$

with the prescribed dimensions q, f, e_j . The quadratic outputs of E_2 lie in D . Write a_j for scalar forms in $S_d Y$ with the same labels as E_j . The background, before adjoining F , is

$$(6.1) \quad W = \begin{cases} \langle Q, E_j + a_j, U + P \rangle, & d \text{ odd}, \\ \langle Q, E_j + a_j, U + P_2 + b \rangle, & d \text{ even}. \end{cases}$$

Here $U \subset S_d X$ has dimension u ; P is a list in $X \otimes S_{d-1} Y$, while P_2 and b are lists in $D \otimes S_{d-2} Y$ and $S_d Y$. Each tail has the labels of U . The terms “even” and “odd” refer to total X -degree modulo two. Every background generator is homogeneous for this parity grading. The families Q and $E_j + a_j$ are even; $U + P$ is odd when d is odd, whereas $U + P_2 + b$ is even when d is even. The family F is odd.

The final deformation adds forms in $S_d Y$ to the generators of F . For a varying multiplication map μ_t , its first-order effect on a relation in $\ker \mu_0$, taken modulo $\text{im } \mu_0$, defines a map from the kernel to the cokernel. We call this the *normal map*. In the present construction it descends through the Koszul relations. Its role is to remove any excess homology and cokernel left after the background has been chosen.

We use the following four properties of this family.

- (i) The background, also with one additional generator $M = M_2 + a_M$ of the quadratic mixed type, has exactly the naturally included homology of Q . This remains true after quotienting the degree- $2d$ target by a complement to $QS_d Y$ in $S_{2d} Y$.
- (ii) The symmetric products of F are independent modulo the background ideal. The construction allows one fixed scalar family containing every scalar tail, including a_M .

- (iii) After adjoining F and projecting as in (i), the even target is filled and the odd first homology is zero. The remaining homology modulo that of Q injects into the space of F coefficients.
- (iv) The scalar-contraction maps on the odd cokernel satisfy the covector-stratum bound of Proposition C.4. Consequently, if the homology in (iii), modulo the included homology of Q , has dimension v and the odd cokernel has dimension j , general scalar motions of the F generators induce a normal map of rank $\min(v, j)$.

Here (i) is Proposition B.5; (ii) is Proposition C.2; (iii) follows from Propositions B.7 and C.6; and (iv) is the covector-incidence calculation in Propositions C.4 and C.6. Appendices B and C prove these assertions with the parameter bounds for $d \geq 9$. Appendix E verifies the same hypotheses for $3 \leq d \leq 8$. The comparison below uses just these four properties.

6.1. Replacing one generator. There are two operations in this step. We first remove a complement to the smaller ring's multiplication image from the target. This makes its cokernel zero without changing its relations. We then apply the hyperplane replacement to pass from the homology of Q to that of its codimension-one subspace Q_- .

Put $q = \lceil \kappa_d(m) \rceil$, $c_0 = C_m(q)$ and $k_0 = H_m(q - 1)$. Choose a general flag $Q_- \subset Q = Q_- \oplus kf_*$ realizing these generic dimensions, and a complement Z_0 to $QS_d(Y)$ in $S_{2d}(Y)$. Set

$$(6.2) \quad T = S(X \oplus Y)/(Z_0), \quad A = S_d(Y) \subset T_d.$$

This leaves degree d unchanged and removes c_0 target dimensions. Since $Z_0 \cap QS_d(Y) = 0$, it changes neither the homology of Q nor that of Q_- with coefficients in $S(Y)$.

Let W be the prepared background and take the additional generator allowed by Proposition B.5, written $M = M_2 + a_M$ with $M_2 \in D \otimes S_{d-2}(Y)$ and $a_M \in S_d(Y)$. Its high component may be chosen independent of those of E_2 . The positive highest- X -degree components of the other families are independent degree by degree. Projecting successively to those degrees gives

$$(6.3) \quad W \cap A = Q, \quad M \notin W + A.$$

In particular, the independent degree- d components first eliminate the pure generators, including any quadratic tails they carry. The background theorem, with its extra-generator and projected versions, gives (3.1), since $H_1(Q; S(Y))_{2d} = \mathcal{K}_T \cap (Q \odot A)$. Take W_0 to consist of Q_- and the non- Q background columns. Lemma 3.1 then gives, for every $\epsilon \neq 0$,

$$(6.4) \quad H_1(W_\epsilon; T)_{2d} = H_1(Q_-; S(Y))_{2d}, \quad \dim W_\epsilon = \dim W.$$

Lemma 6.1 (Compatibility with the final normal map). *For a nonempty open set of nonzero ϵ , adjoining F to W_ϵ gives the product injection of Proposition C.2, zero odd first homology, the covector bound (C.17), and surjectivity onto the even degree- $2d$ target of T . The final scalar motion of F then has the maximal normal rank of Proposition C.6.*

Proof. We preserve the four properties in two ways. Product separation is proved in one quotient that works for every replacement parameter; the remaining rank and covector conditions persist on an open neighborhood of the original family.

Before choosing ϵ , include a_M in the fixed scalar family $\mathcal{S}^+ = \mathcal{S} \cup \{a_M\}$ used for the product-separation map. This single extra column is allowed by the strict estimates. Its product-injection open meets the earlier background, covector and target opens in their common parameter space, so choose the data there. The product injection now holds for every ϵ : in X -degree two, ϵM_2 is killed by the quotient by D , and $f_* + \epsilon a_M$ belongs to the fixed span of $\mathcal{S}^+$. In odd degree the same fixed P family is included in the denominator. The projection by Z_0 affects only X -degree zero. Thus this argument requires no constancy of the even background rank.

The replacement is even in X , so the odd degree- d coefficient space V in (C.4) or (C.7) is fixed. At $\epsilon = 0$ odd homology after adjoining F is zero. While the generators remain independent, the mandatory odd Koszul image has constant dimension. The odd multiplication map therefore has

constant maximal rank on an open neighborhood of zero, and its cokernels form a vector bundle $\mathcal{J}$. Scalar contraction with source V is regular on $\mathbb{P}(\mathcal{J}^*)$. For each $1 \leq k \leq \dim V$, the condition $\dim \ker C_\lambda \geq k$ is closed and its projection to the parameter line is proper. Upper semicontinuity of fiber dimensions preserves (C.17) on an open neighborhood of zero. Intersecting these finitely many neighborhoods preserves every stratum.

At zero, Proposition B.7 fills the positive even rows by decreasing- X -degree elimination, and Q fills the residual pure- Y target after projection by Z_0 . Surjectivity onto the full even target is open, so it persists as well. Choose a nonzero ϵ in these neighborhoods, which is possible over the infinite field k . Equation (6.4) supplies the exact remaining background homology. Product detection then gives the faithful coefficient map (C.23), and the preserved covector bounds give maximal normal rank by Proposition C.6. $\square$

6.2. The two inequalities. The remaining calculation compares the homology and cokernel after replacement. Maximal rank of the normal map cancels their excess dimensions against one another; the Euler identity determines which one can remain.

Theorem 6.2 (Transfer inequalities). *For either exact count $r = \lfloor \kappa_d(m+h) \rfloor$ or $r = \lceil \kappa_d(m+h) \rceil$, put $q = \lceil \kappa_d(m) \rceil$, $c_0 = C_m(q)$ and $k_0 = H_m(q-1)$. The constructed family gives*

$$(6.5) \quad H_{m+h}(r) \leq \max\{k_0, c_0 - \chi_{m+h}(r)\},$$

$$(6.6) \quad C_{m+h}(r) \leq \max\{c_0, k_0 + \chi_{m+h}(r)\}.$$

Proof. Choose the nonzero replacement parameter of Lemma 6.1 and adjoin F . In T the even target is filled and odd homology is zero. Write $k_0 + a_0$ for the even homology dimension, distinguishing the naturally included homology of Q_- , and j_0 for the odd cokernel dimension. Removing the c_0 dimensions of Z_0 from the target gives

$$(6.7) \quad \chi_{m+h}(r) = c_0 - k_0 + j_0 - a_0.$$

The final scalar motion of F has first normal rank $\min(a_0, j_0)$ and keeps Q_- fixed. Lemma C.5 bounds the resulting homology in T by

$$k_0 + \max\{a_0 - j_0, 0\} = \max\{k_0, c_0 - \chi_{m+h}(r)\}.$$

Restoring the unprojected target cannot increase the kernel. These rank minors therefore bound the homology of general independent r -tuples by the same number, proving (6.5). The Euler identity (2.4) gives (6.6). The replacement preserves the number of generators, so the total remains exactly r . The endpoint inputs are those of Proposition B.7; the transfer argument applies whenever the background and its one-generator extension, target maps, product injection and covector estimate hold. $\square$

The transfer inequalities (6.5)–(6.6) have the same form for both critical counts. Since $q-1 \leq r_m^-$, even when $\kappa_d(m)$ is integral, monotonicity gives $k_0 \leq E_d(m)$ and $c_0 \leq E_d(m)$. At r_{m+h}^- the Euler characteristic is nonnegative, and at r_{m+h}^+ it is nonpositive. The two inequalities therefore give

$$(6.8) \quad E_d(m+h) \leq E_d(m) \quad \text{for every sufficiently large } m.$$

In characteristic different from two, the quadratic construction in Appendix D gives the same recurrence with $h = 3$. The characteristic-two quadratic input is established there by a separate induction. The arithmetic argument below applies to every field for which the indicated recurrence has been proved.

7. THE RECURRENCE AND EVENTUAL VANISHING

The recurrence makes the two critical defects bounded. We combine equivariant divisibility with polynomial equidistribution to show that they vanish, using a common threshold over all fields to which the recurrence applies uniformly.

The argument is as follows. A symmetry of the multiplication complex forces each defect to be divisible by certain factors of $\gcd(n, dr)$. Equidistribution supplies a dimension in every residue class modulo h where these factors are larger than the bound on the defects. Both defects must vanish there, and the recurrence propagates vanishing to all later dimensions in that residue class.

Lemma 7.1 (Equivariant divisibility). *For $d \geq 1$, $n \geq 1$, and $0 \leq r \leq N_d(n)$, let g be the largest divisor of $\gcd(n, dr)$ prime to $\text{char } k$; in characteristic zero take $g = \gcd(n, dr)$. The generic dimensions in (2.1) satisfy*

$$(7.1) \quad g \mid 2d H_n(r), \quad g \mid 2d C_n(r).$$

Proof. Extension to an algebraic closure changes none of these dimensions, so assume k algebraically closed. For $r = 0$ the assertions are $H_n(0) = 0$ and

$$2d N_{2d}(n) = n \binom{n + 2d - 1}{2d - 1},$$

and are immediate. We may therefore assume $r \geq 1$.

Let $V = k^n$, $F = k^r$, and $\mathcal{A} = \text{Hom}(F, \text{Sym}^d V)$. The group $G = \text{SL}(V) \times \text{SL}(F)$ acts by

$$(g, a) \cdot B = (\text{Sym}^d g) B a^{-1}.$$

There is a nonempty invariant open set $\mathcal{U} \subset \mathcal{A}$ on which B is injective and the multiplication map has constant maximal rank. On this open set the universal degree- $2d$ complex is

$$\bigwedge^2 F \otimes \mathcal{O}_{\mathcal{U}} \longrightarrow (F \otimes \text{Sym}^d V) \otimes \mathcal{O}_{\mathcal{U}} \longrightarrow \text{Sym}^{2d} V \otimes \mathcal{O}_{\mathcal{U}}.$$

Its first homology and its cokernel are G -equivariant vector bundles of ranks $H_n(r)$ and $C_n(r)$, respectively. Their determinant lines are ordinary trivial line bundles on $\mathcal{U}$.

Here is a direct justification of the last assertion and of the equivariance information it preserves. Every line bundle on a nonempty open subset of affine space has a rational section, hence is represented by a Cartier divisor. The closure in affine space of each prime divisor in that open set is a hypersurface, because the polynomial ring is factorial. The product of their defining polynomials, with the divisor's integer multiplicities, is a rational function having the prescribed divisor on the open set. The line bundle is therefore trivial.

If L is one of our determinant lines and s is a nowhere vanishing section, write its linearization as

$$g \cdot s(x) = a(g, x) s(gx), \quad (g, x) \in G \times \mathcal{U}.$$

The function a is regular and invertible. For fixed x , $a(-, x)$ is a regular unit on G , and every such unit is constant. Indeed, its restriction to any translate of an elementary unipotent subgroup is a unit of $k[t]$, and hence is constant. Elementary unipotent subgroups generate each special linear factor, proving the assertion; a factor SL_1 is trivial. Since $a(1, x) = 1$, we obtain $a(g, x) = 1$. Thus every stabilizer acts trivially on each determinant line.

Let $\ell = g$. Since ℓ is prime to the characteristic, there exists $\zeta \in k$ of order ℓ . The central pair

$$z = (\zeta I_V, \zeta^d I_F)$$

belongs to G and fixes every point of $\mathcal{A}$. It acts on each term of the displayed complex by the same scalar ζ^{2d} : the three weights are respectively $(\zeta^d)^2$, $\zeta^d \zeta^d$, and ζ^{2d} . It follows that it acts on the two determinant lines by

$$\zeta^{2d H_n(r)} \quad \text{and} \quad \zeta^{2d C_n(r)}.$$

Both scalars are 1 by the preceding paragraph. Since ζ has order ℓ , these are precisely the two divisibilities. $\square$

Lemma 7.2. *For each fixed $d \geq 2$ there is a real polynomial $P_d(x)$ of degree d such that*

$$(7.2) \quad \kappa_d(n) = P_d(n) + O(n^{-1}).$$

Its leading coefficient is the irrational number $\rho_d/d!$, where $\rho_d = 1 - \sqrt{1 - 1/H}$ and $H = \binom{2d-1}{d-1}$.

Proof. The expression under the square root in (2.6) is a polynomial $\Delta(n)$ of degree $2d$, with leading coefficient

$$\frac{1}{(d!)^2} - \frac{2}{(2d)!} = \frac{1 - 1/H}{(d!)^2} > 0.$$

Write $\Delta(n) = n^{2d}D(n^{-1})$, where D is a real polynomial and $D(0) > 0$. The positive square root of $D(z)$ is real analytic near $z = 0$. Its Taylor polynomial through degree d , multiplied by n^d , differs from $\sqrt{\Delta(n)}$ by $O(n^{-1})$. Subtracting this polynomial from $N_d(n) + 1/2$ gives P_d and the stated leading coefficient.

Finally, $H \geq 3$ and $H, H - 1$ are coprime. If $\sqrt{(H - 1)/H}$ were rational, the numerator and denominator of this reduced fraction would both be squares. Two positive consecutive integers cannot both be squares: if $b > a \geq 1$, then $b^2 - a^2 \geq 2a + 1 > 1$. This contradiction proves irrationality. $\square$

We recall the precise elementary form of polynomial equidistribution needed here; see [4].

Lemma 7.3 (Weyl's polynomial equidistribution theorem). *If a real polynomial P has positive degree and irrational leading coefficient, the fractional parts of $P(1), P(2), \dots$ are uniformly distributed in $[0, 1)$. The same conclusion holds for $P(b + Ls)/A$, as s tends through the positive integers, whenever b is an integer and L, A are positive integers.*

Proof. The first assertion is Weyl's polynomial equidistribution theorem [4]. For the second, substitution $x = b + Ls$ and division by A multiply the leading coefficient by the nonzero rational number $L^{\deg P}/A$. This preserves irrationality, so the first assertion applies to the resulting polynomial. $\square$

Theorem 7.4 (Eventual vanishing from the recurrence). *Fix $d \geq 2$ and a class $\mathcal{F}$ of infinite fields. Suppose there are integers $h > 0$ and m_0 such that (6.8) holds for every $m \geq m_0$ over every field in $\mathcal{F}$. Then there is an integer N , independent of the field in $\mathcal{F}$, such that $E_d(n) = 0$ for every $n \geq N$. Consequently the degree- $2d$ prediction holds for every number of general degree- d generators in these dimensions.*

Proof. The two initial defects are bounded by the target dimension. Indeed, at $r = r_n^-$ one has $\chi_n(r) \geq 0$, and the dimension of the source after the constant Koszul boundaries are removed is

$$rN_d(n) - \binom{r}{2} = N_{2d}(n) - \chi_n(r) \leq N_{2d}(n).$$

The upper-endpoint cokernel has the same bound. Thus, with $M = N_{2d}(m_0 + h - 1)$, iteration of the recurrence gives, over every field in $\mathcal{F}$,

$$(7.3) \quad 0 \leq E_d(n) \leq M \quad (n \geq m_0).$$

Choose three distinct primes $p_1, p_2, p_3 > \max\{2dM, 2d\}$, none dividing h .

For each pair $p = p_i, q = p_j$ with $i < j$, and each residue class b modulo h , the Chinese remainder theorem gives a positive integer n_b satisfying

$$n_b \equiv b \pmod{h}, \quad n_b \equiv 0 \pmod{pq}.$$

All numbers $n = n_b + hpq s$ satisfy these congruences. Choose $A \in \{0, \dots, pq - 1\}$ characterized by

$$A \equiv 0 \pmod{p}, \quad A \equiv -1 \pmod{q}.$$

By Lemmas 7.2 and 7.3, there are arbitrarily large s such that

$$\left\{ \frac{P_d(n_b + hpq s)}{pq} \right\} \in \left(\frac{A + 1/3}{pq}, \frac{A + 2/3}{pq} \right).$$

Choose one with $n \geq m_0$ and $|\kappa_d(n) - P_d(n)| < 1/6$. Then $\kappa_d(n)$ modulo pq belongs to $(A, A + 1)$, so it is nonintegral and

$$(7.4) \quad r_n^- \equiv 0 \pmod{p}, \quad r_n^+ = r_n^- + 1 \equiv 0 \pmod{q}.$$

Denote this selected dimension by $n(b, i, j)$. These finitely many choices depend only on d, h, m_0 .

For any $k \in \mathcal{F}$, at least one pair p_i, p_j avoids char k . For this pair and each b , Lemma 7.1 applies at $n(b, i, j)$ and gives

$$p_i \mid 2dH_n(r_n^-), \quad p_j \mid 2dC_n(r_n^+).$$

The primes are coprime to $2d$ and exceed M , while the two dimensions are at most M . Both dimensions therefore vanish. The recurrence propagates this vanishing through the residue class b modulo h .

Taking N to be the maximum of all $n(b, i, j)$ works over every field in $\mathcal{F}$ and in every residue class. Proposition 2.2 supplies all generator counts. $\square$

8. COMPLETION OF THE PROOF

Proof of Theorem 1.1. For $d = 1$, the quotient by general linear forms is a polynomial ring in $\max(n - r, 0)$ variables, so the asserted formula is immediate. For $d = 2$, Corollary D.14 gives the quadratic endpoint in all sufficiently large dimensions, with a threshold independent of the field.

Now fix any $d \geq 3$. Choose h and the generator counts from Proposition 5.1. Direct symmetric-power convolution supplies the X -degree-two target map, and the quadratic result supplies the quadratic output maps. All other auxiliary multiplications use the theorem of Boij, Dannetun and Lundqvist below twice the generating degree. The estimates in Appendices B–E give the four properties used in Section 6, with the small-degree parameters when $3 \leq d \leq 8$. Theorem 6.2 gives the recurrence (6.8), and Theorem 7.4 forces $E_d(n) = 0$ eventually. Proposition 2.2 then gives the predicted coefficient in degree $2d$ for every generator count.

For each $d \geq 2$, the coefficients through $2d - 1$ are supplied by [1, Theorem 2]. Take $n_0(d)$ to be the larger of that threshold and the endpoint threshold just obtained. The quadratic endpoint and the cited multiplication estimates have thresholds independent of the field. The product witnesses and all remaining rank arguments work over every infinite field with the same numerical dimensions. For each fixed d , choose h large enough for their strict inequalities and then a common m_0 large enough for the exact counts, rounding terms and convolution bounds. Theorem 7.4 therefore supplies a threshold independent of the characteristic. This proves the statement through degree $2d$ for every $d \geq 1$. $\square$

APPENDIX A. THE QUADRATIC AND QUARTIC PRODUCT SPACES

Both constructions in this appendix use pairs of variables and work over any infinite field k . Write

$$V_i = \langle x_i, y_i \rangle, \quad V = \bigoplus_{i=1}^w V_i.$$

For $s = 2$ or 4 , and for every s -element subset S of $[w]$, choose two general multilinear forms

$$(A.1) \quad f_{S, \epsilon} = \sum_{A \subset S} c_{S, \epsilon, A} x_A y_{S \setminus A}, \quad \epsilon = 0, 1,$$

where $x_A = \prod_{i \in A} x_i$ and the coefficients are independent parameters. Give both x_i and y_i degree $e_i \in \mathbb{Z}^w$. Products then separate according to

$$I = S \cap T, \quad U = S \cup T.$$

We call the products with a fixed pair (I, U) a product component. For the small supports below, strings such as 013 denote subsets $\{0, 1, 3\}$.

The following criterion will be used for both constructions. In each product component, choose one monomial for every form occurring there. If the resulting unordered products, including squares, are distinct monomials, then the corresponding coefficient minor specializes to a permutation matrix. Its determinant is therefore a nonzero polynomial over every field. There are only finitely many product components, so the product of these determinants is nonzero. Since k is infinite, the coefficients can be chosen so that all the determinants are nonzero simultaneously. The different components have disjoint monomial supports, and hence all unordered products are linearly independent. This also implies independence of the forms themselves, by multiplying any proposed linear relation by one of the forms.

Lemma A.1 (A quadratic product space). *For every $w \geq 2$, there is a space $C_2 \subset S_2(k^{2w})$ of dimension $2\binom{w}{2}$ whose symmetric multiplication is injective. Consequently, for every $h \geq 65$, there is a space $H_2 \subset S_2(k^h)$ of dimension $\lfloor 5h^2/32 \rfloor$ with injective symmetric multiplication.*

Proof. Use the forms in (A.1) with $s = 2$. There are three types of product component, distinguished by $|I|$. If $|I| = 2$, the two forms have the same support. Specialize them to x_1x_2 and y_1y_2 ; their two squares and mixed product are distinct. If $|I| = 1$, the supports can be taken to be 12 and 13. Specialize the two forms on 12 to x_1x_2, x_1y_2 , and those on 13 to x_1x_3, x_1y_3 . Their four products are distinct.

For $|I| = 0$, all three pairings of four indices occur in the same component. The following choices treat them simultaneously:

S	$f_{S,0}$	$f_{S,1}$
12	x_1x_2	x_1y_2
13	x_1x_3	x_1y_3
14	y_1x_4	y_1y_4
23	x_2x_3	x_2y_3
24	x_2x_4	y_2x_4
34	x_3y_4	y_3y_4

The four products for 12|34 contain x_1y_4 , those for 13|24 contain x_1x_4 , and those for 14|23 contain y_1x_2 . The three groups are disjoint, and within each group the other two variable choices distinguish its four products. The criterion above therefore gives the required $2\binom{w}{2}$ -dimensional space.

For the stated space in h variables, put $w = \lfloor h/2 \rfloor$. Since $w \geq 32$ and $h \leq 2w + 1$,

$$32w(w-1) - 5h^2 \geq 12w^2 - 52w - 5 > 0.$$

Thus $2\binom{w}{2} \geq \lfloor 5h^2/32 \rfloor$. Embed the $2w$ variables in the h variables and restrict to a subspace of the required dimension. $\square$

Lemma A.2 (A quartic product space). *For every $w \geq 4$, there is a space $C_4 \subset S_4(k^{2w})$ of dimension $2\binom{w}{4}$ whose symmetric multiplication is injective. Consequently, if $h \geq 144$ and $4 \mid h$, there is a space $H_4 \subset S_4(k^h)$ of dimension $\lfloor h^4/256 \rfloor$ with injective symmetric multiplication.*

Proof. Use (A.1) with $s = 4$. Set $r = |I|$. After relabelling the indices, every nonempty product component has

$$I = \{0, \dots, r-1\}, \quad U = \{0, \dots, 7-r\}, \quad 0 \leq r \leq 4.$$

The numbers of unordered products in these five components are, respectively, 140, 40, 12, 4, 3. For $r < 4$, each decomposition $U = S \cup T$, $S \cap T = I$ gives the four products $f_{S,\alpha}f_{T,\beta}$, with $\alpha, \beta \in \{0, 1\}$.

Three cases follow directly from the quadratic construction. If $r = 2$, apply its twelve-product specialization to the four indices of $U \setminus I$ and multiply every quadratic form by x_I . If $r = 3$, write $S = I \cup \{a\}$ and $T = I \cup \{b\}$; take $x_I x_a, x_I y_a$ on S and $x_I x_b, x_I y_b$ on T . If $r = 4$, the two forms x_I, y_I have three distinct unordered products.

For the remaining two cases, explicit monomial choices are given in Tables 1 and 2. In each row set $T = I \cup (U \setminus S)$, and specialize

$$f_{S,\alpha} = x_{A_\alpha} y_{S \setminus A_\alpha}, \quad f_{T,\beta} = x_{B_\beta} y_{T \setminus B_\beta}.$$

Each support occurs in exactly one row, so the tables assign one monomial to each form. To check the products, it suffices to add the characteristic vectors of the displayed subsets: all vectors

$$(A.2) \quad \mathbf{1}_{A_\alpha} + \mathbf{1}_{B_\beta}, \quad \alpha, \beta \in \{0, 1\},$$

are distinct within each table. There are $35 \cdot 4 = 140$ such vectors in the first table and $10 \cdot 4 = 40$ in the second. These are the x -exponent vectors; the fixed product degree determines the y -exponents. Thus the tables give distinct monomials for every unordered product in the two remaining components. The common specialization criterion now yields the space C_4 .

S	A_0	A_1	B_0	B_1	S	A_0	A_1	B_0	B_1
0123	$\emptyset$	013	46	457	0237	023	7	46	1456
0124	02	024	67	567	0245	2	4	1	17
0125	0	15	7	47	0246	6	026	3	137
0126	02	16	347	57	0247	4	024	13	135
0127	0	7	35	356	0256	25	26	34	1347
0134	$\emptyset$	1	25	67	0257	027	0257	36	146
0135	015	135	2	47	0267	7	067	1	13
0136	6	06	2	57	0345	05	35	2	127
0137	3	0137	2	456	0346	36	0346	12	27
0145	$\emptyset$	45	6	236	0347	0	34	6	126
0146	01	4	5	2357	0356	05	036	7	47
0147	014	147	26	356	0357	05	0357	124	1246
0156	01	06	3	24	0367	037	067	4	14
0157	05	07	23	24	0456	045	6	13	127
0167	7	0167	34	25	0457	7	0457	23	36
0234	0	024	$\emptyset$	17	0467	4	47	$\emptyset$	23
0235	0	05	16	146	0567	56	0567	1	123
0236	2	3	$\emptyset$	7					

TABLE 1. Quartic monomial choices when $I = \emptyset$ and $U = \{0, \dots, 7\}$.

For the required count, put $w = h/2$. Since $w \geq 72$, the elementary inequality

$$32 \binom{w}{4} - w^4 = \frac{w}{3} (w^2(w - 24) + 44w - 24) \geq 0$$

gives

$$2 \binom{w}{4} \geq \frac{w^4}{16} = \frac{h^4}{256}.$$

Restricting C_4 to the required dimension proves the assertion. $\square$

Lemma A.3 (Separated coefficient subspaces). *Suppose $O \subset S_j(X)$ and $C \subset S_t(Y)$ have injective symmetric-product maps. There is a subspace $E \subset O \otimes C$ of every dimension at most $(\dim O) \lfloor \dim C/2 \rfloor$ whose symmetric products are independent.*

S	A_0	A_1	B_0	B_1
0123	012	123	45	46
0124	24	024	05	36
0125	01	15	3	4
0126	06	16	034	045
0134	3	14	02	6
0135	03	5	24	06
0136	0	03	0	02
0145	04	5	0	3
0146	1	46	035	235
0156	056	156	023	034

TABLE 2. Quartic monomial choices when $I = \{0\}$ and $U = \{0, \dots, 6\}$.

Proof. Choose a basis $o_1, \dots, o_b$ of O and integers r_i differing by at most one, with prescribed sum and $0 \leq r_i \leq \lfloor \dim C/2 \rfloor$. General r_i -planes $C_i \subset C$ have pairwise zero intersection. Put $E = \bigoplus_i o_i C_i$. Independent products in O separate the unordered pairs of indices. The diagonal products are injective by the hypothesis on C . For $i \neq j$, the map $C_i \otimes C_j \rightarrow S_2(C)$ is injective, as is seen from a decomposition $C = C_i \oplus C_j \oplus C'$. Composing with $S_2(C) \rightarrow S_{2t}(Y)$ proves the assertion for every pair. $\square$

Remark A.4. Product injectivity is an open condition on the Grassmannian of subspaces of a fixed dimension. The flag variety of pairs $H \subset D \subset S_a(X)$ projects onto both Grassmannians. Thus, whenever a product-injective r -plane exists, a general D of dimension at least r contains one: the inverse image of the nonempty rank open has image containing a nonempty open in the Grassmannian of D .

For a prescribed output space $O \subset S_a(X)$ and a subspace D of codimension c , one instead has $\dim(O \cap D) \geq \dim O - c$. Replacing O by $O \cap D$ preserves every multiplication injection already proved on O or on its tensor products, as well as any prescribed bidegree.

APPENDIX B. MULTIPLICATION ESTIMATES FOR THE PREPARED FAMILY

This appendix supplies the multiplication statements used in Section 6. We first prove a normalized multiplication estimate and a stronger estimate for the quotient module of the outer family. We then use the prescribed counts to construct the intermediate families, identify the background homology, and fill the positive target degrees.

The normalized estimate and the outer-module construction apply for every $d \geq 3$. The capacity inequalities and polynomial row maps in Section B.3 are proved here for $d \geq 9$; Appendix E supplies the corresponding bounds for $3 \leq d \leq 8$.

B.1. A normalized multiplication estimate. We begin with the multiplication estimate used to impose the required rank conditions. It applies equally to one set of variables and to a tensor product of polynomial rings.

Lemma B.1 (Normalized multiplication and incidence). *Let $A = S_a V$, $B = S_b V$, and $T = S_{a+b} V$, or let these three spaces be tensor products of such spaces for disjoint sets of variables, with multiplication in each factor. Every subspace $L \subset B$ satisfies*

$$\dim(AL) \geq \frac{\dim T}{\dim B} \dim L.$$

Suppose $A_i B_i \subset T$ are finitely many multiplication maps having bounds $\dim(A_i L) \geq \sigma_i \dim L$. For general lists of r_i elements of A_i , the sum of their coefficient maps from $\bigoplus_i k^{r_i} \otimes B_i$ is injective if

$$(B.1) \quad \sum_i \frac{r_i + \dim B_i}{\sigma_i} < 1.$$

One may replace this sufficient inequality by the sharper rank-by-rank condition obtained by replacing $r_i + \dim B_i$ with $r_i + \dim B_i - k_i$ for coefficient rank k_i .

Proof. Degenerate L to its initial monomial subspace. For uniformly weighted monomials of degrees b and $a + b$ there is a coupling supported on divisibility: conditionally on a target monomial x^β , choose x^α with probability

$$\frac{\prod_\nu \binom{\beta_\nu}{\alpha_\nu}}{\binom{a+b}{b}}.$$

Its source marginal is uniform. This follows, for example, by summing the displayed weights using the coefficient identity for $\prod_\nu (1 - z_\nu)^{-1}$, or by first choosing a uniformly random point of the standard simplex and then independent multinomial samples of sizes a and b . Thus the normalized mass of a set of source monomials is at most the mass of its upper shadow. Products of these couplings prove the tensor-product assertion. Initial monomials cannot increase the dimension of a product space.

For the incidence assertion, stratify each coefficient space by rank. A coefficient tuple of rank k_i in $k^{r_i} \otimes B_i$ has dimension at most $k_i(r_i + \dim B_i - k_i)$. For fixed tuples, their equation on the independently variable lists in A_i has image $\sum_i A_i L_i$, hence codimension at least $\max_i \sigma_i k_i$. Under (B.1) this is greater than the sum of the coefficient-tuple dimensions for every nonzero tuple. Projectivizing the coefficient tuples and taking the incidence projection proves the assertion. This also proves the sharper version. $\square$

B.2. An outer multiplication module with a strict shadow estimate. The following estimate strengthens normalized multiplication for the quotient module associated to the outer family. Its extra term is positive for every nonzero proper source subspace. This strict gain will give both maximal rank for scalar multiplication and the covector bounds needed in the normal-map argument.

Theorem B.2 (Outer module). *Let $d \geq 3$ and $K \geq 1$ be integers. Put $s = d - 1$ and $n = 2d - 1$ for the source and target degrees, respectively, and write $H = \binom{n}{s}$ and $h = KH$. Take $\dim X = h$, $\dim Y = m$, and split $Y = C \oplus Z$ with $\dim C = a = \sigma m + O(1)$, $0 < \sigma < 1$. For each degree- s monomial u in C , choose K general vectors $v_{u,1}, \dots, v_{u,K}$ in X , and let $F \subset X \otimes S_s Y$ be the span of the forms $v_{u,\ell} u$. Its dimension is $f = KN_s(a)$, and multiplication*

$$F \otimes S_d Y \longrightarrow X \otimes S_n Y$$

is injective. For the module $\mathcal{N} = (X \otimes S(Y))/(F \cdot S(Y))$, put $A = \dim \mathcal{N}_s$ and $T = \dim \mathcal{N}_n$. There is $\eta > 0$ such that, for all sufficiently large m and all $M \subset \mathcal{N}_s$,

$$(B.2) \quad \dim(S_d Y \cdot M) \geq \frac{T}{A} \dim M + \eta m^d \min\{\dim M, A - \dim M\}.$$

General scalar forms of every count q act with maximal rank $k^q \otimes \mathcal{N}_s \rightarrow \mathcal{N}_n$. If $qA \leq T$ and $j = T - qA = o(m^n)$, put $J = \text{coker}(k^q \otimes \mathcal{N}_s \rightarrow \mathcal{N}_n)$. For $\lambda \in J^$ define $C_\lambda : \mathcal{N}_s \rightarrow (S_d Y)^*$ by $C_\lambda(v)(p) = \lambda(pv)$. Then, for a constant $\eta' > 0$,*

$$(B.3) \quad \dim\{\lambda \in \mathbb{P}(J^*) : \dim \ker C_\lambda \geq k\} \leq j - \eta' m^d k - 1 \quad (k \geq 1).$$

For the choice $\sigma^s = H\rho$, the same assertions hold for a general $(f + v)$ -plane when v is fixed as m grows, and the limiting slope is $(T/A)/m^d \rightarrow \rho/d!$.

Proof. We first prove the strict estimate for $K = 1$. The argument begins with the monomial fibers of the quotient and a bound on exceptional projections. A transport between the degree profiles then gives a positive gain, which we transfer to the exact quotient dimensions. Finally we deduce the rank and covector statements and treat general K and the extra columns.

Step 1: the monomial fibers. For now write $h = \binom{n}{s}$. The core degree of a monomial is its degree in the variables of C . Each target monomial of core degree j has at most $\binom{j}{s}$ distinct core divisors of degree s . Its attached vectors are therefore independent. This proves polynomial injectivity and

$$A = hN_s(m) - N_s(a), \quad T = hN_n(m) - N_s(a)N_d(m).$$

For each monomial, its fiber is the quotient of X by the span of the vectors attached to its degree- s core divisors. The exact source fiber dimensions are $d_i = h$ for $i < s$ and $d_s = h - 1$. A target fiber of core degree j has dimension at least $c_j = h - \binom{j}{s}$; equality holds when its core monomial is squarefree. Only c_n is zero. We use these lower bounds as coarse target capacities until Step 5.

Step 2: exceptional projections. We record why the estimate for arbitrary fiber subspaces is uniform. After torus degeneration, write M_α for the part of M in the fiber of a degree- s source monomial α . For a fixed source monomial and a proper subspace of its fiber, a general target projection has rank $\min(c_j, \dim M_\alpha)$. Its exceptional locus is a proper determinantal condition on the additional divisor vectors. Disjoint sets of additional vectors give independent conditions. The relevant Grassmannian has dimension at most $\lfloor h^2/4 \rfloor$. Thus, simultaneously for all its points, at most this many pairwise disjoint additional-vector sets can be exceptional. A maximal disjoint family meets every exceptional set and involves at most $h \lfloor h^2/4 \rfloor$ vectors. An additional core divisor has least common multiple of degree at least $s + 1$ with the source monomial, and hence occurs in only $O(m^s)$ targets. After torus degeneration of an arbitrary M , the total loss from all exceptional projections is consequently

$$(B.4) \quad O(m^s) \min\{\ell, A - \ell\}, \quad \ell = \dim M.$$

There are at most $\min(\ell, A - \ell)$ proper nonzero source fibers. The finitely many incidence conditions used here can all be imposed on one nonempty open.

Step 3: transport between degree profiles. We next construct a coupling of the source and target profiles that is positive on every allowed pair. A profile records only the core degree of a monomial. The limiting profile weights are

$$d_i \frac{\sigma^i (1 - \sigma)^{s-i}}{i!(s-i)!} \quad (0 \leq i \leq s), \quad c_j \frac{\sigma^j (1 - \sigma)^{n-j}}{j!(n-j)!} \quad (0 \leq j < n),$$

normalized separately to have mass one. Allowed pairs satisfy $i \leq j \leq i + d$. Put $p = \sigma^s$, and let I have distribution $\text{Bin}(s, \sigma)$ conditioned on $I < s$. Independently, take Bernoulli random variables $B_1, \dots, B_d$ of success probability σ , where $d = s + 1$, and put $R = B_1 + \dots + B_d$. The target $J = I + R$ has the required normalized target distribution, since

$$\binom{j}{s} \binom{n}{j} = \binom{n}{s} \binom{d}{j-s}.$$

Let E be the event $B_1 = \dots = B_s = 1$, and put $t = (h - 1)/(h - p)$, so that $0 < t < 1$. On E , replace the source value I by s with independent probability t , leaving the target J unchanged; otherwise retain I . Denote the resulting source value by I' . Since E has probability p and is independent of I , the new source masses are

$$\Pr(I' = i) = \frac{h}{h - p} \binom{s}{i} \sigma^i (1 - \sigma)^{s-i} \quad (i < s), \quad \Pr(I' = s) = \frac{(h - 1)p}{h - p},$$

which are precisely the normalized source weights. The joint distribution of (I', J) is supported on allowed pairs. Every allowed pair with $i < s$ has positive mass, since its original mass is positive

and a fraction at least $1 - t$ is retained. Every pair (s, j) with $s \leq j \leq n - 1$ also has positive mass: on E , one has $J = I + s + B_d$.

The normalized finite- m weights, using the source dimensions d_i and coarse target capacities c_j , converge to these limits. They admit an equally positive coupling for all sufficiently large m , by the following explicit correction. For each target profile j , correct its marginal discrepancy in row $\max(0, j - d)$. Then correct each remaining source discrepancy in column s . All these entries are allowed; the second step leaves the column marginals unchanged because the remaining source discrepancies sum to zero. Every correction tends to zero, so the resulting coupling retains a common positive lower bound on all allowed pairs.

Step 4: strict growth for the coarse target. Lift this transport to monomials by the divisibility coupling in Lemma B.1, separately in C and Z . Every admissible divisor pair has joint probability at least a constant times m^{-n} , conditional target probability at least a constant times m^{-d} , and conditional divisor probability bounded below by a positive constant. Write $u_\alpha = \dim M_\alpha / d_i$ and $\theta = \ell / A$. The reversible source chain obtained by choosing a target and then another source divisor has a uniform mixing estimate. In fact all degree- s monomials γ in Z may serve as intermediate states: for each α, γ , targets $\alpha\gamma z$, with z a free variable outside their supports, give transition probability at least $\kappa / N_s(\dim Z)$ for some $\kappa > 0$. Summing squared differences over these transitions gives

$$(B.5) \quad \mathbb{E} \text{Var}(u_\alpha \mid \text{target}) \geq (\kappa/2) \text{Var}(u_\alpha).$$

There is also a positive-probability decrease of capacity from every source profile: use target core degree s for $i < s$, and $s + 1$ for $i = s$. On these pairs $0 < c_j < d_i$, and

$$\min(c_j, d_i u) - c_j u \geq \min(c_j, d_i - c_j) u (1 - u).$$

For a target monomial β of core degree j , put $g_\beta = \max_{\alpha|\beta} \min(c_j, \dim M_\alpha)$. This is the largest contribution guaranteed by a single source fiber outside the exceptional projections. Write T_0 for the sum of the coarse capacities over all target monomials. Both the conditional average in the last inequality and $c_j \max_{\alpha|\beta} u_\alpha$ bound g_β from below. Averaging the two bounds, using $\max u - \mathbb{E}u \geq \text{Var}(u)$ on $[0, 1]$, and then (B.5), gives

$$\sum_{\beta} g_{\beta} \geq T_0 \theta + c T_0 \theta (1 - \theta),$$

Subtracting the loss in (B.4) gives

$$(B.6) \quad \dim(S_d Y M) \geq (T_0/A) \ell + c' m^d \min(\ell, A - \ell) - O(m^s) \min(\ell, A - \ell).$$

Step 5: passage to the exact target dimension. We must replace the coarse total T_0 by the exact total T , including when M is close to the full source. Only repeated core variables cause a discrepancy, so $0 \leq T - T_0 = O(m^{n-1})$. Put $k = A - \ell$. After the same monomial degeneration, let B be the set of source monomials whose fibers are not full. Each such fiber contributes at least one to the source codimension, so $|B| \leq k$. If $\alpha \mid \beta$, the relation space in the source fiber is contained in that of the target fiber; hence multiplication gives a surjective quotient map between them. A full source fiber therefore fills every divisible target fiber. Thus every unfilled target monomial has all its degree- s divisors in B .

Every monomial of degree $n = 2s + 1$ can be written as uvy_i , with u, v of degree s and y_i a variable of Y . If its target fiber is not full, then $u, v \in B$. There are therefore at most mk^2 unfilled target monomials. Since each target fiber has dimension at most h , this gives

$$(B.7) \quad \text{codim}_{N_n}(S_d Y \cdot M) \leq h m k^2.$$

The estimate also holds before degeneration, since degeneration cannot increase the dimension of the multiplication image.

The dimension formulas above give $\lambda = \lim_{m \rightarrow \infty} (T/A)m^{-d} > 0$. Choose $\delta > 0$ with $h\delta < \lambda/4$. If $k \leq \delta m^s$, then

$$hmk^2 \leq h\delta m^d k \leq (T/A - \eta m^d)k$$

for all sufficiently large m , with any fixed $0 < \eta \leq \lambda/4$. If $k \geq \delta m^s$, replacing T_0 by T in (B.6) costs only $O_\delta(m^{d-1}) \min(\ell, k)$, which is absorbed by its strict gain. Decreasing η if necessary proves (B.2) in both ranges.

Step 6: scalar multiplication and covectors. We now apply the strict growth estimate to the two incidence problems in the statement. For scalar injectivity, stratify coefficient tuples by rank r . Their projective dimension is at most $r(q + A - r) - 1$, while their condition on the scalar forms has codimension at least the shadow in (B.2). The excess term dominates $r(A - r)$ because $A = O(m^{d-1})$; rank A uses $T \geq qA$. For scalar surjectivity, stratify target covectors by the dimension k of the kernel of their contraction map. Choose this k -plane and impose the equations requiring the covector to annihilate its shadow. The remaining q scalar equations have codimension $q(A - k)$. The same estimate proves surjectivity when $qA \geq T$. For $qA \leq T$, this calculation gives the sharper bound

$$j + k(A - k) - (j/A)k - \eta m^d \min(k, A - k) - 1.$$

For $k \leq A/2$ it implies (B.3) after decreasing the constant; for $k > A/2$, write $t = A - k$ to obtain $t[j/A + k - \eta m^d] - 1 < 0$, so these strata are empty. The full kernel $k = A$ is impossible because $\mathcal{N}_s$ generates $\mathcal{N}_n$.

Step 7: general K and the limiting slope. The proof for general K is unchanged: all source and coarse target capacities multiply by K , the profile transport is the same, and only fixed constants in the exceptional-set argument change. For reference, direct leading-term calculation gives

$$\frac{T/A}{m^d} \longrightarrow \frac{1 - \sigma^s}{d!(H - \sigma^s)};$$

this is $\rho/d!$ when $\sigma^s = H\rho$.

Step 8: finitely many additional columns. Finally append v fixed columns $w_i z_i^s$, where the z_i are distinct free variables and the w_i are their attached vectors in X . Every target involving a private power has at most $Kd + 2 \leq h$ attached vectors, proving polynomial injectivity. Put $A' = A - v$, $T' = T - vN_d(m)$, $\nu = T/A$, and $\nu' = T'/A'$. We compare the extended quotient with the one already constructed, separately for subspaces of dimension at most $A'/2$ and at least $A'/2$. In the first range, discard targets divisible by any private power from the regular-source estimate; this loses only $O_v(m^{d-1})$ per source dimension. On a private source fiber retain targets of core degree at most $s - 1$ and with no other private power. They give the identity on X/kw_i , on a proportion

$$p_{\text{good}} = 1 - \sigma^d - d\sigma^{d-1}(1 - \sigma) > \rho$$

of all multipliers. To check the strict inequality, set $p = \sigma^s = H\rho$. Since $H \geq 10$, $1/2 < p < 13/25$, $\rho < 1/19$, and $s(1 - \sigma) \leq -\log p < 7/10$; hence $p_{\text{good}} > 29/250 > \rho$. These private targets are pairwise disjoint and disjoint from the retained regular targets. For a subspace of dimension $\ell \geq A'/2$, lift it to an $(\ell + v)$ -plane in the old source. Its shadow contains all private column products; subtracting their exact dimension $vN_d(m)$ from (B.2) gives

$$\dim(S_d Y M) \geq \nu' \ell + \left[\eta m^d - \frac{v(N_d(m) - \nu)}{A'} \right] (A' - \ell).$$

The term subtracted from ηm^d is $O_v(m)$. Thus the strict estimate persists. Failure of any required subspace estimate is the projection of a closed rank condition on a relative Grassmannian. Properness therefore promotes the witness to an open set of general $(f + v)$ -planes. Scalar rank and covector statements follow exactly as before. $\square$

B.3. Products of the intermediate families. We now apply the multiplication estimates to the intermediate families. First we check that the prescribed counts fit within the available dimensions; then we construct the row maps with precisely their required Koszul kernels. We work here with $d \geq 9$. Appendix E verifies the same rank conditions for $3 \leq d \leq 8$, so the subsequent coefficient and deformation arguments apply in either range.

For $j \in \mathcal{J}$ let E_j be a labelled e_j -dimensional space of forms in $S_j X \otimes S_{d-j} Y$; for $j = 2$ require its outputs to lie in D . Write a_j for a list of scalar degree- d forms with the same labels. All scalar lists used below, including $Q \subset S_d Y$, are contained in one common list $\mathcal{S}$. Its total length is

$$(B.8) \quad \#\mathcal{S} = (\rho/d!)m^d + O_h(m^{d-1}).$$

Its additional columns beyond Q number $O_h(m^{d-2})$.

For an even integer R , define the formal product space

$$\Pi_R = \bigoplus_{\substack{j < l, j, l \in \mathcal{J} \\ j+l=R}} E_j \otimes E_l \oplus \bigoplus_{\substack{j \in \mathcal{J} \\ 2j=R}} S_2 E_j.$$

The term “formal” specifies its source: no product relation has yet been divided out. We shall prove injectivity of its multiplication map where required.

Lemma B.3 (Capacity inequalities). *Put $t_j = d - j$ and*

$$P_j = \frac{d!}{j!(2d-j)!}, \quad C_j = \frac{1}{2^{d+1}j!t_j!}, \quad B_j = \binom{2d-j}{t_j}, \quad \eta = 253/250.$$

The counts in Proposition 5.1, with boundedly many extra columns, satisfy all the following strict asymptotic inequalities:

$$(B.9) \quad \eta \frac{\gamma_2}{P_2} \frac{5000}{4999} + B_2 \rho < 1, \quad 2\eta \frac{\gamma_j}{P_j} + B_j \rho < 1 \quad (j \geq 4),$$

$$(B.10) \quad \eta \frac{\gamma_2}{C_2} \frac{1250}{1249} < 1, \quad \eta \frac{\gamma_j}{C_j} < 1 \quad (j \geq 4).$$

The size conditions (5.3) hold for large h . For every $4 \leq R \leq d$, $\binom{2d-R}{d-R} \rho < 1/8$.

Proof. We first treat $d \geq 13$ by uniform estimates, and then check $d = 9, 10, 11, 12$ directly. Factorial cancellation gives

$$\frac{\gamma_4}{P_4} = \frac{2d-4}{5d} < \frac{2}{5}, \quad \frac{\gamma_6}{P_6} = \frac{(2d-6)(2d-7)}{28d(d-1)} < \frac{1}{7}.$$

For $j \geq 6$, the successive ratio of γ_j/P_j is

$$\frac{(j+1)(j-3)}{(2j-2)(2j-3)} \frac{(2t_j+4)(2t_j+3)}{(t_j+4)(d+t_j)} < \frac{2}{3},$$

so these ratios decrease. Also $\gamma_2/P_2 = 2\rho_2(2d-3)/d < 92/125$ and $B_2\rho < 501/2000$. For $j \geq 4$, $B_j/H < 2^{-(j-1)}$, hence $B_j\rho < 1/8$. For $d \geq 13$, inequalities (B.9) follow from the rational inequalities

$$\frac{253}{250} \frac{92}{125} \frac{5000}{4999} + \frac{501}{2000} < 1, \quad 2 \frac{253}{250} \frac{2}{5} + \frac{1}{8} < 1.$$

Still assuming $d \geq 13$, every active $j \geq 6$ has $t_j \geq 5$, and

$$\frac{\gamma_j}{C_j} = \frac{2^{d+1}}{\binom{2j-4}{j} \binom{2t_j+4}{t_j}} \leq \frac{10(d+1)^2}{2^d} \leq \frac{245}{1024}.$$

Here one may use $\binom{2j-4}{j} \geq (2/5)2^{2j-4}/(2j-3)$ and $\binom{2t_j+4}{t_j} \geq (1/2)2^{2t_j+4}/(2t_j+5)$; their product and $(2j-3)(2t_j+5) \leq (d+1)^2$ give the bound. For $j = 4$ the same calculation gives $8(d+1)^2/2^d \leq 49/256$.

For $j = 2$ it gives $\gamma_2/C_2 \leq \rho_2(2d - 3)2^{5-d} < 529/32000$. These bounds imply (B.10), including the stated small loss in quadratic outputs.

For clarity, the finite cases are included here as rational checks. Replace ρ by $1/(2H - 1)$ and ρ_2 by $23/125$ in the left sides. Substitution of the displayed factorial formulas gives strict upper bounds for the two scalar dimension ratios, the largest product dimension ratio, and the limiting codimension ratio c/h^2 , respectively. Every decimal in the table denotes an exact terminating rational number.

d	quadratic scalar	other scalar	products	$\lim c/h^2$
9	.856117	.670867	.308420	.000067094
10	.870078	.691024	.149537	.000018586
11	.881489	.707513	.073947	.000005100
12	.890989	.721251	.036974	.000001389

Thus the finite checks require only integer multiplication, with no approximation to a square root. The last column gives the asserted upper bound on c/h^2 in the limit. For $d \geq 13$, the elementary bound $H \geq 2^{2d-1}/(2d)$ gives $2^{d+1}\rho < d2^{3-d} \leq 13/1024$, whose squared half is less than $1/10000$. Finally L/h tends to a fixed number in $(0, 1)$ and c/h tends to infinity, proving the other two size conditions. $\square$

Lemma B.4 (The polynomial row maps). *For the prescribed counts and every positive even integer R , the following maps have, for general parameters, precisely their constant $\mathcal{S}$ - E_R Koszul kernel:*

$$(B.11) \quad (\mathcal{S} \otimes S_R X \otimes S_{d-R} Y) \oplus (E_R \otimes S_d Y) \oplus \Pi_R \longrightarrow S_R X \otimes S_{2d-R} Y.$$

Terms of negative degree and inactive E_R are omitted. When d is odd, one may add to the source $P \otimes S_{R-1} X \otimes S_{d-R+1} Y$, where P is any fixed number of general labelled forms in $X \otimes S_{d-1} Y$. The only additional kernel is $\wedge^2 P$ when $R = 2$. All the row assertions hold simultaneously, and remain valid after adjoining one further column to E_2 together with its corresponding scalar form in $\mathcal{S}$.

Proof. We construct a witness for each row in three stages: separate the formal products, place the scalar image away from them, and then insert the new layer E_R . We treat the fixed odd family P last and intersect the resulting rank opens.

The formal products. Split $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ into equal halves, ignoring at most one Y variable until the end. Each layer occurs in at most one pair in Π_R . For a pair $j < l$, place its two spaces in $S_j X_1 \otimes S_{t_j} Y_1$ and $S_l X_2 \otimes S_{t_l} Y_2$. Disjoint variables make the product map tensor-injective. Each space has asymptotic capacity $2C_j h^j m^{t_j}$ or $2C_l h^l m^{t_l}$, larger than its prescribed count by (B.10). If $j = 2$, intersect its output space with D ; the fractional loss is at most $1/1250 + o(1)$.

For a diagonal pair use the paired-variable subspace of Lemma 4.1 in degree j on $X_1 \oplus X_2$, whose symmetric square is injective by Lemma 4.1. It has $(1/(2^j j!) + o(1))h^j$ independent output directions. Use the same construction in degree t_j on $Y_1 \oplus Y_2$, whose symmetric square is also injective. For each output direction choose a scalar subspace of dimension at most half this latter space, with the subspaces for distinct output directions pairwise disjoint. Such a finite collection exists: general subspaces of dimension at most half the ambient dimension have pairwise zero intersection. Joint direct-sum independence is not required. For two disjoint subspaces A, B the map $A \otimes B \rightarrow S_2(A + B)$ is injective. Thus every pair of output directions gives independent scalar products, while independence of the output products separates the pairs. This proves diagonal injectivity at capacity $C_j h^j m^{t_j}$. The quadratic output restriction has the same fractional loss as above.

Different pairs in Π_R have different X_1 -degrees: the smaller index j for a cross pair and $R/2$ for the diagonal pair. Consequently all formal products are jointly independent. All these degrees are even. Their Y_1 -degrees are $d - j$ for a cross pair and $2\lfloor(d - j)/2\rfloor$ for the diagonal pair. For odd d

they lie in one parity class with at most one additional degree; for even d they lie in a single parity class.

The scalar image. Delete all these Y_1 profiles from the scalar target. In normalized monomial multiplication by $S_d Y$, conditional on a source half-degree, the limiting increment has distribution $\text{Bin}(d, 1/2)$. Thus deleting one parity and at most one additional profile retains at least

$$\frac{1}{2} - 2^{-d} \binom{d}{\lfloor d/2 \rfloor}.$$

This is at least $65/256 > 1/4$ for $d \geq 9$. There are finitely many source profiles, so the same strict lower bounds hold for large finite m after a small decrease. The monomial coupling proof of Lemma B.1, with its target restricted to retained profiles, gives this fraction of the usual normalized shadow for arbitrary subspaces as well. When products occur, $R \geq 4$, and the scalar dimension ratio is $\binom{2d-R}{d-R} \rho < 1/8$. Coefficient-rank incidence therefore makes the entire common scalar list injective in this quotient. Its image is disjoint from all products, simultaneously, in the original target.

The new layer. When a new layer E_R occurs with $R \geq 4$, put its outputs in

$$K_R = \bigoplus_{a \text{ odd}} S_a X_1 \otimes S_{R-a} X_2, \quad \dim K_R = (1/(2R!) + o(1))h^R.$$

This separates its image from all product outputs. On each monomial of degree t_R in Y place b general output vectors, where $b = \lceil \eta \gamma_R t_R! h^R \rceil$; for sufficiently large h this provides more than the required count while satisfying

$$B_R b / \dim K_R + B_R \rho < 1.$$

A target monomial has at most $B_R b$ labels, proving polynomial injectivity of $E_R \otimes S_d Y$. Form the graded quotient module $\mathcal{M}_R = (K_R \otimes S(Y)) / (E_R \cdot S(Y))$, graded by Y -degree. Uniformly for all subspaces $M \subset (\mathcal{M}_R)_{t_R}$, its multiplication satisfies the shadow estimate

$$(B.12) \quad \dim(S_d Y M) \geq \left[\frac{\dim K_R - B_R b}{\dim K_R} \frac{N_{d+t_R}(m)}{N_{t_R}(m)} - O_h(m^{d-1}) \right] \dim M.$$

Indeed, degenerate to monomial fibers and project a source fiber to each target fiber. Apart from exceptional projections, each retains at least the displayed fraction of its dimension. The disjoint-set Grassmannian argument in Theorem B.2 bounds exceptional additional-label sets by a constant. An extra degree- t_R monomial has least common multiple of degree at least $t_R + 1$ with the source, so each exceptional label affects $O_h(m^{d-1})$ targets. Summing the normalized divisibility coupling proves (B.12). The strict scalar inequality and coefficient-rank incidence now give injectivity of $\mathcal{S}$ on this quotient. Before taking the quotient this says exactly that its kernel is the constant $\mathcal{S}$ - E_R Koszul source. Deleting surplus labelled columns preserves this kernel assertion. For $R = 2$ there are no products; use output space D and the first inequality of (B.9). Unused output complements are free modules and satisfy a stronger estimate.

The fixed odd family. It remains to include P . Reserve distinct variables z_i outside the variables used above and put $P_i = w_i z_i^{d-1}$, with general nonzero $w_i \in X$. The scalar/new-layer map has no kernel one degree below the degree just considered. To see this when E_R is present, multiply a hypothetical lower relation by two independent core variables. Both resulting relations are constant Koszul relations. Comparing their scalar-side coefficients gives a linear-coefficient relation among the E_R columns; it vanishes because $E_R \otimes S_d Y$ is injective, hence so is $E_R \otimes Y$. The two constant Koszul matrices therefore vanish. Lower degrees follow by multiplication. When E_R is absent this assertion is immediate. This proves exactness after adjoining the private variables in every component not involving P .

Impose also, for each i , the injection

$$E_R \otimes Y_{\text{core}} \longrightarrow [S_R X / (w_i S_{R-1} X)] \otimes S_{t_R+1} Y_{\text{core}}.$$

The same sparse construction proves it: a target has at most $(t_R + 1)b$ labels, while the output quotient loses only $N_{R-1}(h) = O(h^{R-1})$ dimensions; its strict capacity is weaker than the inequality already imposed. Finitely many such opens intersect. The scalar coefficient degree is $d - R < d - 1$, so scalar images cannot contain a private power. The only possible overlap of P with E_R has private degree $d - 1$ or d , and is eliminated by this projected injection or its constant-coefficient restriction. For $R \geq 4$ a target cannot contain two private powers, since $2(d - 1) > 2d - R$. For $R = 2$ two distinct powers meet only in the target $z_i^{d-1} z_j^{d-1}$; the intersection $w_i X \cap w_j X = k w_i w_j$ gives exactly the alternating P - P relation. Multiplication by a single w_i is injective. This proves precisely the asserted additional kernel.

Simultaneous choice. Every assertion proved is a rank condition on a polynomial matrix with a known, constant-dimensional mandatory kernel. Its witness therefore gives a nonempty open in the same irreducible parameter space. Intersecting the finitely many row opens proves simultaneity. A bounded number of added columns does not affect any strict asymptotic inequality, proving the final assertion. $\square$

B.4. Prepared backgrounds and their exact homology. The row maps now allow us to eliminate all background relations except those already present among the forms in Q . The proposition also records the projected and enlarged versions needed for the replacement argument.

Proposition B.5 (Prepared background). *Let $Q \subset S_d Y$ be general of dimension q , without any assumption that its endpoint homology or cokernel vanishes. There are simultaneous nonempty opens of the preceding parameter spaces on which the following statements hold. For odd d , take $U \subset S_d X$ of dimension $u = \lceil 2h^d / (d + 1)! \rceil$ and an equally labelled family $P \subset X \otimes S_{d-1} Y$, and set*

$$W_{\text{bg}} = \langle Q, E_j + a_j \ (j \in \mathcal{J}), U + P \rangle.$$

For even d , let U be a basis of $S_d X$, take labelled columns $P_2 \in D \otimes S_{d-2} Y$ and $b \in S_d Y$, and set

$$W_{\text{bg}} = \langle Q, E_j + a_j \ (j \in \mathcal{J}), U + P_2 + b \rangle.$$

In both cases

$$(B.13) \quad H_1(W_{\text{bg}}; S)_{2d} = H_1(Q; S(Y))_{2d}$$

by natural inclusion. Equality continues to hold after projecting the degree- $2d$ target by any complement to $Q S_d Y$ in $S_{2d} Y$, and after adding one further generator of the form $E_2 + a_2$. Odd homology is zero both before and after adjoining F . Consequently, in the background quotient $R = S / (W_{\text{bg}})$,

$$(B.14) \quad F \otimes R_{d,\text{even}} \longrightarrow R_{2d,\text{odd}} \quad \text{is injective.}$$

The pure family may simultaneously be chosen with $UX = S_{d+1} X$ in odd degree, and with $U = S_d X$ in even degree.

Proof. We first identify the even homology by eliminating coefficients in increasing X -degree. We then treat projection of the scalar target, prove the odd-parity assertion by a separate witness, and choose the pure family to fill the highest degrees.

Even homology when d is odd. Suppose $d = 2s + 1$. Reindex the even layers by half their X -degrees, so $I = \{j/2 : j \in \mathcal{J}\}$ and $k = \max I$. In an even cycle write $q_t, e_{i,t}$ for its Q, E_{2i} coefficients of X -degree $2t$, and u_t for its $U + P$ coefficient of X -degree $2t + 1$. The row of degree $2r$ is exactly

$$(B.15) \quad Q q_r + \sum_i a_{2i} e_{i,r} + \sum_i E_{2i} e_{i,r-i} + P u_{r-1} + U u_{r-s-1} = 0,$$

with out-of-range terms zero.

Process $r = 1, \dots, \max(2k, s+1)$ in increasing order. The normalization maintained at each processed column $t > 0$ is as follows. Put the Q coefficient equal to zero. If $t \notin I$, all even-family coefficients in that column are zero. If $t \in I$, put $e_{i,t} = 0$ for $i > t$, retain $e_{i,t}$ for $i < t$ as a constant combination of E_{2t} , and choose an upper-triangular representative of the diagonal formal-product class, modulo the constant Koszul boundaries. At row r , previously retained terms in $\sum_i E_{2i}e_{i,r-i}$ are precisely the formal product space Π_{2r} . The term Uu_{r-s-1} has already vanished at an earlier row. Thus (B.15) is exactly a relation in Lemma B.4. That lemma kills the formal product coefficients and the current P coefficient, with its sole exception at $r = 1$: the alternating P - P coefficient is removed by the $(U+P)$ -($U+P$) Koszul boundary. Its other contribution is to the future coefficient u_s .

The remaining coefficients are constant combinations of E_{2r} . Subtract $Q-(E_{2r} + a_{2r})$ boundaries to kill q_r ; for $i > r$ subtract $(E_{2i} + a_{2i})-(E_{2r} + a_{2r})$ boundaries to kill $e_{i,r}$; and subtract boundaries within $E_{2r} + a_{2r}$ to put the diagonal coefficient in the chosen upper-triangular form. Only scalar coefficients and future positive columns change. The coefficients with $i < r$ must be retained: they are killed by the formal product row $i+r$, not by reversing the preceding boundary operation. This completes the row-by-row normalization without disturbing earlier rows. By its last row every positive coefficient, every retained product coefficient, and every u_t has vanished. The residual scalar coefficient of each E_{2i} vanishes by the injection $E_{2i} \otimes S_d Y$ in its row. Only a cycle $Qq_0 = 0$ over $S(Y)$ remains. The homology of Q injects by the formal symmetric-product description in Lemma 2.1, giving equality in even parity.

Even homology when d is even. For $d = 2s$, first omit the pure family. The identical argument, with no P, U terms, runs through $r = \max(s, 2k)$ and proves even homology equal to that from $S(Y)$. At $U = 0$, absorb the fixed u columns P_2 into the quadratic family, with their scalar forms b . Lemma B.4 applies at count $e_2 + u$, hence proves the same equality for $Q, E_j + a_j, P_2 + b$. Now vary the pure columns U freely. The equality just proved means that the polynomial matrix has the largest rank allowed by homology from $S(Y)$ and the constant Koszul kernel. This rank persists on a nonempty open in the full coefficient space. The condition that U is a basis of $S_d X$ is another nonempty open, so the two intersect. This proves even-parity equality for $U + P_2 + b$.

Projection and the additional column. Projecting a complement to $QS_d Y$ changes only the degree-zero row in either argument. At the end this row reads Qq_0 in $S_{2d} Y/Z_0 \simeq QS_d Y$, hence again $Qq_0 = 0$. The additional E_2 column is already allowed in Lemma B.4.

Odd homology. For odd parity use a separate polynomial witness. Set every positive even component and every pure component to zero while retaining their scalar forms, and in odd degree retain P . The tuple is now the independent scalar list $\mathcal{S}$ and the odd list $O = F + P$ (or $O = F$ in even degree). At X -degree one, Theorem B.2, its fixed-column extension, and the positive reserve give exactly the constant $\mathcal{S}$ - O Koszul kernel. At odd degree $b \geq 3$, the two sources are

$$O \otimes S_{b-1} X \otimes S_{d-b+1} Y, \quad \mathcal{S} \otimes S_b X \otimes S_{d-b} Y.$$

Lemma B.1 gives injectivity whenever their leading normalized dimension ratio is less than one. That dimension ratio is

$$\rho \left[\frac{bh}{h+b-1} \binom{2d-b}{d-1} + \binom{2d-b}{d} \right] < H\rho < 1.$$

Indeed, with $B_b = b \binom{2d-b}{d-1} + \binom{2d-b}{d}$, one has $B_3 = H$ and $B_b - B_{b+1} = b \binom{2d-b-1}{d-2} \geq 0$. The coefficient-space overhead is lower order in m . For even d the last odd degree $b = d+1$ has only the first source, whose dimension ratio is $(d+1)h\rho/(h+d) < 1$. There are no higher odd sources. The number of mandatory odd Koszul boundaries, the product of the even and odd generator counts, is constant on the independent-tuple locus. This witness therefore proves an open rank condition after all high components are restored. Homology under inclusion shows that the background alone

also has zero odd homology. Dividing out the ordinary F -background boundaries gives (B.14); all boundaries between two odd families have even parity.

The pure family. Finally, for odd d a general space U of the prescribed dimension has $UX = S_{d+1}X$. Here is a direct incidence proof. For a target covector let k be the dimension of its contraction kernel in X . Choosing that kernel costs $k(h-k)$ parameters; its annihilated shadow has dimension at least $N_{d+1}(h)k/h$ by Lemma B.1. Requiring u general forms to annihilate the remaining contractions costs $u(h-k)$ conditions. The resulting incidence dimension is at most

$$(h-k)[N_{d+1}(h)/h + k - u] - 1 < 0,$$

for large h , since $u > N_{d+1}(h)/h + h$. The full kernel is impossible. This proves the pure cutoff, and it can be intersected with all the preceding nonempty opens. $\square$

B.5. Surjectivity onto the positive target degrees. It remains to fill the positive target rows. The convolution lemma below converts multiplication in the X -variables into vector-valued multiplication in the Y -variables, with an exact number of generators. We then apply it to the prescribed families.

Lemma B.6 (Symmetric-power convolution). *Let $a, e, m \geq 1$. On bases $v_0, \dots, v_{a-1}$ of V , $y_0, \dots, y_{m-1}$ of Y , and $l_0, \dots, l_{m+a-2}$ of L , define $\phi(l_j) = \sum_{i+k=j} v_i \otimes y_k$. Multiplication gives an isomorphism*

$$S_e L \otimes S_{a-1} Y \longrightarrow S_e V \otimes S_{a+e-1} Y.$$

Thus $\binom{m+a+e-2}{e}$ vector forms of degree e , in $\binom{a+e-1}{e}$ output directions, fill the entire indicated target.

Proof. If $N = a + e - 1$ vectors in V are in linear general position, their products over all e -element subsets form a basis of $S_e V$. For a subset I , a functional vanishing on its $a-1$ complementary vectors and nonvanishing on its own vectors, raised to its e th power, separates precisely that product. Write $v(t) = \sum v_i t^i$, $y(t) = \sum y_k t^k$, and $l(t) = \sum l_j t^j$; then $\phi(l(t)) = v(t)y(t)$. A functional annihilating the claimed image annihilates

$$v(t_1) \cdots v(t_e) \otimes y(t_1) \cdots y(t_N)$$

and, by permuting the t_i , the analogous expression for every e -element subset. Distinct t_i give the preceding general-position basis by the Vandermonde determinant. Polynomial continuation then shows that the functional annihilates all $S_e V \otimes S_N Y$. The dimensions agree by the factorial identity

$$\binom{m+a+e-2}{e} \binom{m+a-2}{a-1} = \binom{a+e-1}{e} \binom{m+a+e-2}{a+e-1}. \quad \square$$

Proposition B.7 (The target maps). *Use the quadratic endpoint and the theorem of Boij, Dannetun and Lundqvist, stated in the introduction. For the exact counts of Proposition 5.1, the homogeneous positive components of F, E_j, U fill every target of X -degree $b \geq 2$ in total degree $2d$. Their deformed families in Proposition B.5 consequently fill all positive even target degrees, modulo lower even degrees. After projecting a complement to the image in $S_{2d}Y$, all even targets are filled. These assertions hold simultaneously with every background and product assertion above.*

Proof. We treat the middle rows first, then rows three and four, and finally row two, where two convolution constructions are needed. The pure family fills the remaining highest rows.

The middle rows. Suppose $j < b < 2j$ and a general space of degree- j forms in X , of dimension $\lceil \tau h^j \rceil$, fills $S_b X$, where $\tau > (b-j)!/b!$. This follows from that theorem: there is no degree- b Koszul contribution, and $N_b(h) - \tau h^j N_{b-j}(h) < 0$ for large h . Put $e = d - j$ and $a = d - b + j + 1$. The convolution lemma supplies blocks with $\binom{2d-b}{d-j}$ output directions. Round the number of chosen

output forms up to a multiple of this integer and apply convolution independently to each block. Multiplication by $S_{b-j}X$ then fills $S_bX \otimes S_{2d-b}Y$ using

$$\left[\frac{\tau h^j}{(d-j)! \binom{2d-b}{d-j}} + O_d(1) + O_h(m^{-1}) \right] m^{d-j}$$

generators. The $O_d(1)$ term accounts for rounding the output count and is absorbed by first choosing h sufficiently large. Strictly greater counts preserve surjectivity by extension and rank openness.

Use $j = 4$ for $b = 5, 6, 7$. Use each even $j \geq 6$ for $2j - 4 \leq b \leq \min(2j - 1, d)$ in odd degree, or through $d - 1$ in even degree. These intervals cover every required row from five onwards. The largest threshold in an interval is its first one: the ratio of consecutive thresholds is

$$\frac{b-j+1}{b+1} \frac{2d-b}{d+j-b} < 1 \quad (b \leq 2j-2).$$

At the first row it is precisely γ_j . The prescribed factor $101/100$ therefore gives the desired strict margin. Only target degrees below $2j$ occur in this application.

Rows three and four. For $j = 2$, the quadratic theorem supplies output spaces filling S_3X at density $1/6$ and S_4X at density $\rho_2/2$. Convolution gives thresholds τ_3 and τ_4 for rows three and four, respectively. For $3 \leq d \leq 8$, both are at most γ_2 by definition. For $d \geq 9$, we have $\gamma_2 = \tau_4$ and $\tau_4/\tau_3 = 3\rho_2(2d-3)/(d-1) > 1$. The ratio increases with d , and at $d = 7$ the inequality is exactly $\rho_2 > 2/11$, which follows from $2/3 < (9/11)^2$. Both successful quadratic output spaces have dimension less than $\dim D$. Extend a successful one to a subspace of dimension $\dim D$. The flags with a successful smaller space form a nonempty open incidence locus, dominating a nonempty open in the Grassmannian of such D . Thus a general D admits the required output space, and the convolution construction stays inside D . This proves rows three and four with the imposed output restriction.

Row two. Two convolution constructions fill complementary parts of this row. Write $X = A \oplus B$ with $p = \dim A = h/4$, and put

$$q_1 = \binom{2d-2}{d-1}, \quad b_1 = \left\lceil \frac{p}{q_1} \right\rceil.$$

Apply Lemma B.6 with $a = d$ and $e = d - 1$ to b_1 blocks, and project their $b_1 q_1$ output directions surjectively onto A . This gives

$$(B.16) \quad b_1 \binom{m+2d-3}{d-1}$$

forms in $A \otimes S_{d-1}Y$ whose multiples by $S_{d-1}Y$ fill $A \otimes S_{2d-2}Y$. Their further multiples by X fill $(AX) \otimes S_{2d-2}Y$.

These forms fit inside the prescribed F family. Indeed,

$$(B.17) \quad \frac{1}{4 \binom{2d-2}{d-1}} < \rho_d.$$

To see this, use $H = \binom{2d-1}{d-1} = \frac{2d-1}{d} q_1 < 2q_1$ and $H\rho_d > 1/2$. Since $b_1 = h/(4q_1) + O_d(1)$, first taking h large makes $b_1 < h\rho_d$. Then taking m large makes the count in (B.16) smaller than

$$f = \frac{h\rho_d}{(d-1)!} m^{d-1} + O_h(m^{d-2}),$$

by Proposition 5.1. Extend the constructed family to the exact prescribed dimension f .

The remaining target is the quotient $S_2X/(AX) \simeq S_2B$. Put

$$q_2 = \binom{2d-2}{d-2}, \quad b_2 = \left\lceil \frac{\dim S_2B}{q_2} \right\rceil.$$

Convolution with $e = d-2$ and $a = d+1$, followed by a surjective output projection, fills $S_2B \otimes S_{2d-2}Y$ with $b_2 \binom{m+2d-3}{d-2}$ forms. Its leading coefficient in m is

$$\frac{b_2}{(d-2)!} = \frac{9h^2}{32(d-2)!\binom{2d-2}{d-2}} + O_d(h) = \tau_2 h^2 + O_d(h).$$

This is strictly below the available coefficient αh^2 . For $3 \leq d \leq 8$, this follows from $\gamma_2 \geq \tau_2$ and $\alpha = (101/100)\gamma_2$. For $d \geq 9$, use $\gamma_2 = \tau_4$ and

$$\frac{\tau_2}{\tau_4} = \frac{9d}{32\rho_2(2d-3)} < 1,$$

which follows already for $d \geq 7$ from $\rho_2 > 2/11$. Thus the same order of choices, first h and then m , absorbs all rounding and gives at most the prescribed e_2 generators.

The subspace D can be required to project surjectively onto S_2B . Its codimension c is smaller than $\dim(AX)$: use (5.3) for $d \geq 9$ and (E.1) for $3 \leq d \leq 8$. A general subspace of this dimension therefore has a surjective restriction of the quotient map. Lift the convolution forms through $D \rightarrow S_2B$ and extend to the prescribed e_2 -dimensional family. The lifted forms fill the quotient target, while the chosen F forms fill its kernel $(AX) \otimes S_{2d-2}Y$. Together they fill row two.

Simultaneity and the highest rows. These are witnesses for rank-open conditions on the same irreducible parameter spaces as the background and outer conditions. Their nonempty opens therefore intersect.

For odd d , the pure cutoff $UX = S_{d+1}X$ fills every row $b \geq d+1$. For even d , $U = S_dX$ fills every row $b \geq d$. This completes the homogeneous target assertions. Their nonvanishing rank minors give nonempty opens in the same parameter spaces used above. For the deformed families, their lower components have strictly smaller X -degree. Descending elimination therefore gives surjectivity onto positive even rows modulo the lower rows. The projection of a complement to QS_dY makes the residual degree-zero target equal to QS_dY , which Q fills. All claims follow by finite intersection of the established opens. $\square$

APPENDIX C. PRODUCT SEPARATION AND THE NORMAL MAP

The purpose of this appendix is to prove that a general scalar motion of the outer generators has maximal normal rank. There are two ingredients. First, we separate the symmetric products of those generators from the background relations. This gives an injective coefficient map on the remaining homology. Second, we bound the loci of target covectors for which scalar contraction has a large kernel. An incidence argument then gives the required rank.

We retain the exact counts and the notation of Proposition 5.1. In particular, X and Y have dimensions h and m , respectively, $s = d-1$, and

$$H = \binom{2d-1}{d-1}, \quad \rho = 1 - \sqrt{1 - H^{-1}}, \quad f = \frac{h\rho}{(d-1)!} m^{d-1} + O_h(m^{d-2}).$$

Every assertion below is made for fixed d and h , and sufficiently large m . The numerical estimates in this appendix are for $d \geq 9$; Appendix E supplies their replacements for $3 \leq d \leq 8$, with the same algebraic constructions and comparison arguments. Products of subspaces always mean their images under polynomial multiplication. A tensor product in the source of a map, on the other hand, denotes a formal source and is not identified with its image without an injectivity assertion.

C.1. Separation of products. We will place the symmetric products of F in one bidegree in the Y -variables. The next lemma ensures that the scalar relations remain independent after that bidegree is removed. Thus the two kinds of products can be separated in a quotient.

Lemma C.1 (A deleted bidegree). *Split $Y = A \oplus B$ into two subspaces whose dimensions differ by at most one, and let $\mathcal{P}$ be any one full bidegree summand of $S_{2d-2}(Y)$. For all sufficiently large m and every $L_0 \subset S_{d-2}(Y)$,*

$$(C.1) \quad \dim((S_d(Y)L_0 + \mathcal{P})/\mathcal{P}) \geq \frac{2}{5} \frac{N_{2d-2}(m)}{N_{d-2}(m)} \dim L_0.$$

Consequently, a general scalar family $Q^+ \subset S_d(Y)$ of dimension $(\rho/d! + o(1))m^d$ satisfies

$$Q^+ \otimes S_{d-2}(Y) \hookrightarrow S_{2d-2}(Y)/\mathcal{P}.$$

Proof. We first prove the shadow bound, then use it to exclude scalar relations. Put $s = d - 1$. Give a pair of monomials $y^\alpha \mid y^\beta$, of degrees $s - 1$ and $2s$, the weight $\prod_i \binom{\beta_i}{\alpha_i}$. Every source row has total weight $\binom{m+2s-1}{s+1}$ and every target column total weight $\binom{2s}{s-1}$; their ratio is $N_{2s}(m)/N_{s-1}(m)$. If y^α has A -degree i , the fraction of its row weight in target A -degree j is

$$\binom{s+1}{j-i} \frac{(\dim A + i)_{j-i} (\dim B + s - 1 - i)_{s+1-j+i}}{(m + s - 1)_{s+1}},$$

where $(a)_r = a(a+1) \cdots (a+r-1)$. Uniformly in the finitely many possible i, j , this tends to $2^{-s-1} \binom{s+1}{j-i} \leq 1/2$. Deleting one target bidegree therefore retains at least $2/5$ of every source row for large m . Summing the weights proves (C.1) for a monomial subspace. For general L_0 , pass to leading monomials: the products of a leading monomial which survive the deleted bidegree are leading monomials of the projected product image. This proves the same bound for L_0 .

To deduce scalar injectivity, compare this shadow with the number of scalar parameters. The coefficient of m^d on the right, per source dimension, is $\sigma_d = \frac{2}{5}(d-2)!/(2d-2)!$. It is strictly greater than $\rho/d!$, since

$$d!\sigma_d = \frac{2}{5} \frac{2d-1}{(d-1)H}, \quad H\rho = \frac{1}{1 + \sqrt{1-H^{-1}}} < \frac{3}{5}.$$

If a putative relation has coefficient span of dimension ℓ , its projective coefficient tuples have dimension at most $\ell(\dim Q^+ + N_{d-2}(m) - \ell) - 1$. The equations on the scalar family have rank at least $\frac{2}{5}N_{2d-2}(m)\ell/N_{d-2}(m)$. The strict leading inequality makes this rank larger than the parameter count. Projective incidence excludes every such relation for a general family. $\square$

Proposition C.2 (Separation in the quadratic quotient). *Choose $L \leq h$ with $L > 2^d \rho h$, put $c = L(L+1)/2$, and let*

$$\Lambda : S_2(X) \longrightarrow k^c, \quad D = \ker \Lambda$$

be general. Let Q^+ contain Q and all the scalar families in the prepared background, including one extra auxiliary generator when needed. Then $\text{Sym}^2 F$ injects into

$$(C.2) \quad \frac{S_2(X) \otimes S_{2d-2}(Y)}{D \otimes S_{2d-2}(Y) + Q^+ S_2(X) S_{d-2}(Y)}.$$

If $c \geq 2h - 1$, the same assertion holds after adding $\sum_{i=1}^v P_i(XS_{d-1}(Y))$ to the denominator, where $P_i \in XS_{d-1}(Y)$ are general and v is fixed independently of m . In particular,

$$(C.3) \quad \text{Sym}^2 F \hookrightarrow (S(X \oplus Y)/(W_{\text{bg}}))_{2d}$$

for either prepared background W_{bg} in Proposition B.5.

Proof. We construct a witness for the quotient injection, then include the private columns and pass to the prepared background.

Use Lemma 4.1 on equal-sized subspaces of the two halves of Y , discarding one variable if necessary. It gives a space C of dimension $((2^{d-1}(d-1)!)^{-1} + o(1))m^{d-1}$ with independent symmetric products, all in one bidegree summand $\mathcal{P}$. Choose subspaces $A_1, \dots, A_L \subset C$, of dimensions differing by at most one and adding to f . The inequality $L > 2^d \rho h$ ensures $\dim A_i < \frac{1}{2} \dim C$. General choices have $A_i \cap A_j = 0$ for $i \neq j$. For independent $x_1, \dots, x_L \in X$, set $F = \bigoplus_i x_i A_i$. Its symmetric multiplication is injective: diagonal products use the injection of $\text{Sym}^2 C$, and a cross term uses the injection $A_i \otimes A_j \rightarrow \text{Sym}^2 C$.

Choose Λ to be an isomorphism on $\text{Sym}^2 \langle x_1, \dots, x_L \rangle$. The images of the products are still independent and lie in $k^c \otimes \mathcal{P}$. Lemma C.1 makes the scalar multiplication map injective modulo $\mathcal{P}$. Its image therefore meets $k^c \otimes \mathcal{P}$ trivially, which separates the products from the scalar denominator. The scalar count is unchanged to leading order by all the extra families, so the lemma applies to the entire Q^+ .

For the assertion with P_i , reserve variables $z_1, \dots, z_v$, construct F, Q^+ on the other variables, and specialize $P_i = w_i z_i^{d-1}$. A general Λ is simultaneously injective on every $w_i X$ and every $w_i X + w_j X$: the latter has dimension $2h - 1$ when w_i, w_j are independent. A monomial of degree $2d - 2$ is divisible by at most two of the private powers z_i^{d-1} . When two occur it is exactly $z_i^{d-1} z_j^{d-1}$, and the two detected output spaces meet only in $k\Lambda(w_i w_j)$. Hence the P multiplication map has exactly its constant alternating relations.

It remains to separate these private-column products from the other terms. Every scalar product has total private-variable degree at most $d - 2$, whereas every P product has a private exponent at least $d - 1$. These images are disjoint. Scalar injectivity in each private-variable degree follows from Lemma C.1, or its lower-degree restriction. The F products have private-variable degree zero and are separated from the scalar image by the same lemma. Thus a single polynomial matrix has precisely the obligatory alternating kernel and an independent $\text{Sym}^2 F$ summand. This is its largest possible rank, so the conclusion holds on a nonempty open set of all the parameters.

To obtain the assertion for the actual background quotient, extract X -degree two from an alleged relation in (C.3). Every quadratic high component of the background lies in D ; components of higher X -degree do not contribute. The scalar contributions all belong to Q^+ . For odd d , the only additional term is $\sum P_i(XS_{d-1}(Y))$, already included above. For even d , the quadratic tails of the pure generators lie in D as well. The asserted quotient injection therefore excludes the relation. $\square$

C.2. The odd coefficient and target spaces. All parities in what follows refer to X -degree. By Proposition B.5, the prepared background has only even first homology, namely the included homology from $S(Y)$, and adjoining F creates no odd first homology. The normal map will take the remaining even homology to the odd cokernel.

We now describe the source of scalar contraction and the dual of its target. In even generating degree these spaces split by X -degree. In odd generating degree the relations $U + P$ couple the lowest and top components; we keep that coupling explicit.

Even generating degree. Put $I_0 = (Q, F)$ and $R_0 = S(X \oplus Y)/I_0$. The odd part of its degree- d component is

$$(C.4) \quad V = V_1 \oplus \bigoplus_{\substack{3 \leq t \leq d-1 \\ t \text{ odd}}} V_t, \quad V_1 = \frac{XS_{d-1}(Y)}{F}, \quad V_t = S_t(X)S_{d-t}(Y).$$

Define, for every odd $1 \leq t \leq 2d - 1$,

$$(C.5) \quad T_t = \frac{S_t(X)S_{2d-t}(Y)}{FS_{t-1}(X)S_{d+1-t}(Y) + QS_t(X)S_{d-t}(Y)},$$

omitting any summand having a negative index. Homogeneity gives the exact decomposition

$$(C.6) \quad \Lambda_0 = ((R_0)_{2d}^{\text{odd}})^* = \bigoplus_{\substack{1 \leq t \leq 2d-1 \\ t \text{ odd}}} T_t^*.$$

In particular, the summands with $t > d - 1$ are retained, even though they do not occur in the coefficient space V . Every remaining generator has the form $B_i + p_i$, with B_i of positive even X -degrees and $p_i \in S_d(Y)$. The B_i include the mixed pure generators $U_i + (P_2)_i$.

Odd generating degree. Put instead $I_0 = (Q, F, U + P)$. Choose a vector-space complement C_d of U in $S_d(X)$, and write $B_d = S_d(X)/U$. Eliminating the degree- d relations $U_i = -P_i$ gives

$$(C.7) \quad V = \frac{XS_{d-1}(Y)}{F} \oplus \bigoplus_{\substack{3 \leq t \leq d-2 \\ t \text{ odd}}} S_t(X)S_{d-t}(Y) \oplus C_d.$$

This is a vector-space decomposition of the quotient. The target equations couple its first and last terms. To control those equations, we first establish a shadow estimate after projection modulo U .

Lemma C.3 (The projected top degree). *For odd $d \geq 9$, put $a = N_{d-1}(h)$, $u = \dim U$, and $b = N_d(h) - u$. For general U and sufficiently large h ,*

$$(C.8) \quad \dim(XL_0 \bmod U) \geq \frac{b}{a} \dim L_0 \quad \text{for every } L_0 \subset S_{d-1}(X), \quad \frac{b}{a} \geq \frac{h}{2d}.$$

With $\delta = 1/(100(2d)!)$, the map

$$(C.9) \quad (F \otimes S_{d-1}(X)Y) \oplus (Q^+ \otimes B_d) \longrightarrow B_d \otimes S_d(Y)$$

is generically injective when Q^+ consists of Q and $\lfloor \delta m^d \rfloor$ additional general scalar forms. Consequently, if

$$T_d^{\text{raw}} = \frac{S_d(X)S_d(Y)}{FS_{d-1}(X)Y + QS_d(X)},$$

$$J_d = \frac{B_d \otimes S_d(Y)}{FS_{d-1}(X)Y + QB_d},$$

there is an exact sequence

$$(C.10) \quad 0 \longrightarrow U \otimes (S_d(Y)/Q) \longrightarrow T_d^{\text{raw}} \longrightarrow J_d \longrightarrow 0.$$

Moreover, every $L_0 \subset B_d$ satisfies

$$(C.11) \quad \dim(S_d(Y)L_0 \text{ in } J_d) \geq \lfloor \delta m^d \rfloor \dim L_0.$$

Proof. We first prove the projected shadow estimate. The ordinary normalized multiplication estimate gives $\dim XL_0 \geq N_d(h) \dim L_0 / a$. We use the following elementary projection observation. Suppose a family of subspaces indexed by $\text{Gr}(\ell, a)$ has dimension at least $(u + b)\ell/a$ in an ambient space of dimension $u + b$. For one such subspace of dimension w , the condition that projection modulo a u -plane have rank less than $b\ell/a$ is a Schubert condition of codimension greater than

$$\frac{u\ell}{a} \frac{b(a - \ell)}{a} = \frac{ub}{a^2} \ell(a - \ell).$$

Indeed, writing the intersection dimension as r , its codimension is $r(b - w + r)$, and rank failure gives $r > w - b\ell/a \geq u\ell/a$. If $ub > a^2$, this exceeds the dimension $\ell(a - \ell)$ of the indexing Grassmannian, so a general u -plane has no such failure. The case $\ell = a$ is immediate. In the present situation

$$\frac{ub}{a^2} = \left(\frac{2(d-1)}{d^2(d+1)^2} + o(1) \right) h^2, \quad \frac{b}{ah} \longrightarrow \frac{d-1}{d(d+1)} > \frac{1}{2d}.$$

This proves (C.8). Its nonempty open set meets the open where $UX = S_{d+1}(X)$.

We next use this estimate to prove injectivity of (C.9). Apply the two-family coefficient incidence estimate of Lemma B.1. By torus degeneration in Y and the divisibility coupling, the projected X estimate tensorizes with multiplication in Y . The F parameter action therefore has normalized shadow at least $(b/a)N_d(m)/m$, and the scalar parameter action has shadow $N_d(m)$. With $\sigma = b/a$, the limiting sum of the dimension ratios is

$$\frac{dh\rho}{\sigma} + \rho + d!\delta \leq (2d^2 + 1)\rho + d!\delta < 1.$$

For example, $H > 4d^2 + 3$ for $d \geq 9$ and $\rho < 1/(2H - 1)$ give $(2d^2 + 1)\rho < 1/4$, while $d!\delta < 1/100$. The coefficient-space overhead is of smaller order in m . Thus the displayed polynomial map is injective. At $Q^+ = Q$ this says exactly that its image has no unintended intersection with $U \otimes S_d(Y)$, giving (C.10).

Finally, fix one successful additional scalar family. Its tensor product with every $L_0 \subset B_d$ injects into J_d , proving (C.11) simultaneously for all L_0 . $\square$

We can now describe the coupling between the lowest and top target components. Let T_1 be the outer quotient

$$T_1 = \frac{XS_{2d-1}(Y)}{FS_d(Y) + QXS_{d-1}(Y)}.$$

For odd d , a covector on the quotient by I_0 must satisfy

$$(C.12) \quad \lambda_d(U_i p) = -\lambda_1(P_i p), \quad p \in S_d(Y)/Q.$$

Sequence (C.10) shows that every $\lambda_1 \in T_1^*$ extends across these equations. The space of extensions is an affine translate of J_d^* , and its direction has the scalar shadow (C.11). Thus the top covector is not chosen independently of λ_1 .

Multiplying the same relations by positive even coefficients gives the remaining equations. For every $v \in S_{2r}(X)S_{d-2r}(Y)$, they are

$$(C.13) \quad \lambda_{d+2r}(U_i v) = -\lambda_{1+2r}(P_i v), \quad 1 \leq r \leq (d-1)/2.$$

Their number is at most

$$(C.14) \quad H_0 = u \sum_{r=1}^{(d-1)/2} N_{2r}(h)N_{d-2r}(m) = O_h(m^{d-2}).$$

For the dimension estimate below, we first omit these higher equations. Allow the higher odd target covectors to be free, impose all the low Q, F quotient conditions and (C.12), and call the resulting vector space Λ_{amb} . If $\Lambda_0 = ((S(X \oplus Y)/I_0)_{2d}^{\text{odd}})^*$, then

$$(C.15) \quad \dim \Lambda_{\text{amb}} \leq \dim \Lambda_0 + H_0.$$

Thus omitting the higher equations increases the dimension by at most H_0 . No extension assertion for (C.13) is needed: their possible inconsistency can only reduce the dimension.

C.3. Scalar contraction on the full odd cokernel. In either parity let V be the space just defined and put $D_V = \dim V = O_h(m^{d-1})$. For $\lambda \in \Lambda_0$, define the scalar contraction

$$(C.16) \quad C_\lambda : V \longrightarrow S_d(Y)^*, \quad C_\lambda(v)(p) = \lambda(pv).$$

All products here take place in the quotient by I_0 . The remaining even generators are denoted $B_i + p_i$, $1 \leq i \leq e$. For odd d these are the $E_j + a_j$; for even d they include the mixed pure generators. In both cases $e = O_h(m^{d-2})$. Let J be the odd degree- $2d$ quotient after their addition, and write $j_0 = \dim J$ at general parameters.

The next proposition bounds the covectors for which this contraction has a large kernel. Its proof has two incidence counts: first over the source components, then over the scalar terms p_i .

Keeping the second count in the common parameter space gives the exact cancellation needed in the final bound.

Proposition C.4 (All covector strata). *There is $\xi > 0$ such that, for every integer $k \geq 1$ and all sufficiently large m ,*

$$(C.17) \quad \dim\{[\lambda] \in \mathbb{P}(J^*) : \dim \ker C_\lambda \geq k\} \leq j_0 - \xi m^d k - 1.$$

The assertion holds on a nonempty open set of the common parameter space of the construction.

Proof. Scalar estimates on the separate components. The strong outer estimate (Theorem B.2) supplies $c_1 > 0$ with

$$(C.18) \quad \dim\{[\lambda_1] \in \mathbb{P}(T_1^*) : \dim \ker(C_{\lambda_1}|_{V_1}) = k_1\} \leq \dim T_1 - c_1 m^d k_1 - 1.$$

For every other non-top source layer V_t , the quotient T_t defined by (C.5) has a uniform scalar shadow of order m^d . Indeed, append $\lfloor \delta m^d \rfloor$ general scalar forms to Q , with $\delta = 1/(100(2d)!)$. The two-family incidence estimate has unaugmented limiting dimension ratio

$$\rho \left[\frac{th}{h+t-1} \binom{2d-t}{d-1} + \binom{2d-t}{d} \right] < H\rho < \frac{501}{1000}.$$

The additional scalar dimension ratio is $\delta(2d-t)!/(d-t)! \leq 1/100$. Thus the enlarged polynomial map is injective, and a successful additional family shows that, for every $L_0 \subset V_t$,

$$(C.19) \quad \dim(S_d(Y)L_0 \text{ in } T_t) \geq \lfloor \delta m^d \rfloor \dim L_0.$$

There are only finitely many source layers, so all these additional families can be chosen simultaneously. They establish estimates; they are not added to the generator count.

Combining the source components. Work first on the locus $\lambda_1 \neq 0$. Append the source components in increasing X -degree, and let k_t be the increase in kernel dimension when V_t is appended. The new part of the kernel projects to a k_t -plane L_0 in V_t . Relative to this projection, it is described by a linear map from L_0 to the earlier source modulo its kernel. Choosing this plane and this map requires at most $k_t D_V$ parameters.

Once the graph is fixed, the earlier covector components prescribe the contraction on L_0 . Equivalently, they prescribe the new covector on $S_d(Y)L_0$. By (C.19), this imposes at least $\lfloor \delta m^d \rfloor k_t$ affine conditions. Inconsistent prescriptions give an empty fiber. For odd d , the same argument at the top takes place in the affine fiber (C.12) and uses (C.11).

The count must also include target components that do not occur in the source. In even degree we retain the dimensions of every unused high target summand in (C.6). In odd degree we work in the entire ambient space and allow for the loss (C.15). The parameter costs are of smaller order than the scalar conditions just obtained. Combining these observations with (C.18), and using $D_V = O_h(m^{d-1})$ and $H_0 = O_h(m^{d-2})$, gives $c_2 > 0$ such that

$$(C.20) \quad \dim\{[\lambda] \in \mathbb{P}(\Lambda_0) : \lambda_1 \neq 0, \dim \ker C_\lambda = k\} \leq \dim \Lambda_0 - c_2 m^d k - 1 \quad (k \geq 1).$$

For $k = 0$ we use $\dim \Lambda_0 - 1$.

Imposing the remaining scalar terms. We now pass from Λ_0 to J^* by varying all the scalar terms p_i in their common parameter space. First record the codimension of J^* at general parameters. Odd homology vanishes both before and after adding the e remaining even generators. Their relative odd coefficient map is consequently injective:

$$k^e \otimes V \hookrightarrow (S(X \oplus Y)/I_0)_{2d}^{\text{odd}}.$$

Here all old–new constant Koszul relations have already been removed by passing to V , and products of two new even generators have even parity. We therefore have the exact identity

$$(C.21) \quad \dim \Lambda_0 = j_0 + e D_V.$$

For fixed λ with contraction kernel dimension k , its annihilation of the i th new generator means

$$\lambda(B_i v) + C_\lambda(v)(p_i) = 0 \quad (v \in V).$$

If consistent, this imposes exactly $D_V - k$ independent affine conditions on the scalar form p_i . Thus the entire common scalar family imposes $e(D_V - k)$ conditions. Subtracting these from (C.20) and using (C.21) proves, for general scalar parameters,

$$\dim\{[\lambda] \in \mathbb{P}(J^*) : \lambda_1 \neq 0, \dim \ker C_\lambda = k\} \leq j_0 - (c_2 m^d - e)k - 1.$$

The locus with zero lowest component. To complete the count, we show that there is no nonzero covector in J^* with $\lambda_1 = 0$. Use the target surjections of Proposition B.7 in increasing odd X -degree. If a_i is the largest X -degree of B_i , its equation in target degree t has the form

$$\lambda_t(B_i^{[a_i]} v) + \sum_{0 < b < a_i} \lambda_{t-a_i+b}(B_i^{[b]} v) + \lambda_{t-a_i}(p_i v) = 0.$$

All indices except t are smaller. Vanishing of the earlier covectors thus forces annihilation of the split target image, and the surjection forces $\lambda_t = 0$. In odd degree this deals with $3 \leq t \leq d$; for $t \geq d+2$, equations (C.13) and $UX = S_{d+1}(X)$ give the same conclusion. In even degree, the pure family spanning $S_d(X)$ deals with every $t \geq d+1$, its quadratic tail using only smaller indices. Hence every component vanishes.

We have therefore covered every nonzero target covector. Since $e = O_h(m^{d-2})$, we may replace $c_2 m^d - e$ by ξm^d with fixed $\xi > 0$. Taking the finite union of the exact strata of dimensions at least k gives (C.17). All polynomial rank conditions used above hold on nonempty opens. The scalar-incidence bounds also hold on a nonempty open by projective fiber-dimension semicontinuity. Their finite intersection supplies the final assertion. $\square$

C.4. From coefficient maps to maximal normal rank. We now turn the covector estimate into a rank statement for a first-order deformation. We begin with the elementary fact that rank gained by the normal map lowers the dimension of homology. Given independent degree- d generators g_i and a motion $g_i(\epsilon) = g_i + \epsilon p_i$, the first derivative induces

$$(C.22) \quad \partial_p : H_1(g; S)_{2d} \longrightarrow (S/(g))_{2d}, \quad \left[\sum_i e_i \otimes a_i \right] \longmapsto \left[\sum_i p_i a_i \right].$$

It is well-defined: a constant Koszul boundary maps to an element of the ideal (g) .

Lemma C.5 (First derivative and rank). *If (C.22) has rank r_0 , the degree- $2d$ first homology of a general member of the motion has dimension at most $\dim H_1(g; S)_{2d} - r_0$.*

Proof. The constant Koszul image is a subbundle while the generators stay independent. Divide the coefficient bundle by it and choose local complements to the kernel and image of the resulting matrix at $\epsilon = 0$. An invertible block may be eliminated. The remaining block vanishes at zero and its coefficient of ϵ is the map from the old kernel to the old cokernel in (C.22). A nonzero r_0 -minor of this coefficient gives an r_0 -minor of the remaining block with nonzero leading term. The generic rank increases by at least r_0 , as asserted. $\square$

Proposition C.6 (Maximal rank of the first normal map). *Let K be the naturally retained homology from $S(Y)$ in a prepared background, let $\mathcal{H}$ be the even homology after adjoining F , modulo K , and put $a_0 = \dim \mathcal{H}$. Scalar motions $F_i \mapsto F_i + \epsilon z_i$, $z_i \in S_d(Y)$, induce a map*

$$\partial_z : \mathcal{H} \longrightarrow J$$

of rank $\min(a_0, j_0)$ for general z . The same conclusion holds after the even perturbation in Section 6.

Proof. We first identify the remaining homology with a subspace of the coefficients of F , and then apply the covector estimate to scalar motions of those generators. The coefficient map is

$$(C.23) \quad \mathcal{H} \hookrightarrow k^f \otimes V.$$

To define it, take the coefficients of the F generators in a cycle and reduce them modulo the degree- d relations of the other generators and F . Background cycles and constant Koszul boundaries give zero, so the map is well-defined.

To prove injectivity, suppose that the resulting coefficients vanish. Then subtract old–new boundaries to make all F coefficients constant combinations of the F_i . Modulo alternating combinations the remaining relation is a symmetric F product. Proposition C.2 makes that product zero only when the symmetric coefficient is zero. Subtracting the remaining alternating boundary leaves a background cycle, hence an element of K . This proves injectivity of (C.23).

It remains to prove maximal rank. For $[\lambda] \in \mathbb{P}(J^*)$ with $\dim \ker C_\lambda = k$, the kernel of $\text{id}_{k^f} \otimes C_\lambda$ has dimension fk . Its restriction to (C.23) therefore has rank at least $a_0 - fk$. Equivalently, the equations $\lambda \partial_z = 0$ impose at least $a_0 - fk$ linear conditions on the scalar motion $z = (z_1, \dots, z_f)$. Consider the incidence of scalar motions and nonzero target covectors that annihilate their image. Together with (C.17), the preceding count bounds its dimension by

$$\dim(S_d(Y)^f) + j_0 - a_0 - 1 - (\xi m^d - f)k.$$

The same formula without the last term holds for $k = 0$. Since $f = O_h(m^{d-1})$, its last coefficient is positive for large m . Thus a general motion has no nonzero annihilator when $a_0 \geq j_0$; when $a_0 < j_0$, its projective annihilator has dimension at most $j_0 - a_0 - 1$. These statements say precisely that ∂_z has rank $\min(a_0, j_0)$. Cycles from $S(Y)$ have zero F coefficients and remain unchanged. The compatibility with the perturbation of the distinguished generator is proved in Lemma 6.1. $\square$

APPENDIX D. QUADRATIC GENERATORS

For quadratic generators, the induction adds three variables at a time. We begin with a fixed block in those variables, then construct a multiplication module whose images admit uniform dimension bounds. These bounds give the incidence estimates needed to choose the mixed generators and their deformations simultaneously.

When the characteristic is different from two, the resulting recurrence combines with Theorem 7.4. In characteristic two, we retain a surviving square in the same construction and start from three explicit initial dimensions. The two arguments share the block and multiplication estimates developed first.

D.1. A three-variable quadratic block. The induction uses four fixed quadrics in the new variables. Their quotient is concentrated in degrees zero, one, and two, with a two-dimensional last component. This makes the mixed multiplication maps small enough to describe explicitly. We also record the cubic relations and the coefficient maps on quartic homology, since both enter the deformation after this block is combined with the forms in Y .

Let $X = \langle x, y, z \rangle$ and set

$$(D.1) \quad h_1 = xy, \quad h_2 = xz, \quad h_3 = yz, \quad h_4 = x^2 + y^2 + z^2, \quad Q_X = \langle h_1, h_2, h_3, h_4 \rangle.$$

Write $B = \text{Sym}(X)/(Q_X)$ and $P = B_2$. We use the following coordinates throughout this section:

$$(D.2) \quad P = \langle u, v \rangle, \quad u = [x^2], \quad v = [y^2], \quad [z^2] = -u - v.$$

Lemma D.1. *The Hilbert function of B is $(1, 3, 2)$. The cubic multiplication map $Q_X \otimes X \rightarrow \text{Sym}^3 X$ has kernel basis*

$$(z, -y, 0, 0), \quad (z, 0, -x, 0).$$

The degree-four first Koszul homology H_X of Q_X has dimension three, with basis represented by

$$(D.3) \quad \xi_1 = (z^2, -yz, 0, 0), \quad \xi_2 = (-yz, y^2, 0, 0), \quad \xi_3 = (-xz, 0, x^2, 0).$$

Reduction of the multiplier on h_a modulo Q_X induces a coefficient map $R_a : H_X \rightarrow P$. In the indicated bases, these maps are

$$(D.4) \quad \begin{aligned} R_1 &= \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ R_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_4 = 0. \end{aligned}$$

Proof. Every mixed cubic is divisible by xy , xz , or yz . Multiplying h_4 by each variable then shows that the pure cubes also belong to (Q_X) . This proves the Hilbert function and cubic surjectivity. The cubic kernel has dimension $4 \cdot 3 - 10 = 2$; the two displayed independent relations form a basis.

Quartic multiplication is also onto. Its kernel has dimension $4 \cdot 6 - 15 = 9$, and its incoming Koszul space has dimension $\binom{4}{2} = 6$. Thus $\dim H_X = 3$. The vectors (D.3) are cycles. Their coefficient reductions are (D.4), which are independent as vectors in $P^{\oplus 4}$. Incoming Koszul cycles reduce to zero. The three classes are therefore independent in homology and form a basis. $\square$

We will separate coefficient contributions by applying a scalar functional on the two-dimensional space P . Choose $\omega \in k \setminus \{0, -1\}$, and define $\ell \in P^*$ and $s \in P$ by

$$(D.5) \quad \ell(u) = 1, \quad \ell(v) = \omega, \quad s = \omega u - v.$$

Then $\ell(u) = 1$, $\ker \ell = ks$, and

$$(D.6) \quad \begin{pmatrix} \ell R_1 \\ \ell R_2 \\ \ell R_3 \\ \ell R_4 \end{pmatrix} = \begin{pmatrix} -(1 + \omega) & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the scalar coefficient map is injective: its first three rows have nonzero determinant $-\omega(1 + \omega)$. In particular, scalar contraction by ℓ still detects every class in H_X . At the same time, it kills every contribution in the direction s . These two properties will separate the pure and distinguished contributions in Proposition D.4.

D.1.1. Middle multiplication. We grade products by their degrees in X and Y , in that order. Let Y have dimension m , let $Q \subset \text{Sym}^2 Y$ have dimension q , and put $A_2 = \text{Sym}^2 Y / Q$. The mixed generators form a c -plane $E \subset X \otimes Y$. Their multiplication into bidegree $(2, 2)$, after quotienting by the pure generators, is

$$(D.7) \quad F_{22} : E \otimes X \otimes Y \longrightarrow P \otimes A_2, \quad g \otimes a \otimes b \longmapsto gab.$$

Proposition D.2. Suppose $0 \leq c \leq 3m$ and $0 \leq q \leq \binom{m+1}{2}$. The map (D.7) is surjective for general E, Q if either

- (i) $c \geq \lceil m/2 \rceil$; or
- (ii) $2c \leq m$ and, with $w = m - 2c$,

$$(D.8) \quad q \geq cw + \binom{w+1}{2}.$$

In the first case there is a choice of E that works for every Q .

Proof. We construct E using two-dimensional blocks of Y . First we check the diagonal and pair components of their products, then handle any remaining variables. For $ax + by + dz \in X$, multiplication from X to P is represented by

$$M(a, b, d) = \begin{pmatrix} a & 0 & -d \\ 0 & b & -d \end{pmatrix}.$$

On a two-dimensional block $Y_i = \langle t_i, w_i \rangle$, choose

$$(D.9) \quad g_i = (x + z)t_i + (y + z)w_i.$$

Write

$$M_0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The diagonal component $g_i \otimes X \otimes Y_i \rightarrow P \otimes \text{Sym}^2 Y_i$ has matrix

$$(D.10) \quad D = \begin{pmatrix} M_0 & 0 \\ M_1 & M_0 \\ 0 & M_1 \end{pmatrix}.$$

This square matrix is invertible. A kernel pair $(a, b) \in X \oplus X$ satisfies $M_0 a = M_1 b = 0$, whence $a = \alpha y$ and $b = \beta x$. The middle equation is then $(\beta, \alpha) = 0$.

For two distinct blocks, consider

$$(g_i \otimes X \otimes Y_j) \oplus (g_j \otimes X \otimes Y_i) \longrightarrow P \otimes Y_i \otimes Y_j.$$

A covector in its cokernel has coordinates $\lambda_{ab} = (u_{ab}, v_{ab})$, $a, b \in \{0, 1\}$. Annihilation of the first summand gives

$$u_{0b} = 0, \quad v_{1b} = 0, \quad v_{0b} + u_{1b} = 0,$$

and annihilation of the second gives

$$u_{a0} = 0, \quad v_{a1} = 0, \quad v_{a0} + u_{a1} = 0.$$

The first two equations in each family force all coordinates to vanish except possibly v_{00} and u_{11} . The remaining equations force these to vanish as well. Thus every pair component is onto.

If m is even, these diagonal and pair components prove surjectivity with $c = m/2$ and $Q = 0$. If m is odd, use two-dimensional blocks and one last variable t_* . The additional generator $g_* = (x + y)t_*$ fills its diagonal and every pair involving t_* , since multiplication by $x + y$ maps X onto P . Enlarging E and passing to a quadratic quotient on Y preserve surjectivity. This proves the first assertion.

For the second assertion, decompose

$$Y = Y_1 \oplus \cdots \oplus Y_c \oplus Z, \quad \dim Z = w,$$

and use (D.9) on the c two-dimensional blocks. Their diagonal and pair components are onto. For a basis variable $z_b \in Z$, the component into $P \otimes Y_i z_b$, in coordinates $(ut_i, vt_i, uw_i, vw_i)z_b$, is

$$(D.11) \quad (a, b, d) \longmapsto (a - d, -d, -d, b - d).$$

It has rank three and cokernel functional $(0, 1, -1, 0)$. Including the quadric $t_i z_b$ in Q removes this remaining cokernel direction: $v \otimes t_i z_b$ has cokernel coordinate one.

Take Q to contain all cw monomials $t_i z_b$ and all $\binom{w+1}{2}$ quadratic monomials in Z . The latter remove the entire component $P \otimes \text{Sym}^2 Z$. These quadrics in $S_2 Y$ are independent, so (D.8) allows their extension to a q -plane. Every component is now onto. Surjectivity is open on the irreducible product of Grassmannians parametrizing E, Q , which proves the generic assertion. $\square$

D.1.2. *The coefficient map in bidegree (3, 1).* The next coefficient map records how a relation in bidegree (3, 1) responds when the pure quadrics h_a move. Let K_{31} be the bidegree-(3, 1) first Koszul homology of $Q_X \oplus E$. Cubic surjectivity in Lemma D.1 gives

$$(D.12) \quad \dim K_{31} = 12m + 6c - 10m - 4c = 2m + 2c.$$

Put $P_1 = (X \otimes Y)/E$. A cycle is an equality

$$\sum_j g_j a_j = \sum_{a=1}^4 h_a b_a, \quad a_j \in \text{Sym}^2 X, \quad b_a \in X \otimes Y.$$

Reducing the b_a modulo E defines a coefficient map

$$(D.13) \quad K_{31} \longrightarrow P_1^{\oplus 4}.$$

Lemma D.3. *For $c \leq m$ and general E , the map (D.13) is injective. Consequently, if $D \subset P_1$ has dimension δ , its composition with $(P_1/D)^{\oplus 4}$ has rank at least $2m + 2c - 4\delta$.*

Proof. It suffices that polynomial multiplication

$$E \otimes \text{Sym}^2 X \longrightarrow \text{Sym}^3 X \otimes Y$$

be injective. This is a nonempty open condition when $c \leq m$: take $E = x \otimes L$ for a c -plane $L \subset Y$, and use independence of a basis of L followed by cancellation of x .

If a cycle lies in the kernel of (D.13), write $b_a = \sum_j \beta_{aj} g_j$. Its equation becomes

$$\sum_j g_j \left(a_j - \sum_a \beta_{aj} h_a \right) = 0.$$

Polynomial injectivity gives $a_j = \sum_a \beta_{aj} h_a$. The original cycle is therefore a linear combination of incoming Koszul boundaries between the pure and mixed generators. This proves injectivity. The last assertion follows because $D^{\oplus 4}$ has dimension 4δ . $\square$

D.1.3. *The coefficient of a distinguished quadric.* Later we will move one distinguished generator in $S(Y)$ in the X -quadratic direction s . Its possible coefficients form a subspace $U \subset A_2$. We need those contributions to be independent of both the mixed products and the contributions from H_X . The next statement achieves this for every U at once. Write $\alpha = \dim A_2 = \binom{m+1}{2} - q$.

Proposition D.4. *Suppose $c \leq m$ and*

$$(D.14) \quad \alpha \geq \binom{c+1}{2} + 3.$$

For general Q, E , there are $r_1, r_2, r_3, r_4 \in A_2$ such that, for every subspace $U \subset A_2$, the following map is injective:

$$(D.15) \quad \begin{aligned} &\text{Sym}^2 E \oplus U \oplus H_X \longrightarrow P \otimes A_2, \\ &(b, a, \xi) \longmapsto b + s \otimes a + \sum_{j=1}^4 R_j(\xi) \otimes r_j. \end{aligned}$$

Here b denotes its product image, and $s = \omega u - v$ is fixed as in (D.5). These injectivity conditions are compatible with the nonempty opens in Proposition D.2 and Lemma D.3 whenever their numerical hypotheses hold.

Proof. Fix a c -plane $L \subset Y$. For general Q , the subspace $\text{Sym}^2 L$ maps injectively to A_2 by (D.14). Denote its image by C . Choose r_1, r_2, r_3 independent modulo C , and set $r_4 = 0$. Specialize $E = x \otimes L$. Its symmetric products map injectively to $u \otimes C$.

Apply $\ell \otimes (A_2 \rightarrow A_2/C)$ to a relation in (D.15). The mixed-product terms vanish after quotienting by C , and the distinguished terms vanish because $\ell(s) = 0$. Write $\xi = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$. By (D.6), the remaining equation is

$$-(1 + \omega)a_1\bar{r}_1 + \omega a_2\bar{r}_2 + a_3\bar{r}_3 = 0.$$

It implies $\xi = 0$. The remaining relation lies between $u \otimes C$ and $s \otimes U$, which are disjoint because u, s form a basis of P .

This argument works with $U = A_2$, and hence with every subspace U simultaneously. Injectivity is the nonvanishing of a suitable minor, hence an open condition on the coefficient space. It is nonempty by the displayed specialization, so it intersects the other nonempty opens on their common irreducible parameter space. $\square$

D.2. The three-row convolution module. The incidence argument requires a lower bound for the multiplication image of every subspace of a quotient of $k^3 \otimes Y$. To obtain it, we construct a specialization whose graded pieces and multiplication maps can be described explicitly. Its three rows correspond to the three new variables in the induction. The presentation has $t + 2$ columns supported on t variables; its entries are shifted copies of the same variable sequence, which accounts for the term “convolution.” We then adjoin the remaining variables freely.

The calculation proceeds through the dual graded pieces, encoded below by polynomials in separate pairs of variables. In degree two, these dual pieces are spaces of symmetric bilinear forms. Bounds on their evaluation ranks give bounds on multiplication by linear forms. We also treat multiplication from degree zero before adjoining the free variables used in the induction.

We may work over an algebraic closure of the ground field; rank conclusions descend by extension of scalars. For a graded module M over $\text{Sym}(U)$ and a subspace $L \subset M_j$, write UL and $\text{Sym}^2(U)L$ for its linear and quadratic multiplication images.

D.2.1. The presentation and its inverse system. Let $t \geq 2$, let $U = \langle y_0, \dots, y_{t-1} \rangle$, and let $D = \langle e_0, e_1, e_2 \rangle$. Set $R = \text{Sym}(U)$ and $y_i = 0$ outside $0 \leq i < t$. Define

$$(D.16) \quad M = \text{coker} \left(R(-1)^{t+2} \longrightarrow D \otimes R \right), \quad f_k \longmapsto \sum_{d=0}^2 e_d y_{k-d}, \quad 0 \leq k \leq t+1.$$

Put

$$(D.17) \quad p = t - 1, \quad A = 2p, \quad B = \binom{p+1}{2}.$$

For binary forms we use $V_d = \text{Sym}^d(k^2)$. Thus a binary slot below means a pair of variables in which a polynomial is homogeneous of the indicated degree.

For a vector space Z , write

$$\text{Inv}^j(Z) = (Z^{\otimes j})^{\mathfrak{S}_j}, \quad \text{Inv}^0(Z) = k.$$

In particular, $\text{Inv}^2(Z)$ is the space of symmetric bilinear forms on Z^* and has dimension $\binom{\dim Z + 1}{2}$. Using invariant tensors makes this description valid in every characteristic.

Lemma D.5. *For $0 \leq j \leq 2$ there are identifications*

$$(D.18) \quad M_j^* \simeq \text{Inv}^j(V_{t-2}) \otimes V_{2-j}.$$

Moreover, $M_j = 0$ for $j \geq 3$. Thus the Hilbert function is $(3, A, B, 0, \dots)$, and the presentation (D.16) has no coefficient syzygies in degrees zero, one, or two. Here a coefficient syzygy of degree i is a relation among the displayed columns whose coefficients belong to $\text{Sym}^i U$.

Proof. We may extend scalars to an algebraic closure. For a functional φ on $D \otimes \text{Sym}^j U$, use separate binary slots $s, z_1, \dots, z_j$ to form the polynomial

$$F_\varphi = \sum_{d=0}^2 \sum_{0 \leq i_1, \dots, i_j < t} \varphi(e_d y_{i_1} \cdots y_{i_j}) s_0^{2-d} s_1^d \prod_{\nu=1}^j z_{\nu,0}^{t-1-i_\nu} z_{\nu,1}^{i_\nu}.$$

Every coefficient is exactly a value of φ , including when indices repeat. This gives a linear bijection to the polynomials of the indicated separate degrees invariant under permutation of the z slots: invariance says precisely that the coefficients depend only on the multiset of indices.

Annihilation of the presentation image means

$$\sum_{d=0}^2 \varphi(e_d y_{k-d} y_{i_1} \cdots y_{i_{j-1}}) = 0 \quad \text{for all } k, i_1, \dots, i_{j-1}.$$

These are exactly the coefficient equations obtained by substituting $s = z_\nu$ into F_φ . Bihomogeneity makes the resulting vanishing equivalent to vanishing whenever s, z_ν are proportional. The proportionality locus is defined by the prime polynomial $\det(s, z_\nu)$. These polynomials are relatively prime for distinct slots, so there is a unique factorization

$$(D.19) \quad F_\varphi = \prod_{\nu=1}^j \det(s, z_\nu) H(s; z_1, \dots, z_j).$$

Here H has degree $2 - j$ in s , degree $t - 2$ in each z slot, and is invariant under permutation of those slots. Conversely, every such H supplies a functional annihilating the presentation. This proves (D.18). For $j \geq 3$ the determinant product exceeds the available degree in s , so $F_\varphi = 0$.

The first three dimensions are therefore 3, $2(t - 1)$, and $\binom{t}{2}$. For coefficient degrees $i = 0, 1, 2$, subtracting $\dim M_{i+1}$ from $3 \dim \text{Sym}^{i+1} U$ gives

$$t + 2, \quad (t + 2)t, \quad (t + 2) \binom{t + 1}{2},$$

the full source dimensions. Thus each presentation map is injective. $\square$

D.2.2. Symmetric evaluation. To bound the image UL , we will bound its annihilator in M_2^* . The inverse system identifies that annihilator with a space of symmetric bilinear forms. The following lemma bounds the dimension of such a space from its evaluation ranks; we will apply it to the rational normal curve of binary evaluations.

Lemma D.6. *Let Z be a finite-dimensional vector space and let $\mathcal{X} \subset \mathbb{P}(Z)$ be an irreducible nondegenerate projective variety. Let T be a linear space of symmetric bilinear forms on Z , regarded as symmetric linear maps $Z \rightarrow Z^*$. If $\dim T(v) = b$ for a general $[v] \in \mathcal{X}$, then*

$$(D.20) \quad \dim T \leq \binom{b + 1}{2}.$$

Proof. Induct on b . If $b = 0$, every member of T vanishes on the cone over a dense subset of $\mathcal{X}$, hence on its linear span Z . Thus $T = 0$.

Suppose $b > 0$. Choose $[v] \in \mathcal{X}$ with $\dim T(v) = b$ and put $T_0 = \{H \in T : H(v) = 0\}$. Then $\text{codim}_T T_0 = b$, and every member of T_0 descends to a symmetric form on Z/kv . By symmetry, $T_0(w) \subset v^\perp$ for every w . For general $[w] \in \mathcal{X}$, the space $T(w)$ is not contained in $v^\perp$. Indeed, choose $H \in T$ with $H(v) \neq 0$. Nondegeneracy of $\mathcal{X}$ gives $H(v)(w) = H(w)(v) \neq 0$ for general w . It follows that

$$\dim T_0(w) \leq \dim(T(w) \cap v^\perp) \leq b - 1.$$

If $\dim Z = 1$, then $T_0 = 0$. Otherwise the closure of the projection of $\mathcal{X}$ in $\mathbb{P}(Z/kv)$ is irreducible and nondegenerate. Apply induction to the descended space T_0 on this projection. It gives $\dim T_0 \leq \binom{b}{2}$, and hence $\dim T \leq b + \binom{b}{2}$. $\square$

Proposition D.7. *For every i -dimensional subspace $L \subset M_1$,*

$$(D.21) \quad \dim(UL) \geq b(i) := B - \binom{p - \lceil i/2 \rceil + 1}{2} \geq \frac{B}{A} i.$$

For each $0 \leq i \leq A$, the first bound is attained by some L .

Proof. Put $Z = V_{t-2}$, so $\dim Z = p$. Lemma D.5 identifies $M_1^* = Z \otimes V_1$ and $M_2^* = \text{Inv}^2(Z)$. Let $K = L^\perp \subset Z \otimes V_1$, with $k_0 = \dim K = 2p - i$, and let $T \subset \text{Inv}^2(Z)$ annihilate UL .

Write a member of M_2^* as a symmetric polynomial $H(z, a)$ of degree $t - 2$ in both binary slots. Its inverse expression is $\det(s, z) \det(s, a) H(z, a)$. In the dual of multiplication $U \otimes M_1 \rightarrow M_2$, removing the degree-one inverse factor $\det(s, z)$ leaves

$$\det(s, a) H(z, a).$$

Thus all coefficient contractions in the a slot belong to K when $H \in T$. In particular, for each binary point a ,

$$(D.22) \quad H(z, a) \otimes \det(s, a) \in K.$$

Let $\lambda_a = k \det(s, a) \subset V_1$ and define

$$Z_a = \{f \in Z : f \otimes \det(s, a) \in K\}.$$

Write b_0 for the dimension of Z_a at a general binary point. For two distinct general points a, a' , the quotient maps

$$K \longrightarrow Z \otimes (V_1/\lambda_a), \quad K \longrightarrow Z \otimes (V_1/\lambda_{a'})$$

have rank $k_0 - b_0$ and are jointly injective. Their kernels lie in the disjoint subspaces $Z \otimes \lambda_a$ and $Z \otimes \lambda_{a'}$. Consequently $k_0 \leq 2(k_0 - b_0)$, or $b_0 \leq \lfloor k_0/2 \rfloor$.

The evaluations at binary points form a nondegenerate rational normal curve in $\mathbb{P}(Z^*)$; for $t = 2$ this is the single point of $\mathbb{P}^0$. Regard $T \subset \text{Inv}^2(Z)$ as symmetric maps $Z^* \rightarrow Z$. Equation (D.22) shows that its general evaluation image lies in Z_a . Lemma D.6 therefore gives

$$\dim T \leq \binom{b_0 + 1}{2} \leq \binom{\lfloor (2p - i)/2 \rfloor + 1}{2} = \binom{p - \lceil i/2 \rceil + 1}{2}.$$

Since $\dim(UL) = B - \dim T$, this proves the first inequality. For $a_0 = \lceil i/2 \rceil \leq p$,

$$b(i) = \frac{a_0(2p + 1 - a_0)}{2} \geq \frac{a_0(p + 1)}{2} \geq \frac{p + 1}{4} i = \frac{B}{A} i.$$

For sharpness, choose $Z_0 \subset Z$ of dimension $p - \lceil i/2 \rceil$. Take K to contain $Z_0 \otimes V_1$ and to have dimension $2p - i$, adding one dimension if i is odd. Every $H \in \text{Inv}^2(Z_0)$ has all the displayed contractions in $Z_0 \otimes V_1$, so it annihilates UL for $L = K^\perp$. Its dimension equals the preceding upper bound for T , which proves equality. $\square$

D.2.3. Images of the output space. The preceding estimate starts with a subspace of M_1 . When free variables are adjoined, we will also need the images of subspaces of $M_0 = D$. The next proposition supplies both their linear and quadratic multiplication bounds.

Proposition D.8. *For every subspace $D' \subset D$ of dimension r ,*

$$(D.23) \quad \dim(UD') \geq A \min(r/2, 1).$$

If $D' \neq 0$, then

$$(D.24) \quad \text{Sym}^2(U)D' = M_2.$$

Proof. Let $F = (D')^\perp \subset D^* = V_2$. Project the $(t+2)$ -dimensional column space of (D.16) to $(D/D') \otimes U$. The transpose of this projection is binary multiplication $F \otimes V_{t-1} \rightarrow V_{t+1}$. For $r = 1, 2$, its image has dimension at least $t+2-r$. To see this, choose a basis of F with distinct leading binary exponents and multiply by all t monomial exponents of V_{t-1} . The resulting sumset contains at least $t + \dim F - 1 = t+2-r$ exponents, which give that many independent products. The intersection of the column space with $D' \otimes U$ therefore has dimension at most r . Hence $\dim(UD') \geq rt - r = r(t-1)$. This is (D.23) for $r = 1, 2$; the cases $r = 0, 3$ are immediate.

It remains to prove (D.24) for a nonzero vector $e \in D'$. The inverse expression of $H \in M_2^*$ is $\det(s, z) \det(s, a) H(z, a)$. Contraction in the s slot by e produces

$$B_e(z, a) H(z, a),$$

where B_e is a nonzero symmetric bilinear polynomial. Explicitly, if e takes values α, β, γ on $s_0^2, s_0 s_1, s_1^2$, then

$$B_e(z, a) = \alpha z_1 a_1 - \beta(z_1 a_0 + z_0 a_1) + \gamma z_0 a_0.$$

Thus $B_e \neq 0$ whenever $e \neq 0$. Multiplication by B_e is injective in the polynomial ring, so the dual of $\text{Sym}^2(U)e \rightarrow M_2$ is injective. The original map is therefore onto. $\square$

D.2.4. Adjoining free variables. We now pass from the t variables in the presentation to the full space of m variables. Let W have dimension w , set $Y = U \oplus W$ and $m = t + w$, and put $N = M \otimes \text{Sym}(W)$. The column count is $c = t + 2$, so $t = c - 2$. The relevant pieces are

$$\begin{aligned} \text{(D.25)} \quad N_1 &= M_1 \oplus (D \otimes W), \\ N_3 &= (M_2 \otimes W) \oplus (M_1 \otimes \text{Sym}^2 W) \oplus (D \otimes \text{Sym}^3 W). \end{aligned}$$

In particular,

$$\begin{aligned} a &:= \dim N_1 = A + 3w = 3m - c, \\ \text{(D.26)} \quad T &:= \dim N_3 = Bw + A \binom{w+1}{2} + 3 \binom{w+2}{3}. \end{aligned}$$

Extension by the polynomial ring in W preserves the absence of coefficient syzygies through degree two. Equivalently, decompose each coefficient polynomial by its W -monomial and apply Lemma D.5 to its U -coefficient. Hence

$$\text{(D.27)} \quad T = 3 \binom{m+2}{3} - c \binom{m+1}{2}.$$

D.3. Quadratic coefficients and conditioned covectors. We now convert multiplication-image bounds into generic rank statements. There are two related incidences. The first excludes relations among products with the quadrics in $S_2 Y$. The second bounds the covectors that annihilate those products and have a prescribed coefficient-kernel dimension. This second bound will control the cokernel of the deformation in Section D.5. In both arguments, the quadrics in $S_2 Y$ vary as one common q -dimensional space.

Let X and Y have dimensions 3 and m , respectively, and let $E \subset X \otimes Y$ be a c -plane, with $4 \leq c \leq m$. Its polynomial presentation has cokernel

$$N = \text{coker}(E \otimes \text{Sym}(Y)(-1) \longrightarrow X \otimes \text{Sym}(Y)).$$

Work on the nonempty open where the presentation has no coefficient syzygies in degrees zero, one, or two. The convolution presentation provides such a point. Set

$$\text{(D.28)} \quad a = \dim N_1 = 3m - c, \quad T = \dim N_3 = 3 \binom{m+2}{3} - c \binom{m+1}{2}.$$

Suppose that, for every v -plane $L \subset N_1$, a specified function $\mathcal{B}_m(v)$ satisfies

$$(D.29) \quad \dim(\mathrm{Sym}^2 Y L) \geq \mathcal{B}_m(v).$$

Proposition D.9. *If*

$$(D.30) \quad \mathcal{B}_m(v) \geq v(q + a - v) \quad (1 \leq v \leq \min(q, a)),$$

then a general q -plane $Q \subset \mathrm{Sym}^2 Y$ makes $Q \otimes N_1 \rightarrow N_3$ injective. If also $Q \otimes Y \rightarrow \mathrm{Sym}^3 Y$ is injective, then, writing $A = \mathrm{Sym}(Y)/(Q)$, the induced map

$$(D.31) \quad F_{13} : E \otimes A_2 \longrightarrow X \otimes A_3$$

is injective and has cokernel $J = N_3/(QN_1)$ of dimension

$$(D.32) \quad j = T - qa = 3 \left[\binom{m+2}{3} - mq \right] - c \left[\binom{m+1}{2} - q \right].$$

Proof. Parametrize Q first by ordered quadratic coefficients $f_1, \dots, f_q$. A nonzero tuple $(v_1, \dots, v_q) \in N_1^{\oplus q}$ whose entries span a v -plane has projective parameter dimension $v(q + a - v) - 1$. At a fixed tuple, the equation $\sum_i f_i v_i = 0$ imposes exactly $\dim(\mathrm{Sym}^2 Y \langle v_i \rangle)$ linear conditions on the ordered quadrics. Indeed, after a constant change among the entries, the nonzero v_i include a basis of their span. By (D.30), every such relation incidence has dimension smaller than the quadratic coefficient space. The union over the finitely many possible v is proper. Restricting to linearly independent ordered quadrics proves injectivity for general Q .

Both $Q \otimes N_1 \rightarrow N_3$ and (D.31) have cokernel

$$\frac{X \otimes \mathrm{Sym}^3 Y}{E \mathrm{Sym}^2 Y + X \otimes QY}.$$

Cubic independence gives their common Euler difference (D.32). The first map is injective, so its cokernel has dimension j . The second therefore has zero kernel as well. $\square$

We next refine the incidence count by recording the kernel of a covector's coefficient map. These are the “conditioned covectors” of the subsection title.

Proposition D.10. *Assume (D.29) and (D.30). For a general pair (E, Q) , put $J = N_3/(QN_1)$ and, for $0 \neq \lambda \in J^*$, define*

$$R_\lambda : N_1 \longrightarrow (\mathrm{Sym}^2 Y/Q)^*, \quad R_\lambda(v)(f) = \lambda(fv).$$

Let $\Sigma_v \subset \mathbb{P}(J^)$ be the locus where $\dim \ker R_\lambda = v$. Then*

$$(D.33) \quad \dim \Sigma_v \leq U_m(v) := j - \mathcal{B}_m(v) + qv + v(a - v) - 1 \quad (0 \leq v < a).$$

A negative right side means that the stratum is empty. The stratum Σ_a is empty.

Proof. We first count covectors before imposing the condition that they annihilate QN_1 . Before choosing Q , fix a v -plane $L \subset N_1$. The projective space of covectors in N_3^* annihilating $\mathrm{Sym}^2 Y L$ has dimension at most $T - \mathcal{B}_m(v) - 1$. Varying L adds $v(a - v)$ parameters. On the stratum where the coefficient map has kernel exactly L , its transpose

$$\mathrm{Sym}^2 Y \longrightarrow N_1^*$$

has rank $a - v$. Requiring λ to descend to J^* is equivalent to requiring that Q lie in its kernel. If such q -planes exist, they form a Grassmannian of codimension $q(a - v)$ in $\mathrm{Gr}(q, \mathrm{Sym}^2 Y)$. The generic-fiber dimension theorem subtracts this codimension from the preceding covector bound. Since $j = T - qa$, the result is (D.33).

The construction is relative over the open of mixed coefficients satisfying (D.29). There are finitely many possible v , so their generic-fiber bounds hold simultaneously on a nonempty open

of the pair space. For $v = a$, the covector would annihilate $\text{Sym}^2 Y N_1 = N_3$, which is impossible projectively. $\square$

D.4. A uniform asymptotic surplus. We now establish the multiplication-image bound used in both incidence arguments. Besides the proportional term Tv/a , the bound has a uniform positive surplus away from the zero and full subspaces. This surplus will absorb the parameters needed to choose subspaces and deformations.

The generating degree is two. Put

$$\rho = \frac{3 - \sqrt{6}}{6}, \quad \tau = 6\rho = 3 - \sqrt{6}, \quad q = \lceil \kappa_2(m) \rceil, \quad c = r - q - 4,$$

where r is either critical count in dimension $m + 3$. Here $\rho = \rho_2/2$, with ρ_2 as defined earlier: the factor $1/2$ accounts for the leading term of $N_2(m)$. Then $c = \tau m + O(1)$ and $q = \rho m^2 + m/2 + O(1)$. Use the convolution module above with $t = c - 2$ and $w = m - t$. Thus $t/m \rightarrow \tau$, $w/m \rightarrow 1 - \tau$, and

$$A = 2(t - 1), \quad B = \binom{t}{2}, \quad a = A + 3w, \quad T = Bw + A \binom{w+1}{2} + 3 \binom{w+2}{3}.$$

Lemma D.11. *For every sufficiently large m , every subspace $L \subset N_1$ of dimension v satisfies*

$$(D.34) \quad \dim(S_2 Y L) \geq \frac{T}{a} v + \frac{m^2}{1000} \min\{v, a - v\}.$$

The same simultaneous bounds hold for general c -column presentations in $X \otimes Y$.

Proof. We first reduce to subspaces that split according to the free variables. Their dimensions lead to a finite set of limiting cases, where we verify a strict surplus. Finally we pass back to finite m and then to general presentations.

Scale the free variables independently and take a torus limit. The limiting subspace has the form

$$L_0 = L_A \oplus \bigoplus_{\nu=1}^w D_\nu w_\nu, \quad i = \dim L_A, \quad n_b = \#\{\nu : \dim D_\nu \geq b\} \quad (1 \leq b \leq 3).$$

Here $0 \leq n_3 \leq n_2 \leq n_1 \leq w$ and $v = i + n_1 + n_2 + n_3$. A specialization cannot increase image rank, so it suffices to estimate L_0 . Set $x = i/A$ and

$$H_j(s) = \binom{w+j-1}{j} - \binom{w-s+j-1}{j} \quad (j = 2, 3).$$

For integer s , $H_j(s)$ counts degree- j monomials in the free variables that involve at least one of a specified set of s variables. We use its polynomial extension when taking derivatives below. The multiplication bounds for the convolution module show that the image dimension is at least

$$\begin{aligned} K(x, n_1, n_2, n_3) = & Bwx + B(\max(x, \tfrac{1}{3}) - x)n_1 \\ & + B(\max(x, \tfrac{2}{3}) - \max(x, \tfrac{1}{3}))n_2 + B(1 - \max(x, \tfrac{2}{3}))n_3 \\ & + A \binom{w+1}{2} x + A(\max(x, \tfrac{1}{2}) - x)H_2(n_1) \\ & + A(1 - \max(x, \tfrac{1}{2}))H_2(n_2) + H_3(n_1) + H_3(n_2) + H_3(n_3). \end{aligned}$$

Here the number of free variables means the total degree in W , with repetitions allowed. A component with one free variable contains both UL_A , of dimension at least Bx , and the quadratic image of D_ν , of dimension at least $B \dim D_\nu / 3$. Taking their maximum and summing the contributions counted by n_1, n_2, n_3 gives the first two lines. For two free variables the corresponding lower bounds are Ax , $A/2$, and A , and the contributions counted by $H_2(n_1), H_2(n_2)$ give the next two lines. For three free variables, the image contains the sum of the incident D_ν ; its dimension is at least their

largest dimension, giving the last three terms. Repeated free variables are included in the binomial counts H_j .

All coefficients of H_2 and H_3 are nonnegative, while $H_2''(s) = -1$ and $H_3''(s) = s - w - 1 < 0$ on $[0, w]$. Thus K is concave in the ordered triple (n_1, n_2, n_3) . On each of the four intervals with endpoints

$$0, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad 1,$$

it is affine in x . The region $0 \leq n_3 \leq n_2 \leq n_1 \leq w$ has the four vertices

$$(0, 0, 0), \quad (w, 0, 0), \quad (w, w, 0), \quad (w, w, w).$$

We call these the prefix vertices. As made explicit below using an affine upper bound for the desired surplus, it suffices to check these four vertices at the five displayed values of x .

We verify a strict margin at the limiting vertices; this margin will allow us to return to finite m . At $z = t/m \rightarrow \tau$, put

$$A_0 = 2z, \quad B_0 = z^2/2, \quad w_0 = 1 - z, \quad a_0 = A_0 + 3w_0, \quad T_0 = B_0w_0 + A_0w_0^2/2 + w_0^3/2.$$

For the prefix vertex with b copies of w , define

$$\ell = A_0x + bw_0, \quad I = B_0w_0 \max(x, b/3) + \frac{A_0w_0^2}{2} \max(x, \min(b/2, 1)) + \frac{bw_0^3}{6}.$$

Writing $s = \sqrt{6}$, the values of $a_0I - T_0\ell$ at $z = 3 - s$ are as follows:

$x \backslash b$	0	1	2	3
0	0	$-26 + 32s/3$	$-52 + 64s/3$	$-12 + 5s$
1/3	$-13 + 16s/3$	$-22 + 9s$	$-48 + 59s/3$	$-8 + 10s/3$
1/2	$-39/2 + 8s$	$-13/2 + 8s/3$	$-46 + 113s/6$	$-6 + 5s/2$
2/3	$-26 + 32s/3$	$-13 + 16s/3$	$-44 + 18s$	$-4 + 5s/3$
1	$-39 + 16s$	$-26 + 32s/3$	$-13 + 16s/3$	0

Every nonzero entry is greater than $1/40$; substitution of $s > 2449/1000$ already proves these inequalities. On the first interval use the auxiliary quantity $D = \ell$, on the last use $D = a_0 - \ell$, and on the two middle intervals use $D = a_0/2$. In each case $D \geq \min(\ell, a_0 - \ell)$. On each interval D is affine in x and in the profile coordinates. Subtracting $a_0D/1000$ therefore preserves the concavity and affine dependence used in the vertex reduction above. Also $a_0D < 9$ at all vertices. The strict table therefore implies

$$a_0I - T_0\ell > \frac{a_0D}{1000}$$

at every nontrivial vertex. The empty and full vertices have both sides identically zero, including before passage to the limit.

The normalized finite-dimensional quantities depend continuously on t/m and $1/m$. Thus the same weak inequalities hold for all sufficiently large m . Concavity in the ordered triple, followed by affine interpolation in x on each interval, proves them for every L_0 . Restoring the powers of m gives (D.34).

Finally work on the coefficient open where N_1 and N_3 have dimensions a, T . Failure of any specified image-rank lower bound is closed on the relative Grassmannian of v -planes. Its projection to the coefficient space is closed by properness. The convolution point avoids it. Intersecting the finitely many resulting opens, for $0 \leq v \leq a$, gives the last assertion. $\square$

D.5. The quadratic deformation and the two defects. In this subsection assume $\text{char } k \neq 2$. We now combine the block calculations and incidence bounds to prove the recurrence for the two defects. We first check the numerical hypotheses and record the homology of the split system. We then construct its leading deformation map to J and use the conditioned covector bound to prove that this map has maximal rank. The resulting rank increase gives the recurrence.

D.5.1. *Numerical hypotheses and the mixed maps.* Retain the counts q, c from the preceding subsection. Put $\alpha = N_2(m) - q$, $\beta = N_3(m) - mq$, and

$$j = 3\beta - c\alpha = T - qa, \quad H = 3mc - 2\alpha - \binom{c}{2}, \quad k_{31} = 2m + 2c.$$

For all sufficiently large m the preceding dimensions satisfy

$$(D.35) \quad \frac{m}{2} < c < m, \quad H > 0, \quad j > 0, \quad j = O(m^2), \quad \alpha > \binom{c+1}{2} + 3.$$

We check these inequalities directly from the critical-root formula, which gives

$$\tau m - 4 < c < \tau m - \frac{1}{2}, \quad \rho m^2 + \frac{m}{2} < q < \rho m^2 + \frac{m}{2} + 2$$

eventually. Substitution in j , which decreases in both c and q , gives

$$\frac{m^2}{2\sqrt{6}} + O(m) \leq j \leq \frac{4m^2}{\sqrt{6}} + O(m).$$

The leading coefficients of $\alpha - \binom{c+1}{2}$ and of H are respectively $1/\sqrt{6} - \tau^2/2 > 1/4$ and $3\tau - 2/\sqrt{6} - \tau^2/2 > 2/3$. These prove (D.35).

The cubic multiplication map $Q \otimes Y \rightarrow S_3 Y$ is injective for general Q . We verify this by counting possible linear relations among the quadrics. A nonzero q -tuple of linear coefficients spanning a b -plane has projective dimension $b(m + q - b) - 1$, and its relation imposes

$$N_3(m) - N_3(m - b)$$

linear conditions on the ordered quadrics. If $mq \leq N_3(m)$, the difference between this condition count and the tuple dimension is positive: divided by b , it is

$$F(b) = \frac{m^2}{2} - \frac{mb}{2} + \frac{b^2}{6} + \frac{b}{2} + \frac{1}{3} - q + \frac{1}{b},$$

which decreases on $1 \leq b \leq m$ and satisfies $F(m) = N_3(m)/m - q + 1/m > 0$. The required inequality $mq < N_3(m)$ follows here from $\rho < 1/6$.

Let $D_m = m^2/1000$. By Lemma D.11, the bound $\mathcal{B}_m(v)$ in the incidence results can be taken to be

$$\mathcal{B}_m(v) = \frac{T}{a}v + D_m \min(v, a - v).$$

Because $T/a = q + j/a$, $j > 0$, and $D_m > a$ eventually,

$$\mathcal{B}_m(v) - v(q + a - v) = \frac{j}{a}v - v(a - v) + D_m \min(v, a - v) \geq 0.$$

Thus the outer map F_{13} is injective by Proposition D.9; the middle map F_{22} is onto by Proposition D.2.

D.5.2. *The distinguished coefficient and the split complex.* To isolate the pure Y homology affected by the distinguished generator, choose a general flag $Q_- = \langle q_1, \dots, q_{q-1} \rangle \subset Q = Q_- \oplus kf_*$, and write

$$c_0 = C_m(q), \quad k_0 = H_m(q - 1), \quad \eta = -\chi_m(q).$$

We may require $f_*^2 \notin Q_- S_2 Y$: the latter product space is proper because $q - 1 < \kappa_2(m)$, and squares of quadrics span $S_4 Y$: for quadratic forms a, b , the identity $2ab = (a + b)^2 - a^2 - b^2$ expresses every product as a linear combination of squares. Reading the coefficient of f_* modulo Q gives

$$\theta : H_1(Q)_4 \longrightarrow A_2 = S_2 Y/Q, \quad \ker \theta = H_1(Q_-)_4.$$

Indeed, if that coefficient vanishes modulo Q , constant Koszul boundaries reduce it to a scalar multiple of f_* . The cycle equation and the excluded-square condition force that scalar to be zero.

The formal symmetric-square description in (2.5) makes the remaining inclusion of homology from $S(Y)$ injective. Consequently the image W of θ has dimension

$$w_* = \eta + c_0 - k_0.$$

The next projection removes the pure Y cokernel while we study the deformation; we will restore the full target at the end. Take a complement Z_0 of QS_2Y in S_4Y and first project the degree-four target modulo Z_0 . For the split generator space $Q_X \oplus E \oplus Q$, the projected Koszul complex has the following homology and cokernel:

bidegree	homology	cokernel
$(4, 0)$	3	0
$(3, 1)$	k_{31}	0
$(2, 2)$	H	0
$(1, 3)$	0	j
$(0, 4)$	$\eta + c_0$	0.

For $(3, 1)$ the source, target, and incoming boundary dimensions are $12m + 6c$, $10m$, and $4c$, respectively. For $(2, 2)$, eliminating the pure images leaves F_{22} and the independent alternating source $\bigwedge^2 E$. The outer injection handles $(1, 3)$; the remaining rows are the complexes in the pure X and pure Y degrees. Thus all entries follow from the preceding rank statements.

D.5.3. The leading deformation map. Let $r_1, \dots, r_4 \in A_2$ be the directions in which the pure quadrics $h_1, \dots, h_4$ move, and use the direction $s_X = \omega u - v \in P$ for the distinguished generator. Proposition D.4 applies to this particular W , without a genericity assumption on W . If $W_{\mathbf{r}}$ denotes the three-dimensional image of $\xi \mapsto \sum_b R_b(\xi) \otimes r_b$, it gives an injection

$$\mathrm{Sym}^2 E \oplus W_{\mathbf{r}} \oplus (s_X \otimes W) \longrightarrow P \otimes A_2.$$

Consequently

$$T_{\mathbf{r}, s_X} = F_{22}^{-1}(W_{\mathbf{r}} \oplus (s_X \otimes W)) / \bigwedge^2 E$$

has dimension $t_0 = H + 3 + w_*$ and injects into $E \otimes P_1$, where $P_1 = (X \otimes Y)/E$. To check injectivity, a vector in its kernel is represented in $E \otimes E$. Its product lies both in the symmetric-product image and in the trace image; the displayed directness makes the product zero, leaving exactly the alternating space already divided out.

Choose lifts of the r_b to S_2Y , and lift s_X to $\omega x^2 - y^2 \in S_2X$; we use the same notation for these lifts. Deform the quadratic generators by

$$g_i \mapsto g_i + \varepsilon p_i, \quad h_b \mapsto h_b + \varepsilon r_b, \quad f_* \mapsto f_* + \varepsilon s_X, \quad p_i \in S_2Y,$$

and keep Q_- fixed. After eliminating the split image, the first-order terms on the middle and $(3, 1)$ homology, together with the second-order terms on the pure X homology and the distinguished homology from $S(Y)$, give the map

$$(D.36) \quad T_{\mathbf{r}, s_X} \oplus K_{31} \longrightarrow J, \\ (z, [(a, b)]) \longmapsto \left[\sum_i p_i z_i + \sum_{b=1}^4 r_b b_b \right].$$

Here K_{31} is represented by full relations $\sum_i g_i a_i + \sum_b h_b b_b = 0$, and its map to P_1^4 is injective by Lemma D.3.

To justify that this leading map detects a rank increase, write multiplication as $M_0 + \varepsilon M_1$. A cycle a in the pure X or pure Y degree first maps to the surjective $(2, 2)$ row. Choose a complete source correction a_1 with $M_0 a_1 = -M_1 a$. The coefficient of ε^2 is then $M_1 a_1$, whose only component in J is its mixed coefficient contribution $\sum_i p_i z_i$. The other derivatives have pure X or pure Y bidegree. They are zero or lie in the image of M_0 after the projection; eliminating them contributes

only at a later order. The same calculation for the middle and $(3, 1)$ cycles gives their first-order terms in (D.36). Quotient by the moving constant Koszul image on a fixed nonzero minor chart, eliminate the rows and columns on which M_0 is an isomorphism, and divide the first-order and second-order columns by ε and ε^2 , respectively. Their constant matrix is (D.36), up to signs and additions of first-order columns. Hence each nonzero minor of this map yields a rank increase of the polynomial multiplication matrix. This argument also shows why projecting Z_0 does not change the selected leading terms: a correction entering a pure row can return to the $(1, 3)$ row only after another deformation.

D.5.4. Maximal rank and the two defect bounds. It remains to prove that (D.36) has maximal rank. We bound its annihilating covectors by stratifying them according to the dimension of their coefficient kernels. In the numerical bounds below, $x_+ = \max\{x, 0\}$ denotes the positive part of a real number, distinct from the truncation notation for power series. For a covector in the stratum Σ_v of Proposition D.10, the K_{31} equations impose at least $(k_{31} - 4v)_+$ conditions on the four r_b . After fixing these motions and s_X subject to the injectivity condition above, the remaining equations impose at least $(t_0 - cv)_+$ independent conditions on the p_i , because $T_{\mathbf{r}, s_X} \hookrightarrow E \otimes P_1$. These are successive fiber bounds for the same motions, so their sum is valid even though $T_{\mathbf{r}, s_X}$ depends on $\mathbf{r}$. Put $h_0 = k_{31} + t_0$; the sum is at least $h_0 - (c + 4)v$.

For $0 \leq v < a/2$, the conditioned covector bound therefore gives

$$\dim \Sigma_v - (k_{31} - 4v)_+ - (t_0 - cv)_+ \leq j - h_0 - 1 + v \left(c + 4 + a - v - \frac{j}{a} - D_m \right) \leq \max(j - h_0, 0) - 1$$

for all sufficiently large m . For $a/2 \leq v < a$,

$$\dim \Sigma_v \leq (a - v) \left(\frac{j}{a} + v - D_m \right) - 1 < 0,$$

so these strata are empty; Σ_a is always empty. The incidence of motions and annihilating projective covectors thus has general fiber dimension at most $\max(j - h_0, 0) - 1$. This proves that (D.36) has rank $\min(h_0, j)$.

We now translate this rank increase into bounds for the two defects. The projected split homology dimension is $h_0 + k_0$. Its Euler identity is

$$\chi_{m+3}(r) = c_0 + j - h_0 - k_0.$$

The rank increase just proved leaves projected homology of dimension at most $k_0 + \max(h_0 - j, 0)$. Removing the target projection cannot enlarge a kernel. Using the unprojected Euler identity gives

$$H_{m+3}(r) \leq \max\{k_0, c_0 - \chi_{m+3}(r)\}, \quad C_{m+3}(r) \leq \max\{c_0, k_0 + \chi_{m+3}(r)\}.$$

All conditions used are nonempty opens in the same irreducible space of flags of generator spaces in $S(Y)$, mixed coefficients, and motions. They can therefore be satisfied simultaneously. Applying the two inequalities at the lower and upper counts for $m + 3$ variables yields $E_2(m + 3) \leq E_2(m)$ for every sufficiently large m .

D.6. The quadratic endpoint in characteristic two. In characteristic two the preceding argument needs one additional property: a square must survive at the lower endpoint. We retain that square during the deformation and prove the resulting statement by induction in steps of three variables. The initial dimensions are 28, 29, 30; explicit binary forms for these three cases are given below. Throughout this subsection $\text{char } k = 2$.

At a lower critical count r_n^- with $\chi_n(r_n^-) > 0$, we use the strengthened assertion

$$(D.37) \quad H_n(r_n^-) = 0, \quad \zeta^2 \notin Q - S_2 \quad \text{for some quadratic form } \zeta.$$

At an integral critical root only ordinary endpoint exactness is required. We shall prove the two endpoints and (D.37) in every dimension $n \geq 28$.

The strengthened assertion is open. Indeed, if $R = r_n^- N_2(n) - \binom{r_n^-}{2}$, it is expressed by a rank- $R+1$ minor of the multiplication matrix with the column ζ^2 adjoined. The rank of the ordinary matrix is at most R , so such a minor proves both assertions at once. Nonvanishing of a coefficient of this minor, as a polynomial in ζ , gives an open condition on Q_- . Consequently, on the common flag space, a general extension $Q = Q_- + kf$ satisfies $f^2 \notin Q_-S_2$ whenever (D.37) holds. The coefficient reader used in the preceding subsection then has kernel $H_1(Q_-)_4 = 0$.

Use the scalar choice $s_X = \omega u - v$ and the functional ℓ from (D.5). Their three nonzero trace coefficients are $-(1 + \omega), \omega, 1$.

Lemma D.12. *Suppose $c < m$ and $\alpha \geq \binom{c+1}{2} + 4$. The mixed space E , pure motions $\mathbf{r}$, and a mixed quadratic ζ may be chosen so that the product map*

$$\mathrm{Sym}^2 E \oplus k[\zeta^2] \oplus W_{\mathbf{r}} \oplus (s_X \otimes A_2) \longrightarrow P \otimes A_2$$

is injective, simultaneously with all the preceding nonempty open conditions on these parameters.

Proof. Choose a c -plane $L \subset Y$ and $y_* \notin L$, and specialize $E = xL$ and $\zeta = xy_*$. For general Q , the space $C' = \mathrm{Sym}^2 L \oplus ky_*^2$ injects into A_2 . Choose r_1, r_2, r_3 independent modulo C' and put $r_4 = 0$. The first two source terms have image $u \otimes C'$. Applying $\ell \otimes (A_2 \rightarrow A_2/C')$ to a relation kills those terms and the marked term. The three nonzero scalar trace coefficients force its $W_{\mathbf{r}}$ part to vanish. Independence of u, s_X kills the remaining parts. Injectivity is a nonempty minor-open condition, so it meets the other nonempty opens in the same irreducible parameter space. $\square$

Proposition D.13. *For every $m \geq 28$, exactness at both child endpoints in dimension m , together with (D.37) when the child root is nonintegral, implies the same assertions in dimension $m + 3$.*

Proof. Put $q = r_m^+$, choose either parent critical count r , and define

$$\begin{aligned} c &= r - q - 4, & \alpha &= N_2(m) - q, & \beta &= N_3(m) - mq, \\ a &= 3m - c, & j &= 3\beta - c\alpha, & \delta &= -\chi_m(q), \\ H &= 3mc - 2\alpha - \binom{c}{2}, & k_{31} &= 2m + 2c, & t_0 &= H + 3 + \delta, \end{aligned}$$

and write $h = t_0 + k_{31}$. The split homology has dimensions $3, k_{31}, H, 0, \delta$ in its five bidegrees, and

$$(D.38) \quad \chi_{m+3}(r) = j - h.$$

The child coefficient reader is injective, with image of dimension δ . If the child root is integral, $\delta = 0$ and no marked child class is present.

The preceding deformation gives ordinary endpoint exactness once its leading map has rank $\min(h, j)$. To retain a square at a nonintegral lower parent count, let $e = j - h > 0$ and append the fixed column ζ^2 to the moving multiplication matrix. Its split bidegree is $(2, 2)$, where the quotient is zero, so this does not increase the split rank. Replace the middle preimage space by

$$T^+ = F_{22}^{-1}(W_{\mathbf{r}} \oplus (s_X \otimes W) \oplus k[\zeta^2]) / \bigwedge^2 E.$$

Lemma D.12 gives $\dim T^+ = t_0 + 1$ and an injection $T^+ \hookrightarrow E \otimes P_1$. Indeed, a vector killed by this injection has mixed coefficients in $E \otimes E$. Its product lies both in the symmetric-product image and in the trace-plus-square image; their intersection is zero, leaving only the alternating boundaries.

Write the moving multiplication matrix as $M_0 + \varepsilon M_1$. If $M_0 b = \zeta^2$, the additional split kernel vector is $(b, -1)$, whose response is $\varepsilon M_1 b$. Its $(1, 3)$ component is $\sum_i p_i b_i$. Its other first-response components are in pure rows and are eliminated by the split image. Thus, after the same row elimination as in (D.36), the augmented leading map is

$$T^+ \oplus K_{31} \longrightarrow J, \quad (z, [(a, b)]) \longmapsto \left[\sum_i p_i z_i + \sum_{b=1}^4 r_b b_b \right].$$

The added square column is fixed throughout this calculation.

For a covector with v -dimensional coefficient kernel, impose its equations first on the four pure motions and then, with these fixed, on the mixed motions. Their total codimension is at least

$$(D.39) \quad C^+(v) = (k_{31} - 4v)_+ + (t_0 + 1 - cv)_+.$$

The dependence of T^+ on the pure motions is therefore respected. The numerical verification below shows

$$(D.40) \quad \dim \Sigma_v - C^+(v) \leq e - 2$$

for every nonempty covector stratum. The projective annihilator of the general leading image has dimension at most $e - 2$, so this image has dimension $h + 1$. The augmented polynomial matrix consequently has rank at least $R + 1$, where $R = rN_2(m + 3) - \binom{r}{2}$ is the upper bound for the ordinary matrix. Deleting the fixed column proves that the ordinary rank is R and that ζ^2 is outside its image. This is the strengthened lower assertion. When $e = 0$, and at the upper endpoint, the unaugmented calculation proves ordinary exactness. $\square$

D.6.1. The transfer inequalities. We give the finite bounds and the uniform estimate used in the proposition. In addition to the preceding counts, put

$$t = c - 2, \quad p = t - 1, \quad w = m - t, \quad A = 2p, \quad B = \binom{p+1}{2}, \quad T = Bw + A \binom{w+1}{2} + 3 \binom{w+2}{3}.$$

Thus $a = A + 3w$ and $j = T - qa$. For $k = 2, 3$ write

$$H_k(n) = \binom{w+k-1}{k} - \binom{w-n+k-1}{k}.$$

For each endpoint, the required structural inequalities are

$$(D.41) \quad \begin{aligned} 4 \leq c \leq m, \quad mq \leq N_3(m), \quad \alpha \geq \binom{c+1}{2} + 3, \\ q \geq cw_0 + \binom{w_0+1}{2}, \quad w_0 = (m - 2c)_+, \end{aligned}$$

together with $j > 0$, $H \geq 0$, and $0 \leq \delta \leq \alpha$. At a strengthened lower endpoint, $c < m$ and the last constant 3 in the product inequality is replaced by 4.

For $28 \leq m \leq 40$, scale the free variables separately in the convolution presentation. A limiting subspace is specified by a core dimension i and free dimensions $0 \leq n_3 \leq n_2 \leq n_1 \leq w$, with $v = i + n_1 + n_2 + n_3$. Let

$$\Gamma(i) = \lceil Bi/A \rceil, \quad e_1 = \max(i, p) - i, \quad e_2 = A - \max(i, p).$$

Its quadratic image has dimension at least

$$\Phi = (w - n_1)\Gamma(i) + Bn_1 + i \binom{w+1}{2} + e_1 H_2(n_1) + e_2 H_2(n_2) + \sum_{h=1}^3 H_3(n_h).$$

The three terms of the free-degree decomposition give this bound: a nonzero output direction fills M_2 , the linear output dimensions are at least p and A , and the remaining terms count incident free-variable monomials.

Put $s = n_1 + n_2 + n_3$, $z = n_1 + 3n_2 + 5n_3$, and $V = \sum_h (n_h w - \binom{n_h+1}{2})$. The dimension of the subspace stratum is at most

$$C = i(A - i) + s(A - i) + 3s - z + 3V - (s^2 - z)/2.$$

Indeed, row-echelon coordinates in the ordered blocks $kw_1 \otimes k^3, \dots, kw_w \otimes k^3, M_1$ give this dimension when the free block dimensions decrease; exchanging two adjacent dimensions out of that order increases the dimension.

The exact finite inequalities are

$$(D.42) \quad \Phi \geq qv + \mathcal{C} + \min\{(c+4)v, j\} \quad (0 < v < a).$$

This is a finite integer calculation with the following explicit index set. For each $m = 28, \dots, 40$, take $q = r_m^+$ and $r \in \{r_{m+3}^-, r_{m+3}^+\}$, and compute the parameters above. Run through $i = 0, \dots, A$ and $n_1 = 0, \dots, w$, $n_2 = 0, \dots, n_1$, $n_3 = 0, \dots, n_2$; omit the two cases $v = 0, a$. Evaluating $\Phi - qv - \mathcal{C} - \min\{(c+4)v, j\}$ at these indices gives 26 endpoint configurations and 1 265 842 profiles, with minimum 17. Thus all inequalities in (D.42) hold strictly. For a fixed profile, the covector space has dimension at most $T - \Phi - 1$, and imposing its descent through the same child space Q has codimension $q(a - v)$. Consequently (D.42) gives

$$\dim \Sigma_v \leq j - (c+4)v - 1,$$

with an empty stratum when the right side is negative. Taking unions over exact kernel dimensions at least v gives the same bound on the corresponding closed kernel locus. Proper semicontinuity then extends it from the convolution point to a nonempty open of the full coefficient family. Since $C^+(v) \geq h + 1 - (c+4)v$, this proves (D.40) in this range, as well as the ordinary endpoint bounds. The same calculation gives outer injectivity by proving $\dim J = T - qa$.

For $41 \leq m \leq 129$, use $b(i) = B - \binom{p - \lceil i/2 \rceil + 1}{2}$. A bound $\mathcal{E}(v)$ for the quadratic image of every v -plane is the minimum of

$$\begin{aligned} b(i)w + (B - b(i))n_1 + i \binom{w+1}{2} + \max(p-i, 0)H_2(n_1) + (A - \max(i, p))H_2(n_2) \\ + H_3(n_1) + H_3(n_2) + H_3(n_3) \end{aligned}$$

over $0 \leq i \leq A$ integral and ordered real layers summing to $v - i$. For fixed i this expression is concave in the layers. It therefore suffices to evaluate the edges between the four prefix vertices: for $0 \leq u_0 < v_0 \leq 3$, take

$$n_1 = \dots = n_{u_0} = w, \quad n_{u_0+1} = \dots = n_{v_0} = \frac{v - i - u_0 w}{v_0 - u_0}, \quad n_{v_0+1} = \dots = n_3 = 0,$$

when the middle value lies in $[0, w]$. All resulting denominators divide 1296.

For $130 \leq m \leq 319$, the weaker bound suffices. At $x \in \{0, 1/3, 1/2, 2/3, 1\}$ and $r_0 \in \{0, 1, 2, 3\}$ put

$$L = Ax + r_0 w, \quad I = Bw \max(x, r_0/3) + A \binom{w+1}{2} \max(x, \min(r_0/2, 1)) + r_0 \binom{w+2}{3}.$$

On each consecutive knot interval take the lower convex hull of the eight points $(L, I - (T/a)L)$. The minimum of these four hulls, where defined, added to $(T/a)v$, is a lower bound $\mathcal{E}(v)$. Concavity in the free layers and affine interpolation in each knot interval justify this bound.

In both ranges set

$$U(v) = j - \mathcal{E}(v) + qv + v(a - v) - 1, \quad C(v) = (k_{31} - 4v)_+ + (t_0 - cv)_+.$$

Exact rational evaluation verifies outer injectivity through $\mathcal{E}(v) \geq v(q + a - v)$ for $1 \leq v \leq \min(q, a)$, and verifies

$$U(v) < 0 \quad \text{or} \quad U(v) - C(v) \leq \max(j - h, 0) - 1$$

at both endpoints. At the strengthened lower endpoint it verifies

$$U(v) < 0 \quad \text{or} \quad U(v) - C^+(v) \leq e - 2.$$

For the latter disjunction the minimum of $\max\{-U(v), C^+(v) + e - 2 - U(v)\}$ is 29 in the sharp range and 21819/323 in the hull range. The integral parent root $n = 91$ requires no extra square. For $v = 0$ the required inequality is immediate from $\dim \Sigma_0 \leq j - 1$ and $C^+(0) = h + 1$; the stratum $v = a$ is empty.

For clarity, the finite verification has the following scope.

child dimension	ordinary endpoint configurations	lower configurations with extra square
28–40	26	13
41–129	178	88
130–319	380	190

All entries can be reproduced using integer and rational arithmetic alone. Determine r_n^+ as the least integer r with $N_4(n) - rN_2(n) + \binom{r}{2} \leq 0$, and put $r_n^- = r_n^+$ if equality holds and $r_n^- = r_n^+ - 1$ otherwise. For each child dimension and both parent counts in the table, evaluate the displayed bounds at every integer $v = 1, \dots, a - 1$. In the sharp range, obtain $\mathcal{E}(v)$ by minimizing over $i = 0, \dots, A$ and the six displayed prefix edges for which the intermediate layer lies in $[0, w]$. In the hull range, sort the eight points by their first coordinate, discard all but the lowest point at any repeated coordinate, form the lower convex hull, and evaluate it by linear interpolation on each segment. Take the least value among the four knot intervals containing v .

These operations give the minima just stated and verify (D.41) throughout the table. Among the strengthened lower cases they give $m - c \geq 15$ and $\alpha - \binom{c+1}{2} - 3 \geq 225$. In particular, every finite check used here is specified by its index set and its displayed integer or rational expression.

For $m \geq 320$ the uniform estimate is

$$(D.43) \quad \mathcal{E}(v) \geq (T/a)v + \frac{m^2}{100} \min(v, a - v).$$

Here is a finite polynomial verification of the constant. Write $z = t/m$, $u = 1/m$ and

$$\begin{aligned} A_0 &= 2(z - u), & B_0 &= z(z - u)/2, & w_0 &= 1 - z, \\ a_0 &= A_0 + 3w_0, & b_{20} &= w_0(w_0 + u)/2, & b_{30} &= w_0(w_0 + u)(w_0 + 2u)/6, \\ T_0 &= B_0w_0 + A_0b_{20} + 3b_{30}. \end{aligned}$$

At each knot endpoint and prefix rank put

$$\lambda = A_0x + r_0w_0, \quad I_0 = B_0w_0 \max(x, r_0/3) + A_0b_{20} \max(x, \min(r_0/2, 1)) + r_0b_{30}.$$

Take $D = \lambda$ in the first knot interval, $D = a_0 - \lambda$ in the last, and $D = a_0/2$ in the two middle intervals. Each of the 32 polynomials

$$100(a_0I_0 - T_0\lambda) - a_0D$$

is nonnegative on $1/2 \leq z \leq 14/25$, $0 \leq u \leq 1/64$. To verify this, substitute $z = 1/2 + 3v/50$ and $u = s/64$ and expand in the tensor-product Bernstein basis on $[0, 1]^2$. Every coefficient is nonnegative; outside the identically zero cases the least is 121058433/1024000000. Here the coefficient calculation is explicit: if the substituted polynomial is $P(v, s) = \sum_{i,j} p_{ij} v^i s^j$ of degrees N and M in its two variables, its coefficient of $\binom{N}{a} v^a (1 - v)^{N-a} \binom{M}{b} s^b (1 - s)^{M-b}$ is

$$\sum_{i \leq a, j \leq b} p_{ij} \frac{\binom{a}{i}}{\binom{N}{i}} \frac{\binom{b}{j}}{\binom{M}{j}} \quad (0 \leq a \leq N, 0 \leq b \leq M).$$

Applying this formula to the two endpoints of each of the four knot intervals and to $r_0 = 0, 1, 2, 3$ gives the 32 checks above; identically zero polynomials are omitted from the minimum. Concavity and interpolation extend these inequalities throughout each knot interval, where $D \geq \min(\lambda, a_0 - \lambda)$. Rescaling proves (D.43).

The critical-root bounds give

$$\tau m - 4 < c < \tau m - \frac{1}{2}, \quad \rho m^2 + \frac{m}{2} \leq q \leq \rho m^2 + \frac{m}{2} + 2, \quad \rho = (3 - \sqrt{6})/6, \quad \tau = 3 - \sqrt{6}.$$

They imply $m/2 \leq t \leq 14m/25$, $j > 0$, $j/a \leq 2m/3 + 1$, and $a < 5m/2 + 4$. With $D_m = m^2/100$, for $m \geq 320$ one has

$$D_m > 3m + 4 \geq c + 4 + a - j/a, \quad D_m > 19m/6 + 5 > a + j/a.$$

For $0 < v < a/2$, subtracting $h + 1 - (c + 4)v$ from the covector bound obtained from (D.43) gives

$$e - 2 + v(c + 4 + a - v - j/a - D_m) < e - 2.$$

For $a/2 \leq v < a$ the covector bound is itself negative. This proves (D.40) in the uniform range. The same estimates prove ordinary endpoint exactness and outer injectivity. Finally, the product margin exceeds $m^2/4 - 3m/10 - 2$, $2c > m$, the cubic margin is at least $(1/6 - \rho)m^3 - 5m/3$, and $H > 2m^2/3 - 12m - 10$. These verify the structural conditions, including the extra square.

D.6.2. Three initial dimensions. We give a deterministic construction of the required forms over $\mathbb{F}_2$. The following integer recurrence specifies all their coefficients.

Let $a \oplus b$ denote exclusive-or of the binary digits of nonnegative integers. Define

$$\begin{aligned} C_0 &= 11400714819323198485, \\ C_1 &= 13787848793156543929, \\ C_2 &= 10723151780598845931. \end{aligned}$$

For each $n \in \{28, 29, 30\}$, restart the following recurrence at $s_0 = n$. All residues are chosen in $\{0, \dots, 2^{64} - 1\}$:

$$\begin{aligned} s_j &= (s_{j-1} + C_0) \bmod 2^{64}, \\ a_j &= C_1(s_j \oplus \lfloor s_j/2^{30} \rfloor) \bmod 2^{64}, \\ b_j &= C_2(a_j \oplus \lfloor a_j/2^{27} \rfloor) \bmod 2^{64}, \\ w_j &= b_j \oplus \lfloor b_j/2^{31} \rfloor \quad (j \geq 1). \end{aligned}$$

List the quadratic monomials $m_0, \dots, m_{b-1}$ in the lexicographic order of their nondecreasing index pairs, where $b = N_2(n)$, and put $L_n = \lceil b/64 \rceil$. Write $\text{bit}_t(w) = \lfloor w/2^t \rfloor \bmod 2$. The i th quadratic form is

$$f_i = \sum_{\ell=0}^{b-1} \text{bit}_{\ell \bmod 64}(w_{(i-1)L_n + \lfloor \ell/64 \rfloor + 1}) m_\ell \quad (1 \leq i \leq r_n^+).$$

The lower list is the prefix $f_1, \dots, f_{r_n^-}$ of the upper list. Let M_r be the matrix whose columns are the coefficient vectors of $f_i m_\ell$, $1 \leq i \leq r$, $0 \leq \ell < b$, in the lexicographically ordered quartic monomial basis. Binary elimination gives

n	r_n^-	r_n^+	$\text{rank } M_{r_n^-}$	$\text{rank } M_{r_n^+}$	$\text{rank}[M_{r_n^-} \mid x_1^4]$
28	86	87	31261	31465	31262
29	92	93	35834	35960	35835
30	98	99	40817	40920	40818

The coefficient lists themselves have ranks 87, 93, 99, respectively.

For completeness, the entire elimination procedure is as follows. Start with no pivot columns. Read each product column in the order (i, ℓ) above. If its least nonzero coordinate is already the pivot of a stored column, add that stored column and repeat. A zero remainder is discarded; otherwise store the remainder under its least nonzero coordinate. The number of stored columns is the rank. To obtain the last column of the table, apply the same reduction to the vector of x_1^4 after processing the lower list; in all three cases the remainder is nonzero. Thus the recurrence, the

monomial ordering, and binary column addition specify every operation needed to reproduce the table.

The lower ranks equal $r_n^- N_2(n) - \binom{r_n^-}{2}$, and the upper ranks equal $N_4(n)$. The augmented ranks are one larger than the corresponding lower ranks, so x_1^4 lies outside each lower image. This proves (D.37) with $\zeta = x_1^2$. The ranks hold over every field of characteristic two, since the matrices have entries in $\mathbf{F}_2$.

The three dimensions provide one starting point in each residue class modulo 3. Proposition D.13 proves both endpoints and the strengthened lower assertion successively in dimensions 31, 32, 33 and then in every $n \geq 28$. The endpoint restriction and extension argument therefore gives the degree-four prediction for every generator count in these dimensions.

Corollary D.14. *There is an integer N_2 , independent of the infinite field k , such that for quadratic generators and every $n \geq N_2$, the degree-four prediction holds for every generator count. Together with the lower-degree assertion of [1, Theorem 2], this supplies the quadratic input used in the higher-degree constructions.*

Proof. In characteristic different from two, apply Theorem 7.4 with $d = 2$ and $h = 3$ to the recurrence just proved. Its numerical cutoff is common to all these fields, since the preceding constructions and dimension inequalities have no further characteristic restriction. In characteristic two, Subsection D.6 gives the endpoint for every $n \geq 28$. Taking the larger threshold and the threshold for the lower-degree prefix gives a single bound over every infinite field. $\square$

APPENDIX E. THE ESTIMATES IN DEGREES THREE THROUGH EIGHT

We now verify the construction in degrees $3 \leq d \leq 8$ with a single choice of parameters. There are at most two intermediate families, of X -degrees two and four. The distinction between odd and even d is the same as in Proposition B.5. The product spaces of Lemmas A.1 and A.2 provide the additional room needed in the smallest degrees.

Proposition E.1. *Let $3 \leq d \leq 8$. Use the quadratic endpoint for all generator counts in sufficiently many variables. Then there is a fixed positive integer h such that, for every sufficiently large m and $r \in \{r_{m+h}^-, r_{m+h}^+\}$, putting $q = r_m^+$, $c_0 = C_m(q)$ and $k_0 = H_m(q - 1)$ gives*

$$\begin{aligned} H_{m+h}(r) &\leq \max\{k_0, c_0 - \chi_{m+h}(r)\}, \\ C_{m+h}(r) &\leq \max\{c_0, k_0 + \chi_{m+h}(r)\}. \end{aligned}$$

Consequently $E_d(m + h) \leq E_d(m)$, and the degree- d endpoint holds for every generator count in sufficiently many variables.

The remainder of this section proves the proposition.

E.1. Parameters and the number of generators. Use the parameters and exact counts of Proposition 5.1. Write $E = E_2$, $e = e_2$, $G = E_4$, $g = e_4$, with G absent and $g = 0$ when $d \leq 4$. For the estimates in this section put

$$A_d = \gamma_2, \quad \alpha_d = \alpha, \quad \beta_d = \beta, \quad \gamma_d = \frac{101}{100}\gamma_4, \quad \ell_d = \frac{11}{10}2^d\rho, \quad c_D = c.$$

The first entry of the maximum (5.1) is largest for $d = 3, 4, 5, 6$, and the third is largest for $d = 7, 8$. The divisibility $180180 \mid h$ permits every split used below and makes K integral in all six degrees. We use the full codimension of D :

$$(E.1) \quad \frac{c_D}{h^2} = \frac{\ell_d^2}{2} + O(h^{-1}) < \frac{7}{32}.$$

The outer reserve and its fixed-column extension follow from Proposition 5.1; in particular,

$$(E.2) \quad \alpha_d > \frac{d(d-1)\rho}{2d!}.$$

All the additional scalar lists have size $O_h(m^{d-2})$, and the pure family has a fixed number of labels as m grows.

E.2. The intermediate multiplication maps. Choose $E \subset D \otimes S_{d-2}Y$ of dimension e . If $d \geq 5$, also choose $G \subset S_4X \otimes S_{d-4}Y$ of dimension g . The formal product spaces to be considered are just

$$(E.3) \quad \Pi_4 = S_2E, \quad \Pi_6 = E \otimes G, \quad \Pi_8 = S_2G,$$

with the last two omitted when G is absent. In particular, each product row has a single source.

Put $P_j = d!/[j!(2d-j)!]$ and $B_j = \binom{2d-j}{d-j}$. The parameter inequalities used below are

$$(E.4) \quad \frac{\beta_d}{P_2(1 - \ell_d^2)} + B_2\rho < 1,$$

$$(E.5) \quad \frac{2\gamma_d}{P_4} + B_4\rho < 1 \quad (d \geq 5),$$

$$(E.6) \quad \binom{2d-R}{d-R}\rho < \frac{2}{5} \quad (R \in \{4, 6, 8\}, R \leq d).$$

We also need the product capacities in the following table. A capacity is the leading coefficient after division by h^2m^{d-2} for E , or by h^4m^{d-4} for G .

	product	range	capacity
	S_2E	$d = 3$	$5/64$
	S_2E	$d = 4$	$5/512$
	S_2E	$5 \leq d \leq 8$	$\frac{1/8 - \ell_d^2/2}{2^{d-1}(d-2)!}$
(E.7)	$E \otimes G, E \text{ factor}$	$5 \leq d \leq 8$	$\frac{2/25 - \ell_d^2/2}{2^{d-2}(d-2)!}$
	$E \otimes G, G \text{ factor}$	$5 \leq d \leq 8$	$\frac{(3/5)^4}{24 \cdot 2^{d-4}(d-4)!}$
	S_2G	$d = 5$	$1/512$
	S_2G	$6 \leq d \leq 8$	$\frac{1}{512 \cdot 2^{d-4}(d-4)!}$

Every E capacity is strictly greater than β_d , and every G capacity is strictly greater than γ_d . These, (E.1), (E.2), and (E.4)–(E.6) follow by rational substitution. Explicitly, put $H_d = \binom{2d-1}{d-1}$, $M = 10^8$, and

$$(q_2, q_3, q_4, q_5, q_6, q_7, q_8) = (18350341, 5131670, 1438923, 397615, 108283, 29141, 7770).$$

Then $q_d/M < \rho_d < (q_d + 1)/M$. These bounds are verified by $H_d(M - q_d - 1)^2 < (H_d - 1)M^2 < H_d(M - q_d)^2$. Substitution gives the strict rational bounds in Table 3, and the other stated inequalities.

We explain why these inequalities give all the required polynomial maps. For the row of X -degree two, place general D -valued output vectors on each degree- $(d-2)$ monomial in Y . The sparse quotient estimate (B.12) loses the fraction $\ell_d^2 + o(1)$ of the full quadratic output space. Inequality (E.4) then gives precisely the constant scalar- E Koszul kernel, for the whole common scalar list. The exceptional-projection estimate is $O_h(m^{d-1})$ per coefficient direction, and the leading strict inequality absorbs it. A fixed number of extra columns is absorbed in the same way.

For S_2E in degree three or four, use an output space of dimension $(5/32 + o(1))h^2$ from Lemma A.1 contained in D . Such spaces exist for general D by Remark A.4, since $\dim D > (5/32 + o(1))h^2$. Use Y itself as the scalar space in degree three and the degree-two space of Lemma 4.1 in degree four. Lemma A.3 gives the first two capacities in (E.7).

TABLE 3. Strict rational upper bounds for the multiplication inequalities. The first two columns bound the left sides of (E.4) and (E.5). The remaining columns bound the required density divided by its capacity in (E.7); the cross column is the maximum of the E and G ratios. A dash denotes an absent layer.

d	Sparse E	Sparse G	S_2E	$E \otimes G$	S_2G
3	$\frac{920}{1000}$	—	$\frac{910}{1000}$	—	—
4	$\frac{824}{1000}$	—	$\frac{971}{1000}$	—	—
5	$\frac{803}{1000}$	$\frac{509}{1000}$	$\frac{706}{1000}$	$\frac{624}{1000}$	$\frac{862}{1000}$
6	$\frac{800}{1000}$	$\frac{569}{1000}$	$\frac{355}{1000}$	$\frac{297}{1000}$	$\frac{821}{1000}$
7	$\frac{815}{1000}$	$\frac{613}{1000}$	$\frac{190}{1000}$	$\frac{149}{1000}$	$\frac{411}{1000}$
8	$\frac{837}{1000}$	$\frac{645}{1000}$	$\frac{103}{1000}$	$\frac{81}{1000}$	$\frac{209}{1000}$

When G is present, use the degree-two space $O \subset S_2X$ of Lemma 4.1, associated with equal halves of X , and replace it by $O \cap D$. It still has independent symmetric products, and

$$\dim(O \cap D) \geq \dim O - c_D = \left(\frac{1}{8} - \frac{\ell_d^2}{2} + o(1) \right) h^2.$$

Use the degree- $(d-2)$ space of Lemma 4.1 in Y . Lemma A.3 gives the third capacity. Every E product has bidegree $(2, 2)$ with respect to this splitting of X . Thus the components of S_4X with odd degree in the first summand are available for the new G family.

For $E \otimes G$, split $X = A \oplus B$ with dimensions $2h/5, 3h/5$ and split $Y = C \oplus C'$ into nearly equal halves. Place the two families in

$$(D \cap S_2A) \otimes S_{d-2}C, \quad S_4B \otimes S_{d-4}C',$$

respectively. Their product map is injective because the variable sets are disjoint. The bound $\dim(D \cap S_2A) \geq N_2(2h/5) - c_D$ gives exactly the two cross-product capacities in the table.

For S_2G , use the quartic output space of Lemma A.2, together with the full linear scalar space if $d = 5$ and the corresponding space of Lemma 4.1 otherwise. The separated-product lemma gives the last two capacities.

Whenever the product row also has a scalar source, its product image lies in a single bidegree for the chosen splitting of Y . Delete that bidegree component from the scalar target. The divisibility coupling in the proof of Lemma C.1, now with source degree $d - R$ and multiplier degree d , retains at least $2/5$ of every source row for large m . Indeed the removed fraction tends to an atom of $\text{Bin}(d, 1/2)$, which is at most $1/2$. Passing to initial subspaces proves the same estimate for every coefficient subspace. Inequality (E.6) and coefficient incidence separate the entire scalar image from the product image. The full linear scalar spaces used for $d = 3, 5$ occur only in rows $R > d$, where there is no scalar source.

At row four, put the new G outputs in the components of S_4X with odd degree in the first summand, whose combined dimension is $(1/48 + o(1))h^4$. They are disjoint from the E products. The sparse quotient estimate with (E.5) gives exactly the scalar- G Koszul kernel on these outputs; on their complement use the preceding scalar-avoidance estimate. There are no other new-family/product collisions. We have thus established all the row maps of Lemma B.4 for the present parameters.

E.3. The background and the target maps. Use the two backgrounds

$$(E.8) \quad W_{\text{bg}} = \begin{cases} \langle Q, E + a_2, G + a_4, U + P \rangle, & d \text{ odd}, \\ \langle Q, E + a_2, G + a_4, U + P_2 + b \rangle, & d \text{ even}, \end{cases}$$

omitting the G family when absent. Here a_2, a_4, b are scalar degree- d lists; $P \subset XS_{d-1}Y$ and $P_2 \subset DS_{d-2}Y$ have u labels. All scalar forms belong to the same list $\mathcal{S} = Q, a_2, a_4, b$, with absent terms omitted. Its count is $(\rho/d!)m^d + O_h(m^{d-1})$.

For odd d , the private-variable argument in Lemma B.4 includes P without changing any leading inequality. More explicitly, specialize $P_i = w_i z_i^{d-1}$ and use only the other Y variables in the preceding witnesses. Projecting an output space by $w_i S_{R-1}X$ loses $O(h^{R-1})$ directions. The sparse inequality has a strict margin of order h^R , so the necessary injection of the linear Y multiples of the new family persists in this quotient. At lower coefficient degrees there is no kernel: multiply a putative relation by two coprime core monomials to reach the established row map. The resulting constant Koszul matrices vanish by comparison and the coefficient injection of the new family. The only private-power overlap is consequently $\wedge^2 P$ at row two, as in that lemma.

The coefficient elimination of Proposition B.5 now applies with $\mathcal{J} = \{2, 4\} \cap \{1, \dots, d-1\}$. The retained symmetric and cross coefficients are removed by the three product rows (E.3). The alternating P coefficients are removed by the $(U + P)$ Koszul boundaries; their higher components are removed at the later U column. For even d , first set $U = 0$ and absorb the fixed P_2 list into E with its scalar list b . After proving the required rank, restore U on the open where it is a basis of $S_d X$. This proves

$$(E.9) \quad H_1(W_{\text{bg}}; S)_{2d} = H_1(Q; S(Y))_{2d}.$$

The equality holds also with one extra E generator and after projection of a complement to $QS_d Y$ in $S_{2d} Y$.

For completeness, the odd-parity specialization in that proposition uses, in every row $t \geq 3$, the strict bound

$$\rho \left[\frac{th}{h+t-1} \binom{2d-t}{d-1} + \binom{2d-t}{d} \right] < H\rho < 1.$$

The last row $t = d+1$ in even degree has only the first summand and satisfies the same inequality. Row one uses the outer reserve. Thus odd homology vanishes before and after adjoining F , as required. These are polynomial rank conditions on one irreducible parameter space; the witnesses for different rows therefore give simultaneous nonempty opens.

The target maps have an equally uniform description. In row two, use the two direct convolution constructions in Proposition B.7. The first uses output space A , with $\dim A = h/4$, and fills $(AX) \otimes S_{2d-2}Y$; its generator count fits F by (B.17). The second fills the quotient $S_2 X / (AX) \simeq S_2 B$, whose leading dimension is $9h^2/32$. A general D projects onto this quotient by (E.1), so the forms lift into the actual quadratic output space. Their cost is τ_2 , the first entry in (5.1), and hence is strictly below α_d .

Rows three and four use quadratic output spaces filling $S_3 X$ and $S_4 X$ at densities just above $1/6$ and $\rho_2/2$, respectively. The former follows from [1, Theorem 2], and the latter is the quadratic endpoint. Each successful output space has dimension less than $\dim D$; its rank-open property and the incidence of flags give such a space inside general D . Convolution gives the second and third costs in (5.1). Only rows below the pure cutoff need be used.

If G is present, it fills the remaining rows $R = 5, 6, 7$ up to $R = d$ in odd degree and $R = d-1$ in even degree. Their costs are

$$\frac{(R-4)!}{R!(d-4)!\binom{2d-R}{d-4}}.$$

The largest is the row-five cost $\gamma_d/(101/100)$. These maps use only multiplication of quartics to degrees strictly below eight. Finally, for odd d , the estimate $uh > N_{d+1}(h) + h^2$ and Lemma B.1 give $UX = S_{d+1}X$; for even d we have $U = S_dX$. Every remaining target is filled.

E.4. The normal map and the recurrence. The product injection of Proposition C.2 applies because $L > 2^d \rho h$ and $c_D \geq 2h - 1$. Its denominator contains the entire common scalar list and, in odd degree, the fixed P list. It therefore separates S_2F in the prepared quotient (E.8). Together with (E.9), it gives the injective coefficient map (C.23).

We check the numerical hypotheses for the odd cokernel argument. Its coefficient space is exactly (C.4) or (C.7), according to parity. In odd degree put $a_X = N_{d-1}(h)$, $b_X = N_d(h) - u$. The projection argument of Lemma C.3 applies since $ub_X/a_X^2 \rightarrow \infty$. For $d = 5, 7$, one has $b_X/a_X \geq h/(2d)$ for large h by its displayed limit. For $d = 3$, write $u = h^3/12 + \theta$, $0 \leq \theta < 1$; then

$$b_X - \frac{h}{6}a_X = \frac{5h^2}{12} + \frac{h}{3} - \theta > 0.$$

With $\delta = 1/[100(2d)!]$, the remaining top-degree inequality is

$$(E.10) \quad (2d^2 + 1)\rho + d!\delta < 1 \quad (d = 3, 5, 7).$$

In particular the degree-three case is $19\rho_3 + 1/12000 < 1$. Thus the exact sequence (C.10) and the affine equation (C.12) remain valid; the relation $U = -P$ is retained throughout.

For each intermediate coefficient degree, the inequality needed after adding $[\delta m^d]$ scalar columns is

$$(E.11) \quad H\rho + \delta \frac{(2d-t)!}{(d-t)!} < 1 \quad (3 \leq t \leq d-1, t \text{ odd}).$$

Indeed $H\rho < 13/25$ for every $d \geq 3$, and the second term is at most $1/100$. This supplies the scalar shadow (C.19) on every such layer. The outer strict shadow supplies the first layer; the preceding top-degree calculation supplies the last layer in odd degree. Consequently the proof of Proposition C.4, including the exact common-scalar dimension identity (C.21), gives

$$\dim\{[\lambda] \in \mathbb{P}(J^*) : \dim \ker C_\lambda \geq k\} \leq \dim J - c_1 m^d k - 1$$

for some $c_1 > 0$. The target surjections proved above exclude a nonzero covector with zero outer component. The graph parameters cost only $O_h(m^{d-1})$ per kernel direction, and the higher affine equations cost $O_h(m^{d-2})$, so both are absorbed by the strict m^d shadow.

Proposition C.6 now gives the maximal rank of the first normal map. Lemmas 3.1 and 6.1 apply because the background equality, its one-generator extension and its projected version have all been established. Perturbing the distinguished generator therefore leaves precisely the homology from the lower count in $S(Y)$. Theorem 6.2 gives the two inequalities in Proposition E.1 at the exact counts (5.4). Their signs at r_{m+h}^- and r_{m+h}^+ give the recurrence. Theorem 7.4 then proves the endpoint in the current degree.

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