

A characterization of cut-percolating bigraphs*

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Abstract

Conlon and Lee introduced reflection bigraphs and cut-percolation in their study of graph norms. We prove that, for connected bigraphs, these two notions coincide, confirming their prediction.

1 Introduction

Let $H = (A, B; E)$ be a connected bipartite graph with at least one edge. We distinguish its two vertex parts and call it a *bigraph* (Definition 2.1). For a bounded measurable function, or *kernel*, $W : [0, 1]^2 \rightarrow \mathbb{R}$, its homomorphism density is

$$t_H(W) = \int_{[0,1]^A \times [0,1]^B} \prod_{uv \in E} W(x_u, y_v) dx dy.$$

When W represents the adjacency matrix of a finite graph, this integral is the proportion of vertex maps that preserve edges [23]. The graph H is *weakly norming* if $\|W\|_{r(H)} := t_H(|W|)^{1/|E|}$ is a norm on kernels (Definition 2.13).

Even cycles, complete bipartite graphs, and hypercubes are weakly norming [19]. Every weakly norming graph is also edge-transitive: its symmetries can move any edge to any other [28]. A full classification remains open. Graph norms have close ties to quasirandomness [29, 7], Schatten–von Neumann norms [19, 23], graphon theory [23, 14], and Sidorenko’s inequality [27, 19].

Conlon and Lee [10] introduced a broad class of weakly norming graphs, the *reflection bigraphs*. Their construction included all examples known at the time, and it remains open whether it captures every weakly norming graph. Roughly, take a finite group generated by Euclidean reflections and two points in one closed region between its reflecting hyperplanes. The two vertex parts are distinguished copies of the points’ image sets, with an edge for each pair of images obtained from the same group element; see Figure 3 and Definition 2.10.

Their proof proceeds through a combinatorial property called *cut-percolation*. A *fold* uses a part-preserving symmetry that is its own inverse and exchanges two sides separated by its fixed vertices. It keeps the marked edges on one side and copies that pattern to the other (Definition 2.4). A graph is cut-percolating if a sequence of folds spreads one marked edge to all edges; Figure 1 illustrates this on the six-cycle. Conlon and Lee proved that reflection bigraphs cut-percolate and that cut-percolating bigraphs are weakly norming [10]. Write $\text{Fold}(H)$ for the group generated by the fold symmetries.

Conlon and Lee suggested that cut-percolating bigraphs should in turn be reflection bigraphs [10, Section 6]. Our main result proves this prediction (Theorem 1.1). Figure 2 puts the result in context: reflection bigraphs now have several equivalent structural

*All mathematical work was done by GPT-6. The author takes responsibility for the correctness of the paper.

characterizations, while weakly norming graphs have equivalent analytic characterizations from earlier work [19, 14, 22]. Showing that these two classes coincide would give a powerful structural characterization of weakly norming graphs.

We also obtain further characterizations and recognition results for cut-percolating bigraphs, which we describe below.

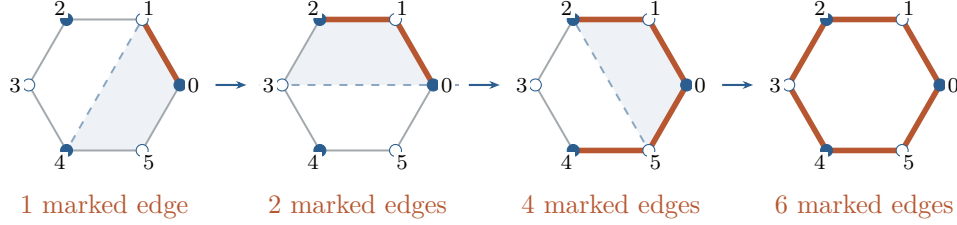


Figure 1: Three folds spread one marked edge over C_6 . At each step, keep the pattern in the shaded half and copy it across the dashed line. The number of marked edges grows from one to two, then four, then six. Filled and empty vertices distinguish the two vertex parts.

1.1 Our results

Classification. Our main theorem gives four equivalent descriptions of the graphs obtained by the Conlon–Lee method (Figure 2). Edge-transitivity of $\text{Fold}(H)$ means that a sequence of fold symmetries can move any edge to any other. If the copying process could only add red edges, this would suffice to make every edge red: we could reach each edge in turn without losing earlier marks. In the actual folding game, however, a fold copies both red and uncolored edges. A red edge becomes uncolored when its counterpart on the

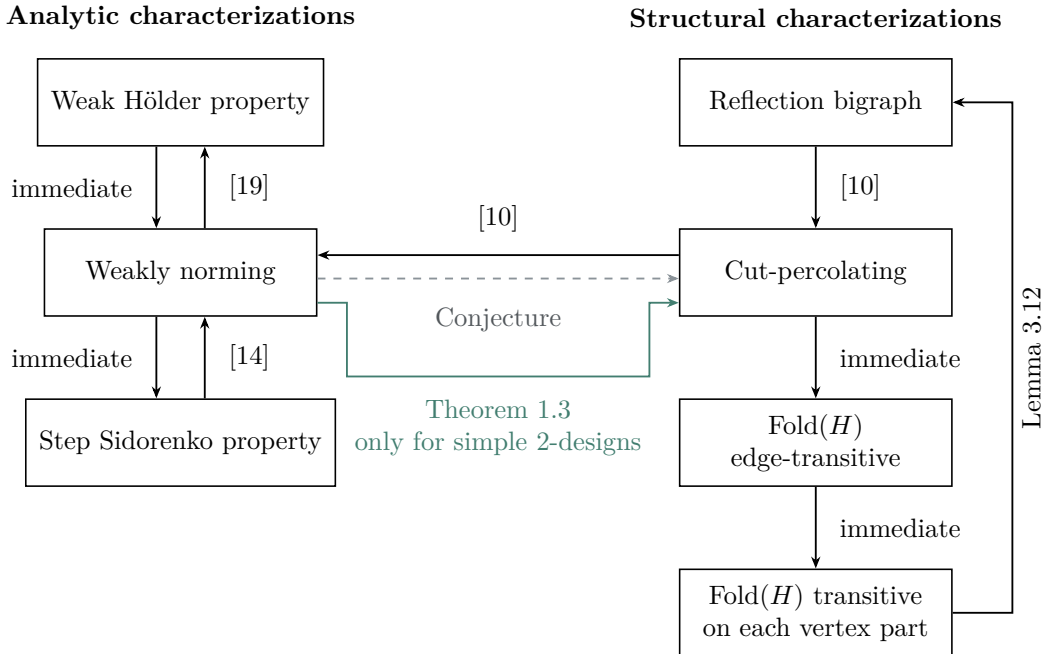


Figure 2: Relations for a finite connected simple bigraph H with an edge. Weak Hölder bounds copy counts with different edge kernels by their monochromatic counts (2); step Sidorenko means that averaging a graphon over partition classes cannot increase its copy count (Definition 2.14). The new implication, Lemma 3.12, closes the right-hand cycle. The dashed arrow remains conjectural. The green arrow holds for incidence bigraphs of simple two-designs and their uniform twin blowups.

retained side is uncolored. Thus, the implication from edge-transitivity to cut-percolation is nontrivial.

Transitivity of $\text{Fold}(H)$ on each vertex part is a weaker condition: any vertex can be moved to any other in the same part, without requiring the same for edges. We show that even this condition forces a reflection representation (Lemma 3.12).

Theorem 1.1 (Classification). *Let $H = (A, B; E)$ be a finite connected simple bigraph with at least one edge. The following conditions are equivalent:*

- (i) $\text{Fold}(H)$ is transitive on each of A and B ;
- (ii) $\text{Fold}(H)$ is transitive on E ;
- (iii) H is a reflection bigraph for a finite Coxeter system;
- (iv) H is cut-percolating.

Thus transitivity on the two vertex parts already forces percolation and a reflection representation. In particular, (ii) $\Rightarrow$ (iv) settles Coregliano’s explicit Conjecture 1.1 [12]. The equivalence with (iii) confirms the stronger reflection prediction stated immediately after Conjecture 6.1 in [10, Section 6]. Together with their theorem that reflection bigraphs are weakly norming, it also settles that conjecture; see Remark 2.9 for the authors’ clarification.

Recognition. The classification raises an algorithmic question: given a graph, can we decide efficiently whether it cut-percolates? We reduce this problem to graph isomorphism, using polynomially many queries on graphs of polynomial size and polynomial work between them. This is a *polynomial-time Turing reduction* (Section 4).

Theorem 1.2 (Recognition). *On finite connected simple bigraphs with at least one edge, recognizing cut-percolation is polynomial-time Turing reducible to graph isomorphism. For an input with n vertices, it can therefore be decided in time $n^{(\log n)^{O(1)}}$, and in polynomial time at any fixed maximum degree.*

The running-time bounds use the graph-isomorphism algorithms of Babai [2] and Luks [24]. Section 4 also gives short, efficiently checkable certificates and an alternative proof independent of the classification.

Incidence designs. For incidence bigraphs of simple two-designs, we can also settle the remaining implication from weak norming to cut-percolation. Recall that a two-design consists of equal-sized subsets, called *blocks*, with every point pair in the same number of blocks; simple means that no block is repeated (Definition A.1).

Theorem 1.3 (Simple two-designs). *Let F be the incidence bigraph of a simple two-design on v points, with block size $2 \leq k < v$ and every point pair in at least one block. Then F is cut-percolating if and only if every k -element subset of the points is a block. Within this family, cut-percolation, being a reflection bigraph, and weak norming are equivalent. The same classification holds after uniformly duplicating vertices independently in the two parts.*

These *complete designs* were already known to give weakly norming reflection bigraphs [10]. Our contribution is to exclude all other designs, even after vertex duplication. The proof in Appendix A is analytic and independent of the structural classification.

The one-sided criterion fails. Coregliano [13, Section 7] suggested a one-sided analogue: if the fold group is transitive on the left part, can folds spread one marked left vertex to the whole left part (Definition 5.1)? We give a counterexample, showing why transitivity on both parts matters in Theorem 1.1.

Theorem 1.4 (One-sided counterexample). *There is a connected bigraph on 256 vertices, each of degree four, whose fold group is transitive on the left part, but no fold sequence spreads one marked left vertex to the whole left part. This bigraph is not weakly norming.*

Section 5 gives the construction and determines all its folds. The obstruction to weak norming is in Appendix C.

Concrete consequences. Two vertices in the same part are *twins* if they have the same neighbors (Definition 2.3). For a connected cut-percolating graph without twins, the full group $\text{Fold}(H)$ itself is a finite reflection group (Theorem B.3). In general, identifying twin vertices gives this intrinsic reflection structure (Corollary B.6). For fixed degrees a, b on the two parts, there are only finitely many connected cut-percolating bigraphs exactly when at least one of a, b is odd (Theorem B.2). At every odd prime degree $p \geq 5$, exactly four such bigraphs occur (Theorem B.9). We also classify those with no cycle of length less than eight (Theorem B.10). Among them, only the edge-triangle incidence bigraph of the regular 24-cell has minimum degree at least three. It has 192 vertices; its two parts index the edges and triangular faces of that four-dimensional polytope, with adjacency by containment. Appendix B derives these lists from the finite Coxeter diagrams.

1.2 Further context and previous work

The homomorphism density introduced above extends to independent probability spaces for the two vertex parts and to a different kernel on each edge (Section 2.4). This extension explains the role of Hölder and Cauchy–Schwarz in the theory.

Analytic characterizations. Hatami’s generalized Hölder inequality [19, Theorem 2.8], stated in (2) and also called *decorated Hölder*, allows a different kernel on each edge. Writing $m = |E|$, it implies the Sidorenko inequality

$$t_H(W) \geq \left(\int_{[0,1]^2} W \right)^m \quad (W \geq 0).$$

Sidorenko’s conjecture asks for this bound for every bipartite H [27]. Progress beyond weakly norming graphs includes graphs with a vertex adjacent to every vertex of the other part [8] and graphs built by suitable gluings of trees [9].

In the usual formulation with *graphons*, symmetric measurable functions $[0, 1]^2 \rightarrow [0, 1]$ [23] (Section 2.4), Král’, Martins, Pach, and Wrochna [21] introduced the step Sidorenko property from Figure 2. Weak norming immediately implies it: the averaging operation cannot increase a relabeling-invariant norm, by convexity; see the explanation after Definition 2.14. Doležal, Grebík, Hladký, Rocha, and Rozhoň [14, Theorem 3.25] proved the converse for connected graphs. Lee and Schülke [22, Theorem 1.5(i)] characterized weak norming by convexity of $W \mapsto t_H(W)$ on nonnegative graphons: the value at a weighted average of kernels is at most the same weighted average of their values. They used Hessians, matrices of second partial derivatives along finite-dimensional kernel perturbations, to exclude several edge-transitive candidates; the same test underlies Lemma A.2.

These characterizations are analytic tests for the property, rather than a description of the graphs that have it. The remaining structural question is whether every connected weakly norming bigraph is cut-percolating. We leave this open. By Theorem 1.1, it is equivalent to asking whether weak norming forces transitivity of the fold group.

Symmetry and percolation. Hatami [19] proved basic restrictions on weakly norming graphs, including *biregularity*: all vertices in each part have the same degree, possibly different between parts (Definition 2.1). Sidorenko [28] proved that a connected weakly norming graph must be edge-transitive under its full automorphism group, the group of all part-preserving graph symmetries (Section 2.1). That group can be larger than the fold group. For instance, the Heawood graph has an edge-transitive automorphism group but is not weakly norming. It is the incidence graph of the Fano plane, the design on seven points whose seven three-point blocks contain each point pair exactly once. The criteria of [21] exclude it; Example A.5 gives an explicit witness.

For reflection bigraphs, Baek, Conlon, and Lee [4, Section 3.1] gave an algebraic proof of percolation on the Coxeter coset spaces of Definition 2.10. They used a stronger percolation property to bound the number of edges in graphs containing no copy of certain incidence graphs. These arguments start with a reflection representation; our classification recovers one from the fold group.

Coregliano [12] proved that a connected bigraph cut-percolates exactly when every *fold-stable* edge coloring is monochromatic. Here a coloring may change under a fold provided the resulting colored bigraph is isomorphic to the original one (Definition 2.15). Monochromatic means that all edges have one color. Our classification supplies a simpler obstruction: when percolation fails, a nonempty proper union of fold-group orbits in one vertex part gives a nonconstant two-color *star coloring*, with all edges at each vertex of one part sharing a color, fixed pointwise by every oriented fold. See Corollary 3.13.

Incidence designs. Our analytic argument builds on the positivity tests for weighted homomorphism counts of Král', Martins, Pach, and Wrochna [21, Theorem 3]. If a design's incidence bigraph is weakly norming, these tests force swaps of points to preserve the blocks. We show that there must be enough such swaps to make every subset of the prescribed size a block. The argument also handles uniform vertex duplication. Together with the known reflection construction [10], this proves Theorem 1.3 independently of the structural classification.

1.3 Proof ideas and organization

The proof of the classification has two steps.

Finding a reflection quotient. The graph's *Laplacian* is the matrix in (5) comparing each vertex value with the values at its neighbors. Place the vertices in its *Fiedler eigenspace*, the eigenspace for the smallest positive eigenvalue, by projecting their coordinate vectors onto it, as in (6). A fold is *atomic* if it exchanges just two components outside its fixed vertices (Definition 2.6). Its *odd functions*, which change sign under the involution, are controlled by the *Dirichlet Laplacian* on one component: the Laplacian restricted to functions zero outside that component. Its lowest eigenvector is strictly positive and unique up to scale. Consequently, the atom acts on the Fiedler eigenspace either trivially or as an orthogonal reflection. The reflecting hyperplane, or *mirror*, contains exactly the images of its fixed vertices. Identifying vertices with the same image gives the *spectral quotient*,

which is a reflection bigraph. Section 3.1 defines the spectral objects; Definition 3.2 defines the quotient.

Recovering the fibers. The vertices above one quotient vertex form its *vertex fiber*; the edges above one quotient edge form its *edge fiber* (Definition 3.2). Fold all image points into one fundamental chamber by a sequence of atomic folds. This sequence gives a bijection from each original vertex fiber to one of two fibers above a chosen *base edge*, and from each whole edge fiber to the base edge fiber, using compatible endpoint bijections. The original graph is therefore exactly a part-respecting tensor product $H = R \otimes F$: its vertices have two coordinates, and adjacency holds in both coordinates (Definition 2.12). Here R is the reflection quotient, and the *residual factor* F is the graph inside the base edge fiber (Theorem 3.6). It inherits fold-group transitivity and has fewer edges. Induction gives the classification.

Recognizing cut-percolation. The Laplacian eigenvectors identify a polynomial list of possible component pairs exchanged by atomic folds (Lemma 4.2). Graph-isomorphism queries then determine which vertex moves these folds allow. By the classification, it suffices to check whether these moves connect each vertex part (Section 4.2).

Excluding incomplete designs. We record, for each point, the labels assigned to its neighboring blocks after deleting one chosen block (Definition A.3). Decorated Hölder imposes a positivity condition on these records (Lemma A.4). For a design, this forces enough swaps of points to generate all point permutations, so every possible block must be present. Deleting all copies of the chosen block extends the argument to uniform vertex duplication.

Separating the one-sided notions. The counterexample encodes its left vertices by bit strings. Some bits can be copied freely. Among vertices agreeing on these bits, differences in the remaining three bits survive every fold sequence. This prevents percolation, although the fold symmetries together can move any left vertex to any other. The key work is to determine all folds, ruling out further symmetries that could erase the protected differences (Section 5).

Section 2 introduces folds, reflection bigraphs, and their connection to graph norms. Section 3 proves the classification, and Section 4 gives the recognition algorithm and an independent alternative. Section 5 constructs the one-sided counterexample. Appendix A contains the design classification and its analytic inequality, and settles Problem 2 of Amarra, Devillers, and Praeger [1] (Corollary A.14). Appendix B describes the intrinsic fold group and derives the consequences for degrees and short cycles. Appendix C proves the analytic obstruction for the one-sided counterexample.

2 Folds, reflection bigraphs, and graph norms

2.1 Graphs, symmetries, and notation

Definition 2.1. A *bigraph* is a finite simple bipartite graph $H = (A, B; E)$ with distinguished vertex parts $A = A(H)$, $B = B(H)$ and $E = E(H) \subseteq A \times B$. Simple means that there are no loops or repeated edges. It is *biregular* if all vertices of A have one common degree and all vertices of B have another common degree.

All maps of bigraphs preserve the specified parts. A *homomorphism* preserves edges; an *isomorphism* is a bijection preserving both edges and nonedges. An *automorphism* is an isomorphism from a graph to itself, and an *endomorphism* is a homomorphism from a graph to itself. Write $\text{Aut}(H)$ for the automorphism group.

Throughout the paper, our definitions and results concern connected bigraphs with at least one edge.¹ Auxiliary graphs and subgraphs need not be connected.

We write $V(H) = A \sqcup B$, where $\sqcup$ denotes disjoint union, $N_H(v)$ for the neighbors of v , and $\deg_H(v) = |N_H(v)|$. For a vertex set T , put $N_H(T) = \bigcup_{v \in T} N_H(v)$. We omit the subscript when the graph is clear. The induced subgraph $H[U]$ keeps the vertices in U and all edges between them; $H - Z = H[V(H) \setminus Z]$. A *vertex cut* is a set whose deletion leaves at least two connected components. The complete bigraph $K_{a,b}$ has parts of sizes a, b and every possible edge; C_ℓ is the cycle with ℓ edges. A *star* is $K_{1,d}$, with one center and d leaves. The star at a vertex means the set of its incident edges. In $\mathbb{R}^{V(H)}$, $\mathbf{e}_v$ is one at v and zero elsewhere. Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 2.2. For a group G acting on a set U , the *orbit* of u is $Gu = \{gu : g \in G\}$ and its *stabilizer* is $\text{Stab}_G(u) = \{g \in G : gu = u\}$. The action is *transitive* if it has one orbit. For $G \leq \text{Aut}(H)$, the induced action on edges sends uv to $(gu)(gv)$; G is *edge-transitive* if any two edges are related by this action. Transitivity on each part means that each of A, B is one vertex orbit.

A permutation fixes a set *pointwise* if it fixes every member, and preserves it *setwise* if it maps the set onto itself. We put $\text{Fix}(f) = \{v : f(v) = v\}$. An *involution* is a nonidentity permutation whose square is the identity. A subgroup *generated* by permutations consists of all finite products of those permutations and their inverses. Conjugating f by g means replacing it by gfg^{-1} . The symmetric group $\text{Sym}(U)$ consists of all permutations of U ; $\text{Sym}(v)$ abbreviates $\text{Sym}(\{1, \dots, v\})$. A *transposition* $(u\ w)$ swaps u, w and fixes everything else.

Definition 2.3. Vertices in the same part with identical neighborhoods are *twins*. A bigraph is *twin-free* if each twin class is a singleton. The *uniform twin blowup* $F(a, b)$, for integers $a, b \geq 1$, replaces each left vertex of F by a independent copies and each right vertex by b independent copies, replacing each original edge by all ab edges between its two clone classes. These copies are also called *clones*.

2.2 Genuine folds and cut-percolation

Definition 2.4. A *fold involution* is a nonidentity automorphism f with $f^2 = 1$ such that no connected component of $H - \text{Fix}(f)$ is preserved setwise by f . Thus those components occur in exchanged pairs. Choose a union L containing one component from each pair and set $Z = \text{Fix}(f)$. The associated *oriented fold* t is

$$t(v) = \begin{cases} v, & v \in L \cup Z, \\ f(v), & v \in f(L). \end{cases}$$

We call L the retained side, $L, f(L)$ the *open sides*, and Z their fixed boundary. The *fold group* $\text{Fold}(H)$ is generated by all fold involutions of H .

There is no edge from L to $f(L)$, so t is a graph endomorphism. Conversely, a side partition $V(H) = L \sqcup Z \sqcup f(L)$ with $Z = \text{Fix}(f)$ and no edge between the two open sides gives a fold in Definition 2.4. This is the fold convention of Coregliano [12, Definition 2.6].

¹For the disconnected case of graph norms, see Garbe, Hladký, and Lee [17, Theorem 3.1].

Definition 2.5. The bigraph H is *cut-percolating* if there are an edge e_0 and a possibly empty sequence of oriented folds $t_1, \dots, t_\ell$ such that

$$E_0 = \{e_0\}, \quad E_i = t_i^{-1}(E_{i-1}), \quad E_\ell = E(H).$$

Here a fold acts on an edge through its two endpoints. Equivalently, a composition of oriented folds sends every edge to one edge.

A *word* of transformations means a finite sequence composed in chronological order; it is *collapsing* if its image is a singleton. The empty word acts as the identity. The order reversal in the equivalent formulation is useful: if $q = t_\ell \circ \dots \circ t_1$ is a collapsing word in chronological order, its inverse-image percolation sequence uses $t_\ell, t_{\ell-1}, \dots, t_1$.

Definition 2.6. A nonidentity fold involution is *atomic* if its nonfixed set has exactly two connected components. Its *moved support* is $V(H) \setminus \text{Fix}(f)$, the union of these two components.

Lemma 2.7. *Every fold involution is a product of commuting atomic fold involutions. Every oriented fold is a composition of commuting oriented atomic folds. Atomic folds are invariant under conjugation by $\text{Aut}(H)$.*

Proof. For each exchanged component pair, restrict the involution to that pair and fix all other vertices. This is an automorphism: every edge leaving either component ends at a vertex fixed by the original involution. It is atomic. The restrictions have disjoint moved supports and multiply to the original involution. Orienting each restriction toward its original retained component gives the oriented decomposition. Conjugation preserves fixed sets and connected components, proving the last assertion. $\square$

Remark 2.8 (Choices of halves). Conlon–Lee [10, Section 3] fix a compatible pair of exchanged halves for each fold involution and allow copying in either direction. Definition 2.4 instead allows the choice to vary between occurrences of the same involution. Both conventions generate exactly the same vertex maps, and hence the same percolating sequences after inserting intermediate steps. Indeed, Lemma 2.7 expresses every oriented fold as a composition of oriented atomic folds. An atomic fold has just one exchanged component pair, so its halves are unique up to reversal and both orientations are available under either convention. Thus Definition 2.5 agrees with Conlon–Lee’s percolation notion, independently of their fixed choices. This argument uses all fold involutions, including the atomic restrictions; it need not hold for a prescribed smaller family.

Remark 2.9 (The broader fixed-cut group). Conlon–Lee’s original definition of a cut involution required only that its fixed set be a vertex cut; their folding maps also required exchanged sides. They later included the latter condition in the definition and explained that it had been used implicitly throughout their earlier work [11, Section 3, footnote 1]. Our assertions about their conjecture use this clarified convention.

The literal fixed-cut condition defines a possibly larger group, which we denote by $\text{Cut}(H)$: it is generated by all involutions $f \in \text{Aut}(H)$ for which $\text{Fix}(f)$ is a vertex cut, even if f preserves some component of $H - \text{Fix}(f)$ setwise. Edge-transitivity of this larger group does not imply cut-percolation.

For example, let F be the incidence bigraph of the Fano plane and put $H = F(2, 1)$, doubling each point vertex. Then H has 21 vertices and 42 edges. Its fold group consists precisely of the independent swaps within the seven point twin pairs, by Corollaries A.12 and B.6. These swaps have 21 edge orbits, so H does not cut-percolate. Nevertheless, $\text{Cut}(H)$ is edge-transitive.

To see this, realize the Fano points as the nonzero vectors of $\mathbb{F}_2^3$, where $\mathbb{F}_2$ is the field with two elements, and its lines as the two-dimensional subspaces, with adjacency given by membership. The linear involution $T(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_3)$ fixes the point $c = (1, 0, 0)$ and every line through it. Lift T to an automorphism $\hat{T}$ of H by retaining clone indices, and let τ_c swap the clones above c . The two maps commute. The moved subgraph of $\hat{T}\tau_c$ consists of the blowup of an eight-cycle and two isolated clones above c . Indeed, T moves four points and four lines; their induced graph has degree two and no four-cycle, so it is an eight-cycle. Thus the fixed set of $\hat{T}\tau_c$ is a vertex cut, and $\hat{T} = (\hat{T}\tau_c)\tau_c$ belongs to $\text{Cut}(H)$. The fixed-cut condition is invariant under conjugation, so conjugating by invertible linear maps supplies all elementary coordinate additions $x_i \mapsto x_i + x_j$ with $i \neq j$. These generate the invertible linear group over $\mathbb{F}_2$, which is transitive on incident point–line pairs. Together with the twin swaps, they give edge-transitivity on H .

2.3 Reflection bigraphs and tensor products

A hyperplane is a linear subspace of codimension one. An orthogonal reflection fixes a hyperplane pointwise and reverses the perpendicular direction. The reflecting hyperplane is also called a *mirror*; its two *closed halfspaces* are the sets with nonnegative and nonpositive scalar product with a normal vector, respectively. A finite real reflection group W is a finite group generated by such reflections. Its reflection hyperplanes divide the orthogonal complement of the common fixed subspace (the *essential space*) into open cones called *chambers*. Choose one, called the *fundamental chamber*; its closure is the *closed fundamental chamber*. Its bounding hyperplanes are its *walls*. The reflections in these walls form a generating set S of *simple reflections*, and (W, S) is a finite Coxeter system. The word simple refers to the chosen chamber; the other reflections need not be wall reflections of that chamber.

The *rank* of the system is $|S|$. For distinct $s, t \in S$, let m_{st} be the order of st . The *Coxeter diagram* has node set S and an edge st when $m_{st} \geq 3$, labeled by m_{st} ; labels three are usually omitted. A system is *irreducible* when this diagram is connected.

We use the following standard facts [20]. For $I \subseteq S$, write $W_I = \langle I \rangle$ for the standard parabolic subgroup, that is, the subgroup generated by I . A point in the closed fundamental chamber has stabilizer W_I , where I consists of the walls containing that point. Every W -orbit meets the closed chamber in exactly one point.

Write W/W_I for the set of left cosets $wW_I = \{wu : u \in W_I\}$, and similarly for $J \subseteq S$.

Definition 2.10. For $I, J \subseteq S$ with $I \cup J = S$, the *reflection bigraph* $\mathcal{R}(W; I, J)$ of Conlon and Lee [10] has distinguished parts W/W_I and W/W_J and edge set

$$\{(wW_I, wW_J) : w \in W\}.$$

Repeated pairs are identified, so the graph is simple. The condition $I \cup J = S$ is exactly the connectedness requirement: paths change a coset representative by right multiplication in W_I or W_J . We allow $I = S$ or $J = S$, direct products of Coxeter systems (*reducible* systems), and the trivial system $W = \{1\}, S = \emptyset$.

For $\text{Sym}(v)$, the *type- A_{v-1} simple system* consists of the adjacent transpositions $(1\ 2), (2\ 3), \dots, (v-1\ v)$. Its geometric action permutes coordinates in $\mathbb{R}^v$; the essential space is the subspace whose coordinate sum is zero.

Geometrically, choose points x, y in the closed chamber with stabilizers W_I, W_J . The two parts are distinguished copies of their orbits, and the edges are (wx, wy) for $w \in W$. This description also explains why an edge never crosses a reflection hyperplane: both endpoints belong to the same closed chamber $w\overline{C}$.

Example 2.11. Let $W = \text{Sym}(3)$, $S = \{s_1, s_2\}$ with $s_1 = (1\ 2)$ and $s_2 = (2\ 3)$, and take $I = \{s_1\}$, $J = \{s_2\}$. Each part has three cosets and every vertex has degree two. The resulting graph is C_6 . In its planar representation the chambers have angle 60° , and the two point orbits form the alternating vertices of a regular hexagon; see Figure 3.

Definition 2.12. The *part-respecting tensor product* $R \otimes F$ has parts $A(R) \times A(F)$ and $B(R) \times B(F)$. The vertices (v, x) and (w, y) are adjacent exactly when $vw \in E(R)$ and $xy \in E(F)$.

Products of reflection bigraphs are reflection bigraphs, using the direct product of the Coxeter systems and the unions of the corresponding parabolic subsets. The complete bigraph $K_{a,b}$ is reflection: use $W = \text{Sym}(a) \times \text{Sym}(b)$ with left stabilizer $\text{Sym}(a-1) \times \text{Sym}(b)$ and right stabilizer $\text{Sym}(a) \times \text{Sym}(b-1)$, under the standard type- A simple systems. The factors with size one are trivial. Consequently $F(a, b) \cong F \otimes K_{a,b}$.

2.4 Weak norming and decorated Hölder

Let (Ω, μ) and (Λ, ν) be probability spaces. For bounded measurable functions of two variables, called *kernels*, $W_e : \Omega \times \Lambda \rightarrow \mathbb{R}$, set

$$t_H((W_e)_{e \in E}) = \int_{\Omega^A \times \Lambda^B} \prod_{uv \in E} W_{uv}(x_u, y_v) d\mu^A(x) d\nu^B(y). \quad (1)$$

The measures μ^A, ν^B are products: each source vertex is assigned an independent target variable in the corresponding space. We call this a *decorated homomorphism density* and write $t_H(W)$ when all edge kernels are equal.

Definition 2.13. The bigraph is *weakly norming* if $W \mapsto t_H(|W|)^{1/|E|}$ is a norm on the space of bounded kernels, modulo equality almost everywhere, for every such pair of probability spaces. This is the bigraph formulation, with independent measures on the two vertex parts.

Here being a norm means positivity away from zero, absolute homogeneity, and the triangle inequality. Functions agreeing outside a set of measure zero are identified.

The *weak Hölder property*, also called *decorated Hölder*, is the following inequality for every family of nonnegative bounded kernels:

$$t_H((W_e)_{e \in E}) \leq \prod_{e \in E} t_H(W_e)^{1/|E|}. \quad (2)$$

The implication to weak norming is elementary. For bounded kernels U, V , we have $|U + V| \leq |U| + |V|$. Expanding the product defining $t_H(|U| + |V|)$ and applying (2) term by term gives the triangle inequality. Taking one edge kernel $W \geq 0$ and all others equal to one gives $t_H(W)^{1/|E|} \geq \int W$, ensuring positivity away from zero; absolute homogeneity is automatic. The converse is Hatami's theorem [19, 12]. We also use two known structural facts. Reflection bigraphs are weakly norming [10, Theorem 1.2], and a connected weakly norming bigraph has a part-preserving edge-transitive automorphism group [28, 12]. The appendix uses this latter fact to obtain transitivity on point-block incidences, called *flags*; see Definition A.1.

Finite models and activities. A *finite-state model* uses finite target sets $\mathcal{I}$ and $\mathcal{J}$, nonnegative vertex weights (α_i) and (β_j) , and nonnegative kernels given by matrices. The weights are called *activities*. Its *partition function* is the unnormalized sum over all assignments:

$$Z = \sum_{x \in \mathcal{I}^A, y \in \mathcal{J}^B} \prod_{u \in A} \alpha_{x_u} \prod_{v \in B} \beta_{y_v} \prod_{uv \in E} W_{uv}(x_u, y_v). \quad (3)$$

When $Z > 0$, dividing each assignment's weight by Z gives a probability measure; expectations in that measure are denoted by $\mathbb{E}_W$ when all kernels equal W . Normalizing the activities multiplies every homomorphism integral by a common factor, which cancels in (2). Zero activities and zero kernel entries are obtained by continuity from strictly positive finite models.

Symmetric kernels. A *graphon* is a symmetric measurable kernel $W : [0, 1]^2 \rightarrow [0, 1]$. For a symmetric kernel on one probability space, $t_H^{\text{sym}}(W)$ denotes the usual graph density: every vertex, in either part, is assigned a variable in that space. For connected bipartite graphs, testing weak norming on symmetric kernels gives the same class as the bigraph formulation above. Approximating kernels by *finite-step kernels*, constant on the rectangles of finite measurable partitions, reduces the question to weighted finite probability spaces, represented by interval partitions. Here is the reduction. Write H^{op} for H with its two parts exchanged. Given a kernel $W : \Omega \times \Lambda \rightarrow \mathbb{R}$, its transpose is $W^T(y, x) = W(x, y)$. Put W and W^T in the two off-diagonal blocks of a symmetric kernel B_W on the equally weighted disjoint union $\Omega \sqcup \Lambda$, with zeros in the diagonal blocks. Connectedness gives, with $n = |V(H)|$,

$$t_H^{\text{sym}}(|B_W|) = 2^{-n} (t_H(|W|) + t_{H^{\text{op}}}(|W|)).$$

If the two terms on the right agree for every nonnegative kernel, the symmetric triangle inequality immediately gives the bigraph one. Otherwise choose a nonnegative kernel Z with $t_H(Z) > t_{H^{\text{op}}}(Z)$, transposing a witness and exchanging its probability spaces if necessary. For kernels U, Z on possibly different pairs of spaces, their tensor product is

$$(U \otimes Z)((x, x'), (y, y')) = U(x, y)Z(x', y'),$$

on the product probability spaces; $Z^{\otimes r}$ is its r -fold tensor power. Apply the symmetric triangle inequality to the embeddings of $U \otimes Z^{\otimes r}$ and $V \otimes Z^{\otimes r}$. Divide by $2^{-n/|E(H)|} t_H(Z)^{r/|E(H)|}$ and let $r \rightarrow \infty$. Multiplicativity of homomorphism densities makes the opposite orientation vanish, leaving the bigraph triangle inequality. Absolute homogeneity is immediate. For positivity, rescale a nonzero nonnegative kernel W so that $W \leq 1$. With $a = |A|$ and $b = |B|$, two applications of Jensen's inequality give

$$t_H(W) \geq t_{K_{a,b}}(W) \geq \left(\int W \right)^{ab} > 0.$$

The converse follows by restricting the bigraph norm to symmetric kernels. This identifies the weak-norming notion used in the step-Sidorenko characterization [14] with the one used here.

Definition 2.14. For a finite measurable partition $\mathcal{P}$ of $[0, 1]$ into sets of positive measure, let $W_{\mathcal{P}}$ be the graphon whose value on $P \times Q$ is the average of W there. The bigraph H has the *step Sidorenko property* if $t_H^{\text{sym}}(W_{\mathcal{P}}) \leq t_H^{\text{sym}}(W)$ for every graphon W and every such partition $\mathcal{P}$.

Weak norming implies this inequality directly. The kernel $W_{\mathcal{P}}$ is an L^1 limit of finite averages of $W(\phi(x), \psi(y))$, where ϕ, ψ independently range over measure-preserving bijections within the partition classes. Each such relabeling preserves the graph norm, so the triangle inequality bounds the norm of every average by $\|W\|_{r(H)}$. Continuity of homomorphism densities on uniformly bounded kernels gives the same bound in the limit. The converse for connected graphs is proved in [14, Theorem 3.25].

2.5 One fold is one Cauchy–Schwarz inequality

Here is the analytic meaning of the copying operation in Figure 1. Give the edges a coloring $c : E(H) \rightarrow \{1, \dots, r\}$ and choose a nonnegative kernel W_i for each color. Write $t_H(c)$ for the integral in which edge e carries $W_{c(e)}$. A coloring is *monochromatic* if all edges have the same color. A *star coloring* assigns the same color to all edges at each vertex of one specified part.

Definition 2.15. An edge coloring c is *fold-stable* if, for every oriented fold t , the colored bigraphs (H, c) and $(H, c \circ t)$ are isomorphic. Here a colored isomorphism preserves the parts and the edge colors. Thus stability allows a relabeling of vertices; the stronger identity $c \circ t = c$ allows none.

Let f exchange the open sides L, R and fix Z , and let t_L, t_R be its two oriented folds. Keep the variables at Z fixed and integrate over L and R separately. This gives nonnegative functions Φ_L, Φ_R of the variables at Z . If K_Z is the product of the kernels on edges with both endpoints in Z , then

$$t_H(c) = \int K_Z \Phi_L \Phi_R.$$

Apply Cauchy–Schwarz with measure K_Z times the product measure on the variables at Z . Since f preserves the parts, the two sides use the same product measures. Copying the decoration on L to R therefore gives $\int K_Z \Phi_L^2$, and copying in the other direction gives $\int K_Z \Phi_R^2$. Thus

$$t_H(c) \leq (t_H(c \circ t_L) t_H(c \circ t_R))^{1/2}. \quad (4)$$

The action on colors is exactly the inverse-image action on marked sets: an edge has color i after the fold precisely when its image had color i before the fold.

For the sequence on C_6 , the retained open sides are $\{0, 5\}$, then $\{1, 2\}$, then $\{0, 1\}$. Their fixed vertex pairs are $\{1, 4\}$, $\{0, 3\}$, and $\{2, 5\}$. Starting with edge 01, copying these sides marks $\{01, 12\}$, then $\{01, 12, 45, 50\}$, then all six edges. Equation (4) explains each step of this process.

A single chosen branch of these inequalities does not yet prove decorated Hölder: the other factors must also be accounted for. Conlon–Lee’s iteration keeps the full Cauchy–Schwarz tree, repeats a percolating sequence, and makes the total weight of monochromatic branches tend to one [10, Section 3]. Here the tree records both factors on the right of each application of (4); each branch receives one half of its parent’s weight. This is the known implication from cut-percolation to weak norming.

3 Spectral classification

We prove the difficult implication in Theorem 1.1: transitivity on the two vertex parts gives a reflection representation. The proof has two stages. First, the lowest nonconstant Laplacian eigenspace turns the fold group into a geometric reflection group. Second, we recover the vertices that this representation identifies. They form a smaller tensor factor to which we apply induction.

3.1 Atomic folds on the Fiedler eigenspace

Let $\Gamma = \text{Fold}(H)$. The *combinatorial Laplacian* is the symmetric matrix $L = D - A$, where D is diagonal with $D_{vv} = \deg_H(v)$ and A_{uv} is one for an edge and zero otherwise. Equivalently, for a real function ϕ on vertices,

$$(L\phi)(v) = \sum_{w \in N_H(v)} (\phi(v) - \phi(w)). \quad (5)$$

It commutes with every graph automorphism. An eigenvalue θ has eigenspace $\ker(L - \theta I)$, the functions satisfying $L\phi = \theta\phi$; $\text{spec}(L)$ denotes the set of these eigenvalues, and I is the identity matrix. Let $\lambda > 0$ be the smallest positive eigenvalue and $\mathcal{E} = \ker(L - \lambda I)$ its *Fiedler eigenspace* [16]. Let P be the orthogonal projection onto $\mathcal{E}$: it is the identity on $\mathcal{E}$ and zero on its orthogonal complement. Define the *spectral vertex map*

$$p : V(H) \longrightarrow \mathcal{E}, \quad p(v) = P\mathbf{e}_v. \quad (6)$$

Equivalently, choose an orthonormal eigenbasis of $\mathcal{E}$ and use its values at v as the coordinates of $p(v)$. The permutation action of Γ on $\mathbb{R}^{V(H)}$ restricts to an *orthogonal representation* $\rho : \Gamma \rightarrow O(\mathcal{E})$: it assigns each group element a linear isometry, compatibly with composition. Here $O(\mathcal{E})$ is the group of linear isometries of $\mathcal{E}$, and $p(gv) = \rho(g)p(v)$.

A function is *f-odd* if $\phi(fv) = -\phi(v)$. For $C \subseteq V(H)$, the *Dirichlet Laplacian* on C is the principal submatrix $L_{C,C}$, obtained by retaining the rows and columns in C ; it is the Laplacian quadratic form restricted to functions zero outside C . The *Rayleigh quotient* of a nonzero function is $\langle \phi, L\phi \rangle / \langle \phi, \phi \rangle$. Connectedness gives its variational characterization

$$\lambda = \min_{\substack{\phi \neq 0 \\ \sum_v \phi(v) = 0}} \frac{\langle \phi, L\phi \rangle}{\langle \phi, \phi \rangle}. \quad (7)$$

The sum condition says that ϕ has *mean zero*.

On one moved component of an atom, Perron–Frobenius gives a positive lowest Dirichlet eigenvector unique up to scale. It will supply the one-dimensional normal direction of a reflection. Using the lowest nonconstant eigenvalue of the whole graph ensures that no other odd direction can appear.

Lemma 3.1 (An atomic fold acts as a reflection or the identity). *For each atomic fold involution f , either $\rho(f) = 1$ or $\rho(f)$ is an orthogonal reflection. In the latter case, with reflecting hyperplane $M_f \subset \mathcal{E}$,*

$$p^{-1}(M_f) = \text{Fix}(f).$$

Moreover no graph edge crosses M_f : its endpoints lie in a common closed halfspace.

Proof. Let C, C' be the two nonfixed components, exchanged by f . An f -odd function on $V(H)$ is determined by its restriction u to C : it is $-u \circ f$ on C' and zero on $\text{Fix}(f)$. Under this identification the Laplacian on odd functions is exactly the principal Dirichlet matrix $L_{C,C}$. At a fixed vertex the contributions from C and C' cancel in pairs, so every Dirichlet eigenvector gives an actual odd Laplacian eigenvector on H .

Since $H[C]$ is connected, the least eigenvalue μ of $L_{C,C}$ is simple, meaning its eigenspace is one-dimensional, and has a strictly positive eigenvector. Indeed, for large c the matrix $cI - L_{C,C}$ is nonnegative and irreducible: its nonzero off-diagonal entries connect C . Perron–Frobenius says that its largest eigenvalue is simple with a positive eigenvector. Its odd extension has mean zero, so (7) gives $\mu \geq \lambda$.

If f acts nontrivially on $\mathcal{E}$, its -1 eigenspace there is nonzero, and hence λ occurs in the Dirichlet spectrum. Then $\lambda \geq \mu$, so $\lambda = \mu$ and this odd eigenspace is one-dimensional. Thus $\rho(f)$ is a reflection.

Choose its odd normal vector $\psi \in \mathcal{E}$, positive on C , negative on C' and zero on $\text{Fix}(f)$. Since $\langle p(v), \psi \rangle = \psi(v)$, the inverse image of its orthogonal hyperplane is exactly $\text{Fix}(f)$. No edge joins C to C' , proving the last assertion. $\square$

3.2 The reflection quotient

The preceding lemma controls each atom separately. We next need control of *every* mirror of the resulting reflection group: two image endpoints must share a chamber. For this purpose it matters that the atomic folds are closed under conjugation.

Definition 3.2. The *spectral quotient* R identifies vertices within each specified part when their images under p agree. Two quotient vertices are adjacent if some original edge joins their classes; repeated edges are identified. The *vertex fiber* over x is its preimage in the appropriate part of H . The *edge fiber* over $xy \in E(R)$ is the set of original edges with endpoints in those two vertex fibers.

Write $T(W)$ for the set of all orthogonal reflections in a finite real reflection group $W \leq O(U)$. A subset T_0 is *conjugacy-invariant* if $wT_0w^{-1} = T_0$ for every $w \in W$. We use the following standard fact to control all mirrors.

Lemma 3.3. *If a conjugacy-invariant subset $T_0 \subseteq T(W)$ generates W , then $T_0 = T(W)$.*

Proof. The Coxeter diagram of (W, S) has vertices S and an edge labeled m_{st} when the order of st is $m_{st} \geq 3$. Reflection conjugacy classes correspond to the components remaining after the even-labeled edges are removed. The *abelianization*, the quotient obtained by requiring all group elements to commute, is a product of two-element groups with one independent generator for each such component. A reflection maps to the generator of its class, so a generating set must meet every class. Conjugacy invariance then gives all reflections. See the proofs of Lemmas 3.6 and 3.7 of Brady–McCammond–Mühlherr–Neumann [6], pp. 7–8. These facts apply to the essential space defined in Section 2.3; a common fixed subspace can be omitted. $\square$

Theorem 3.4 (Spectral reflection quotient). *Suppose Γ acts transitively on each of A and B , and let R be the spectral quotient of Definition 3.2. Then:*

- (i) $W = \rho(\Gamma)$ is a finite real reflection group, allowing the trivial group.
- (ii) R is a connected reflection bigraph for W . More precisely, after choosing a suitable fundamental chamber and corresponding simple system S , there are $I, J \subseteq S$ with $I \cup J = S$ such that

$$R \cong \mathcal{R}(W; I, J).$$

Proof. Atomic folds generate Γ , so Lemma 3.1 shows that their nontrivial images generate W by reflections. Their image set T_0 is invariant under conjugation by W , because atomic folds are invariant under conjugation by Γ and ρ is onto W . Lemma 3.3 gives $T_0 = T(W)$. Consequently every reflecting hyperplane of W is the image of an atomic fold as in Lemma 3.1, and no edge of H crosses any of them.

Fix an edge ab with $a \in A, b \in B$, and write $x = p(a), y = p(b)$. The points x, y belong to a common closed fundamental chamber: for each reflection hyperplane their signed

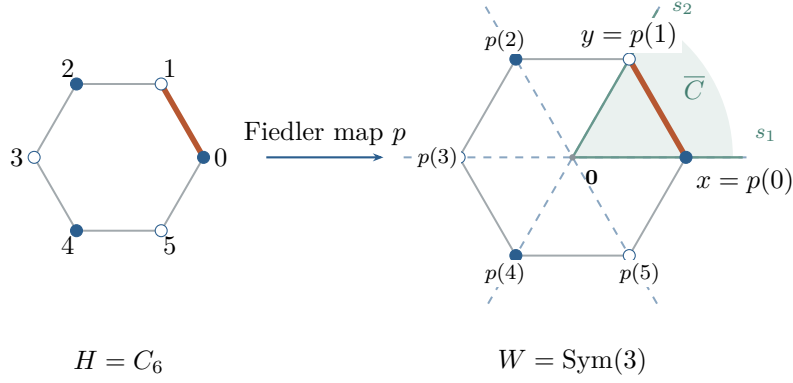


Figure 3: The spectral vertex map p from (6) for C_6 , up to scale. Its two vertex parts form two three-point orbits of the type- A_2 reflection group. The three fold mirrors pass through pairs of opposite vertices. The marked edge has endpoints x, y in the closed fundamental chamber $\bar{C}$; every other edge lies in a translate of this chamber. Here s_1, s_2 are its wall reflections, as in Section 2.3. No edge crosses a mirror.

coordinates have nonnegative product. Any point $tx + (1 - t)y$, $0 < t < 1$, has the same nonzero signs as either endpoint and lies on a hyperplane only when both endpoints do; every closed chamber containing that combination therefore contains both endpoints. The fixed ambient subspace, if present, can be factored off when making this choice.

Let S be the simple system of this chamber, and let I, J index its walls containing x, y , respectively. The standard stabilizer theorem for a finite reflection group (Humphreys [20], Theorem 1.12(a)–(d), pp. 22–23) gives

$$\text{Stab}_W(x) = W_I, \quad \text{Stab}_W(y) = W_J.$$

Transitivity on each of A, B implies that the two parts of R are exactly the two W -orbits of x, y , represented by W/W_I and W/W_J .

Every image edge has its endpoints in some common closed chamber. Move that chamber to the chosen one by an element of W . The resulting endpoints must be x, y , since a W -orbit meets a closed fundamental chamber in exactly one point. Thus the edges of R are exactly the orbit of (x, y) , namely the pairs (wW_I, wW_J) for $w \in W$. The quotient is connected because it is the image of the connected graph H under the quotient homomorphism. Thus $I \cup J = S$, and this is the reflection bigraph of Definition 2.10; original edge transitivity was not assumed. Keeping the two parts distinguished handles the possibility that the same point occurs in both orbit sets. $\square$

An *atomic lift* of a reflection in W is an atomic fold whose image under ρ is that reflection. The proof above shows that every reflection has an atomic lift. The following additional fiber symmetry is useful, although the tensor splitting below does not require it.

Remark 3.5 (Symmetry within vertex fibers). The *representation kernel* $K = \ker \rho = \{g \in \Gamma : \rho(g) = 1\}$ consists of the group elements acting trivially on $\mathcal{E}$. Under the hypotheses of Theorem 3.4, it is transitive on every vertex fiber. Indeed, take v, v' in the same part with $p(v) = p(v')$ and choose $g \in \Gamma$ with $gv = v'$. Then $\rho(g)$ belongs to the parabolic subgroup $\text{Stab}_W(p(v))$, generated by reflections fixing $p(v)$. By Lemma 3.1, each atomic lift of such a reflection fixes every vertex above $p(v)$ pointwise. A product h of these lifts therefore fixes v and satisfies $\rho(h) = \rho(g)$. Thus $gh^{-1} \in K$ sends v to v' .

3.3 Exact tensor splitting

The quotient need not distinguish all vertices of H . Knowing the quotient and the sizes of the fibers from Definition 3.2 would not determine H : we must also recover the incidence pattern inside each edge fiber.

We do this with one common folding word. It moves all image points into a chosen chamber and identifies each original vertex fiber bijectively with one of the two vertex fibers above a chosen edge, called the *base fibers*. The same word identifies each whole edge fiber with the base edge fiber. This compatibility is what gives an exact tensor product.

Theorem 3.6 (The spectral quotient splits as a tensor factor). *Under the hypotheses of Theorem 3.4, there is a connected bigraph F such that*

$$H \cong R \otimes F,$$

where $\otimes$ is the tensor product of Definition 2.12. The graph F is the subgraph of H induced by the two vertex fibers above any quotient edge whose image points lie in a chosen closed fundamental chamber. We call F the residual factor.

Proof. Retain the base edge and closed fundamental chamber $\overline{C}$ chosen in Theorem 3.4. Put $A_0 = p^{-1}(x) \cap A$ and $B_0 = p^{-1}(y) \cap B$ for its two vertex fibers. For a simple reflection s , let T_s be the geometric fold that fixes the closed halfspace containing $\overline{C}$ and reflects the other halfspace. There is a finite sequence $T_{s_m} \cdots T_{s_1}$ taking every point of $p(V(H))$ into $\overline{C}$. Choose c in the interior of the essential chamber. If a current point lies outside $\overline{C}$, choose a simple wall for which its coordinate is negative and fold toward the chamber. The scalar product with c strictly increases for every moved point and stays unchanged for the others. Each point remains in its finite W -orbit, so it can move only finitely many times. Each step moves at least one current point; the procedure therefore terminates with every point in $\overline{C}$.

For each selected s_i choose an atomic lift f_i and orient its graph fold t_i toward the positive halfspace. Lemma 3.1 gives $p \circ t_i = T_{s_i} \circ p$. Define

$$q = t_m \cdots t_1 : H \longrightarrow H.$$

Each t_i is a graph endomorphism, so q is one as well. Every W -orbit meets $\overline{C}$ in exactly one point (Humphreys [20], Theorem 1.12). Consequently q maps each left vertex fiber bijectively onto the base left fiber A_0 and each right vertex fiber bijectively onto the base right fiber B_0 . The asserted bijectivity holds at each step: on a whole vertex fiber, t_i is either the identity or the restriction of the automorphism f_i . For a fiber lying on the wall, f_i fixes it pointwise by Lemma 3.1.

More strongly, on each whole edge fiber, t_i is either the identity or the restriction of the same full automorphism f_i at both endpoints. Indeed, the two image endpoints never have opposite nonzero signs with respect to the wall. If one lies on the wall, f_i fixes its entire vertex fiber pointwise. Thus each t_i , and hence q , maps any complete edge fiber bijectively onto the corresponding image edge fiber. After all steps this image is the base edge fiber.

Let $F = H[A_0 \cup B_0]$. The map q fixes every vertex of F because its image point already lies in $\overline{C}$, so q is a *retraction* onto F : a homomorphism to F that is the identity on F . In particular F is connected as the image of connected H .

For a quotient vertex v , let V_v be its vertex fiber. The map

$$u \longmapsto (p(u), q(u))$$

(with the part label included in the first coordinate) is a bijection from $V(H)$ onto the appropriate part-respecting Cartesian product of vertex sets, since $q|_{V_v}$ is a bijection onto A_0 or B_0 . For every quotient edge vw , its restriction maps the entire edge fiber bijectively onto $E(F)$ using exactly those endpoint bijections. Hence adjacency in H is equivalent to adjacency in both coordinates, proving $H \cong R \otimes F$. $\square$

Example 3.7. The residual factor can be nontrivial even for a complete bigraph. Let $H = K_{a,b}$ with $1 \leq a < b$. Its Fiedler eigenvalue is a , and its Fiedler eigenspace consists of functions supported on the b -vertex part with sum zero. All vertices of the a -vertex part therefore have image zero, while the other images form a regular simplex, with all pairwise distances equal. The spectral quotient is $K_{1,b}$ and the residual factor is $K_{a,1}$, giving $K_{a,b} = K_{1,b} \otimes K_{a,1}$. Permutations of twins in the a -vertex part are invisible on the first Fiedler eigenspace and are recovered by induction on the residual factor.

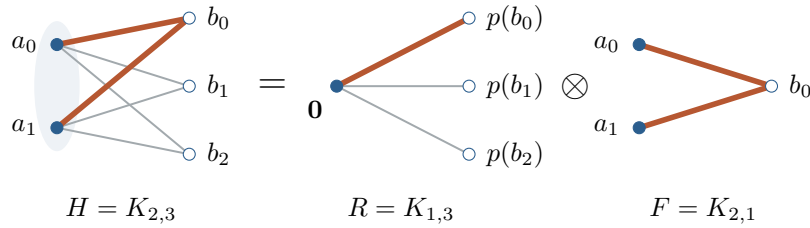


Figure 4: A spectral quotient can lose a whole tensor factor. For $K_{2,3}$, the Fiedler map identifies a_0, a_1 and keeps the three right vertices distinct, giving $R = K_{1,3}$. The colored edges above the selected quotient edge form the entire residual graph $F = K_{2,1}$. Each other quotient edge has the same edge fiber. The tensor splitting proves that these copies have compatible endpoint identifications in general; see Definition 3.2 and Theorem 3.6.

Example 3.8 (Fibers need not be twin classes). The *bipartite adjacency matrix* M has rows indexed by the left part and columns by the right part, with entry one exactly at an edge. Its singular values are the square roots of the eigenvalues of $M^T M$. A matrix tensor product has entries equal to the products of the entries at the corresponding coordinate pairs. Take $H = C_{10} \otimes C_6$. The bipartite adjacency matrices of C_{10} and C_6 have largest singular value 2 and second singular values $2 \cos(\pi/5)$ and 1, respectively. Since the bipartite adjacency matrix of H is their tensor product and H is 4-regular, meaning every vertex has degree four, its Fiedler eigenvalue is $4 - 4 \cos(\pi/5)$. The corresponding functions depend only on the C_{10} coordinate. Thus its spectral quotient is C_{10} and its residual factor is C_6 . A vertex fiber contains three vertices with different neighborhoods.

To continue by induction, we must show that F still has fold-group transitivity. Atoms *invisible* on the Fiedler eigenspace, with $\rho(f) = 1$, restrict to folds inside its edge fibers. *Visible* atoms, with $\rho(f) \neq 1$, require a different argument: their action on the F coordinate consists only of permutations of twins. The next lemma gives this argument.

Lemma 3.9 (Fiber automorphisms with one fixed fiber). *Let R, F be connected bigraphs. Suppose an automorphism of $R \otimes F$ induces an automorphism α on the R coordinate, so that it has the form*

$$(v, x) \longmapsto (\alpha v, \tau_v x),$$

where τ_v is a permutation of the appropriate part of F . If τ_{v_0} is the identity for at least one quotient vertex v_0 , then every τ_v permutes vertices only within their twin classes in F .

Proof. For each edge $vw \in E(R)$, the pair (τ_v, τ_w) is a part-preserving automorphism of F , because it bijects the full edge fiber over vw onto the full edge fiber over $\alpha(vw)$. A permutation τ of one part of F extends to an automorphism fixing the other part pointwise if and only if it preserves every twin class. Call a permutation preserving each twin class a *twin permutation*. Thus if τ_v is a twin permutation, compose (τ_v, τ_w) with $(\tau_v^{-1}, 1)$ to deduce that τ_w is a twin permutation too. Starting from v_0 and following paths in connected R proves the assertion. $\square$

Theorem 3.10 (Fold transitivity passes to the residual tensor factor). *If $\text{Fold}(H)$ is transitive on each part, then $\text{Fold}(F)$ is transitive on each part for the factor F in Theorem 3.6.*

Proof. Work in the tensor coordinates supplied by Theorem 3.6. We show that every atomic fold of H preserves the $\text{Fold}(F)$ -orbit of the F -coordinate of each vertex and each edge of H . Let f be such an atomic fold. If $\rho(f) = 1$, then f preserves every vertex fiber and every edge fiber individually. On the subgraph formed by any edge fiber it is either the identity or a fold involution: intersect its fixed set and its two exchanged open sides with that edge fiber. The intersections still have no edge between the open sides and are exchanged by the restricted involution. Transporting this restriction by the fiber isomorphism to F gives the identity or a fold involution of F . Thus it preserves the claimed vertex and edge orbits. Every vertex lies in an incident edge fiber because H has no isolated vertices.

If $\rho(f) \neq 1$, its image is a reflection α on R . It has the form $f(v, x) = (\alpha v, \tau_v x)$. Since H is connected and f is a nonidentity fold, $\text{Fix}(f)$ is nonempty: otherwise the sole component of $H - \text{Fix}(f)$ would be preserved setwise. Choose a vertex of H fixed by f , and let v_0 be its quotient vertex. Its spectral image lies on the reflecting hyperplane. Lemma 3.1 says that f fixes the entire fiber over v_0 pointwise, so $\tau_{v_0} = 1$.

Lemma 3.9 now implies that every τ_v is a twin permutation of F . A transposition of twins is a fold involution: after its fixed vertices are removed, the two moved vertices are separate components. Hence all twin permutations belong to $\text{Fold}(F)$. For any quotient edge vw , the pair (τ_v, τ_w) is therefore an element of $\text{Fold}(F)$, proving the edge-orbit assertion also in the visible case.

Atomic folds generate $\text{Fold}(H)$. If $\text{Fold}(F)$ had more than one vertex orbit in a part, the corresponding F -coordinate classes of vertices of H would be invariant under $\text{Fold}(H)$, contradicting transitivity in that part. The same argument also shows that edge transitivity descends whenever it is assumed. $\square$

3.4 Completion of the classification

The quotient and the residual factor now have the same symmetry property with which we started, and the quotient is reflection. We finish by checking that the residual factor is strictly smaller. We also recall the geometric proof of the known reverse implication, so that all four equivalences can be followed directly.

Lemma 3.11 (Reflection bigraphs cut-percolate). *Every reflection bigraph is cut-percolating.*

Proof. We recall a geometric proof of the result of Conlon–Lee [10]; an algebraic proof on Coxeter coset spaces is given in [4, Section 3.1]. Choose a fundamental closed chamber $\overline{C}$ for W , and points x, y in its faces whose stabilizers are exactly W_I, W_J . Embed the two parts as the distinguished orbits Wx, Wy . Every edge has endpoints in some $w\overline{C}$, so no edge has endpoints on opposite strict sides of a reflection hyperplane. A reflection fixes exactly the vertices on its hyperplane; its positive and negative vertex sets are exchanged

unions of components. If the reflection acts nontrivially on the graph, it therefore gives a genuine fold. Repeatedly fold across simple walls toward $\overline{C}$. The finite-orbit argument in Theorem 3.6 gives a common finite word taking every vertex into $\overline{C}$. The unique orbit representatives there are x and y , so the word sends every edge to the base edge xy . Reverse the word and take inverse images to obtain cut-percolation. $\square$

The following elementary implications will be useful [12, Section 1]. Every oriented fold maps an edge either to itself or to its image under the underlying fold involution. It therefore preserves each $\text{Fold}(H)$ -edge orbit. If its compositions send all edges to one edge, there can be only one such orbit. Thus cut-percolation implies fold-group edge transitivity. Edge transitivity in turn implies transitivity on each vertex part, since every vertex is incident with an edge.

Lemma 3.12 (Part transitivity forces a reflection representation). *Let H be a finite connected simple bigraph with at least one edge. If $\text{Fold}(H)$ is transitive on each vertex part, then H is a reflection bigraph. Consequently H cut-percolates and $\text{Fold}(H)$ is edge-transitive.*

Proof. Induct on $|E(H)|$ to obtain the reflection representation. The single-edge graph is reflection, using the trivial Coxeter group. For $|E(H)| > 1$, Theorems 3.4–3.10 give $H \cong R \otimes F$, with R reflection and F connected and fold-group transitive on each part. We verify that $|E(R)| \geq 2$, making $|E(F)| < |E(H)|$.

Transitivity on each part makes H biregular, of degrees r, s on the two parts. If R had only one edge, every function in the Fiedler eigenspace would be constant on each part. The only Laplacian eigenvalues on that two-dimensional space are 0 and $r + s$, so $\lambda = r + s$. But at least one part has two vertices u, v . They have the same degree $d \in \{r, s\}$ and are nonadjacent. The mean-zero test function $\mathbf{e}_u - \mathbf{e}_v$ has Rayleigh quotient d , whence $\lambda \leq d < r + s$, a contradiction.

Thus the induction hypothesis makes F reflection. The part-respecting tensor product of reflection bigraphs is reflection, realized by the direct product of their finite Coxeter systems. Therefore H is reflection. Lemma 3.11 gives cut-percolation, and the elementary orbit argument above gives fold-group edge transitivity. $\square$

Proof of Theorem 1.1. The known implications form the chain (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i): the first is Lemma 3.11, and the other two are the elementary implications just recalled. Lemma 3.12 proves (i) $\Rightarrow$ (iii), closing the cycle. $\square$

Corollary 3.13 (A binary star obstruction). *If a finite connected simple bigraph H with an edge is not cut-percolating, it has a nonconstant two-color edge coloring c such that $c \circ t = c$ for every oriented fold t of H . Moreover, all edges at each vertex of one specified part have the same color: the coloring can be chosen by coloring the stars centered in that part.*

Proof. Theorem 1.1 says that $\text{Fold}(H)$ has at least two vertex orbits in one of the specified parts, say A . Choose a nonempty proper union U of these orbits, and color an edge red exactly when its A -endpoint lies in U . There are edges of both colors since H has no isolated vertices. An oriented fold sends a vertex either to itself or to its image under an element of $\text{Fold}(H)$, so it preserves membership in U and therefore preserves the coloring pointwise. $\square$

4 Recognition through graph isomorphism

We first prove Theorem 1.2 using the classification and vertex orbits. We then give an independent recognition proof using pairs of edges. The input is a connected simple bigraph with at least one edge and its bipartition supplied. Put $n = |V(H)|$ and $m = |E(H)|$. For an unpartitioned connected bipartite input, a bipartition is found in linear time; its two possible orders give the same cut-percolation decision.

Decision model and running time. The *graph-isomorphism problem* asks whether two finite graphs are isomorphic. A polynomial-time Turing reduction to it is an algorithm allowed polynomially many such queries on polynomial-sized graphs, with polynomial computation between queries. Running times are measured in the binary input length; this is the *bit complexity* model. Quasipolynomial time is $n^{(\log n)^{O(1)}}$. A decision problem is in NP if every positive instance has a polynomial-length certificate checkable in polynomial time.

Definition 4.1. A *vertex-colored graph* has a specified label at each vertex. A *colored isomorphism* must preserve these labels as well as adjacency. Label names are shared between the two input graphs.

Although there can be many atomic folds, their moved components belong to a polynomial list determined by the Laplacian. Colored graph-isomorphism queries test which vertex images are possible for each component pair. These possible moves determine the vertex orbits of the fold group. By the classification, transitivity on each part decides cut-percolation; see Section 4.2. The edge-pair alternative in Section 4.3 uses the same support list.

4.1 A polynomial list of atomic supports

We enumerate possible moved component pairs before testing their isomorphisms.

Let L be the combinatorial Laplacian from (5) and P_θ its orthogonal projector onto the θ -eigenspace. Here θ ranges over all distinct eigenvalues, not just the Fiedler eigenvalue. For a vector z , its *positive support* is $\{v : z_v > 0\}$, and its *negative support* is $\{v : z_v < 0\}$.

Lemma 4.2. *The two moved components of every atomic fold occur as the positive and negative supports of a vector in the list*

$$\{P_\theta(\mathbf{e}_u - \mathbf{e}_v) : \theta \in \text{spec}(L), u, v \in V(H)\}. \quad (8)$$

The list, including the exact signs of all its coordinates, can be computed in polynomial bit complexity.

Proof. Let f be atomic, with moved components C, D and fixed set Z . As in Lemma 3.1, the Laplacian on the f -odd subspace is the principal Dirichlet matrix $L_{C,C}$. Its least eigenvalue μ is simple and has a strictly positive eigenvector. Its odd extension u is an eigenvector of the full Laplacian, strictly positive on C , strictly negative on D , and zero on Z . On the entire μ -eigenspace, the f -odd subspace is exactly the line $\mathbb{R}u$.

Choose $a \in C$ and $b = f(a)$. The vector $\mathbf{e}_a - \mathbf{e}_b$ is f -odd, and P_μ commutes with f . Therefore

$$P_\mu(\mathbf{e}_a - \mathbf{e}_b) = cu.$$

Its scalar product with u is $2u(a) > 0$, so $c > 0$. This gives exactly the supports C, D in (8).

For completeness, the projector computation uses only one real algebraic number at a time. A real algebraic number is specified by an integer polynomial and a rational interval isolating one of its real roots. Let

$$\chi(t) = \det(tI - L), \quad Q(t) = \chi(t) / \gcd(\chi(t), \chi'(t)).$$

Here χ is the characteristic polynomial, $\gcd$ is the greatest common polynomial divisor, and the prime denotes differentiation. The polynomial Q is squarefree, meaning that it has no repeated roots, and has the distinct Laplacian eigenvalues as its roots. For a root θ , polynomial division gives $R_\theta(t) = Q(t)/(t - \theta)$, and diagonalizability gives

$$P_\theta = \frac{R_\theta(L)}{Q'(\theta)}. \quad (9)$$

Indeed, the right side is one on the θ -eigenspace and zero on every other eigenspace. Each coordinate in the list is thus a rational function of one real algebraic number of degree at most n . The binary lengths of characteristic-polynomial coefficients, polynomial division, and the matrix powers needed in (9) have polynomial bit size. Isolating the real roots and determining the signs of these rational functions compute all signs in polynomial time [5]. There are at most n^3 vectors to inspect. $\square$

For each sign pair (C, D) from the list, put $Z = V(H) \setminus (C \cup D)$. Retain it only if C, D are nonempty, both induced subgraphs are connected, and there is no edge between them. Then C, D are precisely the two components of $H - Z$. Duplicates can be discarded.

Lemma 4.3. *For a retained support pair, its atomic fold involutions correspond exactly to isomorphisms $\phi : H[C] \rightarrow H[D]$ that preserve the bipartition and the neighborhood of each vertex in the named set Z . Named means that each boundary vertex retains its identity, rather than being freely relabeled. Existence, including at most four prescribed vertex images, is a vertex-colored graph-isomorphism query of polynomial size.*

Proof. Give each vertex $x \in C \cup D$ the color

$$(\text{part of } x, N_H(x) \cap Z).$$

The second coordinate is the exact subset of named boundary vertices, encoded as a bit vector. A color-preserving isomorphism ϕ extends to an involution that is ϕ on C , ϕ^{-1} on D , and the identity on Z . All internal and boundary edges are preserved. Its nonfixed components are exactly C, D , so it is atomic. Conversely, an atomic fold with these supports restricts to such an isomorphism.

To prescribe $\phi(x_i) = y_i$, reject inconsistent or noninjective prescriptions and assign a fresh matched individual color to each prescribed pair, retaining its original color. This imposes the prescribed images without changing the underlying induced graphs. $\square$

4.2 Recognition from vertex orbits

The classification lets us work with single vertices rather than pairs of edges. We only need to know which vertices an atom can exchange.

Proposition 4.4. *Cut-percolation can be recognized using $O(n^5)$ colored graph-isomorphism queries of polynomial size and polynomial computation between queries. Each query prescribes just one vertex image.*

Proof. Form an undirected graph Q on $V(H)$. For each retained candidate (C, D) and each $u \in C, v \in D$, add the edge uv when Lemma 4.3 admits an isomorphism with $\phi(u) = v$. Every such edge is an actual move of an atomic fold. Conversely, Lemma 4.2 includes the support of every atom, so all its nonfixed vertex moves occur in Q .

The components of Q are exactly the orbits of the group generated by atoms: a path is a sequence of atom moves, and every atom preserves each component. This group is $\text{Fold}(H)$ by Lemma 2.7. No edge of Q joins the two parts. Theorem 1.1 therefore says that H cut-percolates if and only if $Q[A]$ and $Q[B]$ are both connected. There are at most n^3 candidates and at most n^2 prescribed images for each. All other operations take polynomial time. $\square$

Colored graph isomorphism is polynomial-time reducible to ordinary graph isomorphism by attaching auxiliary graphs encoding the vertex labels. This standard reduction preserves isomorphism answers and has polynomial size. Proposition 4.4 therefore proves the Turing reduction in Theorem 1.2. The quasipolynomial bound follows from the corrected graph-isomorphism algorithm of Babai [2]. At any fixed maximum degree, the colored queries in Lemma 4.3 retain that degree bound, so the colored bounded-degree method of Luks [24] gives polynomial recognition. We use that colored method directly, without attaching degree-increasing color gadgets.

A short certificate. The same classification gives a certificate consisting only of a few atomic involutions. No folding word needs to be supplied.

Proposition 4.5. *A connected n -vertex bigraph with at least one edge is cut-percolating if and only if a list of atomic fold involutions generates a group transitive on each part. A positive instance of degrees d_A, d_B has such a list of length at most $\min\{n - 2, d_A + d_B - 2\}$ and hence a certificate with $O(n \min\{n, d_A + d_B\} \log n)$ bits, verifiable in polynomial time.*

Proof. For a cut-percolating graph, the group generated by all atoms is transitive on each part. Start with no selected atoms and hence n singleton vertex orbits. Suppose the selected atoms generate more than the two part orbits. Some atom must send a vertex between two current orbits: otherwise every atom would preserve every current orbit, contradicting transitivity of the full fold group. Add this atom. The number of orbits decreases by at least one. After at most $n - 2$ additions, the selected atoms are transitive on each part. Corollary B.7 gives the degree bound by describing the full fold group and its atomic generators. Conversely, such atoms imply part-transitivity of $\text{Fold}(H)$ and hence cut-percolation by Theorem 1.1.

Record each involution as its vertex permutation. Check that it is a nonidentity involution preserving parts and adjacency, and that deleting its fixed vertices leaves exactly two connected components which it exchanges. These are precisely the atomic fold conditions. To compute the generated group's vertex orbits, form the graph with edges $vf(v)$ for every recorded atom f and vertex v . Its components must be exactly A and B . All checks take polynomial time, and the list uses $O(n \min\{n, d_A + d_B\} \log n)$ bits. For $n = 2$, the empty list certifies $K_{1,1}$. $\square$

The bound $n - 2$ is sharp for the star $K_{1,n-1}$: its atoms are leaf transpositions, and at least $n - 2$ of them are needed to connect the $n - 1$ leaves. The certificate test above also shows that recognizing cut-percolation is in NP.

4.3 An independent alternative: the pair automaton

View each oriented atomic fold as a transformation of the m -element edge set. The set of available transformations may be large; we test its action on edge pairs without enumerating it. States are unordered edge pairs $\{e_1, e_2\}$, with diagonal states $\{e, e\}$ allowed. There are $\binom{m}{2} + m$ states. A transition sends $\{e_1, e_2\}$ to $\{t(e_1), t(e_2)\}$ for one oriented atomic fold t . This directed state graph is the *pair automaton*. A word gives a path in it, and reaching a diagonal state means merging the original pair.

A three-edge example. On the star $K_{1,3}$, let t_{12} send leaf u_2 to u_1 and let t_{13} send leaf u_3 to u_1 , fixing all other vertices. Each map is an oriented atomic fold. Figure 5 shows their actions on edge pairs: every pair can be merged into $\{e_1, e_1\}$. The same two folds also collapse all three edges at once: $t_{13} \circ t_{12}$ has image $\{e_1\}$. The pair criterion below makes this conclusion available without searching for a word on the whole edge set.

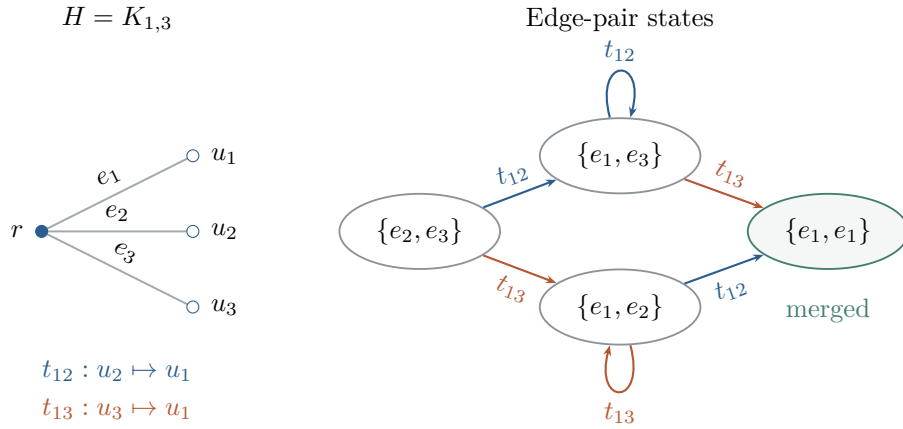


Figure 5: Pairwise merging on a three-edge star. The fold t_{12} sends u_2 to u_1 , and t_{13} sends u_3 to u_1 ; all other vertices are fixed. Every off-diagonal edge-pair state reaches the diagonal state, where both edges have the same image (Section 4.3). The diagram follows edge images; marked-edge percolation uses inverse images in the reverse order.

Testing a transition. For a retained pair (C, D) , consider the orientation that maps C to D by ϕ and fixes $D \cup Z$. To test a transition from $\{e_1, e_2\}$ to $\{e'_1, e'_2\}$, try both matchings of source and target edges. Order the endpoints of each edge by their specified parts. A source endpoint outside C must equal its matched target endpoint. A source endpoint $x \in C$ must have matched target $y \in D$, and imposes $\phi(x) = y$. There are at most four such prescriptions. By Lemma 4.3, the transition exists exactly when one of these colored-isomorphism queries succeeds. Test the other orientation in the same way.

There are $O(n^3)$ retained candidates and $O(m^4)$ pairs of automaton states, and a constant number of queries for each choice. Thus the full directed pair graph is obtained with polynomially many queries of polynomial size. An identity transition can be included without a query; it has no effect on reachability.

From pairs to a single word. The following elementary automata criterion is standard; see, for example, [15].

Lemma 4.6 (Pairwise merging). *A finite family of transformations of a nonempty finite set admits a word whose image is a singleton if and only if every pair can be merged by some word.*

Proof. A collapsing word merges every pair. For the converse, start with the whole set as the current image. If its size exceeds one, choose two distinct elements of that image and apply a word merging them. The image cardinality drops by at least one; applying a function cannot increase it. Repeat until only one element remains. $\square$

Graph search from all diagonal states along reversed transitions decides whether every edge pair can be merged. By Lemma 2.7, oriented atomic folds suffice for every collapsing word. By Lemma 4.6 and Definition 2.5, the test is exactly cut-percolation recognition.

This gives an independent polynomial-time Turing reduction using $O(n^3m^4)$ colored-isomorphism queries. The preceding vertex-orbit procedure uses only $O(n^5)$ queries. Both use the same exact spectral computation and the same boundary-colored induced graphs.

4.4 A single automorphism-group computation

Instead of making a separate isomorphism query for each constrained move, we can compute $\text{Aut}(H)$ once and answer all queries with permutation-group algorithms.

Proposition 4.7. *Given generators of the part-preserving automorphism group of H , cut-percolation can be decided in polynomial time without further graph-isomorphism queries.*

Proof. Let $G = \text{Aut}(H)$ be supplied by generators. For a retained candidate, compute the pointwise stabilizer $G_{(Z)} = \{g \in G : gz = z \text{ for every } z \in Z\}$ by permutation-group algorithms. An element of this stabilizer sending one vertex of C to D interchanges the two components of $H - Z$. Its restriction $\phi : C \rightarrow D$ is the boundary-preserving isomorphism required in Lemma 4.3, even if that element is not an involution: extend ϕ and its inverse separately to obtain the atomic involution.

Consequently each nontrivial transition test asks whether there is $g \in G_{(Z)}$ satisfying $g(x_i) = y_i$ for a tuple of at most four vertices. The set of such g is called the *tuple transporter*. The prescribed images lie in the opposite component, so a nonempty tuple enforces the component interchange. If the tuple is empty, the fold fixes both source edges and the transition is an irrelevant self-loop. For constant tuple length, tuple transporters and successive point stabilizers are computed in polynomial time from generators [26]. Equivalently, the *tuple orbit*, the set of all ordered tuples $(gx_1, \dots, gx_\ell)$ with $g \in G_{(Z)}$, has at most n^4 elements and can be traversed directly. The spectral list and the pair-graph search are already polynomial-time operations. $\square$

Generators of $\text{Aut}(H)$ are found with polynomially many colored graph-isomorphism queries. Successively fix named vertices, compute the orbit of the next vertex under the pointwise stabilizer, and find witnesses by further *individualization*, assigning selected vertices distinct new colors to prescribe their images. Orbit representatives over these levels generate the group [26].

4.5 Certificates consisting of folding words

The reachability argument gives a different certificate, independently of the classification and Proposition 4.5. It bounds the length of a successful folding word, which can also be checked without graph-isomorphism queries.

Proposition 4.8. *If H is cut-percolating, it has a collapsing fold word of length at most $(m-1)\binom{m}{2}$. Hence cut-percolation recognition is in NP.*

Proof. A shortest word merging a given pair has length at most $\binom{m}{2}$, since no nondiagonal pair repeats before the first diagonal is reached. The greedy proof of Lemma 4.6 uses at most $m - 1$ such merges. This gives the claimed word, with the empty word allowed at $m = 1$.

A certificate records every fold's involutory vertex permutation and retained side. In polynomial time one checks nonidentity, part preservation, adjacency preservation, the fixed set, the exchanged side partition, and absence of edges between open sides. Applying the recorded oriented folds to every edge verifies that the final image is a singleton. The total certificate length is polynomial in the input size. $\square$

5 One-sided transitivity does not imply percolation

Theorem 1.1 requires transitivity on both vertex parts. The corresponding statement for just one part is false, even for regular edge-transitive graphs. Coregliano suggested this one-sided analogue in [13, Section 7]. We disprove it with the graph in Theorem 1.4, without using the structural classification.

Definition 5.1. A bigraph *left-cut-percolates* if some word of oriented folds maps its whole left part to one vertex. Its *minimum left image size* is the minimum of $|t(A)|$ over all such words t , including the empty word. Reversing the word identifies left cut-percolation with spreading one marked left vertex to the whole left part by successive inverse images [13, Definition 2.4].

Idea. Separate the binary coordinates into a protected part D and an ordinary part B . There are four fold directions. Each fold reads one bit of B and copies by a translation that may also change D . Products of the fold involutions generate every translation, so their group is transitive on points. But points with the same B coordinates receive the same change in D . Their difference survives every copying operation. This leaves eight unavoidable point images. The main task is to prove that these are all the folds.

5.1 The binary-coset construction

All vector additions are over $\mathbb{F}_2 = \{0, 1\}$, with addition and multiplication modulo two. Let $D = \mathbb{F}_2^3$, choose its standard basis a_1, a_2, a_3 , and put $a_4 = a_1 + a_2 + a_3$. Their only nonempty linear dependence is $a_1 + a_2 + a_3 + a_4 = 0$. Let b_1, b_2, b_3, b_4 be the standard basis of $B = \mathbb{F}_2^4$, and set $X = D \oplus B$. Write $x = (d, b)$ and $b = \sum_{i=1}^4 b_i(x)b_i$, where $b_i(x) \in \mathbb{F}_2$ is an ordinary coordinate. For each $i \in \{1, 2, 3, 4\}$, put

$$U_i = \langle a_i, b_i \rangle.$$

A coset $x + U_i$ has the four points $x, x + a_i, x + b_i, x + a_i + b_i$. Let H_4 be the incidence bigraph between the 128 points of X and all these cosets. Each direction has $128/4 = 32$ cosets, giving 128 right vertices as well. Distinct directions give different point sets, so no repeated coset vertices occur.

Theorem 5.2. *The bigraph H_4 is connected, twin-free, C_4 -free, four-regular, and edge-transitive, with 256 vertices. Its fold involutions are exactly the eight translations by b_i and $b_i + a_i$, $1 \leq i \leq 4$. Its fold group is the full translation group of X , transitive on the 128 points and with four orbits on the coset part. Its minimum left fold image size is eight.*

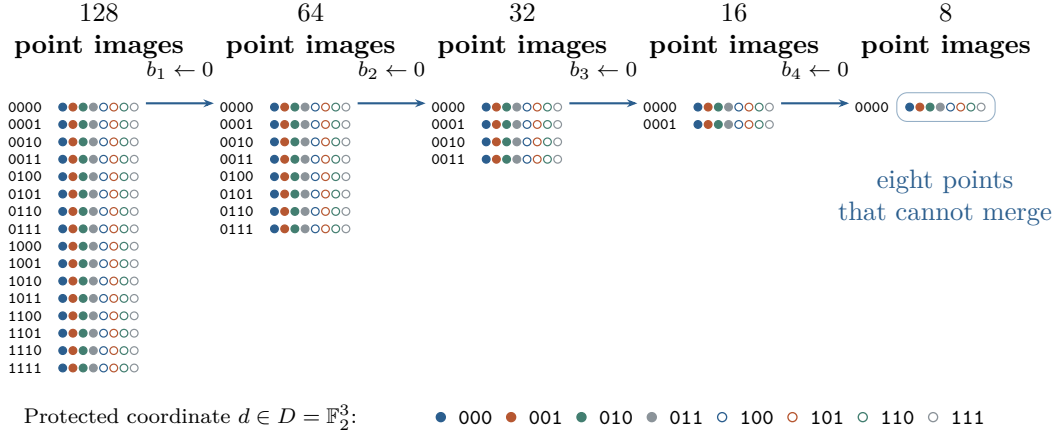


Figure 6: Minimum left images for the binary-coset bigraph H_4 . Only point images are shown; the coset vertices are omitted. Each row is a fiber $D \times \{b\}$, with its four ordinary bits b on the left. The eight protected coordinates $d \in D$ are distinguished by color and by filled or hollow markers. Orienting translations by b_1, b_2, b_3, b_4 successively toward zero leaves 128, 64, 32, 16, 8 point images. Any two points in one row receive the same translation under every oriented fold, so their difference in D cannot disappear. The last eight images are unavoidable, even though the fold involutions generate a point-transitive group (Theorem 5.2).

Connectedness follows since the U_i span X . Distinct points have different incident cosets because $U_i \cap U_j = 0$ for $i \neq j$, and distinct cosets have distinct neighborhoods; thus the graph is twin-free. Two cosets in distinct directions meet in at most one point, while cosets in one direction are disjoint; this also proves C_4 -freeness. Each point is on one coset in each direction. Each coset has four points, proving four-regularity. Translations act on the graph. Every permutation of $a_1, \dots, a_4$ extends to a linear automorphism of D , since the displayed dependence is unique; permuting the b_i in the same way permutes the directions. These automorphisms act transitively on flags, hence on edges.

5.2 All automorphisms

We determine the part-preserving automorphisms before classifying folds. This prevents a choice of coordinates from hiding additional separating symmetries. Call two points *collinear* if they belong to a common coset vertex. A *parallel class* is the family of cosets of one U_i .

Lemma 5.3. *The parallel classes are intrinsic. More precisely, form a graph on coset vertices by joining two disjoint cosets when at least four pairs of their points are collinear. Its connected components are exactly the four parallel classes.*

Proof. For two disjoint cosets of the same U_i , every allowed translation in some U_j , $j \neq i$, taking one coset to the other supplies four collinear point pairs. All such translations generate X/U_i , so these joins connect the whole parallel class.

For different directions, translate one coset to U_i and write the other as $x + U_j$. Disjointness means $x \notin U_i + U_j$. Each collinear pair corresponds uniquely to a vector in $(x + U_i + U_j) \cap U_k$ for $k \neq i, j$, because $U_i \cap U_j = 0$. Any three of the a_i are independent, so each of the two remaining U_k has trivial intersection with $U_i + U_j$. Each contributes at most one pair. Thus different parallel classes have at most two collinear pairs and never reach the threshold four. $\square$

Lemma 5.4. *Every part-preserving automorphism is generated by translations, simultaneous permutations of the a_i, b_i , and the independent shears*

$$b_i \mapsto b_i + a_i.$$

In particular, the full part-preserving automorphism group has order $2^{11}4! = 49152$.

Proof. An automorphism permutes the parallel classes. Delete all coset vertices of direction i . The remaining graph has exactly two components, distinguished by the bit $b_i(x)$: the other U_j span the codimension-one space $\ker b_i$, since $a_i = \sum_{j \neq i} a_j$. Thus an automorphism acts on the b coordinates by a coordinate permutation and independent bit flips. Compose with the corresponding known automorphism so it preserves each direction and every b fiber.

Within a fiber, collinearity in direction i is the matching $d \leftrightarrow d + a_i$. Preserving all these named matchings forces the action on that fiber to be a translation $d \mapsto d + \delta_b$, because the a_i span D . Across two fibers differing only in bit i , the coset incidences force

$$\delta_{b+b_i} + \delta_b \in \langle a_i \rangle.$$

For distinct i, j , a mixed difference of δ therefore lies both in $\langle a_i \rangle$ and $\langle a_j \rangle$, whose intersection is zero. Hence the difference in direction i does not depend on the other bits. Consequently

$$\delta_b = \delta_0 + \sum_i \varepsilon_i b_i(x) a_i, \quad \varepsilon_i \in \mathbb{F}_2.$$

This is a translation followed by the asserted shears. All these choices are distinct: the action on b determines the signed coordinate permutation, and then the four differences and the value at $b = 0$ determine the shear bits and δ_0 . Counting gives $2^7 \cdot 2^4 \cdot 4!$. $\square$

5.3 All fold involutions

Let N be the point-coset incidence matrix. Its product $NN^\top$ is the convolution operator $f(x) \mapsto \sum_i \sum_{u \in U_i} f(x + u)$. The real characters of X are the functions $x \mapsto (-1)^{\ell(x)}$ for linear functionals $\ell : X \rightarrow \mathbb{F}_2$. A character's eigenvalue is four times the number of U_i on which ℓ vanishes. The constant eigenvalue is 16. The largest nonconstant eigenvalue is 12, with eigenspace spanned exactly by the four characters $(-1)^{b_i(x)}$. Indeed, a functional nonzero on D is nonzero on at least two a_i , since their sum is zero. A functional zero on D attains three vanishing subspaces exactly when it is one coordinate b_i .

For the combinatorial Laplacian, the smallest positive eigenvalue is $\theta = 4 - 2\sqrt{3}$, the smaller root of $(4 - \theta)^2 = 12$. All other positive eigenvalue branches are larger. Thus the Fiedler vertex map on points, up to a common positive scale, is

$$(d, b) \mapsto ((-1)^{b_1}, \dots, (-1)^{b_4}).$$

On a direction- i coset its i -th coordinate is zero and all other coordinates are the corresponding signs, multiplied by the common positive factor $4/(4 - \theta)$. The spectral quotient is the subdivision of the four-dimensional cube, obtained by inserting one new vertex in every cube edge: its point vertices are ordinary bit strings, and its coset vertices join the two strings differing in one coordinate. A point fiber is D , while a direction- i coset fiber is $D/\langle a_i \rangle$. Thus the quotient remembers the ordinary bits but loses the protected coordinates. The automorphism action on this eigenspace is the group of signed coordinate permutations. Its reflections are coordinate sign flips and signed transpositions of two coordinates.

Lemma 5.5. *The atomic folds are exactly translation by b_i or $b_i + a_i$.*

Proof. First rule out invisible folds, whose Fiedler action is the identity. By Lemma 5.4, such an involution fixes b and acts by $d \mapsto d + \delta_b$. Fix a moved point (d, b) , so $\delta_b \neq 0$. A direction- j coset through it is fixed exactly when $\delta_b \in \langle a_j \rangle$. If δ_b is none of the a_j , every direction is moved, and their a -vectors span D . If $\delta_b = a_i$, every direction except i is moved, and $a_i = \sum_{j \neq i} a_j$. In either case, add a suitable sequence of a -vectors from moved directions to connect (d, b) to $(d + \delta_b, b)$. All intermediate points keep b and remain moved; the cosets used along the path are moved as well. Thus a moved component contains a point and its image and is preserved by the involution, contrary to the fold condition.

An atom visible on the Fiedler space must be one of its reflections. A signed transposition of coordinates i, j fixes all spectral points with the corresponding equal or opposite bits. The atomic Fiedler lemma would require it to fix every point of each such D fiber. But its action on that fiber has linear part exchanging a_i, a_j , which is nontrivial, followed by a translation. It cannot fix the whole fiber. This rules out these reflections.

For a coordinate flip i , the mirror contains exactly the direction- i coset vertices. The lemma requires all these cosets fixed individually. In the normal form of Lemma 5.4, this forces $\varepsilon_j = 0$ for $j \neq i$ and $\delta_0 \in \langle a_i \rangle$. The involution condition also forces $\varepsilon_i = 0$. The only possibilities are translation by b_i and by $b_i + a_i$.

Both are atoms. Their fixed vertices are exactly the direction- i cosets, and deleting these leaves the two connected components distinguished by $b_i(x)$, which the translation exchanges. $\square$

Proof of Theorem 5.2. Every atom moves the entire point part. A fold cannot decompose into two or more atoms, whose moved supports must be disjoint. Thus every genuine fold is one of the listed atoms. These translations generate every a_i, b_i and hence all of X . Translations are transitive on points and on each direction class of cosets, and never change direction.

Each oriented fold reads only bit $b_i(x)$ and adds one of $b_i, b_i + a_i$ when copying that side. Points differing by a vector in D have identical bits and receive identical additions. Their difference is therefore preserved by every folding word. The eight points in any one D fiber can never merge. Conversely, for each i orient translation by b_i toward $b_i(x) = 0$. These four folds send all points to $D \oplus 0$, attaining exactly eight images. $\square$

A stable two-color witness. Choose a nonzero linear functional $\ell : D \rightarrow \mathbb{F}_2$ and color every edge at point (d, b) by $\ell(d)$. This is a nonconstant two-color star coloring: every edge at a point has its color. It is fold-stable in the sense of Definition 2.15, even though the fold group is point-transitive. For an oriented atom in direction i , the new point color is $\ell(d + g(b_i)a_i)$, where g is either zero or the indicator of the copied bit. The map $(d, b) \mapsto (d + g(b_i)a_i, b)$ is an automorphism from Lemma 5.4; it identifies the new colored bigraph with the original one. All folds of H_4 are atoms, so this covers every oriented fold. The witness records the protected coordinate rather than a point orbit of the fold group.

Analytic scope. Unlike a counterexample lacking edge transitivity, this graph passes Sidorenko's basic necessary symmetry condition. Nevertheless H_4 is not weakly norming. Appendix C proves this by a local distance obstruction and the deleted-root profile inequality of Lemma A.4.

Proof of Theorem 1.4. Take H_4 . Theorem 5.2 gives the combinatorial assertions, and Corollary C.2 gives the analytic one. $\square$

Relation to a known graph. The graph H_4 agrees with the 256-vertex graph on line 687 of the semisymmetric tetravalent catalogue [18, 25]. Our statement concerns its complete fold classification and the failure of the one-sided criterion. The proof uses no catalogue computation.

Acknowledgments

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A Cut-percolating incidence bigraphs of simple two-designs

Definition A.1. A *simple* $2-(v, k, \lambda)$ design $\mathcal{D} = (X, \mathcal{B})$ has v points and a family $\mathcal{B}$ of distinct k -element subsets, called *blocks*, with each pair of distinct points in exactly λ blocks. Its *incidence bigraph* has parts $X, \mathcal{B}$ and joins x to V when $x \in V$. The design is *complete* if $\mathcal{B} = \binom{X}{k}$, *symmetric* if $|\mathcal{B}| = v$, and *nontrivial* here if $2 \leq k < v$ and $\lambda \geq 1$. A *flag* is an incident point–block pair. Design *automorphisms*, denoted $\text{Aut}(\mathcal{D})$, are point permutations preserving $\mathcal{B}$; the design is *flag-transitive* if these act transitively on its flags.

This appendix proves Theorem 1.3 independently of the structural classification. Its main analytic step is that a weakly norming incidence bigraph of a simple two-design, even after a uniform twin blowup, forces the design to be complete. Figure 7 compares the cube, coming from the complete design on four points, with the Heawood graph, coming from the Fano plane.

Complete designs give reflection bigraphs. The known implications from reflection bigraphs to cut-percolation and from cut-percolation to weak norming [10] give one direction. For the converse, we prove that every incomplete design violates decorated Hölder. Thus all the equivalences within this family follow without using Theorem 1.1.

Decorated Hölder gives a positivity condition on neighborhood profiles, the counts of labels on neighboring blocks after a root block and all its copies are deleted (Definition A.3). For a design, two profiles suffice: the identity assignment supplies an off-diagonal entry, and positivity forces a diagonal assignment collapsing two points to one. Such a collapse either contradicts the design’s *closure lines*, intersections of all blocks through two points (Section A.4), or gives an automorphism transposing two points. Applying the argument across a suitable binary partition forces all point transpositions, hence all k -element blocks.

The analytic step must delete a whole right twin class and retain separate activities inside and outside its neighborhood. We give it in full. The underlying weighted rooted homomorphism matrix method comes from Král’, Martins, Pach, and Wrochna [21, Theorem 3].

A.1 A correlation consequence of decorated Hölder

We begin with a local consequence of Hölder. Apply the same perturbation to two groups of edges, with parameters s and t . The logarithmic gap in Hölder is nonnegative and vanishes when $s = t$. Its mixed second derivative is therefore constrained. This gives the following correlation inequality.

Use the finite-state model of (3), with a strictly positive common kernel W and positive activity vectors on each target part. A real *observable* is a function of the assigned target states. Let $\mathbb{E}_W$ denote expectation under the normalized assignment measure, and write $\text{Cov}_W(X, Y) = \mathbb{E}_W[XY] - \mathbb{E}_W[X]\mathbb{E}_W[Y]$ for covariance. The Hessian of a function of



Figure 7: Two three-point-block designs. Left: the Fano plane, with its seven blocks drawn as the three sides, three medians, and the circle; incidences occur only at the marked points. Every closure line, the intersection of the blocks through two points (see (19)), has three points. Its incidence bigraph is not cut-percolating. Right: the incidence bigraph of all three-subsets of a four-point set is the cube Q_3 , a reflection bigraph and hence cut-percolating. The colored line or star represents one block in each picture; A and B are two of its incident points. The classification also holds after every uniform twin blowup.

(s, t) is its matrix of second partial derivatives. A real symmetric matrix M is *positive semidefinite* if

$$z^\top M z \geq 0 \quad \text{for every real vector } z. \quad (10)$$

A *principal submatrix* retains the same selected row and column indices; its determinant is a *principal minor*. Positive semidefiniteness forces every such minor to be nonnegative.

Lemma A.2. Suppose H is weakly norming, $E(H) = E_1 \sqcup E_2$, and F is a real observable on a pair of target states. Under the common-kernel model, put

$$X = \sum_{uv \in E_1} F(x_u, y_v), \quad Y = \sum_{uv \in E_2} F(x_u, y_v).$$

Then $\text{Cov}_W(X, Y) \geq 0$.

Proof. Let $Z(s, t)$ be the partition function with kernels We^{sF} on E_1 and We^{tF} on E_2 . With $p = |E_1|/|E(H)|$ and $q = 1 - p$, decorated Hölder gives

$$D(s, t) := p \log Z(s, s) + q \log Z(t, t) - \log Z(s, t) \geq 0.$$

Moreover $D(r, r) = 0$ for all r . Its Hessian at the origin is positive semidefinite and annihilates $(1, 1)$: the second derivative in that direction is zero, and a positive semidefinite matrix with zero quadratic form on a vector annihilates that vector. Hence the Hessian has the form $c \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ with $c \geq 0$. The mixed derivative of D is $-\text{Cov}_W(X, Y)$, so $c = \text{Cov}_W(X, Y)$. $\square$

There is a useful way to remove the centering. Partition the left source vertices into C_* and D_* . Choose a right target state j and replace it by two identical states j^+, j^- of equal half-weight. Let $\psi(j^+) = 1$, $\psi(j^-) = -1$, and let $\psi = 0$ on every other state. Apply Lemma A.2 to $F(x, y) = f(x)\psi(y)$ and to the edge partition determined by C_*, D_* . Conditional on the unsplit state labels, the signs are independent and centered. Thus $\mathbb{E}X = \mathbb{E}Y = 0$, and terms from different right source vertices vanish. Write ξ_u for the target state of source vertex u . The inequality becomes

$$\sum_{z \in B(H)} \Pr(y_z = j) \mathbb{E} \left[\left(\sum_{u \in N(z) \cap C_*} f(\xi_u) \right) \left(\sum_{u \in N(z) \cap D_*} f(\xi_u) \right) \middle| y_z = j \right] \geq 0. \quad (11)$$

Zero kernels are allowed by approximation.

A.2 Deleted-root profile positivity

The correlation inequality concerns functions of target states. For the design argument we need functions of *neighborhood profiles*: the labels seen by a point after its root block is deleted. The next lemma makes this conversion. We state it for general biregular bigraphs, since the analytic argument uses no design structure.

Definition A.3. Let $G = (X, Y; E)$ be a connected twin-free biregular bigraph whose automorphism group is transitive on Y . Fix $a, b \geq 1$ and a partition $X = C \sqcup D$. Choose a distinguished right vertex $v \in Y$, the *root*, and delete all b right clones above v from $G(a, b)$. Call the left classes in $N_G(v)$ *inner*, and the remaining left classes *outer*.

Use a finite right target alphabet, a set of target states $\mathcal{J}$, a common right activity vector, and a common nonnegative kernel K . Allow separate common left activities for the inner and outer classes. They may also use different finite left alphabets, treated as a disjoint union for the common kernel. For each inner class u , define its *profile* $h_u \in \mathbb{N}^{\mathcal{J}}$ to be the histogram of target labels on all its remaining neighboring right clones. Its total size is $b(\deg_G(u) - 1)$. Every left clone in that class has this same profile. Write $x_{u,i}$ for the target state of the i -th left clone above u . The coordinate $h_u(j)$ counts right clones with target label j adjacent to u after the deletion. Only histograms of this fixed total size index the profile matrix.

Figure 8 illustrates the definition.

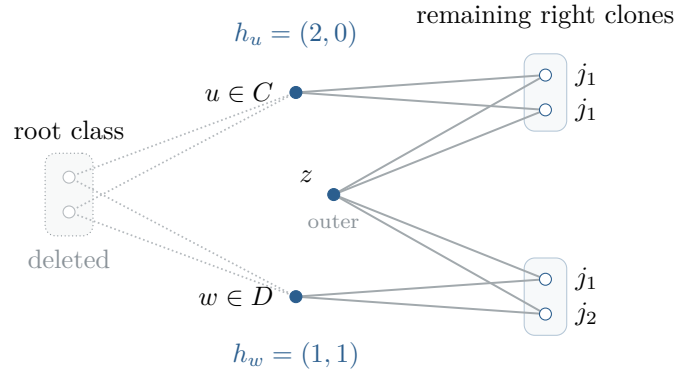


Figure 8: Deleted-root profiles in $C_6(1, 2)$. Dotted edges and the two faded right vertices are removed together. The root neighbors u, w are inner classes, and z is outer. After assigning labels from $\mathcal{J} = \{j_1, j_2\}$ to the remaining right clones, h_u and h_w record their label counts. These are the profiles of Definition A.3. For general multiplicities, all left twins above one base vertex see the same profile.

Let Z_v denote the deleted-root partition function and $\mathbb{E}_v$ its normalized expectation, with $\Pr_v$ the associated probability. Define a symmetric profile matrix by

$$M_{h, h'} = \frac{1}{2} \sum_{v \in Y} Z_v \sum_{\substack{u \in N(v) \cap C \\ w \in N(v) \cap D}} (\Pr_v(h_u = h, h_w = h') + \Pr_v(h_u = h', h_w = h)). \quad (12)$$

If $Z_v = 0$, its contribution means the unnormalized joint weight, which is well-defined as a finite polynomial.

Lemma A.4 (Colored deleted-root profiles). *If $G(a, b)$ is weakly norming, then M in (12) is positive semidefinite for every such finite model. Equivalently, for every real function g on the finite set of possible profiles,*

$$\sum_{v \in Y} Z_v \sum_{\substack{u \in N(v) \cap C \\ w \in N(v) \cap D}} \mathbb{E}_v[g(h_u)g(h_w)] \geq 0. \quad (13)$$

Proof. We use three limits in order. First we remove the root class while fixing the background model. Next, with that class removed, we interpolate profile functions using rare rows and send their mass to zero. Finally we separate the inner and outer alphabets and pass to zero parameters. These are successive limits, not a joint limit. Start with positive old kernels and activities; boundary cases follow at the end. If $|Y| = 1$, twin-freeness and connectedness force $G = K_{1,1}$. Then X is a singleton, one of C, D is empty, and $M = 0$. Assume henceforth that $|Y| \geq 2$.

Deleting the entire right twin class. Duplicate each old left state x into an ordinary state of activity μ_x and a marked state of activity $t\mu_x$. Old right columns ignore the mark. Add a right state j of activity t^{-c} , with kernel value $\ell_x^{1/b}$ on the marked copy of x and zero on its ordinary copy. Here $\ell_x > 0$ is a prescribed inner *unary multiplier*, a factor depending only on the left state. Let d be the degree of a right vertex in G , and choose

$$0 < c < \frac{a}{b(|Y| - 1)}.$$

Condition one named right clone above v to have label j . For a configuration, let k_z be the number of j -clones above each right class z , let $T = \{z \neq v : k_z > 0\}$, and let e be the number of marked left clones beyond the ones forced by these j -clones. Up to a common conditioning factor, its power of t is

$$ad + a|N_G(T) \setminus N_G(v)| + e - c \sum_z k_z. \quad (14)$$

If $T \neq \emptyset$, twin-freeness and equal right degrees imply $|N_G(T) \setminus N_G(v)| \geq 1$. Relative to the configuration $k_v = b$, $T = \emptyset$, $e = 0$, the exponent is larger by at least $a - cb(|Y| - 1) > 0$. If T is empty, either missing a j -clone at v or adding an unnecessary marked clone also strictly increases the exponent. Therefore the unique leading family has all b root clones at j , no other right clone at j , and exactly the inner left clones marked.

As $t \downarrow 0$, the conditional model converges to the model with the whole root class deleted. Each inner clone acquires the factor ℓ_x , since its b incident root entries contribute $(\ell_x^{1/b})^b = \ell_x$. Outer activities remain μ_x . Apply (11) to this new right state, with C_*, D_* the clone unions above C, D , and let the observable on marked left states be an arbitrary function f of their old labels. Right transitivity of G and clone symmetry make $\Pr(y_z = j)$ equal for every right source clone before taking the limit. Dividing out this positive common probability gives, in the limit,

$$\sum_{v \in Y} \sum_{\substack{u \in N(v) \cap C \\ w \in N(v) \cap D}} \mathbb{E}_v \left[\left(\sum_{i=1}^a f(x_{u,i}) \right) \left(\sum_{i=1}^a f(x_{w,i}) \right) \right] \geq 0. \quad (15)$$

The sum over the b clones of each root contributes a common factor b , which has also been divided out. All Z_v are equal, including when inner and outer activities differ. To see this, an automorphism of G taking v to v' takes $N(v)$ to $N(v')$ and its complement to the corresponding complement. Lift it to the clones, retaining each clone index. It gives a

weight-preserving bijection of assignments between the two deleted-root backgrounds: the kernel and right activities are common, and the left activity vectors are common within the inner and outer sets separately. The two vectors need not agree with each other. The source coloring occurs only in the observable sums, so this automorphism need not preserve C, D . Multiplying (15) by the common positive partition function gives its unnormalized version.

Interpolating an arbitrary profile function. For a positive deleted-root background, the conditional partition factor of one inner left clone with profile h is

$$Z(h) = \sum_x \mu_x \ell_x \prod_{j \in \mathcal{J}} K(x, j)^{h_j} > 0.$$

Recall that a kernel row records its values at all right target states. Add finitely many *rare* left rows, whose activities will tend to zero, with strictly positive row vectors $w_i = (w_{i,j})_{j \in \mathcal{J}}$, masses ρc_i with $c_i > 0$, and inner multipliers one. In this proof $O(\rho^d)$ means a quantity bounded by a constant times ρ^d as $\rho \downarrow 0$, with the finite background and the chosen rare rows fixed. Let f be zero on old rows and equal to $f_i \in \mathbb{R}$ on rare row i . Write $w_i^h = \prod_j w_{i,j}^{h_j}$, $B(h) = \sum_i c_i f_i w_i^h$, and $S(h) = \sum_i c_i w_i^h$. The exact conditional mean and its expansion are

$$\mathbb{E}[f(x) \mid h] = \frac{\rho B(h)}{Z(h) + \rho S(h)} = \rho \frac{B(h)}{Z(h)} + O(\rho^2).$$

There are finitely many profiles and $\min_h Z(h) > 0$. The error is therefore uniform over all profiles for this fixed model. The functions $h \mapsto w^h$ with w strictly positive span all functions on the finite profile set. Otherwise a nonzero linear combination of distinct monomials would vanish on the open positive orthant, that is, the open set of vectors with all coordinates positive. A nonzero polynomial cannot vanish on this open set. Choose the rows and coefficients so that $\sum_i c_i f_i w_i^h = Z(h)g(h)$ for all h .

Given all right labels, the left clones are independent. In particular, the two distinct classes $u \in C, w \in D$ in (15) have conditional product mean

$$a^2 \rho^2 g(h_u)g(h_w) + O(\rho^3).$$

The $O(\rho^3)$ error is uniform over the finitely many profiles, roots, and source pairs. The marginal background on right labels changes by $O(\rho)$ uniformly over its finite assignment set, since its weights are polynomials and its normalization is positive at $\rho = 0$. Choose the rows and interpolation coefficients before varying ρ . For each fixed $\rho > 0$, the preceding root-class limit gives the deleted-root inequality. Now divide its unnormalized form by $a^2 \rho^2$ and let $\rho \downarrow 0$. This proves (13) for positive backgrounds with inner activities $\mu_x \ell_x$ and outer activities μ_x .

Independent alphabets and zero entries. To give inner and outer classes disjoint alphabets, assign each desired inner state activity ηm_x and multiplier η^{-1} ; assign each outer state activity m_x and multiplier zero. For each fixed $\eta > 0$, first approach the zero multiplier from positive values, after taking the root-class and rare-row limits. Then let $\eta \downarrow 0$. Inner states retain their desired effective activities, while their outer activities vanish; outer states have zero inner activity. This realizes arbitrary inner and outer activity vectors on disjoint alphabets. The unnormalized entries of M are finite polynomials in the effective activity vectors and kernel entries. A final continuity argument therefore extends the inequality to all nonnegative finite models, including zero partition functions. The earlier error bounds are asserted only for each fixed positive model; no uniformity in these final boundary limits is needed. This proves the lemma. $\square$

Example A.5 (A two-profile witness for the Fano plane). Let $(X, \mathcal{B})$ be the Fano plane, G its incidence bigraph, and V_0 a target line with distinct points A, B . Here *lines* are the design's three-point blocks. The *incidence kernel* is $K(x, J) = 1$ if $x \in J$ and zero otherwise. Take $C = \{A\}$ and $D = X \setminus C$. Use inner target alphabet V_0 , outer alphabet $X \setminus V_0$, right alphabet $\mathcal{B} \setminus \{V_0\}$, and the incidence kernel, with all activities one. Let p_A record b copies of each of the two lines in $N(A) \setminus \{V_0\}$, and define p_B similarly.

Two different inner source classes cannot both have profile p_A . Their left clones would all be at A , since the two target lines in that profile intersect only at A . Any nonroot source line through one class meets any nonroot line through the other at an outer source point. Their target lines must therefore contain both A and a common point outside V_0 , forcing them to coincide. All target lines in both profiles would be equal, contrary to the two distinct lines in p_A . The same argument applies to p_B .

Thus the two diagonal entries are zero. The identity assignment at source root V_0 contributes to the off-diagonal entry, so the principal submatrix is

$$\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \quad c > 0.$$

It has negative determinant. Since the Fano plane is flag-transitive, Lemma A.4 applies under a weak-norming assumption and excludes every $G(a, b)$.

A.3 A two-profile minor in a design

We now choose the finite model that will test a design. The two profiles belong to two target points A, B on a root block V_0 . Give both points a small activity ε . Then the identity assignment contributes to the off-diagonal entry at order ε^{2a} . If the corresponding diagonal entry has higher order, a 2-by-2 principal minor is negative. The rest of the proof determines what a diagonal assignment of order ε^{2a} would force.

Let $\mathcal{D} = (X, \mathcal{B})$ be a design with the parameters of Theorem 1.3, in the notation of Definition A.1. Every point lies in

$$r = \frac{\lambda(v-1)}{k-1} > \lambda$$

blocks. Its incidence bigraph F is connected and biregular. It is twin-free: distinct points have only $\lambda < r$ common blocks, and distinct blocks are distinct point sets.

Assume that $F(a, b)$ is weakly norming. Its known necessary part-preserving edge transitivity descends through the canonical twin classes to F . Hence the design is flag-transitive; in particular its block part is transitive. Lemma A.4 applies.

Fix a partition $X = C \sqcup D$, a target block V_0 , and distinct points $A \in V_0 \cap C$, $B \in V_0 \cap D$. The deleted-root model uses inner target alphabet V_0 , outer alphabet $X \setminus V_0$, right alphabet $\mathcal{B} \setminus \{V_0\}$, and the incidence kernel. All activities are one, except that the inner points A, B have activity $\varepsilon > 0$.

Let p_A contain b copies of each block in $N(A) \setminus \{V_0\}$, and define p_B analogously. The profiles are distinct. If an inner source class has profile p_A , every one of its left clones is at A : after restoring the deleted root class with every clone labeled V_0 , a *virtual root block*, its target point is incident with every member of $N(A)$, and equal point degrees and twin-freeness force that point to be A . The same holds for p_B and B .

The assignment sending every point clone to its base point and every nonroot block clone to its base block, at source root V_0 , contributes to the off-diagonal entry for the source pair A, B . Consequently, for some constant $c_0 > 0$ independent of ε ,

$$M_{p_A, p_B} \geq c_0 \varepsilon^{2a}, \quad M_{p_B, p_B} = O(\varepsilon^{2a}). \quad (16)$$

If no p_A, p_A assignment has degree exactly $2a$ in ε , then $M_{p_A, p_A} = O(\varepsilon^{2a+1})$. The determinant of this two-profile principal submatrix is then negative for small ε , contradicting Lemma A.4.

Call an assignment contributing at degree exactly $2a$ in ε a *minimal assignment*. We record what it would entail. At some source root V , there would be distinct classes $u \in V \cap C$, $w \in V \cap D$ all of whose clones are at A . All other point clones would avoid A, B . Selecting one actual target image from each point class gives a map $f : X \rightarrow X$ with

$$f^{-1}(A) = \{u, w\}, \quad f^{-1}(B) = \emptyset. \quad (17)$$

Select actual target images of all block classes as well, and restore a virtual root row $V \mapsto V_0$. This gives an incidence-preserving map h with point restriction f .

More information comes from the clone counts on $N(u)$. Divide these counts by b to form a matrix P with rows $N(u)$ and columns $N(A)$; the virtual root row has its single entry at V_0 . Every row and column sum is one because the prescribed profile contains exactly b copies of every nonroot column. Thus P is *doubly stochastic*: its entries are nonnegative and each row and column sums to one. Complete adjacency between clone classes implies

$$P_{E,J} > 0 \implies f(x) \in J \text{ for every } x \in E. \quad (18)$$

The selected image of each block in $N(u)$ can be chosen from its positive columns.

A.4 Closure lines and saturated transport

For distinct points u, x , put

$$S_x^u = N(u) \cap N(x), \quad L(u, x) = \{z \in X : S_x^u \subseteq N(z)\}. \quad (19)$$

Thus S_x^u is the set of blocks through both points and $L(u, x)$ is their intersection, called the *closure line*. Each S_x^u has size λ . If z, t are distinct members of $L(u, x)$, then the λ defining blocks contain both, and therefore $N(z) \cap N(t) = S_x^u$. It follows that $L(z, t) = L(u, x)$. These *closure lines* form a linear space: every pair belongs to a unique line, and distinct lines meet in at most one point. Any design block containing two points of a closure line contains the entire line. Call a line *thick* if it has at least three points, and *thin* if it has two.

For $x \notin \{u, w\}$, (17) gives $f(x) \neq A$. By (18), the λ rows in S_x^u send all their mass to the λ columns in $S_{f(x)}^A$. The latter columns have total capacity λ , so they are saturated and no other row can send mass to them. Hence

$$P_{E,J} > 0 \implies (E \in S_x^u \iff J \in S_{f(x)}^A). \quad (20)$$

In particular, distinct source patterns S_x^u, S_y^u have distinct target patterns $S_{f(x)}^A, S_{f(y)}^A$. If the latter sets were equal, their saturated preimages would be equal.

Lemma A.6 (A collapse creates a thin flag). *Any minimal assignment described in (17)–(18) produces a target flag (A, J) such that every closure line through A contained in J has size two.*

Proof. Choose a block E containing w but not u , using $r > \lambda$. Its $k - 1$ other points lie on distinct closure lines through u . Otherwise E would contain two points on such a line and hence the whole line, including u . Their patterns S_x^u are thus distinct. By (20), their images lie on $k - 1$ distinct closure lines through A .

The block $J = h(E)$ contains $A = f(w)$ and all these $k - 1$ images, which are distinct and different from A . They exhaust its k points. If a closure line through A in J had a third point, that point would be one of these images and would duplicate an already represented closure line. This contradicts their distinctness. $\square$

If the design has any thick line, a defining block and a point on that line give a flag whose block contains a thick line through its point. Flag transitivity forces this property at every flag, contrary to Lemma A.6. Thus no minimal assignment exists. Choose any two distinct target points in a block and any partition placing them in opposite classes. The minor (16) contradicts weak norming. We conclude: under our weak-norming assumption, every closure line has size two. Equivalently, for fixed u the patterns S_x^u , $x \neq u$, are all distinct.

A.5 Thin closure lines force a point transposition

Two-point closure lines alone do not force a complete design. For example, complementing the seven blocks of the Fano plane gives a 2 -($7, 4, 2$) design. Two distinct complemented blocks intersect in two points, so every closure line has size two. The additional information comes from the collapse assignment forced by profile positivity. We now show that it produces a point transposition preserving all blocks.

Suppose now that all closure lines have size two and that a minimal assignment exists. By (20), f is injective outside $\{u, w\}$. Together with (17), this gives a bijection

$$f : X \setminus \{u, w\} \longrightarrow X \setminus \{A, B\}. \quad (21)$$

Distinct target blocks through A are distinguished by their incidence with $X \setminus \{A, B\}$. Indeed, two equal-sized sets both containing A cannot become identical after deleting B unless they were identical already. By (20)–(21), all positive columns of any row of P have the same such signature. Each row therefore has only one positive column. Since P is doubly stochastic, it is a *permutation matrix*, with one entry one in each row and column and all others zero. All right classes in $N(u)$ are monochromatic, meaning that every clone in a class has the same target label, and h maps $N(u)$ bijectively to $N(A)$.

Lemma A.7. *For a minimal assignment in a design with only two-point closure lines, the transposition $(u\ w)$ preserves the entire block family.*

Proof. Change only $f(w) = A$ to $\pi(w) = B$. By (21), the resulting map π is a point permutation, with $\pi(u) = A$.

There are $r - \lambda$ blocks through u avoiding w . On such a block E , f is injective, so the k points of $f(E)$ exhaust the image block $h(E)$. Thus $h(E) = f(E) = \pi(E)$ is a block through A avoiding B . The star bijection shows that these images exhaust all $r - \lambda$ target blocks through A avoiding B . The remaining λ blocks through both u, w map bijectively to the target blocks through both A, B . Their image point sets have $k - 1$ points and avoid B , so in that case $h(E) = f(E) \cup \{B\} = \pi(E)$ as well.

For a block E' through w avoiding u , f is again injective on its point set. Therefore $h(E') = f(E') = \pi(u\ w)(E')$. These $r - \lambda$ images are distinct and exhaust the same target family through A avoiding B . Pairing the two image families shows that for every block E through u avoiding w , the set $(u\ w)(E)$ is a block through w avoiding u . The transposition exchanges these two families, and fixes every block containing both points or neither. It preserves all blocks. $\square$

The corresponding involution of the incidence bigraph is a genuine atomic fold. Its nonfixed subgraph consists of two disjoint stars, centered at u, w , whose leaves are the blocks containing exactly their respective center. The two stars are exchanged. The following argument needs only the design automorphism, but this observation explains its connection with the main theorem.

A.6 A binary partition forces the complete design

Let $T \leq \text{Aut}(\mathcal{D})$ be generated by all point transpositions preserving the design. Suppose T has at least two point orbits. Choose one orbit C and put $D = X \setminus C$. Since $\lambda \geq 1$, there is a target block V_0 containing $A \in C$ and $B \in D$.

Apply the colored profile matrix with this partition and these target profiles. Any minimal p_A, p_A assignment must collapse a source pair $u \in C, w \in D$. By Lemma A.7, $(u \ w) \in T$, contrary to their different T -orbits. Thus no minimal assignment exists, and the two-profile determinant from (16) is negative. Weak norming forces T to be transitive.

Form a graph on X whose edges are the pairs whose transpositions preserve the design. Its components are exactly the T -orbits. Transpositions on the edges of a connected graph generate the full symmetric group: swaps along a path generate a transposition between its endpoints. Therefore $T = \text{Sym}(X)$. The nonempty block family is invariant under $\text{Sym}(X)$ and consists of k -sets, so it must be all of $\binom{X}{k}$.

Proof of Theorem 1.3. If $F(a, b)$ cut-percolates, it is weakly norming by [10]. The preceding argument then forces $\mathcal{B} = \binom{X}{k}$.

Conversely, suppose $\mathcal{B} = \binom{X}{k}$. Take $W = \text{Sym}(v)$ with its standard type- A_{v-1} simple system. The stabilizers of $\{1\}$ and $\{1, \dots, k\}$ are standard parabolic subgroups, and their orbit pairs are exactly incidences between singletons and k -subsets. Thus F is reflection. As $F(a, b) = F \otimes K_{a,b}$, every indicated blowup is reflection. The known reflection-to-cut-percolation implication of [10] shows that this blowup cut-percolates and is weakly norming. The preceding analytic argument shows that weak norming forces the design to be complete. Thus completeness implies reflection, reflection implies cut-percolation, cut-percolation implies weak norming, and weak norming implies completeness. This proves all the claimed equivalences independently of Theorem 1.1. $\square$

Corollary A.8. *For a nontrivial simple symmetric 2 -(v, k, λ) design, its incidence bigraph and every uniform twin blowup are cut-percolating exactly when $k = v - 1$, equivalently $\lambda = k - 1$. The same criterion holds for weak norming. These incidence bigraphs are the crown graphs: $K_{v,v}$ with a perfect matching deleted, that is, with v disjoint edges covering all vertices removed.*

Proof. The complete design has $\binom{v}{k}$ blocks. Under $2 \leq k < v$, $\binom{v}{k} = v$ holds exactly at $k = v - 1$. In a symmetric design $r = k$, so $k(k - 1) = \lambda(v - 1)$ and the stated parameter equivalence follows. $\square$

A finite projective plane of order q is a symmetric 2 -($q^2 + q + 1, q + 1, 1$) design. The Fano plane is the order-two example used above.

Corollary A.9. *The incidence bigraph of a finite projective plane of order $q \geq 2$, and every uniform twin blowup of it, is not cut-percolating and is not weakly norming.*

Proof. Its parameters are $v = q^2 + q + 1$, $k = q + 1$, and $\lambda = 1$. For $q \geq 2$ one has $k \neq v - 1$, so Corollary A.8 excludes both properties. $\square$

A.7 The fold group and the exact one-sided obstruction

There is a complementary combinatorial description of the folds of every design in Definition A.1, regardless of whether it is weakly norming. The incidence spectrum puts all atomic folds into the ordinary permutation representation on points. The result below needs only Lemma 3.1; it does not use the structural classification or the analytic argument above. The *minimum left image size* is the least number of point vertices remaining in the image of a word of oriented folds, as in Definition 5.1. Assume throughout this subsection that $2 \leq k < v$ and $\lambda \geq 1$. Write G for the incidence bigraph of the design. The point subspace $\mathbf{1}_X^\perp$ below consists of real functions on X whose values sum to zero.

For a concrete example, partition sixteen points into four classes of four and take every block choosing two points in each of three classes. Swapping two points in one class always preserves the blocks, but swapping points from different classes does not. The fold group is the product of the four within-class symmetric groups, and its minimum left image size is four. Figure 9 also shows the support family used in the following subsection to recover the full automorphism group.

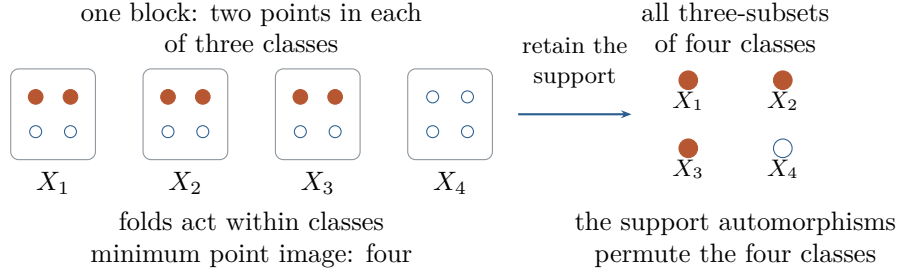


Figure 9: Partition sixteen points into four classes of four, and take all blocks choosing two points in each of three classes. This is the 2 -($16, 6, 108$) design from [1, Construction 4.2] with $(e_1, e_2; k) = (4, 4; 6)$. Its incidence bigraph has 880 vertices. Its fold group is $\text{Sym}(4)^4$, so folds leave at least one point image per class (Theorem A.10). Retaining only which classes a block meets gives the complete three-subset design on four classes. Recovering both levels shows that the full design automorphism group is $\text{Sym}(4) \wr \text{Sym}(4)$, acting independently within classes and permuting the classes (Theorem A.13). The orange points indicate one block and its support.

Call distinct points x, y *exchangeable* if their transposition, fixing every other point, preserves $\mathcal{B}$. Declare every point exchangeable with itself. This is an equivalence relation: conjugating $(x y)$ by $(y z)$ gives $(x z)$. Let $X_1, \dots, X_c$ be its classes.

Theorem A.10. *The atomic folds of G are exactly the automorphisms induced by transpositions of exchangeable points. In particular, as an action on the points and their blocks,*

$$\text{Fold}(G) = \mathfrak{S}_{X_1} \times \cdots \times \mathfrak{S}_{X_c},$$

where $\mathfrak{S}_{X_i}$ means all permutations within X_i , fixing the other points. For every uniform twin blowup $G(a, b)$, the minimum possible image size of its left (point) part under a word of oriented folds is exactly c .

Proof. Every point has degree $r = \lambda(v - 1)/(k - 1)$ and every block has degree k . Let N be the point–block incidence matrix, with entry one for an incidence and zero otherwise. Then

$$NN^\top = (r - \lambda)I + \lambda J,$$

where I is the identity and J the all-ones matrix. Since $k < v$, we have $r > \lambda$. On the point subspace $\mathbf{1}_X^\perp$ the matrix $NN^\top$ is the positive scalar $r - \lambda$. On the constants its eigenvalue is rk .

Put θ for the smaller root of

$$(r - \theta)(k - \theta) = r - \lambda.$$

It lies strictly between zero and $\min\{r, k\}$. The Fiedler eigenspace, the lowest nonconstant Laplacian eigenspace from Section 3.1, is exactly

$$\left\{ \left(u, \frac{N^\top u}{k - \theta} \right) : u \in \mathbf{1}_X^\perp \right\}.$$

Indeed, the two nonconstant eigenvalue branches come from that quadratic equation; its larger root exceeds $\max\{r, k\}$. The remaining eigenvalues are $0, r + k$ on the constant vectors and possibly k on $\ker N$. Restriction to the point coordinates thus identifies the Fiedler representation of every graph automorphism with its ordinary permutation action on $\mathbf{1}_X^\perp$. That action is faithful, meaning that only the identity acts as the identity: fixing every point fixes every block because the design is simple.

Lemma 3.1 says an atom acts as a reflection or the identity on this space. It cannot act as the identity by faithfulness. An involutive point permutation has one negative direction on $\mathbf{1}_X^\perp$ for each disjoint point transposition in its cycle decomposition. Hence an atom must transpose exactly two points.

Conversely, if $(x y)$ preserves the design, the moved vertices of the induced incidence automorphism are x, y and the blocks containing exactly one of them. The moved subgraph consists of two exchanged stars, centered at x, y . Both stars have $r - \lambda > 0$ leaves. Thus this automorphism is an atomic fold. All permutations within each exchangeability class are generated by these transpositions, giving the formula.

Corollary B.6, whose group formula needs no cut-percolation assumption, shows that every fold of $G(a, b)$ preserves the union of left clones above each X_i . No word can merge points from different classes, so its left image has at least c vertices. In the base graph, orient transposition folds within X_i toward one selected point to collapse that class. Lift these folds while retaining clone indices; then collapse the remaining clones by twin transpositions. This attains exactly one image vertex per class, hence c . $\square$

Corollary A.11. *For every such design and every $a, b \geq 1$, the following are equivalent: left cut-percolation of $G(a, b)$; transitivity of its fold group on the point part; completeness of the design.*

Proof. The first two properties both mean $c = 1$. Then all point transpositions preserve the design, so the full symmetric group preserves it. This group is transitive on the k -subsets; the nonempty block family must contain them all. Conversely, a complete design has every transposition, so $c = 1$. $\square$

Corollary A.12. *If $\lambda = 1$ and $k \geq 3$, the base incidence bigraph has no nonidentity fold. The same holds for every incomplete symmetric two-design, where symmetric means $|\mathcal{B}| = v$.*

Proof. In a Steiner two-design, meaning $\lambda = 1$, a point transposition $(x y)$ would take a block containing x but not y to a different block sharing $k - 1 \geq 2$ points. Those common points would occur together twice, contradicting $\lambda = 1$.

For a symmetric design, $r = k$ and N is square and invertible. The induced point and block permutation matrices are conjugate by N , so a point transposition would move exactly two blocks. They differ only in exchanging x, y and thus intersect in $k - 1$ points. Distinct blocks in a symmetric design meet in λ points, so $\lambda = k - 1$. The parameter identity $k(k - 1) = \lambda(v - 1)$ gives $k = v - 1$, the complete design. Apply Theorem A.10 in both cases. $\square$

A.8 An automorphism question for nested designs

The exchangeability classes also recover a prescribed partition without knowing it in advance. Repeating this operation recovers an entire nested chain. Amarra, Devillers, and Praeger [1, Problem 2] asked whether the full automorphism group of their flag-transitive two-designs is the iterated wreath product used in their construction. They proved this for chains of length two and three. We give the equality for all lengths. The reconstruction itself applies to a larger class of set families and needs no two-design assumption.

Recovering a nested partition. Let $s \geq 1$, and let $e_1, \dots, e_s$ and $t_1, \dots, t_s$ be integers with

$$2 \leq t_i < e_i \quad (1 \leq i \leq s).$$

Put $X = [e_1] \times \dots \times [e_s]$, where $[e] = \{1, \dots, e\}$. For $0 \leq i \leq s$, the partition $\mathcal{C}_i$ fixes the coordinates $i + 1, \dots, s$, so its classes have size $c_i = \prod_{j=1}^i e_j$. Set $y_0 = 1$ and $y_i = \prod_{j=1}^i t_j$. Let $\mathcal{B}(e; t)$ be all y_s -element subsets $B \subseteq X$ such that

$$|B \cap C| \in \{0, y_i\} \quad (C \in \mathcal{C}_i, 0 \leq i \leq s).$$

Equivalently, at the top choose t_s of the e_s classes; inside each chosen class choose t_{s-1} children, and continue independently to the bottom. The group preserving the partition chain is the iterated imprimitive wreath product $W = \text{Sym}(e_1) \wr \dots \wr \text{Sym}(e_s)$: arbitrary permutations within each bottom class, together with arbitrary permutations of sibling classes at every level. Every element of W preserves $\mathcal{B}(e; t)$.

Theorem A.13. *Every permutation of X preserving $\mathcal{B}(e; t)$ preserves the whole partition chain. In particular,*

$$\text{Aut}(X, \mathcal{B}(e; t)) = W.$$

Proof. For $s = 1$ the family consists of all t_1 -subsets and has automorphism group $\text{Sym}(e_1)$. Assume $s \geq 2$. The exchangeability relation defined above makes sense for any nonempty set family, with the same conjugation proof. It is invariant under every automorphism of that family.

For our family, its classes are exactly $\mathcal{C}_1$. Every transposition within such a class preserves the family, because it does not change any of the required intersection sizes. Now let x, y be in different bottom classes C, D . There is a member of $\mathcal{B}(e; t)$ meeting both C, D : at the first parent containing them, choose both relevant children, which is possible because every $t_i \geq 2$; continue the required choices above and below that parent. Within C choose its t_1 points to contain x , and within D choose them to avoid y , which is possible since $t_1 < e_1$. The resulting block B has $x \in B$, $y \notin B$, and meets both classes in t_1 points. Its image under $(x y)$ meets C in $t_1 - 1$ points. Since $1 \leq t_1 - 1 < t_1$, that image is not a block. Thus x, y are not exchangeable.

Consequently every automorphism preserves $\mathcal{C}_1$. For each block retain only its *support*, the set of bottom classes it meets. The resulting family on the $\mathcal{C}_1$ -classes is precisely $\mathcal{B}((e_2, \dots, e_s); (t_2, \dots, t_s))$. Conversely, every support in this family and every independent

choice of t_1 points within its classes give an original block. Thus the kernel of the induced automorphism action is $\text{Sym}(e_1)^{\prod_{i=2}^s e_i}$, and every support-family automorphism lifts by arbitrary bijections of the bottom classes. Apply induction to the support family. It recovers all the remaining levels and gives the stated iterated wreath product. $\square$

The published design problem. In Construction 4.2 of [1], set $v = \prod_i e_i$, choose $1 < k < v$, and put

$$y_i = 1 + \frac{k-1}{v-1}(c_i - 1), \quad y_0 = 1.$$

The arithmetic hypotheses there imply, by [1, Lemma 4.1], that all ratios $t_i = y_i/y_{i-1}$ are integers with $1 < t_i < e_i$. Lemma 3.2 of [1] identifies the orbit of the specified generating block under W with all the sets having the required intersection sizes. This is exactly the family in Theorem A.13. Hence:

Corollary A.14. *For every two-design $\mathcal{D}^{\text{ft}}(e; k)$ in [1, Construction 4.2],*

$$\text{Aut}(\mathcal{D}^{\text{ft}}(e; k)) = \text{Sym}(e_1) \wr \cdots \wr \text{Sym}(e_s).$$

Thus the equality in [1, Problem 2] holds for every chain length.

For a two-design in this construction, the first partition is also the point-orbit partition of its fold group by Theorem A.10. The fold group is therefore $\text{Sym}(e_1)^{v/e_1}$, and its minimum left image size is v/e_1 , including uniform twin blowups. Thus these known designs provide explicit examples with many stable point classes, although the full automorphism group is flag-transitive.

B Some consequences of the classification

The classification turns questions about degrees and short cycles into restrictions on finite Coxeter diagrams. We record a rank bound, a finiteness criterion, the intrinsic fold group, and three explicit lists. The positive examples in the lists were already known to Conlon and Lee [10]; the new deduction is that the lists are exhaustive under their stated hypotheses. These statements concern cut-percolating bigraphs, not all weakly norming bigraphs.

We use the Coxeter diagram and rank from Section 2.3. The finite irreducible diagrams have the following standard names [20]. The diagram A_r is a path with all labels three; B_r is a path with one terminal label four; D_r has three paths of lengths 1, 1, $r-3$ meeting at one node, with all labels three. The diagrams E_6, E_7, E_8 instead have path lengths 1, 2, $r-4$. The diagram F_4 is a four-node path with labels 3, 4, 3; H_3, H_4 have a terminal label five and all other labels three. Finally, $I_2(m)$ consists of two nodes joined with label m . An isolated node is A_1 .

The standard parabolic intersection identity is $W_I \cap W_J = W_{I \cap J}$ [20]. The subgroup index $[U : V]$ is the number of cosets of V in U . Our reflection presentations have $S = I \cup J$, as required in Definition 2.10. Any entire diagram component in $I \cap J$ cancels from both coset spaces. We call a presentation with these components removed *effective*.

B.1 Degree bounds and finiteness

Degrees (a, b) mean that every left vertex has degree a and every right vertex has degree b .

Lemma B.1. *If (W, S) is irreducible of rank r and $T \subsetneq S$, then $[W : W_T] \geq r + 1$.*

Proof. In the real reflection representation, choose a nonzero point x in the closed fundamental chamber with stabilizer exactly W_T : take a point on the walls indexed by T and strictly inside all other walls. Its orbit has $[W : W_T]$ points. The linear span of the orbit is a nonzero invariant subspace, so it is all of $\mathbb{R}^r$ by irreducibility. The sum of the orbit points is invariant under W and hence zero. Therefore these spanning points are linearly dependent, and their number is at least $r + 1$. $\square$

Theorem B.2 (Rank bound and degree finiteness). *Let $a, b \geq 1$ be fixed integers.*

- (i) *Every connected cut-percolating bigraph of degrees (a, b) has an effective reflection presentation of rank at most $a + b - 2$.*
- (ii) *There are finitely many such bigraphs up to isomorphism if and only if at least one of a, b is odd.*

Proof. Choose an effective presentation with $S = I \cup J$ and put $K = I \cap J$. The degrees are

$$a = [W_I : W_K], \quad b = [W_J : W_K].$$

Call a component of the diagram induced on I *active* if it contains a node outside K . If these active components have ranks $r_1, \dots, r_t$ and relative parabolic indices $d_1, \dots, d_t$, then

$$a = \prod_{i=1}^t d_i, \quad \sum_{i=1}^t r_i \leq \sum_{i=1}^t (d_i - 1) \leq a - 1.$$

The first inequality follows from Lemma B.1. The second follows by repeatedly using $(u - 1) + (v - 1) \leq uv - 1$ for $u, v \geq 1$. The analogous sum of active ranks for J is at most $b - 1$. Every node of $I \setminus K$ or $J \setminus K$ belongs to an active component. Every component of the diagram on K must touch one of these nodes, since otherwise it would be an entire component of S discarded from the effective presentation. Thus the active I - and J -components together cover S , giving $|S| \leq a + b - 2$.

At bounded rank, the only unbounded parameter in the finite Coxeter classification is the label m in a dihedral component $I_2(m)$. Consider its two nodes. If one lies in $I \setminus J$ and the other in $J \setminus I$, this component contributes a factor two to each of a, b . In every other effective assignment with $I \cup J$ containing both nodes, one parabolic contains both nodes and the other contains zero or one. The nontrivial local degree is then $2m$ or m , so $m \leq \max\{a, b\}$. If at least one degree is odd, the unbounded assignment is impossible. There are then only finitely many effective diagrams of bounded rank and bounded labels, and only finitely many choices of I, J in each. This proves finiteness.

If both degrees are even, start with C_{2m} and replace every left vertex by $b/2$ twins and every right vertex by $a/2$ twins. The resulting bigraph has degrees (a, b) . It is a reflection bigraph: the cycle uses $I_2(m)$, and uniform twin duplication is obtained by taking its tensor product with the complete bigraph $K_{b/2, a/2}$, itself a reflection bigraph for a product of symmetric groups. Products of Coxeter systems realize these tensor products. The number of vertices grows with m , giving infinitely many pairwise nonisomorphic examples. $\square$

B.2 The intrinsic fold group

The reflection representation in Theorem 1.1 need not use the entire fold group. The following refinement identifies that group once twins are removed. Atomic folds are the reflections themselves, rather than merely lifts of reflections in a quotient.

Theorem B.3. *Let H be a finite connected twin-free cut-percolating bigraph with an edge. Then $\text{Fold}(H)$ has a faithful finite real reflection representation whose full reflection set is exactly the set of atomic fold involutions. Moreover, there is a Coxeter system $(\text{Fold}(H), S)$ and $I, J \subseteq S$ realizing the given vertex and edge action of H as the reflection bigraph $\mathcal{R}(\text{Fold}(H); I, J)$.*

Lemma B.4. *The part-respecting tensor product of two connected bigraphs with an edge is connected. Every fold of one factor lifts to a fold of the tensor product by fixing the other coordinate. If both factors are twin-free, their atomic folds lift to atomic folds of the product.*

Proof. For any two product vertices choose paths in the factors. Their lengths have the same parity. Insert backtracks along an edge into the shorter path until the lengths agree, and then walk in both coordinates simultaneously. This proves connectivity.

For a fold f of one factor, use the involution $(v, x) \mapsto (v, fx)$. The two exchanged sides are the products with the original sides; there is no edge between them. Hence each component of the moved induced subgraph is sent to a different component. If f is atomic, each of its two moved components contains an edge: an isolated moved vertex and its mate would be twins, because all their neighbors are fixed. The connectivity argument shows that their products with the other factor are connected, so the lifted fold is atomic. $\square$

Proof of Theorem B.3. Induct on the number of edges. For one edge the fold group and its reflection representation are trivial. Otherwise Theorem 3.6 gives $H = R \otimes F$, where R is the Fiedler quotient and F the residual factor. Both are connected, both are twin-free (a twin pair in either factor would yield a twin pair in H), and F cut-percolates by Theorem 3.10. The quotient has at least two edges, so F is smaller. Write W for the image of $\text{Fold}(H)$ on the Fiedler eigenspace. It is a finite real reflection group. Its action on R is faithful because the spectral vertex images span that eigenspace.

Let f be an atomic fold of H . If it is visible on the Fiedler eigenspace, the proof of Theorem 3.10 gives

$$f(v, x) = (\alpha v, \tau_v x),$$

where α is a reflection in W and every τ_v is a twin permutation of F . Twin-freeness gives $\tau_v = 1$. Every reflection in W has such an atomic lift, so these folds generate $W \times 1$ on the tensor coordinates.

If f is invisible, it has the form $f(v, x) = (v, \tau_v x)$. For every edge vw of R , (τ_v, τ_w) is an automorphism of F . If v, v' share the neighbor w , composing these pairs yields an automorphism of F fixing the opposite part pointwise. Such an automorphism only permutes twins, so $\tau_v = \tau_{v'}$. Connectedness propagates this equality: all left τ_v agree, and all right τ_w agree. Thus $f = 1 \times \beta$ for one automorphism β of F . Restricting f to an edge fiber, as in Theorem 3.10, shows that β is a genuine fold. Moreover it is atomic; otherwise its distinct moved component pairs would lift to more than one pair for f .

Conversely, every fold of F lifts by Lemma B.4. Therefore

$$\text{Fold}(H) = W \times \text{Fold}(F).$$

The two factors commute and intersect trivially. The induction hypothesis supplies a faithful reflection representation of $\text{Fold}(F)$ with atomic folds exactly its reflections. Take its direct sum with the representation of W . A reflection in this direct sum acts as a reflection in exactly one factor and as the identity in the other. The visible lifts above and the atomic lifts from F show that these are precisely the atomic folds of H .

Finally, R is a reflection bigraph for W with its given permutation action, and the induction hypothesis says the same for F and $\text{Fold}(F)$. The product of the two Coxeter systems and the union of the respective parabolics give exactly the action on $R \otimes F = H$. $\square$

Adding twins. Use the uniform twin blowup of Definition 2.3. The symmetric group $\mathfrak{S}_a$ consists of all permutations of a elements. A *semidirect product* $N \rtimes Q$ below means that Q permutes the independent symmetric-group factors of N by its action on twin classes.

Lemma B.5. *An atomic fold in a connected bigraph either transposes two twins and fixes all other vertices, or descends to an atomic fold of its twin quotient, fixing every vertex in each fixed quotient class pointwise.*

Proof. Let f be atomic, and suppose a twin class T is preserved by f but not fixed pointwise. Choose $u \in T$ with $fu \neq u$. If a neighbor w of u were moved, twinhood gives the two edges uw and $(fu)w$ inside the moved induced subgraph. They connect u to fu , forcing that component to be preserved by f , contrary to the fold condition. Hence all neighbors of T are fixed. The moved vertices of T are isolated components. Atomicity now forces f to transpose only u, fu , which are twins.

Otherwise every quotient-fixed class is fixed pointwise, and every moved class is exchanged with a different twin class. The moved subgraph is a blowup of the moved subgraph in the quotient. Components containing an edge lift to connected components, since all adjacencies between classes are complete. An isolated moved quotient vertex would have an isolated exchanged mate with the same fixed neighborhood, contradicting twin-freeness of the quotient. Thus all moved components contain an edge, and the two components exchanged by f project to exactly two quotient components exchanged by its induced involution. This is an atomic quotient fold. $\square$

Corollary B.6 (Uniform twins). *Let $G = (A_0, B_0; E_0)$ be a finite connected twin-free bigraph with an edge. For every $a, b \geq 1$, with the natural permutation action,*

$$\text{Fold}(G(a, b)) \cong \left(\mathfrak{S}_a^{A_0} \times \mathfrak{S}_b^{B_0} \right) \rtimes \text{Fold}(G).$$

Every finite connected cut-percolating bigraph H with an edge has the form $G(a, b)$ for a twin-free cut-percolating quotient G . In this case, modulo independent permutations inside twin classes, the full fold group is the finite Coxeter group in Theorem B.3.

Proof. By Lemma B.5, every atomic fold of $G(a, b)$ induces a quotient fold or the identity. All transpositions inside a twin class are atomic folds, so the full product of symmetric groups lies in $\text{Fold}(G(a, b))$. Conversely, choose clone indices consistently and lift every fold of G by retaining its clone index. A quotient fold's fixed classes are fixed pointwise, and its moved components lift to exchanged components. Thus these lifts belong to the fold group and give a group section. The quotient-map kernel is precisely the independent twin permutations, proving the formula.

For cut-percolating H , part transitivity makes the twin classes on each specified part have a common size, say a, b . Thus $H = G(a, b)$. The formula implies that $\text{Fold}(G)$ is transitive on each quotient part; the classification gives cut-percolation of G . Apply Theorem B.3. $\square$

Corollary B.7 (Atomic generators bounded by the degrees). *Let H be a finite connected cut-percolating bigraph of degrees d_A, d_B . Its full fold group is generated by at most $d_A + d_B - 2$ atomic fold involutions. Consequently a positive recognition certificate needs at most $\min\{|V(H)| - 2, d_A + d_B - 2\}$ atoms.*

Proof. Write $H = G(a, b)$ as in Corollary B.6, and let G have degrees r, s . By Theorem B.3, the intrinsic Coxeter system of $\text{Fold}(G)$ realizes G . It is effective: any diagram component in both vertex stabilizers would act trivially, contradicting faithfulness; and connectedness requires the union of the two parabolics to contain the simple system. The rank argument in the proof of Theorem B.2 gives at most $r + s - 2$ simple generators, all atoms. Their clone-index-preserving lifts are atomic by the proof of Lemma B.5, since a moved quotient component contains an edge. Add $a - 1$ transpositions connecting the clones in one left twin class and $b - 1$ such transpositions in one right twin class. The quotient group is transitive on each part. Conjugating these transpositions by its lifts gives the corresponding generators in every twin class. Thus they generate the full semidirect product, using at most

$$r + s + a + b - 4 \leq br + as - 2 = d_A + d_B - 2$$

atoms. The inequality is equivalent to $(b - 1)(r - 1) + (a - 1)(s - 1) \geq 0$. The $|V(H)| - 2$ certificate bound follows by adding atoms that merge two current vertex orbits until the two part orbits remain. $\square$

Why the twin-free hypothesis is needed. For $H = C_6(3, 1)$, the group above is $\mathfrak{S}_3^3 \rtimes \mathfrak{S}_3$, where the last factor permutes the three left twin classes. Its order is 1296. Its subgroup $C_3^3 \rtimes C_3$ of order 81 is nonabelian; here C_3 is a cyclic group of order three. It is a Sylow three-subgroup, meaning its order is the largest power of three dividing the full group order. In contrast, every finite Coxeter group of order 1296 has abelian Sylow three-subgroups. Indeed, the only irreducible finite types whose orders can divide $1296 = 2^4 3^4$ are A_1, A_2, A_3, B_2, B_3 and dihedral types; all have cyclic Sylow three-subgroups. Their direct products have abelian Sylow three-subgroups. Thus $\text{Fold}(H)$ itself need not be a Coxeter group before twins are removed, even for this twelve-vertex example.

B.3 Odd prime degree

The following small part of the finite Coxeter classification is enough for a complete list at every odd prime degree.

Lemma B.8. *Let (W, S) be a finite irreducible Coxeter system and let $T \subsetneq S$ have odd prime index p . Then one of the following holds:*

- (i) *W has type A_{p-1} and T omits an endpoint of its path diagram;*
- (ii) *W has type $I_2(p)$ and T contains one of its two nodes.*

The alternatives coincide for $p = 3$.

Proof. The parabolic must be maximal among standard parabolics. Otherwise an intermediate standard parabolic factors its index into two integers greater than one. It therefore omits a single node. In type A_{n-1} , the resulting indices are $\binom{n}{k}$ for $1 \leq k < n$. For $2 \leq k \leq n - 2$, this number exceeds n . If it were prime p , the identity $k \binom{n}{k} = n \binom{n-1}{k-1}$ would force p to divide the positive integer $\binom{n-1}{k-1} = (k/n)p < p$, a contradiction. Only endpoint omissions remain, of index $n = p$. The indices in B_r are $2^k \binom{r}{k}$ for $1 \leq k \leq r$. In D_r they are $2^k \binom{r}{k}$ for $1 \leq k \leq r - 2$ and 2^{r-1} at the two fork endpoints. All are even. For the exceptional types, the distinct indices are:

Type	Maximal standard parabolic indices
E_6	27, 72, 216, 720
E_7	56, 126, 576, 756, 2016, 4032, 10080
E_8	240, 2160, 6720, 17280, 60480, 69120, 241920, 483840
F_4	24, 96
H_3	12, 20, 30
H_4	120, 600, 720, 1200

None is prime. These integers follow directly by dividing the standard finite Coxeter group orders by the orders after one node is deleted [20]. Finally, a maximal standard parabolic in $I_2(m)$ has index m , giving $m = p$. $\square$

The four families. The *hypercube* Q_p has vertices $\{0, 1\}^p$, with an edge when two strings differ in one coordinate; its parts are the strings of even and odd coordinate sum. The *crown* on $2q$ vertices is $K_{q,q}$ with one perfect matching removed. The *middle-layer inclusion bigraph* M_p has parts $\binom{[2p-1]}{p-1}$ and $\binom{[2p-1]}{p}$, with adjacency by containment. Here $[n] = \{1, \dots, n\}$ and $\binom{X}{k}$ denotes the family of all k -element subsets of X .

Theorem B.9 (Odd prime degree). *For every odd prime $p \geq 5$, the connected p -regular cut-percolating bigraphs are exactly*

$$K_{p,p}, \quad Q_p, \quad K_{p+1,p+1} \setminus \text{a perfect matching}, \quad M_p.$$

For $p = 3$, the complete list is $K_{3,3}$, Q_3 , M_3 (also called the Desargues graph), and the 192-vertex edge-triangle incidence bigraph of the regular 24-cell, described in Section B.4. The crown with four vertices in each part coincides with Q_3 .

Proof. Take an effective connected presentation and put $K = I \cap J$. Since $[W_I : W_K] = [W_J : W_K] = p$, each of I, J has exactly one active component. Lemma B.8 shows that each is either A_{p-1} or $I_2(p)$ and that

$$I \setminus J = \{s\}, \quad J \setminus I = \{t\}.$$

The active component on I consists of s and exactly one component L of K : either $L = A_{p-2}$ with s joined to an endpoint by an edge labeled three, or $L = A_1$ with its edge to s labeled p . Similarly t attaches to one component L' of K . No other component of K remains in an effective presentation.

For $p \geq 5$, suppose that an active component is $I_2(p)$. If $L = L'$, the other active component would also have to be $I_2(p)$, because its part in K must be the same one-node component. Two edges labeled $p \geq 5$ meeting at that node are incompatible with positive definiteness of the Coxeter Gram matrix, the matrix of inner products of the unit normals to its simple walls; see Lemma B.13. If $L \neq L'$ and s, t were joined, the whole diagram would be irreducible. For $p \geq 7$, no finite irreducible diagram of rank at least three contains an edge labeled p . For $p = 5$, the only possibilities are H_3, H_4 ; a second $I_2(5)$ gives two edges labeled five, while an A_4 component gives total rank six. Neither is possible. Hence s, t commute, the two active components are separate, and the bigraph is $K_{p,p}$.

It remains to consider two active components of type A_{p-1} . If $L = L'$, then s, t attach to the same or opposite endpoints of one A_{p-2} path. They cannot be joined directly, since finite Coxeter diagrams have no cycle. Attachment to the same endpoint gives D_p , and attachment to opposite endpoints gives A_p . These presentations yield respectively Q_p and the crown with $p + 1$ vertices in each part. For the first identification, D_p acts by

coordinate permutations and changes of an even number of coordinate signs. Its two vertex orbits are the sign vectors in $\{-1, 1\}^p$ with an even or odd number of negative coordinates, and adjacency changes one coordinate. For the second, A_p is the permutation group on $p + 1$ points: the two parts are singletons and p -subsets, with containment as adjacency. Identifying a p -subset with its missing point gives the crown.

If $L \neq L'$, then s, t are endpoints of two disjoint A_{p-1} paths. If they commute, the bigraph is again $K_{p,p}$. If they are joined by a label three, the diagram is A_{2p-2} and the two coset spaces are the $(p-1)$ - and p -subsets of a $(2p-1)$ -point set, yielding M_p . A larger joining label would be internal in a path of rank $2p-2 \geq 8$, impossible in the finite Coxeter classification. This exhausts $p \geq 5$.

When $p = 3$, the two alternatives in Lemma B.8 coincide as $A_2 = I_2(3)$. The common component $L = L'$ consists of one node, giving A_3 and Q_3 , also the four-by-four crown. For $L \neq L'$, the joining labels two, three, and four give respectively $A_2 \times A_2$, A_4 , and F_4 . Labels at least five do not give a finite diagram of the form $3-m-3$. The three resulting graphs are $K_{3,3}$, M_3 , and the 24-cell edge-triangle incidence graph. All listed presentations are reflection bigraphs and hence cut-percolating by Conlon–Lee [10]. $\square$

B.4 Graphs without short cycles

The *girth* of a graph is the length of a shortest cycle, with girth infinity if it has no cycle. The *subdivision* $S(G)$ replaces each edge uv of G by a path u, x_{uv}, v of two edges. Its specified parts are the original vertices and the new edge vertices. The d -dimensional *hypercube* Q_d has vertex set $\{0, 1\}^d$ and joins two binary strings differing in one coordinate. The dodecahedral graph uses the twenty vertices and thirty edges of the regular dodecahedron. The 120-cell graph uses the vertices and edges of the regular four-dimensional polytope with 120 dodecahedral three-dimensional faces. The 24-cell is the convex hull of the twenty-four coordinate permutations of $(\pm 1, \pm 1, 0, 0)$. The *edge-triangle incidence bigraph of the 24-cell* has one vertex for each edge and one for each triangular face, with adjacency meaning containment.

Theorem B.10. *Up to exchanging the specified parts, the connected cut-percolating bigraphs of minimum degree at least two and girth at least eight are exactly the following:*

<i>Bigraph</i>	<i>Vertices</i>	<i>Degrees in the two parts</i>	<i>Girth</i>
$C_{2m}, m \geq 4$	$2m$	$(2, 2)$	$2m$
$S(Q_d), d \geq 3$	$(d+2)2^{d-1}$	$(d, 2)$	8
$S(\text{dodecahedral graph})$	50	$(3, 2)$	10
$S(120\text{-cell graph})$	1800	$(4, 2)$	10
<i>Edge-triangle incidence of the 24-cell</i>	192	$(3, 3)$	8

Corollary B.11. *The edge-triangle incidence bigraph of the 24-cell is the only connected cut-percolating bigraph of minimum degree at least three and girth at least eight. Every other connected cut-percolating bigraph of minimum degree at least three contains a four-cycle or a six-cycle.*

Barnhill’s angle formula for standard parabolic subgroups [3, Proposition 6.8] gives the exact girth $2 \min\{m_{st} : s \in I \setminus J, t \in J \setminus I\}$ of a reflection bigraph when neither parabolic contains the other. For the exhaustion argument, we only need the following elementary upper bound, which we prove directly.

Lemma B.12 (A cycle from a cross pair). *For $s \in I \setminus J$ and $t \in J \setminus I$, $\mathcal{R}(W; I, J)$ contains a cycle of length $2m_{st}$.*

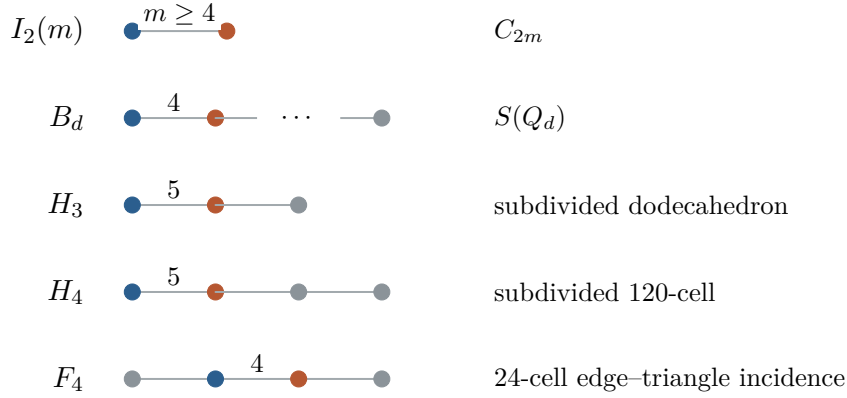


Figure 10: The effective Coxeter diagrams for girth at least eight. Blue marks $I \setminus J = \{s\}$, orange marks $J \setminus I = \{t\}$, and gray nodes belong to $I \cap J$. The two distinguished nodes must be joined by a label at least four. The finite Coxeter classification leaves just these five possibilities; see Theorem B.10. Each marked endpoint is omitted from the opposite parabolic. Unlabeled edges have label three.

Proof. The rank-two parabolic group $D = W_{\{s,t\}}$ is dihedral of order $2m_{st}$. By the parabolic intersection property,

$$D \cap W_I = \langle s \rangle, \quad D \cap W_J = \langle t \rangle.$$

Consequently, the left and right cosets of these two order-two subgroups embed in the two vertex parts of $\mathcal{R}(W; I, J)$. The edges indexed by D are distinct: their common stabilizer is $D \cap W_{I \cap J} = \{1\}$. They form the rank-two reflection bigraph, which is the cycle $C_{2m_{st}}$. Extra edges in the ambient graph do not remove this cycle. $\square$

Lemma B.13 (No two large labels meet). *In a finite Coxeter diagram, no vertex is incident with two edges whose labels are at least four.*

Proof. Choose unit inward normals to the simple mirrors. Their Gram matrix is positive definite and has entries $-\cos(\pi/m_{st})$ off the diagonal. Suppose a vertex has two neighbors with labels at least four. Put $a, b \geq 1/\sqrt{2}$ for the corresponding cosines and $c \geq 0$ for the cosine between those neighbors. The relevant three-by-three principal determinant is

$$1 - a^2 - b^2 - c^2 - 2abc \leq 0,$$

contrary to positive definiteness. $\square$

Proof of Theorem B.10. Choose a connected reflection presentation with $S = I \cup J$, and put $K = I \cap J$. Neither $I \setminus J$ nor $J \setminus I$ is empty: otherwise one part would have degree one. Lemma B.12 and girth at least eight give $m_{st} \geq 4$ for every pair $s \in I \setminus J$, $t \in J \setminus I$. Lemma B.13 therefore forces

$$I \setminus J = \{s\}, \quad J \setminus I = \{t\}.$$

These two nodes are joined by a label at least four. All other nodes belong to K . Every irreducible component avoiding s, t lies in both parabolics and cancels from both coset spaces, so it can be discarded.

The finite irreducible Coxeter diagrams containing an edge labeled at least four are precisely $I_2(m)$ for $m \geq 4$, B_d for $d \geq 3$, F_4 , H_3 , and H_4 . In every case this edge is unique. Our parabolics omit its opposite endpoints:

$$I = S \setminus \{t\}, \quad J = S \setminus \{s\}.$$

For $I_2(m)$ this gives C_{2m} . For B_d , marking the terminal edge labeled four gives the vertex–edge incidence of the d -cube, namely $S(Q_d)$. The diagrams H_3, H_4 have a terminal edge labeled five; they give respectively vertex–edge incidence of the dodecahedron and of the 120-cell. The middle edge labeled four in F_4 gives edge–triangle incidence of the 24-cell. The rank-two case B_2 is already $I_2(4)$.

All these incidence graphs are reflection bigraphs and hence cut-percolating by Conlon–Lee. Their girths are as listed. For subdivisions, girth doubles: the hypercube has girth four, and the dodecahedral and 120-cell graphs have girth five. The edge–triangle incidence of the 24-cell has girth eight. The geometric verification of these girths is given below. $\square$

Girths of the positive examples. The dodecahedron and the 120-cell have pentagonal two-faces. The directions of the incident edges at a vertex form a simplex, called its *vertex figure*. Thus any two incident edges lie in a pentagonal face and meet at angle 108° . In a cycle of length three or four, let $u_1, \dots, u_k$ be its successive unit displacement vectors. Adjacent vectors have scalar product $c = \cos 72^\circ > 0$, and their sum is zero. For $k = 3$ this would give $0 = 3 + 6c$. For $k = 4$ it would give

$$0 = 4 + 8c + 2(\langle u_1, u_3 \rangle + \langle u_2, u_4 \rangle) \geq 8c > 0.$$

Thus neither skeleton has a triangle or a square; pentagonal faces give girth five. Their subdivisions have girth ten. A hypercube has no triangle and has square faces, giving girth four and eight before and after subdivision.

For the 24-cell, use the 24 coordinate permutations of $(\pm 1, \pm 1, 0, 0)$, joining pairs at squared distance two. Its triangular faces are precisely its three-vertex cliques, and the graph induced on the neighbors of any vertex is a cube Q_3 . Two different edges belong together to at most one triangular face, ruling out a four-cycle in edge–triangle incidence. A six-cycle would use three distinct edges that pairwise lie in three distinct triangles. These three edges are either the sides of a single triangle, which would repeat that face, or have a common endpoint. In the latter case their other endpoints form a triangle in its cube neighborhood, also impossible. A square in that cube neighborhood gives four incident edges and four triangular faces, hence an eight-cycle in the incidence bigraph. This proves its girth is eight.

B.5 Odd degree without four-cycles

The same diagram argument gives a sharper conclusion at odd regular degree, with no primality assumption. Recall that M_d is the inclusion bigraph between the $(d-1)$ - and d -subsets of $[2d-1]$, defined in Section B.3.

Theorem B.14. *For every odd integer $d \geq 3$, the connected d -regular cut-percolating bigraphs without a four-cycle are exactly M_d , and, when $d = 3$, the edge–triangle incidence bigraph of the 24-cell.*

Proof. Choose an effective connected reflection presentation, with $S = I \cup J$ and $K = I \cap J$. By Lemma B.12, every node of $I \setminus J$ is adjacent in the diagram to every node of $J \setminus I$: a nonadjacent pair would have label two and give a four-cycle. Finite irreducible Coxeter diagrams are trees. The complete bipartite subgraph between these two nonempty exclusive sets therefore has a singleton side. Suppose $I \setminus J = \{s\}$.

If $J \setminus I$ has at least two nodes, they are neighbors of s . If s has no other neighbor, its active component in I is A_1 , giving degree two, contrary to oddness. Thus s has degree three in the diagram, with two neighbors in $J \setminus I$ and one in K . It is the unique branch

node of a diagram of type D or E . Let ℓ_1, ℓ_2 be the lengths of the two arms starting in $J \setminus I$, and let ℓ_3 be the length of the arm entirely in K . The local degrees are

$$[W_I : W_K] = \ell_3 + 2, \quad [W_J : W_K] = (\ell_1 + 1)(\ell_2 + 1).$$

For their common value to be odd, both ℓ_1, ℓ_2 must be even. A type- D diagram has arms $1, 1, r - 3$, so this is impossible. The type- E arms are $1, 2, L$, where $L \in \{2, 3, 4\}$. The only possibility would use the two even arms, leaving $\ell_3 = 1$; but then the two degrees are 3 and $3(L + 1)$, unequal. We conclude that $J \setminus I$ is also a singleton, say $\{t\}$.

The adjacent nodes s, t are omitted from opposite maximal parabolics. Regularity says these parabolics have equal orders. In type A_r , adjacent omissions have indices $\binom{r+1}{i}, \binom{r+1}{i+1}$; equality forces $r = 2i$, and the resulting graph is M_{i+1} . In type B_r , number the nodes $0, \dots, r - 1$ with the edge labeled four between 0 and 1. Omitting adjacent nodes $k, k + 1$ gives local degrees $2(k + 1)$ and $r - k$; the first is always even. In type D_r , number the long arm edges starting at the branch node by $k = 0, 1, \dots$. Their local degrees are $2(k + 3)$ and $r - k - 2$, while a fork edge has one local degree two. Thus neither type contributes an odd regular example. The maximal parabolic indices in E_7, E_8, H_3, H_4 are all distinct. In E_6 , the equal-index nodes are in opposite arms and are not adjacent. The only adjacent equal-index pair in F_4 is its middle edge, giving degree three and the 24-cell graph. Dihedral diagrams give degree two. This exhausts the finite list.

Conversely, M_d is a reflection bigraph with the two adjacent central nodes of A_{2d-2} distinguished. Its girth is six by Barnhill's formula, or directly by subset containment: two distinct $(d - 1)$ -subsets have at most one common d -superset, excluding four-cycles, and the three $(d - 1)$ -subsets extending a fixed $(d - 2)$ -subset by three different points give a six-cycle. The 24-cell example was verified above. Both are cut-percolating. $\square$

C A local obstruction that survives twins

The profile inequality in Lemma A.4 also lifts a local distance obstruction to arbitrary uniform twin blowups. The unblown weighted-root method of [21, Theorem 3] uses the same principle: weight vertices by their distance from a root, and compare the leading terms of a two-by-two minor. Here the profiles retain enough information to choose two neighborhood bijections even when the source vertices have twins.

Write $\text{dist}_F(x, y)$ for graph distance, the minimum number of edges in a path from x to y . A rooted homomorphism $\phi : F \rightarrow F$ fixing v is *distance-preserving from v* if $\text{dist}_F(v, \phi(z)) = \text{dist}_F(v, z)$ for every vertex z .

Proposition C.1. *Let $F = (A, B; E)$ be a connected C_4 -free biregular bigraph, with left degree at least three, right degree at least two, and $\text{Aut}(F)$ transitive on B . Fix $v \in B$ and $u_1 \in N_F(v)$. Suppose that for no distinct $x, y \in N_F(v)$ is there a distance-preserving rooted homomorphism $\phi : F \rightarrow F$ such that*

$$\phi(v) = v, \quad \phi(x) = \phi(y) = u_1,$$

and such that both maps $\phi : N_F(x) \rightarrow N_F(u_1)$ and $\phi : N_F(y) \rightarrow N_F(u_1)$ are bijections. Then $F(a, b)$ is not weakly norming for any $a, b \geq 1$.

Proof. The degree and C_4 -freeness assumptions imply twin-freeness. Choose $u_2 \in N_F(v) \setminus \{u_1\}$, which exists by the right-degree assumption. Suppose that $F(a, b)$ is weakly norming. First we obtain a profile matrix at one fixed source root, with no source coloring. Average the matrices of Lemma A.4 over all partitions $A = C \sqcup D$, assigning each vertex independently

to either side with probability $1/2$. Every ordered pair of distinct inner classes appears with the same positive factor. The average is positive semidefinite. Source automorphisms taking one root to another preserve the uncolored matrix, including the inner and outer activity vectors, by the same relabeling argument as in that lemma. Dividing out the common root count and the positive averaging factor shows that the following matrix at source root v is positive semidefinite:

$$J_{p,q} = \sum_{\substack{x,y \in N_F(v) \\ x \neq y}} \text{unnormalized weight of } (h_x = p, h_y = q).$$

It is symmetric because the source pairs are ordered.

Delete all right clones above v . Use the inner left target alphabet $N_F(v)$, the outer left alphabet $A \setminus N_F(v)$, and the right alphabet $B \setminus \{v\}$, with the incidence kernel of F . Give every target state z activity $\gamma^{\text{dist}_F(v,z)}$, where $\gamma > 0$. Lemma A.4 allows these separate alphabets and zero kernel entries.

Every inner source class is monochromatic in a positive-weight assignment. Indeed, two different labels in $N_F(v)$ have no common right neighbor other than v , by C_4 -freeness; that label has been deleted, and a surviving neighboring right clone exists.

For $u \in N_F(v)$, let p_u be the profile containing b copies of each label in $N_F(u) \setminus \{v\}$. These profiles are distinct. They contain at least two different labels, whose unique common left neighbor is u , so a class with profile p_u has every left clone at u . Moreover, this profile permits a bijective choice of one label from each remaining neighboring right base class. To see this, form a bipartite graph between these base classes and the labels in $N_F(u) \setminus \{v\}$, joining a class to every label appearing on one of its clones. Each class has b clones, and each target label occurs b times in the profile. For any set of source classes, its b times as many slots fit only into the labels adjacent to it, each with capacity b . Every set of classes thus has at least as many adjacent labels. This is Hall's matching condition, and it gives the desired bijection.

Reinsert a virtual source root mapped to v . Every target label assigned to a clone above z is at distance at most $\text{dist}_F(v,z)$: lift a shortest source path to the clones and apply the assignment to it. Consequently the total exponent of γ is at most

$$D = a \sum_{z \in A} \text{dist}_F(v,z) + b \sum_{z \in B \setminus \{v\}} \text{dist}_F(v,z).$$

Equality holds exactly when every clone preserves distance. The identity assignment attains this exponent. Let L_{ij} be the coefficient of γ^D in $J_{p_{u_i}, p_{u_j}}$. The identity gives $L_{12} > 0$. We claim that $L_{11} = 0$.

A contribution to L_{11} would give two distinct inner classes x, y , both assigned u_1 , with the prescribed profiles. Their remaining right neighborhoods are disjoint, since a shared class would give a four-cycle through v . We can therefore make the two matching choices above independently, and choose one arbitrary clone label at every other base vertex. The complete edges between clone classes make the selected map a homomorphism of F . Reinserting $v \mapsto v$ gives bijections on both marked neighborhoods, and maximality of the exponent D makes the map distance-preserving. This contradicts the hypothesis.

The leading minor is consequently strictly negative:

$$L_{11}L_{22} - L_{12}^2 = -L_{12}^2 < 0.$$

It is the coefficient of γ^{2D} in the corresponding minor of J . For sufficiently large finite γ , that minor is negative, contradicting positive semidefiniteness. The finite boundary model with zero incidence entries is already permitted by Lemma A.4. $\square$

Corollary C.2. *The binary-coset bigraph H_4 is not weakly norming. The same holds for every uniform twin blowup of H_4 .*

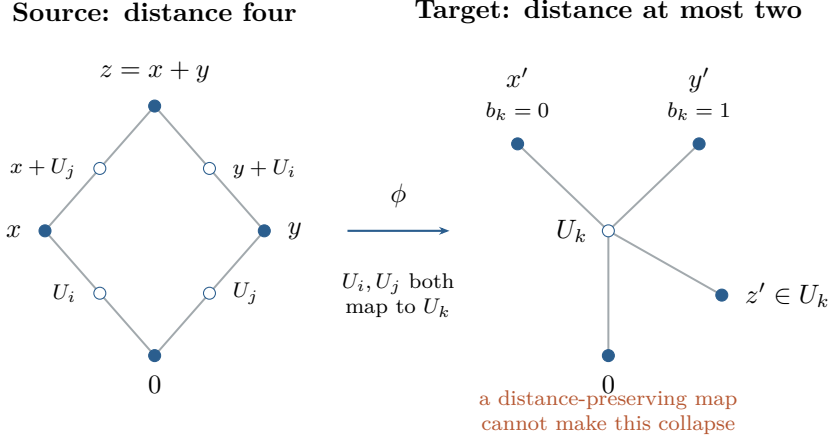


Figure 11: The local distance obstruction for H_4 . Points are filled and coset vertices are hollow. Choose $x \in U_i$, $y \in U_j$ so that $z = x + y$ is at root distance four. A putative collapse of both root cosets to U_k , bijective on their point neighborhoods, allows a choice with $x' = \phi(x)$ and $y' = \phi(y)$ having different ordinary coordinate b_k values. Any point collinear with both must lie in U_k , since all other directions preserve b_k . Thus $z' = \phi(z)$ has root distance at most two. The target diagram shows membership in this one coset; target points may coincide. Corollary C.2 makes this obstruction analytic and retains it under twins.

Proof. Apply Proposition C.1 to H_4 with the parts reversed: the left vertices are cosets and the right vertices are points. Both degrees are four, the graph is C_4 -free, and translations are transitive on points. Take the root point to be 0. Suppose a distance-preserving homomorphism ϕ maps two distinct root cosets U_i, U_j to the same root coset U_k , bijectively on both point neighborhoods.

The three nonzero points of each source coset map bijectively to $U_k \setminus \{0\}$. Choose $x \in U_i \setminus \{0\}$ and $y \in U_j \setminus \{0\}$ with images a_k and b_k , respectively. The point $z = x + y$ is collinear with both x and y and has root distance four. To see the latter, it suffices to show that z lies in no U_ℓ . We have $z \notin U_i \cup U_j$ because $U_i \cap U_j = \{0\}$. If either x or y has a nonzero ordinary coordinate, that coordinate forces direction i or j , already excluded. Otherwise $z = a_i + a_j$, which is none of the a_ℓ : the only nonempty dependence among the four protected directions is $a_1 + a_2 + a_3 + a_4 = 0$.

Put $x' = \phi(x)$, $y' = \phi(y)$, and $z' = \phi(z)$. The point z' shares a coset with both x' and y' . If $z' \notin U_k$, neither of these cosets has direction k . Every other direction preserves coordinate b_k , forcing $b_k(z') = b_k(x') = b_k(y')$, a contradiction. Hence $z' \in U_k$, so its distance from 0 is at most two, contrary to distance preservation. The proposition excludes all uniform twin multiplicities. $\square$

Corollary C.3. *For H_4 , decorated Hölder already fails with two strictly positive kernels: one on edges in a chosen coset direction and one on all remaining edges. These two edge classes are preserved by every oriented fold.*

Proof. Work with the reversed parts as above. Partition the left coset vertices into direction one and all other directions, and choose the target profiles of U_1, U_2 at root 0. Translations preserve this source partition and act transitively on roots, so the colored profile matrices at every root agree. The two-profile proof of Proposition C.1 now applies to this colored matrix

directly. The identity still contributes to its off-diagonal entry, and the distance-preserving collapse is still impossible for every pair of distinct source cosets. Thus its leading minor is negative.

Choose a profile function giving a strictly negative quadratic form. The rare-row interpolation and successive finite approximations in Lemma A.4 then give a finite model with strictly positive common kernel W and a real edge observable F for which $\text{Cov}(X, Y) < 0$. Here X, Y are the sums of F over the two edge classes, as in Lemma A.2. Strict negativity persists when zero entries are replaced by sufficiently small positive entries; all parameters can be chosen finite.

Let $Z(s, t)$ be the partition function with We^{sF} on the chosen direction and We^{tF} on the others, and put $\alpha = 1/4$. For the logarithmic Hölder gap

$$D(s, t) = \alpha \log Z(s, s) + (1 - \alpha) \log Z(t, t) - \log Z(s, t)$$

edge transitivity gives $\nabla D(0, 0) = 0$. Also $D(s, s) = 0$ and $D_{st}(0, 0) = -\text{Cov}(X, Y)$. Consequently

$$\left. \frac{d^2}{d\varepsilon^2} D(\varepsilon, -\varepsilon) \right|_{\varepsilon=0} = 4 \text{Cov}(X, Y) < 0.$$

For sufficiently small $\varepsilon > 0$ the two positive kernels $We^{\varepsilon F}$ and $We^{-\varepsilon F}$ therefore violate decorated Hölder on these two edge classes. Coset direction is preserved by every atom, hence by every oriented fold, proving the last assertion. $\square$

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