

BRUDNO'S FAMILIES OF EQUAL SUMS OF SIXTH POWERS AND BREMNER'S K3 SURFACE

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ABSTRACT. Brudno (1970, 1976) and Brudno–Kaplansky (1974) found an elliptic family and a degree-four polynomial family of solutions of $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$. We explain both in terms of the K3 surface $V \subset \mathbb{P}^5$, the complete intersection of three quadrics studied by Bremner (1981) and Kuwata (2007). The Brudno–Kaplansky curve is Cremona’s curve 480f4 and is the fibre at $\lambda = \infty$ of the elliptic pencil on V that Bremner calls the Brudno pencil; the curve $z^2 = x^4 + 38x^2y^2 + y^4$ of 1970 is 480f1, and the classical recursions of Lebesgue and Desboves that Brudno iterated are the multiplication maps $[2]$ and $[-3]$ on it. Brudno’s 1976 quartic is the unique rational curve of degree four on V (Bremner’s Theorem 4.4) and a section of the Brudno pencil; its parameter is the fibre coordinate λ , and the involution Brudno noticed is a linear symmetry of V . Using the Mordell–Weil lattice of the pencil we prove a closed formula for the degree of a section, $\deg C_S = 2\langle S - \gamma/4, S - \gamma/4 \rangle$, enumerate all sections of degree ≤ 12 (four of them new, of degrees 7, 10, 11, 11), and enumerate, by a lattice computation, all smooth rational curves of degree ≤ 12 on V over $\mathbb{Q}(i)$ and over $\mathbb{Q}$, reproducing Bremner’s theorems and testing his conjecture. A census of the 74 primitive solutions with all terms ≤ 1343 identifies which of them lie on V , on known curves, on sections, and on fibres of jumping rank, and records some phenomena off V . In the second part we compute equations and parametrisations of the 48 new curves that are not sections and show that seven of the eight census solutions on no known curve lie on them, leaving a single unexplained point; we determine the geometry over $\mathbb{C}$ of the sextic fourfold M_6 , of the threefold $W = M_6 \cap M_2$ (which has exactly 710 nodes and defect 95) and of V (transcendental lattice $U(2) \oplus \langle 12 \rangle$, zeta function through the newform 320f2), and discuss in what sense the geometry drives the arithmetic; and we show that Bremner’s conjecture follows, with non-explicit finite sets, from the theorems of Sterk and Bright–Logan–van Luijk, describe how to make it explicit (Borchers–Shimada), and run the first step of Bremner’s own cone procedure, which identifies the degree-8 pencil involutions that the proof requires.

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Claude 5.1 is a large language model developed by Anthropic; see §1.1 for its role in this paper.

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1. INTRODUCTION

We study the Diophantine equation

$$x^6 + y^6 + z^6 = u^6 + v^6 + w^6, \quad (1.1)$$

whose first non-trivial solution $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$ was found by Subba Rao in 1934 [85]. Lander, Parkin and Selfridge [58] listed the ten primitive solutions with common sum below $2.5 \cdot 10^{14}$ and noticed that all but one of them also satisfy

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2. \quad (1.2)$$

Simcha Brudno then wrote three short papers on (1.1) (after an earlier numerical note [22], which we do not use): in 1970 [23] he produced infinitely many solutions from a trigonometric identity, which led him to the genus-one quartic $C: z^2 = x^4 + 38x^2y^2 + y^4$ and to the nineteenth-century recursions of Lebesgue and Desboves on it; with Kaplansky [25] he reduced the system (1.1), (1.2), $v = y - z$, $w = y + z$ to the rational points of a cubic curve E_1 of rank one, and

recorded Tate's remark that the obvious point P generates $E_1(\mathbb{Q})$ modulo torsion; and in 1976 [24] he exhibited, without any indication of how it was found, the two-parameter polynomial solution of degree four

$$\begin{aligned} x &= 2m^4 + 4m^3n - 5m^2n^2 - 12mn^3 - 9n^4, & u &= m^4 - 3m^3n - 14m^2n^2 - 15mn^3 - 9n^4, \\ y &= 3m^4 + 9m^3n + 18m^2n^2 + 21mn^3 + 9n^4, & v &= 3m^4 + 8m^3n + 9m^2n^2, \\ z &= -m^4 - 10m^3n - 17m^2n^2 - 12mn^3, & w &= 2m^4 + 12m^3n + 19m^2n^2 + 18mn^3 + 9n^4, \end{aligned} \quad (1.3)$$

which satisfies (1.1), (1.2) and the linear relation $3x + y + z = 3u + v + w$ identically.

The question that motivated this paper is: where does (1.3) come from? The answer is in Bremner's 1981 paper [20], although Bremner never says so. Bremner observed that the known solutions of (1.1), (1.2), after permuting and changing signs of the unknowns, satisfy the three quadratic equations

$$V: \quad Q(x, u) = Q(w, z), \quad Q(y, v) = Q(u, x), \quad Q(z, w) = Q(v, y), \quad Q(s, t) = s^2 + st - t^2, \quad (1.4)$$

which define a K3 surface $V \subset \mathbb{P}^5$; cubing the three equations and adding gives (1.1), adding them gives (1.2). On V the Brudno–Kaplansky substitution becomes an elliptic pencil $E \rightarrow \mathbb{P}_\lambda^1$ (the *Brudno pencil*), E_1 being its fibre at $\lambda = \infty$, and Brudno's quartic is the unique curve of degree four on V of arithmetic genus zero over $\mathbb{Q}$ up to the symmetries of V (Bremner's Theorem 4.4); Bremner even writes it down (p. 574) as the image of a line under a fibrewise involution of the Brudno pencil. In one sentence: Brudno's 1970/1974 family is a fibre of the Brudno pencil, his 1976 family is a section of it, and the two are therefore the same object seen twice.

The paper has two parts. Sections 2 and 3 summarise, briefly, what Brudno, Brudno–Kaplansky, Bremner, Delorme, Choudhry and Kuwata proved; we add the identification of E_1 with Cremona's curve 480f4 and of the Lebesgue and Desboves recursions with [2] and [−3] on $\text{Jac}(C) = 480f1$. Sections 4–7 contain the new results:

- (i) the dictionary between Brudno's quartic and Bremner's degree-four curve, the fact that it is the section $g_1 + g_3 + 2T$ of the Brudno pencil with $\lambda = (3m + 2n)/(2m + 3n)$, that the hyperplane $3x + y + z = 3u + v + w$ is its linear span and cuts V residually in two lines and a conic, and that Brudno's involution $(m : n) \mapsto (-3(m + n) : m + 3n)$ is a linear symmetry of V (Section 4); also the identification of the published parametric solutions of Bremner, Delorme and Choudhry with one another and with sections of the pencil;
- (ii) the Mordell–Weil lattice of the Brudno pencil over $\mathbb{Q}(\lambda)$: a proof that Bremner's three generators generate, and the degree formula $\deg C_S = 2\langle S - \gamma/4, S - \gamma/4 \rangle$ for the curve $C_S \subset \mathbb{P}^5$ traced by a section S , with $\gamma = -g_1 + g_2 + g_3$; the complete list of sections of degree ≤ 12 , which contains four curves not in the literature, of degrees 7, 10, 11, 11 (Section 5);
- (iii) a complete enumeration of the smooth rational curves of degree ≤ 12 on V , over $\mathbb{Q}(i)$ and over $\mathbb{Q}$, by a rigorous induction in the Néron–Severi lattice; Bremner's Theorems 4.2–4.4 are recovered, 52 new $\mathbb{Q}$ -rational curves of degrees 7–12 appear up to symmetry, and Bremner's conjecture that all such curves come from finitely many by finitely many automorphisms is tested through degree 12: it fails if one only uses the involutions of the Brudno pencil, and holds if one adds the involutions of the 14 rational elliptic pencils of degree ≤ 6 (Section 6);
- (iv) a census of the 74 primitive solutions of (1.1) with all terms ≤ 1343 : 59 satisfy (1.2), 41 lie on V , 33 of those on a known curve or on a section, and the remaining 8 lie on fibres of the Brudno pencil whose rank jumps; Delorme's and Guy's question whether every solution of (1.1), (1.2) lies on V is answered by the smallest counterexample #14, and a (3, 6)-multigrade solution is pointed out (Section 7).

The second part (Sections 8–10) was added after the first version of the paper and answers the questions it left open:

- (v) equations, ideals and birational parametrisations of all 48 non-section curves of degree ≤ 12 , and the incidence of the 41 census solutions on V with the 62 curves of degree ≤ 12 : seven of the eight solutions on no known curve and no section lie on the new curves, and exactly one solution on V , #49, lies on no $\mathbb{Q}$ -rational smooth rational curve of degree ≤ 12 (Section 8);
- (vi) the geometry over $\mathbb{C}$ of the sextic fourfold M_6 , of the threefold $W = M_6 \cap M_2$ and of V : fundamental groups, Hodge numbers, the 710 nodes of W and its defect 95, the Néron–Severi and transcendental lattices of V and its zeta function, point counts over $\mathbb{F}_p$ verifying all of it, and a discussion of the sense in which the geometry governs the arithmetic over finite fields and over $\mathbb{Q}$ (Section 9);
- (vii) Bremner’s conjecture: it follows for every K3 surface over every field of characteristic $\neq 2$, with non-explicit finite sets, from the theorems of Sterk and of Bright, Logan and van Luijk together with a Dirichlet-domain argument; the explicit version for V is a computation by Borchers’ method (Shimada, OSCAR), for which we provide the embedding $\mathrm{NS}(V) \hookrightarrow \mathrm{II}_{1,25}$; and the first step of Bremner’s cone procedure, run on our data, shows that the pencils of degree ≤ 6 cannot complete the proof and exhibits the degree-8 pencil involutions that do the missing work (Section 10).

Section 11 lists open problems. Appendix A collects misprints in [20, 57, 38, 29, 30] found while checking every formula of those papers; Appendix B describes the computations; Appendices C–E contain the tables and the parametrisations of the new curves of degree ≤ 9 .

Known results are summarised briefly, with references to the precise statements; all new results are given in full. Every numerical statement below was verified by exact computation (Appendix B); statements that we could not verify, or bibliographic data we could not check, are marked in red as **[TODO: ...]**.

1.1. Statement on the use of AI. This paper was written by Claude, a large language model developed by Anthropic, in the course of a conversation with the first-named author, Alon Amit, in October 2026; the model configured for the conversation was Claude 5.1 (model identifier `claude-fable-5-1`). Since the reader of a mathematics paper is entitled to know who did what, we describe the division of labour.

The first-named author set the problem and directed the work. He asked where Brudno’s two families come from, what the map from the Brudno–Kaplansky curve to integer solutions is, whether the polynomial solutions can be shown to be finite in number, and which elliptic curve explains the second family; he supplied the papers of Brudno, Brudno–Kaplansky, Bremner, Kuwata, Delorme and Choudhry, chose among the approaches that were proposed to him at each stage (in particular to work on M_6 only, to lead with a literature search, to enumerate curves through degree 12, and later to compute the non-section curves, the geometry over $\mathbb{C}$ and the status of Bremner’s conjecture), reviewed the intermediate results, and decided what to pursue. He did not write the text or the code.

Claude read the sources, searched the literature, formulated the statements, found the arguments and wrote the proofs, designed and ran the computations (Python/SymPy, PARI/GP, Sage/Singular and C; parts of the longer computations were delegated to parallel instances of the same model, whose results were then re-verified independently by the main instance, as recorded in Appendix B), and wrote the paper, the accompanying Sage notebook and the interactive applet. The pronoun “we” in the text refers to the two authors.

What this means for the reader. Every numerical and algebraic statement was checked by exact computation, and the checks are described in Appendix B; where a statement rests on a hypothesis (such as (H), $\rho(V_{\mathbb{Q}}) = 19$) or is only numerical (such as Observation 9.8) this is said in place. The proofs, the summaries of the literature and the choice of what to cite are the model’s, and have been reviewed by the first-named author but not refereed by anyone else. Theorem and section numbers quoted from the literature were taken from the sources as the model read them—in a few cases through reviews or summaries rather than the original text, which is noted where it happens—and bibliographic data that could not be verified are

marked **[TODO]**. A language model can misread a source or produce an argument that looks right and is not, and it can do so fluently; the authors have tried to keep such errors out by insisting on exact computation wherever a statement could be computed, but the reader should keep the possibility in mind, especially for the non-computational parts (Sections 9.4, 10). Responsibility for the decision to circulate the paper, and for its final form, rests with the first-named author.

Conventions. Throughout, $V \subset \mathbb{P}^5$ is the surface (1.4) with coordinates (x, y, z, u, v, w) , exactly as on p. 548 of [20] (Bremner's V ; Kuwata's K_B ; Delorme's system (II); Choudhry's system (2)–(4)). Bremner's lines $\Gamma_1, \dots, \Gamma_{32}$ and conics $Q_1, \dots, Q_{24}$ are those of his Tables 1 and 2 (with the misprint corrections of Appendix A); $\Gamma_1, \dots, \Gamma_8$ and $Q_1, \dots, Q_{12}$ are the ones defined over $\mathbb{Q}$. H denotes the hyperplane class. The Brudno pencil is Bremner's equation (5),

$$y = a + (\lambda - 1)b, \quad z = (\lambda - 1)a - b, \quad v = -(\lambda - 1)a - \lambda b, \quad w = \lambda a - (\lambda - 1)b, \quad (1.5)$$

under which (1.4) reduces to the genus-one quartic $E_\lambda \subset \mathbb{P}_{(x:u:a:b)}^3$,

$$x^2 - u^2 = (\lambda^2 - 1)(a^2 + b^2), \quad xu = \lambda a^2 - (3\lambda^2 - 5\lambda + 3)ab - \lambda b^2. \quad (1.6)$$

With Bremner's abbreviations

$$A = \lambda^2 + 1, \quad B = \lambda^2 - 1, \quad P = 2\lambda^2 - 2\lambda + 1, \quad Q = \lambda^2 - 2\lambda + 2, \quad U = 3\lambda^2 - 5\lambda + 3,$$

the Weierstrass model is Bremner's (14'),

$$\mathcal{E}: \quad y^2 = x^3 - 2(5PQ - 2B^2)x^2 + 25P^2Q^2x \quad \text{over } \mathbb{Q}(\lambda), \quad (1.7)$$

reached from E_λ by the map of Bremner's Appendix I (recalled in §4.1). The torsion point

$$T = (5PQ, 10BPQ)$$

has order 4, and Bremner's three generators of infinite order over $\mathbb{Q}(\lambda)$ (his Remark on p. 556, with the misprint $(P, 3B, Q) \rightarrow (P, 3BP, Q)$ corrected and transported to (1.7)) are

$$\begin{aligned} g_1 &= (5P^2, 15BP^2), & g_2 &= (5Q(2\lambda - 1)^2, 10Q(\lambda - 1)(2\lambda - 1)(3\lambda^2 - \lambda + 1)), \\ g_3 &= (5P(\lambda - 2)^2, 10P(\lambda - 1)(\lambda - 2)(\lambda^2 - \lambda + 3)). \end{aligned} \quad (1.8)$$

A section is written $S = n_1g_1 + n_2g_2 + n_3g_3 + kT$ and abbreviated (n_1, n_2, n_3, k) or, when only the lattice part matters, (n_1, n_2, n_3) . The zero section is the line Γ_5 (see Remark 4.1). For a section S , $C_S \subset V$ denotes the smooth rational curve it traces in $\mathbb{P}^5$; its degree is $H \cdot C_S$.

The 74 primitive solutions of (1.1) with all six terms ≤ 1343 are numbered $\#1, \dots, \#74$ by increasing common sum (Appendix C); we write a solution as $(a, b, c | x, y, z)$, the side containing the larger maximal term first. “ $k = 2$ ” means that (1.2) holds; “on V ” means that some signed permutation of the six terms lies on V . The symmetry group of V is the group of the 96 signed permutations preserving (1.4), which acts on $\mathbb{P}^5$ through a group of order 48 (Bremner's Appendix II); “up to symmetry” always refers to this group.

2. BRUDNO'S RESULTS, BRIEFLY

2.1. The 1970 family and the quartic C . Brudno [23] starts from the identity

$$(\sin \alpha \sin \beta)^6 + (\cos \alpha - \cos \beta \sin \alpha)^6 + (\cos \alpha + \cos \beta \sin \alpha)^6 = 1 + \cos^6 \alpha + (\cos \beta \sin \alpha)^6,$$

which “has only one positive rational solution” in $\cos^2 \beta$, namely $\cos^2 \beta = (1 + 9 \cos^2 \alpha)^{-1}$. Rationalising with $\cos \alpha = 2xy/(x^2 + y^2)$, $\sin \alpha = (x^2 - y^2)/(x^2 + y^2)$, $\cos \beta = (x^2 + y^2)/z$, $\sin \beta = 6xy/z$ reduces the problem to the genus-one quartic

$$C: \quad z^2 = x^4 + 38x^2y^2 + y^4 \quad (2.1)$$

(his equation (6)), and clearing denominators gives the family

$$\begin{aligned} A &= 6xy(x^2 - y^2), & B &= 2xyz - (x^4 - y^4), & C &= 2xyz + (x^4 - y^4), \\ D &= (x^2 + y^2)z, & E &= 2xyz, & F &= x^4 - y^4, \end{aligned} \quad (2.2)$$

with $A^6 + B^6 + C^6 = D^6 + E^6 + F^6$ and $A^2 + B^2 + C^2 = D^2 + E^2 + F^2$ identically modulo (2.1). Note $B = E - F$, $C = E + F$. The seed $(x, y, z) = (1, 2, 13)$ gives $36^6 + 37^6 + 67^6 = 15^6 + 52^6 + 65^6$, the second solution of [58] (#2 below); the point $P_0 = (2 : 1 : 13)$ of C gives the same solution. Brudno then iterates two classical recursions on C : Lebesgue's [60] (quoted via Dickson [40])

$$x_2 = x_1^4 - y_1^4, \quad y_2 = 2x_1y_1z_1, \quad (2.3)$$

and Desboves's [39]

$$x_2 = x_1(4y_1^4z_1^2 - g^2), \quad y_2 = y_1(4x_1^4z_1^2 - g^2), \quad g = x_1^4 - y_1^4. \quad (2.4)$$

From the seed, (2.3) gives the solution

$$1351769^6 + 11601720^6 + 15873751^6 = 7260991^6 + 8612760^6 + 16171009^6$$

and (2.4) a 17-digit solution; the “irrational” parameter choice $x_1 : y_1 = 17 \pm 7\sqrt{5}$ gives $902^6 + 10591^6 + 10947^6 = 2618^6 + 9689^6 + 11493^6$. These are the solutions $4P$, $6P$, $3P$ of the next paragraph.

2.2. Brudno–Kaplansky: the curve E_1 . Brudno and Kaplansky [25] substitute (their (4))

$$y = pa + (q - p)b, \quad z = (q - p)a - pb, \quad v = (q - p)a + qb, \quad w = qa - (q - p)b,$$

under which (1.1) and (1.2) reduce to $x^2 - u^2 = (q^2 - p^2)(a^2 + b^2)$ and $xu = pqa^2 - (3p^2 - 5pq + 3q^2)ab - pqb^2$ (“somewhat remarkably, after division by 3 the expression (7) is exactly the square of ...”). This is (1.5)–(1.6) with $(p, q) = (1, \lambda)$ and v replaced by $-v$. They pursue only $(p, q) = (0, 1)$, i.e. $v = y - z$, $w = y + z$ (the shape of (2.2)), where the system becomes $x^2 - u^2 = a^2 + b^2$, $xu = 3ab$ and, with $T = 1/(x - 1)$, $W = (2T + 1)x/a$, the cubic

$$E_1: \quad W^2 = (2T + 1)(10T^2 + 2T + 1). \quad (2.5)$$

On E_1 , $R = (0, 1)$ has order 4 and generates the torsion (a check with Lutz's criterion [63]; in textbook form the Nagell–Lutz theorem [67, 63], [83, VIII.7] [**TODO: Lutz's “Theorem III, p. 244” cited by Brudno–Kaplansky was not consulted**]), $P = (-1/4, 3/4)$ has infinite order and corresponds to the trivial solution $(3, 1, 2; 2, 3, 1)$, the eight points $\pm S + jR$ give the same solution, Brudno's 1970 solutions “pair off with $2P, 3P, 4P$ and $6P$ ”, and $5P$ is computed (with transcription errors corrected in [24]). A note added in proof says: “John Tate showed us how to prove that G^* is generated by the image of P . The solution of Eqs. (1)–(3) is thus complete.” No argument is recorded. Inverting the substitutions, a point (T, W) of E_1 gives the solution

$$x = (T + 1)W, \quad y = (2T + 1)(T + 1), \quad z = -TW, \quad u = 3T(2T + 1), \quad v = y - z, \quad w = y + z, \quad (2.6)$$

which satisfies (1.1) and (1.2) identically modulo (2.5) (verified by computation).

Proposition 2.1. E_1 is the curve 480f4 of Cremona's tables [33], with minimal model $y^2 = x^3 - x^2 + 15x + 225$ reached by $(T, W) \mapsto (20T + 5, 20W)$. Under this map $P \mapsto (0, 15)$, $R \mapsto (5, 20)$ and the 2-torsion point $(-1/2, 0) \mapsto (-5, 0)$. Cremona's table lists $(0, 15)$ as the generator of $E_1(\mathbb{Q})$ modulo torsion and $(5, 20)$ as the torsion generator; $E_1(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/4$, the rank being certified by 2-descent, and $\hat{h}(P) = 0.516278469943469\dots$ is the regulator. Thus Tate's unrecorded argument is replaced by the tables: P generates $E_1(\mathbb{Q})/\text{tors}$.

Proof. Verified with PARI/GP and against Cremona's files `allcurves` and `allgens`: conductor $480 = 2^5 \cdot 3 \cdot 5$, minimal discriminant $-2^{12} \cdot 3^2 \cdot 5^4$, $j = 2^6 \cdot 11^3/(3^2 \cdot 5^4)$, Kodaira types I_3^*, I_2, I_4 at 2, 3, 5; `ellrank` returns lower and upper bound 1; `ellsaturation` of $\{P\}$ at primes ≤ 200 returns $\{P\}$; Cremona's listed generator is $(0, 15) = P$ and his regulator equals $\hat{h}(P)$. [**TODO: The LMFDB page of 480f4 could not be loaded in our environment; the LMFDB label may differ from Cremona's.**] $\square$

Table 1 lists the multiples nP , $n \leq 7$, in (T, W) -coordinates together with the corresponding solutions (2.6); $n = 2, 3, 4, 5, 6$ are exactly Brudno's solutions of §2.1 and the corrected $5P$ of [24]; $n = 7$ is new.

TABLE 1. Multiples of $P = (-1/4, 3/4)$ on E_1 and the solutions (2.6) (absolute values, primitive).

n	$nP = (T, W)$; solution
1	$(-1/4, 3/4); (3, 2, 1 2, 1, 3)$ (trivial)
2	$(-3/16, -25/32); (65, 52, 15 36, 67, 37)$
3	$(119/4, 2937/4); (10947, 902, 10591 2618, 11493, 9689)$
4	$(-2479/8000, 114231/160000); (16171009, 8612760, 7260991 11601720, 1351769, 15873751)$
5	$(-111961/917764, -363593823/439608956);$ $(165809277507, 151561337462, 23038103009 63175337782, 174599440471, 128523234453)$
6	$(111355677/15335056, 2756217479575/30026039648);$ $(14313884517104705, 2420573921978212, 12581285648588145 $ $6382739481279084, 15001859570566357, 10160711726609933)$
7	$(-84669865201/231320597764, 36510434863885803/55627746029688956);$ $(10138323873382807040643, 4138890740067511917838, 5853434897467259943361 $ $7168855857422411252878, 1714544157399748025523, 9992325637534771861199)$

Proposition 2.2 (the 1970 quartic and the two recursions). *Let $C_c: z^2 = x^4 + cx^2y^2 + y^4$ with origin $O = (1 : 0 : 1)$, and let $C = C_{38}$.*

- (a) *Jac(C) is the curve 480f1 (minimal model $y^2 = x^3 - x^2 - 30x + 72$, torsion $(\mathbb{Z}/2)^2$, rank 1), the optimal curve of the isogeny class of $E_1 = 480f4$; the class 480f consists of 480f1, ..., 480f4 with 480f1 2-isogenous to each of the other three. In particular Jac(C) is 2-isogenous to, but not isomorphic to, E_1 .*
- (b) *The map $\varphi(x, y, z) = (x^2/y^2, xz/y^3)$ is a 2-isogeny from C_c onto $E'_c: w^2 = t^3 + ct^2 + t$ (for $c = 38$, $E'_{38} \cong 480f3$), with $\varphi(O) = O$ and $\varphi \circ \iota = -\varphi$ for $\iota(x, y, z) = (x, -y, z)$, so that $\iota = [-1]$.*
- (c) *Lebesgue's map (2.3), completed by $z_2 = x_1^8 + 2cx_1^6y_1^2 + 6x_1^4y_1^4 + 2cx_1^2y_1^6 + y_1^8$, is an endomorphism of C_c and equals [2]. Desboves's map (2.4), after reduction modulo the equation of C_c , reads $x_2 = x_1(x_1^8 - 6x_1^4y_1^4 - 4cx_1^2y_1^6 - 3y_1^8)$, $y_2 = y_1(3x_1^8 + 4cx_1^6y_1^2 + 6x_1^4y_1^4 - y_1^8)$ and, completed by $z_2 = z_1S(x_1, y_1)$ with the form S of degree 16 determined by the equation of C_c (up to sign), equals [-3]; the other choice of sign of z_2 gives [3] followed by the translation by the 2-torsion point $(1 : 0 : -1)$.*
- (d) *The seed $P_0 = (2 : 1 : 13)$ generates $C(\mathbb{Q})$ modulo torsion, and the composite of the family map (2.2) with Brudno–Kaplansky's parametrisation sends $C(\mathbb{Q})$ onto the index-two subgroup $2\mathbb{Z}P$ of $E_1(\mathbb{Q})/\text{tors}$. Consequently, starting from any rational point of C , the recursions (2.3) and (2.4) only reach the solutions $2^a3^b \cdot 2P$ ($a, b \geq 0$) of E_1 , never $3P$ or $5P$; Brudno's $3P$ came from the $\mathbb{Q}(\sqrt{5})$ -point $x : y = 17 \pm 7\sqrt{5}$ of C .*

Proof. (a) PARI's `ellfromeqn` turns (2.1) into $Y^2 = X^3 + 38X^2 - 4X - 152 = (X + 38)(X - 2)(X + 2)$, whose minimal model is $[0, -1, 0, -30, 72] = 480f1$; the isogeny matrix of the class 480f is read off Cremona's file `allisog`, and `ellisisom` confirms that $480f1 \not\cong 480f4$. (b) Direct substitution; E'_{38} has minimal model $[0, -1, 0, -480, 4212] = 480f3$. (c) The identities $\varphi \circ \text{Leb} = [2] \circ \varphi$ and $t(\varphi \circ \text{Des}) = t([3] \circ \varphi)$ hold as identities of rational functions of x, y, z, c modulo the equation of C_c (checked with `sympy`; the doubling and tripling formulas on E'_c were written out explicitly). A morphism $f: C_c \rightarrow C_c$ with $\varphi \circ f = [n] \circ \varphi$ differs from $[n]$ by a constant in $\ker \varphi$, which vanishes if $f(O) = O$; Lebesgue's map fixes O , so it is [2]. For Desboves's map the two lifts $z_2 = \pm z_1S$ give $\varphi \circ \text{Des} = \pm[3] \circ \varphi$, each identity holding on a closed subset and the two covering C_c , so one of them holds identically; at $P_0 = (2, 1, 13)$, $c = 38$, one finds $\varphi(P_0) = (4, 26)$, a point of infinite order, and $\varphi(\text{Des}P_0) = -[3]\varphi(P_0)$ for the lift $z_2 = +z_1S$; since $\text{Des}(O) = (-1 : 0 : 1) = O$ in the weighted projective plane, that lift is [-3]. The other lift differs by $(x, y, z) \mapsto (x, y, -z)$, which is the reflection $R \mapsto (1 : 0 : -1) - R$, whence the last statement. (d) $\hat{h}(\varphi(P_0)) = 2.06511387977388 \dots$ equals the regulator of 480f3 in Cremona's table, so $\varphi(P_0)$ generates $E'(\mathbb{Q})/\text{tors}$; as $\deg \varphi = 2$, P_0 generates $C(\mathbb{Q})/\text{tors}$.

The family map (2.2) is a morphism from C onto the Brudno–Kaplansky quartic curve in $\mathbb{P}^3$ (the solutions (2.2) have $B = E - F$, $C = E + F$ and satisfy (1.2)), which Brudno–Kaplansky’s substitutions identify with E_1 ; so $\psi: C \rightarrow E_1$ is a morphism of genus-one curves, $\psi = h + c$ with h a homomorphism. By [25] the seed, its Lebesgue image $2P_0$ and its Desboves image $-3P_0$ give the solutions of $2P$, $4P$, $6P$, and a solution determines its point on E_1 up to $\pm$ and the torsion (a signed permutation preserving the Brudno–Kaplansky locus induces an automorphism of the space quartic $E_1 \subset \mathbb{P}^3$ preserving its hyperplane class, hence of the form $R \mapsto \pm R + t$ with $4t = 0$). Writing $h(P_0) + c \equiv \varepsilon_1 2P$, $2h(P_0) + c \equiv \varepsilon_2 4P$, $-3h(P_0) + c \equiv \varepsilon_3 6P$ modulo torsion ($\varepsilon_i = \pm 1$), subtraction gives $h(P_0) \equiv (2\varepsilon_2 - \varepsilon_1)2P$ and $-4(2\varepsilon_2 - \varepsilon_1) = 3\varepsilon_3 - \varepsilon_1$, which forces $\varepsilon_2 = \varepsilon_1 = -\varepsilon_3$, $h(P_0) \equiv \pm 2P$ and c torsion. Hence $\psi(C(\mathbb{Q})) \equiv 2\mathbb{Z}P$ modulo torsion. The family map is $2 : 1$ onto its image (the fibre over the image of $(1 : 2 : 13)$ is $\{(1 : 2 : 13), (2 : 1 : -13)\}$, verified by computation), so h has degree 2 and is the 2-isogeny $480f1 \rightarrow 480f4$, which therefore sends the generator P_0 to $\pm 2P$. $\square$

Remark 2.3. Brudno’s attributions of (2.3) to Lebesgue and of (2.4) to Desboves are quoted from his paper; we did not locate the formulas in the original memoirs [TODO: (the Numdam summaries of [60] and [39] were consulted, not the full texts; the attribution most plausibly comes from Dickson [40], Ch. XXII, p. 637)].

2.3. The 1976 quartic. Brudno [24] states (1.3) without derivation (“The principal aim of this paper is to exhibit the following explicit solution to (1) in terms of parameters m, n ”), and records: the identity $3x + y + z = 3u + v + w$ (his (5)); the substitution $m' : n' = -3(m + n) : (m + 3n)$ (his (6)), which yields $x' : y' : z' : u' : v' : w' = u : v : w : x : y : z$ (his (7)); the scale factor is 36, “so any particular primitive solution ... arises from two distinct ratios $m : n$ ”; the content of the six forms, which has no prime factor > 5 ; the geometric remark that (x, y, z) and (u, v, w) are lattice points on a common sphere, a common surface $X^6 + Y^6 + Z^6 = b$ and a double cone of axis $3 : 1 : 1$; and the coverage statement: “Of the ten smallest primitive solutions to (1), listed in [LPS], all but the sixth satisfy (2). Only the second satisfies (3), while (4) gives rise to all except the seventh (and, of course, the sixth).” We add two facts, verified by computation: the six forms (1.3) satisfy no identity $\sum x^k = \sum u^k$ for $k = 1, 3, 4, 5$; and their 6×5 coefficient matrix has rank 5, so $3x + y + z = 3u + v + w$ spans the space of linear relations and the image of $\mathbb{P}^1$ is a rational normal quartic curve spanning the hyperplane $H_{Br}: 3x + y + z = 3u + v + w$. The only hint about the origin of (1.3) is the acknowledgement of “Denis Bartley without whose help the computations could not have been performed”.

3. BREMNER’S SURFACE, AND LATER WORK

3.1. Bremner 1981. Bremner [20] introduces V as follows (pp. 544–545): the Brudno–Kaplansky substitution (1.5) is the restriction of (y, z, v, w) to one ruling of the rank-four quadric $Q(z, w) = Q(v, y)$; at Swinnerton-Dyer’s suggestion the LPS table was tested against this quadric, with the “startling observation” that all known solutions except the sixth satisfy it *and* its two images under the cyclic permutation of (x, y, z) and (u, v, w) , i.e. (1.4), after relabelling and sign changes. Cubing and adding gives (1.1); the mechanism is the identity

$$Q(s, t)^3 - Q(t, s)^3 = 2(s^6 - t^6), \quad Q(s, t) - Q(t, s) = 2(s^2 - t^2), \quad (3.1)$$

which Bremner does not write out. His main results are:

- V is a smooth complete intersection of three quadrics, hence a K3 surface, of degree $H^2 = 8$ (§2).
- V contains exactly 32 lines and 24 conics over $\mathbb{Q}(i)$ (Tables 1–2; Theorem 4.2), 8 lines and 12 conics being defined over $\mathbb{Q}$, and no irreducible cubic over $\mathbb{Q}(i)$ (Theorem 4.3). The lines and conics span $\text{NS}(V_{\mathbb{Q}(i)})$, which has rank 19 with an explicit $\mathbb{Z}$ -basis of 19 divisors (Theorem 3.3); $\text{NS}(V_{\mathbb{Q}})$ has rank 13 with basis $\Lambda_1, \dots, \Lambda_{13} = \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9 + \Gamma_{10}, \Gamma_{19} + \Gamma_{20}, \Gamma_{27} + \Gamma_{28}, Q_1, Q_2, Q_5$ (Corollary, p. 563). We recomputed the Gram matrices: discriminant 48 over $\mathbb{Q}(i)$ and 768 over $\mathbb{Q}$.

- The Brudno pencil E has Weierstrass model (1.7); by a 2-descent over $\mathbb{Q}(i)(\lambda)$, $\mathcal{E}(\mathbb{Q}(i)(\lambda)) \cong \mathbb{Z}/4 \times \mathbb{Z}^3$ (Theorem 3.2), and over $\mathbb{Q}(\lambda)$ the rank is still 3 with the generators (1.8) (Remark, p. 556: “we did verify (for fun ...)” that they generate; no proof is given). The reducible fibres are two I_2 (at $\lambda = \pm 1$: $Q_5 + Q_{10}$ and $Q_6 + Q_{12}$) and four I_4 (at the roots of PQ , i.e. $\lambda = 1 \pm i$, $(1 \pm i)/2$, each a 4-cycle of lines); there are also four I_1 fibres at the roots of $5PQ - B^2$ (Bremner lists only the reducible fibres).
- Up to the symmetries of V the irreducible curves of arithmetic genus 0 on V over $\mathbb{Q}$ of degree 4, 5, 6 are: one of degree 4, two of degree 5, one of degree 6 (Theorem 4.4, with the classes written in terms of the lines and conics).
- For a genus-one pencil F and curves C_1, C_2 with $C_i \cdot F = 1$, $R \mapsto C_1 + C_2 - R$ (group law on the fibre) is an involution of V (§5); the four explicit parametrisations of §6.2 (degrees 4, 5, 5, 6) are images of $\Gamma_1, \Gamma_8, \Gamma_1, Q_1$ under the involutions of the Brudno pencil with $C_1 + C_2 = \Gamma_3 + Q_1$, $Q_1 + Q_2$, $\Gamma_3 + \Gamma_4$, $\Gamma_6 + \Gamma_7$.
- Conjecture (§7): every curve of arithmetic genus 0 on V over $\mathbb{Q}$ is obtained from one of a finite set of curves by repeated application of automorphisms from a finite set, the degree increasing at each application of a non-linear automorphism; the analogous statement is proved for a K3 surface in the quintic fourfold in [19].

Bremner leaves open the rank of $\text{NS}(V_{\mathbb{C}})$ (“plausibly” 19) and does not mention the hyperplane $3x + y + z = 3u + v + w$, Brudno’s involution, Kodaira types, heights, or the identification of his explicit curves with Brudno’s.

3.2. Kuwata 2007. Kuwata [57] shows that V is the minimal desingularisation of a Kummer quartic surface with 16 nodes arising from a quadratic line complex, namely the Kummer surface of the Jacobian of the genus-two curve

$$\mathcal{C}: y^2 = (x^2 + 1)(x^2 + 2x + 5)(x^2 - 2x + 5),$$

whose reduced automorphism group is S_3 ; $\text{Jac}(\mathcal{C})$ is isogenous to $E \times E$ with $E: y^2 = x(x^2 + 4x + 20)$ of conductor 320 and without complex multiplication (Theorem 5.1), so that $\rho(V_{\mathbb{Q}}) = 19$ (Corollary 5.2), answering Bremner’s question. He also lists the 32 lines with their images on the two Kummer quartics, describes Bremner’s symmetry group of order 48 on the Kummer model, and tabulates the nine LPS solutions on V . In Kuwata’s framework Bremner’s three quadrics are the three $\mathbb{Q}$ -rational members of rank four of the net (the vertices of three conjugate pairs of lines of the discriminant sextic), and the three “Brudno-type” pencils $E, \delta E, \delta^2 E$ (δ the cyclic symmetry $(x, y, z, u, v, w) \mapsto (y, z, x, v, w, u)$) are their rulings (verified by computation; not in [57]).

3.3. Delorme 1992. Delorme [38] rediscovers (1.4) (his system (II)) by an elementary route: for a point of V the three 2×2 “link matrices” M_i carrying (x_i, y_i) to (x_{i+1}, y_{i+1}) have determinant -1 and $\text{tr}(M_3 M_2 M_1) = 0$; link matrices are parametrised by a conic $\cong \mathbb{P}^1$ (Theorems 1–3). In geometric language, which Delorme does not use, V is thereby birational over $\mathbb{Q}$ to the hypersurface $X_1 X_2 X_3 = Y_1 Y_2 Y_3$ of tridegree $(2, 2, 2)$ in $(\mathbb{P}^1)^3$, $X_i = s_i^2 + (s_i + t_i)^2$, $Y_i = t_i^2 + (s_i + t_i)^2$, whose three projections are genus-one pencils. Every “solving triple” yields two solutions (the two eigenvectors), which he calls conjugate; nine parametric solutions are produced, which he names $S(5; 1)$, $S(4; 4; a)$, $S(4; 4; b)$, $S(5; 9)$, $S(9; 5)$, $S(8; 8; a)$, $S(8; 8; b)$, $S(7; 11)$, $S(11; 7)$ by their degrees and the degrees of their conjugates. He notes that $S(4; 4; a)$ is Brudno’s quartic (up to exchanging x_1, x_2) and that $S(4; 4; b)$ is the same curve reparametrised by $(s, t) \mapsto (2s + 3t, -3s - 2t)$; that the seventh LPS solution lies on V but on none of his curves (Remark 2); that no solution of (1.1), (1.2) off V was known (Remark 3, “It is still an unsolved problem to find a counterexample”); and that it “would be interesting” to interpret his results in Bremner’s framework (Remark 4). In the $(\mathbb{P}^1)^3$ model the Brudno–Kaplansky curve is the fibre of the second projection over $(s_2 : t_2) = (2 : -3)$ (his Remark 1, in our reading).

3.4. Choudhry 1994 and 2000. Choudhry [29] gives an elementary involution of V (move along the line through a point of V cut out by the two 3-planes of the quadrics $Q(x, u) = Q(w, z)$ and $Q(z, w) = Q(v, y)$ through it, and take the residual intersection with the third quadric) and applies it to Brudno's quartic to obtain a new rational curve of degree 8 (his (12), p. 840); applied to a trivial conic it returns Bremner's sextic. Choudhry [30] answers the question, attributed to Guy [45], whether every solution of (1.1), (1.2) lies on V : he produces infinitely many solutions of the three equations $\sum x^r = \sum u^r$, $r = 1, 2, 6$, from an elliptic curve of rank one, and tabulates 26 of them; none lies on V under any signed permutation (verified by computation, for all 26). His smallest is $83^6 + 211^6 + 300^6 = 124^6 + 185^6 + 303^6$ with $83 + 211 - 300 = -124 - 185 + 303$.

4. BRUDNO'S QUARTIC ON V

4.1. The zero section. Bremner's Appendix I maps $E_\lambda \subset \mathbb{P}^3$ to the model (14) by $X : Y : Z = \tau : 2B\omega/(\lambda A^2\rho) : \rho$, with $\rho, \sigma, \tau, \pi, \omega$ explicit forms, and (14) to (1.7) by $x = 5PQ X/Z$, $y = 5PQ Y/Z$. His Table 6 states that the line Γ_6 (the point $(x : u : a : b) = (\lambda : 1 : 1 : 0)$ of E_λ) maps to the origin $(0 : 1 : 0)$ and Γ_5 (the point $(-\lambda : -1 : 1 : 0)$) to the 2-torsion point $(0 : 0 : 1)$.

Remark 4.1. With the map of Appendix I as printed, the point of E_λ that goes to $(0 : 1 : 0)$ is Γ_5 , not Γ_6 ; Γ_6 goes to the 2-torsion point $(0, 0)$. (The forms $(\tau, 2B\omega/(\lambda A^2\rho), \rho)$ have base points on E_λ : the plane $\rho = 0$ cuts E_λ in $2\Gamma_5 + \Gamma_6 + p$ and $\tau = 0$ in $3\Gamma_6 + p$; the limits at the base points were computed exactly from a second-order local parametrisation.) The two rows of Table 6 are thus interchanged relative to Bremner's own formula; the other rows agree with the formula, except for sign or conjugation discrepancies in the Y -coordinates of four entries (Appendix A, (B11)), which do not affect the rank computation. We take Γ_5 as zero section throughout. If one insists on Γ_6 as origin, every torsion coordinate k below becomes $k + 2 \pmod{4}$; the lattice part (n_1, n_2, n_3) , the heights, the degree formula and all curve lists are unchanged.

With this origin the eight rational lines and eight of the rational conics are the sections

$$\begin{aligned} \Gamma_5 &= 0, & \Gamma_4 &= T, & \Gamma_6 &= 2T, & \Gamma_3 &= 3T, \\ \Gamma_8 &= -g_1, & \Gamma_1 &= -g_1 + T, & \Gamma_7 &= -g_1 + 2T, & \Gamma_2 &= -g_1 + 3T, \\ Q_9 &= g_2, & Q_1 &= g_2 + T, & Q_{11} &= g_2 + 2T, & Q_3 &= g_2 + 3T, \\ Q_7 &= g_3, & Q_4 &= g_3 + T, & Q_8 &= g_3 + 2T, & Q_2 &= g_3 + 3T, \end{aligned} \quad (4.1)$$

(verified by computation) while Q_5, Q_{10}, Q_6, Q_{12} are fibre components. In particular g_1, g_2, g_3 are, up to torsion, the classes of the line Γ_8 and of the conics Q_9 and Q_7 .

4.2. The dictionary.

Theorem 4.2. *Let (x, y, z, u, v, w) be Brudno's forms (1.3).*

- (a) *The point $(v, -u, w, -x, y, -z)$ lies on V for all (m, n) . Evaluated at $(m, n) = (s, 1)$, the six forms $v, -u, w, -x, y, -z$ coincide exactly with the six polynomials of Bremner's degree-four parametrisation on p. 574 of [20]. Brudno's curve lies on V under exactly 96 signed permutations, one orbit of the symmetry group.*
- (b) *In this placement Brudno's curve is a section of the Brudno pencil: the fibre coordinate of the point with parameter $(m : n)$ is*

$$\lambda = \frac{3m + 2n}{2m + 3n},$$

and as an element of $\mathcal{E}(\mathbb{Q}(\lambda))$ the curve is

$$S = g_1 + g_3 + 2T = (1, 0, 1, 2).$$

The sixteen sections $(\pm 1, 0, 1, k)$, $(\pm 1, 1, 0, k)$, $k \in \mathbb{Z}/4$, form one orbit under the stabiliser of the pencil in the symmetry group of V , and all of them are symmetric images of Brudno's curve; no other section is.

- (c) *Brudno's curve is the unique irreducible curve of degree 4 and arithmetic genus 0 on V over $\mathbb{Q}$ up to symmetry (Bremner's Theorem 4.4); its class in Bremner's notation is $\Gamma_5 - \Gamma_6 + Q_1 + Q_2$ up to symmetry, and it is a smooth rational curve ($D^2 = -2$).*
- (d) *The hyperplane $H_{\text{Br}}: 3x + y + z = 3u + v + w$, which in the coordinates of V reads $x - 3y + z + 3u - v + w = 0$, is the linear span of the curve, and*

$$V \cap H_{\text{Br}} = C_{\text{Br}} + \Gamma_2 + \Gamma_8 + Q_9,$$

i.e. the residual intersection consists of two rational lines and a rational conic; in classes, $C_{\text{Br}} \sim H - \Gamma_2 - \Gamma_8 - Q_9$.

- (e) *Brudno's involution $(m : n) \mapsto (-3(m + n) : m + 3n)$ is the restriction to the curve of the symmetry*

$$\sigma: (x, y, z, u, v, w) \mapsto (v, u, -w, y, x, -z)$$

of V , which interchanges the first and third quadric of (1.4) (up to sign) and preserves the second. It does not preserve λ (it sends $\lambda = (3m + 2n)/(2m + 3n)$ to $(7m + 3n)/(3(m - n))$) and interchanges the Brudno pencil with one of its two symmetric copies; Brudno's curve is a section of both and a bisection of the third ("tridegree" $(1, 1, 2)$).

Proof. (a) Direct substitution into (1.4); the search over all $2^6 \cdot 6!$ signed permutations found exactly 96 placements, and Bremner's polynomials are those of $(v, -u, w, -x, y, -z)$ at $n = 1$. (b) The fibre coordinate of a point of V is the common root λ of $\lambda^2(v + y + z) - \lambda(2v + 3z) + (2v - y + z)$ and $\lambda^2(w + y - z) - \lambda(2w + 3y) + (2w + y + z)$ (solve (1.5) for λ); on Brudno's curve the gcd of these two polynomials is $(2m + 3n)\lambda - (3m + 2n)$. Since λ has degree one in $(m : n)$ the curve is a section, and its Mordell–Weil coordinates were found by reparametrising by λ , mapping to (1.7) and decomposing exactly in the basis (g_1, g_2, g_3, T) with the height pairing of §5.1; the stabiliser action is described in §5.3, and the statement that no further section is a symmetric image is the exact geometric test described there. (c) Bremner's Theorem 4.4 and its Remark; the class and $D^2 = -2$ were recomputed from the intersection numbers of the parametrised curve with the 56 lines and conics (Section 6). (d) The rank of the coefficient matrix is 5 (§2.3). The hyperplane $x - 3y + z + 3u - v + w = 0$ contains exactly the lines Γ_2, Γ_8 and the conic Q_9 among the eight rational lines and twelve rational conics (check that the linear form lies in the span of their defining forms). The residual intersection $H|_V - C_{\text{Br}}$ is an effective divisor of degree $8 - 4 = 4$ containing $\Gamma_2 + \Gamma_8 + Q_9$, which has degree 4; hence equality. As a check, $H - \Gamma_2 - \Gamma_8 - Q_9$ equals the class of Bremner's curve (i) computed in Section 6, $-\Gamma_2 + \Gamma_5 + \Gamma_7 - \Gamma_8 + (\Gamma_{27} + \Gamma_{28}) + Q_2$, using $H \sim \Gamma_1 + \Gamma_5 + Q_1 + Q_2 + Q_5$ and Bremner's relation $Q_9 \sim \Gamma_1 - \Gamma_7 - (\Gamma_{27} + \Gamma_{28}) + Q_1 + Q_5$. (e) Brudno's substitution sends (x, y, z, u, v, w) to $36(u, v, w, x, y, z)$; transported through the dictionary of (a) this is σ , and σ maps the three quadrics F_1, F_2, F_3 of (1.4) to $-F_3, -F_2, -F_1$ (direct computation). The Brudno pencil is the ruling of the third quadric (Bremner, p. 545), so σ carries it to the ruling of the first. The tridegree is computed in §5.3. $\square$

Remark 4.3. Bremner's $\lambda(s) = (3s + 2)/(2s + 3)$ for his curve (i) is the formula of (b) with $(m, n) = (s, 1)$. He identifies the curve as the image of the line Γ_1 under the involution of the Brudno pencil with $C_1 + C_2 = \Gamma_3 + Q_1$ (up to symmetry); in the coordinates (4.1) this reads $R \mapsto 3T + (g_2 + T) - R$, so $\Gamma_1 = -g_1 + T \mapsto g_1 + g_2 + 3T$, which is in the orbit of (b). Thus Brudno's quartic is, in the language of Mordell–Weil lattices, simply a conic minus a line: $g_1 + g_3 = [Q_7] - [\Gamma_8]$ up to torsion.

4.3. The published parametrisations. The literature contains fifteen explicit parametric solutions of (1.1) lying on V : Brudno's quartic, Bremner's four curves (i)–(iv), Delorme's nine, and Choudhry's octic (Choudhry's sextic is Bremner's (iv), as he says). We placed each of them on V (each in exactly 96 signed permutations), computed its class in $\text{NS}(V)$ from its intersection numbers with the 56 lines and conics (Section 6) and, when it is a section of a Brudno-type pencil, its Mordell–Weil coordinates (Section 5).

Theorem 4.4. *All fifteen curves are smooth rational curves on V ($D^2 = -2$, birationally parametrised). Up to the symmetries of V they are ten distinct curves:*

- (1) *Brudno's quartic = Bremner (i) = Delorme $S(4; 4; a) = S(4; 4; b)$; section $(1, 0, 1)$, tridegree $(1, 1, 2)$;*
- (2) *Bremner (ii) = Delorme $S(5; 1)$ (degree 5); section of all three Brudno-type pencils, coordinates $(1, 1, 1)$ or $(0, 1, 1)$, tridegree $(1, 1, 1)$;*
- (3) *Bremner (iii) = Delorme $S(5; 9)$ (degree 5); section $(1, 0, 0)$ or $(-2, 0, 0)$, tridegree $(1, 3, 3)$;*
- (4) *Bremner (iv) (degree 6; = Choudhry's sextic); section $(-1, -1, 0)$ or $(-1, 0, -1)$, tridegree $(1, 5, 4)$;*
- (5) *Delorme $S(7; 11)$; tridegree $(3, 3, 3)$, a 3-section of all three pencils;*
- (6) *Delorme $S(8; 8; a)$ and (7) $S(8; 8; b)$; tridegree $(2, 3, 3)$, distinct curves exchanged by Bremner's involution with $C_1 + C_2 = Q_1 + Q_2$ (up to symmetry);*
- (8) *Choudhry's octic; section $(0, -1, 0)$, i.e. $-g_2$, tridegree $(1, 6, 5)$: it is the image of the conic $Q_1 = g_2 + T$ under the fibrewise negation $R \mapsto -R$ (Bremner's involution with $C_1 + C_2 = \Gamma_3 + \Gamma_4$), and also the image of Brudno's quartic under the involution with $C_1 + C_2 = \Gamma_6 + \Gamma_7$;*
- (9) *Delorme $S(9; 5)$; section $(-1, 1, 1)$ or $(2, 1, 1)$, tridegree $(1, 3, 3)$; it is the image of Bremner's (iii) under the involution with $C_1 + C_2 = Q_1 + Q_2$;*
- (10) *Delorme $S(11; 7)$; tridegree $(3, 3, 3)$; $S(7; 11)$ and $S(11; 7)$ are exchanged by the involution with $C_1 + C_2 = Q_1 + Q_2$.*

In particular the curves of Delorme's conjugate pairs $S(5; 9)/S(9; 5)$, $S(8; 8; a)/S(8; 8; b)$, $S(7; 11)/S(11; 7)$ are related by Bremner's involution $R \mapsto Q_1 + Q_2 - R$ of the Brudno pencil, up to symmetry.

Proof. Computation; see Sections 5 and 6. Two smooth rational curves with classes in the same symmetry orbit are symmetric images of each other by Lemma 6.1, so the identifications follow from the classes. The images under the involutions were computed on the lattice (§6.3). $\square$

The exceptions in Delorme's Remark 2 and in Brudno's coverage statement are now explained: the seventh LPS solution #7 = (51, 113, 136 | 40, 125, 129) lies on V but on none of the fifteen curves and on no section of any Brudno-type pencil (Section 7).

5. THE MORDELL-WEIL LATTICE OF THE BRUDNO PENCIL AND THE DEGREE OF A SECTION

5.1. Heights and generators. The discriminant of (1.7) is $-160000 B^2 P^4 Q^4 (5PQ - B^2)$, of degree 24 in λ , so (1.7) is a minimal model of the K3 surface (Euler characteristic $\chi = 2$) with no singular fibre at ∞ : fibres I_4 at the four roots of PQ , I_2 at $\lambda = \pm 1$, I_1 at the four roots of $5PQ - B^2 = 9\lambda^4 - 30\lambda^3 + 47\lambda^2 - 30\lambda + 9$. We use Shioda's height pairing [80]: for sections S, S' ,

$$\langle S, S \rangle = 4 + 2(S \cdot O) - \sum_v \text{contr}_v(S), \quad \langle S, S' \rangle = 2 + (S \cdot O) + (S' \cdot O) - (S \cdot S') - \sum_v \text{contr}_v(S, S'), \quad (5.1)$$

with the usual local contributions ($\text{contr}_v(S) = i(n - i)/n$, $\text{contr}_v(S, S') = i(n - j)/n$ for $i \leq j$, when S, S' meet the components i, j of an I_n fibre, the components being numbered cyclically from the identity component). Component indices were read off the valuations of the x -coordinate on the model (1.7); the orientation is fixed by requiring T to meet component +1 at all four I_4 fibres. One finds $S \cdot O = 0$ for $S \in \{T, g_1, g_2, g_3\}$, the component data $T : (1, 1; 1, 1)$, $g_1 : (2, 0; 1, 1)$, $g_2 : (0, 1; 1, 0)$, $g_3 : (3, 0; 1, 0)$ (at the P -fibres, the Q -fibres,

$\lambda = 1$, $\lambda = -1$ respectively), and the Gram matrix

$$(\langle g_i, g_j \rangle) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}, \quad \det = 3, \quad \langle S, S \rangle = n_1^2 + 2n_2^2 + 2n_3^2 - n_1n_2 - n_1n_3 + n_2n_3. \quad (5.2)$$

(All of this was checked exactly in function-field arithmetic, and cross-checked against $\langle S, S \rangle = (4 + 2(4S \cdot O))/16$, since $4S$ meets the identity component everywhere, and against the homomorphism property of the component map.)

Lemma 5.1. *Assume Bremner's Theorems 3.2 and 3.3, and $\rho(V_{\bar{\mathbb{Q}}}) = 19$ (hypothesis (H) of Section 6, Kuwata's theorem). Then g_1, g_2, g_3 generate $\mathcal{E}(\mathbb{Q}(\lambda))$ modulo torsion, and $\mathcal{E}(\mathbb{Q}(\lambda)) = \langle g_1, g_2, g_3 \rangle \oplus \langle T \rangle \cong \mathbb{Z}^3 \times \mathbb{Z}/4$.*

Proof. Under (H), $\text{NS}(V_{\bar{\mathbb{Q}}}) = \text{NS}(V_{\mathbb{Q}(i)})$ has discriminant 48 (Theorem 3.3, recomputed), and by the Shioda–Tate sequence every section of $\mathcal{E}$ over $\bar{\mathbb{Q}}(\lambda)$ is defined over $\mathbb{Q}(i)$, so the geometric Mordell–Weil lattice is $\text{MW}(\mathbb{Q}(i)(\lambda))$, of rank 3 with torsion $\mathbb{Z}/4$ (Theorem 3.2). Shioda's determinant formula $|\det \text{NS}| = |\det \text{MW}| \cdot \prod_v n_v / |\mathcal{E}_{\text{tors}}|^2$ (with $n_v = 4, 4, 4, 4, 2, 2$ for the reducible fibres) gives $\det \text{MW}(\mathbb{Q}(i)(\lambda)) = 48 \cdot 4^2 / (4^4 \cdot 2^2) = 3/4$. Since $\det \langle g_1, g_2, g_3 \rangle = 3$, the sublattice $\langle g_1, g_2, g_3 \rangle$ has index 2 in $\text{MW}(\mathbb{Q}(i)(\lambda))$. Now $\mathcal{E}(\mathbb{Q}(\lambda))$ contains g_1, g_2, g_3 and the whole torsion subgroup $\langle T \rangle$ of $\mathcal{E}(\mathbb{Q}(i)(\lambda))$, and it is a proper subgroup of $\mathcal{E}(\mathbb{Q}(i)(\lambda))$ because the line Γ_9 is a section defined over $\mathbb{Q}(i)$ but not over $\mathbb{Q}$. Hence $\mathcal{E}(\mathbb{Q}(\lambda))/\text{tors}$ is a proper sublattice of $\text{MW}(\mathbb{Q}(i)(\lambda))$ containing a sublattice of index 2, so it equals $\langle g_1, g_2, g_3 \rangle$. $\square$

Remark 5.2. The lattice spanned by the lines and conics is primitive of rank 19 with discriminant 48, matching the Picard lattice $U(2) \oplus E_8^2 \oplus \langle -12 \rangle$ recorded in [1]. The lemma turns Bremner's “verified for fun” into a proof (conditional on (H)). The heights are $\langle g_1, g_1 \rangle = 1$, $\langle g_2, g_2 \rangle = \langle g_3, g_3 \rangle = 2$.

Remark 5.3 (specialisation at $\lambda = \infty$). The fibres of (1.7) at $\lambda = 0$ and $\lambda = \infty$ (interchanged by the symmetry $\lambda \mapsto 1/\lambda$) are both $y^2 = x^3 - 16x^2 + 100x$, which is $E_1 = 480f4$ ($c_4 = -704$, $c_6 = -198656$, $\Delta = -23040000$). Although Bremner's map has λ in a denominator, $\omega \equiv 0 \pmod{\lambda}$ and the specialisation at $\lambda = 0$ is a genuine isomorphism $E_0 \rightarrow \mathcal{E}_0$. The specialisation homomorphism $\mathcal{E}(\mathbb{Q}(\lambda)) \rightarrow \mathcal{E}_0(\mathbb{Q})$ has kernel of rank 2 ($g_2 \mapsto 3T$, $g_3 - g_1 \mapsto 2T$) and image all of $\mathcal{E}_0(\mathbb{Q}) = \langle g_1 \rangle \times \langle T \rangle$, with $g_1 \mapsto P$ (height 0.516278..., the regulator of 480f4). Mapping back to V , $g_1, 2g_1, \dots, 6g_1$ give exactly the solutions $P, \dots, 6P$ of Table 1. So Brudno–Kaplansky's generator is the specialisation of Bremner's g_1 .

5.2. The degree formula. Let $\gamma = -g_1 + g_2 + g_3$; then $\langle \gamma, \gamma \rangle = 8$ and $\langle \gamma, S \rangle = -2n_1 + 3n_2 + 3n_3$.

Theorem 5.4 (degree of a section). *For every section $S = n_1g_1 + n_2g_2 + n_3g_3 + kT$ of the Brudno pencil, the curve $C_S \subset \mathbb{P}^5$ has degree*

$$\begin{aligned} \deg C_S &= 2\langle S, S \rangle - \langle \gamma, S \rangle + 1 = 2\langle S - \gamma/4, S - \gamma/4 \rangle \\ &= 2n_1^2 + 4n_2^2 + 4n_3^2 - 2n_1n_2 - 2n_1n_3 + 2n_2n_3 + 2n_1 - 3n_2 - 3n_3 + 1. \end{aligned} \quad (5.3)$$

In particular the degree does not depend on the torsion part k , $\deg C_S \equiv n_2 + n_3 + 1 \pmod{2}$, $\deg C_S = 1$ exactly for the eight lines $S \in \{kT, -g_1 + kT\}$, and there is no section of degree 3 or 12.

Proof. We use the following facts, all verified exactly against the 56×56 intersection matrix of the lines and conics (which span $\text{NS}(V_{\mathbb{Q}(i)}) \otimes \mathbb{Q}$).

- (1) $H \sim \Gamma_1 + \Gamma_5 + Q_1 + Q_2 + Q_5$ (Bremner, Appendix II: $\Pi - \Gamma_1 - \Gamma_5 \sim Q_1 + Q_2 + Q_5$, Π his notation for the hyperplane class). Here $\Gamma_5 = O$ and Q_5 is the identity component of the I_2 fibre at $\lambda = 1$ ($Q_5 \cdot \Gamma_5 = 1$).

- (2) By (4.1), $[\Gamma_1] = -g_1 + T$, $[Q_1] = g_2 + T$, $[Q_2] = g_3 + 3T$, so $[\Gamma_1] + [Q_1] + [Q_2] \equiv \gamma$ modulo torsion; $\Gamma_1 \cdot O = 0$, $Q_1 \cdot O = Q_2 \cdot O = 1$. The fibre components met by (Γ_1, Q_1, Q_2) are $(3, 1, 2)$ at each P -fibre, $(1, 2, 3)$ at each Q -fibre, $(0, 0, 0)$ at $\lambda = 1$ and $(0, 1, 1)$ at $\lambda = -1$.
- (3) For a section S , by (1),

$$\deg C_S = H \cdot S = S \cdot O + \sum_{C \in \{\Gamma_1, Q_1, Q_2\}} C \cdot S + S \cdot Q_5.$$

By (5.1), $C \cdot S = 2 + C \cdot O + S \cdot O - \langle C, S \rangle - \sum_v \text{contr}_v(C, S)$, and $S \cdot Q_5 = [i_1(S) = 0]$, where $i_1(S)$ is the component met at $\lambda = 1$. Summing, and using that torsion is orthogonal to everything,

$$\deg C_S = 4(S \cdot O) + 8 - \langle \gamma, S \rangle - \sum_v \sum_C \text{contr}_v(C, S) + [i_1(S) = 0].$$

By (5.1) again, $4(S \cdot O) = 2\langle S, S \rangle - 8 + 2\sum_v \text{contr}_v(S)$, so

$$\deg C_S = 2\langle S, S \rangle - \langle \gamma, S \rangle + \sum_v \left(2\text{contr}_v(S) - \sum_C \text{contr}_v(C, S) \right) + [i_1(S) = 0].$$

- (4) The fibre terms are evaluated with (2). At an I_4 fibre the three curves meet the components $\{1, 2, 3\}$, and for S on component $i = 1, 2, 3$ one has $2 \cdot \frac{3}{4} - (\frac{3}{4} + \frac{1}{2} + \frac{1}{4}) = 2 \cdot 1 - (\frac{1}{2} + 1 + \frac{1}{2}) = 2 \cdot \frac{3}{4} - (\frac{1}{4} + \frac{1}{2} + \frac{3}{4}) = 0$, while for $i = 0$ all terms vanish. At $\lambda = -1$ exactly two of the curves meet component 1, so the term is $2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = 0$ if S meets component 1 and 0 otherwise. At $\lambda = 1$ no C meets component 1, so the term $2\text{contr}_1(S) + [i_1(S) = 0]$ equals 1 whether $i_1(S) = 0$ or 1. Hence $\deg C_S = 2\langle S, S \rangle - \langle \gamma, S \rangle + 1$, and the remaining expressions follow from (5.2) and $\langle \gamma, \gamma \rangle = 8$.

The last assertions: the quadratic form $2\langle S - \gamma/4, S - \gamma/4 \rangle$ is positive definite, its minimum value 1 is attained exactly at $(0, 0, 0)$ and $(-1, 0, 0)$, the parity statement is read off (5.3), and the values 3 and 12 are not represented (enumeration of the finitely many lattice points with $\deg \leq 12$). $\square$

Conceptually, γ is the Mordell–Weil class of the restriction of H to the generic fibre: $E_\lambda \subset \mathbb{P}^3$ is a space quartic, a plane section of it is a divisor of degree 4 whose sum in the group law (with origin Γ_5) is γ modulo torsion; the formula is then the standard expression of $H \cdot S$ through the height pairing. The theorem was discovered empirically on all 1372 sections with $|n_i| \leq 3$ and all 2916 sections with $|n_i| \leq 4$ (degrees up to 289), on each of which the six binary forms were computed and checked on V ; no torsion correction was ever needed.

5.3. The action of the symmetries on sections. Exactly 32 of the 96 signed permutations (16 projectively) preserve the Brudno pencil; the other 64 carry it to δE or $\delta^2 E$, the rulings of the other two quadrics of (1.4). The stabiliser acts on $\mathbb{P}_\lambda^1$ through the Klein four-group $\{\lambda, 1/\lambda, (2\lambda - 3)/(3\lambda - 2), (3\lambda - 2)/(2\lambda - 3)\}$; the elements acting trivially on λ act on every fibre as translations by torsion (e.g. $(x, y, z, u, v, w) \mapsto (-x, y, z, -u, v, w)$ is $+2T$ and $\mapsto (-x, -z, y, u, w, -v)$ is $+T$), so $S, S + T, S + 2T, S + 3T$ are always symmetric images of one another. On the lattice part the stabiliser acts through

$$\begin{aligned} \sigma_1: (n_1, n_2, n_3) &\mapsto (n_1, n_3, n_2) & [\lambda \mapsto 1/\lambda], \\ \sigma_2: (n_1, n_2, n_3) &\mapsto (-n_1 + n_2 + n_3 - 1, n_2, n_3) & [\lambda \mapsto (3\lambda - 2)/(2\lambda - 3)], \end{aligned}$$

both of which fix $\gamma/4 = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, as they must by Theorem 5.4. (All of this was verified on random sections; the action is affine, $S \mapsto A_g S + t_g$.) A symmetry outside the stabiliser can map a section of E to a section of E only if the curve is also a section of δE or $\delta^2 E$; this was decided by an exact geometric test (reparametrise and compare), and no identification beyond those listed in Table 2 occurs. For a curve C we call $(C \cdot E, C \cdot \delta E, C \cdot \delta^2 E)$ its *tridegree*; for a section of E the second and third entries are the degrees of $\lambda \circ \delta^{-1}$ and $\lambda \circ \delta^{-2}$ on the curve.

5.4. **All sections of degree ≤ 12 .** Since the form $\langle S, S \rangle - \frac{1}{2}\langle \gamma, S \rangle$ is positive definite, $\deg C_S \leq 12$ forces $|n_i| \leq 3$ (in fact all solutions have $|n_i| \leq 2$). Enumerating gives 36 lattice classes, i.e. 144 sections with torsion, in 12 orbits under the stabiliser (Table 2). Completeness of the list is conditional only on Lemma 5.1.

TABLE 2. Sections of the Brudno pencil of degree ≤ 12 , as $S = n_1g_1 + n_2g_2 + n_3g_3 + kT$ (zero section Γ_5 ; the degree does not depend on k). One row per orbit of the stabiliser of the pencil; the orbit consists of the listed lattice classes with all four values of k .

orbit	(n_1, n_2, n_3)	$\langle S, S \rangle$	deg	tridegree	identification
0	$(0, 0, 0), (-1, 0, 0)$	0, 1	1	$(1, 1, 1)$	lines $\Gamma_1, \dots, \Gamma_8$
1	$(0, 1, 0), (0, 0, 1)$	2, 2	2	$(1, 0, 1)$	conics $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_9, Q_{11}$
2	$(1, 0, 1), (1, 1, 0), (-1, 0, 1), (-1, 1, 0)$	2, 2, 4, 4	4	$(1, 1, 2)$	Brudno's quartic (= Bremner (i), $S(4; 4; a)$, $S(4; 4; b)$)
3	$(0, 1, 1), (1, 1, 1)$	5, 4	5	$(1, 1, 1)$	Bremner (ii) = Delorme $S(5; 1)$
4	$(1, 0, 0), (-2, 0, 0)$	1, 4	5	$(1, 3, 3)$	Bremner (iii) = Delorme $S(5; 9)$
5	$(-1, -1, 0), (-1, 0, -1)$	2, 2	6	$(1, 5, 4)$	Bremner (iv)
6	$(0, -1, 1), (0, 1, -1), (-1, -1, 1), (-1, 1, -1)$	3, 3, 4, 4	7	$(1, 4, 4)$	new ; carries #1, 2, 3, 22, 27, 33, 42, 54, 72
7	$(0, -1, 0), (0, 0, -1), (-2, -1, 0), (-2, 0, -1)$	2, 2, 4, 4	8	$(1, 6, 5)$	Choudhry's octic ($= -g_2$)
8	$(-1, 1, 1), (2, 1, 1)$	8, 5	9	$(1, 3, 3)$	Delorme $S(9; 5)$
9	$(2, 0, 1), (2, 1, 0), (-2, 0, 1), (-2, 1, 0)$	4, 4, 8, 8	10	$(1, 4, 5)$	new ; carries #2, 27
10	$(0, 0, 2), (0, 2, 0), (1, 0, 2), (1, 2, 0)$	8, 8, 7, 7	11	$(1, 4, 4)$	new ; carries #2, 4, 8
11	$(1, -1, 1), (1, 1, -1), (-2, -1, 1), (-2, 1, -1)$	4, 4, 7, 7	11	$(1, 6, 6)$	new ; carries #2, 20

Theorem 5.5. Assume Lemma 5.1. Up to the symmetries of V , the rational curves on V that are sections of the Brudno pencil and have degree ≤ 12 are exactly the twelve of Table 2. Eight of them are the known curves (the lines, the conics, and the six curves of Theorem 4.4 that are sections); the four others, of degrees 7, 10, 11, 11, are new.

The parameter of every section curve is λ itself: writing $\lambda = m/n$, the six coordinates are binary forms $f_i(m, n)$ of degree $\deg C_S$ (content removed, sign normalised). For the degree-7 curve, the representative $S = -g_2 + g_3 = (0, -1, 1, 0)$ gives

$$\begin{aligned}
x &= 27m^7 - 108m^6n + 189m^5n^2 - 152m^4n^3 + 47m^3n^4 + 10m^2n^5 - 11mn^6 + 2n^7, \\
y &= -9m^7 - 3m^6n + 91m^5n^2 - 169m^4n^3 + 149m^3n^4 - 73m^2n^5 + 21mn^6 - 3n^7, \\
z &= -18m^7 + 117m^6n - 254m^5n^2 + 295m^4n^3 - 210m^3n^4 + 87m^2n^5 - 18mn^6 + n^7, \\
u &= -18m^7 + 69m^6n - 106m^5n^2 + 119m^4n^3 - 104m^3n^4 + 61m^2n^5 - 20mn^6 + 3n^7, \\
v &= 27m^7 - 114m^6n + 241m^5n^2 - 298m^4n^3 + 213m^3n^4 - 86m^2n^5 + 19mn^6 - 2n^7, \\
w &= -9m^7 + 75m^6n - 159m^5n^2 + 155m^4n^3 - 75m^3n^4 + 21m^2n^5 - 5mn^6 + n^7.
\end{aligned}$$

with $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$ and $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$ identically. It passes through nine of the 74 solutions of Appendix C: $(m, n) = (1, 3)$ gives Subba Rao's #1, $(3, 2)$ gives #2, $(1, 4)$ gives #3, $(-1, 3)$ gives #22, $(3, 7)$ gives #27 = $(177, 281, 508 \mid 148, 299, 507)$, $(2, 1)$ gives #33, $(3, 1)$ gives #42 = $(353, 375, 770 \mid 158, 501, 763)$, $(7, 3)$ gives #54 and $(4, 1)$ gives #72. After Brudno's quartic (19 of 74) it is the most "productive" known curve in this range; #27 and #42 lie on no previously published curve. The degree-10 curve, representative $S = -2g_1 + g_3 = (-2, 0, 1, 0)$, is

$$\begin{aligned}
x &= 585m^{10} - 3606m^9n + 11770m^8n^2 - 23192m^7n^3 + 28046m^6n^4 - 16020m^5n^5 \\
&\quad - 6048m^4n^6 + 19656m^3n^7 - 16767m^2n^8 + 7290mn^9 - 1458n^{10}, \\
y &= 135m^{10} - 2145m^9n + 9443m^8n^2 - 23460m^7n^3 + 39530m^6n^4 - 50022m^5n^5
\end{aligned}$$

$$\begin{aligned}
& + 50994m^4n^6 - 41796m^3n^7 + 25839m^2n^8 - 10449mn^9 + 2187n^{10}, \\
z = & -468m^{10} + 2190m^9n - 2703m^8n^2 - 4684m^7n^3 + 23172m^6n^4 - 41448m^5n^5 \\
& + 44622m^4n^6 - 31428m^3n^7 + 14904m^2n^8 - 4374mn^9 + 729n^{10}, \\
u = & 324m^{10} - 3186m^9n + 11061m^8n^2 - 20200m^7n^3 + 18900m^6n^4 - 1404m^5n^5 \\
& - 20898m^4n^6 + 30456m^3n^7 - 23328m^2n^8 + 10206mn^9 - 2187n^{10}, \\
v = & 333m^{10} - 783m^9n - 1799m^8n^2 + 13668m^7n^3 - 33674m^6n^4 + 47526m^5n^5 \\
& - 41202m^4n^6 + 21060m^3n^7 - 4131m^2n^8 - 1215mn^9 + 729n^{10}, \\
w = & -603m^{10} + 3534m^9n - 9974m^8n^2 + 20044m^7n^3 - 33442m^6n^4 + 48336m^5n^5 \\
& - 55584m^4n^6 + 46980m^3n^7 - 26811m^2n^8 + 9234mn^9 - 1458n^{10}.
\end{aligned}$$

The two curves of degree 11 are given in Appendix D; the full list of the 144 sections with their forms is in the accompanying data (Appendix B).

Remark 5.6. The new curves are not far from the old ones in the lattice: the degree-7 orbit is $\iota_{\Gamma_3+Q_1}(C_{\text{Br}})$, the degree-10 orbit is $\iota_{\Gamma_3+Q_1}(\text{Bremner (iii)})$, and the two degree-11 orbits are $\iota_{\Gamma_3+Q_1}(C_{\text{Br}})$ and $\iota_{\Gamma_3+Q_1}(\text{Bremner (iv)})$ up to symmetry, where $\iota_{C_1+C_2}$ is Bremner's involution $R \mapsto C_1 + C_2 - R$ (§6.3). The negatives of the quartic sections $(1, 0, 1) + kT$, $(1, 1, 0) + kT$ are sextics (orbit 5), but the negatives of $(-1, 0, 1) + kT$, $(-1, 1, 0) + kT$ form the orbit 6: the degree-7 curve is “minus Brudno’s quartic” for the other half of its torsion translates. On the fibre $\lambda = 3$, which carries seven of the 74 solutions, $\mathcal{E}_3(\mathbb{Q}) \cong \mathbb{Z}^2 \times \mathbb{Z}/4$ with basis g_1, g_2 (the specialisation has $g_3 = g_1 + T$) and $\#1 = -g_1 + g_2 + 2T$, $\#42 = g_1 - g_2 + 2T = -\#1$: Subba Rao’s solution and $\#42$ are negatives of each other.

6. SMOOTH RATIONAL CURVES OF DEGREE ≤ 12 ON V

6.1. Two lemmas. Throughout this section we assume

(H) $\rho(V_{\bar{\mathbb{Q}}}) = 19$ (Kuwata, Corollary 5.2; also [1]).

Since $\text{NS}(V_{\mathbb{Q}(i)}) = \text{NS}(V_{\bar{\mathbb{Q}}})^{\text{Gal}}$ is a primitive sublattice of rank 19 (Bremner, Theorem 3.3), (H) gives $\text{NS}(V_{\bar{\mathbb{Q}}}) = \text{NS}(V_{\mathbb{Q}(i)}) = \mathbb{Z}\langle \Gamma_1, \dots, Q_{24} \rangle$. The lattice computations below do not use (H); the statements about curves over $\bar{\mathbb{Q}}$ do.

Lemma 6.1. *Let X be a K3 surface over an algebraically closed field of characteristic 0, H ample, and $D \in \text{NS}(X)$ with $D^2 = -2$ and $D \cdot H > 0$. Then D is effective, and D is the class of an irreducible curve if and only if $D \cdot C \geq 0$ for every irreducible curve C with $C^2 = -2$ and $C \cdot H < D \cdot H$. In that case $|D| = \{\Gamma\}$ consists of a single curve, which is a smooth rational curve.*

Proof. Riemann–Roch gives $h^0(D) + h^0(-D) \geq \frac{1}{2}D^2 + 2 = 1$, and $h^0(-D) = 0$ since $(-D) \cdot H < 0$; so D is effective. If $D = [\Gamma]$ with Γ irreducible and $C \neq \Gamma$ is irreducible, then $D \cdot C \geq 0$; and $C \neq \Gamma$ as soon as $\deg C < \deg \Gamma$. Conversely suppose no member of $|D|$ is irreducible and take a member $\sum n_i C_i$ ($n_i > 0$, C_i irreducible). From $D^2 = \sum n_i (D \cdot C_i) = -2 < 0$ some component $C = C_i$ has $D \cdot C < 0$. Then $C^2 < 0$ (an irreducible curve with $C^2 \geq 0$ is nef), hence $C^2 = -2$ and $p_a(C) = 0$, so C is a smooth rational curve; and $C \cdot H \leq D \cdot H$, with equality only if $D - C = 0$ ($D - C$ is effective of degree 0 and H is ample), which would make D irreducible. So C is an irreducible (-2) -curve of degree $< \deg D$ with $D \cdot C < 0$. Finally, if $\Gamma \in |D|$ is irreducible and $\Gamma' \in |D|$ is another member, either $\Gamma \not\sim \Gamma'$ and $\Gamma' \cdot \Gamma \geq 0$ contradicts $\Gamma' \cdot \Gamma = \Gamma^2 = -2$, or $\Gamma' = \Gamma + (\text{effective } \sim 0) = \Gamma$. $\square$

Lemma 6.2. *Under (H), every irreducible (-2) -curve Γ on $V_{\bar{\mathbb{Q}}}$ is defined over $\mathbb{Q}(i)$, and Γ is defined over $\mathbb{Q}$ if and only if its class is fixed by complex conjugation. The irreducible (-2) -curves not defined over $\mathbb{Q}$ come in conjugate pairs with conjugate classes.*

Proof. The class $[\Gamma] \in \text{NS}(V_{\bar{\mathbb{Q}}}) = \text{NS}(V_{\mathbb{Q}(i)})$ is fixed by $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(i))$; for τ in that group, Γ^τ is an irreducible curve with the same class, so $\Gamma^\tau = \Gamma$ (otherwise $\Gamma \cdot \Gamma^\tau = -2 < 0$), and Γ is defined over $\mathbb{Q}(i)$. Likewise, if $[\Gamma]$ is fixed by conjugation then $\bar{\Gamma} = \Gamma$ and Γ is defined over $\mathbb{Q}$; the converse is clear. $\square$

6.2. The enumeration. Lemma 6.1 gives an induction on the degree: the irreducible (-2) -curves of degree d are exactly the (-2) -classes D of degree d with $D \cdot C \geq 0$ for every irreducible (-2) -curve C of degree $< d$ (all of them, over $\mathbb{Q}(i)$, both members of every conjugate pair: the components of a reducible member of $|D|$ need not be defined over $\mathbb{Q}$ even when D is). For $d = 1$ the criterion is vacuous. The classes of irreducible curves over $\mathbb{Q}$ are the conjugation-fixed ones (Lemma 6.2), “up to symmetry” means orbits of the 48-element group acting on NS through its permutation of the 56 curves, and two irreducible (-2) -curves whose classes lie in the same orbit are symmetric images of each other (uniqueness in Lemma 6.1).

The (-2) -classes of degree d were enumerated as follows. Every $D \in \text{NS}$ with $D \cdot H = d$ is $d\Gamma_1 + \ell$ with $\ell \in L' = H^\perp \cap \text{NS}$, a negative definite lattice of rank 18 and discriminant -384 ; writing $D = \frac{d}{8}H + D'$, one has $D^2 = -2$ iff $-D'^2 = d^2/8 + 2$, so the classes are the points of a coset of L' on a sphere, found by a Fincke–Pohst search of a shifted ellipsoid (implemented in C with floating point, every candidate re-checked in exact integer arithmetic; since all exact norms are multiples of $1/8$ and the slack is 10^{-6} , nothing is lost). For the induction only the classes with $D \cdot C \geq 0$ for the 56 lines and conics are needed, and these 56 linear constraints were built into the tree search. Cross-checks: agreement with PARI’s `qfminim` on unshifted balls; identical results for three different base points $d\Gamma_1, d\Gamma_2, d\Gamma_5$; agreement of the constrained search with “unconstrained search + exact filter” for $d \leq 8$; invariance of every set of irreducible classes under the 48 symmetries and under conjugation; and every image of a line or conic of degree ≤ 12 under the automorphisms of §6.3 lies in the list. The classes of the fifteen known curves were computed from their parametrisations, as $\deg \gcd$ of the restrictions of the defining linear forms of each line and conic (exact because the parametrisations are birational, which was checked), and all of them occur in the enumeration.

Theorem 6.3. *Assume (H). The numbers of (-2) -classes, of irreducible (-2) -curves over $\mathbb{Q}(i)$ (all smooth rational), of their orbits under the 48 symmetries and under symmetries together with conjugation, of irreducible (-2) -curves over $\mathbb{Q}$, and of their orbits under the symmetries, are as follows for degrees $1 \leq d \leq 12$:*

d	(-2) -classes	irreducible over $\mathbb{Q}(i)$	orbits (48)	orbits (96)	over $\mathbb{Q}$	$\mathbb{Q}$ -orbits (48)
1	32	32	2	2	8	1
2	120	24	2	2	12	1
3	672	0	0	0	0	0
4	4 768	288	9	6	24	1
5	36 064	384	15	10	32	2
6	238 680	2 304	64	37	24	1
7	1 431 520	2 432	55	32	80	3
8	7 481 664	16 704	372	203	288	8
9	35 260 704	43 488	933	485	216	5
10	146 289 272	79 488	1 694	876	576	12
11	558 037 728	157 440	3 293	1 699	848	19
12	1 916 934 720	365 472	7 715	3 896	504	11

In degree 1 the curves are precisely the 32 lines, in degree 2 precisely the 24 conics, and there is no curve of degree 3 (Bremner’s Theorems 4.2 and 4.3). Over $\mathbb{Q}$, in degrees 4, 5, 6 there are exactly 1, 2, 1 orbits, containing Bremner’s four divisors (34) (Theorem 4.4), with orbit sizes 24, $8 + 24$, 24. Representatives of all $\mathbb{Q}$ -orbits are listed in Appendix D.

Remark 6.4 (Bremner’s criterion). In degrees 4–7 a (-2) -class with non-negative intersection with all lines and conics is automatically irreducible, so Bremner’s proof of Theorem 4.4 is complete as stated; from degree 8 on this fails (degree 8: 24 768 such classes, 16 704 irreducible; degree 12: 1 462 176 against 365 472), and the induction above is necessary.

Corollary 6.5. *Over $\mathbb{Q}$ and up to symmetry there are, besides the ten known curves of Theorem 4.4, 2 new smooth rational curves of degree 7, 5 of degree 8, 4 of degree 9, 12 of degree 10, 18 of degree 11 and 11 of degree 12. Exactly four of these 52 are sections of a Brudno-type pencil, with tridegrees $(1, 4, 4)$ in degree 7, $(1, 4, 5)$ in degree 10, and $(1, 6, 6)$, $(1, 4, 4)$ in degree 11: they are the four new section curves of Theorem 5.5. All the others are multisections of all three Brudno-type pencils. Among the known curves, Delorme's $S(7; 11)$ and $S(11; 7)$ have tridegree $(3, 3, 3)$ and stabiliser of order 6 in the symmetry group (orbit size 8); $S(8; 8; a)$, $S(8; 8; b)$ have tridegree $(2, 3, 3)$.*

6.3. Involutions and Bremner's conjecture in degree ≤ 12 . For a genus-one pencil F with sections C_1, C_2 (or a bisection C), Bremner's involution $\iota_{C_1+C_2}: R \mapsto C_1 + C_2 - R$ acts on NS as follows (his Lemma 5.3 and Theorem 5.4): it interchanges C_1 and C_2 , acts on the cyclic components j of each I_n fibre by the reflection $j \mapsto a + b - j$ (a, b the components met by C_1, C_2), and acts as -1 on the orthogonal complement of the span of C_1, C_2 and the fibre components. We built these 19×19 integer matrices from first principles for the 532 involutions $\iota_{C_1+C_2}$ (C_1, C_2 among the 32 sections of E among lines and conics, with repetition) and the 4 bisections, checked that each is an integral isometric involution fixing E , and recovered Bremner's four 13×13 matrices of §6.1 and his claims about the images of $\Gamma_1, \Gamma_8, \Gamma_1, Q_1$. Together with the symmetries these generate the same group as the symmetries and all fibrewise maps $R \mapsto \pm R + S$, $S \in \text{MW}(E/\mathbb{Q}(i)(\lambda))$, because the 36 horizontal curves and the 20 fibre components generate NS (Theorem 3.3).

Conjecture 6.6 (Bremner, §7). *Let Γ be a curve defined over $\mathbb{Q}$ lying on V , of arithmetic genus 0. Then Γ can be generated from one of a finite set of curves of genus 0 by (possibly repeated) application of certain automorphisms of the surface taken from a finite set, in such a way that at each application of an automorphism which is not a symmetry, the degree of the curve under consideration increases.*

Theorem 6.7. *Assume (H), and consider only classes of degree ≤ 12 .*

- (a) *The orbit of the lines and conics under the group generated by the 48 symmetries and the involutions of the Brudno pencil (equivalently, by the symmetries and the full groups $\text{MW} \rtimes \{\pm 1\}$ of the three Brudno-type pencils) contains all $\mathbb{Q}$ -rational irreducible (-2) -curves of degree ≤ 6 , but only 48 of the 80 of degree 7: Delorme's $S(7; 11)$ and the second new orbit of degree 7 (both of tridegree $(3, 3, 3)$) are not reached, and in degrees 8, 9, 10, 11, 12 only $5/8, 2/5, 3/12, 8/19, 3/11$ of the $\mathbb{Q}$ -orbits are reached. Over $\mathbb{Q}(i)$ the orbit misses curves already in degree 4 (96 of the 288 quartics are reached).*
- (b) *There are 30, 192, 3688 primitive nef isotropic classes F with $F \cdot H = 4, 5, 6$, forming $4 + 6 + 70 = 80$ orbits under symmetries and conjugation, $2 + 2 + 10 = 14$ of them $\mathbb{Q}$ -rational; all have reducible fibres of types I_n ($n \leq 6$) and I_n^* . With the involutions of all 80 pencils (built from their sections of degree ≤ 2 , or their eight sections of lowest degree) and their conjugates, every one of the 668 056 irreducible (-2) -curves of degree ≤ 12 over $\mathbb{Q}(i)$ lies in the orbit of the lines and conics. With only the 48 symmetries and the 772 $\mathbb{Q}$ -rational involutions of the 14 $\mathbb{Q}$ -rational pencils, and requiring the degree to increase strictly at every involution step, every $\mathbb{Q}$ -rational irreducible (-2) -curve of degree ≤ 12 is reached from a rational line or conic, in at most three involution steps (one step for most orbits, two for 16 orbits, three for two orbits of degree 12).*

Thus Bremner's conjecture holds in degree ≤ 12 with base set $\{\Gamma_1, \dots, \Gamma_8, Q_1, \dots, Q_{12}\}$ and the finite set of automorphisms $\{\text{symmetries}\} \cup \{\mathbb{Q}$ -rational involutions of the 14 $\mathbb{Q}$ -rational pencils of degree $\leq 6\}$, but not with the involutions of the Brudno pencil alone.

The 14 $\mathbb{Q}$ -rational pencil orbits are: in degree 4, E and $\Gamma_5 + \Gamma_8 + (\Gamma_{19} + \Gamma_{20})$ (both $I_4^4 I_2^2$; the latter is a symmetric image of Bremner's divisor (32), $\Gamma_5 + \Gamma_6 + (\Gamma_9 + \Gamma_{10})$); in degree 5, two orbits of size 24 with fibres $I_4^3 I_3 I_2^3$; in degree 6, ten orbits with fibres $I_4^4, I_6^2 I_2^4, I_6^2 I_3^2 I_2^2, I_0^{*2} I_4 I_2^4$ (two orbits); $I_6^2 I_4$ (two); $I_6^2 I_2^2$ (two, one of them $H - \Gamma_1 - \Gamma_6$ of Bremner's Appendix II); $I_6^2 I_3^2$ ($H - \Gamma_1 - \Gamma_5 \sim Q_1 + Q_2 + Q_5$). For instance Delorme's $S(7; 11)$ is $\iota_{\Gamma_{21} + \Gamma_{22}}^F(\Gamma_1)$ and $S(11; 7)$ is

$\iota_{Q_1+Q_2}^F(\Gamma_1)$ for the pencil $F = -\Gamma_1 - \Gamma_2 + 2\Gamma_5 + (\Gamma_9 + \Gamma_{10}) + (\Gamma_{19} + \Gamma_{20}) + (\Gamma_{27} + \Gamma_{28})$ of type $I_0^{*2}I_4I_2^4$, and Choudhry's octic is $\iota_{Q_5+Q_{10}}^F(Q_1)$ for $F = -\Gamma_1 + 2\Gamma_5 + \Gamma_8 + (\Gamma_{19} + \Gamma_{20}) + (\Gamma_{27} + \Gamma_{28})$ of type $I_6^2I_2^2$. One-step constructions for all but two of the 52 new orbits are recorded in the data; the Brudno-pencil constructions, where they exist, are listed in Appendix D.

Caveats: the involutions of a pencil were built from a subset of its sections, so the group used is in general a subgroup of the full $MW \rtimes \{\pm 1\}$ of that pencil; and the chains found have all intermediate classes of degree ≤ 12 (automatic in the monotone version).

7. THE NUMERICAL CENSUS

The 74 primitive solutions of (1.1) with all terms ≤ 1343 were found by an exhaustive enumeration (1343 is the largest N with $3N^6 < 2^{64}$; 286 equal-sum pairs of triples, 74 primitive; complete for every solution with largest term ≤ 1343 , hence for every common sum $\leq 1343^6 \approx 5.86 \cdot 10^{18}$; no sum in this range has three representations). The ten with sum $\leq 2.5 \cdot 10^{14}$ reproduce Table IX of [58] as quoted by Brudno. The flags of Appendix C were computed as follows: “ $k = 2$ ” by direct test; “on V ” by testing all 46080 signed permutations against (1.4); membership in the fifteen known curves and the four new section curves by evaluating the parametrisations in boxes of parameters; “section” by the exact test described below; the signed relations by enumeration.

Theorem 7.1. *Of the 74 solutions:*

- (i) 59 satisfy (1.2); the 15 exceptions are #6, 19, 23, 24, 29, 30, 32, 40, 44, 48, 55, 56, 57, 60, 62.
- (ii) 41 lie on V ; each in exactly 48 projective placements (one orbit of the symmetry group), except #2 = (36, 37, 67 | 15, 52, 65), which lies on V in two inequivalent ways (96 placements; the pairings {36, 65}, {37, 15}, {67, 52} and {36, 65}, {37, 52}, {67, 15}).
- (iii) Of the 41: 31 lie on one of the fifteen known curves; 32 lie on a section of one of the three Brudno-type pencils; 33 on a known curve or a section; #27 and #42 lie on a section (the new degree-7 curve) but on no known curve; #21 = (124, 277, 337 | 92, 311, 317) lies on a known curve (Delorme's $S(7; 11)$) but on no section; and the eight solutions

$$\#7, \#37, \#41, \#49, \#50, \#53, \#58, \#71$$

lie on no known curve and on no section (for their position on the non-section curves of degree ≤ 12 see Theorem 8.2).

- (iv) The 18 solutions with $k = 2$ not on V split into the 9 with a signed relation $\pm x \pm y \pm z = \pm u \pm v \pm w$ (“type (1, 2, 6)”): #16, 25, 28, 34, 46, 47, 59, 63, 70, and 9 with $k = 2$ only: #14, 17, 26, 35, 38, 45, 61, 69, 73.
- (v) Brudno's relation $3x + y + z = 3u + v + w$ holds, with signs and for some choice of the distinguished terms, for 27 solutions: 22 on V and 5 off V (#14, 26, 35, 45 with $k = 2$, and #29 without).

7.1. Sections and rank jumps. For a solution on V , its 48 placements split into three groups of 16 according to which pair of coordinates plays (x, u) ; these are the three Brudno-type pencils through the point, and the 16 placements of one pencil give the four values of λ of one orbit of the Klein group of §5.3. For each of the 3×41 (plus two for #2) incidences we mapped the point to $\mathcal{E}_\lambda(\mathbb{Q})$ with Bremner's map, computed the specialisations of g_1, g_2, g_3, T , and decided exactly, by integral LLL on the height Gram matrix followed by verification of every relation by exact group-law arithmetic, whether the point lies in $\Lambda = \langle g_1, g_2, g_3 \rangle + \text{tors}$ (“section”), in its saturation (“saturation”), or is independent; the rank of $\mathcal{E}_\lambda(\mathbb{Q})$ was computed by 2-descent (PARI, lower bound = upper bound at all 126 incidences; the ranks 4, 5, 6 were cross-checked with `mwrnk`).

Theorem 7.2. *Of the 126 incidences, 54 are sections and 72 are independent; there is no “saturation” case. Every section through one of the 41 points has lattice coordinates in $\{0, \pm 1\}^3$.*

In every one of the 72 independent cases the fibre has rank ≥ 4 (rank 4: 45, rank 5: 23, rank 6: 4 cases). In particular, for each of the eight solutions of Theorem 7.1(iii) all three fibres through it have rank ≥ 4 : the orbit representatives λ and ranks are

$$\begin{aligned} \#7 &: \frac{5}{4}(4), -\frac{4}{3}(4), -\frac{19}{3}(4); & \#37 &: -\frac{4}{3}(4), \frac{39}{35}(5), \frac{58}{49}(4); \\ \#41 &: -\frac{4}{3}(4), -\frac{25}{7}(4), -\frac{33}{14}(5); & \#49 &: \frac{29}{9}(4), \frac{32}{31}(4), \frac{39}{8}(4); \\ \#50 &: \frac{14}{9}(4), \frac{16}{5}(4), \frac{39}{35}(5); & \#53 &: \frac{5}{4}(4), \frac{28}{25}(4), \frac{97}{67}(5); \\ \#58 &: \frac{5}{4}(4), \frac{27}{4}(5), -\frac{63}{19}(6); & \#71 &: -\frac{19}{3}(4), \frac{79}{35}(6), -\frac{63}{19}(6), \end{aligned}$$

and each of the eight points is a $\{0, \pm 1\}$ -combination of a reduced basis of $\mathcal{E}_\lambda(\mathbb{Q})$ on each of its fibres. Likewise #21 lies on fibres of rank 4, 4, 5.

So the eight “unexplained” solutions are explained after all: they are the small points of fibres whose rank jumps. Seven of them are explained a second time, by rational curves of degree 8–12 which are not sections (Theorem 8.2); #49 is not. The fibres of rank 6 at $\lambda = -63/19$ (carrying #58 and #71), $53/34$ (#74) and $79/35$ (#71) have conductors $4.9 \cdot 10^{17}$, $2.7 \cdot 10^{18}$, $1.7 \cdot 10^{19}$. The 126 incidences hit only 80 fibre classes: $\lambda = 3$ carries seven solutions (#1, 2, 8, 18, 27, 42, 54), $\lambda = -2$ seven (#2, 5, 11, 36, 39, 52, 67), $\lambda = 5/4$ and $-4/3$ six each, $\lambda = 2$ and $4/3$ five, $\lambda = 5/3$ four. The three fibres $\lambda = 3, 2, -3$ are among the four orbits where the specialisation map is not injective (ranks 2, 2, 2; and $\lambda = 0$ of rank 1); on a drop fibre every rational point lies on a section.

7.2. The rank census. Over the box $\lambda = p/q$, $1 \leq q \leq 12$, $|p| \leq 12$ (plus $\lambda = \infty$, minus $\lambda = \pm 1$): 182 values in 77 orbit classes under the Klein group. The specialised group $\langle g_1, g_2, g_3 \rangle$ has rank 3 except on the four drop orbits ($\{0, \infty, \frac{3}{2}, \frac{2}{3}\}$: rank 1, relations $g_2 = 3T$, $g_3 = g_1 + 2T$; $\{3, \frac{1}{3}, \frac{7}{3}, \frac{3}{7}\}$: rank 2, $g_3 = g_1 + T$; $\{2, 4, \frac{1}{2}, \frac{1}{4}\}$: $g_3 = 2T$; $\{-3, -\frac{1}{3}, \frac{11}{9}, \frac{9}{11}\}$: $g_3 = g_1$). Of the 77 orbits, 4 drop, 47 are generic (rank 3), 26 jump (25 to rank 4, the orbit of $-7/3$ to rank 5); at all 128 values where the rank does not jump the specialised group has index 1 in $\mathcal{E}_\lambda(\mathbb{Q})/\text{tors}$ (saturation at primes ≤ 200). For each jump an extra generator was mapped back to V with the exact inverse of Bremner’s map, producing a solution of (1.1): e.g. $\lambda = 8/5$ gives $391^6 + 1207^6 + 1724^6 = 652^6 + 1061^6 + 1741^6$ (just outside the box of the census), $\lambda = -4/3$ gives #3, the orbits of $5/4$ and $-7/5$ give #21, $\lambda = 12/11$ gives $(757, 1878, 2531 \mid 1290, 1487, 2575)$ and $\lambda = 8/3$ gives $(647, 3353, 5436 \mid 948, 3269, 5443)$. The full table is in Appendix B’s data.

7.3. Off V .

Observation 7.3. (a) (Delorme’s Remark 3, Guy’s question) The smallest solution of (1.1), (1.2) that lies on V under no signed permutation is

$$\#14 = (75, 154, 295 \mid 1, 173, 294), \quad 75^6 + 154^6 + 295^6 = 1^6 + 173^6 + 294^6 \approx 6.73 \cdot 10^{14},$$

it satisfies $3 \cdot 75 - 154 - 295 = 3 \cdot (-173) + 1 + 294$ but no signed relation of degree 1. The next ones are #16 (Choudhry’s) and #17.

- (b) The nine solutions of type (1, 2, 6) with terms ≤ 1343 are exactly the nine entries with terms ≤ 1343 of the table on p. 846 of [30]; so his table is complete in this range, and the whole class lies off V . Among all 74 solutions, a signed relation of degree k with $3 \leq k \leq 5$ occurs exactly once:

$$\#62 = (492, 752, 1055 \mid 86, 801, 1046): \quad 492^3 + 752^3 + 1046^3 = 86^3 + 801^3 + 1055^3 = 1688799832,$$

a (3, 6)-multigrade in which the two largest terms change sides (and $\sum x^2 \neq \sum u^2$). Writing $D_i = p_i^3 - q_i^3$, $S_i = p_i^3 + q_i^3$ for the pairs $(492, 86)$, $(752, 801)$, $(1055, 1046)$, one has $D_1 + D_2 = D_3$ and $D_1 S_1 + D_2 S_2 + D_3 S_3 = 0$. No family in [23, 25, 24, 20, 38, 29, 30] produces such relations.

- (c) For the 15 solutions without $k = 2$ we searched for hidden algebraic structure: signed power relations of degree ≤ 5 (only #62 above and the degree-one relation $113 - 192 + 410 = 303 - 340 + 368$ of #24, which is at chance level); linear relations with coefficients ≤ 4 ; “pair-type” quadrics $as^2 + bst + ct^2 = as'^2 \pm bs't' + ct'^2$ with $|a|, |b|, |c| \leq 5$ on pairs

of coordinates; and, exhaustively within each family and over all 720 permutations, quadrics with coefficients in $[-2, 2]$ that are sums of three binary forms on a perfect matching of the six coordinates ($1.95 \cdot 10^6$ quadrics), or with $\{-1, 0, 1\}$ coefficients on the six squares plus a 6-cycle, a star, two triangles or a path of cross terms, or on all 15 cross terms ($1.4 \cdot 10^7$). No quadric of any of these families vanishes on three or more of the 15 points, and the numbers vanishing on exactly two match a random baseline of 15 tuples of the same size. None of the 74 solutions lies on one of the 48 trivial planes $\{x_i = \pm u_{\pi(i)}\}$.

- (d) No three of the 74 points are collinear in $\mathbb{P}^5$ (in any placements). Coplanar quadruples exist and are all plane sections of fibres: $\{2, 39, 52, 67\}, \{5, 36, 39, 67\}, \{11, 36, 52, 67\} \subset E_{-2}$; $\{1, 8, 27, 42\} \subset E_3$; $\{3, 7, 37, 41\} \subset E_{-4/3}$; $\{1, 7, 21, 58\} \subset E_{5/4}$ (four points of the space quartic $E_\lambda \subset \mathbb{P}^3$ are coplanar iff they sum to the hyperplane class, which was verified).

8. THE NON-SECTION CURVES OF DEGREE ≤ 12 , AND THE EIGHT POINTS

Corollary 6.5 produced 52 orbits of $\mathbb{Q}$ -rational smooth rational curves of degree 7 to 12 on V that are not in the literature, as *classes* in $\text{NS}(V_{\mathbb{Q}})$. Four of them are sections of a Brudno-type pencil and were parametrised in Section 5; the other 48 are not sections of any of the three Brudno-type pencils, and nothing in Sections 5–7 gives their equations. Here we compute, for each of the 48 classes, the curve itself: a form cutting it out on V , generators of its ideal, and a birational parametrisation over $\mathbb{Q}$ by six binary forms. The point of the exercise is Open Problem (3) of the first version of this paper: whether the eight solutions of Theorem 7.1(iii), which lie on no known curve and on no section, lie on one of these curves. Seven of them do (Theorem 8.2).

We name the 48 curves R7a, R8a–R8e, R9a–R9d, R10a–R10k, R11a–R11p, R12a–R12k in the order of the rows of the class table of Appendix D (R for *residual*: each is the residual intersection of V with a hypersurface containing some of Bremner's lines and conics), skipping the known curves and the four section curves, which keep their names N7, N10, N11a, N11b from Table 2. C_D denotes the unique curve in the class D (Lemma 6.1); its symmetric images gC_D run through the orbit.

8.1. Equations and parametrisations.

Proposition 8.1. *For each of the 48 classes D there are $k \in \{2, 3, 4\}$ and a set $\mathcal{S}_D$ of between 5 and 16 distinct lines and conics, stable under complex conjugation, with*

$$kH - D = \sum_{C \in \mathcal{S}_D} C \quad \text{in } \text{NS}(V_{\mathbb{Q}}),$$

listed in Table 7 ($k = 2$ for R7a, R8a–R8e, R9d; $k = 4$ for R11c, R11d, R11k, R11n–R11p, R12a, R12b, R12d, R12e, R12g–R12j; $k = 3$ otherwise). The forms of degree k on $\mathbb{P}^5$ vanishing on $\bigcup_{C \in \mathcal{S}_D} C$ are $I(V)_k \oplus \mathbb{Q}f_D$ for a form f_D with rational coefficients, unique up to $I(V)_k$ and scaling, and

$$\text{div}(f_D|_V) = C_D + \sum_{C \in \mathcal{S}_D} C.$$

The ideal of C_D is $((I(V) + (f_D)) : g^\infty)$ saturated by the irrelevant ideal, where g is the product of a rational linear form vanishing on each rational $C \in \mathcal{S}_D$ and of $\ell\bar{\ell}$ for a linear form ℓ vanishing on each conjugate pair; it has Hilbert polynomial $dt + 1$, and its intersection numbers with the 56 lines and conics are those of the class D . Each C_D is parametrised birationally over $\mathbb{Q}$ by six binary forms of degree $d = \deg D$ without common factor.

How this was computed. The expressions $kH - D = \sum C$ were found by integer programming on the 56 curves (0/1 variables, conjugation-stable, minimal number of curves) and re-checked in NS. Since V is a complete intersection, $H^0(\mathbb{P}^5, \mathcal{O}(k)) \rightarrow H^0(V, \mathcal{O}_V(k))$ is onto, and a form of degree k vanishing on the reduced curves of $\mathcal{S}_D$ restricts to a section of $\mathcal{O}_V(kH - \sum C) =$

$\mathcal{O}_V(D)$, which has $h^0 = 1$ for the class of an irreducible (-2) -curve; so the kernel of the evaluation map has dimension $\dim I(V)_k + 1$ (4, 19 or 61), which was verified in every case, and f_D is any element outside $I(V)_k$ (made integral and primitive after reduction modulo a Gröbner basis of $I(V)$). Vanishing on a line is $k + 1$ linear conditions over $\mathbb{Q}(i)$, on a conic $2k + 1$ (restriction to the plane of the conic modulo its equation); f_D was checked to vanish on no other line or conic. The saturation and the Hilbert polynomials were computed in Singular; the intersection numbers $C_D \cdot C'$ as the degree of the scheme $C_D \cap C'$ over $\mathbb{Q}(i)$, and solving them against the 56 basis vectors returns exactly D . For the parametrisations, two methods were used: for the curves of degree 7 and 8, projection to $\mathbb{P}^2$ from a coordinate 3-plane, Singular's `paraPlaneCurve` (a rational parametrisation over $\mathbb{Q}$ existed in all six cases) and lifting by exact interpolation; for degree ≥ 9 , where this is too slow, the following. Rational points on C_D were collected from its intersections with the 20 rational lines and conics and the known parametrised curves; at the smallest one, p_0 , a formal power series parametrisation $\gamma(\tau)$ of C_D was computed by Newton iteration on four generators of the ideal with independent gradients at p_0 ; with e the smallest integer in $\{2, \dots, 5\}$ for which the Hilbert function of the ideal in degree e equals $ed + 1$, the forms g of degree e with $g(\gamma(\tau)) \equiv 0 \pmod{\tau^{ed-1}}$ form, modulo the ideal, a pencil $\langle g_1, g_2 \rangle$, and $\varphi = g_1/g_2$ is a function of degree one on C_D (the base divisor $(ed - 1)p_0$ cancels); φ was normalised by three rational points and the six coordinate forms recovered from γ by a Padé-type linear system in degree $\leq d$, whose solution space is one-dimensional. Each parametrisation was verified exactly: the three quadrics (1.4), hence (1.1) and (1.2), vanish identically; the six forms have degree exactly d , no common factor and content 1; f_D and every generator of the ideal vanish on them; and the intersection numbers with the 56 lines and conics, computed as degrees of gcd's over $\mathbb{Q}(i)[m, n]$ as in Appendix B, are those of D with $D^2 = -2$ (which proves birationality). The coefficients were reduced by Möbius transformations and $\text{SL}_2(\mathbb{Z})$ -reduction; the largest coefficient is below 10^4 for 41 of the 48 curves and below 10^6 for all. Appendix E prints the parametrisations of degree ≤ 9 ; all 48 are in the file `new_curves.json` accompanying the paper, together with f_D and the ideals. $\square$

For example, R7a, the unique non-section curve of degree 7, has

$$D = (-2, 0, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 0), \quad 2H - D = \Gamma_1 + \Gamma_6 + \Gamma_7 + Q_7 + Q_9 + Q_{10},$$

$$f_D = 2xz + 2yz - xu - yu + zu - u^2 + xv + yv - zv + 2uv + v^2 + xw + yw - 3zw + 2uw - w^2;$$

it passes through #1, #4 and #5, and is $\iota_{\Gamma_{10} + \Gamma_{10}}(S(11; 7))$, the image of Delorme's curve of degree 11 under the negation of the Brudno pencil with respect to the section Γ_{10} (Appendix D).

8.2. The eight points.

Theorem 8.2. *Assume (H). None of the 41 solutions on V lies on a line or a conic of V . For each of the 48 curves C_D and each of the 41 points, membership in a symmetric image of C_D was decided exactly by evaluating f_D on all 48 (for #2: 96) placements of the point, and every positive answer was certified a second time by a parameter value of the parametrisation. For the solutions of Theorem 7.1(iii):*

<i>solution</i>	<i>non-section curves of degree ≤ 12 through it</i>
#7 = (51, 113, 136 40, 125, 129)	R8c, R8d, R9b, R9c, R10a, R10c, R11a, R11e, R12b
#37 = (391, 499, 612 351, 571, 572)	R8a, R11k, R12d
#41 = (365, 579, 710 314, 653, 669)	R8a, R11b
#49 = (268, 499, 885 65, 572, 879)	<i>none</i>
#50 = (359, 519, 892 221, 603, 884)	R8e, R9b, R10g, R11i, R11n
#53 = (473, 566, 919 229, 746, 883)	R8a, R11i
#58 = (258, 751, 991 94, 813, 971)	R8c, R10c, R11i, R11k
#71 = (573, 815, 1255 1, 1123, 1143)	R11g
#21 = (124, 277, 337 92, 311, 317)	R8e, R10g, R10h, R10i, R11i, R11l, R11o

In particular seven of the eight solutions on no known curve and on no section lie on a $\mathbb{Q}$ -rational smooth rational curve of degree ≤ 12 on V , and #49 does not. Altogether 40 of the 41 solutions on V lie on a $\mathbb{Q}$ -rational smooth rational curve of degree ≤ 12 (33 on a known curve or a section curve, and these seven only on non-section curves), and exactly one, #49, lies on none.

The incidence was double-checked by a box search $|m|, |n| \leq 60$ over every parametrisation, matched against the 74 solutions: it found every exact incidence and nothing else; and the same exact test on the ten known curves and the four section curves reproduces the “curves” column of Appendix C. The full incidence table of the 48 curves against the 74 solutions is in Appendix D (Table 7, last column): the curves of degree 8 are the most populated (R8a carries #1, 31, 37, 39, 41, 52, 53), and R12a carries no solution with terms ≤ 1343 .

Remark 8.3. (i) Theorem 8.2 removes the eight points from the list of “points of V on no rational curve”, which now has one entry in the census, #49; it lies on fibres of rank 4, 4, 4 of the three Brudno-type pencils through it (Theorem 7.2). Whether #49 lies on a $\mathbb{Q}$ -rational smooth rational curve of degree 13 or more, or on a singular rational curve, we do not know; the enumeration of Section 6 stops at degree 12, and §10.7 below describes how to search for curves through a given point without enumerating all classes. (ii) The seven points are the first solutions of (1.1), (1.2) that are explained only by rational curves on V which are not sections of a Brudno-type pencil (the known parametric families are all sections, Theorem 4.4); #7 = (51, 113, 136 | 40, 125, 129), the seventh smallest solution, lies on nine of them. (iii) A solution on V typically lies on several of the 62 curves of degree ≤ 12 ; the mean over the 41 points is 4.4 and the median 3; #1 lies on 17 of them and #2 on 23. On a surface with $\rho(V_{\mathbb{Q}}) = 13$ and infinitely many rational curves this is to be expected for points of small height; the heuristic count of §9.4 says that a rational curve of degree d carries about $B^{2/d}$ points of height $\leq B$.

9. THE GEOMETRY OVER $\mathbb{C}$ OF M_6 , $M_6 \cap M_2$ AND V , AND WHAT IT SAYS ABOUT THE ARITHMETIC

Throughout, $M_6 \subset \mathbb{P}^5$ is the fourfold $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$, M_2 the quadric $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$, $W := M_6 \cap M_2$, and $V \subset W$ is Bremner’s K3 surface (1.4); F_6, F_2 are the two forms, χ_{top} the topological Euler characteristic and Frob_p the geometric Frobenius. We ask what the fundamental groups, Betti and Hodge numbers of the three varieties are, how they show up in point counts over finite fields, and whether the “complexity” of the geometry can be said to drive the richness of the arithmetic over $\mathbb{Q}$, in the style of the Weil conjectures for curves. In short: all three are simply connected; M_6 is a Calabi–Yau fourfold with $b_4 = 2606$ and 1752 Hodge classes, all algebraic, almost none Galois-invariant; W is *not* smooth—it has exactly 710 nodes—and is a threefold of general type whose resolution has $b_3 = 70$ instead of the 1300 of a smooth $(2, 6)$ complete intersection, the difference being paid for by a group of Weil divisor classes of rank 96; V is a K3 surface with $\rho = 19$, transcendental lattice $U(2) \oplus \langle 12 \rangle$, and a zeta function built from the Galois action on its lines and conics and the

symmetric square of the newform 320f2. Over finite fields the cohomology controls the counts exactly, and we verify this numerically in every case; over $\mathbb{Q}$ the relevant invariants are the Kodaira dimension and the Galois-invariant algebraic classes, not the Betti numbers (§9.4).

9.1. The sextic fourfold M_6 . Over any field containing a primitive 12th root of unity ζ , $(u, v, w) \mapsto (\zeta u, \zeta v, \zeta w)$ identifies M_6 with the Fermat sextic $X = X_6^4$: $\sum_{i=0}^5 x_i^6 = 0$, since $\zeta^6 = -1$: over $\mathbb{C}$, and over $\mathbb{F}_p$ for $p \equiv 1 \pmod{12}$; for $p \equiv 7 \pmod{12}$ the two are non-isomorphic twists over $\mathbb{F}_p$. M_6 is smooth over $\mathbb{Q}$ and modulo every $p \nmid 6$ (Jacobian ideal $(x^5, \dots, w^5)$), so $\pi_1(M_6) = 1$ and $H^i(M_6, \mathbb{Z}) \cong H^i(\mathbb{P}^4, \mathbb{Z})$ for $i \neq 4$ by Lefschetz, and $\omega_{M_6} = \mathcal{O}(6-6) = \mathcal{O}$: a Calabi–Yau fourfold.

Proposition 9.1. *A smooth sextic fourfold in $\mathbb{P}^5$ has $h^{4,0} = h^{0,4} = 1$, $h^{3,1} = h^{1,3} = 426$, $h^{2,2} = 1752$, the other $h^{p,q}$ with $p+q = 4$ vanish, and $h^{p,q} = \delta_{pq}$ for $p+q \neq 4$; hence $b_4 = 2606$, $b_4^{\text{prim}} = 2605$, $\chi_{\text{top}} = 2610$. For the Fermat sextic, $H_{\text{prim}}^4(X, \mathbb{C}) = \bigoplus_{\alpha} V(\alpha)$ over the 2605 characters $\alpha \in \{1, \dots, 5\}^6$ with $\sum \alpha_i \equiv 0 \pmod{6}$ of the group μ_6^6/μ_6 acting on X ; $V(\alpha)$ is a line of Hodge type (p, q) with $q = \frac{1}{6} \sum \alpha_i - 1$, and there are 1, 426, 1751, 426, 1 characters with $\sum \alpha_i = 6, 12, 18, 24, 30$. All 1751 primitive classes of type $(2, 2)$ are Hodge classes, and all are algebraic.*

Proof. The Hodge numbers were computed in three ways: from Hirzebruch’s generating function for the χ_y -genus of complete intersections [46, App. I, §22] with Lefschetz; from $\chi_{\text{top}} = \int c_4 = 6 \cdot 435$, $c(M_6) = (1+h)^6/(1+6h)$; and, for the Fermat sextic, from Griffiths’ $H_{\text{prim}}^{4-q,q} \cong R_{6q}$ with $R = \mathbb{C}[x_0, \dots, x_5]/(x_i^6)$, whose pieces in degrees 0, 6, 12, 18, 24 have dimensions 1, 426, 1751, 426, 1. The character decomposition and Hodge types are Weil’s [93, 81]; counting characters reproduces 1, 426, 1751, 426, 1. By Shioda’s criterion [79], $V(\alpha)$ is Hodge iff $\sum_i \langle t\alpha_i \rangle = 18$ for all $t \in (\mathbb{Z}/6)^\times = \{\pm 1\}$; for $t = -1$ this is $\sum (6 - \alpha_i) = 36 - 18$, automatic, so every $(2, 2)$ -character is Hodge. Algebraicity of all of them is the Hodge conjecture for X_6^4 , which is known: Braun, Fortin, Lopez Garcia and Villaflor Loyola [18, Thm. 1] exhibit a basis of $H^{2,2}(X) \cap H^4(X, \mathbb{Q})_{\text{prim}}$ consisting of the primitive parts of 1001 planes, 720 Aoki–Shioda cycles and 30 further cycles, and prove the $\mathbb{Z}[1/6]$ -integral Hodge conjecture for X [18, Thm. 2]; see Shioda [79, 78] for the inductive structure of the Hodge classes of Fermat varieties. $\square$

The 1001 planes $\{x_{\sigma(0)} = \varepsilon_1 x_{\sigma(1)}, x_{\sigma(2)} = \varepsilon_2 x_{\sigma(3)}, x_{\sigma(4)} = \varepsilon_3 x_{\sigma(5)}\}$, $\varepsilon_i^6 = -1$, of X are defined over $\mathbb{Q}(\zeta_{12})$; on M_6 they become $\{x = \varepsilon u', y = \varepsilon' v', z = \varepsilon'' w'\}$ with $\varepsilon^6 = \varepsilon'^6 = \varepsilon''^6 = 1$ and (u', v', w') a permutation of (u, v, w) , together with planes like $\{x = \zeta y, u = \zeta' v, z = \pm w\}$, $\zeta^6 = \zeta'^6 = -1$; only the 48 “trivial” planes $\{x = \pm u', y = \pm v', z = \pm w'\}$ are defined over $\mathbb{Q}$. Galois permutes the characters (Frob $_p$ maps $V(\alpha)$ to $V(p\alpha)$: it fixes every $V(\alpha)$ for $p \equiv 1 \pmod{6}$ and swaps $V(\alpha), V(-\alpha)$ for $p \equiv 5 \pmod{6}$), and the Galois-invariant part of $H^4(M_6, \mathbb{Q}_\ell)$ is comparatively small: by the computation below, the Frobenius eigenvalue on $V(\alpha)$ equals p^2 for all $p \equiv 1 \pmod{6}$, $p < 200$, for exactly 665 characters, all of plane type $(\alpha_a + \alpha_b \equiv 0 \pmod{6})$ on a perfect matching), forming 332 pairs $\{\alpha, -\alpha\}$ plus $(3, \dots, 3)$; so at most 333 primitive classes are Galois-invariant, and these are spanned by Galois-orbit sums of planes— $\mathbb{Q}$ -rational classes that carry no rational points.

Proposition 9.2 (Weil). *Let $p \equiv 1 \pmod{6}$, ω a character of order 6 of $\mathbb{F}_p^\times$ and $g(\chi) = \sum_{t \in \mathbb{F}_p^\times} \chi(t) e^{2\pi i t/p}$. For the diagonal sextic $\sum_{i=0}^5 a_i x_i^6 = 0$ in $\mathbb{P}^5$,*

$$\#\{\sum a_i x_i^6 = 0\}(\mathbb{F}_p) = \frac{p^5-1}{p-1} + \frac{1}{p} \sum_{\alpha} \prod_{i=0}^5 \omega^{-\alpha_i}(a_i) g(\omega^{\alpha_i}), \quad (9.1)$$

the sum over the 2605 characters of Proposition 9.1; the α -term is the eigenvalue of Frob $_p$ on $V(\alpha)$ and has absolute value p^2 .

This is [93] (from $\#\{x : ax^6 = b\} = \sum_{\chi^6=1} \chi(b) \bar{\chi}(a)$ and the value $\frac{p-1}{p} \prod g(\chi_i)$ of $\sum_{\sum b_i=0} \prod \chi_i(b_i)$ when all $\chi_i \neq 1$, $\prod \chi_i = 1$). For M_6 , $a = (1, 1, 1, -1, -1, -1)$ and $\prod \omega^{-\alpha_i}(a_i) = \omega(-1)^{\alpha_3+\alpha_4+\alpha_5}$, with $\omega(-1) = 1$ for $p \equiv 1 \pmod{12}$ and -1 for $p \equiv 7$

(mod 12): this is the twist. For $p \equiv 5 \pmod{6}$, $x \mapsto x^3$ is a bijection of $\mathbb{F}_p$, so x^6 and x^2 are equidistributed and $\#M_6(\mathbb{F}_p) = (p^2 + 1)(p^2 + p + 1)$ is the count of a split quadric fourfold; in Weil's language Frob_p permutes the characters by $\alpha \mapsto -\alpha$ and only $\alpha = (3, \dots, 3)$, the pull-back of the quadric's class under $x_i \mapsto x_i^3$, contributes. We implemented (9.1) and compared it with an exact count $(N_3(c) = \#\{x^6 + y^6 + z^6 = c\})$ by convolution, $\#M_6(\mathbb{F}_p) = (\sum_c N_3(c)^2 - 1)/(p - 1)$ for all $7 \leq p \leq 109$: they agree, as do the quadric formulas for $p \equiv 5 \pmod{6}$ (Table 3). The 1751 eigenvalues on the $(2, 2)$ -characters are p^2 times sixth roots of unity (Stickelberger), as they must be for algebraic classes, and none of the 852 eigenvalues on the $(3, 1)$ - and $(1, 3)$ -characters is. The count $\#M_6(\mathbb{F}_p)$ thus “sees” all 2606 classes, whether or not they are defined over $\mathbb{Q}$; e.g. for $p \equiv 1 \pmod{12}$, $p < 400$, exactly 1751 or 1001 of the $(2, 2)$ -eigenvalues of X equal p^2 according as 2 is or is not a cube modulo p —1001 being the number of planes, which are rational over $\mathbb{F}_p$, and $750 = 720 + 30$ the number of non-linear generators of [18], consistent with those being defined over a cubic extension of $\mathbb{Q}(\zeta_{12})$ (observed, not proved).

TABLE 3. Point counts over $\mathbb{F}_p$ by exact enumeration. $\#M_6(\mathbb{F}_p)$ agrees with (9.1) for $p \equiv 1 \pmod{6}$ and with $(p^2 + 1)(p^2 + p + 1)$ for $p \equiv 5 \pmod{6}$; the Fermat sextic X has the same count iff $p \equiv 1 \pmod{4}$. $\#V(\mathbb{F}_p)$ is Theorem 9.12; $R(p)$ is defined in (9.4). For $p = 3, 5$, $x^6 = x^2$ on $\mathbb{F}_p$ and $\#M_6 = \#W = \#M_2$; V has bad reduction at 5.

p	$p \pmod{12}$	$\#M_6(\mathbb{F}_p)$	$\#X(\mathbb{F}_p)$	$\#W(\mathbb{F}_p)$	$\#V(\mathbb{F}_p)$	$R(p)/(p + p^2)$
3	3	130	112	130	32	—
5	5	806	806	806	182	—
7	7	9 774	0	1 562	96	36.00
11	11	16 226	15 984	4 442	188	32.73
13	1	87 570	87 570	11 750	408	102.20
17	5	89 030	89 030	23 030	632	97.65
19	7	258 750	234 720	18 842	492	36.63
23	11	293 090	292 032	28 250	704	33.22
29	5	733 382	733 382	85 670	1 400	93.10
31	7	1 235 142	1 119 744	60 986	1 164	34.26
37	1	3 160 530	3 160 530	160 070	2 040	95.51
41	5	2 898 086	2 898 086	204 374	2 456	94.56
43	7	4 263 462	4 084 560	144 122	2 208	35.92
47	11	4 987 970	4 983 552	178 586	2 528	34.68
53	5	8 045 030	8 045 030	377 990	3 800	92.20
59	11	12 329 762	12 322 800	325 274	3 980	34.90
61	1	16 227 954	16 227 954	552 614	4 824	96.55

9.2. The threefold $W = M_6 \cap M_2$.

Proposition 9.3. *W is an irreducible, normal, simply connected complete intersection of type $(2, 6)$ with $\omega_W = \mathcal{O}_W(2)$, and it is singular: its singular locus consists of exactly 710 ordinary double points, the points all of whose nonzero coordinates are fourth roots of unity ε_i (after scaling) with $\sum_{x,y,z} \varepsilon_i^2 = \sum_{u,v,w} \varepsilon_i^2$: 30 with two nonzero coordinates, 360 with four, 320 with six; 122 are rational (coordinates in $\{0, \pm 1\}$, e.g. $(1:0:0:1:0:0)$), the other 588 form 294 conjugate pairs over $\mathbb{Q}(i)$. The discriminant of the tangent cone at a node is a square in the node's field of definition ($\mathbb{Q}$ or $\mathbb{Q}(i)$); hence, for $p \neq 2, 3$, the exceptional quadric surface over an $\mathbb{F}_p$ -rational node of $W_{\mathbb{F}_p}$ is split over $\mathbb{F}_p$.*

Proof. $\nabla F_2 = 2(x, y, z, -u, -v, -w) \neq 0$ on $\mathbb{P}^5$, so W is singular at P iff $\nabla F_6 = 6(x^5, \dots, -w^5)$ is proportional to ∇F_2 , i.e. iff all nonzero coordinates have the same fourth power; scaling them to μ_4 and using $\varepsilon^6 = \varepsilon^2$, both equations become $\sum_{x,y,z} \varepsilon_i^2 = \sum_{u,v,w} \varepsilon_i^2$. Enumerating $(\{0\} \cup \mu_4)^6$ modulo μ_4 over $\mathbb{Q}(i)$ and checking the Jacobian rank gives the 710 points; independently, the ideal $(F_6, F_2) + (2 \times 2 \text{ minors})$, saturated, is zero-dimensional of degree 710 (over $\mathbb{Q}$ and

over two finite fields), so the singular scheme is reduced. Near a node P , W is the hypersurface $G = F_6 - \lambda F_2 = 0$ of the smooth quadric M_2 with $dG(P) = 0$, and the Hessian of G on $T_P M_2$ modulo the Euler direction has rank 4 at all 710 points: A_1 singularities. Its determinant modulo squares is $\pm 36^2, \pm 144^2$ or $\pm 576^2$, positive at the rational nodes and a square in $\mathbb{Q}(i)$ at all of them; over a finite field a 4-dimensional form with square discriminant is hyperbolic, which gives the last claim. Normality and irreducibility: Cohen–Macaulay, connected, singular locus of codimension 3. $\pi_1 = 1$ by Lefschetz for local complete intersections [44, Part II]. $\square$

Remark 9.4 (W covers the Segre cubic). The squaring map $x_i \mapsto y_i = x_i^2$ sends W onto $\{\sum \pm y_i = 0, \sum \pm y_i^3 = 0\}$, which after $y_i \mapsto -y_i$ on u, v, w is the *Segre cubic* $S: \sum_{i=1}^6 y_i = \sum_{i=1}^6 y_i^3 = 0$ in $\mathbb{P}^4$, the cubic threefold with 10 nodes and 15 planes [42, §9.4.4]; $W \rightarrow S$ is a $(\mathbb{Z}/2)^5$ -cover branched along the six coordinate sections. The 320 nodes with six nonzero coordinates map 32:1 onto the 10 nodes of S (the points $(\pm 1, \dots, \pm 1)$ with three signs of each kind), the 360 with four map 8:1 onto the 45 points of S with two zero coordinates and the others ± 1 (the three vertices of each of the 15 triangles of lines $S \cap \{y_a = y_b = 0\}$), and the 30 with two map 2:1 onto the 15 points with four zero coordinates. The 15 planes of S pull back to 120 planes on W : the 48 rational planes $\{x = \pm u', y = \pm v', z = \pm w'\}$ and 72 planes $\{x = \pm iy, u = \pm iv, z = \pm w\}$ and symmetric images, over $\mathbb{Q}(i)$. (All verified by computation.)

A smooth $(2, 6)$ complete intersection W_t in $\mathbb{P}^5$ (a smoothing of W , e.g. $M_6 \cap$ a general quadric) is simply connected with $H^2 = \mathbb{Z}H$ and

$$h^{3,0}(W_t) = h^0(\mathcal{O}_{W_t}(2)) = 20, \quad h^{2,1}(W_t) = 630, \quad b_3(W_t) = 1300, \quad \chi_{\text{top}}(W_t) = -1296$$

(generating function and Chern classes agree); it is of general type, $K = \mathcal{O}(2)$. For the nodal W these numbers change by an amount measured by the *defect* $\sigma(W) = \text{rk Cl}(W) - \text{rk Pic}(W)$ (Weil divisors modulo Cartier divisors): with $\delta = 710$ nodes, whose Milnor fibres are 3-spheres [31, 94, 34],

$$\chi_{\text{top}}(W) = \chi_{\text{top}}(W_t) + \delta, \quad b_3(\widetilde{W}) = b_3(W_t) - 2\delta + 2\sigma, \quad b_2(\widehat{W}) = b_4(\widehat{W}) = 1 + \sigma, \quad (9.2)$$

where $\widehat{W}$ is a small resolution (a complex manifold; each node replaced by a $\mathbb{P}^1$), $\widetilde{W} \rightarrow W$ is the blow-up of the nodes, with $b_2 = b_4 = 1 + \delta + \sigma$ and the same b_3 , and $h^{3,0}(\widetilde{W}) = h^0(\omega_W) = 20$ because nodes are canonical. The defect is given by Werner's formula and its generalisation to a nodal hypersurface X in a smooth fourfold Y [94, 34, 36, 35]: it is the number of *dependent* conditions imposed by the nodes on $H^0(Y, \omega_Y(2X))$,

$$\sigma(X) = \#\Sigma - \text{rk}(H^0(Y, \omega_Y(2X)) \rightarrow \mathbb{C}^\Sigma), \quad (9.3)$$

provided $h^2(\Omega_Y^3) = h^1(\Omega_Y^3(X)) = h^1(\mathcal{O}_Y(X + K_Y)) = h^2(\mathcal{O}_Y(-X)) = h^3(\mathcal{O}_Y(-X)) = 0$. We take $Y = M_2$, $\omega_Y = \mathcal{O}_Y(-4)$, $X = W \in |\mathcal{O}_Y(6)|$, so $\omega_Y(2X) = \mathcal{O}_Y(8)$ and $h^0 = 825$. The hypotheses hold: the last three by Kodaira and Lefschetz, the first since $h^{3,2}(Y) = 0$, and $h^1(\Omega_Y^3(6)) = h^1(T_Y(2)) = 0$ because, by the normal-bundle and Euler sequences, $H^1(T_Y(2))$ is the cokernel of $H^0(T_{\mathbb{P}^5}(2)|_Y) \rightarrow H^0(\mathcal{O}_Y(4))$, $\xi \mapsto \xi(F_2)$, whose image is the degree-4 part of the Jacobian ideal of F_2 , i.e. everything. (With the roles exchanged, $Y = M_6$, the hypothesis $h^1(\Omega_{M_6}^3(2)) = h^1(T_{M_6}(2)) = 0$ fails—this group is the degree-8 part of the Jacobian ring of F_6 —and the formula is not applicable with degree-4 forms.)

Proposition 9.5. $\sigma(W) = 95$: the 710 nodes impose exactly 615 independent conditions on $H^0(M_2, \mathcal{O}(8))$. Hence $\chi_{\text{top}}(W) = -586$; the resolutions have $b_3(\widetilde{W}) = b_3(\widehat{W}) = 70$, $h^{3,0} = 20$, $h^{2,1} = 15$, $h^{1,1}(\widetilde{W}) = 96$, $h^{1,1}(\widehat{W}) = 806$; and W itself has $b_2(W) = 1$, $b_4(W) = 96$, $b_3(W) = 685$, of which 70 dimensions are pure of weight 3 and $615 = \delta - \sigma$ are of weight 2.

Proof. The 710×1287 matrix of the degree-8 monomials evaluated at the nodes (entries in $\{0, \pm 1, \pm i\}$) has rank 615 over $\mathbb{Q}(i)$, computed exactly as the rank of a 1420×2574 integer matrix and re-checked modulo three primes; since F_2 vanishes at the nodes this is the rank in (9.3). (The same code applied to Schoen's quintic $\sum x_i^5 - 5 \prod x_i$, 125 nodes, quintic forms,

returns the known defect 24.) The rest follows from (9.2), from $b_3 = 2h^{3,0} + 2h^{2,1}$ and, for W , from the sequence of the pair $(\widetilde{W}, \bigcup E_P)$ with $H^i(\mathbb{P}^5) \cong H^i(W)$ for $i < 3$ [44], giving $b_3(W) = b_3(\widetilde{W}) + \delta - \sigma$; consistently, $\chi_{\text{top}}(W) = 1 + 1 + 96 + 1 - 685 = -586$. $\square$

The cover of Remark 9.4 makes $h^{3,0} = 20$ plausible from the other side: $H^3(\widetilde{W})$ splits into eigenspaces indexed by the even subsets $T \subset \{1, \dots, 6\}$, the T -part being the anti-invariant cohomology of the double cover of S branched along $\sum_{i \in T} \{y_i = 0\}$, whose 3-forms are sections of $\omega_S(\frac{|T|}{2}) = \mathcal{O}_S(\frac{|T|}{2} - 2)$: 5 for $|T| = 6$, 1 for each of the 15 subsets with $|T| = 4$, 0 for $|T| = 2$, total 20. (A consistency check, not a proof: the pieces are singular.) We did not identify the 70-dimensional Galois representation $H^3(\widetilde{W})$ (Remark 9.9).

Remark 9.6 (Choudhry's surface). Choudhry's solutions of type $(1, 2, 6)$ [30] lie on $W \cap \{x + y + z = u + v + w\}$, which is *reducible*: with $e_1 = x + y + z$, $e_2 = xy + yz + zx$ and Newton's identity $p_6 = e_1^6 - 6e_1^4e_2 + 9e_1^2e_2^2 - 2e_2^3 + 6e_1^3e_3 - 12e_1e_2e_3 + 3e_3^2$, one has $F_6 = 3(xyz - uvw)(xyz + uvw + 2e_1^3 - 4e_1e_2)$ modulo the hyperplane and the quadric. So $W \cap \{x + y + z = u + v + w\}$ is the union of the six planes $\{(u, v, w) = \text{a permutation of } (x, y, z)\}$ and the smooth K3 surface of degree 6

$$S_{\text{Ch}}: \quad x + y + z = u + v + w, \quad x^2 + y^2 + z^2 = u^2 + v^2 + w^2, \quad xyz + uvw = -2e_1^3 + 4e_1e_2,$$

a $(1, 2, 3)$ complete intersection (smoothness checked by a Gröbner computation); Choudhry's rank-one elliptic curve lies on it. We did not compute $\rho(S_{\text{Ch}})$.

Proposition 9.7 (symmetry orbits in W). *The group G of signed permutations of the coordinates preserving $\{F_6 = 0\}$ and $\{F_2 = 0\}$ is $(S_3 \times S_3 \times \mathbb{Z}/2) \ltimes (\mathbb{Z}/2)^6$, of order 4608, acting on $\mathbb{P}^5$ through $G/\{\pm 1\}$. The stabiliser of V has order 96 (Bremner's group), so W contains exactly 48 images gV ; it contains 32 images of S_{Ch} (stabiliser of order 144), 48 rational planes and 72 planes over $\mathbb{Q}(i)$. Of the 59 census solutions with $\sum x^2 = \sum u^2$ (Theorem 7.1), 40 lie on exactly one of the 48 surfaces gV and one ($\#2$) on two of them, 9 (type $(1, 2, 6)$) lie on exactly one surface gS_{Ch} , and 9 lie on none of these 80 K3 surfaces and on none of the 120 planes.*

Point counts of W and the trace formula for a nodal threefold. Let $p \neq 2, 3, 5$. The nodes reduce to 710 distinct nodes of $W_{\mathbb{F}_p}$, all $\mathbb{F}_p$ -rational if $p \equiv 1 \pmod{4}$ and 122 of them if $p \equiv 3 \pmod{4}$; call this number $\delta(p)$. Let $\widetilde{W} \rightarrow W$ be the blow-up of the nodes, defined over $\mathbb{F}_p$, with exceptional quadrics E_P , split over $\mathbb{F}_p$ for rational P (Proposition 9.3), so $\#W(\mathbb{F}_p) = \#\widetilde{W}(\mathbb{F}_p) - \delta(p)(p^2 + 2p)$. For $\widetilde{W}$ the trace formula and Deligne's theorem [37] give $\#\widetilde{W}(\mathbb{F}_p) = 1 + p^3 + \text{tr}(\text{Frob}_p | H^2) + \text{tr}(\text{Frob}_p | H^4) - \text{tr}(\text{Frob}_p | H^3)$ with $H^2(\widetilde{W}) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell(-1) \oplus \mathbb{Q}_\ell[\Sigma](-1) \oplus D$, $\mathbb{Q}_\ell[\Sigma]$ the permutation representation on the nodes and D the 95-dimensional representation on the defect classes—classes of Weil divisors, so with eigenvalues p times roots of unity—and H^4 dual to H^2 . Writing $\text{tr}(\text{Frob}_p | D) = p t(p)$, $t(p) \in \mathbb{Z}$, $|t(p)| \leq 95$:

$$R(p) := \#W(\mathbb{F}_p) - (1 + p + p^2 + p^3) + p\delta(p) = (p + p^2)t(p) - \text{tr}(\text{Frob}_p | H^3(\widetilde{W})), \quad (9.4)$$

with $|\text{tr}(\text{Frob}_p | H^3(\widetilde{W}))| \leq 70p^{3/2}$ by Deligne. We computed $\#W(\mathbb{F}_p)$ exactly for all 316 primes $7 \leq p < 2600$ (convolution of the distribution of (x^2, x^6) on $\mathbb{F}_p^2$; sparse exact convolution for $p < 300$, FFT with checked rounding for all p). The quotient $R(p)/(p + p^2)$, equal to $t(p)$ up to at most $70/\sqrt{p}$, is in Table 3 for small p ; over the 155 primes $p \equiv 1 \pmod{4}$ with $300 < p < 2600$ its mean is 94.971 and it never differs from 95 by more than 1.75; over the 161 primes $p \equiv 3 \pmod{4}$ in that range its mean is 34.999 and it never differs from 35 by more than 0.80; the means do not depend on p modulo 8, 3 or 5.

Observation 9.8. The counts are consistent with $t(p) = 95$ for $p \equiv 1 \pmod{4}$ and $t(p) = 35$ for $p \equiv 3 \pmod{4}$, and with no other constant values (statistical errors of the means about 0.05 and 0.02). That is: all 95 defect classes are defined over $\mathbb{Q}(i)$, complex conjugation has trace 35 on them, so 65 are defined over $\mathbb{Q}$ and 30 are anti-invariant, $\text{rk Cl}(W_{\mathbb{Q}}) = 66$; and the

point counts confirm $\sigma(W) = 95$ independently of (9.3). With these values, (9.4) gives the exact traces of Frob_p on $H^3(\widetilde{W})$: divided by $p^{3/2}$ they have root mean square 20.9 (maximum 53.8) for $p \equiv 1 \pmod{4}$ and 6.6 (maximum 16.1) for $p \equiv 3 \pmod{4}$, inside the Weil bound 70 and far below what a 1300-dimensional H^3 could produce.

Remark 9.9. This is the Weil-conjecture statement for W : the error term of $\#W(\mathbb{F}_p)$ is governed by b_3 of the resolution, which is 70 and not 1300, because the 710 nodes have killed $2(710 - 95) = 1230$ dimensions of H^3 —the price of the 95 extra Weil divisor classes (among the divisors: the 48 copies of V , the 32 copies of S_{Ch} , the 120 planes). The traces for $p \equiv 1 \pmod{4}$ are all divisible by 10 (by 4 for $p \equiv 3 \pmod{4}$) and their root mean square 20.9 exceeds the $\sqrt{70} \approx 8.4$ of 35 independent Sato–Tate pairs: $H^3(\widetilde{W})$ presumably decomposes, under the cover structure and the symmetries, into a few motives with multiplicities. We did not pursue this; the traces are in the data accompanying the paper.

9.3. The K3 surface V . V is a smooth complete intersection of three quadrics (Gröbner check), hence a K3 surface: $\pi_1 = 1$, $b_2 = 22$, $h^{2,0} = 1$, $h^{1,1} = 20$, $\chi_{\text{top}} = 24$. On each affine chart a Gröbner basis over $\mathbb{Z}$ of the singular ideal contains 100, and V is singular modulo 2 and 5: V has good reduction exactly at $p \neq 2, 5$. By [57, Cor. 5.2], $\rho(V_{\mathbb{Q}}) = 19$, and under hypothesis (H) of Section 6 $\text{NS}(V_{\mathbb{Q}}) = \text{NS}(V_{\mathbb{Q}(i)})$ is spanned by the 32 lines and 24 conics. The 56×56 intersection matrix was recomputed here from scratch (degrees of the schemes $C \cap C'$ over $\mathbb{Q}(i)$) and agrees with the one used in Section 6.

Proposition 9.10. (a) $\text{NS}(V_{\mathbb{Q}}) \cong U(2) \oplus E_8(-1)^2 \oplus \langle -12 \rangle$ (as in [1, Ex. 3.3]), with discriminant group $(\mathbb{Z}/2)^2 \times \mathbb{Z}/12$ and discriminant form $v(2) \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{2}{3} \rangle \cong u(2) \oplus \langle \frac{5}{4} \rangle \oplus \langle \frac{2}{3} \rangle$.¹
 (b) The transcendental lattice is $T_V \cong U(2) \oplus \langle 12 \rangle \cong \langle -4 \rangle \oplus A_2(2)$: even, of signature $(2, 1)$ and discriminant -48 , unique in its genus.
 (c) Complex conjugation acts on $\text{NS}(V_{\mathbb{Q}})$ with characteristic polynomial $(x-1)^{13}(x+1)^6$; the invariant sublattice is $\text{NS}(V_{\mathbb{Q}})$ (rank 13, discriminant 768), the anti-invariant one is negative definite of rank 6 and discriminant 256.

Proof. (a) The Gram matrix of Bremner’s $\mathbb{Z}$ -basis (17) has determinant 48, signature $(1, 18)$ and the stated discriminant form (Sage), and all 56 curves are integral in it; $U(2) \oplus E_8(-1)^2 \oplus \langle -12 \rangle$ has the same genus, and an even indefinite lattice with $\text{rk} \geq \ell(A) + 2$ is unique in its genus [71, Thm. 1.14.2] ($19 \geq 3 + 2$). (b) $T_V = \text{NS}^\perp$ in $\Lambda_{K3} = U^3 \oplus E_8(-1)^2$ is even of signature $(2, 1)$ with discriminant form $-q_{\text{NS}}$ [71, §1.6]. Of the even genera of signature $(2, 1)$ and determinant -48 exactly one has discriminant form $-q_{\text{NS}}$; it is the genus of $U(2) \oplus \langle 12 \rangle$ and contains one class (Sage’s enumeration; or [71, Thm. 1.14.2], as the 2-part of q_{NS} splits off $u(2)$). Explicitly, an anti-isometry $A_{\text{NS}} \rightarrow A_{U(2) \oplus \langle 12 \rangle}$ was found whose graph defines an even unimodular overlattice of $\text{NS} \oplus (U(2) \oplus \langle 12 \rangle)$ of signature $(3, 19)$ —necessarily Λ_{K3} —with $\text{NS}^\perp$ of Gram matrix $\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix}$. (c) From the permutation of the 56 curves induced by $i \mapsto -i$. $\square$

Remark 9.11 (Kummer and Shioda–Inose structures). By [57, Thm. 5.1], $V = \text{Km}(\text{Jac } \mathcal{C})$ for $\mathcal{C}: y^2 = (x^2 + 1)(x^2 + 2x + 5)(x^2 - 2x + 5)$, and $\text{Jac}(\mathcal{C})$ is $\mathbb{Q}$ -isogenous (degree 4) to $E \times E$, $E: y^2 = x(x^2 + 4x + 20)$, which is 320f2 in Cremona’s tables: conductor $2^6 \cdot 5$, $j = 21296/25$, rank one, no complex multiplication. Since $T(\text{Km } A) \cong T(A)(2)$ [69, 65], (b) gives $T(\text{Jac } \mathcal{C}) \cong U \oplus \langle 6 \rangle$ and $\text{NS}(\text{Jac } \mathcal{C}) \cong U \oplus \langle -6 \rangle$. For $A = E_1 \times E_2$ with $\text{Hom}(E_1, E_2) = \mathbb{Z}\varphi$, $\deg \varphi = n$, the classes of $E_1 \times 0$, $0 \times E_2$ and the graph of φ have Gram matrix $\begin{pmatrix} 0 & 1 & n \\ 1 & 0 & 1 \\ n & 1 & 0 \end{pmatrix}$, so $\text{NS}(A) \cong U \oplus \langle -2n \rangle$, $T(A) \cong U \oplus \langle 2n \rangle$; conversely, a primitive isotropic class of positive degree on an abelian surface is an elliptic curve [49], and two elliptic curves meeting once split the

¹Here $u(2)$ and $v(2)$ are the discriminant forms of $U(2)$ and of D_4 on $(\mathbb{Z}/2)^2$, taking the values 0, 0, 0, 1 and 0, 1, 1, 1 in $\mathbb{Q}/2\mathbb{Z}$, and $\langle a \rangle$ is the form $x \mapsto ax^2$ on a cyclic group; $v(2) \oplus \langle \theta/4 \rangle \cong u(2) \oplus \langle 5\theta/4 \rangle$ by Nikulin’s relations. Sage’s normal form gives $v(2) \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{2}{3} \rangle$ for NS and $v(2) \oplus \langle \frac{7}{4} \rangle \oplus \langle \frac{4}{3} \rangle$ for T_V ; the signatures modulo 8 are $7 \equiv 1 - 18$ and $1 \equiv 2 - 1$, as they must be.

surface. Hence $\text{Jac}(\mathcal{C}) \cong E_1 \times E_2$ over $\mathbb{C}$ with E_1, E_2 cyclically 3-isogenous: not $E \times E$ (whose T is $U \oplus \langle 2 \rangle$), but what Kuwata's Remark 5.3 predicts, with $E': y^2 = x(x^2 - 44x + 500) = 320f4$ and an explicit 3-isogeny $E \rightarrow E'$. Since $E_8(-1)^2 \subset \text{NS}(V)$ is primitive, V also has a Shioda-Inose structure [65, §6], with partner abelian surface A' , $T(A') \cong U(2) \oplus \langle 12 \rangle$.

Let $a_p = p + 1 - \#E(\mathbb{F}_p)$ for $E = 320f2$ and let α_p, β_p be the roots of $x^2 - a_p x + p$.

Theorem 9.12. *For every prime $p \neq 2, 5$ and $q = p^k$,*

$$Z(V/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-p^2T)P_{\text{NS}}(T)P_T(T)}, \quad (9.5)$$

$$P_T(T) = (1 - \alpha_p^2 T)(1 - pT)(1 - \beta_p^2 T) = (1 - pT)(1 - (a_p^2 - 2p)T + p^2 T^2),$$

with $P_{\text{NS}}(T) = (1 - pT)^{19}$ if $p \equiv 1 \pmod{4}$ and $(1 - pT)^{13}(1 + pT)^6$ if $p \equiv 3 \pmod{4}$; equivalently

$$\#V(\mathbb{F}_q) = 1 + q^2 + q t_{\text{NS}}(q) + \alpha_p^{2k} + p^k + \beta_p^{2k}, \quad t_{\text{NS}}(q) = \begin{cases} 19, & q \equiv 1 \pmod{4}, \\ 7, & q \equiv 3 \pmod{4}. \end{cases} \quad (9.6)$$

Proof. $Z^{-1} = (1-T)(1-p^2T) \det(1 - \text{Frob}_p T | H_{\text{et}}^2(V_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$ and $H^2 = \text{NS}(V_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell(-1) \oplus T_\ell$. On the first summand Frob_p acts by p times its permutation of the 56 curves, trivial for $p \equiv 1 \pmod{4}$ and complex conjugation for $p \equiv 3 \pmod{4}$, so Proposition 9.10(c) gives P_{NS} (we also checked that modulo $p \equiv 3 \pmod{4}$ exactly 8 lines and 12 conics are $\mathbb{F}_p$ -rational and Frobenius swaps the 18 conjugate pairs). The Kummer quartic and $V \rightarrow \text{Km}(\text{Jac } \mathcal{C})$ are defined over $\mathbb{Q}$ [57, Thm. 5.1], so $T_\ell(V) \cong T_\ell(\text{Jac } \mathcal{C})$ as Galois modules; $\text{Jac}(\mathcal{C}) \sim E \times E$ over $\mathbb{Q}$, and $H^2(E \times E) = \text{Sym}^2 H^1(E) \oplus (\text{three algebraic classes})$, so $T_\ell = \text{Sym}^2 H^1(E)$ with eigenvalues $\alpha_p^2, \alpha_p \beta_p = p, \beta_p^2$. A quadratic twist of E , of $\mathcal{C}$ or of the torsor underlying the Kummer construction does not change Sym^2 ; a twist of V by a character acting on T is excluded by the counts below. $\square$

Independently of Kuwata's identification we counted $\#V(\mathbb{F}_q)$ by the $O(q^3)$ formula $\#V_{\text{aff}}(\mathbb{F}_q) = \text{tr}(N^3)$, $N_{ab} = \#\{(s, t) : Q(s, t) = a, Q(t, s) = b\}$ (from the shape of (1.4); cross-checked against a direct enumeration of $\mathbb{F}_p^6$ for $p \leq 13$), for all 44 primes $3 \leq p < 200$, $p \neq 5$, over $\mathbb{F}_{p^2}$ for $p \leq 23$ and over $\mathbb{F}_{p^3}$ for $p \leq 11$, which tests the full characteristic polynomial. Every count equals (9.6) with trivial twist, and the twists by $(\frac{-1}{p}), (\frac{\pm 2}{p}), (\frac{\pm 5}{p}), (\frac{\pm 10}{p})$ are refuted. Table 4 shows the small primes; e.g. $\#V(\mathbb{F}_3) = 32$, all 32 points on the 8 rational lines; and for $a_p = 0$ ($p = 11, 131$) $P_T = (1 - pT)(1 + pT)^2$: V is supersingular.

TABLE 4. $\#V(\mathbb{F}_p)$ by exact enumeration against (9.6), $a_p = a_p(E)$ for $E = 320f2$.

p	$p(4)$	a_p	$\#V(\mathbb{F}_p)$	$1 + p^2 + p t_{\text{NS}}$	$a_p^2 - p$	p	$p(4)$	a_p	$\#V(\mathbb{F}_p)$	$1 + p^2 + p t_{\text{NS}}$	$a_p^2 - p$
3	3	-2	32	31	1	43	3	-10	2 208	2 151	57
7	3	-2	96	99	-3	47	3	6	2 528	2 539	-11
11	3	0	188	199	-11	53	1	6	3 800	3 817	-17
13	1	-2	408	417	-9	59	3	12	3 980	3 895	85
17	1	-6	632	613	19	61	1	-2	4 824	4 881	-57
19	3	-4	492	495	-3	67	3	2	4 896	4 959	-63
23	3	-6	704	691	13	71	3	12	5 612	5 539	73
29	1	-6	1 400	1 393	7	73	1	2	6 648	6 717	-69
31	3	4	1 164	1 179	-15	79	3	-8	6 780	6 795	-15
37	1	-2	2 040	2 073	-33	83	3	6	7 424	7 471	-47
41	1	6	2 456	2 461	-5						

Remark 9.13. (a) The factor $1 - pT$ of P_T says that for every good p the Frobenius-invariant part of H^2 is one dimension larger than $\text{NS}(V_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell$ (resp. $\text{NS}(V_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell$ for $p \equiv 1 \pmod{4}$): by the Tate conjecture for K3 surfaces [27, 64], $\rho(V_{\mathbb{F}_p}) = 14$ or 20 and $\rho(V_{\overline{\mathbb{F}_p}}) \geq 20$, the usual jump

of the Picard number of a K3 surface with $\rho = 19$ upon reduction. (b) P_T is the Euler factor of $L(\text{Sym}^2 f, s)$ for the weight-two newform f of level 320 attached to E : the transcendental motive of V is $\text{Sym}^2 h^1(E)$, with no twist. (c) The algebraic part of $H^2(V)$ contributes $19p$ or $7p$ to $\#V(\mathbb{F}_p)$ according to the Galois action on the lines and conics, the transcendental part contributes $a_p^2 - p \in [-p, 3p]$, and the “Weil conjectures for V ” say that nothing else does.

9.4. Geometry versus arithmetic. For a curve $C/\mathbb{Q}$ of genus g there are two separate statements: over $\mathbb{F}_q$ the genus controls the point count exactly, $|\#C(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}$; over $\mathbb{Q}$ it controls the *qualitative* behaviour ($g = 0$: none or a parametrised infinity; $g = 1$: a finitely generated group of non-geometric rank; $g \geq 2$: finitely many), not the number of points. The picture for our three varieties separates along the same lines, with two different “complexities” in play.

Over finite fields, cohomological complexity controls the counts. This is §§9.1–9.3: $\#M_6(\mathbb{F}_p)$ is the sum of 2606 Frobenius eigenvalues and all 2605 non-trivial ones are needed (Proposition 9.2); $|\#V(\mathbb{F}_p) - 1 - p^2 - pt_{\text{NS}}(p)| \leq 3p$ because $\text{rk } T_V = 3$ (Theorem 9.12); and the fluctuation of $\#W(\mathbb{F}_p)$ is at most $70p^{3/2}$, observed to be about $20p^{3/2}$, rather than the $1300p^{3/2}$ a smooth $(2, 6)$ complete intersection could show (Observation 9.8). Conversely the counts detect the algebraic classes and their fields of definition: the 19 classes of V over $\mathbb{Q}(i)$, 13 over $\mathbb{Q}$; the 95 Weil divisor classes of W over $\mathbb{Q}(i)$, 65 over $\mathbb{Q}$; the 1001 planes of X_6^4 as Tate classes modulo p . Complexity of the cohomology drives the arithmetic over finite fields, but in both directions: large b_i means large fluctuations, not more points.

Over $\mathbb{Q}$, two invariants matter, and neither is a Betti number. (i) *Kodaira dimension*: M_6 , V and S_{Ch} have $\kappa = 0$, W has $\kappa = 3$ ($\omega_W = \mathcal{O}(2)$ is ample and the nodes are canonical). The conjectures of Bombieri–Lang [59] and Campana [26] predict that rational points on a variety of general type are not Zariski dense, while “special” varieties, in particular those with $\kappa = 0$, have potentially dense rational points; for elliptic K3 surfaces potential density is a theorem [11], and for V density over $\mathbb{Q}$ itself follows from the positive rank of the Brudno pencil (Bremner’s Theorem 3.2 [20]; Lemma 5.1). The census is exactly the Bombieri–Lang picture for W : of the 59 points of $W(\mathbb{Q})$ with coordinates ≤ 1343 , 41 lie on the 48 K3 surfaces gV , 9 on the 32 K3 surfaces gS_{Ch} , and 9 are unexplained (Proposition 9.7); the conjecture says these too should lie on finitely many special subvarieties of W (Open Problem (2) of Section 11). For M_6 one expects $M_6(\mathbb{Q})$ to be Zariski dense, and the 15 solutions with $\sum x^2 \neq \sum u^2$ are the beginning of its “generic” part. None of this is read off from $b_4 = 2606$ or $b_3 = 1300$. (ii) *Algebraic classes defined over $\mathbb{Q}$* : inside the special world, what produces rational points is the algebraic cohomology that Galois fixes. For M_6 all 1752 Hodge classes are algebraic, but the cycles live over $\mathbb{Q}(\zeta_{12})$ and a cubic extension, at most 334 classes are Galois-invariant and those are sums of conjugate planes; the only cycles of M_6 we know over $\mathbb{Q}$ are the hyperplane class and the 48 trivial planes, which carry no interesting solutions: over $\mathbb{Q}$ the huge $H^{2,2}$ of M_6 is essentially invisible. For V it is the opposite: $\rho(V_{\mathbb{Q}}) = 19$ is as large as it can be short of the singular ($\rho = 20$, CM) case, 13 classes are defined over $\mathbb{Q}$ and all 19 over $\mathbb{Q}(i)$, and each is realised by explicit lines and conics. This is the source of the arithmetic: every primitive nef isotropic class in $\text{NS}(V_{\mathbb{Q}})$ is a genus-one fibration over $\mathbb{Q}$ (the Brudno pencil and its two images, the 14 rational pencils of degree ≤ 6 of Theorem 6.7, ...), each has a Mordell–Weil group (rank 3 for the Brudno pencil) whose elements are rational curves on V , and the fibrewise involutions $R \mapsto C_1 + C_2 - R$ generate an infinite group of automorphisms moving rational points around. A K3 surface with $\rho(V_{\mathbb{Q}}) = 1$ has none of this. So the precise sense in which geometry drives the arithmetic of (1.1) is: the Picard number over $\mathbb{Q}$ and the Galois action on NS, inside the $\kappa = 0$ world—the same two invariants, $\rho(V_{\mathbb{Q}}) = 13$ and $\rho(V_{\mathbb{Q}(i)}) = 19$, that the finite-field counts display in P_{NS} .

Counting by height (heuristic). A rational curve of degree d on $V \subset \mathbb{P}^5$ carries $\asymp B^{2/d}$ points of height $\leq B$: the 8 rational lines $\asymp B^2$, the 12 conics $\asymp B$, Brudno’s quartic $\asymp B^{1/2}$, the quintics $\asymp B^{2/5}$, the 52 curves of degrees 7–12 of Corollary 6.5 $\asymp B^{2/7}, \dots, B^{1/6}$; the sections of the Brudno pencil of degree $\leq d$ number $\asymp d^{3/2}$ (Theorem 5.4). Off the rational

curves, the Batyrev–Manin philosophy for trivial canonical class [9] predicts growth slower than any power of B , and for the orbit of a point under an infinite group of automorphisms of a K3 surface the counts are of the order of a power of $\log B$, with a generally non-integral exponent [82, 2, 8] (we use only the qualitative statement; the exponents of those papers were not re-derived here). So in any box almost all solutions of (1.1), (1.2) on V should lie on rational curves of small degree—the phenomenon behind the parametric families of Brudno, Bremner, Delorme and Choudhry and behind the census (33 of 41 points of V on known curves or sections, the other 8 on fibres of jumping rank, seven of them on non-section curves of degree ≤ 12 by Theorem 8.2). On W , by Bombieri–Lang, all but finitely many points should lie on the special subvarieties (gV , gS_{Ch} , further K3 surfaces or curves), each contributing according to its own geometry, and on the Calabi–Yau fourfold M_6 one expects, besides the trivial planes, a dense but thin set of points whose counting function is an open question (Open Problem (6) of Section 11).

9.5. Status of the statements. *Computed* (exact arithmetic; scripts and logs accompany the paper): the Hodge numbers and Euler characteristics of the three complete-intersection types by three methods; the Jacobian-ring dimensions, the 2605 characters with Hodge types and Shioda’s criterion; smoothness of M_6 and V and the bad primes $\{2, 5\}$ of V ; the 710 nodes, their fields of definition, type A_1 , tangent-cone discriminants, the Segre-cubic book-keeping and the 120 planes; the rank 615, hence $\sigma = 95$ given (9.3) and the verification of its hypotheses, with Schoen’s quintic as control; the reducibility of Choudhry’s surface and the smoothness of S_{Ch} ; the orbit sizes 48, 32, 48, 72 and the incidences of the census; the 56×56 intersection matrix, the Gram determinants 48 and 768, the discriminant form of NS, the genus identifications, the uniqueness of T_V in its genus and the explicit gluing; the conjugation character $(x - 1)^{13}(x + 1)^6$; all counts in the tables (two methods for V and for M_6), the $44 + 10$ verifications of (9.6) and the refutation of seven twists, Weil’s formula for $p \leq 109$, the 316 counts of W and the 665 Frobenius-fixed characters. *Cited*: Weil’s formula; Shioda, Shioda–Katsura and [18] on Fermat varieties; Hirzebruch’s generating function; Lefschetz for singular complete intersections; the nodal-threefold relations (9.2) and the defect formula (9.3); Kuwata’s theorem; Nikulin’s theory; $T(\text{Km } A) = T(A)(2)$ and the Shioda–Inose criterion; Kani’s theorem; Deligne; the Tate conjecture for K3 surfaces. *Observed, not proved*: $t(p) = 95, 35$ and everything deduced from it in Observation 9.8 and Remark 9.9; the Tate-class counts 1751/1001. *Heuristic*: the paragraph “Counting by height” and the conjectures (Bombieri–Lang, Campana, Batyrev–Manin) invoked in §9.4.

10. APPROACHES TO BREMNER’S CONJECTURE

Bremner’s conjecture ([20, §7]; §6.3) says that every curve of arithmetic genus 0 defined over $\mathbb{Q}$ on V is obtained from one of a finite set of such curves by repeated application of automorphisms taken from a finite set, the degree increasing at every application of an automorphism that is not a symmetry. The Theorem 6.7 verified it through degree 12 with the base set $\{\Gamma_1, \dots, \Gamma_8, Q_1, \dots, Q_{12}\}$ and the 772 $\mathbb{Q}$ -rational involutions of the 14 $\mathbb{Q}$ -rational elliptic pencils of degree ≤ 6 . This section answers two questions: what the literature on automorphisms of K3 surfaces already implies, and how one could try to prove the conjecture in full. The short answers are: (i) the conjecture, including the degree clause, follows for *every* K3 surface over *every* field of characteristic $\neq 2$ from Sterk’s theorem in the form of Bright–Logan–van Luijk, combined with Looijenga’s Dirichlet-domain construction (Theorem 10.9), but with finite sets that are not explicit; (ii) making the sets explicit for V is a computation which is now within reach (Borchers’ method as implemented by Shimada and in OSCAR, or Bremner’s own cone-cutting method), and a first run of Bremner’s method on our data (§10.5) already shows that the pencils of degree ≤ 6 do not suffice for the *proof*, identifies exactly what is missing (involutions of pencils of degree 8, among them the negation of the Kummer pencil), and shows that the missing automorphisms are again of Bremner’s type.

Throughout, Σ_k denotes the set of irreducible curves $\Gamma \subset V$ defined over a field $k \supset \mathbb{Q}$ with $p_a(\Gamma) = 0$; by adjunction these are exactly the smooth rational curves, $\Gamma^2 = -2$, and we call them (-2) -curves. Their classes are the (-2) -classes $D \in \text{NS}(V_k)$ with $D \cdot H > 0$ that are irreducible (Lemma 6.1). $\deg \Gamma = \Gamma \cdot H$. $\text{Aut}(V_k)$ is the group of automorphisms defined over k ; it acts on $\text{NS}(V_{\bar{\mathbb{Q}}})$, on $\text{NS}(V_k)$ and on Σ_k . The *symmetries* are the elements of $\text{Aut}(V_{\bar{\mathbb{Q}}})$ fixing H , i.e. the linear automorphisms of $V \subset \mathbb{P}^5$ over $\mathbb{Q}$; by Bremner's Appendix II they form the group of order 48. We use hypothesis (H) of Section 6, $\rho(V_{\bar{\mathbb{Q}}}) = 19$ (Kuwata [57]), so that $\text{NS}(V_{\bar{\mathbb{Q}}}) = \text{NS}(V_{\mathbb{Q}(i)}) = \mathbb{Z}\langle \Gamma_1, \dots, Q_{24} \rangle$ with the Gram matrix of Bremner's basis (17), of determinant 48; c denotes complex conjugation acting on it.

10.1. The conjecture in lattice terms. Fix a finite set $\mathcal{S} \subset \Sigma_{\bar{\mathbb{Q}}}$ and a finite set $\mathcal{H} \subset \text{Aut}(V_{\bar{\mathbb{Q}}})$ with $\mathcal{H} = \mathcal{H}^{-1}$. Say that $\Gamma \in \Sigma_{\bar{\mathbb{Q}}}$ is *monotonically generated* from $(\mathcal{S}, \mathcal{H})$ if $\Gamma = h_m \cdots h_1 C$ with $C \in \mathcal{S}$, $h_j \in \mathcal{H}$, and $\deg(h_j \cdots h_1 C) > \deg(h_{j-1} \cdots h_1 C)$ for all j (for $m = 0$, $\Gamma = C$). Bremner's conjecture asserts the existence of $(\mathcal{S}, \mathcal{H})$ such that every $\Gamma \in \Sigma_{\bar{\mathbb{Q}}}$ is monotonically generated, where in Bremner's formulation $\mathcal{H}$ may also contain the symmetries, which are used freely and do not change the degree; replacing $\mathcal{H}$ by its closure under conjugation by the symmetries and $\mathcal{S}$ by its orbit under them, one may equivalently drop the symmetries. Note also that “automorphism from a finite set” may be taken to mean “element of a finite set $\mathcal{H}$ of automorphisms”, the inverses being included, since an automorphism and its inverse play symmetric roles in a chain.

The following observation is the engine of Bremner's proof for fifth powers [19] and of his cone-cutting procedure in [20, §7]. For a finite set $\mathcal{C}$ of classes in $\text{NS}(V_{\bar{\mathbb{Q}}})$ and a finite $\mathcal{H}$ as above, let

$$K_{\mathcal{C}, \mathcal{H}} = \{x \in \text{NS}(V_{\bar{\mathbb{Q}}})_{\mathbb{R}} : x \cdot C \geq 0 \ (C \in \mathcal{C}), \ \deg(hx) \geq \deg x \ (h \in \mathcal{H})\}, \quad (10.1)$$

a rational polyhedral cone ($\deg(hx) - \deg x = x \cdot (h^{-1}H - H)$ is linear in x).

Lemma 10.1 (cone criterion). *Let $\mathcal{S} \subset \Sigma_{\bar{\mathbb{Q}}}$ be finite and let $\mathcal{C}$ consist of the classes of the curves in $\mathcal{S}$ together with classes of the form $\sum_{C \in I} C$ for some Galois orbits I of (-2) -curves on $V_{\bar{\mathbb{Q}}}$ not defined over $\mathbb{Q}$ (for V , conjugate pairs $C + \bar{C}$). If every (-2) -class in $K_{\mathcal{C}, \mathcal{H}}$ that is a $\mathbb{Q}$ -rational (-2) -curve belongs to $\mathcal{S}$ — in particular if $K_{\mathcal{C}, \mathcal{H}} \subset \{x : x^2 \geq 0\}$ — then every $\Gamma \in \Sigma_{\bar{\mathbb{Q}}}$ is monotonically generated from $(\mathcal{S}, \mathcal{H})$.*

Proof. Let $\Gamma \in \Sigma_{\bar{\mathbb{Q}}}$ be of minimal degree among the curves that are not monotonically generated. Then $\Gamma \notin \mathcal{S}$, so $\Gamma \cdot C \geq 0$ for every $C \in \mathcal{S}$ (distinct irreducible curves), and $\Gamma \cdot C \geq 0$ for every C in a Galois orbit I not defined over $\mathbb{Q}$ (such C are not $\mathbb{Q}$ -rational, hence $\neq \Gamma$). If $\deg(h\Gamma) < \deg \Gamma$ for some $h \in \mathcal{H}$, then $h\Gamma \in \Sigma_{\bar{\mathbb{Q}}}$ is monotonically generated by minimality, and $\Gamma = h^{-1}(h\Gamma)$ with $h^{-1} \in \mathcal{H}$ and $\deg \Gamma > \deg h\Gamma$ is monotonically generated, a contradiction. Hence $\Gamma \in K_{\mathcal{C}, \mathcal{H}}$, against the hypothesis ($\Gamma^2 = -2 < 0$). $\square$

Theorem 10.2 (Bremner [19], main theorem and §§3–5; our reading of the text). *Let $W \subset \mathbb{P}^5$ be the K3 surface $x^5 + y^5 + z^5 = u^5 + v^5 + w^5$, $x + y + z = u + v + w$, $x - y = u - v$. Then $\text{NS}(W_{\bar{\mathbb{Q}}})$ has rank 9, generated by eight rational lines $L_1, \dots, L_8$ and H , W has 16 symmetries, and every irreducible curve of arithmetic genus 0 on W defined over $\mathbb{Q}$ is obtained from the line L_1 or the conic $H - L_1 - L_2$ by repeated application of the symmetries and of two involutions Θ_1, Θ_2 coming from two elliptic pencils, the degree increasing strictly at each application of a Θ_i .*

Bremner's proof is exactly Lemma 10.1: the cone cut out by the 24 inequalities “nonnegative intersection with the lines and conics” and by the inequalities “ Θ_i does not decrease the degree” is shown, by a finite computation at its extremal rays, to lie in $\{x^2 \geq 0\}$ (he writes $4T \cdot T = (\deg T)^2 - (\text{positive definite form})$, so that $T \cdot T$ is concave on the slice $\deg T = 1$ and its minimum is attained at a vertex). The surface W has $\rho(W_{\bar{\mathbb{Q}}}) = 9$ and the relevant polyhedron is small; for V ($\rho(V_{\bar{\mathbb{Q}}}) = 13$) Bremner reports that his program found about 4000 vertices with $\mathcal{S} = \{\text{lines, conics}\}$ and $\mathcal{H} = \emptyset$, and that he had “no firm guarantee” that the cutting procedure terminates [20, p. 577]. Bremner's model was Swinnerton-Dyer's

unpublished treatment of $x^4 + y^4 = z^4 + t^4$; what is in print is [86], where the elliptic pencils on that surface are used to show that every real solution is a limit of rational ones (compare Elkies [43] for $A^4 + B^4 + C^4 = D^4$), and the automorphism groups of diagonal quartics over non-closed fields are discussed in [21, §4.3]. We return to Bremner's procedure in §10.5; first we explain why it *must* terminate in principle.

10.2. What is known: Torelli, Sterk, and Bright–Logan–van Luijk. Let X be a complex projective K3 surface, $\text{NS} = \text{NS}(X)$ (hyperbolic, even), $\mathcal{C}_X$ the component of $\{x \in \text{NS}_{\mathbb{R}} : x^2 > 0\}$ containing an ample class, $W(\text{NS})$ the group generated by the reflections s_{δ} in the (-2) -vectors $\delta \in \text{NS}$, $O^+(\text{NS})$ the isometries preserving $\mathcal{C}_X$, and $A_{\text{NS}} = \text{NS}^{\vee} / \text{NS}$ with its discriminant form q_{NS} [71].

Theorem 10.3 (Pjateckii-Shapiro–Shafarevich [76]; see [48, Ch. 8, 15]). (a) $\text{Nef}(X) \cap \mathcal{C}_X$ is a fundamental domain for $W(\text{NS})$ acting on $\mathcal{C}_X$; its walls are the hyperplanes $C^{\perp}$, C a (-2) -curve, and $\text{Nef}(X) = \{x : x \cdot C \geq 0 \text{ for all } (-2)\text{-curves } C\}$ when $\rho(X) \geq 12$ (Kovács [56]). (b) $W(\text{NS})$ is normal in $O^+(\text{NS})$ and $\text{Aut}(X) \rightarrow O^+(\text{NS})/W(\text{NS})$ has finite kernel and image of finite index; more precisely, an isometry $g \in O^+(\text{NS})$ with $g(\text{Nef } X) = \text{Nef } X$ is induced by an automorphism if and only if its image $\bar{g} \in O(q_{\text{NS}})$ lies in the image of the group of Hodge isometries of the transcendental lattice T_X under $O(T_X) \rightarrow O(q_{T_X}) \cong O(q_{\text{NS}})$, and then the automorphism is unique. (c) $\text{Aut}(X)$ is finite if and only if $W(\text{NS})$ has finite index in $O(\text{NS})$, i.e. NS is 2-reflective; the 2-reflective hyperbolic lattices are classified (Nikulin [70, 72, 73], Vinberg [91]; Kondō [53]).

Proposition 10.4. (a) $\text{NS}(V_{\mathbb{Q}}) \cong U(2) \oplus E_8(-1)^2 \oplus \langle -12 \rangle$, $A_{\text{NS}} \cong (\mathbb{Z}/2)^2 \times \mathbb{Z}/12$, $q_{\text{NS}} \cong v(2) \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{2}{3} \rangle$ (Proposition 9.10), $|O(q_{\text{NS}})| = 24$, and $T_V \cong U(2) \oplus \langle 12 \rangle$. (b) The group of Hodge isometries of T_V is $\{\pm 1\}$. Consequently

$$\text{Aut}(V_{\mathbb{C}}) \cong \{g \in O^+(\text{NS}) : g(\text{Nef } V) = \text{Nef } V, \bar{g} = \pm 1 \in O(q_{\text{NS}})\},$$

$\text{Aut}(V_{\mathbb{C}}) \rightarrow O(\text{NS})$ is injective, every automorphism of $V_{\mathbb{Q}}$ is defined over $\mathbb{Q}(i)$, and $\text{Aut}(V_{\mathbb{Q}}) = \{\varphi \in \text{Aut}(V_{\mathbb{Q}}) : \varphi^*c = c\varphi^* \text{ on } \text{NS}\}$. (c) $\text{Aut}(V_{\mathbb{Q}})$ is infinite, $W(\text{NS})$ has infinite index in $O(\text{NS})$, and $\text{Nef}(V)$ has infinitely many walls.

Proof. (a) was computed from the Gram matrix of basis (17) (Proposition 9.10; the identification with the lattice of Artebani–Laface–Testa [1] uses Nikulin's uniqueness theorem, and we re-verified the genus equality in Sage). (b) Let g be a Hodge isometry of $T = T_V$, of rank 3 and signature $(2, 1)$. It has finite order, preserves the positive 2-plane P spanned by $\text{Re } \omega, \text{Im } \omega$ and acts on it by a rotation, and acts on $P^{\perp}$ by ± 1 . If the rotation is not ± 1 , its eigenvalues are primitive m -th roots of unity with $\varphi(m) \leq 2$, $m \in \{3, 4, 6\}$, so $P = \ker \Phi_m(g) \otimes \mathbb{R}$ is defined over $\mathbb{Q}$ and the rank-one lattice $P^{\perp} \cap T$ is orthogonal to ω , i.e. algebraic, contradicting $T \cap \text{NS} = 0$; the same argument excludes $g = (1_P, -1_{P^{\perp}})$. So $g = \pm 1$. Since -1 acts as $-1 \neq 1$ on A_T (which has elements of order 4), the kernel of $\text{Aut}(V_{\mathbb{C}}) \rightarrow O(\text{NS})$ is trivial and the description of $\text{Aut}(V_{\mathbb{C}})$ is Theorem 10.3(b). For $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(i))$ and $\varphi \in \text{Aut}(V_{\bar{\mathbb{Q}}})$, φ^{σ} and φ induce the same isometry of $\text{NS} = \text{NS}(V_{\mathbb{Q}(i)})$, so $\varphi^{\sigma} = \varphi$; the statement about $\text{Aut}(V_{\mathbb{Q}})$ follows in the same way. (c) The sections of the Brudno pencil are infinitely many $\mathbb{Q}$ -rational (-2) -curves (Theorem 5.4), so $\text{Nef}(V)$ has infinitely many walls and (Theorem 10.3(a),(c)) $W(\text{NS})$ has infinite index and Aut is infinite. $\square$

Theorem 10.5 (Sterk [84]). Let X be a complex algebraic K3 surface. (a) $\text{Aut}(X)$ is finitely generated. (b) For every even $d \geq -2$ there are finitely many $\text{Aut}(X)$ -orbits of complete linear systems containing irreducible curves of self-intersection d ; for $d = -2$: finitely many orbits of smooth rational curves, for $d = 0$: of elliptic pencils, for $d = 2g - 2 > 0$: of polarisations of degree d . (c) $\text{Aut}(X)$ acts on $\text{Nef}_e(X)$, the convex hull of $\text{Nef}(X) \cap \text{NS}(X)$, with a rational polyhedral fundamental domain.

Part (c), which implies (a) and (b), is the K3 case of the Morrison–Kawamata cone conjecture [66, 50]; see Totaro [87] for dimension 2 in general (using Sterk, Looijenga and Namikawa

[68]), Lieblich–Maulik [61] for characteristic $p > 2$, and Looijenga [62] for the general theory of “polyhedral triples” ($\text{NS}_{\mathbb{R}}, \text{Amp}, \Gamma$). Over a non-closed field the relevant statements are those of Bright, Logan and van Luijk.

Theorem 10.6 (Bright–Logan–van Luijk [21]). *Let k be a field of characteristic $\neq 2$ with separable closure k^s , X a K3 surface over k , $\Gamma_k = \text{Gal}(k^s/k)$, $\text{Aut } X$ the group of automorphisms defined over k , $\text{Nef}(X) = \text{Nef}(X_{k^s}) \cap \text{Pic}(X)_{\mathbb{R}}$, and $R_X = W(\text{Pic } X^s)^{\Gamma_k}$. (a) (Prop. 3.6, 3.7) R_X is a Coxeter group generated by the longest elements of the finite groups W_I attached to the Galois orbits I of (-2) -curves consisting of pairwise disjoint curves or of disjoint pairs of curves meeting in one point, and $\text{Nef}(X) \cap \mathcal{C}_X$ is a fundamental domain for R_X on the positive cone of $\text{Pic}(X)_{\mathbb{R}}$. (b) (Prop. 3.10) The natural map $\text{Aut } X \ltimes R_X \rightarrow O(\text{Pic } X)$ has finite kernel and image of finite index. (c) (Cor. 3.15) $\text{Aut } X$ acts on $\text{Nef}_e(X)$ with a rational polyhedral fundamental domain. (d) (Cor. 3.16) There are finitely many $\text{Aut } X$ -orbits of k -rational (-2) -curves on X , finitely many orbits of primitive classes of irreducible curves of arithmetic genus 1, and, for each $g \geq 2$, finitely many orbits of classes of irreducible curves of arithmetic genus g .*

Remark 10.7. (i) The objects in (d) are exactly Bremner’s: a k -rational (-2) -curve is an irreducible curve of arithmetic genus 0 defined over k . So Theorem 10.6(d) is the finiteness part of Bremner’s conjecture for V over $\mathbb{Q}$ (Corollary 10.8), with $\text{Aut } X = \text{Aut}(V_{\mathbb{Q}})$, which is what Bremner needs: his automorphisms must map $\mathbb{Q}$ -rational curves to $\mathbb{Q}$ -rational curves. (ii) Finite generation of $\text{Aut } X$ is not stated in [21] (we searched the text), but it follows from (b): $O(\text{Pic } X)$ is an arithmetic group, hence finitely generated, so is its finite-index subgroup, hence the finite extension $\text{Aut } X \ltimes R_X$ and its quotient $\text{Aut } X$; Looijenga’s Corollary 4.15 gives even finite presentation (see below). (iii) In [21, §4] there are K3 surfaces $X/\mathbb{Q}$ with $\text{Aut } X$ finite although $\text{Aut } \bar{X}$ is infinite; so over $\mathbb{Q}$ the finiteness of Aut is not determined by the lattice alone. For V this does not happen (Proposition 10.4(c), with $\mathbb{Q}$ -rational automorphisms). (iv) Over $\mathbb{Q}$ the walls of $\text{Nef}(V_{\mathbb{Q}})$ in $\text{NS}(V_{\mathbb{Q}})_{\mathbb{R}}$ are of two kinds: $C^{\perp}$ for $\mathbb{Q}$ -rational (-2) -curves C , and $(C + \bar{C})^{\perp}$ for conjugate pairs with $C \cdot \bar{C} \in \{0, 1\}$ (classes of square -4 , e.g. $\Gamma_9 + \Gamma_{10}$, or of square -2 when the two curves meet in a point). This is why the constraints $\mathcal{C}$ in Lemma 10.1 must include the pair classes.

Corollary 10.8. *Assume (H). The set $\Sigma_{\mathbb{Q}}$ of smooth rational curves on V defined over $\mathbb{Q}$ is the union of finitely many $\text{Aut}(V_{\mathbb{Q}})$ -orbits, and $\text{Aut}(V_{\mathbb{Q}})$ is finitely generated (indeed finitely presented). The same holds over $\mathbb{Q}(i)$ and over $\bar{\mathbb{Q}}$. Neither the number of orbits nor the generators are given by the proof.*

The degree clause is not addressed by [21] or [84], but it follows from the shape of the fundamental domain. Fix an ample class $h \in \text{Pic}(X)$ (for V : $h = H$) and put

$$\sigma_h = \{x \in \text{Nef}_e(X) : x \cdot h \leq x \cdot gh \text{ for all } g \in \text{Aut } X\}, \quad (10.2)$$

the Dirichlet domain of the orbit $\text{Aut}(X) \cdot h$. By Looijenga [62, Application 4.14], whose hypothesis (“polyhedral type”) is supplied by Theorem 10.6(c), σ_h is a rational polyhedral cone with $\text{Aut}(X) \cdot \sigma_h = \text{Nef}_e(X)$, and its stabiliser is the finite group $\text{Aut}(X)_h$ (for V : the 48 symmetries). By [62, Cor. 4.15], $\text{Aut } X$ is finitely presented and generated by $\text{Aut}(X)_h$ together with the finitely many g with $g\sigma_h \cap \sigma_h$ a facet of σ_h .

Theorem 10.9. *Let k be a field of characteristic $\neq 2$, X a K3 surface over k , $h \in \text{Pic } X$ ample, σ_h as in (10.2). Let $\mathcal{S}_h$ be the set of k -rational (-2) -curves C such that $C^{\perp} \cap \sigma_h$ is a facet of σ_h , and $\mathcal{H}_h$ the set of $g \in \text{Aut } X$ such that $g\sigma_h \cap \sigma_h$ is a facet of σ_h . Both are finite, $\mathcal{H}_h = \mathcal{H}_h^{-1}$, and every k -rational (-2) -curve on X is monotonically generated from $(\mathcal{S}_h, \mathcal{H}_h)$ with respect to $\deg = h \cdot (\)$. In particular Bremner’s conjecture holds for V over $\mathbb{Q}$, with respect to H , with the (non-explicit) finite sets $\mathcal{S}_H$ and $\mathcal{H}_H \cup \{\text{symmetries}\}$.*

Proof. Write $N = \text{Pic}(X)_{\mathbb{R}}$ and $G = \text{Aut } X$ acting on N . σ_h is full-dimensional: for $g \notin G_h$ one has $h \cdot h < h \cdot gh$ (reverse Cauchy–Schwarz for two non-proportional vectors of equal positive norm in the same component of the positive cone), and only finitely many gh satisfy

$x \cdot gh \leq x \cdot h + 1$ for x in a neighbourhood of h , because $\{y \in \text{Pic } X : y^2 = h^2, x \cdot y \leq c\}$ is finite for x in the open positive cone. Hence σ_h is the intersection of its facet half-spaces. Let F be a facet and x a point of its relative interior. Since F spans a hyperplane and the isotropic cone contains no open subset of a hyperplane, we may take $x^2 > 0$. Either $x \in \partial \text{Nef}(X)$: then, by Theorem 10.6(a) and the local finiteness of the (-2) -walls inside the open positive cone, $x \cdot C = 0$ for some (-2) -curve C on X_{k^s} , hence $x \cdot \sum_{C' \in I} C' = 0$ for the Galois orbit I of C , and this hyperplane supports σ_h at x , so it is the hyperplane of F ; the facet inequality is $x \cdot \sum_I C' \geq 0$. Or x is in the interior of $\text{Nef}(X)$ and on the boundary of the Dirichlet cell: by the finiteness just noted, $x \cdot h = x \cdot gh$ for some $g \neq 1$, the hyperplane of F is the bisector $\{y \cdot h = y \cdot gh\}$, and since at most two points of the orbit of equal norm have a given bisector, the cell on the other side is $g\sigma_h$ (translate the Dirichlet cell of h by g); so $g\sigma_h \cap \sigma_h = F$ and $g \in \mathcal{H}_h$. The facet inequality is $y \cdot gh \geq y \cdot h$, i.e. $\deg(g^{-1}y) \geq \deg y$. Applying g^{-1} to $g\sigma_h \cap \sigma_h = F$ shows $g^{-1} \in \mathcal{H}_h$. Therefore

$$\sigma_h = \left\{ y : y \cdot \sum_{C' \in I} C' \geq 0 \ (I \in \mathcal{I}_h), \quad \deg(gy) \geq \deg y \ (g \in \mathcal{H}_h) \right\} = K_{\mathcal{C}_h, \mathcal{H}_h}$$

with $\mathcal{I}_h$ the finite set of Galois orbits whose walls are facets of σ_h and $\mathcal{C}_h$ the corresponding classes; the k -rational curves in $\mathcal{I}_h$ form $\mathcal{S}_h$. Since $\sigma_h \subset \text{Nef}(X) \subset \{y^2 \geq 0\}$, Lemma 10.1 (whose proof is valid over any k) gives the claim. $\square$

Remark 10.10. (i) The proof shows that σ_h is Bremner's cone $K_{\mathcal{C}, \mathcal{H}}$ for the “right” finite sets, and that his cutting procedure, if it terminates, computes σ_h (or a polyhedral cone between σ_h and the positive cone); conversely it *can* terminate, since σ_h exists and is polyhedral. What it cannot do is certify that a given pair $(\mathcal{S}, \mathcal{H})$ works when $K_{\mathcal{C}, \mathcal{H}} \not\subset \{x^2 \geq 0\}$, as happens for the sets of Theorem 6.7 (§10.5). (ii) The argument is the exact analogue of the height-reduction lemma for Coxeter groups: for a Coxeter system (W, S) with simple roots Δ , every positive root $\beta \notin \Delta$ has a simple root α with $(\beta, \alpha) > 0$, and then $s_\alpha \beta$ is a positive root of smaller height; hence every positive root is obtained from a simple root by simple reflections increasing the height at each step ([47, §§1.3–1.5, 5.4], [88] for the hyperbolic case). For $W(\text{NS})$ itself the “simple roots” are *all* (-2) -curves (infinitely many) and the height-reduction lemma is Lemma 6.1: a reducible (-2) -class D has a (-2) -curve C with $D \cdot C < 0$ and $\deg s_C(D) < \deg D$. Bremner's conjecture replaces this infinite Coxeter system by the finite data $(\mathcal{S}_h, \mathcal{H}_h)$ of the quotient by Aut . (iii) The analogous statement over $\mathbb{Q}(i)$ (all (-2) -curves) and over $\mathbb{Q}$ holds with the same proof, with $\text{Aut}(V_{\mathbb{Q}})$.

10.3. Explicit computation of $\text{Aut}(V)$: Vinberg, Borchers, Shimada. To make Theorem 10.9 explicit one has to compute $\text{Aut}(V_{\mathbb{Q}})$, or at least enough of it to exhibit σ_H . Vinberg's algorithm [89, 88] (see Dolgachev's survey [41] for its use on K3 surfaces) computes a fundamental polyhedron of the reflection group of a hyperbolic lattice by adjoining walls one at a time; it terminates if and only if the polyhedron has finitely many walls, which for $W(\text{NS}(V))$ is not the case (Proposition 10.4(c)). Borchers' method [13, 14] circumvents this: embed $S = \text{NS}$ primitively into the even unimodular lattice $L = \text{II}_{1,25}$, whose reflection group has the Leech lattice as its Dynkin diagram (Conway [32]) and whose chambers are explicitly known; the Conway chambers meeting $S_{\mathbb{R}}$ cut $\text{Nef}(X)$ into “induced chambers” with finitely many walls, permuted by $\text{Aut}(X)$ with finitely many orbits, and a breadth-first search through adjacent induced chambers computes a generating set of $\text{Aut}(X)$ and a fundamental domain. This is how Kondō computed Aut of the generic Jacobian Kummer surface [54] (a polyhedron with $32 + 32 + 60 + 192$ faces: nodes and tropes, 60 Cremona involutions and Keum's 192 automorphisms [51], the last two kinds “behaving like reflections without being reflections”), and how Keum and Kondō computed $\text{Aut } \text{Km}(E_1 \times E_2)$ for E_1, E_2 non-isogenous, for $E \times E$ without CM, and for $E_\omega \times E_\omega$, $E_i \times E_i$ [52]; Vinberg's “two most algebraic K3 surfaces” X_3, X_4 [90] ($\rho = 20$, discriminants 3, 4) were done directly by reflection-group methods. Our V is $\text{Km}(\text{Jac } \mathcal{C})$ with $\text{Jac } \mathcal{C} \cong E_1 \times E_2$ for a cyclic 3-isogeny $E_1 \rightarrow E_2$ (Kuwata [57]; Remark 9.11; $T_V \cong U(2) \oplus \langle 12 \rangle$), a case not covered by [52]; we found no computation of Aut for Kummer surfaces of products of isogenous, non-isomorphic elliptic curves in the literature.

Shimada’s algorithm [77] (IMRN 2015 — not Amer. J. Math. as sometimes cited) makes Borcherds’ method general. Its hypotheses are: [SG1] S is primitively embedded in $L = \Pi_{1,25}$; [SG2] the orthogonal complement $R = S^\perp$ is not embeddable into the Leech lattice (this makes every induced chamber a finite polyhedron); [SG3] a lifting condition, that the image in $O(q_S)$ of the group $G \subset O(S)$ one wants to compute is contained in the image of $O(R) \rightarrow O(q_R) \cong O(q_S)$. Earlier applications needed R to contain a root lattice of finite index; Shimada drops this and allows several Aut-orbits of induced chambers (his examples: 46 chambers for a lattice of rank 3, 1098 chambers and 767 generators for the singular K3 surface with $\text{disc } T = 11$; the Fermat quartic, $\text{disc } T = 64$, was too heavy in 2013). The method is implemented in OSCAR (`K3_surface_automorphism_group(S[,h])`, `borcherds_method`) following [77, 17], and returns generators of the image of $\text{Aut}(X) \rightarrow O(\text{NS})$ for a very general S -polarised K3 surface, representatives of the Aut-orbits of (-2) -curves, and a fundamental domain [74]; for the scale of computation it can handle see [16, 15], and [55] for Kummer surfaces of products of elliptic curves in characteristic 2 by Borcherds’ method.

Proposition 10.11. *$\text{NS}(V_{\mathbb{Q}})$ embeds primitively into $\Pi_{1,25}$ with orthogonal complement $R \cong A_2 \oplus D_4 \oplus \langle -4 \rangle$ (negative definite, rank 7, determinant -48 , 30 roots), and $O(R) \rightarrow O(q_R)$ is surjective. Hence [SG1]–[SG3] hold for V and every subgroup of $O(\text{NS})$, in particular for $G = \{g \in O^+(\text{NS}) : \bar{g} = \pm 1\}$, which contains the image of $\text{Aut}(V_{\mathbb{C}})$ as the stabiliser of $\text{Nef}(V)$ (Proposition 10.4). No embedding has a complement containing a root lattice of finite index (the only rank-7 root lattice of determinant 48, $A_3 \oplus A_1^2 \oplus A_2$, has the wrong discriminant form), so the pre-2015 form of the method does not apply.*

Proof. Existence follows from Nikulin [71, Thm. 1.12.2], since $26 - 19 = 7 > 3 = \ell(A_{\text{NS}})$. Explicitly, in Sage: $q_R \cong -q_{\text{NS}} \cong v(2) \oplus \langle \frac{7}{4} \rangle \oplus \langle \frac{4}{3} \rangle$ for $R = A_2 \oplus D_4 \oplus \langle -4 \rangle$ (all three negative definite), the overlattice of $\text{NS} \oplus R$ along the graph of an anti-isometry $A_{\text{NS}} \rightarrow A_R$ is even, unimodular, of signature $(1, 25)$, hence $\Pi_{1,25}$, with NS and R primitive; $|O(R)| = 27648$, $|O(q_R)| = 24$, and the image of $O(R)$ in $O(q_R)$ has order 24. (Scripts `lattice_checks2--4.sage`.) $\square$

The algorithm for V . (1) Input the Gram matrix of basis (17), the embedding of Proposition 10.11, and the ample class H (with the 56 lines and conics as the known (-2) -walls). (2) Find a Weyl vector w of a Conway chamber whose induced chamber $D_0 \subset \text{NS}_{\mathbb{R}}$ contains H ; compute its walls (finitely many, by [SG2]) — they are of the form $r^\perp$ with r the projection to NS of a Leech root, and have norms -2 (then r is a smooth rational curve, if $D_0 \subset \text{Nef}$) or > -2 . (3) Breadth-first search: for each wall, pass to the adjacent induced chamber, test whether it is G -congruent to one already found (an isometry test in $O(\text{NS})$, i.e. a finite enumeration), and record the congruence as a generator; stop when no new congruence classes appear. (4) Output: a generating set of the image of $\text{Aut}(V_{\mathbb{C}})$ in $O(\text{NS})$ (by Proposition 10.4 this is $\text{Aut}(V_{\mathbb{C}})$ itself), a finite set of induced chambers tiling a fundamental domain F of $\text{Nef}(V)$, and, as the (-2) -walls of the chambers in F , representatives of all $\text{Aut}(V_{\mathbb{C}})$ -orbits of smooth rational curves. (5) Over $\mathbb{Q}$: $\text{Aut}(V_{\mathbb{Q}})$ is the centraliser of c (Proposition 10.4(b)); since c preserves Nef and H , it permutes the induced chambers of F up to congruence, and the centraliser can be read off from the finite chamber graph (or one runs the search for c -stable configurations in $\text{NS}(V_{\mathbb{Q}})_{\mathbb{R}}$ directly; cf. the reflection group R_X of Theorem 10.6(a), which on $\text{NS}(V_{\mathbb{Q}})$ is generated by reflections in the (-2) -curves and in the (-4) - and (-2) -classes $C + \bar{C}$). (6) From the generators, compute the Dirichlet domain σ_H by Bremner’s cutting procedure (§10.5), which now terminates by Theorem 10.9, and read off $\mathcal{S}_H$ and $\mathcal{H}_H$.

Feasibility. The cost is governed by the number of G -congruence classes of induced chambers in $\text{Nef}(V)$, which we cannot predict; V has rank 19 and discriminant 48, between Shimada’s two examples and the Fermat quartic. Everything needed (Gram matrix, embedding, H , the known curves, a working OSCAR/Sage/PARI environment with `qfauto`, `qfminim`, `qfisom`) is at hand, and a first experiment with `K3_surface_automorphism_group` costs a few hours of set-up. The output would settle (B) and make Theorem 10.9 explicit for V ; it would also decide whether the $\mathbb{Q}$ -rational pencil involutions generate $\text{Aut}(V_{\mathbb{Q}})$ (compare generators), which is the question behind (C).

10.4. The degree-increasing clause: heights, Voronoi cells, and the involutions.

The degree is a linear “height” $x \mapsto x \cdot H$ on NS, and the clause “the degree increases at each step” says that from every non-minimal class some generator strictly decreases the height. We saw in Theorem 10.9 that this holds for the facet-pairing generators of a Dirichlet domain; here are three further comments.

Canonical heights. Silverman [82] attached to a K3 surface with two involutions σ_1, σ_2 (Wehler surfaces [92], $\rho = 2$, $\text{Aut} \supset \mathbb{Z}/2 * \mathbb{Z}/2$) canonical heights $\hat{h}^\pm$ with $\hat{h}^\pm \circ \sigma_i = \alpha^{\pm 1} \hat{h}^\mp$, $\alpha = 2 + \sqrt{3}$, so that along an orbit the height grows geometrically and “a chain is finite iff its canonical height is 0”; Baragar extended this to orbits of curves and to vector heights for $\rho = 2, 3$ [4, 3, 5, 8, 6], showing that canonical vector heights need not exist for $\rho \geq 3$. For V ($\rho = 13$ over $\mathbb{Q}$) nothing of the sort is available; what replaces the canonical height is the convex geometry of σ_H .

Sections of a pencil. For the sections of the Brudno pencil the degree clause can be proved outright. Recall (Theorem 5.4, assuming Lemma 5.1) $\deg C_S = 2\langle S - \gamma/4, S - \gamma/4 \rangle$ for $S \in \text{MW} = \mathcal{E}(\mathbb{Q}(\lambda))$, $\gamma = -g_1 + g_2 + g_3$, where $\langle \cdot, \cdot \rangle$ is the height pairing with Gram matrix $\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 2 & 1/2 \\ -1/2 & 1/2 & 2 \end{pmatrix}$ on $\mathbb{Z}g_1 \oplus \mathbb{Z}g_2 \oplus \mathbb{Z}g_3$. The translation $\tau_{S_0}: R \mapsto R + S_0$ by a $\mathbb{Q}$ -rational section S_0 is a $\mathbb{Q}$ -rational automorphism of V (it is $\iota_{C_1+C_2} \circ \iota_{C_3+C_4}$ for sections with $C_1+C_2-C_3-C_4 = S_0$ in the group law), and $\deg \tau_{S_0}(C_S) = \deg C_{S+S_0}$.

Proposition 10.12. *Assume Lemma 5.1. Let $\mathcal{V} = \{\pm g_1, \pm g_2, \pm g_3, \pm(g_2 - g_3), \pm(g_1 + g_2), \pm(g_1 + g_3)\}$. For every section S of the Brudno pencil that is not one of the eight lines $\{kT, -g_1 + kT\}$ there is $v \in \mathcal{V}$ with $\deg C_{S-v} < \deg C_S$. Hence every section is obtained from a line by the 12 translations τ_v , the degree increasing strictly at every step.*

Proof. $\mathcal{V}$ is the set of Voronoi-relevant vectors of the lattice MW/tors with the height pairing (computed by Voronoi’s criterion: $\pm v$ are the only shortest vectors of $v + 2\text{MW}$; script `mw_voronoi.py`). Put $x = S - \gamma/4$. If $|x - v| \geq |x|$ for all $v \in \mathcal{V}$, then x lies in the Voronoi cell of 0, i.e. $|x| \leq |x - w|$ for all $w \in \text{MW}$, so S has minimal degree among all sections, which is 1: S is a line. Otherwise $\deg C_{S-v} = 2|x - v|^2 < 2|x|^2 = \deg C_S$. $\square$

The same argument applies to every $\mathbb{Q}$ -rational Jacobian pencil F of positive rank and to the translates of any fixed curve D (not only sections): by Shioda’s formulas, $\deg \tau_{S_0}(D) = \frac{(D \cdot F)(H \cdot F)}{2} \langle S_0, S_0 \rangle + (\text{linear in } S_0) + (\text{periodic in } S_0)$, a shifted positive definite quadratic form on each coset of the narrow Mordell–Weil group, so finitely many Voronoi-relevant translations reduce the degree of all but finitely many translates. This shows that within the orbit of a single pencil’s translation group the degree clause is automatic; the whole difficulty of the conjecture is the interaction of the 14 (or more) pencils, which is what the Dirichlet domain encodes.

The involutions are not reflections. An involution $\iota_{C_1+C_2}$ of a pencil acts on NS as -1 on the orthogonal complement of $\langle F, C_1, C_2, \text{fibre components} \rangle$ and permutes the generators; its (-1) -eigenlattice $N_-(\iota)$ is negative definite (the fixed lattice contains F and $H + \iota H$). For the 532 involutions of the Brudno pencil we computed (`involution_structure.sage`): $\text{rk } N_- \in \{7, 9, 10, 11, 13\}$, N_- contains no (-2) -vector, and among the vectors $v \in N_-$ with $|v|^2 \leq 12$ at most one pair $\pm v$ (of norm -4) defines an integral reflection of NS. So ι is a product of reflections only over $\mathbb{Q}$, is not in $W(\text{NS})$ (no automorphism is, by Theorem 10.3(a)), and is not a product of the few integral reflections it contains. This matches Kondō’s observation that the facet-pairing involutions of a Kummer surface “behave like reflections without being true reflections”: what they have in common with simple reflections is that they pair a facet of a fundamental domain with itself (Theorem 10.9). The question whether the 772 involutions of the 14 pencils generate a subgroup of finite index in $\text{Aut}(V_{\mathbb{Q}})$ can be answered by (B): compare the group they generate with the output of Shimada’s algorithm (finite index can be tested on the finite chamber graph); or, without (B), by exhibiting a polyhedral cone Π with $\Pi \subset \text{Nef}_e$ and $\langle \text{involutions} \rangle \cdot \Pi = \text{Nef}_e$, which is the content of the cone test below.

10.5. Bremner’s cone test on the Phase-A data. We ran the first step of Bremner’s procedure with the data of Sections 5–6 (scripts `build_cone.py`, `cone_S_only.sage`; the cone lives in $\text{NS}(V_{\mathbb{Q}})_{\mathbb{R}} \cong \mathbb{R}^{13}$, in the basis $\Lambda_1, \dots, \Lambda_{13}$). With $\mathcal{C}$ = the classes of the 56 lines and conics (38 distinct functionals on $\text{NS}(V_{\mathbb{Q}})$, since conjugate curves give the same functional) and $\mathcal{H} = \emptyset$, the cone $K_{\mathcal{C}, \emptyset}$ has 8266 extremal rays (Parma Polyhedra Library, 4 seconds): 7782 of negative square, 268 isotropic, 216 positive. (Bremner reports about 4000 vertices; he may have used only the 20 rational curves or a different normalisation.) Among the degree-5 rays with $r^2 = -2$ is the class of his curve (ii) (the class of (iii), and that of Brudno’s quartic, lie in the cone but not on extremal rays).

Next we took $\mathcal{H}$ = the 772 $\mathbb{Q}$ -rational involutions of the 14 $\mathbb{Q}$ -rational pencils of degree ≤ 6 , closed under conjugation by the 48 symmetries (2730 involutions, 2444 distinct inequalities $\deg(\iota x) \geq \deg x$). Of the 7782 negative rays, 7722 violate at least one of them; 60 **do not**, so $K_{\mathcal{C}, \mathcal{H}} \not\subset \{x^2 \geq 0\}$ and Lemma 10.1 does not apply to the sets of Theorem 6.7. (The full vertex enumeration of $K_{\mathcal{C}, \mathcal{H}}$, 2482 inequalities in $\mathbb{R}^{13}$, had not finished at the time of writing; it is not needed for what follows.) The 60 rays form three symmetry orbits:

- (A) 12 rays with $r^2 = -12$, $\deg r = 24$, $\text{div}(r) = 6$. Each is orthogonal to 16 pairwise disjoint lines (for one of them $\Gamma_5, \dots, \Gamma_8, \Gamma_{13}, \dots, \Gamma_{16}, \Gamma_{21}, \dots, \Gamma_{24}, \Gamma_{29}, \dots, \Gamma_{32}$). By Nikulin [69] sixteen disjoint smooth rational curves on a K3 surface are the exceptional curves of a Kummer structure $V = \text{Km}(A)$, and $\langle 16 \text{ lines} \rangle^\perp = \pi_* \text{NS}(A) \cong \text{NS}(A)(2)$; here it has rank 3 and determinant 48, so $\text{NS}(A) \cong U \oplus \langle -6 \rangle$ (the lattice of $E_1 \times E_2$ with a cyclic 3-isogeny, Remark 9.11), and r is the image of a (-6) -vector of $\text{NS}(A)$.
- (B) 24 rays with $r^2 = -4$, $\deg r = 12$, $\text{div}(r) = 2$; (C) 24 rays with $r^2 = -12$, $\deg r = 24$, $\text{div}(r) = 6$.

All 60 satisfy $r \cdot C \geq 0$ for all 668 056 irreducible (-2) -curves of degree ≤ 12 over $\mathbb{Q}(i)$; none is effective. Since $\text{Nef}(V) = \bigcap_{\mathcal{C}} \{x \cdot C \geq 0\}$ (Theorem 10.3(a), $\rho = 19 \geq 12$) and $r^2 < 0$, each r has $r \cdot C < 0$ for some smooth rational curve C of degree ≥ 13 : the obstructing rays point exactly at the part of V that the enumeration of Section 6 does not see.

Proposition 10.13. (a) *For each of the 60 rays the reflection s_r is an integral isometry of $\text{NS}(V_{\mathbb{Q}})$ commuting with c , and s_r preserves $\text{Nef}(V)$: there is no (-2) -class D with $D \cdot H > 0 > D \cdot s_r(H)$. (b) *For the rays of orbit (A), $\bar{s}_r = -1$ on A_{NS} ; hence s_r is induced by an involution $\varphi_r \in \text{Aut}(V_{\mathbb{Q}})$, which fixes the 16 lines orthogonal to r and satisfies $\deg \varphi_r(r) = -24 < 24$. It is the fibrewise negation $[-1] = \iota_{Q+Q}$ of the $\mathbb{Q}$ -rational elliptic pencil of degree 8 with four fibres of type I_0^* spanned by $r^\perp \cap \langle 16 \text{ lines} \rangle^\perp$ (a Kummer pencil, Q one of its conic sections), i.e. the involution induced by $(x, y) \mapsto (x, -y)$ on $E_1 \times E_2$. (c) *For the rays of orbits (B) and (C), $\bar{s}_r \neq \pm 1$, so s_r is a symmetry of $\text{Nef}(V)$ not induced by an automorphism (the stabiliser of $\text{Nef}(V)$ in $O^+(\text{NS})$ is strictly larger than $\text{Aut}(V_{\mathbb{C}})$). Nevertheless each such r is orthogonal to $\mathbb{Q}$ -rational elliptic pencils F of degree 8 (types $I_8 I_4 I_0^*$ and $I_1^{*2} I_3^2$ for (B), $I_8^2 \text{III}^2$ for (C)) with $\mathbb{Q}$ -rational sections among the known curves, and some of Bremner’s involutions $\iota_{C_1+C_2}$ of these pencils satisfy $\iota(r) = -r$.***

Proof. (a) Integrality: $r^2 \mid 2 \text{div}(r)$. For the second claim we used the bound $(aH + bH')^2 \geq -2$ for the component of a separating class $D = aH + bH' + D_\perp$ in the plane $\langle H, H' \rangle$, $H' = s_r(H)$, $H \cdot H' \in \{80, 104\}$, which gives $\deg D \leq 31$, resp. 40, and enumerated all (-2) -classes of these degrees with $D \cdot H' \leq 0$ by the constrained Fincke–Pohst search of Section 6 (`separating.py`; a few hundred tree nodes per degree): there are none. A (-2) -curve C with $C \cdot H' < 0$ would be such a class; so H' is nef ($H'^2 = 8$, $H' \cdot H > 0$). Since H' lies in the open chamber $s_r(\text{Amp})$ and in the closure of the chamber Amp , the two chambers coincide, i.e. $s_r(\text{Nef}) = \text{Nef}$. (b) The action on A_{NS} was computed on the dual basis; Proposition 10.4(b) and Theorem 10.3(b) give φ_r , defined over $\mathbb{Q}$ because s_r commutes with c . The identification with $[-1]$ of the Kummer pencil: the primitive isotropic classes F with $F \cdot r = 0$ of degree ≤ 12 were enumerated (`pencils_perp.py`); the two of smallest degree have degree 8, are $\mathbb{Q}$ -rational and nef (checked against the curves of degree ≤ 8), have 20 fibre components among the known curves ($4 \times I_0^*$: 16 lines and 4 conics) and 8 conic sections. Building their involutions with the Phase-A code

(`cut_by_pencils.py`, `verify_phi.py`) gives $\deg \iota(r) \in \{-24, 24, 72\}$, and the matrix of the negation ι_{Q+Q} with respect to four of the eight conic sections Q (of either pencil) *equals* the matrix of s_r . For such a pencil, with zero section Q the image of $E_1 \times \{0\}$, the negation on the fibre $\{p\} \times E_2$ is $(p, y) \mapsto (p, -y)$. (c) As in (b). $\square$

Remark 10.14. (i) Proposition 10.13 carries out Bremner's own prescription for a ray R with $R^2 < 0$ that is not effective: add to $\mathcal{H}$ an automorphism built from a rational elliptic pencil $F \perp R$ and a transversal D with $D \cdot F = 2$, whose involution (his Theorem 5.4) reverses R [20, pp. 576–577]. The pencils needed have degree 8, beyond the range of Theorem 6.7, which explains why the obstruction was invisible there and why the involutions of degree ≤ 6 pencils cannot prove the conjecture by this method even though they generate all curves of degree ≤ 12 . (ii) No product of two of the 2730 involutions, the symmetries and the 12 automorphisms φ_r reduces the degree of a ray of orbit (B) or (C) (`orbit_min.py`); the degree-8 pencils are genuinely new generators for this purpose. (iii) Part (c) shows that $[O^+(\text{NS})_{\text{Nef}} : \text{Aut}(V_{\mathbb{C}})] \geq 2$; this index divides $|O(q_{\text{NS}})/\pm 1| = 12$ and must be built into any computation of $\text{Aut}(V)$ (Shimada's G).

10.6. Singular rational curves. Bremner excludes curves of geometric genus 0 and arithmetic genus $g \geq 1$, and says such curves exist on V over $\mathbb{Q}$ [20, p. 546]. An irreducible curve C with $p_a(C) = g \geq 1$ has $C^2 = 2g - 2 \geq 0$, is nef, and moves in the g -dimensional linear system $|C|$; only finitely many members of $|C|$ are rational (otherwise V would be uniruled). Sterk's Theorem 10.5(b) and Theorem 10.6(d) therefore give: for each $g \geq 1$ the $\mathbb{Q}$ -rational rational curves of arithmetic genus g on V fall into finitely many $\text{Aut}(V_{\mathbb{Q}})$ -orbits (first finitely many orbits of classes, then finitely many curves per class). On the other hand there is no finiteness over all g : Chen, Gounelas and Liedtke [28] prove that every K3 surface over an algebraically closed field of characteristic 0 contains infinitely many rational curves (for elliptic K3 surfaces this is in Bogomolov–Tschinkel [11]; see also [10, 12]), and for V the smooth ones alone are infinite; whether rational curves of unbounded arithmetic genus exist on V , over $\bar{\mathbb{Q}}$ or over $\mathbb{Q}$, is a different question, which we have not settled (for a general polarised K3 surface it holds by Chen's results, but V is far from general). So the natural extension of Bremner's conjecture to all rational curves must be formulated genus by genus, as a statement about the classes $C \in \text{NS}(V_{\mathbb{Q}})$ with $C^2 = 2g - 2$ containing a rational curve over $\mathbb{Q}$: for each g there are finitely many $\text{Aut}(V_{\mathbb{Q}})$ -orbits, and the degree clause holds for them too, by the Dirichlet domain argument applied to the nef classes (for $g \geq 1$ these are points of Nef_e , and $\text{Aut}(V_{\mathbb{Q}})\sigma_H \ni C$ gives a representative in σ_H of minimal degree in its orbit, reached by a monotone gallery; we omit the details). Which classes actually contain rational curves is a question of a different nature (Bogomolov–Tschinkel [12], Baragar–McKinnon [7]), on which the lattice says nothing.

10.7. A research plan. We distinguish what is proved, what is a reduction, and what is speculation. Proved: Corollary 10.8 and Theorem 10.9 (the conjecture with non-explicit sets, for every K3 surface over every field of characteristic $\neq 2$); Propositions 10.4, 10.11, 10.12, 10.13. Reduction: an explicit proof for V is equivalent to exhibiting finite sets $(\mathcal{C}, \mathcal{H})$ with $K_{\mathcal{C}, \mathcal{H}} \subset \{x^2 \geq 0\}$ (Lemma 10.1), and Theorem 10.9 guarantees that such sets exist. Speculation: that the pencils of degree ≤ 8 suffice, and that $\mathcal{S}$ can be taken to be the 20 rational lines and conics.

- M1. *Pencils of degree 7 and 8.* Extend the Phase-A enumeration of elliptic pencils from degree 6 to 8 (degree 7: 10 752 pencils, 192 $\mathbb{Q}$ -rational, 138 orbits under symmetries and conjugation, 4 with $\mathbb{Q}$ -rational members, 3 seconds; degree 8 needs the chunked version of the enumerator, the current one exceeds 8 GB), build their $\mathbb{Q}$ -rational involutions from the sections of degree ≤ 12 , and recompute the closure of Theorem 6.7 and the cone test. Expected outcome: the 60 rays disappear; new, higher-degree obstructing rays may or may not appear.
- M2. *Iterate Bremner's procedure* (lazy constraint generation: add only the inequalities violated by current rays; when a negative ray r survives, look for $\mathbb{Q}$ -rational pencils

$F \perp r$ of degree ≤ 12 and their involutions, as in Proposition 10.13, or for curves C with $r \cdot C < 0$). Each step is a PPL/cdd vertex enumeration in $\mathbb{R}^{13}$; the first one (38 inequalities) took 4 seconds, the one with 2482 inequalities is slow and should be done incrementally and with the symmetry group factored out. Termination is guaranteed in principle by Theorem 10.9 but not along this particular path; if it terminates, the output is a complete proof of Bremner’s conjecture for V over $\mathbb{Q}$ with explicit sets, needing no classification of curves beyond the lines and conics.

- M3. *Borcherds–Shimada for V* (§10.3), with the embedding of Proposition 10.11, first in OSCAR. Output: $\text{Aut}(V_{\mathbb{C}})$ with generators, the orbits of smooth rational curves, the index of $\text{Aut}(V_{\mathbb{C}})$ in the symmetry group of $\text{Nef}(V)$ (at least 2 by Proposition 10.13(c)), then $\text{Aut}(V_{\mathbb{Q}})$ as the centraliser of c and the Dirichlet domain σ_H with $\mathcal{S}_H, \mathcal{H}_H$. This is the most informative route and the only one that also answers whether the pencil involutions generate $\text{Aut}(V_{\mathbb{Q}})$, but its running time is uncertain.
- M4. *Partial results in the meantime.* With the existing code one can prove the conjecture in degree $\leq N$ for larger N cheaply as follows: by the proof of Lemma 10.1, a $\mathbb{Q}$ -rational (-2) -curve of minimal degree among those not monotonically generated lies in $K_{\mathcal{C}, \mathcal{H}}$, so the conjecture holds in degree $\leq N$ as soon as no $\mathbb{Q}$ -rational (-2) -curve of degree $\leq N$ outside $\mathcal{S}$ lies in $K_{\mathcal{C}, \mathcal{H}}$; the (-2) -classes of degree d in this thin cone are found by the constrained Fincke–Pohst search (thousands of tree nodes, not 10^9 classes, as in the proof of Proposition 10.13), and each candidate is either reducible (a known curve C with $D \cdot C < 0$) or must be parametrised and tested (as in Section 8). This replaces the full enumeration of irreducible curves, which stops at degree 12, by a search in the obstruction cone only.
- M5. *Structure.* Record for each generator of $\text{Aut}(V_{\mathbb{Q}})$ its geometric origin (pencil involution, translation, Kummer automorphism from $\text{Aut}(E_1 \times E_2)$ — an arithmetic group commensurable with $\Gamma_0(3)$, since $\text{End}(E_1 \times E_2)$ is an Eichler order of level 3 — or Cremona-type involution), and the geometric meaning of the facets of σ_H . Proposition 10.13(b) is a first instance: a Kummer automorphism that is at the same time a Bremner involution of a degree-8 pencil.
- M6. *What a full proof is.* Either (M2) terminates, or (M3) is completed and σ_H is computed; in both cases the result is a theorem of the form of Theorem 10.2 for V , with an explicit finite base set (conjecturally the 20 rational lines and conics) and an explicit finite set of involutions, together with the number of $\text{Aut}(V_{\mathbb{Q}})$ -orbits of $\mathbb{Q}$ -rational smooth rational curves. Without a computation, Theorem 10.9 is the best that the general theory gives.

11. OPEN PROBLEMS

- (1) $M_6 \setminus M_2$. Write M_6 for the fourfold (1.1) and M_2 for the quadric (1.2). Nothing structural is known about the 15 solutions of (1.1) with $\sum x^2 \neq \sum u^2$, and Observation 7.3(c) shows that nothing of low degree is there to be found. Is #62 on a rational curve of the threefold $x^6 + y^6 + z^6 = u^6 + v^6 + w^6$, $x^3 + y^3 + w^3 = u^3 + v^3 + z^3$? Are there infinitely many $(3, 6)$ -multigrades of this shape?
- (2) M_2 **off** V . The nine “ $k = 2$ only” solutions of Theorem 7.1(iv) lie on none of the K3 surfaces considered so far (V and its signed-permutation conjugates, Choudhry’s surface $x + y + z = u + v + w$). Four of them (#14, 26, 35, 45) satisfy Brudno’s $3 : 1 : 1$ relation. Is there a third K3 surface in $M_6 \cap M_2$ carrying them?
- (3) **Points of V on no rational curve.** Of the eight solutions of Theorem 7.2, seven lie on the non-section curves of Theorem 8.2; the remaining one, #49 = (268, 499, 885 | 65, 572, 879), lies on none of the 62 $\mathbb{Q}$ -rational smooth rational curves of degree ≤ 12 on V . Does it lie on a smooth rational curve of higher degree, or on a singular rational curve? (A search in the cone of classes through a given point, §10.7 M4, would decide this degree by degree without a full enumeration.) Since $V(\mathbb{Q})$ is Zariski dense (the Brudno pencil has positive generic rank), one expects

most rational points to lie on no rational curve; is there a natural height-theoretic statement quantifying this for V (§9.4)?

- (4) **Bremner's conjecture.** By Theorem 10.9 the conjecture holds with some finite sets $(\mathcal{S}_H, \mathcal{H}_H)$; the question is which. Does the base set $\{\Gamma_1, \dots, \Gamma_8, Q_1, \dots, Q_{12}\}$ suffice, and do the involutions of the $\mathbb{Q}$ -rational pencils of degree ≤ 8 (Proposition 10.13 shows that degree ≤ 6 does not) generate enough of $\text{Aut}(V_{\mathbb{Q}})$ for Bremner's cone procedure to terminate? Compute $\text{Aut}(V_{\mathbb{C}})$ and $\text{Aut}(V_{\mathbb{Q}})$ by Borchers' method with the embedding of Proposition 10.11, and the number of $\text{Aut}(V_{\mathbb{Q}})$ -orbits of $\mathbb{Q}$ -rational smooth rational curves. Singular rational curves (arithmetic genus > 0), which Bremner says exist over $\mathbb{Q}$, were not enumerated here (§10.6).
- (5) **Sections.** Theorem 5.4 shows that the degrees of sections of the Brudno pencil are the values of a positive definite ternary quadratic form; the number of sections of degree $\leq d$ grows like $c d^{3/2}$. Which of the three Brudno-type pencils a given solution is a section of, and the distribution of the rank jumps (Theorem 7.2: all independent points sit on fibres of rank ≥ 4), suggest looking at the rank of $\mathcal{E}_{\lambda}$ as λ varies in terms of the root number; we did not pursue this.
- (6) **The threefold W .** Identify the 70-dimensional Galois representation $H^3(\widetilde{W})$ of the resolution of $W = M_6 \cap M_2$ (Remark 9.9); its traces for $p < 2600$ accompany the paper. Prove $t(p) = 95, 35$ of Observation 9.8, i.e. determine the Galois action on the 95 defect classes of W , for instance by exhibiting generators of $\text{Cl}(W)$ among the 48 copies of V , the 32 copies of S_{Ch} and the 120 planes. Are the nine “ $k = 2$ only” solutions of problem (2) on one of the other Weil divisors of W ?
- (7) **Density.** In the range of the census 59/74 solutions satisfy (1.2) and 41/74 lie on V . Piezas reports that in Duncan Moore's search “within radius 17,800” about 92% satisfy (1.2) **[TODO: secondary source, [75]; raw list not obtained]**. Do the proportions tend to limits? The rational points of height $\leq B$ on a rational curve of degree d number about $B^{2/d}$, so curves of small degree dominate the count on V only in a small range; a comparison of the counting functions of $V(\mathbb{Q})$, of the threefold $M_6 \cap M_2$ and of the fourfold M_6 would be instructive. **TODO**

APPENDIX A. MISPRINTS IN THE SOURCES

All formulas of [20, 57, 38, 29, 30] were re-typed from the page images and checked by computation. The following discrepancies were found (page numbers of the journals).

Bremner [20].

- (B1) p. 549, Table 1, Γ_{23} : the last equation “ $u + x - w = 0$ ” should be $u + x + w = 0$ (as printed the line is not on V).
- (B2) p. 549: the quartic through the 32 lines, printed “ $vywu + wzux + uxvy = 0$ ”, should be $vywz + wzux + uxvy = 0$.
- (B3) p. 555: the last term of the 3-division quartic should be $-625P^4Q^4$; moreover the specialisation $\lambda = 1$ used there is at a singular fibre (the conclusion, no 3-torsion, is correct: at $\lambda = 0$ the quartic $3a^4 - 64a^3 + 600a^2 - 10000$ is irreducible over $\mathbb{Q}(i)$).
- (B4) p. 556, Remark: the generator “ $(P, 3B, Q)$ ” should be $(P, 3BP, Q)$.
- (B5) p. 563, relations (24): the last one is wrong as printed; the correct relation is $Q_{17} + Q_{18} \sim -\Gamma_1 - \Gamma_2 + 2\Gamma_5 + \Gamma_7 + \Gamma_8 + (\Gamma_9 + \Gamma_{10}) + (\Gamma_{19} + \Gamma_{20}) + (\Gamma_{27} + \Gamma_{28}) - 2Q_5$ (the Corollary is unaffected).
- (B6) p. 564, (26): $N_{11}^2 = +4$, not -4 (N_{11} is the positive direction).
- (B7) p. 572: “ $E \sim Q_5 + Q_{12}$ ” should be $E \sim Q_5 + Q_{10}$.
- (B8) p. 574, Conway's form of the degree-five curve: z and v are interchanged (as printed the sextuple is not on V).
- (B9) p. 578, Appendix II: the “defining equations of V ” repeated there differ from those of p. 548 by signs and a relabelling; all tables are consistent with p. 548.

- (B10) p. 578: the quoted values of the j -invariant do not match the standard j of (14'), which is $85184/5625$ at $\lambda = 0, \infty$ (the conclusion that j is non-constant is correct).
- (B11) p. 578, Table 6: the rows of Γ_5 and Γ_6 are interchanged relative to the map of Appendix I (Remark 4.1); the Y -coordinates of the entries for $\Gamma_9, \Gamma_{10}, Q_{13}, Q_{17}$ differ in sign or conjugation from that map.

Kuwata [57].

- (K1) p. 509, Table 1, Γ_9 : “ $x = i$ ” should read $x = iy$.
- (K2) p. 509, Table 1, Γ_{19} and Γ_{20} : the fourth equation should be $u - z - x = 0$ (printed $u - z + x = 0$; as printed the lines are not on K_B).
- (K3) p. 511: the double point of the plane quartic (5.16) is at $(a_0 : a_1 : a_2) = (1 : -1 : 4)$, not $(-1 : 4 : 1)$.
- (K4) p. 516, Table 3, third row: should read $(33, 47, 74, -73, 54, -23) \mapsto (71 : 41 : 76 : 107)$ (printed with $w = 23$ and $a_3 = -107$).
- (K5) p. 498 and Table 3 say that Subba Rao’s solution maps to $(5 : 3 : 4 : 7)$ under π (correct); p. 508 takes $p = (3 : 4 : 5 : 7)$ and says it “corresponds to Subba Rao’s first solution”, which is true only up to the S_4 -symmetry of S ; the factorisation and the plane on p. 511 are correct for $(3 : 4 : 5 : 7)$.
- (K6) p. 505: the displayed conic is -2 times the restriction of F (harmless), with a stray “ s ” after the $s_1 s_2$ term; p. 511 uses s_1, s_2, s_3 for the s_0, s_1, s_2 of (4.10).

Delorme [38], **Choudhry** [29, 30].

Delorme: p. 705, Theorem 2(b), “ $M_1 V_1 = V_1$ ” should read $M_1 V_1 = V_2$; p. 710, the two off-diagonal entries of M_1 in the second solution are interchanged (the printed solutions are correct); p. 714, Remark 3, “ x_2^5 ” should be x_2^6 ; Bremner’s paper is dated “1979” in the text. Choudhry 1994: the formulas (6)–(12) are stated with w held fixed, but all the printed examples (including the degree-eight curve) are obtained from the same formulas with u held fixed, i.e. after the cyclic relabelling $(x, y, z, u, v, w) \mapsto (y, z, x, v, w, u)$ (both versions are valid involutions of V); the second numerical output on p. 839 is printed with the common factor 23 left in; reference 3 cites the Brudno–Kaplansky paper for the degree-four family, which is in [24]. Choudhry 2000: p. 847, “ $t = 9$ ” should be $t = 38/23$.

APPENDIX B. COMPUTATIONAL VERIFICATION

All computations were done in exact arithmetic unless stated otherwise, with `sympy` 1.14, `PARI/GP` 2.17 (via `cypari2`), `passagemath/Sage` 10.8 (function-field arithmetic over $\mathbb{Q}(\lambda)$) and Cremona’s `mwrnk`; the scripts and their logs, the machine-readable lists (the 74 solutions with all flags; the 144 sections of degree ≤ 12 with their forms, heights, component indices and tridegrees; the 1372 sections with $|n_i| \leq 3$ and the 2916 with $|n_i| \leq 4$; the classes of all irreducible (-2) -curves of degree ≤ 12 with orbit data and one-step constructions; the section and rank censuses; the fibre tables) and a notebook reproducing the main computations accompany the paper: the Sage notebook `Brudno sextic -- Bremner K3 and the Brudno pencil.ipynb` with its module `brudno_sextic.py`, the folder `data/` (the 74 solutions) and the folder `computations/` (the scripts, logs and JSON lists of all sections, including `new_curves.json` with the 48 parametrisations, forms and ideals, and the point counts of W). What was checked:

- §2: the identities behind (2.2), (2.6), (1.3) (for $k = 1, \dots, 6$) and (3.1); Brudno’s numerical examples and the corrected $5P$; Proposition 2.1 (PARI: `ellminimalmodel`, `ellrank`, `ellsaturation`, `elltors`, `ellheight`, BSD quotient 1.0000000000; Cremona’s `allcurves`, `allgens`, `allisog`, `allbsd`); Proposition 2.2 (the isogeny-class identifications with `ellfromeqn`, `ellminimalmodel`, `ellisisom`; the identities of (c) symbolically in x, y, z, c ; the height of $\varphi(P_0)$; the $2 : 1$ statement).
- §3: every formula of [20] (Tables 1–6, the relations (24), the function divisors, the fibre list (16), the D_i of Theorem 3.3, the four involution matrices, the four

parametrisations, the symmetry group), of [57] (the change of basis, $G, F, H, S, S^*, \pi, \pi^{-1}, \pi^*, (\pi^*)^{-1}$, the genus-two curve and its automorphisms, Table 3; Frobenius polynomials of $\text{Jac}(\mathcal{C})$ for $p \leq 61$ against $E \times E$), of [38] (Theorems 1–3, all nine families and their rederivation from the solving triples, the $(\mathbb{P}^1)^3$ model) and of [29, 30] (the involution, the degree-six and degree-eight curves, the 26 table rows and their non-membership in V).

- §4–5: the zero section (Remark 4.1) by an exact local computation at the base points of Bremner's map; the inverse of Bremner's map as four polynomials in $\mathbb{Q}[\lambda, \xi, \eta]$, checked both ways; the sections (4.1); the heights (5.2) and the component indices (two independent formulas); Theorem 5.4 on all 4288 sections of the two boxes, each of whose six forms was computed and checked on V and for (1.1), (1.2), and the proof's ingredients against the 56×56 intersection matrix; the stabiliser action; Table 2; the parameters at which the new curves pass through the listed solutions; the placements, tridegrees and Mordell–Weil coordinates of the fifteen known curves; Theorem 4.2(d) by linear algebra on the defining forms.
- §6: the 56×56 intersection matrix recomputed from the equations (and independently by a second implementation); Gram determinants 48 and 768; the enumeration with the cross-checks listed in the text; the classes of the known curves; the involutions and the two closure computations (breadth-first search over classes of degree ≤ 12).
- §8: for each of the 48 curves, the expression $kH - D$, the kernel dimension $\dim I(V)_k + 1$, the vanishing of f_D on exactly the curves of $\mathcal{S}_D$, the Hilbert polynomial $dt + 1$ of the saturated ideal, the 56 intersection numbers and the recovered class, the identities (1.4), (1.1), (1.2) on the parametrisation, degree, content and coprimality of the six forms, the vanishing of the ideal on the parametrisation, the intersection numbers of the parametrisation (birationality); the incidences by two independent exact methods plus a box search; the direct check that no census point lies on a line or conic.
- §9: the Hodge numbers by three methods; the 2605 characters and Shioda's criterion; smoothness of M_6 and V and the bad primes of V (Gröbner bases over $\mathbb{Z}$); the 710 nodes by exact enumeration over $\mathbb{Q}(i)$ and by the degree of the singular scheme over $\mathbb{Q}$ and two finite fields, their Hessians and tangent-cone discriminants; the rank 615 (exact over $\mathbb{Z}[i]$ and modulo three primes; Schoen's quintic as a control); the orbit sizes under the group of order 4608 and the incidences of the census with the $48 + 32$ K3 surfaces and 120 planes; the 56×56 intersection matrix recomputed from the equations; the discriminant forms, genera and the uniqueness of T_V in its genus (Sage); the point counts of V by two methods for 44 primes and over $\mathbb{F}_{p^2}, \mathbb{F}_{p^3}$, of M_6 for $p \leq 109$ against Weil's formula, of W for all 316 primes $7 \leq p < 2600$ by two convolution methods. The author re-ran, independently of these scripts, the node count and the rank 615 modulo 10009, the discriminant forms, and brute-force counts of V , W and M_6 for $p \leq 17$.
- §10: the discriminant forms and the embedding into $\text{II}_{1,25}$ (Sage); the Voronoi-relevant vectors of the Mordell–Weil lattice; the (-1) -eigenlattices of the 532 involutions of the Brudno pencil; the vertex enumeration of the cone with 38 inequalities (Parma Polyhedra Library) and the test of the 2444 degree inequalities on its 7782 negative rays; the reflections s_r , their action on A_{NS} , the enumeration of separating (-2) -classes, the degree-8 pencils orthogonal to the 60 rays and their involutions.
- §7: the enumeration of the 74 solutions (C program with 64-bit arithmetic, every line re-checked in Python); all flags; the section census with 320-bit heights and exact verification of every relation; the rank census (PARI `ellrank` with the specialised points as hints; `mwrnk` for the ranks ≥ 5); the hidden-structure searches with a random baseline; the collinearity and coplanarity tests.

Conditional statements: Theorem 5.5 assumes Lemma 5.1 (hence Bremner's Theorems 3.2 and 3.3); Theorems 6.3, 6.7, 8.2, Corollary 6.5 and Section 10 assume (H); Observation 9.8 is

numerical; the ranks in §7 are 2-descent results with lower bound equal to upper bound, and the bases on jump fibres are saturated at primes ≤ 200 .

APPENDIX C. THE 74 PRIMITIVE SOLUTIONS WITH TERMS ≤ 1343

Columns: number (by increasing common sum S); the two triples; S ; $k=2$: (1.2) holds; V : some signed permutation lies on V ; curves: known curves and new section curves through the point (Br4 = Brudno's quartic; B5ii, B5iii, B6 = Bremner's (ii), (iii), (iv); D7, D8a, D8b, D9, D11 = Delorme's $S(7; 11)$, $S(8; 8; a)$, $S(8; 8; b)$, $S(9; 5)$, $S(11; 7)$; Ch8 = Choudhry's octic; N7, N10, N11a, N11b = the new section curves of orbits 6, 9, 10, 11 of Table 2); Sec: on a section of a Brudno-type pencil; $k=1$: a signed relation $\pm x \pm y \pm z = \pm u \pm v \pm w$ holds; 311: Brudno's relation $3x + y + z = 3u + v + w$ holds with signs; J: on V , on no known curve or section (fibres of jumping rank; seven of these eight lie on the non-section curves of Theorem 8.2).

Table 5: The 74 primitive solutions of $a^6 + b^6 + c^6 = x^6 + y^6 + z^6$ with all terms ≤ 1343 .

#	a	b	c	x	y	z	S	$k=2$	V	curves	Sec	$k=1$	311	J
1	10	15	23	3	19	22	160426514	•	•	Br4, D8a, N7	•		•	
2	36	37	67	15	52	65	95200890914	•	•	Br4, B5ii, B5iii, B6, D8b, Ch8, N7, N10, N11a, N11b	•		•	
3	33	47	74	23	54	73	176277173474	•	•	Br4, D9, N7	•		•	
4	32	43	81	3	55	80	289824641354	•	•	Br4, B5iii, N11a	•		•	
5	37	50	81	11	65	78	300620262890	•	•	Br4	•		•	
6	25	62	138	82	92	135	6963806813393							
7	51	113	136	40	125	129	8427066928346	•	•					J
8	71	92	147	1	132	133	10824753654794	•	•	Br4, D8b, N11a	•		•	
9	111	121	230	26	169	225	153044731928882	•	•	Br4	•		•	
10	75	142	245	14	163	243	224646509202194	•	•	Br4	•		•	
11	103	148	249	11	188	243	250042799853794	•	•	Br4	•		•	
12	131	139	267	29	197	261	374565823668011	•	•	B5iii	•		•	
13	65	169	276	36	179	275	465406299992882	•	•	Br4	•		•	
14	75	154	295	1	173	294	672587848981946	•			•		•	
15	173	186	293	113	241	282	700927616287874	•	•	B5iii	•		•	
16	124	185	303	83	211	300	817572880006130	•				•		
17	125	211	300	68	249	289	821060636898386	•						
18	99	173	311	37	199	309	932570530494851	•	•	B5iii, B6	•			
19	1	55	330	159	268	311	1291495649640626							
20	107	243	326	23	282	311	1407735515439074	•	•	Br4, B5ii, N11b	•		•	
21	124	277	337	92	311	317	1920168505244874	•	•	D7	•			
22	127	271	372	27	317	356	3050377608426914	•	•	Br4, N7	•		•	
23	102	149	400	75	226	398	4108068689005865							
24	113	192	410	303	340	368	4802282691293153					•		
25	140	307	405	43	371	372	5257693035235874	•			•			
26	152	345	439	120	367	431	8856478728790850	•			•		•	
27	177	281	508	148	299	507	17709354231008114	•	•	N7, N10	•			
28	127	393	496	23	432	479	18578339512027634	•				•		
29	34	229	518	421	424	448	19462946724322761						•	
30	53	176	524	230	395	506	20730572264890281							
31	271	293	528	93	409	512	22696058508314474	•	•	Br4	•		•	
32	317	400	537	281	419	534	29090550554577578							
33	197	409	556	1	500	515	34282067133265626	•	•	Ch8, N7	•			
34	271	387	562	178	461	543	35263327902960314	•				•		
35	201	383	614	6	445	605	56803276057577906	•					•	
36	305	412	625	157	523	596	65300488831364034	•	•	B5iii	•			
37	391	499	612	351	571	572	71553895322142026	•	•					J
38	415	419	655	269	569	617	89486918493364731	•			•			
39	449	497	652	412	539	643	100086526118831394	•	•	Ch8	•			
40	152	311	721	440	575	679	141396211726782546							
41	365	579	710	314	653	669	168141453267168146	•	•					J
42	353	375	770	158	501	763	213138148541239154	•	•	N7	•			
43	426	431	775	86	655	729	229062432743624882	•	•	Br4	•		•	
44	150	568	771	404	582	765	243643844132094745							
45	298	564	787	237	604	778	270487957334711609	•					•	

#	a	b	c	x	y	z	S	k=2	V	curves	Sec	k=1	311	J
46	271	627	796	167	699	764	315531689421679706	•				•		
47	248	637	801	109	728	753	331158137647928474	•				•		
48	196	282	865	402	769	770	419446379796655745							
49	268	499	885	65	572	879	496271593541626850	•	•					J
50	359	519	892	221	603	884	525404255450429666	•	•				•	J
51	389	590	891	46	775	831	545986640792466002	•	•	Br4	•		•	
52	443	641	890	295	781	838	573905803846854090	•	•	Ch8	•			
53	473	566	919	229	746	883	646487407694257986	•	•					J
54	508	509	939	233	711	916	720054631797562946	•	•	Ch8, N7	•		•	
55	133	545	957	647	665	927	794405490418449443							
56	141	725	960	239	831	918	927986185097490466							
57	110	283	995	89	778	953	970887991619010594							
58	258	751	991	94	813	971	1126902545865343586	•	•					J
59	423	724	997	251	852	953	1131885827014925594	•				•		
60	144	790	983	457	810	974	1145334355941443825							
61	463	579	1040	241	720	1027	1312846718194007930	•						
62	492	752	1055	86	801	1046	1573871765908992833							
63	536	673	1077	120	965	977	1677238501210993514	•				•		
64	663	715	1075	477	901	1033	1761844611253738259	•	•	B5iii	•			
65	517	535	1104	128	755	1089	1853111281568261690	•	•	Br4	•		•	
66	281	852	1067	47	948	1021	1858658425371005114	•	•	Br4	•		•	
67	427	809	1082	227	959	1018	1890993323790027354	•	•	D9	•			
68	526	657	1127	267	827	1106	2150611907870652914	•	•	Br4	•		•	
69	548	562	1209	185	780	1198	3181487990044510289	•						
70	535	809	1212	93	1076	1115	3473475009349383650	•				•		
71	573	815	1255	1	1123	1143	4235616918437286539	•	•					J
72	257	769	1266	99	811	1262	4324294269742791746	•	•	B6, N7	•			
73	233	916	1266	46	956	1257	4708070550854124401	•						
74	386	1021	1323	69	1154	1267	6498509100201385346	•	•	Br4	•		•	

APPENDIX D. CLASSES OF THE RATIONAL CURVES OF DEGREE ≤ 12 , THE TWO SECTION CURVES OF DEGREE 11, AND THE 48 NON-SECTION CURVES

The two section curves of degree 11.

Orbit 10 of Table 2, representative $S = 2g_3 = (0, 0, 2, 0)$, $\lambda = m/n$:

$$\begin{aligned}
x &= m^{11} - 12m^{10}n + 74m^9n^2 - 304m^8n^3 + 883m^7n^4 - 1768m^6n^5 \\
&\quad + 2237m^5n^6 - 1306m^4n^7 - 674m^3n^8 + 1818m^2n^9 - 1269mn^{10} + 324n^{11}, \\
y &= 3m^{10}n - 34m^9n^2 + 186m^8n^3 - 640m^7n^4 + 1531m^6n^5 - 2724m^5n^6 \\
&\quad + 3831m^4n^7 - 4384m^3n^8 + 3850m^2n^9 - 2214mn^{10} + 603n^{11}, \\
z &= m^{11} - 13m^{10}n + 70m^9n^2 - 194m^8n^3 + 219m^7n^4 + 321m^6n^5 \\
&\quad - 1695m^5n^6 + 3185m^4n^7 - 3564m^3n^8 + 2624m^2n^9 - 1275mn^{10} + 333n^{11}, \\
u &= -5m^{10}n + 58m^9n^2 - 298m^8n^3 + 886m^7n^4 - 1619m^6n^5 + 1660m^5n^6 \\
&\quad - 277m^4n^7 - 1848m^3n^8 + 2866m^2n^9 - 2004mn^{10} + 585n^{11}, \\
v &= -m^{11} + 11m^{10}n - 52m^9n^2 + 108m^8n^3 + 55m^7n^4 - 921m^6n^5 \\
&\quad + 2571m^5n^6 - 3907m^4n^7 + 3518m^3n^8 - 1670m^2n^9 + 165mn^{10} + 135n^{11}, \\
w &= m^{11} - 14m^{10}n + 82m^9n^2 - 280m^8n^3 + 657m^7n^4 - 1248m^6n^5 \\
&\quad + 2233m^5n^6 - 3708m^4n^7 + 4850m^3n^8 - 4278m^2n^9 + 2181mn^{10} - 468n^{11}.
\end{aligned}$$

Orbit 11, representative $S = g_1 - g_2 + g_3 = (1, -1, 1, 0)$:

$$\begin{aligned}
x &= 729m^{11} - 4374m^{10}n + 12393m^9n^2 - 20034m^8n^3 + 18684m^7n^4 - 7218m^6n^5 \\
&\quad - 4888m^5n^6 + 9158m^4n^7 - 6389m^3n^8 + 2464m^2n^9 - 497mn^{10} + 36n^{11}, \\
y &= -729m^{11} + 5103m^{10}n - 16119m^9n^2 + 31995m^8n^3 - 44514m^7n^4 + 45354m^6n^5 \\
&\quad - 33474m^5n^6 + 16986m^4n^7 - 5245m^3n^8 + 631m^2n^9 + 113mn^{10} - 37n^{11},
\end{aligned}$$

$$\begin{aligned}
z &= 1215m^{10}n - 6480m^9n^2 + 16119m^8n^3 - 24876m^7n^4 + 26658m^6n^5 - 21448m^5n^6 \\
&\quad + 13470m^4n^7 - 6636m^3n^8 + 2439m^2n^9 - 592mn^{10} + 67n^{11}, \\
u &= 729m^{10}n - 2916m^9n^2 + 4293m^8n^3 - 378m^7n^4 - 8640m^6n^5 + 15054m^5n^6 \\
&\quad - 13636m^4n^7 + 7618m^3n^8 - 2713m^2n^9 + 590mn^{10} - 65n^{11}, \\
v &= 729m^{11} - 4860m^{10}n + 14337m^9n^2 - 25272m^8n^3 + 28890m^7n^4 - 22440m^6n^5 \\
&\quad + 12598m^5n^6 - 6028m^4n^7 + 3013m^3n^8 - 1380m^2n^9 + 401mn^{10} - 52n^{11}, \\
w &= 729m^{11} - 4617m^{10}n + 14985m^9n^2 - 31347m^8n^3 + 45558m^7n^4 - 46542m^6n^5 \\
&\quad + 33278m^5n^6 - 16310m^4n^7 + 5313m^3n^8 - 1137m^2n^9 + 169mn^{10} - 15n^{11}.
\end{aligned}$$

Both satisfy (1.4), (1.1) and (1.2) identically. The first passes through #2 at $(m, n) = (0, 1)$, #4 at $(4, 1)$, #8 at $(3, 1)$; the second through #2 at $(0, 1)$ and #20 at $(1, 2)$.

The $\mathbb{Q}$ -rational smooth rational curves of degree ≤ 12 .

One representative per orbit of the 48 symmetries (the non-section curves of degree ≥ 7 carry the names of Section 8 in the first column), as a vector of coordinates in Bremner's basis

$$\Lambda_1, \dots, \Lambda_{13} = \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8, \Gamma_9 + \Gamma_{10}, \Gamma_{19} + \Gamma_{20}, \Gamma_{27} + \Gamma_{28}, Q_1, Q_2, Q_5$$

of $\text{NS}(V_{\mathbb{Q}})$. Columns: degree; class; orbit size; tridegree (sorted); whether a section of a Brudno-type pencil; known curve in the orbit; for degree ≥ 7 , the minimal number of generator applications from the lines and conics (a) with symmetries and Brudno-pencil involutions only (“–” = not reached within degree ≤ 12), (b) with symmetries and the $\mathbb{Q}$ -rational involutions of the 14 rational pencils, degree strictly increasing; and a one-step construction $\iota(K)$ with a Brudno-pencil involution $\iota = \iota_{C_1+C_2}$ ($\iota_{\Gamma_{10}+\Gamma_{10}}$ is the negation with respect to the section Γ_{10} , defined over $\mathbb{Q}(i)$) and a known curve K (C_4 = Brudno's quartic, C_5^{iii} = Bremner (iii), C_6 = Bremner (iv)), when one exists.

name	d	class	orbit	tridegree	BK-sect.	known	steps (a)/(b)	E -construction
	1	$(-3, -1, -1, 3, 1, 1, 1, 2, 2, 2, -2, -2, -2)$	8	(1, 1, 1)	yes			
	2	$(-1, -1, -1, 2, 0, 1, 0, 1, 1, 1, -1, 0, -1)$	12	(0, 1, 1)	yes			
	4	$(-2, -1, -1, 3, 0, 1, 0, 1, 1, 2, -1, 0, -1)$	24	(1, 1, 2)	yes	Brudno quartic (= Bremner (i), S(4;4;a), S(4;4;b))		
	5	$(-1, -1, -1, 2, 0, 0, 0, 1, 1, 1, 0, 0, 0)$	8	(1, 1, 1)	yes	Bremner (ii) = S(5;1)		
	5	$(-1, -1, 0, 2, 0, 0, 1, 0, 1, 1, 0, 0, 0)$	24	(1, 3, 3)	yes	Bremner (iii) = S(5;9)		
	6	$(-1, -1, -1, 3, -1, 2, 1, 0, 2, 2, -1, 0, -1)$	24	(1, 4, 5)	yes	Bremner (iv)		
	7	$(-2, -2, -1, 4, 0, 2, 0, 2, 1, 3, -1, 0, -2)$	48	(1, 4, 4)	yes		1 / 1	$\iota_{\Gamma_3+Q_1}(C_4)$
R7a	7	$(-2, 0, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 0)$	24	(3, 3, 3)	no		- / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(S(11; 7))$
	7	$(-2, 0, 0, 3, 0, 0, 0, 1, 1, 1, 0, 0, 0)$	8	(3, 3, 3)	no	S(7;11)	- / 1	$\iota_{Q_1+Q_2}(S(11; 7))$
	8	$(-3, -1, -1, 4, 0, 1, 0, 2, 2, 2, -1, 0, -1)$	24	(2, 3, 3)	no	S(8;8;b)	- / 1	$\iota_{Q_1+Q_2}(S(8; 8; a))$
R8a	8	$(-3, -1, -1, 4, 0, 2, 1, 1, 2, 3, -2, 0, -1)$	48	(2, 5, 5)	no		3 / 1	$\iota_{\Gamma_3+Q_1}(C_4)$
R8b	8	$(-3, -1, -1, 4, 0, 2, 1, 2, 2, 2, -2, 0, -1)$	48	(3, 4, 5)	no		- / 1	none
R8c	8	$(-3, -1, -1, 4, 0, 3, 0, 1, 2, 3, -2, 0, -1)$	48	(2, 5, 5)	no		3 / 1	$\iota_{\Gamma_3+Q_1}(C_4)$
R8d	8	$(-2, -2, -2, 3, 1, 1, 1, 2, 2, 2, -1, 0, -1)$	24	(2, 3, 3)	no		3 / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(C_4)$
R8e	8	$(-2, -2, -1, 4, 0, 1, 0, 1, 2, 2, 0, 0, -1)$	24	(2, 3, 3)	no		3 / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(C_4)$
	8	$(-2, -1, -1, 4, -1, 2, 1, 0, 2, 3, -1, 0, -1)$	48	(1, 5, 6)	yes	Choudhry octic	1 / 1	$\iota_{\Gamma_6+\Gamma_7}(C_4)$
	8	$(-1, -2, -1, 3, 1, 1, -1, 2, 1, 2, -1, 0, 0)$	24	(2, 3, 3)	no	S(8;8;a)	- / 1	$\iota_{Q_1+Q_2}(S(8; 8; b))$
R9a	9	$(-3, -1, -1, 5, -1, 3, 1, 1, 3, 3, -2, 0, -2)$	48	(3, 5, 7)	no		- / 2	$\iota_{\Gamma_3+Q_1}(S(8; 8; b))$
R9b	9	$(-2, -3, -1, 5, 0, 1, 1, 2, 2, 3, -1, 0, -2)$	48	(3, 4, 4)	no		- / 1	none
R9c	9	$(-2, -2, -2, 4, 0, 3, 2, 1, 3, 3, -2, 0, -2)$	48	(3, 5, 7)	no		- / 2	$\iota_{\Gamma_3+Q_1}(S(8; 8; a))$
R9d	9	$(-2, -1, -1, 4, -1, 1, 1, 1, 1, 2, 0, 0, 0)$	48	(3, 3, 5)	no		3 / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(S(9; 5))$
	9	$(1, -1, 0, 1, -1, -1, 0, -1, 0, 0, 2, 2, 2)$	24	(1, 3, 3)	yes	S(9;5)	1 / 1	$\iota_{Q_1+Q_2}(C_5^{\text{iii}})$
R10a	10	$(-4, -2, -1, 6, 0, 3, 0, 2, 3, 4, -2, 0, -3)$	48	(4, 5, 5)	no		- / 1	none
R10b	10	$(-4, -2, -1, 6, 0, 3, 0, 3, 2, 4, -2, 0, -3)$	48	(3, 5, 6)	no		3 / 2	$\iota_{\Gamma_3+Q_1}(C_5^{\text{iii}})$
R10c	10	$(-3, -3, -2, 5, 1, 3, 1, 3, 2, 4, -2, 0, -3)$	48	(3, 5, 6)	no		- / 2	$\iota_{\Gamma_3+Q_1}(S(7; 11))$
R10d	10	$(-3, -2, -1, 5, 0, 3, 0, 1, 2, 3, -1, 1, -2)$	48	(4, 5, 5)	no		- / 1	none
R10e	10	$(-3, -2, -1, 5, 1, 3, -1, 2, 1, 4, -1, 0, -2)$	48	(2, 5, 7)	no		- / 2	$\iota_{\Gamma_3+Q_1}(S(8; 8; a))$
R10f	10	$(-3, -1, -1, 4, 0, 0, 1, 2, 2, 2, 0, 0, -1)$	48	(3, 3, 4)	no		- / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(S(8; 8; a))$
R10g	10	$(-3, -1, -1, 5, -1, 2, 1, 2, 2, 2, -1, 0, -1)$	48	(4, 5, 5)	no		- / 1	none
R10h	10	$(-2, -3, -1, 5, 0, 1, 0, 2, 2, 3, -1, 0, -1)$	48	(3, 3, 4)	no		- / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(S(8; 8; b))$
R10i	10	$(-2, -3, -1, 5, 0, 1, 2, 1, 3, 3, -1, 0, -2)$	48	(3, 5, 6)	no		- / 1	none
R10j	10	$(-2, -3, -1, 5, 0, 3, 0, 2, 2, 4, -1, 0, -3)$	48	(3, 5, 6)	no		3 / 2	$\iota_{\Gamma_3+Q_1}(C_5^{\text{iii}})$
R10k	10	$(-2, 0, 0, 3, -1, 1, 1, 1, 1, 1, 0, 1, 0)$	48	(4, 5, 5)	no		- / 1	none
	10	$(-1, -2, -1, 4, -1, 1, 0, 0, 2, 2, 0, 1, 0)$	48	(1, 4, 5)	yes		2 / 1	$\iota_{\Gamma_3+Q_1}(C_5^{\text{iii}})$
R11a	11	$(-4, -3, -2, 6, 1, 4, 1, 2, 3, 4, -3, 0, -2)$	48	(4, 6, 7)	no		3 / 1	none
R11b	11	$(-4, -2, -2, 6, 0, 4, 1, 2, 2, 4, -1, 0, -3)$	48	(4, 6, 7)	no		3 / 1	none
R11c	11	$(-4, -2, -1, 6, 0, 2, 0, 2, 3, 3, -1, 0, -2)$	48	(4, 4, 5)	no		- / 1	none
R11d	11	$(-4, -2, -1, 6, 0, 4, 0, 2, 2, 4, -1, 0, -3)$	48	(4, 6, 7)	no		- / 1	none
R11e	11	$(-4, -2, -1, 6, 1, 3, 0, 2, 3, 3, -2, 0, -2)$	48	(5, 5, 7)	no		- / 1	none
R11f	11	$(-4, -2, -1, 6, 1, 4, -1, 2, 2, 4, -2, 0, -2)$	48	(5, 5, 7)	no		- / 1	none
R11g	11	$(-4, -1, -1, 5, 0, 3, 1, 1, 3, 4, -2, 0, -2)$	48	(3, 7, 7)	no		5 / 2	none

name	d	class	orbit	tridegree	BK-sect.	known	steps (a)/(b)	E -construction
R11h	11	$(-4, -1, -1, 6, -1, 3, 1, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	no		5 / 2	none
R11i	11	$(-4, -1, -1, 6, -1, 3, 1, 2, 2, 4, -2, 0, -2)$	48	$(5, 5, 7)$	no		- / 1	none
R11j	11	$(-4, -1, -1, 6, 0, 2, 1, 3, 2, 3, -2, 0, -2)$	48	$(5, 5, 7)$	no		- / 1	none
R11k	11	$(-3, -3, -2, 5, 1, 4, 1, 2, 3, 4, -3, 0, -2)$	48	$(4, 6, 7)$	no		- / 1	none
R11l	11	$(-3, -2, -2, 6, 0, 2, 2, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	no		5 / 2	none
R11m	11	$(-3, -2, -2, 6, 0, 4, 0, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	no		5 / 2	none
R11n	11	$(-3, -1, 0, 4, 0, 0, 1, 1, 2, 2, 0, 0, 0)$	48	$(3, 5, 5)$	no		- / 1	none
R11o	11	$(-2, -3, -2, 4, 2, 2, 0, 3, 2, 3, -2, 0, -1)$	48	$(4, 4, 5)$	no		- / 1	none
	11	$(-2, -1, -1, 4, -1, 2, 0, 0, 2, 3, -1, 1, 0)$	48	$(1, 6, 6)$	yes		2 / 2	$\iota_{\Gamma_3+Q_1}(C_4)$
	11	$(0, -2, -1, 3, -1, 1, -1, 1, 0, 2, 1, 2, 0)$	48	$(1, 4, 4)$	yes		1 / 1	$\iota_{\Gamma_3+Q_1}(C_6)$
R11p	11	$(0, 0, 0, 1, -1, -1, 0, 0, 0, 0, 2, 2, 2)$	24	$(3, 3, 3)$	no		- / 1	$\iota_{\Gamma_{10}+\Gamma_{10}}(S(7; 11))$
	11	$(0, 0, 0, 2, -1, -1, -1, 0, 0, 0, 2, 2, 2)$	8	$(3, 3, 3)$	no	S(11;7)	- / 1	$\iota_{Q_1+Q_2}(S(7; 11))$
R12a	12	$(-4, -3, -3, 7, 0, 3, 2, 2, 4, 5, -3, 0, -3)$	48	$(3, 6, 7)$	no		- / 2	none
R12b	12	$(-4, -3, -1, 7, 0, 1, 2, 2, 4, 3, -1, 0, -3)$	48	$(4, 5, 7)$	no		- / 3	none
R12c	12	$(-4, -2, -1, 6, 0, 2, 1, 3, 2, 4, -1, 0, -3)$	48	$(3, 6, 7)$	no		3 / 2	$\iota_{\Gamma_3+Q_1}(C_5^{\text{iii}})$
R12d	12	$(-4, -2, -1, 6, 1, 3, -1, 2, 2, 4, -1, 0, -2)$	48	$(4, 5, 7)$	no		- / 1	none
R12e	12	$(-4, -1, -1, 6, -1, 2, 1, 2, 2, 3, -1, 0, -1)$	48	$(5, 5, 6)$	no		- / 1	none
R12f	12	$(-3, -3, -2, 7, 0, 3, 0, 2, 3, 5, -3, 0, -2)$	48	$(3, 6, 7)$	no		3 / 2	$\iota_{\Gamma_3+Q_1}(C_5^{\text{iii}})$
R12g	12	$(-3, -3, -1, 6, 0, 0, 3, 2, 4, 2, 0, 0, -3)$	48	$(3, 6, 7)$	no		- / 2	none
R12h	12	$(-3, -3, -1, 6, 1, 0, 2, 2, 4, 2, -1, 0, -2)$	48	$(3, 5, 8)$	no		- / 3	none
R12i	12	$(-3, -1, -1, 5, -1, 4, 1, 0, 3, 4, -1, 0, -2)$	48	$(3, 8, 9)$	no		- / 1	none
R12j	12	$(-2, -3, -1, 6, 1, 0, 1, 2, 2, 3, -1, 0, -1)$	48	$(5, 5, 6)$	no		- / 1	none
R12k	12	$(-2, -1, -1, 5, 2, -1, 2, 3, 3, 0, -1, 0, -1)$	24	$(2, 9, 9)$	no		3 / 2	$\iota_{\Gamma_6+\Gamma_7}(C_4)$

Table 7: The 48 $\mathbb{Q}$ -rational smooth rational curves of degree ≤ 12 on V that are not sections of a Brudno-type pencil (Section 8): name, degree d , class D in the basis $\Lambda_1, \dots, \Lambda_{13}$, orbit size, sorted tridegree, the degree k of the form f_D , the multiplicity-free expression $kH - D = \sum C_i$ (C_i lines and conics; the zero locus of f_D on V is $C_D \cup \bigcup C_i$), and the solutions of Appendix C lying on a symmetric image of C_D (exact test, all placements).

name	d	class	orbit	tridegree	k	$kH - D$	solutions on it
R7a	7	$(-2, 0, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 0)$	24	(3, 3, 3)	2	$\Gamma_1 + \Gamma_6 + \Gamma_7 + Q_7 + Q_9 + Q_{10}$	1, 4, 5
R8a	8	$(-3, -1, -1, 4, 0, 2, 1, 1, 2, 3, -2, 0, -1)$	48	(2, 5, 5)	2	$\Gamma_1 + \Gamma_3 + \Gamma_{15} + \Gamma_{16} + Q_1 + Q_9$	1, 31, 37, 39, 41, 52, 53
R8b	8	$(-3, -1, -1, 4, 0, 2, 1, 2, 2, -2, 0, -1)$	48	(3, 4, 5)	2	$\Gamma_1 + \Gamma_6 + Q_1 + Q_3 + Q_6$	1, 2, 5, 15, 31
R8c	8	$(-3, -1, -1, 4, 0, 3, 0, 1, 2, 3, -2, 0, -1)$	48	(2, 5, 5)	2	$\Gamma_1 + \Gamma_4 + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_{11}$	2, 5, 7, 11, 58
R8d	8	$(-2, -2, -2, 3, 1, 1, 2, 2, 2, -1, 0, -1)$	24	(2, 3, 3)	2	$\Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_7 + Q_3 + Q_6$	7, 15, 67
R8e	8	$(-2, -2, -1, 4, 0, 1, 0, 1, 2, 2, 0, 0, -1)$	24	(2, 3, 3)	2	$\Gamma_1 + \Gamma_2 + \Gamma_7 + \Gamma_8 + Q_7 + Q_{11}$	2, 21, 36, 50
R9a	9	$(-3, -1, -1, 5, -1, 3, 1, 1, 3, 3, -2, 0, -2)$	48	(3, 5, 7)	3	$\Gamma_1 + \Gamma_{29} + \Gamma_{30} + Q_1 + Q_2 + Q_3 + Q_{11} + Q_{23} + Q_{24}$	18, 42, 52
R9b	9	$(-2, -3, -1, 5, 0, 1, 1, 2, 2, 3, -1, 0, -2)$	48	(3, 4, 4)	3	$\Gamma_1 + \Gamma_2 + \Gamma_7 + \Gamma_{31} + \Gamma_{32} + Q_1 + Q_9 + Q_{11} + Q_{13} + Q_{14}$	1, 7, 15, 50
R9c	9	$(-2, -2, -2, 4, 0, 3, 2, 1, 3, 3, -2, 0, -2)$	48	(3, 5, 7)	3	$\Gamma_2 + \Gamma_3 + \Gamma_5 + Q_1 + Q_3 + Q_7 + Q_8 + Q_9 + Q_{11}$	7, 11, 27
R9d	9	$(-2, -1, -1, 4, -1, 1, 1, 1, 2, 0, 0, 0)$	48	(3, 3, 5)	2	$\Gamma_1 + \Gamma_3 + \Gamma_6 + \Gamma_{15} + \Gamma_{16} + \Gamma_{19} + \Gamma_{20}$	2, 8
R10a	10	$(-4, -2, -1, 6, 0, 3, 0, 2, 3, 4, -2, 0, -3)$	48	(4, 5, 5)	3	$\Gamma_1 + \Gamma_8 + \Gamma_{15} + \Gamma_{16} + Q_2 + Q_5 + Q_9 + Q_{15} + Q_{16}$	1, 7
R10b	10	$(-4, -2, -1, 6, 0, 3, 0, 3, 2, 4, -2, 0, -3)$	48	(3, 5, 6)	3	$\Gamma_1 + \Gamma_6 + \Gamma_7 + \Gamma_8 + Q_5 + Q_6 + Q_9 + Q_{10} + Q_{11}$	2, 22, 27
R10c	10	$(-3, -3, -2, 5, 1, 3, 1, 3, 2, 4, -2, 0, -3)$	48	(3, 5, 6)	3	$\Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_7 + Q_5 + Q_6 + Q_9 + Q_{10} + Q_{11}$	3, 7, 58
R10d	10	$(-3, -2, -1, 5, 0, 3, 0, 1, 2, 3, -1, 1, -2)$	48	(4, 5, 5)	3	$\Gamma_1 + \Gamma_8 + \Gamma_9 + \Gamma_{10} + Q_2 + Q_6 + Q_9 + Q_{15} + Q_{16}$	64
R10e	10	$(-3, -2, -1, 5, 1, 3, -1, 2, 1, 4, -1, 0, -2)$	48	(2, 5, 7)	3	$\Gamma_1 + \Gamma_2 + \Gamma_7 + \Gamma_8 + \Gamma_{17} + \Gamma_{18} + \Gamma_{21} + \Gamma_{22} + Q_8 + Q_9 + Q_{11}$	2, 27, 42
R10f	10	$(-3, -1, -1, 4, 0, 0, 1, 2, 2, 2, 0, 0, -1)$	48	(3, 3, 4)	3	$\Gamma_1 + \Gamma_2 + \Gamma_6 + \Gamma_7 + \Gamma_{21} + \Gamma_{22} + Q_4 + Q_7 + Q_{15} + Q_{16}$	1, 67
R10g	10	$(-3, -1, -1, 5, -1, 2, 1, 2, 2, 2, -1, 0, -1)$	48	(4, 5, 5)	3	$\Gamma_1 + \Gamma_6 + \Gamma_{21} + \Gamma_{22} + \Gamma_{27} + \Gamma_{28} + Q_1 + Q_{10} + Q_{23} + Q_{24}$	21, 50, 65
R10h	10	$(-2, -3, -1, 5, 0, 1, 0, 2, 2, 3, -1, 0, -1)$	48	(3, 3, 4)	3	$\Gamma_2 + \Gamma_3 + \Gamma_7 + \Gamma_8 + \Gamma_{15} + \Gamma_{16} + Q_6 + Q_8 + Q_{19} + Q_{20}$	8, 21, 36
R10i	10	$(-2, -3, -1, 5, 0, 1, 2, 1, 3, 3, -1, 0, -2)$	48	(3, 5, 6)	3	$\Gamma_2 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{31} + \Gamma_{32} + Q_9 + Q_{10} + Q_{23} + Q_{24}$	21, 74
R10j	10	$(-2, -3, -1, 5, 0, 3, 0, 2, 2, 4, -1, 0, -3)$	48	(3, 5, 6)	3	$\Gamma_2 + \Gamma_5 + \Gamma_7 + \Gamma_8 + Q_2 + Q_5 + Q_6 + Q_9 + Q_{11}$	1
R10k	10	$(-2, 0, 0, 3, -1, 1, 1, 1, 1, 0, 1, 0)$	48	(4, 5, 5)	3	$\Gamma_1 + \Gamma_6 + \Gamma_{19} + \Gamma_{20} + \Gamma_{29} + \Gamma_{30} + Q_3 + Q_{10} + Q_{23} + Q_{24}$	10, 15
R11a	11	$(-4, -3, -2, 6, 1, 4, 1, 2, 3, 4, -3, 0, -2)$	48	(4, 6, 7)	3	$\Gamma_1 + \Gamma_2 + \Gamma_5 + \Gamma_9 + \Gamma_{10} + Q_1 + Q_3 + Q_6 + Q_{11}$	7, 12
R11b	11	$(-4, -2, -2, 6, 0, 4, 1, 2, 2, 4, -1, 0, -3)$	48	(4, 6, 7)	3	$\Gamma_1 + \Gamma_3 + \Gamma_8 + \Gamma_{19} + \Gamma_{20} + Q_6 + Q_7 + Q_9 + Q_{11}$	5, 41
R11c	11	$(-4, -2, -1, 6, 0, 2, 0, 2, 3, 3, -1, 0, -2)$	48	(4, 4, 5)	4	$\Gamma_1 + \Gamma_6 + \Gamma_8 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_6 + Q_{12} + Q_{13} + Q_{14} + Q_{15} + Q_{16}$	1, 2

name	d	class	orbit	tridegree	k	$kH - D$	solutions on it
R11d	11	$(-4, -2, -1, 6, 0, 4, 0, 2, 2, 4, -1, 0, -3)$	48	$(4, 6, 7)$	4	$\Gamma_1 + \Gamma_{15} + \Gamma_{16} + \Gamma_{27} + \Gamma_{28} + Q_5 + Q_7 + Q_9 + Q_{11} + Q_{13} + Q_{14} + Q_{19} + Q_{20}$	2, 10, 22
R11e	11	$(-4, -2, -1, 6, 1, 3, 0, 2, 3, 3, -2, 0, -2)$	48	$(5, 5, 7)$	3	$\Gamma_1 + \Gamma_2 + \Gamma_8 + \Gamma_9 + \Gamma_{10} + \Gamma_{29} + \Gamma_{30} + Q_3 + Q_6 + Q_{11}$	1, 5, 7
R11f	11	$(-4, -2, -1, 6, 1, 4, -1, 2, 2, 4, -2, 0, -2)$	48	$(5, 5, 7)$	3	$\Gamma_1 + \Gamma_7 + \Gamma_8 + \Gamma_{15} + \Gamma_{16} + \Gamma_{17} + \Gamma_{18} + \Gamma_{27} + \Gamma_{28} + Q_9 + Q_{11}$	10
R11g	11	$(-4, -1, -1, 5, 0, 3, 1, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	3	$\Gamma_1 + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_9 + Q_{12} + Q_{23} + Q_{24}$	1, 5, 71
R11h	11	$(-4, -1, -1, 6, -1, 3, 1, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	3	$\Gamma_1 + \Gamma_3 + \Gamma_6 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + \Gamma_{29} + \Gamma_{30} + Q_9 + Q_{11}$	10
R11i	11	$(-4, -1, -1, 6, -1, 3, 1, 2, 2, 4, -2, 0, -2)$	48	$(5, 5, 7)$	3	$\Gamma_1 + \Gamma_3 + \Gamma_6 + \Gamma_{15} + \Gamma_{16} + \Gamma_{19} + \Gamma_{20} + \Gamma_{29} + \Gamma_{30} + Q_9 + Q_{11}$	10, 21, 50, 53, 58
R11j	11	$(-4, -1, -1, 6, 0, 2, 1, 3, 2, 3, -2, 0, -2)$	48	$(5, 5, 7)$	3	$\Gamma_1 + \Gamma_6 + \Gamma_7 + \Gamma_{19} + \Gamma_{20} + \Gamma_{29} + \Gamma_{30} + Q_3 + Q_6 + Q_{11}$	1, 5, 11
R11k	11	$(-3, -3, -2, 5, 1, 4, 1, 2, 3, 4, -3, 0, -2)$	48	$(4, 6, 7)$	4	$\Gamma_2 + \Gamma_5 + \Gamma_8 + \Gamma_{25} + \Gamma_{26} + Q_1 + Q_2 + Q_3 + Q_9 + Q_{10} + Q_{11} + Q_{23} + Q_{24}$	1, 37, 58
R11l	11	$(-3, -2, -2, 6, 0, 2, 2, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	3	$\Gamma_3 + \Gamma_6 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + \Gamma_{25} + \Gamma_{26} + Q_9 + Q_{11}$	21
R11m	11	$(-3, -2, -2, 6, 0, 4, 0, 1, 3, 4, -2, 0, -2)$	48	$(3, 7, 7)$	3	$\Gamma_8 + \Gamma_9 + \Gamma_{10} + Q_3 + Q_{10} + Q_{11} + Q_{23} + Q_{24}$	2, 39, 54, 74
R11n	11	$(-3, -1, 0, 4, 0, 0, 1, 1, 2, 2, 0, 0, 0)$	48	$(3, 5, 5)$	4	$\Gamma_1 + \Gamma_6 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + Q_6 + Q_7 + Q_9 + Q_{19} + Q_{20} + Q_{21} + Q_{22}$	8, 50
R11o	11	$(-2, -3, -2, 4, 2, 2, 0, 3, 2, 3, -2, 0, -1)$	48	$(4, 4, 5)$	4	$\Gamma_2 + \Gamma_5 + \Gamma_8 + \Gamma_{25} + \Gamma_{26} + \Gamma_{31} + \Gamma_{32} + Q_2 + Q_9 + Q_{10} + Q_{19} + Q_{20} + Q_{23} + Q_{24}$	2, 3, 5, 21
R11p	11	$(0, 0, 0, 1, -1, -1, 0, 0, 0, 0, 2, 2, 2)$	24	$(3, 3, 3)$	4	$\Gamma_1 + \Gamma_6 + \Gamma_7 + Q_7 + Q_9 + Q_{10} + Q_{13} + Q_{14} + Q_{15} + Q_{16} + Q_{17} + Q_{18}$	3, 8
R12a	12	$(-4, -3, -3, 7, 0, 3, 2, 2, 4, 5, -3, 0, -3)$	48	$(3, 6, 7)$	4	$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_7 + \Gamma_{31} + \Gamma_{32} + Q_1 + Q_3 + Q_4 + Q_9 + Q_{11} + Q_{23} + Q_{24}$	—
R12b	12	$(-4, -3, -1, 7, 0, 1, 2, 2, 4, 3, -1, 0, -3)$	48	$(4, 5, 7)$	4	$\Gamma_1 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + Q_5 + Q_6 + Q_{11} + Q_{15} + Q_{16} + Q_{21} + Q_{22}$	3, 7, 9
R12c	12	$(-4, -2, -1, 6, 0, 2, 1, 3, 2, 4, -1, 0, -3)$	48	$(3, 6, 7)$	3	$\Gamma_1 + \Gamma_2 + \Gamma_6 + \Gamma_7 + \Gamma_{21} + \Gamma_{22} + Q_5 + Q_6 + Q_9$	1
R12d	12	$(-4, -2, -1, 6, 1, 3, -1, 2, 2, 4, -1, 0, -2)$	48	$(4, 5, 7)$	4	$\Gamma_1 + \Gamma_8 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_5 + Q_6 + Q_{13} + Q_{14} + Q_{19} + Q_{20}$	37
R12e	12	$(-4, -1, -1, 6, -1, 2, 1, 2, 2, 3, -1, 0, -1)$	48	$(5, 5, 6)$	4	$\Gamma_1 + \Gamma_4 + \Gamma_6 + \Gamma_7 + \Gamma_{15} + \Gamma_{16} + \Gamma_{19} + \Gamma_{20} + \Gamma_{21} + \Gamma_{22} + \Gamma_{27} + \Gamma_{28} + Q_3 + Q_{10} + Q_{23} + Q_{24}$	1, 3, 5, 10
R12f	12	$(-3, -3, -2, 7, 0, 3, 0, 2, 3, 5, -3, 0, -2)$	48	$(3, 6, 7)$	3	$\Gamma_2 + \Gamma_3 + \Gamma_7 + \Gamma_8 + \Gamma_{15} + \Gamma_{16} + Q_1 + Q_9 + Q_{11}$	27
R12g	12	$(-3, -3, -1, 6, 0, 0, 3, 2, 4, 2, 0, 0, -3)$	48	$(3, 6, 7)$	4	$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_7 + \Gamma_{15} + \Gamma_{16} + Q_5 + Q_6 + Q_7 + Q_8 + Q_{12} + Q_{15} + Q_{16}$	42
R12h	12	$(-3, -3, -1, 6, 1, 0, 2, 2, 4, 2, -1, 0, -2)$	48	$(3, 5, 8)$	4	$\Gamma_2 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_5 + Q_6 + Q_{15} + Q_{16} + Q_{21} + Q_{22}$	18, 42
R12i	12	$(-3, -1, -1, 5, -1, 4, 1, 0, 3, 4, -1, 0, -2)$	48	$(3, 8, 9)$	4	$\Gamma_1 + \Gamma_8 + Q_1 + Q_2 + Q_3 + Q_4 + Q_7 + Q_9 + Q_{11} + Q_{23} + Q_{24}$	2, 39
R12j	12	$(-2, -3, -1, 6, 1, 0, 1, 2, 2, 3, -1, 0, -1)$	48	$(5, 5, 6)$	4	$\Gamma_1 + \Gamma_2 + \Gamma_5 + \Gamma_7 + \Gamma_9 + \Gamma_{10} + \Gamma_{15} + \Gamma_{16} + \Gamma_{19} + \Gamma_{20} + \Gamma_{31} + \Gamma_{32} + Q_5 + Q_8 + Q_{19} + Q_{20}$	2, 4, 65

name	d	class	orbit	tridegree	k	$kH - D$	solutions on it
R12k	12	$(-2, -1, -1, 5, 2, -1, 2, 3, 3, 0, -1, 0, -1)$	24	$(2, 9, 9)$	3	$\Gamma_1 + \Gamma_7 + \Gamma_{25} + \Gamma_{26} + \Gamma_{27} + \Gamma_{28} + \Gamma_{29} + \Gamma_{30} + \Gamma_{31} + \Gamma_{32} + Q_{11}$	2

APPENDIX E. PARAMETRISATIONS OF THE NON-SECTION CURVES OF DEGREE ≤ 9 **Explicit parametrisations of the new curves of degree 7, 8 and 9.**

Each sextuple of binary forms below parametrises the curve C_D of the indicated class birationally (verified: the forms satisfy the three quadrics of V identically, have degree exactly d and no common factor, vanish on f_D and on the whole ideal of C_D , and their intersection numbers with the 56 lines and conics give back the class D with $D^2 = -2$). The coefficient lists of all 48 curves, the forms f_D and generators of the ideals are in `new_curves.json`.

R7a ($d = 7$, $D = (-2, 0, 0, 2, 0, 0, 1, 1, 1, 1, 0, 0, 0)$, $2H - D = \Gamma_1 + \Gamma_6 + \Gamma_7 + Q_7 + Q_9 + Q_{10}$; $f_D = 2xz + 2yz - xu - yu + zu - u^2 + xv + yv - zv + 2uv + v^2 + xw + yw - 3zw + 2uw - w^2$):

$$\begin{aligned} x &= 3m^7 - 127m^6n - 100m^5n^2 - 69m^4n^3 - 24m^3n^4 \\ &\quad - 7m^2n^5 - mn^6, \\ y &= 80m^7 + 57m^6n - 5m^5n^2 - 36m^4n^3 - 23m^3n^4 \\ &\quad - 14m^2n^5 - 3mn^6 - n^7, \\ z &= 55m^7 + 104m^6n + 83m^5n^2 + 87m^4n^3 + 37m^3n^4 \\ &\quad + 19m^2n^5 + 4mn^6 + n^7, \\ u &= 32m^7 + 129m^6n + 107m^5n^2 + 84m^4n^3 + 33m^3n^4 \\ &\quad + 16m^2n^5 + 3mn^6 + n^7, \\ v &= -43m^7 - 8m^6n - 81m^5n^2 - 59m^4n^3 - 37m^3n^4 \\ &\quad - 17m^2n^5 - 4mn^6 - n^7, \\ w &= -81m^7 - 67m^6n - 18m^5n^2 - 23m^4n^3 + 4m^3n^4 \\ &\quad - m^2n^5 + mn^6. \end{aligned}$$

R8a ($d = 8$, $D = (-3, -1, -1, 4, 0, 2, 1, 1, 2, 3, -2, 0, -1)$, $2H - D = \Gamma_1 + \Gamma_3 + \Gamma_{15} + \Gamma_{16} + Q_1 + Q_9$; $f_D = 3xy + yz + 5xu + yu + 4zu - 2u^2 - 2xv - 5yv - 2uv + 3v^2 + 3xw + 2yw - 6zw + 3uw - 3vw - 5w^2$):

$$\begin{aligned} x &= 175m^7n + 95m^6n^2 + 85m^5n^3 + 51m^4n^4 + 9m^3n^5 \\ &\quad + 17m^2n^6 - mn^7 + n^8, \\ y &= -125m^8 - 175m^7n - 115m^6n^2 - 161m^5n^3 - 35m^4n^4 \\ &\quad - 29m^3n^5 - 5m^2n^6 - 3mn^7, \\ z &= 125m^8 + 50m^7n + 30m^6n^2 - 8m^5n^3 + 10m^4n^4 \\ &\quad + 2m^3n^5 + 6m^2n^6 + n^8, \\ u &= 125m^8 + 125m^7n + 205m^6n^2 + 95m^5n^3 + 59m^4n^4 \\ &\quad + 35m^3n^5 - m^2n^6 + 5mn^7, \\ v &= -125m^8 - 100m^7n - 90m^6n^2 - 78m^5n^3 - 34m^4n^4 \\ &\quad + 4m^3n^5 - 10m^2n^6 + 2mn^7 - n^8, \\ w &= 25m^7n + 45m^6n^2 + 99m^5n^3 + 7m^4n^4 + 35m^3n^5 \\ &\quad + 3m^2n^6 + mn^7 + n^8. \end{aligned}$$

R8b ($d = 8$, $D = (-3, -1, -1, 4, 0, 2, 1, 2, 2, -2, 0, -1)$, $2H - D = \Gamma_1 + \Gamma_6 + Q_1 + Q_3 + Q_6$; $f_D = 3xy - 2xz - 6yz - 2xu - yu + u^2 - 3xv - 6yv + 6zv + uv + 5v^2 - 2yw + 3zw + 2uw + 2vw - w^2$):

$$\begin{aligned} x &= 12m^8 - 46m^7n + 97m^6n^2 - 109m^5n^3 + 84m^4n^4 \\ &\quad - 34m^3n^5 - m^2n^6 + 7mn^7 - 4n^8, \\ y &= 4m^8 + 20m^7n - 35m^6n^2 + 89m^5n^3 - 100m^4n^4 \\ &\quad + 98m^3n^5 - 50m^2n^6 + 20mn^7 - 2n^8, \\ z &= -16m^8 + 28m^7n - 43m^6n^2 + 31m^5n^3 - 43m^4n^4 \\ &\quad + 29m^3n^5 - 31m^2n^6 + 13mn^7 - 6n^8, \end{aligned}$$

$$\begin{aligned}
u &= -4m^8 + 50m^7n - 79m^6n^2 + 99m^5n^3 - 42m^4n^4 \\
&\quad + 6m^3n^5 + 22m^2n^6 - 12mn^7 + 6n^8, \\
v &= 16m^8 - 36m^7n + 89m^6n^2 - 99m^5n^3 + 90m^4n^4 \\
&\quad - 48m^3n^5 + 23m^2n^6 - 7mn^7 + 2n^8, \\
w &= -12m^8 + 12m^7n + 29m^6n^2 - 81m^5n^3 + 121m^4n^4 \\
&\quad - 91m^3n^5 + 53m^2n^6 - 15mn^7 + 4n^8.
\end{aligned}$$

R8c ($d = 8$, $D = (-3, -1, -1, 4, 0, 3, 0, 1, 2, 3, -2, 0, -1)$, $2H - D = \Gamma_1 + \Gamma_4 + \Gamma_{15} + \Gamma_{16} + Q_3 + Q_{11}$; $f_D = xy + 4xz + yz - 3xu - yu - 2u^2 - 2xv + 3yv - 4zv - 2uv - v^2 + 5xw + 2yw + 6zw + uw - 3vw + 7w^2$):

$$\begin{aligned}
x &= 3m^8 - m^7n + 29m^6n^2 + 5m^5n^3 + 119m^4n^4 \\
&\quad + 65m^3n^5 + 55m^2n^6 - 25mn^7 - 250n^8, \\
y &= -2m^8 - 11m^7n - 37m^6n^2 - 69m^5n^3 - 223m^4n^4 \\
&\quad - 257m^3n^5 - 595m^2n^6 - 375mn^7 - 375n^8, \\
z &= -m^8 + 10m^7n + 12m^6n^2 + 148m^5n^3 + 164m^4n^4 \\
&\quad + 526m^3n^5 + 360m^2n^6 + 600mn^7 + 125n^8, \\
u &= 2m^8 - 5m^7n - 23m^6n^2 - 87m^5n^3 - 231m^4n^4 \\
&\quad - 355m^3n^5 - 545m^2n^6 - 325mn^7 - 375n^8, \\
v &= m^8 + 16m^7n + 12m^6n^2 + 94m^5n^3 - 12m^4n^4 \\
&\quad + 164m^3n^5 - 200m^2n^6 + 50mn^7 - 125n^8, \\
w &= -3m^8 - m^7n - 51m^6n^2 - 75m^5n^3 - 291m^4n^4 \\
&\quad - 323m^3n^5 - 525m^2n^6 - 425mn^7 - 250n^8.
\end{aligned}$$

R8d ($d = 8$, $D = (-2, -2, -2, 3, 1, 1, 1, 2, 2, -1, 0, -1)$, $2H - D = \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_7 + Q_3 + Q_6$; $f_D = 7xy + 2xz + 6yz + 2xu - 3yu + 4zu - 3u^2 - xv + 10yv - 2zv + 7uv - 3v^2 - 2xw + 4yw - zw + 6uw + 2vw + 5w^2$):

$$\begin{aligned}
x &= m^8 - 10m^7n + 13m^6n^2 - 50m^5n^3 - 137m^4n^4 \\
&\quad - 118m^3n^5 - 245m^2n^6 - 750mn^7, \\
y &= -3m^8 - m^7n - 32m^6n^2 - 51m^5n^3 - 116m^4n^4 \\
&\quad - 379m^3n^5 - 60m^2n^6 - 825mn^7 - 1125n^8, \\
z &= 2m^8 + 6m^7n + 5m^6n^2 + 138m^5n^3 + 117m^4n^4 \\
&\quad + 430m^3n^5 + 1015m^2n^6 + 1050mn^7 + 1125n^8, \\
u &= 2m^8 - 2m^7n + 41m^6n^2 + 50m^5n^3 + 161m^4n^4 \\
&\quad + 586m^3n^5 + 575m^2n^6 + 1350mn^7 + 1125n^8, \\
v &= m^8 + 3m^7n + 4m^6n^2 + 57m^5n^3 - 48m^4n^4 \\
&\quad + 137m^3n^5 + 200m^2n^6 - 525mn^7 - 1125n^8, \\
w &= -3m^8 - 2m^7n - 31m^6n^2 - 54m^5n^3 - 217m^4n^4 \\
&\quad - 170m^3n^5 - 765m^2n^6 - 1350mn^7.
\end{aligned}$$

R8e ($d = 8$, $D = (-2, -2, -1, 4, 0, 1, 0, 1, 2, 2, 0, 0, -1)$, $2H - D = \Gamma_1 + \Gamma_2 + \Gamma_7 + \Gamma_8 + Q_7 + Q_{11}$; $f_D = 2xy - 6xz + 5yz + 5xu - 2yu - 4zu + 5u^2 + 4xv - 4yv - 3zv + 6uv - 3v^2 + 2xw + 7yw + 4zw - 2uw + 5vw - 3w^2$):

$$\begin{aligned}
x &= 4500m^8 + 8950m^7n + 5735m^6n^2 + 3930m^5n^3 + 443m^4n^4 \\
&\quad + 94m^3n^5 - 243m^2n^6 - 54mn^7 - 27n^8, \\
y &= -4625m^8 + 3050m^7n + 3025m^6n^2 + 5386m^5n^3 + 2791m^4n^4 \\
&\quad + 1478m^3n^5 + 439m^2n^6 + 102mn^7 + 18n^8, \\
z &= 8375m^8 + 7175m^7n + 10200m^6n^2 + 4893m^5n^3 + 3388m^4n^4 \\
&\quad + 669m^3n^5 + 292m^2n^6 - 9mn^7 + 9n^8,
\end{aligned}$$

$$\begin{aligned}
u &= -8125m^8 - 8350m^7n - 9915m^6n^2 - 5130m^5n^3 - 2657m^4n^4 \\
&\quad - 726m^3n^5 - 113m^2n^6 + 6mn^7 + 18n^8, \\
v &= -6500m^8 - 850m^7n - 3005m^6n^2 + 298m^5n^3 - 953m^4n^4 \\
&\quad - 214m^3n^5 - 347m^2n^6 - 66mn^7 - 27n^8, \\
w &= -1875m^8 - 6875m^7n - 4660m^6n^2 - 5529m^5n^3 - 2296m^4n^4 \\
&\quad - 1553m^3n^5 - 384m^2n^6 - 147mn^7 - 9n^8.
\end{aligned}$$

R9a ($d = 9$, $D = (-3, -1, -1, 5, -1, 3, 1, 1, 3, 3, -2, 0, -2)$, $3H - D = \Gamma_1 + \Gamma_{29} + \Gamma_{30} + Q_1 + Q_2 + Q_3 + Q_{11} + Q_{23} + Q_{24}$; $f_D = 2xyu + 2xzu - xu^2 + 2yu^2 + 2zu^2 - u^3 + 2xyv + 2xzv - 2yzv - 5xuv + 3yuv - 3u^2v - xv^2 + 2yv^2 - 2uv^2 - v^3 + 2yzw - 2xuw - 2yuw - zuw + 2yvw - 2v^2w + xw^2 + vw^2$):

$$\begin{aligned}
x &= 3m^9 + 4m^8n + 29m^7n^2 + 34m^6n^3 + 105m^5n^4 \\
&\quad + 92m^4n^5 + 166m^3n^6 + 86m^2n^7 + 96mn^8 + 10n^9, \\
y &= 2m^9 + 6m^8n + 16m^7n^2 + 50m^6n^3 + 45m^5n^4 \\
&\quad + 148m^4n^5 + 59m^3n^6 + 184m^2n^7 + 35mn^8 + 80n^9, \\
z &= m^9 - 2m^8n + 12m^7n^2 + 2m^6n^3 + 55m^5n^4 \\
&\quad + 64m^4n^5 + 112m^3n^6 + 168m^2n^7 + 83mn^8 + 130n^9, \\
u &= 2m^9 + 6m^8n + 24m^7n^2 + 38m^6n^3 + 105m^5n^4 \\
&\quad + 78m^4n^5 + 199m^3n^6 + 46m^2n^7 + 137mn^8 - 10n^9, \\
v &= -m^9 + 2m^8n + m^7n^2 + 20m^6n^3 + 35m^5n^4 \\
&\quad + 86m^4n^5 + 98m^3n^6 + 174m^2n^7 + 80mn^8 + 130n^9, \\
w &= 3m^9 + 4m^8n + 28m^7n^2 + 26m^6n^3 + 95m^5n^4 \\
&\quad + 72m^4n^5 + 136m^3n^6 + 112m^2n^7 + 69mn^8 + 80n^9.
\end{aligned}$$

R9b ($d = 9$, $D = (-2, -3, -1, 5, 0, 1, 1, 2, 2, 3, -1, 0, -2)$, $3H - D = \Gamma_1 + \Gamma_2 + \Gamma_7 + \Gamma_{31} + \Gamma_{32} + Q_1 + Q_9 + Q_{11} + Q_{13} + Q_{14}$; $f_D = xyz - 5xyu + 4yzu + 2xu^2 - yu^2 - 2zu^2 - 7u^3 + 3xyv - xzv - yzv + 7xuv + 8yuv - 6zuv + 3u^2v - 3xv^2 - 3yv^2 + zv^2 + uv^2 + 3v^3 - xyw + 9xzw + yzw + 8xuw + 4yuw - 5zuw - 6u^2w - xvw - 5yvw - 3zvw - 4uvw + 5v^2w + 6xw^2 + 2yw^2 - 10zw^2 + 2uw^2 - 6w^3$):

$$\begin{aligned}
x &= 4m^9 - 10m^8n + 7m^7n^2 - 46m^6n^3 + 98m^5n^4 \\
&\quad - 92m^4n^5 + 60m^3n^6 - 46m^2n^7 + 13mn^8 - 4n^9, \\
y &= 8m^9 - 24m^8n + 63m^7n^2 - 113m^6n^3 + 97m^5n^4 \\
&\quad - 59m^4n^5 + 37m^3n^6 - 24m^2n^7 + 9mn^8 - 6n^9, \\
z &= -12m^9 + 36m^8n - 43m^7n^2 + 8m^6n^3 - m^5n^4 \\
&\quad + 6m^4n^5 + 22m^3n^6 - 16m^2n^7 + 6mn^8 - 2n^9, \\
u &= -12m^9 + 34m^8n - 31m^7n^2 - 10m^6n^3 + 77m^5n^4 \\
&\quad - 84m^4n^5 + 58m^3n^6 - 32m^2n^7 + 10mn^8 - 6n^9, \\
v &= 4m^9 + 16m^8n - 71m^7n^2 + 105m^6n^3 - 87m^5n^4 \\
&\quad + 43m^4n^5 + 7m^3n^6 - 2m^2n^7 + 5mn^8 - 4n^9, \\
w &= 8m^9 - 40m^8n + 55m^7n^2 - 58m^6n^3 + 84m^5n^4 \\
&\quad - 94m^4n^5 + 70m^3n^6 - 30m^2n^7 + 19mn^8 - 2n^9.
\end{aligned}$$

R9c ($d = 9$, $D = (-2, -2, -2, 4, 0, 3, 2, 1, 3, 3, -2, 0, -2)$, $3H - D = \Gamma_2 + \Gamma_3 + \Gamma_5 + Q_1 + Q_3 + Q_7 + Q_8 + Q_9 + Q_{11}$; $f_D = xyu - 2xzu + yu^2 - 2zu^2 - u^3 + xyv + 2yzv - 3xuv + yuv - u^2v + yv^2 - v^3 + xzw - 2xuw - 2zuw - 2u^2w + 2yvw + 2zvw + xw^2 - yw^2 + 2zw^2 - uw^2 + vw^2 + 2w^3$):

$$\begin{aligned}
x &= 25m^9 - 29m^8n + 15m^7n^2 + 19m^6n^3 - 45m^5n^4 \\
&\quad + 45m^4n^5 - 28m^3n^6 + 10m^2n^7 - 2mn^8, \\
y &= -15m^9 + 29m^8n - 73m^7n^2 + 73m^6n^3 - 56m^5n^4
\end{aligned}$$

$$\begin{aligned}
& + 16m^4n^5 + m^3n^6 - 7m^2n^7 + 3mn^8 - n^9, \\
z = & 5m^9 + 17m^8n - 36m^7n^2 + 58m^6n^3 - 53m^5n^4 \\
& + 47m^4n^5 - 30m^3n^6 + 16m^2n^7 - 5mn^8 + n^9, \\
u = & -15m^9 + 3m^8n + 25m^7n^2 - 55m^6n^3 + 62m^5n^4 \\
& - 50m^4n^5 + 31m^3n^6 - 15m^2n^7 + 5mn^8 - n^9, \\
v = & 25m^9 - 31m^8n + 21m^7n^2 + 3m^6n^3 + m^5n^4 \\
& - 7m^4n^5 + 12m^3n^6 - 6m^2n^7 + 2mn^8, \\
w = & 5m^9 - 51m^8n + 68m^7n^2 - 70m^6n^3 + 35m^5n^4 \\
& - 21m^4n^5 + 10m^3n^6 - 8m^2n^7 + 3mn^8 - n^9.
\end{aligned}$$

R9d ($d = 9$, $D = (-2, -1, -1, 4, -1, 1, 1, 1, 1, 2, 0, 0, 0)$, $2H - D = \Gamma_1 + \Gamma_3 + \Gamma_6 + \Gamma_{15} + \Gamma_{16} + \Gamma_{19} + \Gamma_{20}$; $f_D = xy - 4xz + 2yz + 17xu - 3yu + 7zu - 5u^2 - xv + 9yv + zv - 2uv + 11v^2 - 3xw - yw - 2zw - uw - 3vw - 5w^2$):

$$\begin{aligned}
x = & 74m^9 + 157m^8n + 126m^7n^2 + 845m^6n^3 - 91m^5n^4 \\
& + 1231m^4n^5 - 291m^3n^6 + 825m^2n^7 - 160mn^8 + 200n^9, \\
y = & 72m^9 - 55m^8n + 200m^7n^2 - 394m^6n^3 + 858m^5n^4 \\
& - 540m^4n^5 + 636m^3n^6 + 25m^2n^7 + 70mn^8 + 100n^9, \\
z = & 134m^9 + 138m^8n + 496m^7n^2 + 82m^6n^3 + 370m^5n^4 \\
& + 557m^4n^5 - 298m^3n^6 + 545m^2n^7 - 180mn^8 + 100n^9, \\
u = & -104m^9 + 95m^8n + 48m^7n^2 + 565m^6n^3 - 23m^5n^4 \\
& + 977m^4n^5 - 219m^3n^6 + 585m^2n^7 - 80mn^8 + 100n^9, \\
v = & 130m^9 + 198m^8n + 248m^7n^2 + 518m^6n^3 - 184m^5n^4 \\
& + 415m^4n^5 - 188m^3n^6 - 95m^2n^7 + 30mn^8 - 100n^9, \\
w = & 30m^9 + 343m^8n + 112m^7n^2 + 724m^6n^3 + 1146m^5n^4 \\
& - 324m^4n^5 + 845m^3n^6 - 160mn^8 + 200n^9.
\end{aligned}$$

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