

Bruck loops with abelian inner mapping groups: a short proof

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Abstract

Phillips and Stanovský proved that a Bruck loop with abelian inner mapping group has central nilpotency class at most two and asked for a human-friendly proof. We give a short self-contained proof: a displacement cocycle produces a commutant-valued endomorphism, right alternativity for commutant elements forces its image to be fixed, and cocycle rigidity makes that image central. This yields the abelian center quotient.

1 Introduction and elementary structure

Phillips and Stanovský's proof uses over 16,000 lines of Waldmeister output, or over 1,000 pages of structured equational reasoning [1, §2]. Their Problem 5 asks for a human-friendly proof.

Definition 1. A loop Q has an identity 1 and bijective translations $L_x(y) = xy$, $R_x(y) = yx$ for $x \in Q$. A Bruck loop here satisfies, for all $x, y, z \in Q$, the left Bol identity and automorphic inverse property (AIP):

$$x(y(xz)) = (x(yx))z, \quad (xy)^{-1} = x^{-1}y^{-1}. \quad (1)$$

Compose permutations from right to left. Put $H = \text{Inn}(Q) = \text{Mlt}(Q)_1$, the stabilizer of 1 in $\text{Mlt}(Q) = \langle L_x, R_x : x \in Q \rangle$. The commutant $C(Q)$ consists of $c \in Q$ with $ct = tc$ for every $t \in Q$. The center $Z(Q)$ consists of those $c \in C(Q)$ satisfying $(cx)y = c(xy)$, $(xc)y = x(cy)$, and $(xy)c = x(yx)$ for all $x, y \in Q$.

Theorem 1 (Phillips–Stanovský [1]). *If Q is a Bruck loop and $\text{Inn}(Q)$ is abelian, then $Q/Z(Q)$ is an abelian group. Thus Q has central nilpotency class at most two.*

No finiteness or divisibility is assumed. We recall Kreuzer's automorphism argument [2].

Lemma 1 (Bruck structure). *Let Q be a Bruck loop. For $u, v \in Q$, the map $\lambda(u, v) = L_{uv}^{-1}L_uL_v$ is an automorphism in H . For all $x, y, u, v \in Q$,*

$$L_{x^{-1}} = L_x^{-1}, \quad L_{x^2} = L_x^2, \quad L_xL_yL_x = L_{x(yx)}, \quad (2)$$

$$L_{uv}^2 = L_uL_v^2L_u, \quad \lambda(u, v)^{-1} = L_vL_uL_{uv}^{-1}, \quad (3)$$

$$\lambda(u, v)(vu) = uv. \quad (4)$$

Proof. For $x \in Q$, choose $a \in Q$ with $ax = 1$. Bol gives $L_xL_aL_x = L_x$, so $L_a = L_x^{-1}$ and $xa = 1$. Thus inverses are two-sided and $x^{-1}(xz) = z = x(x^{-1}z)$ for $z \in Q$ (LIP). Bol with $y = 1$ gives left alternativity, proving (2). By AIP and LIP, $u(uv)^{-1} = u(u^{-1}v^{-1}) = v^{-1}$. Thus right multiplication by $(uv)^{-1}$ gives

$$(u((vv)u))(uv)^{-1} = u((vv)(u(uv)^{-1})) = u((vv)v^{-1}) = uv.$$

Left alternativity gives $(ww)w^{-1} = w(ww^{-1}) = w$; applying this with $w = uv$ and cancelling gives $u((vv)u) = (uv)^2$. Equation (2) now gives the square formula, and multiplication verifies the inverse word in (3).

For $\alpha = \lambda(u, v)$ and any $z \in Q$, that inverse word expresses

$$\alpha L_z \alpha^{-1} = L_{uv}^{-1}(L_u(L_vL_zL_v)L_u)L_{uv}^{-1}$$

as three Bol sandwiches, hence as a left translation. Evaluating at 1 identifies it as $L_{\alpha(z)}$, since $\alpha(1) = 1$; hence α is an automorphism. Evaluating the square formula at 1 and cancelling on the left gives $\alpha(vu) = uv$. $\square$

2 The displacement cocycle

Lemma 2 (Displacement cocycle). *Let Q be a Bruck loop with H abelian, and fix an automorphism $\alpha \in H$. Define its displacement $f : Q \rightarrow Q$ by $f(x) = \alpha(x)x^{-1}$ and put $D_x = L_{\alpha(x)}L_x^{-1}$ for $x \in Q$. For all $x, y, z \in Q$,*

$$D_x = L_{f(x)} = R_{\alpha(x)}R_x^{-1}, \quad \alpha(x) = f(x)x, \quad (5)$$

$$f(xy)(zy) = f(y)((f(x)z)y). \quad (6)$$

Proof. Fix arbitrary $x, y \in Q$. The permutation $\rho = R_{x^{-1}y}^{-1}R_yR_{x^{-1}}$ fixes 1, so lies in H . It also fixes x , since $xx^{-1} = 1$ and $x(x^{-1}y) = y$. As ρ commutes with α , it fixes $\alpha(x)$ too. Thus $\alpha(x)(x^{-1}y) = (\alpha(x)x^{-1})y$, giving $L_{f(x)} = D_x$. Naturality under α , and its commutation with $T_x = L_x^{-1}R_x \in H$, give $L_{\alpha(x)}^{-1}R_{\alpha(x)} = L_x^{-1}R_x$. Together these give (5), with its last assertion obtained by evaluating at x .

Now $R_yR_x = R_{xy}\rho$ for $\rho = R_{xy}^{-1}R_yR_x \in H$. Conjugate by α , use $\alpha\rho = \rho\alpha$, and substitute $R_{\alpha(t)} = D_tR_t$ (valid for every $t \in Q$) to obtain $D_yR_yD_xR_x = D_{xy}R_yR_x$. Cancellation in the permutation group gives $D_{xy} = D_yR_yD_xR_y^{-1}$; evaluation at zy gives (6). $\square$

Lemma 3 (Cocycle calculus). *In any loop Q with two-sided inverses, LIP and AIP, a map $f : Q \rightarrow Q$ satisfying (6) for all $x, y, z \in Q$ has $f(1) = 1$, preserves multiplication, and has image in $C(Q)$.*

Proof. At $x = y = z = 1$, the identity makes $f(1)$ idempotent, hence $f(1) = 1$. For arbitrary $x, y, t \in Q$, its case $z = 1$ is

$$f(xy)y = f(y)(f(x)y). \quad (7)$$

Set $x = y^{-1}$; left inverse cancellation and then right cancellation give $f(y^{-1}) = f(y)^{-1}$. Now apply (6) at $(y^{-1}, y, f(y)t)$ to obtain

$$f(y)(ty) = (f(y)t)y. \quad (8)$$

Equations (7) and (8) yield $f(xy) = f(y)f(x)$ by right cancellation. Consequently $f((xy)y^{-1}) = f(y)^{-1}(f(y)f(x)) = f(x)$, an equality of images under f . Now

$$\begin{aligned} (f(x)y)y^{-1} &= (f(y^{-1})(f(xy)y))y^{-1} && \text{by (7) and LIP,} \\ &= f(y^{-1})((f(xy)y)y^{-1}) && \text{by (8) with } y^{-1}, \\ &= f((xy)y^{-1})(yy^{-1}) = f(x) && \text{by (6) at } (xy, y^{-1}, y). \end{aligned}$$

Thus $(cy)y^{-1} = c$ for $c \in f(Q)$. Set $y = c^{-1}t^{-1}$: LIP gives $cy = t^{-1}$ and AIP gives $y^{-1} = ct$, so $t^{-1}(ct) = c$. Multiplying on the left by t gives $ct = tc$. Thus $f(Q) \subseteq C(Q)$, and reversed multiplicativity becomes $f(xy) = f(x)f(y)$. $\square$

3 Fixed displacement values and rigidity

Lemma 4 (Rigidity at a fixed value). *Let Q be a Bruck loop and let $f : Q \rightarrow Q$ satisfy (6) for all $x, y, z \in Q$. Fix $y \in Q$ with $f(f(y)) = 1$. Then, for every $x, z \in Q$,*

$$f(xy)z = f(x)(f(y)z), \quad f(x)(zy) = (f(x)z)y. \quad (9)$$

Proof. Fix $x, z \in Q$. By Lemma 3, f is multiplicative and commutant-valued. Equation (6) at $(x, f(y), z)$ gives $f(x)(zf(y)) = (f(x)z)f(y)$, hence $R_{f(x)}R_{f(y)} = R_{f(y)}R_{f(x)}$. For $a \in C(Q)$ and $b \in Q$, the argument of [3, Lemma 3.2] gives the following slight strengthening:

$$a \in C(Q), \quad R_aR_b = R_bR_a \implies R_{ab} = R_bR_a. \quad (10)$$

To see this, take arbitrary $t \in Q$ and cancel a in the following Bol calculation:

$$a(t(ab)) = (a(ta))b = ((ta)a)b = ((ta)b)a = a((ta)b).$$

Since $f(x)f(y) = f(xy) \in C(Q)$, all three corresponding right translations are left translations. Hence $L_{f(xy)} = L_{f(x)}L_{f(y)} = L_{f(y)}L_{f(x)}$. This proves the first assertion of (9). Substitution into (6) gives $f(y)(f(x)(zy)) = f(y)((f(x)z)y)$; left cancellation by $f(y)$ proves the second. $\square$

Proposition 5 (Fixed displacement image). *Let Q be a Bruck loop with H abelian. Fix $u, v \in Q$, put $\alpha = \lambda(u, v)$, and define $f : Q \rightarrow Q$ by $f(x) = \alpha(x)x^{-1}$. Then $\alpha(f(z)) = f(z)$ for every $z \in Q$.*

Proof. By Lemmas 2 and 3, f preserves multiplication, identity and inverses. Put $a = f(u)$ and $b = f(v)$, both in $C(Q)$. Applying f to the definition of λ gives $f\alpha = \lambda(a, b)f$; also $\alpha f = f\alpha$ by the definition of f . Thus $\alpha(f(z)) = \lambda(a, b)(f(z))$ for every $z \in Q$. We first fix the single value b . Taking equal factors in (10) gives $(tb)b = t(bb)$ for every $t \in Q$, whence

$$\lambda(a, b)(b) = (ab)^{-1}(a(bb)) = (ab)^{-1}((ab)b) = b.$$

Thus $\alpha(b) = b$ and $f(b) = 1$. This one value suffices: Lemma 4, applied at (u, v) , gives $L_{ab} = L_a L_b$, hence $\lambda(a, b) = \text{id}$. Therefore $\alpha(f(z)) = f(z)$ for every z . $\square$

4 The full center and the quotient

Lemma 6 (Center criterion). *Let Q be a left Bol loop. An element $c \in Q$ belongs to the center if $ct = tc$ for every $t \in Q$ and $(cx)y = c(xy)$ for all $x, y \in Q$.*

Proof. Bol and the assumed left nuclearity give $c(x(cy)) = (c(xc))y = c((xc)y)$; cancellation proves middle nuclearity. Commuting c through $(xy)c = c(xy) = (cx)y = (xc)y = x(cy) = x(y c)$ proves right nuclearity. $\square$

For $c, d \in Z(Q)$, nuclearity gives

$$(cd)x = c(dx) = c(xd) = (cx)d = (xc)d = x(cd), \quad ((cd)x)y = c(d(xy)) = (cd)(xy).$$

Hence cd is central by Lemma 6; so is 1. Also $c(xc^{-1}) = (cx)c^{-1} = (xc)c^{-1} = x(cc^{-1}) = x = c(c^{-1}x)$, proving that c^{-1} commutes with x . Multiplying the desired left nuclear identity for c^{-1} on the left by c reduces both sides to xy , by left nuclearity of c and LIP. Thus c^{-1} is central.

For $x, y \in Q$, write $x \sim y$ if $y = cx$ for some $c \in Z(Q)$. The closure just proved makes this an equivalence relation. For $c, d \in Z(Q)$ and $x, y \in Q$, $(cx)(dy) = (cd)(xy)$ by nuclearity, and $(cx)^{-1} = c^{-1}x^{-1}$ by AIP. Hence the class operations $[x][y] = [xy]$ and $[x]^{-1} = [x^{-1}]$ are well defined.

Proof of Theorem 1. Fix arbitrary $u, v \in Q$, put $\alpha = \lambda(u, v)$ and define its displacement $f = f_{u,v} : Q \rightarrow Q$ by $f(x) = \alpha(x)x^{-1}$. For every $y \in Q$, Proposition 5 gives $\alpha(f(y)) = f(y)$, hence $f(f(y)) = \alpha(f(y))(f(y))^{-1} = f(y)(f(y))^{-1} = 1$. Lemma 4 yields $(f(x)t)y = f(x)(ty)$ for all $x, t, y \in Q$. Since $f(Q) \subseteq C(Q)$ by Lemma 3, Lemma 6 gives $f(Q) \subseteq Z(Q)$.

Associativity. For every $z \in Q$, recovery (5) gives $[\alpha(z)] = [f(z)z] = [z]$. Taking classes in $(uv)\alpha(z) = u(vz)$, which defines λ , gives $([u][v])[z] = [u]([v][z])$.

Commutativity. Equation (4) and recovery (5) give $uv = \alpha(vu) = f(vu)(vu)$. The witness $f(vu) \in Z(Q)$ gives $[u][v] = [uv] = [vu] = [v][u]$.

Every class has a representative; the identity and inverses descend, so $Q/Z(Q)$ is an abelian group. Its projection has kernel $Z(Q)$, giving the central series $1 \leq Z(Q) \leq Q$. $\square$

Remark. Abelianness of H is used only in Lemma 2, to commute the chosen α with elements of H . Thus the theorem also holds if every $\lambda(u, v)$ belongs to the group center $Z(H)$.

AI disclosure and formalization. GPT-Astra generated the exposition and Lean development under the author's direction. Inspection of the original proof suggested the displacement approach. The cocycle calculation was reconstructed from automated proof-search output and checked in Lean. Sources: [nasqret/bruck-palomar](#) (revision [2b9313a](#)). Palomar submission [sv12zwiscn4f](#): mechanically verified; registry review pending.

References

- [1] J. D. Phillips and D. Stanovský, *Bruck loops with abelian inner mapping groups*, Comm. Algebra **40** (2012), no. 7, 2449–2454.
- [2] A. Kreuzer, *Inner mappings of Bruck loops*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 53–57.
- [3] M. K. Kinyon, J. D. Phillips, and P. Vojtěchovský, *When is the commutant of a Bol loop a subloop?*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2393–2408.