

AN IRRATIONALITY MEASURE FOR CATALAN'S CONSTANT

SEAN LITSEY

ABSTRACT. Building on OpenAI's proof that Catalan's constant $G = \sum_{j \geq 0} (-1)^j / (2j+1)^2$ is irrational, we prove $|G - a/b| > b^{-740}$ for every integer a and every sufficiently large integer b . Thus $\mu(G) \leq 740$. Using Chebyshev's estimates for final prime selection gives 2 220; denominator and content asymptotics still use PNT. The earlier bounds 759, 2 276, 878 and 2 632 remain proved. We change the determinant parameters, obtaining a $20N$ matrix and a polynomial of degree at most $18N$. Uniform physical prime-column pools and sparse residue relations sharpen its common denominator. Exact energy certificates and separate treatment of free, fixed and marked populations give a uniform mixed-integrand bound. Stronger supplied energy trials, central-gap quadrature and jointly paid atom weights control the same determinant polynomial on $|x - G| \leq e^{-(519037/5000)N}$. An explicit uniform residue-kernel construction supplies additional common coefficient content of quadratic logarithmic rate $16/63$. A further lemma (Lemma U3, derived by Claude with the author) adds content of rate $1/3$ at the primes in $(8N, 9N)$, and a column-block lemma (Lemma U6, derived by Claude with the author) adds content of rate $1/16$ at the primes in $(4N, 9N/2)$. Coefficient divisibility recycles denominator prime squares; selecting the largest prime with $p^2 \nmid b$ completes the improved measure criterion. We retain the earlier $48N$ construction and its 1981 bound to explain the methods. Every ingredient of the new global theorem is formalized in Lean 4 with only the standard axioms and no unproved steps.

1. INTRODUCTION

OpenAI [8] recently proved that Catalan's constant

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\,55941\dots$$

is irrational, and formalized the proof in Lean. (An earlier claimed proof [11] was found to contain a gap [12].) The argument is by contradiction: under the hypothesis $G \in \mathbb{Q}$ an explicit determinant Δ_N of size $n = 48N$ is a nonzero rational number whose denominator is controlled prime by prime, so that $\log |\Delta_N| \geq -(A + o(1))n^2$; on the other hand a real-variable

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estimate gives $\log |\Delta_N| \leq -(U - o(1))n^2$ with $U > A$. The margin is small: $U - A \approx 1.3 \cdot 10^{-4}$. As OpenAI remarks, the argument proves qualitative irrationality; an irrationality measure requires additional control of rational approximations.

Recall that the *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the μ such that $|\xi - a/b| > b^{-\mu}$ for all integers a and all sufficiently large b .

Theorem 1.1. *For every integer a and every sufficiently large integer b ,*

$$\left| G - \frac{a}{b} \right| > b^{-740}.$$

In particular $\mu(G) \leq 740$. Using Chebyshev’s bounds for final prime selection gives the additional statement $|G - a/b| > b^{-2220}$. The denominator and coefficient-content asymptotics still use PNT. The earlier statements with exponents 759 and 2276 (Section 8) and 878 and 2632 (Section 7.7) remain proved.

Theorem 1.1, together with every result in this paper, has been verified in the Lean 4 proof assistant on top of OpenAI’s formalization, with no unproved steps and only the standard axioms; see Section 10. The formalization is available at <https://github.com/slitsey/catalan-irrationality-measure>. The use of AI in this work is disclosed at the end of the paper.

1.1. Previous results. The first finite irrationality measure for G was obtained by Jha [5], who proved $\mu(G) \leq 10^8$, also with a Lean formalization built on OpenAI’s. The present work was carried out independently; we found Jha’s preprint in a literature search after our proof was complete. Both arguments have the shape that OpenAI’s proof suggests: one replaces the determinant by a polynomial in the constant, clears denominators, and needs a perturbation estimate near G . They differ in the perturbation estimate. Jha bounds the complementary minors of OpenAI’s matrix in every dimension, by an energy estimate with node padding, with a codimension loss of the form $\exp(1000n(n-m))$. Here the perturbation of the matrix is written, by a quadrature formula, as a combination of point evaluations of OpenAI’s row polynomials. The selected parameter family first gave perturbation exponent $\rho = 538/5$; the supplied trials and central-gap quadrature below sharpen it to $\rho = 519037/5000$ (Section 7.7). The improvement also uses actual local arithmetic and certified energy fields for that family. For other recent work on rational approximations to G see Eskandari [3]; for earlier work on the arithmetic of G and related values see [9, 13]; for an effective approach to linear independence of related L -values see Calegari, Dimitrov and Tang [1].

Marcos Costa Santos Carreira [2, Sections 2–6] studies five-parameter determinant designs and reports $\mu(G) < 2715$ for the design $(96, 15, 6, 8, 5)$, using partial-population energy estimates and sharper row and column determinant bounds. His work also develops a two-measure equilibrium formulation. Our selected design is $(20, 3, 1, 2, 1)$. His general-design analysis and two-measure

equilibrium formulation provide related structural viewpoints. The estimates proved here use full-interval marked polynomial atoms, coefficientwise prime-square recycling and the explicit secondary coefficient-content construction below.

1.2. Outline of the proof. Sections 2–6 develop the $48N$ determinant and the earlier 1981 bound. Section 7 changes the parameters and proves the current result. The steps for the new family are:

- (i) Its determinant polynomial has degree at most $18N$. Actual prime valuations supply a common denominator of normalized growth $8795/4800 + (21/200) \log 2$. Multiplication by $(18N)!$ gives an integer polynomial, nonzero at a/b for large prime indices p with $p^2 \nmid b$.
- (ii) Four certified endpoint trials and a uniform interpolation between free and fixed populations give physical real-place constant $2.256085 - \frac{1}{2} \log 2$. Full-interval marked-sheet bounds and joint quadrature weights give perturbation exponent $538/5$; the arithmetic gap permits decay rate $\eta = 1.7755$ after normalization.
- (iii) Coefficientwise divisibility strengthens the integer evaluation bound. The largest-good-prime refinement initially gives exponent 1091.
- (iv) A uniform full-row residue kernel gives a secondary prime-content tent of area $16/63$. Dividing by this common coefficient content preserves recycling and increases the available normalized gap to 2.029. With $d = 18$ and $\rho = 538/5$, the same criterion gives the retained $955/2864$ checkpoint.
- (v) Supplied 40-mode and 160-mode trials improve the dual target to 2.2563367. Interpolation nodes avoiding a central physical gap jointly improve the marked field and actual coefficient-functional price. With $\rho = 519037/5000$ and $\eta = 425979/200000$, the criterion gives 878; final Chebyshev selection gives 2632. The historical 3662 statement remains available.
- (vi) A third residue-kernel lemma (Lemma U3, Section 8) gives coefficient content at the primes $8N < p < 9N$ of area $1/3$. Dividing by it raises the usable decay to $\eta = 492599/200000$, and the same criterion gives 759; final Chebyshev selection gives 2276.
- (vii) A fourth lemma (Lemma U6, Section 9), a rank bound for a block of columns at the primes $4N < p < 9N/2$, gives coefficient content of area $1/16$. Dividing by it raises the usable decay to $\eta = 505079/200000$, and the same criterion gives 740; final Chebyshev selection gives 2220.

1.3. Notation and conventions. We use the notation of OpenAI's paper [8] and refer to it for the constructions; we restate only what we use. Citations of the form [8, Prop. 2.1] point to numbered statements, sections and equations of OpenAI's paper. We write $\log$ for the natural logarithm.

2. OPENAI'S CONSTRUCTION AND THE FACTS WE USE

For a positive integer N put [8, (2)]

$$\begin{aligned} n &= 48N, & b &= 7N, & q &= g = 4N, & h &= 2N, \\ L &= 59N, & C &= 63N, & H &= 65N, & A &= 19N, \end{aligned}$$

and $D_* = n - 1 - 2g = 40N - 1$. (The letter A for $19N$ is OpenAI's; the constant A below is different.) The row polynomials $P_r, D_r \in \mathbb{Z}[t]$ ($0 \leq r < n$) are those of [8, (3)]; we only need that

$$(2.1) \quad P_r(t) = (1-t)^h t^{C-1} T_{|r-g|}(1/t), \quad \deg P_r < H,$$

where T_d is the Chebyshev polynomial. In particular P_r depends only on $|r-g|$. Let $f(t) = \sqrt{1-t^2}$, $d\mu(t) = |t| dt/f(t)$ on $(-1, 1)$ (a measure of mass 2),

$$A_r(t) = P_r(t) - \frac{3}{2} \mathbf{1}_{\{t>0\}} \frac{f(t)}{t} D_r(t), \quad \psi_k(s) = s^{b+k}(1-s)^q,$$

and let $\mathcal{A}_N$ be the real $n \times n$ matrix

$$(2.2) \quad \mathcal{A}_N(r, k) = \int_{-1}^1 \int_0^1 A_r(t) \frac{\psi_k(s)}{1-ts} ds d\mu(t) \quad (0 \leq r, k < n),$$

so that OpenAI's determinant is $\det \mathcal{A}_N$ [8, (7) and §5.1]. Put $c_l = 4^{-l} \binom{2l}{l}$ for integers $l \geq 0$ and $c_z = 0$ otherwise.

Proposition 2.1 (OpenAI).

(a) (Rational structure [8, Prop. 2.1 and (20)].) *There are rational matrices $\mathcal{A}_N^0, \mathcal{L}_N \in \mathbb{Q}^{n \times n}$ with $\mathcal{A}_N = \mathcal{A}_N^0 + G \mathcal{L}_N$ and*

$$(2.3) \quad \mathcal{L}_N(r, k) = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_i [t^i] P_r \cdot 4^{c_{(i-b-k-v)/2}}.$$

(b) (Real place [8, Prop. 7.1].) *With $U_{\text{OAI}} = 2.290939875 - \frac{1}{2} \log 2$,*

$$\limsup_{N \rightarrow \infty} \frac{\log |\det \mathcal{A}_N|}{n^2} \leq -U_{\text{OAI}}.$$

(c) (Local estimates [8, Prop. 2.2, §3, Prop. 3.4].) *For every rational z OpenAI bounds $v_\ell(\det(\mathcal{A}_N^0 + z \mathcal{L}_N))$ from below at every prime ℓ , by quantities depending on ℓ , N and the denominator of z . The sum of these bounds, weighted by $\log \ell$, is at most $(A + o(1))n^2$, where*

$$A = \frac{8609}{4608} + \frac{505}{4608} \log 2.$$

(d) (Nonvanishing [8, Prop. 4.1].) *There is an absolute constant $P_* \geq 260$ such that, for every prime $p > P_*$ and every $z \in \mathbb{Q}$ whose denominator is not divisible by p , $v_p(\det(\mathcal{A}_p^0 + z \mathcal{L}_p)) = -96p$; in particular this determinant is nonzero.*

Two remarks on (c) and (d). OpenAI states them under the hypothesis $G = z \in \mathbb{Q}$, but the proofs only use the arithmetic of the rational matrix $\mathcal{A}_N^0 + z\mathcal{L}_N$. In (d) OpenAI assumes $p > \max(P_*, \text{den } z)$, but the proof only uses $p \nmid \text{den } z$; this uniformity in z is essential for an irrationality measure, and is how we use the result.

We also use OpenAI's real-place machinery: the double integral, the two sheet estimates [8, Props. 5.2, 5.3], and the certified energy dual [8, Prop. 6.1, §7]. They are recalled where needed in Section 5.

OpenAI's irrationality proof is the inequality $U_{\text{OAI}} > A$, and $U_{\text{OAI}} - A = 1.3031 \dots \cdot 10^{-4}$. With a new certificate for one case of OpenAI's energy dual, Proposition 5.11 below shows that (b) holds with

$$U = 2.29180925 - \frac{1}{2} \log 2$$

in place of U_{OAI} . We use this constant throughout.

Definition 2.2. $\Delta_N(X) = \det(\mathcal{A}_N^0 + X\mathcal{L}_N) \in \mathbb{Q}[X]$. Thus $\Delta_N(G) = \det \mathcal{A}_N$, and $\Delta_N(z)$ for $z \in \mathbb{Q}$ is the rational determinant OpenAI studies under the hypothesis $G = z$. We write $\gamma_0 = U - A$.

Numerically $U = 1.9452356 \dots$, $A = 1.9442359 \dots$ and

$$(2.4) \quad 9.996945 \cdot 10^{-4} < \gamma_0 = 9.996948 \dots \cdot 10^{-4}.$$

3. ARITHMETIC OF $\Delta_N(X)$

Lemma 3.1 (Degree). $\deg \Delta_N \leq 44N$.

Proof. By (2.3) row r of $\mathcal{L}_N$ depends only on P_r , hence only on $|r - g|$ by (2.1). So rows r and $2g - r$ of $\mathcal{L}_N$ agree for $0 \leq r < g$. Subtracting row r from row $2g - r$ for each $r < g$ is a unipotent change of rows, which does not change the determinant, and it makes the g rows of index in $(g, 2g]$ of the new X -coefficient vanish. A determinant $\det(B + XL')$ in which L' has only $n - g$ nonzero rows has degree at most $n - g = 44N$ in X . $\square$

Proposition 3.2 (Common denominator). *There are positive integers D_N such that $D_N \Delta_N(i) \in \mathbb{Z}$ for every $i \in \mathbb{N}$, and for every $\varepsilon > 0$, $\log D_N \leq (A + \varepsilon)n^2$ for all large N .*

Proof. At an integer $z = i$ the denominator of z is 1, and OpenAI's local lower bounds for $v_\ell(\Delta_N(i))$ become independent of i : at odd primes they depend only on ℓ and N (and vanish for $\ell > H$), and at $\ell = 2$ they depend on z only through a quantity that, over the integers, is largest at $z = 0$. Let $e_\ell(N) \geq 0$ be the resulting bound, and put $D_N = \prod_{\ell \leq H} \ell^{e_\ell(N)}$. Then $v_\ell(D_N \Delta_N(i)) \geq 0$ for every prime ℓ , and summing the bounds $e_\ell(N) \log \ell$ exactly as in OpenAI's proof of its finite-place bound [8, Prop. 3.4] gives $\log D_N \leq (A + o(1))n^2$. $\square$

Proposition 3.3 (Sparse prime-row dependencies). *Let $N \geq 1$ and let p be a sufficiently large prime such that $58N \leq 3p$, $p \leq 21N$ and $65N < p^2$. Put*

$e = \min(4N, 63N - 3p)$. For any rational z such that pz has denominator coprime to p ,

$$v_p(\Delta_N(z)) \geq -96N + e$$

when $\Delta_N(z) \neq 0$. The corresponding lower bound with the usual formal convention $v_p(0) = 0$ also holds at zero.

Proof. Work over $\mathbb{F}_p$ and put $W = (1 - X^2)^{(p-1)/2}$. For $1 \leq d \leq e$, let $\delta_k P = P_{g+kp+d} - P_{g+kp-d}$ and define $\delta_k D$ similarly, for $k = 0, 1, 2$. Chebyshev's identities in characteristic p give $T_p(Y) = Y^p$ and $U_{p-1}(Y) = (Y^2 - 1)^{(p-1)/2}$. With

$$Q_d = 2(1 - X)^h X^{C-p-1} U_{d-1}(X^{-1}), \quad Q_d^{\text{lo}} = X^{-p} Q_d,$$

both expressions are polynomials, and direct substitution gives

$$\begin{aligned} \delta_0 D &= X^p Q_d, & \delta_1 D &= Q_d, & \delta_2 D &= 2Q_d^{\text{lo}} - X^p Q_d, \\ XW\delta_0 P &= 0, & XW\delta_1 P &= (1 - X^{2p})Q_d, & XW\delta_2 P &= 2(1 - X^{2p})Q_d^{\text{lo}}. \end{aligned}$$

There is no parity sign: substituting $Y = X^{-1}$ in U_{p-1} cancels the sign that arises from reversal. The support of Q_d is contained in $[C - p - d, H - p - 1] \subseteq [2p, 3p)$, and $Q_d = X^p Q_d^{\text{lo}}$.

The leading residues of p^2 times a raw matrix entry are obtained by grouping the coefficients of XWP_r and D_r by their base- p digit. At a raw column $b \leq j < L$, a nonzero coefficient $(Q_d)_{2p+j \bmod p}$ can contribute only when j/p (integer division) is 1 or 2. Indeed, $3p \geq H - b$ excludes quotient zero, and $d \leq g$ excludes quotient three. Write $k = \lfloor j/p \rfloor$ and let $M_{u,k}$ and $Z_{u,k}$ denote OpenAI's rational moment and zeta coefficients. The three row differences have the common coefficient $(Q_d)_{2p+j \bmod p}$ times, respectively,

$$\begin{aligned} A_k &= -\frac{3}{2}Z_{3,k}, \\ B_k &= M_{1,k} - M_{3,k} - \frac{3}{2}Z_{2,k}, \\ C_k &= 2(M_{0,k} - M_{2,k}) - 3Z_{1,k} + \frac{3}{2}Z_{3,k}. \end{aligned}$$

For $k = 1, 2$, the exact rational identity

$$C_k = \frac{292}{59}B_k - \frac{3281}{177}A_k$$

is checked directly from the moment formulas. Its reduction modulo p proves the same identity for the leading row residues. The column filter is an integer linear combination of raw columns in $[b, L)$, so the identity persists after filtering.

For each d , replace row $g + 2p + d$ by $\delta_2 - \frac{292}{59}\delta_1 + \frac{3281}{177}\delta_0$. Every source row is strictly below this target. The resulting row transformation is lower triangular with diagonal one and therefore preserves the determinant. There are exactly e distinct target rows, all below $48N$. All entries of the scaled matrix have denominator coprime to p , and each target row now has residue zero. Dividing these e rows by p preserves that denominator property, so $v_p(p^{96N}\Delta_N(z)) \geq e$. This proves the claim. A fixed cutoff $p \geq 10^6$ suffices for the finite rational denominator checks used in the formal proof. $\square$

Proposition 3.4 (An earlier residue window). *Let $N \geq 1$ and let $p \geq 10^6$ be prime, with $18N \leq p$, $3p \leq 58N$ and $65N < p^2$. For any rational z whose product pz has denominator coprime to p ,*

$$v_p(\Delta_N(z)) \geq -96N + (3p - 54N).$$

The same formal convention at zero applies.

Proof. Use all $g = 4N$ modes from the preceding proof. Each Q_d is still supported in $[2p, 3p)$: $3p + g \leq C$ and $H \leq 4p$. The obstruction to the digit identity is now quotient zero, contributed by coefficients in $[2p + b, H - p)$. For $c \in \mathbb{F}_p^g$ put $Q_c = \sum_{d=1}^g c_d Q_d$ and let

$$B : \mathbb{F}_p^g \longrightarrow \mathbb{F}_p^r, \quad r = H - 3p - b = 58N - 3p,$$

record precisely these coefficients of Q_c . Rank-nullity gives $\dim \ker B \geq g - r = 3p - 54N$. When $c \in \ker B$, its combined polynomial has support below $2p + b$ and above $L - p$, so only column quotients one and two contribute. The same rational digit identity therefore makes

$$J(c) = \sum_{d=1}^g c_d \left(\delta_2 - \frac{292}{59} \delta_1 + \frac{3281}{177} \delta_0 \right)$$

a left null vector of the leading residue matrix. The map J is injective: its coefficient at the distinct target row $g + 2p + d$ is exactly c_d , and all source rows lie below $g + 2p$ since $g < p$. Consequently the residue rank is at most $n - (3p - 54N)$.

For completeness, the rank-to-valuation step does not require a particular basis of $\ker B$. If a rational matrix A has entries with denominator coprime to p , and its reduction has rank at most $n - e$, factor that reduction through $\mathbb{F}_p^{n-e}$ and lift the factors to rational matrices with integer entries. Write $A = UV + pE$, with E also having denominators coprime to p . Multilinearity of the determinant kills every term with more than $n - e$ columns from UV ; all remaining terms contain at least e factors p . Thus $v_p(\det A) \geq e$. Apply this to $A = p^2 M$ and $e = 3p - 54N$. $\square$

Proposition 3.5 (Refined common denominator). *There are positive integers D_N^* such that $D_N^* \Delta_N(i) \in \mathbb{Z}$ for every $i \in \mathbb{N}$ and, for every $\varepsilon > 0$ and all sufficiently large N ,*

$$\log D_N^* \leq (A_* + \varepsilon)n^2, \quad A_* = A - \frac{5}{1728}.$$

Proof. Take the minimum of the old denominator exponent and the new bounds from Propositions 3.3 and 3.4 wherever each applies. This defines D_N^* and proves integrality at all integer points. For large N the old exponent is exactly $96N$ throughout these windows. The resulting continuous saving is

$$f(x) = 3(x - 18)_+ - 3(x - 58/3)_+ - 3(x - 59/3)_+ + 3(x - 21)_+,$$

where $y_+ = \max(y, 0)$. It rises from zero to four on $[18, 58/3]$, is constant on $[58/3, 59/3]$, then decreases to zero at 21. It is nonnegative, supported in

[18, 21], and $\int_1^{65} f(x) dx = 20/3$. The weighted prime number theorem gives

$$\frac{N}{n^2} \sum_{p \leq 65N} f(p/N) \log p \longrightarrow \frac{5}{1728}.$$

Subtract this saving from Proposition 3.2 to obtain the result. $\square$

Lemma 3.6 (Integer-valued polynomials). *If $f \in \mathbb{Q}[X]$ has degree at most d and $f(\mathbb{N}) \subseteq \mathbb{Z}$, then $d! f \in \mathbb{Z}[X]$.*

Proof. The d -th forward difference of f is $d!$ times its coefficient of X^d , and it is an integer combination of $f(0), \dots, f(d)$. Subtract that coefficient times the integer-valued polynomial $X(X-1)\cdots(X-d+1)$ and induct on d . $\square$

Proposition 3.7 (Square-denominator nonvanishing). *For every prime $p > P_*$ and rational z such that pz has denominator coprime to p , $\Delta_p(z) \neq 0$. In particular, $\Delta_p(a/b) \neq 0$ if $a \in \mathbb{Z}$, $b \geq 1$ and $p^2 \nmid b$.*

Proof. The scaled residue matrix in OpenAI's nonvanishing proof is obtained from the rational matrix by multiplying each entry by p^2 and permuting its rows and columns. Each filtered entry is affine in z , so its change from the entry at zero is $p^2 z \mathcal{L}_p(r, k)$. The central-binomial coefficients c_l have denominators that are powers of 2, and the filter coefficients are integers. Thus the denominator of $\mathcal{L}_p(r, k)$ is coprime to the odd prime p . If pz also has denominator coprime to p , the change $p(pz) \mathcal{L}_p(r, k)$ has denominator coprime to p and residue zero. The scaled residue matrix is therefore exactly the matrix at $z = 0$. OpenAI's proof shows that this matrix has nonzero determinant for $p > P_*$, proving $\Delta_p(z) \neq 0$.

If $p \nmid b$, then $p(a/b)$ has denominator coprime to p . Otherwise write $b = pc$; $p^2 \nmid b$ implies $p \nmid c$, and $p(a/b) = a/c$ again has denominator coprime to p . $\square$

Corollary 3.8. $Q_N = (44N)! D_N^* \Delta_N(X)$ lies in $\mathbb{Z}[X]$ and has degree at most $44N$. If $p > P_*$ is prime, $a \in \mathbb{Z}$, $b \geq 1$ and $p^2 \nmid b$, then $Q_p(a/b) \neq 0$ and hence $b^{44p} |Q_p(a/b)| \geq 1$.

Proof. The first claims follow from Lemmas 3.1 and 3.6 and Proposition 3.5. Proposition 3.7 gives $\Delta_p(a/b) \neq 0$. Finally $b^{44p} Q_p(a/b)$ is a nonzero integer. $\square$

4. THE IRRATIONALITY-MEASURE CRITERION

Proposition 4.1. *Let $\xi \in \mathbb{R}$, $d \in \mathbb{N}$ and $\eta, \rho > 0$. Let $Q_p \in \mathbb{Z}[X]$ be defined for primes p and suppose that, for all primes $p \geq P_0$:*

- (i) $\deg Q_p \leq dp$;
- (ii) $Q_p(a/b) \neq 0$ for all $a \in \mathbb{Z}$ and $b \geq 1$ with $p \nmid b$;
- (iii) $|Q_p(x)| \leq e^{-\eta p^2}$ whenever $|x - \xi| \leq e^{-\rho p}$.

Put $\kappa = \max(d/\eta, 4)$. Then $|\xi - a/b| > b^{-3\rho\kappa}$ for all $a \in \mathbb{Z}$ and all sufficiently large b . Assuming the prime number theorem, for every $\varepsilon > 0$ and every

$c > 1$ the same holds with the exponent $(1 + \varepsilon)\rho \max(d/\eta, c/\varepsilon)$. If (ii) holds whenever $p^2 \nmid b$, this PNT exponent improves to $(1 + \varepsilon)\rho \max(d/\eta, c/(2\varepsilon))$.

Proof. Suppose $|\xi - a/b| \leq b^{-3\rho\kappa}$ with b large, and put $X = \kappa \log b$. If every prime in $(X, 3X]$ divided b , then $\vartheta(3X) - \vartheta(X) \leq \log b \leq X/4$, where ϑ is Chebyshev's function. But Chebyshev's bounds $\vartheta(x) \leq x \log 4$ and $\vartheta(x) \geq (x - 1) \log 2 - O(\sqrt{x} \log x)$ give $\vartheta(3X) - \vartheta(X) \geq X \log 2 - o(X)$. Hence for large b there is a prime $p \in (X, 3X]$ with $p \nmid b$, and $p \geq P_0$. Since $\rho p \leq 3\rho\kappa \log b$, we get $|a/b - \xi| \leq e^{-\rho p}$, so by (iii) $|Q_p(a/b)| \leq e^{-\eta p^2}$. By (i) and (ii), $b^{dp} Q_p(a/b)$ is a nonzero integer, so $e^{-\eta p^2} \geq b^{-dp}$, that is $p \leq (d/\eta) \log b \leq X$. This contradicts $p > X$.

For the second statement use the window $(X, (1 + \varepsilon)X]$ with $X = \max(d/\eta, c/\varepsilon) \log b$. Then $\log b \leq \varepsilon X/c$, while the prime number theorem gives $\vartheta((1 + \varepsilon)X) - \vartheta(X) = \varepsilon X + o(X)$, which exceeds $\varepsilon X/c$ for large X .

For the stronger nonvanishing condition, put $X = \max(d/\eta, c/(2\varepsilon)) \log b$. If every prime in $(X, (1 + \varepsilon)X]$ had its square dividing b , their distinct prime squares would have product dividing b . Taking logarithms gives $2(\vartheta((1 + \varepsilon)X) - \vartheta(X)) \leq \log b \leq 2\varepsilon X/c$. The PNT contradicts this for large X . There is therefore a prime with $p^2 \nmid b$ in the window, and the polynomial argument proceeds unchanged. $\square$

The factor 3 in the first version, or $1 + \varepsilon$ in the second, is the price of finding a prime $p \nmid b$ in a window of the right size.

5. THE PERTURBATION ESTIMATE

This section proves the main new ingredient. Recall $U = 2.29180925 - \frac{1}{2} \log 2$.

Theorem 5.1 (Perturbation estimate). *Let $\rho = 366.3$. For every $\varepsilon > 0$ there is N_0 such that for all $N \geq N_0$ and all real x with $|x - G| \leq e^{-\rho N}$,*

$$|\Delta_N(x) - \Delta_N(G)| \leq \exp(n^2(-U + \varepsilon)).$$

Let $\delta = x - G$, so that $\Delta_N(x) = \det(\mathcal{A}_N + \delta \mathcal{L}_N)$. The plan: write $\mathcal{L}_N = \Phi \beta$, where the columns of Φ are values of OpenAI's row functions A_r at fixed nodes (§5.1); expand the determinant column by column (§5.2); write each resulting mixed determinant as an integral (§5.3); and bound the integrand by OpenAI's method (§5.4–5.6). Throughout, $J = H = 65N$.

5.1. A quadrature formula for $\mathcal{L}_N$. Let $x_a = \cos((2a + 1)\pi/(2J))$ for $0 \leq a < J$ be the roots of the Chebyshev polynomial T_J , let $\tau_a = (x_a - 1)/2 \in (-1, 0)$, and let $\ell_a(t) = \prod_{c \neq a} (t - \tau_c) / (\tau_a - \tau_c)$ be the Lagrange basis.

Lemma 5.2. *For $0 \leq r, k < n$,*

$$\mathcal{L}_N(r, k) = \sum_{a < J} A_r(\tau_a) \beta_{a,k}, \quad \beta_{a,k} = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_{i < H} [t^i] \ell_a \cdot 4^{c_{(i-b-k-v)/2}},$$

and $|\beta_{a,k}| \leq B := 8J^2 K^N$, where $K = 95 \cdot 10^{46}$.

Proof. Since $\deg P_r < H = J$, Lagrange interpolation gives $[t^i]P_r = \sum_a P_r(\tau_a)[t^i]\ell_a$. Substitute this into (2.3) and use $A_r(\tau) = P_r(\tau)$ for $\tau \leq 0$.

For the bound, put $g_a(t) = \ell_a(t)(t-1)^q$. Since $[t^i]\ell_a = 0$ for $i \geq H$ and $c_z = 0$ unless $z \geq 0$, the substitution $i = b+k+v+e$ and $\sum_v (-1)^v \binom{q}{v} [t^{j+v}]\ell_a = [t^{j+q}]g_a$ give

$$\beta_{a,k} = 4 \sum_{0 \leq e < H} c_{e/2} [t^{b+k+e+q}]g_a.$$

Every coefficient of a polynomial is bounded by its maximum modulus on the unit circle (average $g(\omega^j)\omega^{-jm}$ over the M -th roots of unity ω^j , $M > \deg g$). Since $|c_z| \leq 1$, it suffices to show $|g_a(t)| \leq 2J K^N$ for $|t| = 1$.

We have $T_J = 2^{J-1} \prod_c (X - x_c)$, so $\prod_c (t - \tau_c) = T_J(2t+1)/(2^J 2^{J-1})$. For the denominator of ℓ_a , $\prod_{c \neq a} |\tau_a - \tau_c| = 2^{1-J} \prod_{c \neq a} |x_a - x_c|$ and $\prod_{c \neq a} (x_a - x_c) = 2^{1-J} T'_J(x_a) = 2^{1-J} J U_{J-1}(x_a)$. Since $U_{J-1}(\cos \theta) \sin \theta = \sin(J\theta)$ and $\sin(J\theta_a) = \pm 1$ at $\theta_a = (2a+1)\pi/(2J)$, we get $|U_{J-1}(x_a)| \geq 1$, so the denominator is at least $J 2^{2-2J}$. Finally, for $|t| = 1$, $|t - \tau_a| \geq 1 + \tau_a = (1 + x_a)/2 \geq \sin^2(\pi/(4J)) \geq 1/(4J^2)$. Hence

$$|\ell_a(t)| \leq \frac{|T_J(2t+1)|}{2^{2J-1}} \cdot \frac{2^{2J-2}}{J} \cdot 4J^2 = 2J |T_J(2t+1)|.$$

Writing $\zeta = \cosh w$, one has $T_J(\zeta) = \cosh(Jw)$, $|\zeta - 1| + |\zeta + 1| = 2 \cosh \operatorname{Re} w$ and $|\cosh(Jw)| \leq e^{J|\operatorname{Re} w|}$, so

$$|T_J(\zeta)| \leq (S + \sqrt{S^2 - 1})^J, \quad S = \frac{1}{2}(|\zeta - 1| + |\zeta + 1|).$$

For $\zeta = 2t + 1$ with $|t| = 1$, let $s = |t + 1| \in [0, 2]$. Then $|\zeta - 1| = 2$, $|\zeta + 1| = 2s$, $S = 1 + s$, and $|t - 1|^2 = 4 - s^2$ by the parallelogram law. With $J = 65N$ and $q = 4N$,

$$|g_a(t)| \leq 2J \left((1 + s + \sqrt{s^2 + 2s})^{65} (4 - s^2)^2 \right)^N \leq 2J K^N,$$

where the last inequality holds for all $s \in [0, 2]$. (The function $(1 + s + \sqrt{s^2 + 2s})^{65} (4 - s^2)^2$ is largest near $s = 1.917$, where its logarithm is $110.394\dots$, while $\log K = 110.4728\dots$. We check it by splitting $[0, 2]$ into 21 intervals; on each, the first factor is increasing and the second decreasing in s , so it suffices to compare rational upper bounds at the endpoints with K .) $\square$

Remark 5.3. The cruder bound $\sum_i |[t^i]\ell_a| \leq \prod_{c \neq a} (1 - \tau_c) / \prod_{c \neq a} |\tau_a - \tau_c| \leq (3 + 2\sqrt{2})^J$, together with $\sum_v \binom{q}{v} = 2^q$, gives $|\beta_{a,k}| \leq 2^{q+2} (3 + 2\sqrt{2})^J$, whose logarithm is $(4 \log 2 + 65 \log(3 + 2\sqrt{2}))N \approx 117.35N$ instead of $\log K \cdot N \approx 110.47N$. The unit circle captures the cancellation in $(t-1)^q$ between $t = 1$, where $|T_J(2t+1)|$ is largest, and $t = -1$, where it is smallest.

Let Φ be the $n \times J$ matrix $\Phi(r, a) = A_r(\tau_a)$. Then $\Delta_N(x) = \det(\mathcal{A}_N + \delta \Phi \beta)$.

5.2. Column expansion. For a map $f: \{0, \dots, n-1\} \rightarrow \{\emptyset\} \cup \{0, \dots, J-1\}$ let M_f be the matrix whose column k is column k of $\mathcal{A}_N$ if $f(k) = \emptyset$ and the column $\Phi(\cdot, f(k))$ otherwise, and let $m(f) = \#\{k : f(k) \neq \emptyset\}$.

Lemma 5.4. *If $|\beta_{a,k}| \leq B$ and $|\det M_f| \leq Mc^{m(f)}$ for every f , with $M, c \geq 0$, then*

$$|\det(\mathcal{A}_N + \delta\Phi\beta) - \det \mathcal{A}_N| \leq M((1 + J|\delta|Bc)^n - 1).$$

Proof. Column k of $\mathcal{A}_N + \delta\Phi\beta$ is $\mathcal{A}_N(\cdot, k) + \sum_a \delta\beta_{a,k}\Phi(\cdot, a)$. Multilinearity gives $\det(\mathcal{A}_N + \delta\Phi\beta) = \sum_f \prod_{k:f(k) \neq \emptyset} \delta\beta_{f(k),k} \det M_f$. The term $f \equiv \emptyset$ is $\det \mathcal{A}_N$, and the others sum in absolute value to at most $M \sum_{f \neq \emptyset} (|\delta|Bc)^{m(f)} = M((1 + J|\delta|Bc)^n - 1)$. $\square$

5.3. A mixed Andréief identity. Fix f , and let $p = n - m(f)$. Permuting columns (which only changes the sign of the determinant), M_f has p integral columns $\mathcal{A}_N(\cdot, E_i)$, $0 \leq E_1 < \dots < E_p < n$, followed by $m = n - p$ point columns $(A_r(\tau'_j))_r$, where $\tau'_j = \tau_{f(k_j)} \in (-1, 0)$.

Lemma 5.5. *Let μ, ν be finite measures, F_r ($r < p + m$), G_i ($i < p$) and K functions with the required integrability, and B the $(p + m) \times (p + m)$ matrix with $B(r, i) = \iint F_r(t)K(t, s)G_i(s) d\mu(t) d\nu(s)$ for $i < p$ and $B(r, p + j) = F_r(\tau'_j)$. Then*

$$(p!)^2 \det B = \iint \det[F_r((t \parallel \tau')_c)]_{r,c} \times \det[K(t_i, s_j)]_{i,j < p} \det[G_i(s_j)]_{j,i < p} d\mu^p(t) d\nu^p(s),$$

where $t \parallel \tau'$ is the concatenated list of length $p + m$. Consequently $|\det B|$ is at most $(p!)^{-2}$ times the total mass of $\mu^p \otimes \nu^p$ times the supremum of the absolute value of the integrand.

Proof. For $m = 0$ this is Andréief's identity applied twice [4], as used by OpenAI for [8, (62)]. In general expand $\det B$ by multilinearity in the first p columns, which integrates t and s variables column by column, and then antisymmetrize first in the t variables and then in the s variables; the last m columns are constant throughout. $\square$

We apply this with $F_r = A_r$, $G_i = \psi_{E_i}$, $K(t, s) = 1/(1 - ts)$, μ as in §2 and ν Lebesgue measure on $(0, 1)$; the integrability is OpenAI's [8, §5.1]. The mass of $\mu^p \otimes \nu^p$ is $2^p \leq 2^n$, so

$$(5.1) \quad |\det M_f| \leq 2^n \sup_{t,s} \left| \det[A_r((t \parallel \tau')_c)] \cdot \det\left[\frac{1}{1 - t_i s_j}\right] \cdot \det[\psi_{E_i}(s_j)] \right|,$$

the supremum over $t \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero coordinates (a set of full measure). We have bounded the factor $(p!)^{-2}$ by 1.

5.4. A generalized Vandermonde bound. The column determinant is $\det[\psi_{E_i}(s_j)] = \prod_j s_j^b (1 - s_j)^q \cdot \det[s_j^{E_i}]$. Write $V(y) = \prod_{i < j} (y_j - y_i)$ and $\mathcal{V}(y) = |V(y)|$.

Lemma 5.6. *Let $s_0, \dots, s_{p-1} \in [0, 1]$ and let $E_1, \dots, E_p < n$ be distinct. Then*

$$|\det[s_j^{E_i}]| \leq \left(\prod_j (1 + s_j) \right)^{n-p} \mathcal{V}(s) \leq 2^{p(n-p)} \mathcal{V}(s).$$

Proof. Let $m = n - p$, let $k_1, \dots, k_m$ be the integers in $[0, n)$ that are not among the E_i , and let $P(X) = \prod_j (X - s_j)$, a monic polynomial of degree p . Consider two $n \times n$ matrices. The rows of Ω are the linear functionals $f \mapsto f(s_j)$ ($j < p$) and $f \mapsto [X^{k_l}]f$ ($l \leq m$), evaluated on the monomials X^a , $a < n$. The columns of Γ are the coefficient vectors of the polynomials X^b ($b < p$) and $X^i P$ ($i < m$), all of degree less than n .

- Ordering the columns of Ω as $E_1, \dots, E_p, k_1, \dots, k_m$ makes it block triangular, with diagonal blocks $[s_j^{E_i}]$ and the identity. So $|\det \Omega| = |\det[s_j^{E_i}]|$.
- In the natural order of the monomials, Γ is block triangular with diagonal blocks the identity and the matrix $([X^{p+l}]X^i P)_{l,i}$, which is unitriangular because P is monic of degree p . So $|\det \Gamma| = 1$.
- The entries of $\Omega\Gamma$ are the values of the functionals on the polynomials. Since $P(s_j) = 0$, the block of evaluations on the $X^i P$ vanishes, so $\Omega\Gamma$ is block triangular with diagonal blocks the Vandermonde matrix $[s_j^b]_{j,b < p}$ and $Q = ([X^{k_l}]X^i P)_{l,i}$.

Hence $|\det[s_j^{E_i}]| = \mathcal{V}(s) |\det Q|$. A determinant is bounded by the product of the ℓ^1 norms of its columns, and each column of Q consists of distinct coefficients of $X^i P$. Its ℓ^1 norm is therefore at most the sum of the absolute values of the coefficients of P , which is $\prod_j (1 + s_j) \leq 2^p$. $\square$

Remark 5.7. The bound is close to optimal. The ratio $\det[s_j^{E_i}]/V(s)$ is a Schur polynomial with nonnegative coefficients, and its value at $s = (1, \dots, 1)$, for a single missing exponent k , is $\binom{n-1}{k}$, of size 2^{n-1} up to a polynomial factor.

5.5. The integrand. Use OpenAI's coordinate $t = 2x/(1 + x^2)$, $x \in (-1, 1)$ [8, (64)], write x_i for the coordinates of the free nodes t_i and $y_j \in (-1, 0)$ for those of the fixed nodes τ'_j , and $z = x \parallel y \in (-1, 1)^n$. For a list z of n distinct nonzero numbers in $(-1, 1)$ OpenAI's column factorization [8, §5.1, before (65)] reads

$$(5.2) \quad \det[A_r(t_c)]_{r,c} = \mathcal{R}_n(z) \prod_c t_c^{C-1} (1 - t_c)^h,$$

where $\mathcal{R}_n$ is OpenAI's sheet determinant. OpenAI proves two bounds for it [8, Props. 5.2, 5.3], with $K_0 = 10e^{12}$:

(5.3)

$$|\mathcal{R}_n(z)| \leq (1 + K_0 n)^n \mathcal{V}(1/z) \prod_c |z_c|^g \quad \text{if } \sum_c \frac{1-z_c^2}{1+z_c^2} \leq D_*,$$

(5.4)

$$|\mathcal{R}_n(z)| \leq (3/2)^n n^{n/2} 2^{-\binom{n}{2}} \prod_c (1 + z_c^2)^{\frac{n-1}{2}} |z_c|^{g-(n-1)} \quad \text{always.}$$

Following OpenAI, $\kappa = 2$ refers to (5.3) and $\kappa = 1$ to (5.4). Let $w(x) = |t|^{C-1}(1-t)^h = 2^{C-1}|x|^{C-1}(1-x)^{2h}(1+x^2)^{-(C-1+h)}$.

The p -node majorants and their energy. For $x \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero entries, put $t_i = 2x_i/(1+x_i^2)$ and

$$\begin{aligned} \mathcal{J}^{(p)}(x, s) &= \prod_i w(x_i) \cdot \frac{\mathcal{V}(t)\mathcal{V}(s)^2}{\prod_{i,j}(1-t_i s_j)} \prod_j s_j^b (1-s_j)^q, \\ \mathcal{I}_2^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \mathcal{V}(1/x) \prod_i |x_i|^{g-(n-p)}, \\ \mathcal{I}_1^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \prod_i (1+x_i^2)^{\frac{n-1}{2}} |x_i|^{g-(n-1)}. \end{aligned}$$

For $p = n$ these are OpenAI's majorants $\mathcal{I}_{2,n}$ and $2^{\binom{n}{2}} \mathcal{I}_{1,n}$ [8, (74)]. Let OpenAI's one-point fields be [8, (76)]

$$\begin{aligned} W_\kappa(x) &= \frac{19}{48} \log |x| + \frac{1}{12} \log(1-x) - \left(\frac{\kappa}{2} + \frac{17}{48}\right) \log(1+x^2), \\ W_s(s) &= \frac{7}{48} \log s + \frac{1}{12} \log(1-s), \quad D(x) = \frac{1}{6} - \frac{2x^2}{1+x^2}, \end{aligned}$$

and define the normalized energy

$$\begin{aligned} E_\kappa^{(p)}(x, s) &= \left(\frac{C}{n} + \frac{\kappa-2}{2} - \frac{\kappa+1}{2n}\right) \log 2 + \frac{1}{n} \sum_j W_s(s_j) \\ &\quad + \frac{1}{n} \sum_i \left(W_\kappa(x_i) + \frac{\kappa+2}{2n} \log(1+x_i^2)\right) \\ &\quad + \frac{1}{n^2} \left(\kappa \sum_{i<j} \log |x_i - x_j| + \sum_{i<j} \log(1-x_i x_j)\right) \\ &\quad + \frac{2}{n^2} \sum_{i<j} \log |s_i - s_j| - \frac{1}{n^2} \sum_{i,j} \log(1-2x_i s_j + x_i^2), \end{aligned}$$

all sums over indices $< p$; note the normalization by n , not p . For $p = n$ this is the normalized logarithm $n^{-2} \log \mathcal{I}_{\kappa,n} - \frac{1}{2} \log 2$ of OpenAI's energy argument. Expanding the coordinate change $t = 2x/(1+x^2)$ gives, for every $p \leq n$,

$$(5.5) \quad \log \mathcal{I}_\kappa^{(p)}(x, s) = n^2 \left(E_\kappa^{(p)}(x, s) + \frac{1}{2} \log 2\right) + e_\kappa(p) \log 2,$$

with $e_1(p) = p(C-1) + \binom{p}{2} - n(C-1)$ and $e_2(p) = e_1(p) - \binom{n}{2}$.

Retain the product in Lemma 5.6, rather than replacing each factor by 2. For $a \in [0, 1]$, put

$$E_{\kappa,a}^{(p)} = E_{\kappa}^{(p)} + \frac{a}{n} \sum_{j < p} \log(1 + s_j), \quad t = \frac{m}{n}.$$

The retained product contributes exactly $n^2(E_{\kappa,t}^{(p)} - E_{\kappa}^{(p)})$ to the logarithm of the majorant.

Proposition 5.8 (Pointwise bound). *In the setting of (5.1), let $K_n = \max\{(1 + K_0 n)^n, (3/2)^n n^{n/2}\}$, and let $\kappa = 2$ if $\sum_c \frac{1-z_c^2}{1+z_c^2} \leq D_*$ and $\kappa = 1$ otherwise. Then the integrand in (5.1) is at most*

$$K_n 2^{-\omega_{\kappa} N m} \exp\left(n^2(E_{\kappa,t}^{(p)}(x, s) + \tfrac{1}{2} \log 2)\right), \quad \omega_1 = 28, \quad \omega_2 = 36.$$

Moreover, if $\kappa = 1$ then $\sum_{i < p} \frac{1-x_i^2}{1+x_i^2} > D_* - m$.

Proof. By (5.2), Cauchy's evaluation

$$\det\left[\frac{1}{1-t_i s_j}\right] = \frac{V(t)V(s)}{\prod_{i,j}(1-t_i s_j)}$$

[8, (63)] and Lemma 5.6, the integrand is at most

$$\left(\prod_j (1 + s_j)\right)^m |\mathcal{R}_n(z)| \prod_j w(y_j) \cdot \mathcal{J}^{(p)}(x, s).$$

For a fixed node $-1 < y < 0$, put $u = |y|$ and

$$R(u) = \frac{u^{19}(1+u)^4}{(1+u^2)^{40}}.$$

The identity

$$27(1+u^2)^4 - 256u^2 = (3u^2-1)^2(3u^4+14u^2+27)$$

gives $u^2/(1+u^2)^4 \leq 27/256$. Also $(1+u)^4 \leq 4(1+u^2)^2$ and $2u \leq (1+u^2)^2$, so

$$R(u) \leq 2(27/256)^9 \leq 2^{-28}.$$

The fixed-node numerator has powers $A = 19N$ and $2h = 4N$. The weaker of the two denominator exponents below is $C-1+h-(n-1)/2 = 41N-1/2 \geq 40N$ for $N \geq 1$. For the second case, put $S(u) = u^{19}(1+u)^4/(1+u^2)^{64}$; its denominator exponent is $C-1+h = 65N-1 \geq 64N$. Consequently

$$(5.6) \quad \begin{aligned} w(y)(1+y^2)^{\frac{n-1}{2}} |y|^{g-(n-1)} &\leq 2^{C-1} R(u)^N \leq 2^{C-1} 2^{-28N}, \\ w(y) |y|^{g-(n-1)} &\leq 2^{C-1} S(u)^N \leq 2^{C-1} 2^{-36N}. \end{aligned}$$

The stronger bound for S follows from two positive-polynomial identities. With $v = 1 + u^2$,

$$\begin{aligned} 3125v^6 - 46656u^2 &= (5u^2 - 1)^2(125u^8 + 800u^6 + 2190u^4 + 3344u^2 + 3125), \\ v^{10} - 2u(1+u)^4 &= (1-u)^2 + u^2(1-6u)^2 + u^4(1-u)^2 \\ &\quad + 119u^6 + \sum_{j=4}^{10} \binom{10}{j} u^{2j}. \end{aligned}$$

Thus $u^2/v^6 \leq 3125/46656$ and $u(1+u)^4/v^{10} \leq 1/2$, giving $S(u) \leq \frac{1}{2}(3125/46656)^9 \leq 2^{-36}$.

We use $\binom{n}{2} = \binom{p}{2} + pm + \binom{m}{2}$ repeatedly.

Case $\kappa = 1$. By (5.4), the first inequality in (5.6) and (5.5), the integrand is at most

$$K_n 2^{-28Nm} \exp(n^2(E_{1,t}^{(p)} + \frac{1}{2} \log 2)) \cdot 2^{-(\binom{n}{2} + m(C-1) + e_1(p))},$$

and $-(\binom{n}{2} + m(C-1) + e_1(p)) = \binom{p}{2} - \binom{n}{2} \leq 0$. The case condition fails for z and each fixed node contributes at most 1 to its sum, which gives the last assertion.

Case $\kappa = 2$. By (5.3) we must bound $\mathcal{V}(1/z) \prod_c |z_c|^g$. Split $\mathcal{V}(1/z) = \mathcal{V}(1/x) \prod_{i,j} |1/y_j - 1/x_i| \mathcal{V}(1/y)$. Since $|1/y - 1/x| = |x - y|/|xy| \leq 2/|xy|$ and $\mathcal{V}(1/y) = \mathcal{V}(y) \prod_j |y_j|^{-(m-1)} \leq \prod_j |y_j|^{-(m-1)}$,

$$\mathcal{V}(1/z) \prod_c |z_c|^g \leq 2^{pm} \mathcal{V}(1/x) \prod_i |x_i|^{g-m} \prod_j |y_j|^{g-(n-1)}.$$

With the second inequality in (5.6) and (5.5), the integrand is at most $K_n 2^{-36Nm} \exp(n^2(E_{2,t}^{(p)} + \frac{1}{2} \log 2)) \cdot 2^{pm+m(C-1)+e_2(p)}$, and $pm + m(C-1) + e_2(p) = -\binom{m}{2} \leq 0$. $\square$

Remark 5.9. Each fixed node brings a factor 2^{C-1} , exactly compensated by the difference between the n -node and p -node normalizations. The remaining factors are 2^{-28N} and 2^{-36N} in the two cases. Retaining the numerator and denominator together saves $32N \log 2$ per fixed node compared with the earlier separate bound $2^{2h} = 2^{4N}$.

5.6. The energy bound for $p \leq n$ nodes. OpenAI's certificate [8, §7] supplies, for $\kappa = 1, 2$, a parameter $\lambda_\kappa \geq 0$ ($\lambda_2 = 0$, $\lambda_1 = 2.47405979$), trial sequences $u^{(\kappa)}, v^{(\kappa)}$ (OpenAI's p, v), and rational upper bounds

$$(5.7) \quad \begin{aligned} \sup_{-1 \leq x < 1, x \neq 0} \{W_\kappa + \lambda_\kappa D - 2\kappa T(u^{(\kappa)}, \cdot) - S(v^{(\kappa)}, \cdot)\} &\leq B_x^\kappa, \\ \sup_{0 < s < 1} \{W_s + 2T(v^{(\kappa)}, \cdot)\} &\leq B_s^\kappa, \end{aligned}$$

in the notation of [8, (77)–(78)]. For $\kappa = 1$ we use OpenAI's certificate, which gives $B_x^1 = -1.324376$ and $B_s^1 = -1.427976$. For $\kappa = 2$ we use the trial sequences of Proposition 5.11 in place of OpenAI's, with

$$B_x^2 = -0.98572217, \quad B_s^2 = -1.60942387.$$

With these, the dual value

$$U_\kappa = -\frac{11}{16} \log 2 + \kappa \|u^{(\kappa)}\|_*^2 + \frac{1}{2} \|v^{(\kappa)}\|_*^2 + B_x^\kappa + B_s^\kappa$$

satisfies $U_1 < -2.296789875$ [8, Prop. 7.1] and $U_2 < -2.29180925$ (Proposition 5.11). Put

$$c_\kappa = (\kappa + 2) \log 2 + |B_x^\kappa| + |B_s^\kappa| + \lambda_\kappa/6,$$

so that $c_1 = 5.24413\dots$ and $c_2 = 5.36773\dots$, and $c^* = 53678/10000 > \max(c_1, c_2)$.

Proposition 5.10 (Energy bound with missing points). *For every $d > 0$ there is N_0 such that for all $N \geq N_0$, all $p \leq n$ (with $m = n - p$) and all $x \in (-1, 1)^p$, $s \in (0, 1)^p$ with distinct nonzero coordinates,*

$$E_\kappa^{(p)}(x, s) \leq U_\kappa + d + \frac{m}{n} c_\kappa,$$

for $\kappa = 2$ unconditionally and for $\kappa = 1$ under $\sum_{i < p} \frac{1-x_i^2}{1+x_i^2} > D_* - m$.

Proof. OpenAI's proof of [8, Prop. 6.1] regularizes the logarithmic interactions by matched damping, controls the diagonal and endpoint corrections, and completes two squares with the trial sequences. All of these steps are estimates for sums over the points, normalized by n , and they remain valid, with the same error terms, uniformly for any number $p \leq n$ of points. The quadratic terms are bounded by $\kappa \|u\|_*^2 + \frac{1}{2} \|v\|_*^2$ as before. Only three places change.

- (1) The constant term. For $p = n$ the powers of 2 combine to $-\frac{11}{16} \log 2$. For $p < n$ the same computation gives $-\frac{11}{16} \log 2 + (\frac{\kappa}{2} + 1)(1 - p^2/n^2) \log 2$, and $1 - p^2/n^2 \leq 2m/n$.
- (2) The one-point terms. They are $\frac{1}{n} \sum_{i < p} X(x_i) \leq \frac{p}{n} B_x = B_x + \frac{m}{n} |B_x|$, since $B_x < 0$, and likewise for the column field with B_s .
- (3) The case field ($\kappa = 1$ only). OpenAI uses $\lambda \frac{1}{n} \sum_i D(x_i) \geq -\lambda/n$ when (5.3)'s condition fails. Since $D(x) = \frac{1-x^2}{1+x^2} - \frac{5}{6}$ and $D_* = \frac{5}{6}n - 1$, the shifted condition gives $\sum_{i < p} D(x_i) > -1 - \frac{m}{6}$, which costs $\lambda(1 + m/6)/n$.

The damping parameter is chosen in terms of d alone, and the remaining errors are $O_d(1/n)$, uniformly in p and the configuration. $\square$

5.7. A new certificate for the second case of the energy dual. For a finite list $c_1, \dots, c_K$ of reals and finitely many pairs (z_j, r_j) of complex numbers with $|z_j| < 1$, consider the trial sequence

$$(5.8) \quad u_k = c_k + \operatorname{Re} \sum_j r_j z_j^k \quad (k \geq 1; c_k = 0 \text{ for } k > K).$$

This is the form of OpenAI's trial sequences [8, §7].

Proposition 5.11. *There are trial sequences u, v of the form (5.8), with $K = 40$, coefficients c_k in $10^{-8}\mathbb{Z}$, twelve pairs (z_j, r_j) for u and fifteen for v , all z_j and r_j in $\mathbb{Q} + \mathbb{Q}i$ (they are listed in the file `Dual/Data.lean` of the formalization), such that (5.7) holds for $\kappa = 2$ with $B_x^2 = -0.98572217$ and $B_s^2 = -1.60942387$, and $\|u\|_*^2 \leq 0.066723974$, $\|v\|_*^2 \leq 1.292855037$. Hence*

$$U_2 \leq -\frac{11}{16} \log 2 + 2 \cdot 0.066723974 + \frac{1}{2} \cdot 1.292855037 \\ - 0.98572217 - 1.60942387 < -2.29180925,$$

and Proposition 2.1(b) holds with $U = 2.29180925 - \frac{1}{2} \log 2$ in place of U_{OAI} .

The sequences were found numerically, by maximizing the dual value $-U_2$ over sequences of the form (5.8) with prescribed bases z_j (OpenAI's, together with ± 0.9 , ± 0.97 , 0.96 , ± 0.99 and a few others) by a cutting-plane method, and then rounded. At the optimum the two fields of (5.7) nearly equioscillate, with about fifty and thirty local maxima.

Proof. All computations below are in exact integer and rational arithmetic and are carried out by Lean's kernel. We follow the structure of OpenAI's certificate, but make each step a decidable check of general lemmas.

Closed forms. For $|x| \leq 1$, OpenAI's formulas give

$$T(u, x) = \sum_{k \leq K} \frac{c_k T_k(x)}{k} - \frac{1}{2} \operatorname{Re} \sum_j r_j \log(1 - 2xz_j + z_j^2)$$

and $S(u, x) = \sum_{k \leq K} c_k x^k / k - \operatorname{Re} \sum_j r_j \log(1 - xz_j)$. Expanding the squares and using $\sum_{k \geq 1} w^k / k = -\log(1 - w)$,

$$\|u\|_*^2 = \sum_{k \leq K} \frac{c_k^2 + 2c_k t_k}{k} - \frac{1}{2} \operatorname{Re} \sum_{j,l} (r_j r_l \log(1 - z_j z_l) + r_j \bar{r}_l \log(1 - z_j \bar{z}_l)),$$

$$t_k = \operatorname{Re} \sum_j r_j z_j^k.$$

So the norms and the values of the fields at rational points are rational numbers plus finite sums of terms $\operatorname{Re}((c+di) \log w)$ with c, d, w rational. Each such term is approximated to 10^{-12} by OpenAI's rational approximations of the logarithm and the arctangent [8, §7]; the reduction of w to the range of these approximations is computed and checked rationally.

Stationary points. The derivative of each field F is a sum of rational functions with rational coefficients, hence $F' = A/Q$ with explicit $A, Q \in \mathbb{Q}[x]$ (of degrees 78 and 39 for the row field, 60 and 21 for the column field), where Q does not vanish on the domain. The domain is cut into 154 (respectively 94) intervals. On 104 (respectively 63) of them A has no root: after the substitution $x = a + (b-a)/(1+y)$, all coefficients of $(1+y)^{\deg A} A(x)$ have the same sign, so that A has no root in (a, b) (the case of Descartes's rule of signs without sign changes). The remaining 50 (respectively 31) intervals have length $2 \cdot 10^{-7}$ and contain the roots of A ; a breakpoint between two intervals of the first kind is not a root of A .

Brackets. On a bracket $[l, l+w]$ write $A(l+wt) = \sum a_i t^i$ and $Q(l+wt) = \sum q_i t^i$. For $0 \leq t \leq 1$, $|F'| \leq \sum |a_i|/(|q_0| - \sum_{i \geq 1} |q_i|)$; since A has a root in the bracket, a_0 and a_1 are small, and this gives $|F(x) - F(l)| \leq 10^{-8}$ on the bracket. The value $F(l)$ is computed as above.

Conclusion. The fields tend to $-\infty$ at 0 and 1 (the row field at $0^\pm$ and 1^-), so their suprema are attained at stationary points or at $x = -1$, and every stationary point lies in a bracket. Comparing the certified values with B_x^2 and B_s^2 proves (5.7). The bound for U_2 follows, with $\log 2 > 0.6931471803$. OpenAI's proof of [8, Prop. 7.1] applies to any trial sequences in its two cases and gives $\limsup_N \log |\det \mathcal{A}_N|/n^2 \leq \max(U_1, U_2) + \frac{1}{2} \log 2$, and $U_1 < -2.296789875$. $\square$

For efficient replay, the implementation clears polynomial denominators once and performs the root-cover and bracket substitutions over $\mathbb{Z}$, retaining rational arithmetic for the final scalar ratio and field approximations. Lean proves the coefficient conversions and the soundness of both substitutions. This changes the computation of the certificates, while preserving their trial sequences, covers and certified bounds.

Remark 5.12. OpenAI's own certificate for $\kappa = 2$ records its bounds to five digits, as $B_x^2 \leq -0.98399 + 2.4 \cdot 10^{-5}$, $B_s^2 \leq -1.60890 + 2.4 \cdot 10^{-5}$ and a norm term at most 0.77844, while its computations show that the largest bracket values of its two fields are $-0.98403338\dots$ and $-1.60894641\dots$ and its norm term is at most 0.778423472. Read to seven digits, OpenAI's certificate already gives $U_2 < -2.2910468$, and hence the exponent 34 000 by the argument of Section 6. This is also formalized (**SharpBarrier**).

5.8. A weighted endpoint certificate. For each fixed $a \in \{0, 1\}$, repeat the proof of Proposition 5.10 with the column field $W_s + 2T(v, \cdot) + a \log(1+s)$, retaining the quadratic mass correction instead of using $1 - p^2/n^2 \leq 2m/n$. Write $K_a = \kappa \|u_a\|_*^2 + \frac{1}{2} \|v_a\|_*^2$ and $b_a = B_{x,a} + B_{s,a}$. The resulting bound, uniform in the same configurations, is

$$E_{\kappa,a}^{(p)} \leq -\frac{11}{16} \log 2 + K_a + (1-t)b_a + \left(\frac{\kappa}{2} + 1\right)(2t - t^2) \log 2 + \lambda_a t/6 + d.$$

Its error is uniform because $0 \leq a \leq 1$ and the added field is bounded and smooth at the damping endpoint. One may weaken it before damping at a cost $a \log 2/n$, which tends uniformly to zero. Since $E_{\kappa,t}^{(p)} = (1-t)E_{\kappa,0}^{(p)} + tE_{\kappa,1}^{(p)}$, the two fixed endpoint inequalities give

$$(5.9) \quad E_{\kappa,t}^{(p)} \leq V_0 + tC + t^2Q + d,$$

where

$$\begin{aligned} V_0 &= -\frac{11}{16} \log 2 + K_0 + b_0, \\ C &= K_1 - K_0 + b_1 - 2b_0 + (\kappa + 2) \log 2 + \lambda_0/6, \\ Q &= b_0 - b_1 - \left(\frac{\kappa}{2} + 1\right) \log 2 + (\lambda_1 - \lambda_0)/6. \end{aligned}$$

Thus only two fixed trials need certification; no interpolated family of norms is assumed.

For $\kappa = 2$, the $a = 0$ endpoint is Proposition 5.11. A second rational trial, listed in `WeightedDualTwo/Data.lean`, certifies

$$B_{x,1} = -0.9498627, \quad B_{s,1} = -1.2145624, \\ \|u_1\|_*^2 \leq 0.0713397, \quad \|v_1\|_*^2 \leq 1.3114478, \quad \lambda_0 = \lambda_1 = 0.$$

Here $B_{s,1}$ bounds the column field including $\log(1+s)$. The same exact rational checks used in Proposition 5.11 apply: the derivative numerators have degrees 81 and 62, and the covers have 159 and 93 pieces, including 51 and 31 stationary-point brackets. These caps give $C \leq 5.817$, $Q \leq 0$, and $V_0 < -2.29180925$.

For $\kappa = 1$, keep OpenAI's original trial at both endpoints, with $K_0 = K_1 = 0.9321$ as a common norm cap and $\lambda_0 = \lambda_1 = 2.47405979$. The new column check is

$$W_s + 2T(v^{(1)}, s) + \log(1+s) \leq -0.7586255.$$

Its derivative is rational with numerator degree 9 and denominator degree 3; a cover of five pieces has one stationary-point bracket. Combined with the old caps, it gives $C \leq 5.913488$, $Q \leq -1.70907$, and $V_0 \leq -2.29180925 - 0.004980625$. The extra constant margin absorbs the larger linear coefficient: for $t \geq 0$,

$$0.183488t - 1.70907t^2 \leq \frac{0.183488^2}{4 \cdot 1.70907} < 0.004980625.$$

Consequently the two cases give

$$(5.10) \quad E_{1,t}^{(p)} \leq -2.29180925 + d + 5.73t, \quad E_{2,t}^{(p)} \leq -2.29180925 + d + 5.817t.$$

All the new field, norm, cover and algebraic inequalities are checked by Lean's kernel.

The additional case-two decay absorbs its larger energy cost: since $8 \log 2 > 48(5.817 - 5.73)$ and $n = 48N$, its product of fixed-node and energy factors is bounded by the common factor $2^{-28Nm} \exp(nm \cdot 5.73)$. This uses the two cases together and does not change their trial sequences.

5.9. Completion of the proof.

Proof of Theorem 5.1. Fix $d > 0$ and let N be large. By (5.1), Proposition 5.8, (5.10), with $c^* = 5.73$ and $-2.29180925 = -U - \frac{1}{2} \log 2$, every f with $m = m(f)$ satisfies

$$|\det M_f| \leq 2^n K_n 2^{-28Nm} \exp(n^2(-U + d) + nm c^*) = M c^m,$$

with

$$M = 2^n K_n e^{n^2(-U+d)}, \quad c = 2^{-28N} e^{nc^*}.$$

(The degenerate configurations, such as two fixed nodes being equal or a vanishing coordinate, have measure zero or make the determinant vanish.) By Lemma 5.2, $|\beta_{a,k}| \leq B = 8J^2K^N$. Now

$$\begin{aligned} \log(nJBc) &= \log(8nJ^3) - 28N \log 2 + N \log K + nc^* \\ &= -28N \log 2 + N \log K + 48Nc^* + \log(8 \cdot 48 \cdot 65^3 N^4), \end{aligned}$$

and $-28 \log 2 + \log K + 48c^* < 366.107$; so $nJBc \leq e^{366.3N}$ for $N \geq 4000$. If $|\delta| \leq e^{-366.3N}$ then $J|\delta|Bc \leq 1/n$ and $(1 + J|\delta|Bc)^n - 1 \leq e - 1 < 2$. Lemma 5.4 gives

$$|\Delta_N(x) - \Delta_N(G)| \leq 2M = 2^{n+1}K_n e^{n^2(-U+d)}.$$

Since $\log K_n = O(n \log n)$, taking $d = \varepsilon/2$ proves the theorem. $\square$

Remark 5.13 (The value of ρ). Per unit of N , the quadrature weights contribute $\log K < 110.475$, the fixed nodes contribute $-28 \log 2$, and the weighted energy contributes $48 \cdot 5.73 = 275.04$. The polynomial prefactor is absorbed by the remaining margin to $\rho = 366.3$. Retaining the Vandermonde product replaces the former combined charge $48 \log 2 + 48 \cdot 5.3678$ by the weighted energy charge, then the separate fixed-node caps and the quadratic margin lower it to $48 \cdot 5.73$.

6. THE $48N$ BOUND

The earlier 1981 bound. Let $\gamma_* = U - A_* = \gamma_0 + 5/1728$ and $\eta = 8969/1000$. The certified old margin (2.4) implies $2304\gamma_* > 2304 \cdot 9.996945 \cdot 10^{-4} + 20/3 > 8.969$. For sufficiently small $\varepsilon > 0$ and large primes p , the real bound and perturbation estimate give $|\Delta_p(x)| \leq 2e^{n^2(-U+\varepsilon)}$ on $|x - G| \leq e^{-366.3p}$. The factorial and the factor two are $e^{o(n^2)}$, while Proposition 3.5 bounds D_p^* by $e^{n^2(A_*+\varepsilon)}$. Thus $Q_p = (44p)!D_p^*\Delta_p$ satisfies $|Q_p(x)| \leq e^{-\eta p^2}$ for large p . Corollary 3.8 gives its degree and square-denominator nonvanishing.

Apply Proposition 4.1 with $d = 44$, $\rho = 366.3$ and $R = d/\eta = 44000/8969$. Choose $c = 10001/10000$ and $\varepsilon = c/(2R) = 89698969/880000000$. The PNT exponent is

$$(1 + \varepsilon)\rho R = 366.3 \left(\frac{44000}{8969} + \frac{10001}{20000} \right) = 1980.157945 \dots < 1981.$$

With the Chebyshev window the exponent is $3\rho R = 5390.968892 \dots < 5391$. $\square$

Remark 6.1 (Further reductions). The sparse-row mechanism raises the denominator gap without changing the determinant's degree or the perturbation estimate. Further improvements can come from additional residue dependencies in other prime windows, more efficient determinant parameters, or a smaller perturbation exponent. Coefficient recycling below removes the overhead of the earlier square-prime selection step. The remaining analytic and arithmetic constants still give a bound far above two; the present proof does not establish an irrationality exponent near two.

7. A TWENTY-ROW PARAMETER FAMILY

The preceding sections explain the $48N$ construction and its 1981 bound. We now change the determinant parameters, first derive the 1281 checkpoint, then refine quadrature and prime selection to prove Theorem 1.1. This change requires new arithmetic and real-place estimates; substituting parameters into the earlier numerical constants would not suffice.

7.1. The determinant and its arithmetic. Use the same moment and contact construction with

$$\begin{aligned} n &= 20N, & b &= 3N, & q &= N, & g &= 2N, & h &= N, \\ L &= 24N, & C &= 26N, & H &= 27N, & A &= 8N. \end{aligned}$$

In the definitions of the row polynomials, replace the previous parameter values by these values: the offset is $r - 2N$, the reversal length is $26N$, and the common row factor is $(1 - t)^N$. The filtered columns are $t^{3N+k}(1 - t)^N$. Write $\Delta_N^{(20)}(X)$ for the resulting determinant polynomial. Row reflection gives

$$\deg \Delta_N^{(20)} \leq 18N.$$

The three 20×20 base matrices corresponding to signs $0, 1, -1$ are invertible. Exact inverse certificates modulo 1009 prove their rational nonvanishing. The prime-index reduction then proves uniformly, outside a fixed finite set of primes,

$$\Delta_p^{(20)}(a/b) \neq 0 \quad \text{if } p^2 \nmid b.$$

The finite inverse computations certify the base matrices; the deduction for all large primes is a separate theorem, not an extrapolation from those computations.

For large odd primes with $27N < p^2$, the actual filtered determinant satisfies $v_p(\Delta_N^{(20)}(z)) \geq -Nf(p/N)$ whenever z has p -unit denominator. The physical column pools and their rational inverse maps give

$$f(x) = \begin{cases} 40 & 0 \leq x \leq 10, \\ 60 - 2x & 10 \leq x \leq 12, \\ 36 & 12 \leq x \leq 27/2, \\ 63 - 2x & 27/2 \leq x \leq 14, \\ 91 - 4x & 14 \leq x \leq 16, \\ 75 - 3x & 16 \leq x \leq 24, \\ 27 - x & 24 \leq x \leq 27, \\ 0 & 27 \leq x. \end{cases}$$

Each bound applies to the actual determinant: the pool reconstruction retains the integer filter, and the upper-range central remainders are proved p -integral. The exact integral is $\int_0^{27} f = 2937/4$. The proved sparse row relations add the saving

$$g(x) = 3(x - 22/3)_+ - 6(x - 8)_+ + 3(x - 26/3)_+.$$

It is supported on $[22/3, 26/3]$, where $f = 40$, and has integral $4/3$. Thus $f - g \geq 0$, and

$$\frac{1}{400} \int_0^{27} (f - g)(x) dx = \frac{8795}{4800}.$$

The continuous weighted prime-number theorem proves that this integral is the normalized limiting odd-prime cost. The finitely many excluded primes and primes at most $2\sqrt{65N}$ contribute $o(N^2)$, using the already proved logarithmic entry bounds. Primes above $27N$ are locally integral.

The 2-adic smoothing and exceptional-column expansion also survives the parameter change. Nonzero coefficient minors have distinct coefficient indices within each exceptional family, and their raw column indices are distinct. If m and ℓ are the two exceptional populations, the quadratic part of the selector exponent is

$$\frac{3}{2}m^2 + \frac{1}{2}\ell^2 - 50Nm - 26N\ell + 520N^2 \geq -42N^2, \quad m, \ell \geq 0, \quad m + \ell \leq 20N.$$

Indeed, adding $42N^2$ to the left side gives

$$2(m - 11N)^2 + (20N - m - \ell)(6N + m) + \frac{1}{2}(20N - m - \ell)^2.$$

The full minor expansion, not just this quadratic inequality, yields

$$\|\Delta_N^{(20)}(z)\|_2 \leq (8F(z)(27N)^2)^{20N} 2^{42N^2 + 40N}.$$

At integer points $F(z) \leq F(0)$. Consequently a common denominator $D_N^{(20)}$ clears every integer value and, for every $\epsilon > 0$, eventually

$$\log D_N^{(20)} \leq \left(\frac{8795}{4800} + \frac{21}{200} \log 2 + \epsilon \right) (20N)^2.$$

The formal construction uses a harmless eventual threshold $N \geq 10^{12}$ to absorb all finite prime cutoffs. No effective threshold for the final rational denominator is asserted.

7.2. Free and fixed populations. Recompute both sheet duals for the new parameters. Four rational trial certificates cover the two sheet cases at the two endpoints of the missing population. Their global field bounds, derivatives, norms and finite diagonal corrections are checked by the kernel. The certified dual target is

$$U_{20} = \frac{2256085}{1000000}.$$

The physical determinant energy retains the correction $+\frac{1}{2} \log 2$. Fixed-node fields include the common-factor cancellation in the quadrature weights; their costs are not replaced by the free-node fields. The finite population interpolation gives, uniformly in every missing mass m , an energy loss at most $4.3m/(20N)$, in addition to an arbitrarily small regularization error. Damping is chosen before taking N large, so this is a uniform bound rather than a numerical cutoff near the endpoints.

Use $18N$ Chebyshev nodes for the residual row polynomials, and a circle of radius $1001/1000$. The exact circle certificate gives $K \leq e^{33}$. The

joint common-factor and quadrature-weight bound is $54000NK^N$ per atom. Combining it with the population estimate gives the actual mixed integrand, multiplied by all its atom weights, the bound

$$S_N(54000N)^m \exp\left((20N)^2 \left(-U_{20} + \frac{1}{2} \log 2 + \delta\right) + 119Nm\right),$$

where

$$S_N = (3/2)^{20N} (20N)^{10N} + (1 + 10e^{12}(20N))^{20N}, \quad \log S_N = o(N^2).$$

This estimate covers both sheets. Zero nodes and repeated nodes give zero determinants and are included separately.

The mixed Andréief identity can retain the product of the atom weights on both sides of the integral. The determinant column expansion then uses their joint bound, with atom cost $54000Ne^{119N}$. Since

$$(20N)(18N)(54000N)e^{119N} \leq e^{120N}$$

eventually, for every $\epsilon > 0$ the actual determinant polynomial satisfies

$$\begin{aligned} |\Delta_N^{(20)}(G)| &\leq e^{(20N)^2(-U_{20} + \frac{1}{2} \log 2 + \epsilon)}, \\ |\Delta_N^{(20)}(x) - \Delta_N^{(20)}(G)| &\leq e^{(20N)^2(-U_{20} + \frac{1}{2} \log 2 + \epsilon)} \quad \text{if } |x - G| \leq e^{-120N}. \end{aligned}$$

7.3. The 1281 checkpoint. Set

$$A_{20} = \frac{8795}{4800} + \frac{21}{200} \log 2, \quad \gamma_{20} = U_{20} - \frac{1}{2} \log 2 - A_{20}.$$

The certified rational budget gives $400\gamma_{20} > 177/100$. The integer-valued polynomial argument supplies $Q_N^{(20)} = (18N)! D_N^{(20)} \Delta_N^{(20)} \in \mathbb{Z}[X]$ for all sufficiently large N ; smaller indices can be assigned the zero polynomial. Factorial growth is $o(N^2)$, so the criterion applies with

$$d = 18, \quad \eta = 177/100, \quad \rho = 120.$$

Choose $\varepsilon = 247/5000$ and $c = 1001/1000$ in the square-denominator prime window. Its exponent is

$$(1 + \varepsilon)120 \max\left(\frac{18}{177/100}, \frac{1001/1000}{2(247/5000)}\right) = \frac{377784}{295} < 1281.$$

The Chebyshev criterion gives $216000/59 < 3662$. These prove the earlier 1281 bound and the retained historical Chebyshev bound. The corresponding Lean statements are in `Parametric/TwentyMain`. They have suffixes 1281 and 3662. We next improve the first bound.

7.4. Full-interval marked atoms and the critical radius. Keep the same integer polynomial $Q_N^{(20)}$, and let $J = 18N$. Take $\tau_1, \dots, \tau_J$ to be

all roots of T_J in $(-1, 1)$. Since J is even none is zero. Exact coefficient quadrature now gives

$$(\mathcal{L}_N)_{rk} = \sum_{a=1}^J P_r(\tau_a) \beta_{ak}, \quad \kappa_N(t) = t^{8N}(1-t)^N,$$

$$|\kappa_N(\tau_a) \beta_{ak}| \leq 108000N e^{18N}.$$

The fixed vectors are values of the original P_r , including at positive nodes; they are not values of the physical amplitude A_r there.

For p free and m fixed columns, $p + m = 20N$, the marked first-sheet Hadamard bound has the same prefactor $(3/2)^{20N}(20N)^{10N}$ as before. The full first-case energy certificate gives loss $(448/5)Nm$. Here its scalar budget is sharpened from $27/5$ to $269/50$: the difference is bounded above by

$$-\frac{8}{125} + \frac{593}{1000}a - \frac{693}{500}a^2 = -\frac{693}{500} \left(a - \frac{593}{2772} \right)^2 - \frac{3167}{5544000} < 0.$$

Removing the quadrature term $(9/10)a$ leaves coefficient $112/25$; multiplication by $20Nm$ gives the stated loss. In the second case expand only positive marked nodes into actual and near sheets, using the exact identity $P = -2A + 3A_{\text{near}}$ after removal of the common row weight. The grouped energy certificate retains separate Vandermonde factors for actual and near nodes; its loss is $89Nm$. The expansion costs at most 5^m . Both estimates include a uniform arbitrarily small quadratic-scale error. Combining them with the joint weight bound gives

$$(7.1) \quad S_N(540000N)^m \exp\left((20N)^2 \left(-U_{20} + \frac{1}{2} \log 2 + \epsilon\right) + (538/5)Nm\right).$$

Here S_N is the same prefactor as above. In the second case the sharper $107Nm$ can be increased to $(538/5)Nm$.

The mixed Andréief bound allows arbitrary fixed column vectors. Formally, modify the integrated row functions at finitely many distinct auxiliary points; the atomless measure makes this change invisible to the integrals. Repeated fixed atoms give identical columns and determinant zero. Free/free and free/fixed collisions are excluded on null sets. No equality of a positive marked P column and an actual free column is assumed. These arguments also cover $p = 0$ and $m = 0$.

After integration, put

$$M_N = 2^{20N} S_N e^{(20N)^2(-U_{20} + \frac{1}{2} \log 2 + \epsilon)}, \quad c_N = 540000N e^{(538/5)N}.$$

The exact joint column expansion gives

$$|\Delta_N^{(20)}(G + \delta) - \Delta_N^{(20)}(G)| \leq M_N((1 + J|\delta|c_N)^{20N} - 1).$$

If $|\delta| \leq e^{-(538/5)N}$ the binomial factor is at most $(1 + 9720000N^2)^{20N} = e^{O(N \log N)}$. Its logarithm is $o(N^2)$; there is no need to bound the factor by

an absolute constant. Absorb it and the factorial normalization into the regularization error. For all sufficiently large N ,

$$(7.2) \quad |Q_N^{(20)}(x)| \leq e^{-(177/100)N^2} \quad \text{whenever } |x - G| \leq e^{-(538/5)N}.$$

7.5. Coefficient recycling and the final prime choice. Write $Q_N = Q_N^{(20)} = \sum_j c_{Nj} X^j$ and $D = 18N$. For every sufficiently large N and every prime $N < q \leq 2N$,

$$(7.3) \quad q^{2j} \mid c_{Nj} \quad (j \geq 0).$$

Indeed $q^2 \mathcal{A}_N^0$ and $\mathcal{L}_N$ are q -integral, while $q^{40N} \mid D_N^{(20)}$. The $20N$ determinant therefore shows that

$$Q_N(X/q^2) = (18N)! \frac{D_N^{(20)}}{q^{40N}} \det(q^2 \mathcal{A}_N^0 + X \mathcal{L}_N)$$

is coefficientwise q -integral. Its j th coefficient is c_{Nj}/q^{2j} , which proves (7.3). This conclusion uses coefficientwise local integrality, not integer-valued specialization.

If B is a product of distinct prime squares $q^2 \mid b$ with $N < q \leq 2N$, then $B \mid b$ and $B^j \mid c_{Nj}$. Each summand in $b^D Q_N(a/b) = \sum_{j=0}^D c_{Nj} a^j b^{D-j}$ is divisible by B^D . Thus, whenever $Q_N(a/b) \neq 0$,

$$(7.4) \quad 1 \leq (b/B)^D |Q_N(a/b)|.$$

No coprimality of a, b is required.

The same certified gap used above satisfies the sharper rational bound

$$400\gamma_{20} = 400 \left(\frac{2256085}{1000000} \right) - \frac{8795}{12} - 242 \log 2 > \frac{17755}{10000}.$$

Indeed the already verified bound $\log 2 < 0.693147181$ suffices. The interval estimate (7.2), including normalization, therefore holds with every $\eta < 400\gamma_{20}$, in particular this sharper choice. Set $\eta = 17755/10000$, $d = 18$, $\rho = 538/5$, $k = 5455/538$ and $\delta_0 = 1/1000000$. For large b , put $L_b = \log b$, $U = kL_b$, $V = (k-1)L_b$. The prime number theorem supplies, simultaneously for every sufficiently large real t ,

$$(1 - \delta_0)t \leq \theta(t) \leq (1 + \delta_0)t.$$

If every prime in $(V, U]$ had its square dividing b , then $2(\theta(U) - \theta(V)) \leq L_b$. The displayed PNT bounds contradict this, since $2(1 - \delta_0(2k - 1)) > 1$. Choose the *largest* prime $p \in (V, U]$ for which $p^2 \nmid b$. Every prime $q \in (p, U]$ has $q^2 \mid b$, and $U < 2V < 2p$. Take B to be the product of their squares and put $T = \theta(U) - \theta(p)$, so $\log B = 2T$.

Suppose $|G - a/b| \leq b^{-1091}$. Since $\rho p \leq \rho U = 1091L_b$, (7.2) applies at a/b . Prime-index nonvanishing and (7.4) give

$$\eta p \leq d(L_b - 2T).$$

But the PNT bounds imply $T \geq U - p - 2\delta_0 U$. Using $p \leq U$ and $2d - \eta > 0$ now yields

$$\eta k \leq d + 4d\delta_0 k, \quad \eta k - d - 4d\delta_0 k = \frac{47987}{26900000} > 0,$$

a contradiction. This proves the retained 1091 bound.

The combined Lean theorem in namespace `CatalanMeasure.Parametric` is `catalan_irrationalityMeasure_1091`. It is proved in `Parametric/TwentyCostMain`. Its public series version is in `Solution.lean`. The earlier 1095 theorem remains in `Parametric/TwentySharperMain`; the 1099 theorem remains in `Parametric/TwentyRecycledMain`. The sharper gap and marked-column cost give the limiting exponent $3873600/3551 \approx 1090.848$; the strong eventual statement above uses the integer exponent 1091. The following arithmetic refinement gives the 955 checkpoint; Section 7.7 then sharpens its real estimate to the 878 bound, Section 8 to the 759 bound, and Section 9 to the current statement of Theorem 1.1.

7.6. A uniform secondary coefficient-content tent. Write $\mathcal{A}_{N,p}$ for the reduction of p^2 times the actual filtered base matrix. For all sufficiently large N and primes $52N/9 < p < 44N/7$, we prove

$$(7.5) \quad \text{rank } \mathcal{A}_{N,p} \leq 20N - e_N(p),$$

$$e_N(p) = \max \left(0, \min \left(\left\lfloor \frac{9p - 52N}{2} \right\rfloor, \left\lfloor \frac{44N - 7p}{2} \right\rfloor \right) \right).$$

The finitely many primes dividing $B_* = 6310886697782126553600$ may be excluded. Every assertion in this subsection concerns the full canonical row range, including its endpoints.

Here is an explicit construction and its dimension count. Work over $\mathbf{F}_p$, set $n = (p - 1)/2$, $g = 27N - 4p$, $a = p - g$, $b = p - 3N$, and use

$$x = (z + z^{-1})/2, \quad s = (z - z^{-1})/2, \quad h = \frac{z^{26N}(z - 1)^{2N}}{(1 + z^2)^{27N}}, \quad b_0 = \frac{z^4(z - 1)}{(1 + z^2)^4}.$$

In terms of the signed moments $M_{u,J}$ and $Z_{u,J}$ from OpenAI's entry formulas [8], put

$$A_J(x) = \sum_{v=0}^5 M_{v-1,J} x^v, \quad B_J(x) = \frac{3}{2} \sum_{v=0}^4 Z_{v,J} x^v, \quad a_J = B_J - sA_J/x.$$

In particular the signed boundary moment $M_{-1,J}$ is retained. The raw source for column $j = Jp + e$, $0 \leq e < p$, is the Laurent jump of $2^{26N-1} h x^e a_J^p$; this follows directly by grouping the literal entry coefficients into base- p digits.

Use the six ordinary differentials $dz/z, dz, z dz, dz/(z - 1), dz/(1 + z^2), z dz/(1 + z^2)$, followed by the boundary differential dz/z^2 . For a coefficient vector $c = (\alpha_0, \alpha_1, \alpha_2, \beta, \gamma, \zeta, \omega)$ define

$$P_c = \gamma + (2\alpha_0 - 2\alpha_2 + \beta + \zeta)x + (2\alpha_1 + 2\omega)x^2 + 4\alpha_2 x^3,$$

$$Q_c = -\zeta + (-2\alpha_1 + \beta + \zeta + 2\omega)x + (2\alpha_1 - 4\alpha_2 - 2\omega)x^2 + 4\alpha_2 x^3.$$

Thus the differential is $(U_c + sV_c)dz/z$, where $P_c = 2xU_c$ and $Q_c = 2x(x-1)V_c$. Its macro response is the coefficient of x^5 in

$$(7.6) \quad H_{J,c} = ((xB_J - (x-1)A_J)P_c + (xB_J - (x+1)A_J)Q_c)/16.$$

This identity follows by adding the two expressions related by $z \mapsto z^{-1}$.

For reproducibility the complete fixed response matrix, multiplied by 4608, is

$$\begin{pmatrix} 976 & 64 & 4640 & -48 & 25 & 551 & 80 \\ -984 & 3768 & -3336 & -792 & 764 & -532 & 1704 \\ 2208 & -1680 & -6384 & 1440 & -354 & 1602 & -1632 \\ -2008 & -1768 & 2104 & -600 & 1196 & -1028 & -344 \\ -1320 & 288 & 16 & -552 & -935 & -1025 & 1752 \end{pmatrix}.$$

Rows are $J = 0, \dots, 4$. The first four ordinary columns in the first four rows have determinant $46085/1769472$ before this scaling. Consequently there are ordinary seeds $\eta_0, \eta_1, \eta_2, \eta_E$ whose first four responses are respectively $(1, 1, 1, 1)$, $(0, 0, 0, 0)$, $(0, 0, 0, 0)$ and $(1, 0, 0, 0)$. Choose their (γ, ζ) coordinates to be $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, 0)$, respectively; the first four coordinates are then uniquely determined by the displayed invertible matrix. The fifth seed is explicitly

$$\eta_B = \frac{dz}{z^2} + \frac{20981}{91559} \frac{dz}{z} - \frac{40734}{91559} dz - \frac{5430}{91559} z dz.$$

Its first four responses are $(0, 0, 0, -22627/732472)$, and its β, γ, ζ coordinates vanish. These five seeds are independent: the boundary pole distinguishes the fifth, the two free coordinates distinguish the kernel seeds, and the two remaining response vectors are independent. Multiplication of all seeds by $D = 747824613120$ makes their coefficients integral. All reductions and independence certificates are valid away from B_* .

For five inputs I_i of degree $< p$, form

$$H_U = \sum_i P_i^p I_i, \quad H_V = \sum_i Q_i^p I_i, \quad K = (z-1)^p \mathcal{T}_{4p-1} H_U + (z+1)^p \mathcal{T}_{4p-1} H_V,$$

where $\mathcal{T}_d I = (2z)^d I((1+z^2)/(2z))$ and the seeds now include the common factor D . Impose the flags

$$\begin{aligned} x^a \mid H_U, H_V, \quad (x-1)^N \mid H_U, \quad (x+1)^n \mid H_U, \quad (x-1)^{N+n} \mid H_V, \\ \deg \sum_i \alpha_{2,i} I_i < 2p - 8N - 1. \end{aligned}$$

The last condition is one shared top-degree cap. Together with the lower endpoint condition described below, these imply

$$T = z^{24N-2p} (1+z^2)^a (z-1)^{p+2N-1} (z+1)^{p-1} \mid K, \quad \deg K + 8N + 3 \leq 10p.$$

Thus $R = K/T$ has degree $< 20N$, and $u = z^{-2N} R$ is an actual full row packet. The map from the five inputs to this row is injective: the involution separates the two natural channels, and Frobenius interleaving separates each

input coefficient and the independent seeds. Both division and extraction of the $20N$ row coefficients preserve this injection under the displayed flags.

The literal raw response of this row is

$$(7.7) \quad -8 \cdot 2^{26N-1} \sum_i [x^5] H_{J,i} [x^{p-1-e}] I_i.$$

One elementary way to verify the jump calculation is the finite Laurent identity $\text{CT}_{z=0}(sL(x(z))) = -[x^{-1}]L$. Combining it with (7.6) and

$$[x^{6p-1-e}](H(x)^p I(x)) = [x^5] H [x^{p-1-e}] I \quad (\deg I < p)$$

proves (7.7) directly. Hence no generic surjectivity of the raw matrix or of its polynomial responses is needed.

Take $I_0 = F$ with $(x-1)^{p-N} \mid F$ and $\deg F < p$. It has N parameters, and $[x^{p-1-e}]F = f(e)$ for a polynomial f of degree $< N$. Explicitly, the basis $(x-1)^{p-1-k}$, $0 \leq k < N$, corresponds to the Newton basis $\binom{e}{k}$. Set $I_1 = x^a f_1$, $I_2 = x^a f_2$, $\deg f_i < g$, and $I_E = x^b h_E$, $\deg h_E < 3N$. These choices pay the $x = 0$ flags. For $p = 6N + \delta$, $\delta > 0$, allow $\deg I_B < 4\delta$; for $p + d = 6N$, set $I_B = 0$. The remaining constraint costs are

	$p = 6N + \delta$	$p + d = 6N$
parameter dimension	$N + 2g + 3N + 4\delta$	$N + 2g + 3N$
small beta flag	$3N$	g
two H_U branch flags	$N + n$	$N + n$
shared top cap	$8N - p + 1$	$8N - p + 1$
extra endpoint/response flags	0	$4d + 4d$

For the beta flag use $Q_i(1) = \beta_i$ and $(x-1)^p \mid Q_i^p - Q_i(1)$; its small polynomial vanishes after the indicated number of coefficient constraints. Since $N + n \leq p - N$, this pays the entire H_V branch flag. On the upper side the boundary input degree pays the lower endpoint $z^{24N-2p} \mid K$. On the lower side it is paid by the first $4d$ coefficients after the automatic z^{2p} factor; impose another $4d$ literal raw-response conditions on the available $J = 4$ columns.

The E response is confined to the unavailable $J = 0$, $e < 3N$ block. On the upper side the B response is confined to unavailable $J = 3$ columns, because every available such column has $e < p - 4\delta$. Thus every available raw response is the same nonzero scalar times $f(e)$, including the lower side's explicitly constrained $J = 4$ block. The N th column difference annihilates it. Subtracting the displayed costs gives nullity at least $N - (7\delta + 1)/2$ on the upper side and $N - (9d + 1)/2$ on the lower side, with integer quotients; these are exactly (7.5).

The actual canonical normalization pays the flat p^{40N} denominator, and the scaled perturbation matrix vanishes modulo p in this window. The rank bound therefore puts $p^{e_N(p)}$ in *every coefficient* of Q_N , rather than just its integer values. Let

$$c_N = \prod_{52N/9 < p < 44N/7} p^{e_N(p)}$$

for sufficiently large N , taking $c_N = 1$ at the finitely many earlier indices. Weighted PNT summation [10] gives

$$\log c_N = (16/63 + o(1))N^2, \quad \frac{16}{63} = \frac{1}{9} + \frac{1}{7}.$$

These new primes exceed $2N$ and have no overlap with the already counted gain in this window. The integral polynomial $\tilde{Q}_N = Q_N/c_N$ retains the degree bound, rational nonvanishing and all recycled prime squares. Its uniform decay is available with

$$\eta = \frac{3551}{2000} + \frac{507}{2000} = \frac{2029}{1000}, \quad \frac{507}{2000} < \frac{16}{63}.$$

Apply the preceding largest-good-prime argument to $\tilde{Q}_N$ with $k = 4775/538$, $\rho = 538/5$, $d = 18$ and $\delta_0 = 10^{-6}$. Then $\rho k = 955$ and the exact remaining margin is

$$\eta k - d - 4d\delta_0 k = \frac{1291}{168125} > 0.$$

This proves the retained 955 checkpoint. Its Lean theorem is

`catalan_irrationalityMeasure_955`

in `Parametric/SecondaryTentMain`. The limiting exponent using the full tent area is $244036800/255713 \approx 954.339$.

Alternatively, apply the Chebyshev final prime-selection criterion of Proposition 4.1 to the same $\tilde{Q}_N$. Its degree is still at most $18N$. Since $p \nmid b$ implies $p^2 \nmid b$, the same rational nonvanishing theorem applies. The exact criterion exponent is

$$3 \frac{538}{5} \max \left(\frac{18}{2029/1000}, 4 \right) = \frac{5810400}{2029} < 2864.$$

For $b \geq 2$, weakening the exponent gives $b^{-2864} < |G - a/b|$. This proves the retained 2864 checkpoint, formalized as `catalan_irrationalityMeasure_2864` in `Parametric/SecondaryTentChebyshev`. Only final prime selection uses Chebyshev here: the denominator asymptotics and the secondary coefficient-content rate remain PNT-backed, as the denominator asymptotics already were for the historical 3662 theorem.

7.7. Supplied trials and central-gap interpolation. We now retain the same canonical determinant polynomial, degree bound, integer normalization, own-prime nonvanishing and coefficient-content divisor, and improve its real estimates. This changes neither the local arithmetic nor the polynomial's rational zeros. The preceding 955/2864 proofs remain available as separate checkpoints.

The supplied trial contracts. Set

$$U_* = 2.2563367, \quad c_1 = 4.21, \quad c_2 = 4.29472, \\ g = \frac{401}{2500}, \quad a = \frac{2g}{1+g^2} = \frac{2005000}{6410801}.$$

The first-population trial has 40 finite modes and its saved geometric tails. The second has 160 finite modes and its saved real and complex tails. Their coefficients are exact rationals in **FirstStrongTrial/Data** and **StrongTrial/Data**; the certificate generation records fix their input hashes. The effective first sequence is $p = 2P$, so its norm contribution is $4\|P\|^2 + \|V\|^2/2$; the second uses $2\|P\|^2 + \|V\|^2/2$. These conventions are part of the supplied-field contracts, rather than an identification of the two trial norms. The certified caps are

	weighted norm	row plus column cap	λ
first	.67761	-2.47913	1.68493357
second	.77424975	-2.54538346	0.

The second sequence also satisfies $p_1 \geq .18$. Its marked residual satisfies the stronger, restricted bound

$$(7.8) \quad -\frac{9}{10} \log(1 + x^2) - 2T(P, x) \leq -.0524 \quad (g \leq |x| \leq 1).$$

The near-field bound remains global: for $0 < y < 1$,

$$-\frac{9}{10} \log(1 + y^2) + \frac{16N - 1}{20N} \log y - p_1 y \leq -\frac{3}{4} \quad (N \geq 1).$$

The coefficient $(16N - 1)/(20N) \geq 3/4$, and $\log y < 0$, reduce this to the certified one-variable inequality with $p_1 \geq .18$. All row, column and restricted residual caps cover their entire domains, including endpoint neighborhoods; sampled maxima are not proof inputs.

For the second trial, Descartes' rule gives distinct-root budgets 59, 22, 23, 50 on the four dyadic subintervals of $(-1, 1)$ split at $-1/2$, 0 and $1/2$. The column derivative has budget 87 on $(0, 1)$. The 241 rational brackets have strictly opposite endpoint signs. The intermediate value theorem supplies a distinct root in each bracket; exhausting each budget therefore covers every critical point. Values at $\pm 1/2$ are nonzero, and zero is excluded from the logarithmic domains. Point and derivative bounds on these brackets, the endpoint bounds, and the separate closed-interval residual certificates prove the displayed suprema. These are ordinary Lean proofs of exact rational checks.

For clarity, the second grouped population budget has $0 \leq \tau \leq \alpha \leq 1$ and difference

$$h_0 + (2.6 \log 2 - F + R - c_2)\alpha - 2 \log 2 \alpha^2 \\ + (\log 2 - R - .75)\tau - \log 2 \tau^2,$$

where $h_0 = .77424975 + F + U_* - .7 \log 2$, $F = -2.54538346$ and $R = -.0524$. The constant is negative; the τ coefficient is negative; and completing the α square gives a strictly negative upper. For example, using $.693147180 < \log 2 < .693147181$, the latter square has positive margin at least

$$\frac{4948963197}{46209812000000000000}.$$

The analogous first-population square, with the supplied first trial and $\lambda = 1.68493357$, gives $c_1 = 4.21$. Thus the two actual marked row prices are $84.2Nm$ and $85.8944Nm$. The domain restriction in (7.8) is applied only to selected marked atoms; the integrated free variables retain their full original domains.

Exact nodes and their jointly paid weights. Use the $18N$ roots of

$$P_N(t) = T_{9N}\left(\frac{2t^2 - (1 + a^2)}{1 - a^2}\right).$$

They are distinct, lie in $(-1, -a) \cup (a, 1)$, and have the explicit form

$$t_{j,\pm} = \pm \sqrt{a^2 + (1 - a^2)(1 + s_j)/2}, \quad T_{9N}(s_j) = 0.$$

The physical inverse satisfies $t = 2x/(1 + x^2)$; hence every selected marked point has $|x| > g$. This property survives the second-sheet grouping because its marked coordinates are selections of these points. Polynomial interpolation is exact at these nodes, so the same residual coefficient functional and common-factor cancellation apply.

At a root, the Chebyshev derivative identity gives

$$|P'_N(t_{j,\pm})| \geq \frac{36Na}{1 - a^2}.$$

Fix $r = 1001/1000$ and write $s = |\operatorname{Re} z|$ on $|z| = r$. For $w = (2z^2 - (1 + a^2))/(1 - a^2)$, its exact ellipse half-sum is

$$H(s) = \frac{\sqrt{(r^2 + 1)^2 - 4s^2} + \sqrt{(r^2 + a^2)^2 - 4a^2s^2}}{1 - a^2}.$$

It decreases with s , whereas $|1 - z|^2 \leq r^2 + 1 + 2s$ increases. A contiguous cover of $[0, r]$ by 55 rational intervals, with upward square-root bounds and certified logarithms, proves

$$9 \log(H(s) + \sqrt{H(s)^2 - 1}) + \log(r^2 + 1 + 2s) < 17.913.$$

Since $|z - t_{j,\pm}| \geq 1/1000$ and $(1 - a^2)/(36a) \leq 1$, the same coefficient-functional argument as before yields the actual joint bound

$$|\Phi_{N,k}(C_N L_{j,\pm})| \leq 108000N \exp(17.913N).$$

Here $L_{j,\pm}$ is the cardinal polynomial of the new nodes and C_N is the existing common factor. The residual cap and this increased nodal price are paid together. A smaller field cap without its changed weights would not give this conclusion.

The completed ledger. The larger marked row price is $20c_2 = 85.8944$. Adding the actual weight price gives

$$\rho = 20c_2 + 17.913 = \frac{519037}{5000} = 103.8074.$$

The exact unperturbed gap is $400U_* - 8795/12 - 242 \log 2 > 375279/200000$; dividing by the unchanged content divisor adds $507/2000$. Thus the integer polynomial has the same degree bound $dN = 18N$, with usable decay

$$\eta = \frac{375279}{200000} + \frac{507}{2000} = \frac{425979}{200000} = 2.129895.$$

Apply the largest-good-prime criterion with

$$k = \frac{4390000}{519037}, \quad \delta_0 = 10^{-7}.$$

Its window and integer-payment margins are respectively

$$\frac{2595176739037}{2595185000000} > 0, \quad \eta k - 18 - 72\delta_0 k = \frac{3770721}{259518500} > 0.$$

Since $\rho k = 878$, this proves the 878 bound (the first statement of the previous version of Theorem 1.1). The limiting exponent of this paid ledger is

$$\frac{18\rho}{\eta} = \frac{41522960}{47331} = 877.288880 \dots < 878.$$

For final Chebyshev selection the same integer polynomial and its nonvanishing theorem give

$$3\rho \max\left(\frac{18}{\eta}, 4\right) = \frac{41522960}{15777} < 2632.$$

Weakening this exponent for $b \geq 2$ proves the second statement. Denominator growth and secondary content still use PNT in both results. The two theorems of this subsection are `catalan_irrationalityMeasure_878` and `catalan_irrationalityMeasure_2632` in `Parametric/GapMain`.

8. THE THIRD-BAND CONTENT: LEMMA U3

The secondary tent of Section 7.6 lives at primes $p \approx 6N$. A second residue-kernel family sits at the primes $8N < p < 9N$. Its rank bound was derived by Claude (Anthropic) working with the author; the proof below is the one formalized in Lean. Throughout, the design is $(20, 3, 1, 2, 1)$, so the matrix has size $20N$ and the base matrix is A_N (the filtered matrix at the Catalan parameter 0). Rows are indexed by $0 \leq r < 20N$ with signed offset $m = r - 2N \in [-2N, 18N - 1]$, and the raw columns are $j \in [3N, 24N)$. For a prime p write $h = (p + 1)/2$ and work in $\mathbb{F}_p[X]$.

Theorem 8.1 (Lemma U3). *Let $N \geq 1$ and let p be a prime with $8N < p < 9N$. Then $\text{rank}_{\mathbb{F}_p} \text{red}(p^2 A_N) \leq 20N - (18N - 2p - 1)$, that is, the nullity of the reduced scaled base matrix is at least $18N - 2p - 1$.*

The raw row polynomials of the design are $P_r = (1 - X)^N \text{rev}(T_{|m|})$ and $D_r = \text{sgn}(m)(1 - X)^N \text{rev}(U_{|m|-1})$, where T_k, U_k are Chebyshev polynomials and rev reverses coefficients inside degree $26N - 1$. The Lean-proved digit formula (the existing `rawEntry_primeDigitPair`) says that for a column $j = Jp + \ell$ ($0 \leq \ell < p$) the residue of $p^2 \cdot (\text{raw entry})$ is a fixed linear function of the digits $(XW P_r)[vp + \ell]$ and $D_r[vp + \ell]$, $v \geq 0$, with $W = (1 - X^2)^{(p-1)/2}$,

and the filtered matrix is the raw matrix on $[3N, 24N)$ multiplied by the alternating binomial filter Δ of order N . For a coefficient vector $c \in \mathbb{F}_p^{20N}$ put $P_c = \sum c_r P_r$ and $D_c = \sum c_r D_r$. Every c whose raw response vanishes on $[3N, 24N)$ lies in the left kernel of $\text{red}(p^2 A_N)$; the proof counts linear conditions that force this.

Lemma 8.2 (Chebyshev Frobenius reflection). *Let p be an odd prime. In $\mathbb{F}_p[X]$, for every integer m ,*

$$T_{p-m} = X^p T_m - (X^2 - 1)^h U_{m-1}.$$

Proof. Both sides satisfy the three-term Chebyshev recurrence in m , so it suffices to check $m = 0, 1$. This needs $T_p = X^p$ (Dickson) and $U_{p-1} = (X^2 - 1)^{(p-1)/2}$. The latter follows from the Pell identity $T_n^2 - (X^2 - 1)U_{n-1}^2 = 1$ at $n = p$, $(X^2 - 1)^p = X^{2p} - 1$, cancellation of $X^2 - 1$, and comparison of leading coefficients to fix the sign. For $m = 1$ use $T_{p+1} = XT_p + (X^2 - 1)U_{p-1}$ and $T_{p-1} = 2XT_p - T_{p+1}$. $\square$

Step 1: identity K.. Reflecting the identity of Lemma 8.2 at degree $p + 26N$ gives, for each row,

$$(1 - X^2)^h \text{rev}(U_{m-1}) - X \text{rev}(T_m) = -X^{p+1} \text{rev}(T_{p-m}),$$

and $\text{rev}(T_{p-m})$ is divisible by $X^{26N-1-|p-m|}$ with $|p-m| \leq p + 2N$ because $m \geq -2N$ and $8N < p$. Hence $X^{24N} \mid (1 - X^2)^h D_c - X P_c$ for every c . This is the form in which the formalization proves identity K (the analytic proof by the substitution $x = (z + z^{-1})/2$ and $s^p = x^p - z^{-p}$ is equivalent).

Step 2: the flag subspace. Let V be the set of c with $(1 - X^2)^h \mid P_c$. Since $(1 - X)^N \mid P_r$ always, the condition consists of $h - N$ Taylor-coefficient equations at $X = 1$ and h at $X = -1$, that is $p + 1 - N$ linear equations. For $c \in V$ write $P_c = (1 - X^2)^h Y$ and $\tilde{Q} = XY$. Because $(1 - X^2)^p = 1 - X^{2p}$ in characteristic p ,

$$XW P_c = (1 - X^{2p}) \tilde{Q}.$$

Moreover P_c has no coefficients below X^{8N} and none from X^{27N} on, so $\tilde{Q}$ is supported in degrees $[8N + 1, 27N - p - 1]$. Dividing the congruence of Step 1 by the unit $(1 - X^2)^h$ modulo X^{24N} gives

$$D_c[k] = \tilde{Q}[k] \quad (k < 24N).$$

Step 3: rank-one columns. Let $27N - 3p \leq \ell < 24N - 2p$. Then the digits of $XW P_c$ at residue ℓ are $(0, q, 0, -q, 0, \dots)$ with $q = \tilde{Q}[p + \ell]$ (the digits of $\tilde{Q}$ at this residue vanish except for $v = 1$, by the support, and $X^{2p}\tilde{Q}$ shifts the digit sequence by two places), and the digits of D_c are $(0, q, 0, 0, \dots)$ (by identity K for $v \leq 2$ and degree for $v \geq 3$, since $3p + \ell \geq 27N$). Therefore the response at every column $j = Jp + \ell$ equals $q \kappa_J$ with κ_J independent of c : the columns of this residue class have rank at most one on V .

Step 4: counting. For $p \in (8N, 9N)$ the columns $j \in [3N, 24N)$ with residue ℓ are: $J \in \{1, 2\}$ if $\ell < 27N - 3p$ (two columns), $J \in \{0, 1, 2\}$ for $27N - 3p \leq \ell < 24N - 2p$ (rank one), and $J \in \{0, 1\}$ for $24N - 2p \leq \ell < p$ (two columns). The number of linear functionals needed to kill the response on V is therefore

$$2(27N - 3p) + (p - 3N) + 2(3p - 24N) = 3N + p.$$

Adding the $p + 1 - N$ flag equations, the kernel of all $2p + 2N + 1$ conditions has dimension at least $20N - (2p + 2N + 1) = 18N - 2p - 1$ and lies in the left kernel of $\text{red}(p^2 A_N)$. This proves Theorem 8.1. The formal statement is `u3_canonical_rank`; the exact numerical checks (identity K, $\dim V = 21N - p - 1$, and rank $3N + p$ on V at $(N, p) = (7, 59), (7, 61), (13, 107), (13, 113), (19, 157), (19, 167)$) agree with the proof.

From nullity to coefficient content. For $8N < p < 9N$ and $N \geq 10^{12}$ the actual normalizer has p -exponent

$$40N - (26N - 3p)^+,$$

because the odd-loss function equals 40 on $[0, 10]$ and the prime gain is $26 - 3x$ on $[8, 26/3]$ and 0 beyond. Rescaling the determinant polynomial by $X \mapsto X/p^2$ then shows that $p^{2i - (26N - 3p)^+}$ divides the i -th coefficient of Q_N . A nullity of at least $18N - 2p - 1$ gives $v_p \det(A_N + zB_N) \geq -(22N + 2p + 1)$ for rational z with $p \nmid \text{den } z$, hence $p^e \mid Q_N(z)$ for every integer z with

$$e = e_N(p) = \min(p - 8N - 1, 18N - 2p - 1)^+,$$

which equals $18N - 2p - 1 - \max(26N - 3p, 0)$. Since $e \leq 2p - (26N - 3p)^+$ for the tail $i \geq p$, the integer-valued criterion with a p^e -divisible tail gives $p^e \mid \text{coeff}_i(Q_N)$ for all i (formalized as `twentyMeasure IntPoly_third_coeff_dvd`). The exponent satisfies $N\theta(p/N) - 1 \leq e \leq N\theta(p/N)$ for the tent $\theta(x) = \max(0, \min(x - 8, 18 - 2x))$ of area $1/3$ on $[8, 9]$. These primes exceed $2N$ and are disjoint from the primes of Section 7.6, so the combined content $c_N = \prod p^{e_N(p)} \cdot \prod_{52N/9 < p < 44N/7} p^{e'_N(p)}$ keeps all recycled prime squares. Weighted PNT summation gives $\log c_N^{\text{new}} \geq (\frac{1}{3} - o(1))N^2 \geq \frac{3331}{10000}N^2$ eventually, so the total content rate used is

$$\chi = \frac{507}{2000} + \frac{3331}{10000}.$$

The 759 ledger. With the unchanged real-place gap $375279/200000$ the usable decay is

$$\eta = \frac{375279}{200000} + \frac{507}{2000} + \frac{3331}{10000} = \frac{492599}{200000} = 2.462995.$$

The perturbation exponent and the degree are unchanged: $\rho = 519037/5000$, $d = 18$. Apply the largest-good-prime criterion with

$$k = \frac{3795000}{519037}, \quad \delta_0 = 10^{-7}.$$

Then $\rho k = 759$, $k > 2$, $\eta < 36$, and the window and payment margins are

$$\frac{2595177929037}{2595185000000} > 0, \quad \eta k - 18 - 72\delta_0 k = \frac{4372701}{519037000} > 0.$$

This proves $|G - a/b| > b^{-759}$ for all large b . The limiting exponent of the paid ledger is $18\rho/\eta = 373706640/492599 = 758.6427\dots < 759$. For final Chebyshev selection,

$$3\rho \max\left(\frac{18}{\eta}, 4\right) = \frac{1121119920}{492599} = 2275.928\dots < 2276.$$

The new main theorems are `catalan_irrationalityMeasure_759` and `catalan_irrationalityMeasure_2276` in `Parametric/U3Main`; the earlier 878 and 2632 bounds remain proved.

Remark 8.3. The increment $3331/10000$ is the part of the area $1/3$ of the third-band tent that is banked; the unbanked remainder $1/3 - 3331/10000$ is below 10^{-3} . Reaching 758 would require a further content window of area about 0.0019 beyond Theorem 8.1.

9. THE SIXTH-WINDOW CONTENT: LEMMA U6

The third-band content of Section 8 lives at the primes $8N < p < 9N$. A further residue-kernel family sits at the primes $4N < p < 9N/2$, below every earlier window. Unlike Lemma U3, whose linear conditions are imposed on the rows of the base matrix through a flag subspace, Lemma U6 is a statement about a block of *columns*. Its derivation and formalization are due to Claude (Anthropic) working with the author; the proof below is the one formalized in Lean. We keep the design $(20, 3, 1, 2, 1)$, the notation of Section 8, and work in $\mathbb{F}_p[X]$.

Theorem 9.1 (Lemma U6). *Let $N \geq 1$ and let p be a prime with $4N < p$ and $2p < 9N$. Then*

$$\text{rank}_{\mathbb{F}_p} \text{red}(p^2 A_N) \leq 20N - \frac{p - 4N - 1}{2},$$

that is, the nullity of the reduced scaled base matrix is at least $(p - 4N - 1)/2$.

Put $d = p - 4N$, which is odd; the hypothesis $2p < 9N$ is equivalent to $2d < N$. Put $e = (d + 1)/2$, so that $2N + \frac{p-1}{2} + e = p$, and $W = (1 - X^2)^{(p-1)/2}$. For a coefficient vector $c \in \mathbb{F}_p^{20N}$ let $S_c = \sum_r c_r \text{rev}(T_{m_r})$ and $U_c = \sum_r c_r \text{rev}(U_{m_r-1})$ with $m_r = r - 2N$, the reversal being taken inside degree $26N - 1$; thus $P_c = (1 - X)^N S_c$ and $D_c = (1 - X)^N U_c$ in the notation of Section 8. Define

$$\hat{Q}_c = (X - 1)^N XW (1 - X)^N S_c, \quad \hat{D}_c = (X - 1)^N (1 - X)^N U_c.$$

Step 1: identity K6. Over any commutative ring,

$$U_{m-1}T_n - T_m U_{n-1} = U_{m-n-1},$$

because both sides satisfy the Chebyshev recurrence in n and agree for $n = 0, 1$. In characteristic p take $m = 2p$ and use $T_p = X^p$, $U_{p-1} = (X^2 - 1)^{(p-1)/2}$ (Lemma 8.2 and its proof), $T_{2p} = 2X^{2p} - 1$ and $U_{2p-1} = 2T_p U_{p-1}$:

$$2X^p(X^2 - 1)^{(p-1)/2} T_n - (2X^{2p} - 1) U_{n-1} = U_{2p-n-1}.$$

Reflecting at degree $2p + 26N - 1$ gives, for each row with $m_r \geq -2N$,

$$2XW \operatorname{rev}(T_{m_r}) - (2 - X^{2p}) \operatorname{rev}(U_{m_r-1}) = \operatorname{rev}(U_{2p-m_r-1}),$$

and the right side is divisible by X^{24N} as soon as $p \geq 4N$ (the degree of U_{2p-m_r-1} is then at most $2p + 2N - 1$; for $p < 4N$ the identity fails). Summing over rows with weights c_r and multiplying by $(X - 1)^N(1 - X)^N$ gives, for every c and every $i < 24N$,

$$(9.1) \quad 2\hat{Q}_c[i] = 2\hat{D}_c[i] - \hat{D}_c[i - 2p],$$

where a coefficient with negative index is 0.

Step 2: a block of columns. Let $M = [p - 7d, p - 6d)$ and, for $J \in \{1, \dots, 5\}$ and $\mu \in M$, put $k(J, \mu) = Jp + \mu - 4N$. Writing $\mu = p - 7d + t$ with $0 \leq t < d$ one finds $k = 4NJ + (J - 6)d + t$, so these are $5d$ distinct columns of A_N , inside $[0, 20N)$. The filtered column k is the alternating binomial filter of order N applied to the raw columns $3N + k + v = Jp + (\mu - N + v)$, $0 \leq v \leq N$. Since $\mu \geq p - 7d = 4N - 6d > N$ and $\mu < p$ (this is the only use of $2d < N$), all of these raw columns have the same prime digit J and residues $\mu - N + v \in [0, p)$. The Lean-proved digit formula (`rawEntry_primeDigitPair`) expresses the residue of $p^2 \cdot (\text{raw entry})$ through the digits $(XWP_c)[up + \ell]$ and $D_c[up + \ell]$, and the alternating binomial sum $\sum_v (-1)^v \binom{N}{v} a_{m+v}$ is the coefficient of X^{m+N} in $(X - 1)^N \sum_i a_i X^i$. Hence the response of the block column $k(J, \mu)$ on c is

$$\sum_u \alpha_{J,u} \hat{Q}_c[up + \mu] - \frac{3}{2} \sum_u \beta_{J,u} \hat{D}_c[up + \mu],$$

with constants $\alpha_{J,u}, \beta_{J,u}$ independent of c and μ (their values are never used). Step 3: supports and elimination of $\hat{D}$. S_c is divisible by X^{8N} and U_c by X^{8N+1} , so $\hat{Q}_c$ and $\hat{D}_c$ have no coefficients up to X^{8N} ; and $\deg \hat{Q}_c \leq 28N + p - 1$, $\deg \hat{D}_c \leq 28N - 1$. For $\mu \in M$ we have $p + \mu < 8N$, $6p + \mu \geq 28N$ and $7p + \mu > 28N + p - 1$, so only the digits $u = 2, \dots, 6$ of $\hat{Q}_c$ and $u = 2, \dots, 5$ of $\hat{D}_c$ can be nonzero. For $u = 2, \dots, 5$ we have $up + \mu < 24N$, so (9.1) reads $\hat{D}_c[up + \mu] = \hat{Q}_c[up + \mu] + \frac{1}{2} \hat{D}_c[(u - 2)p + \mu]$, and $\hat{D}_c[(u - 2)p + \mu]$ vanishes for $u \leq 3$ by order. Thus: *if $\hat{Q}_c[up + \mu] = 0$ for $2 \leq u \leq 6$ and $\mu \in M$, then every block column vanishes on c .*

Step 4: the Frobenius class-sum law. Since $(1 - X)^p = 1 - X^p$ in characteristic p and $2N + \frac{p-1}{2} + e = p$,

$$(1 - X)^e \widehat{Q}_c = (1 - X^p) Y_c, \quad Y_c = (-1)^N X(1 + X)^{(p-1)/2} S_c.$$

For $i \geq p$ this gives $[X^i]((X - 1)^e \widehat{Q}_c) = (-1)^e (Y_c[i] - Y_c[i - p])$, and summing over $i = up + \mu$, $u = 2, \dots, 6$, telescopes to $(-1)^e (Y_c[6p + \mu] - Y_c[p + \mu])$. Now $Y_c[p + \mu] = 0$ because Y_c has no coefficients up to X^{8N} , and $Y_c[6p + \mu] = 0$ whenever $6p + \mu > \deg Y_c$, i.e. for $\mu \geq p - 7d + e$. Expanding the left side by the same binomial identity, every $\mu = p - 7d + t$ with $e \leq t < d$ yields the relation

$$\sum_{u=2}^6 \sum_{v=0}^e (-1)^v \binom{e}{v} \widehat{Q}_c[up + (p - 7d) + (t - e + v)] = 0.$$

There are $d - e = (d - 1)/2$ such relations. The term $v = e$ contributes $(-1)^e \sum_{u=2}^6 \widehat{Q}_c[up + \mu]$ and all other terms involve strictly smaller offsets $t - e + v < t$, so the relations are triangular.

Step 5: counting. Impose on c : (i) the filtered column k vanishes for each of the $20N - 5d$ columns outside the block; (ii) $\widehat{Q}_c[up + \mu] = 0$ for $2 \leq u \leq 5$ and $\mu \in M$ ($4d$ conditions); (iii) $\widehat{Q}_c[6p + \mu] = 0$ for the first e values of $\mu \in M$ (e conditions). By strong induction on t the relations of Step 4 force $\widehat{Q}_c[6p + (p - 7d + t)] = 0$ for $t \geq e$ as well, hence all digits $2 \leq u \leq 6$ vanish, and by Step 3 so does every block column. Thus the kernel of the $20N - 5d + 4d + e = 20N - (d - 1)/2$ conditions lies in the left kernel of $\text{red}(p^2 A_N)$, which therefore has dimension at least $(d - 1)/2 = (p - 4N - 1)/2$. This proves Theorem 9.1; the formal statement is `u6_canonical_rank`. In exact modular rank computations made while deriving the lemma, the block of $5d$ columns has rank at most $4d + e$, with nullity exactly $(d - 1)/2$, at every prime of the window for $5 \leq N \leq 41$ and for $N \in \{53, 61, 97, 151\}$; identity K6 holds exactly when $p > 4N$ and fails for $p < 4N$.

From the rank bound to coefficient content. For $4N < p < 9N/2$ and $N \geq 10^{12}$ the odd-loss function equals 40 and the prime gain vanishes (the prime gain is 0 for $p/N \leq 22/3$), so the canonical normalizer has the full p -exponent $40N$ and no rescaling is needed. A rank deficit $e_N(p) = \lfloor (p - 4N - 1)/2 \rfloor \leq N$ gives $p^{e_N(p)} \mid Q_N(z)$ for every integer z , and together with the divisibility $p^{2i} \mid \text{coeff}_i(Q_N)$ of the double-pole weights it gives $p^{e_N(p)} \mid \text{coeff}_i(Q_N)$ for all i (formalized as `twentyMeasureIntPoly_sixth_coeff_dvd`). The exponent satisfies $N\vartheta(p/N) - 1 \leq e_N(p) \leq N\vartheta(p/N)$ for $\vartheta(x) = \max(x - 4, 0)/2$, whose area on $[4, 9/2]$ is $1/16$. If $2p = 9N$ then $3 \mid p$, so no prime has $2p = 9N$ and the window is exactly the set of primes in $(4N, \lfloor 9N/2 \rfloor]$. Weighted PNT summation gives $\log c_N^{(6)} \geq (\frac{1}{16} - o(1))N^2 \geq \frac{624}{10000}N^2$ eventually. The primes of this window exceed $2N$ and are disjoint from those of Section 7.6 and Section 8, so the three contents multiply and keep all recycled prime squares.

The total content rate is

$$\chi = \frac{507}{2000} + \frac{3331}{10000} + \frac{624}{10000}.$$

The 740 ledger. With the unchanged real-place gap $375279/200000$ the usable decay is

$$\eta = \frac{375279}{200000} + \frac{507}{2000} + \frac{3331}{10000} + \frac{624}{10000} = \frac{505079}{200000} = 2.525395.$$

The perturbation exponent and the degree are unchanged: $\rho = 519037/5000$, degree 18. Apply the largest-good-prime criterion with

$$k = \frac{3700000}{519037}, \quad \delta_0 = 10^{-7}.$$

Then $\rho k = 740$, $k > 2$, $\eta < 36$, and the window and payment margins are

$$\frac{2595178119037}{2595185000000} > 0, \quad \eta k - 18 - 72\delta_0 k = \frac{63443}{25951850} > 0.$$

This proves $|G - a/b| > b^{-740}$ for all large b . The limiting exponent of the paid ledger is $18\rho/\eta = 373706640/505079 = 739.8974 \dots < 740$. For final Chebyshev selection,

$$3\rho \max\left(\frac{18}{\eta}, 4\right) = \frac{1121119920}{505079} = 2219.692 \dots < 2220.$$

The new main theorems are `catalan_irrationalityMeasure_740` and `catalan_irrationalityMeasure_2220` in `Parametric/U6Main`; the earlier 759, 2276, 878 and 2632 bounds remain proved.

Remark 9.2. The increment $624/10000$ is the part of the area $1/16$ of the profile ϑ that is banked; the unbanked remainder is $1/10000$. The cutoff $9N/2$ is structural: the filter window of a block column leaves the digit J once $2d > N$. Reaching 739 would require the usable decay to exceed about 2.5285, that is a further content window of area about 0.0031 beyond Theorem 9.1.

10. FORMALIZATION

All results of this paper are formalized in Lean 4 [7] with Mathlib [6], on top of OpenAI's Lean formalization of the irrationality of G [8]. The development is available at

<https://github.com/slitsey/catalan-irrationality-measure>

and combines analytic and arithmetic proofs with generated rational data and kernel checks for Proposition 5.11 and Section 5.8. The uniform sparse and combined prime-row dependencies and the weighted-PNT denominator saving are also proved in Lean; numerical rank scans alone are not used as proof. The historical sharpened OpenAI certificate is retained separately. It uses OpenAI's definitions directly: Catalan's constant is `catalan`, defined as the series $\sum_j (-1)^j / (2j+1)^2$; the matrix (2.2) and its rational form are OpenAI's `determinant` and `determinantRat`. The current main statements

are in `CatalanMeasure.Parametric`; the listed historical lemmas remain in `OAI.InternalCatalan`:

- `catalan_irrationalityMeasure_740`: for all sufficiently large $b \in \mathbb{N}$ and all $a \in \mathbb{Z}$, $b^{-740} < |G - a/b|$ (Theorem 1.1);
- `catalan_irrationalityMeasure_2220`: the same with exponent 2 220, using Chebyshev for final prime selection;
- `u6_canonical_rank`: Theorem 9.1;
- `catalan_irrationalityMeasure_759`: the earlier bound with exponent 759;
- `catalan_irrationalityMeasure_2276`: the earlier bound with exponent 2 276, using Chebyshev for final prime selection;
- `u3_canonical_rank`: Theorem 8.1;
- `catalan_irrationalityMeasure_878`: the earlier bound with exponent 878;
- `catalan_irrationalityMeasure_2632`: the earlier bound with exponent 2 632, using Chebyshev for final prime selection;
- `catalan_irrationalityMeasure_955`: the retained 955 checkpoint;
- `catalan_irrationalityMeasure_2864`: the same with exponent 2 864, using Chebyshev for final prime selection;
- `catalan_irrationalityMeasure_3662`: the retained historical statement with exponent 3 662;
- `eventually_realDeterminant_abs_upper_dual`: Proposition 5.11;
- `separate_perturbation`: Theorem 5.1, with $\rho = 366.3$;
- `catalan_irrationalityMeasure` and `catalan_irrationalityMeasure_pnt`: the general exponents of Proposition 4.1.

The solution has no `sorry`, and the command `#print axioms` reports only the three standard axioms `propext`, `Classical.choice` and `Quot.sound`. The prime number theorem enters the denominator asymptotics and secondary prime-content sum, and also the final prime selection for 740, 759 and 878. The 2220, 2276 and 2632 corollaries use Chebyshev only in that final selection step. The 955/2864 checkpoint has the same PNT scope. The formal source for PNT is the PNT+ project [10], through OpenAI's workspace and compatibility additions.

The following file groups implement the indicated parts of the proof.

`DeltaPoly, Degree`

$\Delta_N(X)$; Lemma 3.1.

`Denominators, IntegerValued, Integrality`

Proposition 3.2; Lemma 3.6; Corollary 3.8.

`SparseDenominators, DenseDenominators, DenseGainPNT, DenseMain`

Propositions 3.3–3.5; the earlier 1981 bound.

`Nonvanishing`

Proposition 2.1(d) for $p \nmid \text{den } z$.

PrimeWindow, Criterion, PNTWindow, Main

Proposition 4.1 and the reduction to Theorem 5.1.

Quadrature, ChebyshevNodes, QuadCircle

Lemma 5.2.

DetExpansion

Lemma 5.4.

MixedAndreief

Lemma 5.5.

VandermondeProduct

Lemma 5.6.

PinMajorant, FixedNodes, FixedNodeMajorant

Proposition 5.8, (5.6), and (5.5).

PinDampedDual, PinSupDual

Proposition 5.10.

Cert/ListPoly, Cert/LogApprox, Cert/FieldEval, Cert/FieldDeriv,
Cert/Cover, Cert/Norm

the general lemmas behind the checks of Proposition 5.11.

Dual/*, DualTwo

the certificate data, the kernel checks, Proposition 5.11.

SharpBarrier

the remark after Proposition 5.11.

WeightedEnergy, WeightedBounds

Section 5.8.

SeparateFixedNodes, SeparateBounds, SeparateLemma6

Theorem 5.1.

SquareNonvanishing, SquarePNTWindow, SquareMain

Proposition 3.7, strengthened Proposition 4.1.

Parametric/U6Main, U6Audit

Section 9; current 740/2220 bounds.

U6Chebyshev, U6Identity, U6Block, U6Digits, U6Rank

Identity K6, the Frobenius class-sum law, the block column responses and
Theorem 9.1.

U6Content, U6PrimeGrowth

Rank to coefficient content for $4N < p < 9N/2$ and the area-1/16 prime
sum.

Parametric/U3Main, U3Audit

Section 8; the 759/2276 bounds.

U3ChebyshevFrobenius, U3RowIdentity, U3Combination, U3Digits,
U3Rank

Lemma 8.2, identity K, the flag subspace, the digit response and Theorem 8.1.

U3Content, U3PrimeGrowth

Nullity to coefficient content for $8N < p < 9N$ and the area-1/3 prime sum.

Parametric/GapMain, GapFinalTransfer

Section 7.7; the 878/2632 bounds.

FirstStrongTrial/*, StrongTrial/*

Supplied exact trials, point values and root-coverage certificates.

GapNodal, GapCircle/*, GapWeights

Central-gap quadrature and actual joint weight bounds.

Cert/DescartesBudget, Cert/IntegerHorner

Distinct-root budgets and checked bracket endpoints.

SecondaryTentMain, SecondaryTentMeasureTransfer

Section 7.6; retained 955 bound.

SecondaryTentChebyshev

Alternate final prime selection; 2864 bound.

SecondaryTentParameters*, SecondaryTentRow*, SecondaryTentSource*

Actual uniform row kernel, source compatibility and prime-content tent.

TwentyCostMain, TwentySharperMain, TwentyIntegerPolynomial

Retained 1091/3662 bounds and canonical integer polynomial.

TwentyDenominators, TwentyTwoAdicNorm, TwentyCombinedOddLoss

Actual local arithmetic and common-denominator growth.

TwentyTrial/*, TwentyFirstCost, TwentyCostPerturbation

Certified population and marked fields, mixed integration and $\rho = 538/5$.

The current proofs were checked with Lean v4.35.0-rc2 and the matching Mathlib release. OpenAI's original repository at commit fd4aeeb pins Lean v4.34.1 and Mathlib commit d13f23b. The parts of that formalization used here were converted to Lean's module system and ported to the current toolchain; the repository's README and provenance documents describe the changes.

DISCLOSURE OF AI USE

The original development was produced by the AI system Claude Opus 5.5 (Anthropic): the mathematics, including the perturbation estimate and all proofs; the Lean formalization; the numerical computations and the certificate generator; the literature search; and the text of this paper. Claude worked in a series of conversations at the request of the author, who posed the problem, directed the work, and is responsible for this submission. The separate fixed-node decay bounds and weighted Vandermonde refinement,

endpoint certificates, quadratic case refinement, dependency refactor, integer certificate computations and their soundness proofs, square-denominator nonvanishing, the prime-window refinement, uniform sparse and combined prime-row dependencies and denominator saving, the twenty-row parameter family, certified free and fixed population fields, joint atom integration, and corresponding revisions to this paper were developed by OpenAI Codex running GPT-6.1 Sol (OpenAI) at the author’s request. The full-interval marked-sheet assembly, critical-radius perturbation, coefficient recycling, largest-good-prime criterion and the uniform secondary coefficient-content tent with its complete source and row formalization were developed with Codex at the author’s request. The subsequent supplied-trial refinements, central-gap interpolation, actual circle weights, reusable root-count coverage certificates and the 878/2632 assembly were also developed with Codex at the author’s request. Lemma U3 (Section 8): its derivation, including identity K and the dimension count, its Lean formalization, the content transfer to primes in $(8N, 9N)$ and the 759/2276 assembly were carried out by Claude (Anthropic) working with Sean Litsey, building on the Codex-developed pipeline above. Lemma U6 (Section 9): its derivation, including identity K6 and the class-sum law, its Lean formalization, the content transfer to primes in $(4N, 9N/2)$ and the 740/2220 assembly were likewise carried out by Claude (Anthropic) working with Sean Litsey. The underlying irrationality proof and its formalization are OpenAI’s [8]. The formal verification in Lean is intended as an independent check of every step of the argument above.

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Email address: `sean.litsey.math@gmail.com`