

AN IRRATIONALITY MEASURE FOR CATALAN'S CONSTANT

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ABSTRACT. Building on OpenAI's recent proof that Catalan's constant $G = \sum_{j \geq 0} (-1)^j / (2j + 1)^2$ is irrational, we prove that

$$\left| G - \frac{a}{b} \right| > b^{-8.127}$$

for all integers a and all sufficiently large integers b . Thus the irrationality exponent of G is at most 8.127; without the prime number theorem in the final step the same argument gives 23.168. The previous best bound, due to Jha, is 10^8 . The new ingredient is a perturbation estimate for OpenAI's determinant, viewed as a polynomial $\Delta_N(X)$ in the constant: on the interval $|x - G| \leq e^{-404.25N}$ the polynomial moves by no more than OpenAI's bound for $|\Delta_N(G)|$. It is proved by a quadrature formula for the coefficient of the constant, an Andréief identity with fixed columns, a generalized Vandermonde bound, and OpenAI's energy method rerun for configurations with fewer points. We also give a new certificate, with re-optimized trial sequences, for one of the two cases of OpenAI's logarithmic-energy dual; it raises the margin of the argument by a factor of almost eight. Every step, including the final theorem, is formalized in Lean 4 on top of OpenAI's formalization, without `sorry` and with only the standard axioms.

1. INTRODUCTION

OpenAI [7] recently proved that Catalan's constant

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\,55941\dots$$

is irrational, and formalized the proof in Lean. (An earlier claimed proof [9] was found to contain a gap [10].) The argument is by contradiction: under the hypothesis $G \in \mathbb{Q}$ an explicit determinant Δ_N of size $n = 48N$ is a nonzero rational number whose denominator is controlled prime by prime, so that $\log |\Delta_N| \geq -(A + o(1))n^2$; on the other hand a real-variable estimate gives $\log |\Delta_N| \leq -(U - o(1))n^2$ with $U > A$. The margin is small: $U - A \approx 1.3 \cdot 10^{-4}$. As OpenAI remarks, the argument proves qualitative

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irrationality; an irrationality measure requires additional control of rational approximations.

Recall that the *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the μ such that $|\xi - a/b| > b^{-\mu}$ for all integers a and all sufficiently large b .

Theorem 1.1. *For every integer a and every sufficiently large integer b ,*

$$\left| G - \frac{a}{b} \right| > b^{-8127}.$$

In particular $\mu(G) \leq 8127$. If in the last step the prime number theorem is replaced by Chebyshev's bounds, the same argument gives $|G - a/b| > b^{-23168}$.

Theorem 1.1, together with every result in this paper, has been verified in the Lean 4 proof assistant on top of OpenAI's formalization, with no unproved steps and only the standard axioms; see Section 7. The formalization is available at <https://github.com/slitsey/catalan-irrationality-measure>. The use of AI in this work is disclosed at the end of the paper.

1.1. Previous results. The first finite irrationality measure for G was obtained by Jha [4], who proved $\mu(G) \leq 10^8$, also with a Lean formalization built on OpenAI's. The present work was carried out independently; we found Jha's preprint in a literature search after our proof was complete. Both arguments have the shape that OpenAI's proof suggests: one replaces the determinant by a polynomial in the constant, clears denominators, and needs a perturbation estimate near G . They differ in the perturbation estimate. Jha bounds the complementary minors of OpenAI's matrix in every dimension, by an energy estimate with node padding, with a codimension loss of the form $\exp(1000n(n-m))$. Here the perturbation of the matrix is written, by a quadrature formula, as a combination of point evaluations of OpenAI's row functions, and each perturbed column costs a factor $e^{\rho N}$ with $\rho = 404.25$ (Theorem 5.1). Our exponent is smaller than Jha's by a factor of about 12 000; the difference comes from this step and from the new energy certificate described below. For other recent work on rational approximations to G see Eskandari [2]; for earlier work on the arithmetic of G and related values see [8, 11]; for an effective approach to linear independence of related L -values see Calegari, Dimitrov and Tang [1].

1.2. Outline of the proof. OpenAI's matrix has entries in $\mathbb{Q} + \mathbb{Q}G$ (Proposition 2.1(a)). Replacing G by a variable gives a polynomial $\Delta_N(X) \in \mathbb{Q}[X]$, and we proceed in four steps.

- (i) *Arithmetic* (Section 3). $\deg \Delta_N \leq 44N$ (Lemma 3.1). OpenAI's local estimates, applied at integer points, give a common denominator D_N with $\log D_N \leq (A + o(1))n^2$ (Proposition 3.2). So $Q_N = (44N)! D_N \Delta_N(X)$ has integer coefficients. OpenAI's nonvanishing argument gives $Q_p(a/b) \neq 0$ for primes $p \nmid b$ (Proposition 2.1(d)).

- (ii) *Perturbation* (Section 5). For $|x - G| \leq e^{-404.25N}$, $|\Delta_N(x) - \Delta_N(G)| \leq e^{-(U-\varepsilon)n^2}$ (Theorem 5.1). Together with OpenAI's real bound this makes $|Q_p(x)| \leq e^{-\eta p^2}$ on that interval, for any $\eta < 2304\gamma_0$, where $\gamma_0 = U - A$.
- (iii) *Criterion* (Section 4). A good approximation a/b of G lies in the small interval for a prime $p \nmid b$ of size about $\log b$, which contradicts $|Q_p(a/b)| \geq b^{-44p}$.
- (iv) *The real-place constant* (Section 5.7). OpenAI's real-place bound comes from a dual of a logarithmic energy, with certified trial sequences, in each of two cases. For the second case we construct new trial sequences and certify them with a computation checked by Lean's kernel. This improves the real-place constant from $2.290939875 - \frac{1}{2} \log 2$ to $U = 2.29180925 - \frac{1}{2} \log 2$ (Proposition 5.11) and raises γ_0 from $1.30 \cdot 10^{-4}$ to $1.00 \cdot 10^{-3}$.
- (v) *Numerics* (Section 6). The exponent is about $\rho \cdot 44/(2304\gamma_0) \approx 404.25 \cdot 19.1$, up to the factor coming from the size of the prime window.

1.3. Notation and conventions. We use the notation of OpenAI's paper [7] and refer to it for the constructions; we restate only what we use. Citations of the form [7, Prop. 2.1] point to numbered statements, sections and equations of OpenAI's paper. We write $\log$ for the natural logarithm.

2. OPENAI'S CONSTRUCTION AND THE FACTS WE USE

For a positive integer N put [7, (2)]

$$\begin{aligned} n &= 48N, & b &= 7N, & q &= g = 4N, & h &= 2N, \\ L &= 59N, & C &= 63N, & H &= 65N, & A &= 19N, \end{aligned}$$

and $D_* = n - 1 - 2g = 40N - 1$. (The letter A for $19N$ is OpenAI's; the constant A below is different.) The row polynomials $P_r, D_r \in \mathbb{Z}[t]$ ($0 \leq r < n$) are those of [7, (3)]; we only need that

$$(2.1) \quad P_r(t) = (1 - t)^h t^{C-1} T_{|r-g|}(1/t), \quad \deg P_r < H,$$

where T_d is the Chebyshev polynomial. In particular P_r depends only on $|r - g|$. Let $f(t) = \sqrt{1 - t^2}$, $d\mu(t) = |t| dt/f(t)$ on $(-1, 1)$ (a measure of mass 2),

$$A_r(t) = P_r(t) - \frac{3}{2} \mathbf{1}_{\{t>0\}} \frac{f(t)}{t} D_r(t), \quad \psi_k(s) = s^{b+k} (1 - s)^q,$$

and let $\mathcal{A}_N$ be the real $n \times n$ matrix

$$(2.2) \quad \mathcal{A}_N(r, k) = \int_{-1}^1 \int_0^1 A_r(t) \frac{\psi_k(s)}{1 - ts} ds d\mu(t) \quad (0 \leq r, k < n),$$

so that OpenAI's determinant is $\det \mathcal{A}_N$ [7, (7) and §5.1]. Put $c_l = 4^{-l} \binom{2l}{l}$ for integers $l \geq 0$ and $c_z = 0$ otherwise.

Proposition 2.1 (OpenAI).

(a) (Rational structure [7, Prop. 2.1 and (20)].) *There are rational matrices $\mathcal{A}_N^0, \mathcal{L}_N \in \mathbb{Q}^{n \times n}$ with $\mathcal{A}_N = \mathcal{A}_N^0 + G \mathcal{L}_N$ and*

$$(2.3) \quad \mathcal{L}_N(r, k) = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_i [t^i] P_r \cdot 4^{c_{(i-b-k-v)/2}}.$$

(b) (Real place [7, Prop. 7.1].) *With $U_{\text{OAI}} = 2.290939875 - \frac{1}{2} \log 2$,*

$$\limsup_{N \rightarrow \infty} \frac{\log |\det \mathcal{A}_N|}{n^2} \leq -U_{\text{OAI}}.$$

(c) (Local estimates [7, Prop. 2.2, §3, Prop. 3.4].) *For every rational z OpenAI bounds $v_\ell(\det(\mathcal{A}_N^0 + z \mathcal{L}_N))$ from below at every prime ℓ , by quantities depending on ℓ , N and the denominator of z . The sum of these bounds, weighted by $\log \ell$, is at most $(A + o(1))n^2$, where*

$$A = \frac{8609}{4608} + \frac{505}{4608} \log 2.$$

(d) (Nonvanishing [7, Prop. 4.1].) *There is an absolute constant $P_* \geq 260$ such that, for every prime $p > P_*$ and every $z \in \mathbb{Q}$ whose denominator is not divisible by p , $v_p(\det(\mathcal{A}_p^0 + z \mathcal{L}_p)) = -96p$; in particular this determinant is nonzero.*

Two remarks on (c) and (d). OpenAI states them under the hypothesis $G = z \in \mathbb{Q}$, but the proofs only use the arithmetic of the rational matrix $\mathcal{A}_N^0 + z \mathcal{L}_N$. In (d) OpenAI assumes $p > \max(P_*, \text{den } z)$, but the proof only uses $p \nmid \text{den } z$; this uniformity in z is essential for an irrationality measure, and is how we use the result.

We also use OpenAI's real-place machinery: the double integral, the two sheet estimates [7, Props. 5.2, 5.3], and the certified energy dual [7, Prop. 6.1, §7]. They are recalled where needed in Section 5.

OpenAI's irrationality proof is the inequality $U_{\text{OAI}} > A$, and $U_{\text{OAI}} - A = 1.3031 \dots \cdot 10^{-4}$. With a new certificate for one case of OpenAI's energy dual, Proposition 5.11 below shows that (b) holds with

$$U = 2.29180925 - \frac{1}{2} \log 2$$

in place of U_{OAI} . We use this constant throughout.

Definition 2.2. $\Delta_N(X) = \det(\mathcal{A}_N^0 + X \mathcal{L}_N) \in \mathbb{Q}[X]$. Thus $\Delta_N(G) = \det \mathcal{A}_N$, and $\Delta_N(z)$ for $z \in \mathbb{Q}$ is the rational determinant OpenAI studies under the hypothesis $G = z$. We write $\gamma_0 = U - A$.

Numerically $U = 1.9452356 \dots$, $A = 1.9442359 \dots$ and

$$(2.4) \quad 9.996945 \cdot 10^{-4} < \gamma_0 = 9.996948 \dots \cdot 10^{-4}.$$

3. ARITHMETIC OF $\Delta_N(X)$

Lemma 3.1 (Degree). $\deg \Delta_N \leq 44N$.

Proof. By (2.3) row r of $\mathcal{L}_N$ depends only on P_r , hence only on $|r - g|$ by (2.1). So rows r and $2g - r$ of $\mathcal{L}_N$ agree for $0 \leq r < g$. Subtracting row r from row $2g - r$ for each $r < g$ is a unipotent change of rows, which does not change the determinant, and it makes the g rows of index in $(g, 2g]$ of the new X -coefficient vanish. A determinant $\det(B + XL')$ in which L' has only $n - g = 44N$ nonzero rows has degree at most $n - g = 44N$ in X . $\square$

Proposition 3.2 (Common denominator). *There are positive integers D_N such that $D_N \Delta_N(i) \in \mathbb{Z}$ for every $i \in \mathbb{N}$, and for every $\varepsilon > 0$, $\log D_N \leq (A + \varepsilon)n^2$ for all large N .*

Proof. At an integer $z = i$ the denominator of z is 1, and OpenAI's local lower bounds for $v_\ell(\Delta_N(i))$ become independent of i : at odd primes they depend only on ℓ and N (and vanish for $\ell > H$), and at $\ell = 2$ they depend on z only through a quantity that, over the integers, is largest at $z = 0$. Let $e_\ell(N) \geq 0$ be the resulting bound, and put $D_N = \prod_{\ell \leq H} \ell^{e_\ell(N)}$. Then $v_\ell(D_N \Delta_N(i)) \geq 0$ for every prime ℓ , and summing the bounds $e_\ell(N) \log \ell$ exactly as in OpenAI's proof of its finite-place bound [7, Prop. 3.4] gives $\log D_N \leq (A + o(1))n^2$. $\square$

Lemma 3.3 (Integer-valued polynomials). *If $f \in \mathbb{Q}[X]$ has degree at most d and $f(\mathbb{N}) \subseteq \mathbb{Z}$, then $d! f \in \mathbb{Z}[X]$.*

Proof. The d -th forward difference of f is $d!$ times its coefficient of X^d , and it is an integer combination of $f(0), \dots, f(d)$. Subtract that coefficient times the integer-valued polynomial $X(X-1)\cdots(X-d+1)$ and induct on d . $\square$

Corollary 3.4. $Q_N = (44N)! D_N \Delta_N(X)$ lies in $\mathbb{Z}[X]$ and has degree at most $44N$. If $p > P_*$ is prime, $a \in \mathbb{Z}$, $b \geq 1$ and $p \nmid b$, then $Q_p(a/b) \neq 0$ and hence $b^{44p} |Q_p(a/b)| \geq 1$.

Proof. The first claims follow from Lemmas 3.1 and 3.3 and Proposition 3.2. The denominator of a/b divides b , so Proposition 2.1(d) gives $\Delta_p(a/b) \neq 0$. Finally $b^{44p} Q_p(a/b)$ is a nonzero integer. $\square$

4. THE IRRATIONALITY-MEASURE CRITERION

Proposition 4.1. *Let $\xi \in \mathbb{R}$, $d \in \mathbb{N}$ and $\eta, \rho > 0$. Let $Q_p \in \mathbb{Z}[X]$ be defined for primes p and suppose that, for all primes $p \geq P_0$:*

- (i) $\deg Q_p \leq dp$;
- (ii) $Q_p(a/b) \neq 0$ for all $a \in \mathbb{Z}$ and $b \geq 1$ with $p \nmid b$;
- (iii) $|Q_p(x)| \leq e^{-\eta p^2}$ whenever $|x - \xi| \leq e^{-\rho p}$.

Put $\kappa = \max(d/\eta, 4)$. Then $|\xi - a/b| > b^{-3\rho\kappa}$ for all $a \in \mathbb{Z}$ and all sufficiently large b . Assuming the prime number theorem, for every $\varepsilon > 0$ and every $c > 1$ the same holds with the exponent $(1 + \varepsilon)\rho \max(d/\eta, c/\varepsilon)$.

Proof. Suppose $|\xi - a/b| \leq b^{-3\rho\kappa}$ with b large, and put $X = \kappa \log b$. If every prime in $(X, 3X]$ divided b , then $\vartheta(3X) - \vartheta(X) \leq \log b \leq X/4$, where ϑ is Chebyshev's function. But Chebyshev's bounds $\vartheta(x) \leq x \log 4$ and $\vartheta(x) \geq (x-1) \log 2 - O(\sqrt{x} \log x)$ give $\vartheta(3X) - \vartheta(X) \geq X \log 2 - o(X)$. Hence for large b there is a prime $p \in (X, 3X]$ with $p \nmid b$, and $p \geq P_0$. Since $\rho p \leq 3\rho\kappa \log b$, we get $|a/b - \xi| \leq e^{-\rho p}$, so by (iii) $|Q_p(a/b)| \leq e^{-\eta p^2}$. By (i) and (ii), $b^{dp} Q_p(a/b)$ is a nonzero integer, so $e^{-\eta p^2} \geq b^{-dp}$, that is $p \leq (d/\eta) \log b \leq X$. This contradicts $p > X$.

For the second statement use the window $(X, (1+\varepsilon)X]$ with $X = \max(d/\eta, c/\varepsilon) \log b$. Then $\log b \leq \varepsilon X/c$, while the prime number theorem gives $\vartheta((1+\varepsilon)X) - \vartheta(X) = \varepsilon X + o(X)$, which exceeds $\varepsilon X/c$ for large X . $\square$

The factor 3 in the first version, or $1+\varepsilon$ in the second, is the price of finding a prime $p \nmid b$ in a window of the right size.

5. THE PERTURBATION ESTIMATE

This section proves the main new ingredient. Recall $U = 2.29180925 - \frac{1}{2} \log 2$.

Theorem 5.1 (Perturbation estimate). *Let $\rho = 404.25$. For every $\varepsilon > 0$ there is N_0 such that for all $N \geq N_0$ and all real x with $|x - G| \leq e^{-\rho N}$,*

$$|\Delta_N(x) - \Delta_N(G)| \leq \exp(n^2(-U + \varepsilon)).$$

Let $\delta = x - G$, so that $\Delta_N(x) = \det(\mathcal{A}_N + \delta \mathcal{L}_N)$. The plan: write $\mathcal{L}_N = \Phi \beta$, where the columns of Φ are values of OpenAI's row functions A_r at fixed nodes (§5.1); expand the determinant column by column (§5.2); write each resulting mixed determinant as an integral (§5.3); and bound the integrand by OpenAI's method (§5.4–5.6). Throughout, $J = H = 65N$.

5.1. A quadrature formula for $\mathcal{L}_N$. Let $x_a = \cos((2a+1)\pi/(2J))$ for $0 \leq a < J$ be the roots of the Chebyshev polynomial T_J , let $\tau_a = (x_a - 1)/2 \in (-1, 0)$, and let $\ell_a(t) = \prod_{c \neq a} (t - \tau_c) / (\tau_a - \tau_c)$ be the Lagrange basis.

Lemma 5.2. *For $0 \leq r, k < n$,*

$$\mathcal{L}_N(r, k) = \sum_{a < J} A_r(\tau_a) \beta_{a,k}, \quad \beta_{a,k} = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_{i < H} [t^i] \ell_a \cdot 4 c_{(i-b-k-v)/2},$$

and $|\beta_{a,k}| \leq B := 8J^2 K^N$, where $K = 95 \cdot 10^{46}$.

Proof. Since $\deg P_r < H = J$, Lagrange interpolation gives $[t^i] P_r = \sum_a P_r(\tau_a) [t^i] \ell_a$. Substitute this into (2.3) and use $A_r(\tau) = P_r(\tau)$ for $\tau \leq 0$.

For the bound, put $g_a(t) = \ell_a(t)(t-1)^q$. Since $[t^i] \ell_a = 0$ for $i \geq H$ and $c_z = 0$ unless $z \geq 0$, the substitution $i = b+k+v+e$ and $\sum_v (-1)^v \binom{q}{v} [t^{j+v}] \ell_a = [t^{j+q}] g_a$ give

$$\beta_{a,k} = 4 \sum_{0 \leq e < H} c_{e/2} [t^{b+k+e+q}] g_a.$$

Every coefficient of a polynomial is bounded by its maximum modulus on the unit circle (average $g(\omega^j)\omega^{-jm}$ over the M -th roots of unity ω^j , $M > \deg g$). Since $|c_z| \leq 1$, it suffices to show $|g_a(t)| \leq 2J K^N$ for $|t| = 1$.

We have $T_J = 2^{J-1} \prod_c (X - x_c)$, so $\prod_c (t - \tau_c) = T_J(2t+1)/(2^J 2^{J-1})$. For the denominator of ℓ_a , $\prod_{c \neq a} |\tau_a - \tau_c| = 2^{1-J} \prod_{c \neq a} |x_a - x_c|$ and $\prod_{c \neq a} (x_a - x_c) = 2^{1-J} T'_J(x_a) = 2^{1-J} J U_{J-1}(x_a)$. Since $U_{J-1}(\cos \theta) \sin \theta = \sin(J\theta)$ and $\sin(J\theta_a) = \pm 1$ at $\theta_a = (2a+1)\pi/(2J)$, we get $|U_{J-1}(x_a)| \geq 1$, so the denominator is at least $J 2^{2-2J}$. Finally, for $|t| = 1$, $|t - \tau_a| \geq 1 + \tau_a = (1 + x_a)/2 \geq \sin^2(\pi/(4J)) \geq 1/(4J^2)$. Hence

$$|\ell_a(t)| \leq \frac{|T_J(2t+1)|}{2^{2J-1}} \cdot \frac{2^{2J-2}}{J} \cdot 4J^2 = 2J |T_J(2t+1)|.$$

Writing $\zeta = \cosh w$, one has $T_J(\zeta) = \cosh(Jw)$, $|\zeta - 1| + |\zeta + 1| = 2 \cosh \operatorname{Re} w$ and $|\cosh(Jw)| \leq e^{J|\operatorname{Re} w|}$, so

$$|T_J(\zeta)| \leq (S + \sqrt{S^2 - 1})^J, \quad S = \frac{1}{2}(|\zeta - 1| + |\zeta + 1|).$$

For $\zeta = 2t + 1$ with $|t| = 1$, let $s = |t + 1| \in [0, 2]$. Then $|\zeta - 1| = 2$, $|\zeta + 1| = 2s$, $S = 1 + s$, and $|t - 1|^2 = 4 - s^2$ by the parallelogram law. With $J = 65N$ and $q = 4N$,

$$|g_a(t)| \leq 2J \left((1 + s + \sqrt{s^2 + 2s})^{65} (4 - s^2)^2 \right)^N \leq 2J K^N,$$

where the last inequality holds for all $s \in [0, 2]$. (The function $(1 + s + \sqrt{s^2 + 2s})^{65} (4 - s^2)^2$ is largest near $s = 1.917$, where its logarithm is $110.394\dots$, while $\log K = 110.4728\dots$. We check it by splitting $[0, 2]$ into 21 intervals; on each, the first factor is increasing and the second decreasing in s , so it suffices to compare rational upper bounds at the endpoints with K .) $\square$

Remark 5.3. The cruder bound $\sum_i |[t^i] \ell_a| \leq \prod_{c \neq a} (1 - \tau_c) / \prod_{c \neq a} |\tau_a - \tau_c| \leq (3 + 2\sqrt{2})^J$, together with $\sum_v \binom{q}{v} = 2^q$, gives $|\beta_{a,k}| \leq 2^{q+2} (3 + 2\sqrt{2})^J$, whose logarithm is $(4 \log 2 + 65 \log(3 + 2\sqrt{2}))N \approx 117.35N$ instead of $\log K \cdot N \approx 110.47N$. The unit circle captures the cancellation in $(t - 1)^q$ between $t = 1$, where $|T_J(2t+1)|$ is largest, and $t = -1$, where it is smallest.

Let Φ be the $n \times J$ matrix $\Phi(r, a) = A_r(\tau_a)$. Then $\Delta_N(x) = \det(\mathcal{A}_N + \delta \Phi \beta)$.

5.2. Column expansion. For a map $f: \{0, \dots, n-1\} \rightarrow \{\emptyset\} \cup \{0, \dots, J-1\}$ let M_f be the matrix whose column k is column k of $\mathcal{A}_N$ if $f(k) = \emptyset$ and the column $\Phi(\cdot, f(k))$ otherwise, and let $m(f) = \#\{k : f(k) \neq \emptyset\}$.

Lemma 5.4. *If $|\beta_{a,k}| \leq B$ and $|\det M_f| \leq M c^{m(f)}$ for every f , with $M, c \geq 0$, then*

$$|\det(\mathcal{A}_N + \delta \Phi \beta) - \det \mathcal{A}_N| \leq M((1 + J|\delta|Bc)^n - 1).$$

Proof. Column k of $\mathcal{A}_N + \delta\Phi\beta$ is $\mathcal{A}_N(\cdot, k) + \sum_a \delta\beta_{a,k}\Phi(\cdot, a)$. Multilinearity gives $\det(\mathcal{A}_N + \delta\Phi\beta) = \sum_f \prod_{k:f(k) \neq \emptyset} \delta\beta_{f(k),k} \det M_f$. The term $f \equiv \emptyset$ is $\det \mathcal{A}_N$, and the others sum in absolute value to at most $M \sum_{f \neq \emptyset} (|\delta|Bc)^{m(f)} = M((1 + J|\delta|Bc)^n - 1)$. $\square$

5.3. A mixed Andréief identity. Fix f , and let $p = n - m(f)$. Permuting columns (which only changes the sign of the determinant), M_f has p integral columns $\mathcal{A}_N(\cdot, E_i)$, $0 \leq E_1 < \dots < E_p < n$, followed by $m = n - p$ point columns $(A_r(\tau'_j))_r$, where $\tau'_j = \tau_{f(k_j)} \in (-1, 0)$.

Lemma 5.5. *Let μ, ν be finite measures, F_r ($r < p + m$), G_i ($i < p$) and K functions with the required integrability, and B the $(p + m) \times (p + m)$ matrix with $B(r, i) = \iint F_r(t)K(t, s)G_i(s) d\mu(t) d\nu(s)$ for $i < p$ and $B(r, p + j) = F_r(\tau'_j)$. Then*

$$(p!)^2 \det B = \iint \det[F_r((t \parallel \tau')_c)]_{r,c} \times \det[K(t_i, s_j)]_{i,j < p} \det[G_i(s_j)]_{j,i < p} d\mu^p(t) d\nu^p(s),$$

where $t \parallel \tau'$ is the concatenated list of length $p + m$. Consequently $|\det B|$ is at most $(p!)^{-2}$ times the total mass of $\mu^p \otimes \nu^p$ times the supremum of the absolute value of the integrand.

Proof. For $m = 0$ this is Andréief's identity applied twice [3], as used by OpenAI for [7, (62)]. In general expand $\det B$ by multilinearity in the first p columns, which integrates t and s variables column by column, and then antisymmetrize first in the t variables and then in the s variables; the last m columns are constant throughout. $\square$

We apply this with $F_r = A_r$, $G_i = \psi_{E_i}$, $K(t, s) = 1/(1 - ts)$, μ as in §2 and ν Lebesgue measure on $(0, 1)$; the integrability is OpenAI's [7, §5.1]. The mass of $\mu^p \otimes \nu^p$ is $2^p \leq 2^n$, so

$$(5.1) \quad |\det M_f| \leq 2^n \sup_{t,s} \left| \det[A_r((t \parallel \tau')_c)] \cdot \det\left[\frac{1}{1 - t_i s_j}\right] \cdot \det[\psi_{E_i}(s_j)] \right|,$$

the supremum over $t \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero coordinates (a set of full measure). We have bounded the factor $(p!)^{-2}$ by 1.

5.4. A generalized Vandermonde bound. The column determinant is $\det[\psi_{E_i}(s_j)] = \prod_j s_j^b (1 - s_j)^q \cdot \det[s_j^{E_i}]$. Write $V(y) = \prod_{i < j} (y_j - y_i)$ and $\mathcal{V}(y) = |V(y)|$.

Lemma 5.6. *Let $s_0, \dots, s_{p-1} \in [0, 1]$ and let $E_1, \dots, E_p < n$ be distinct. Then*

$$|\det[s_j^{E_i}]| \leq 2^{p(n-p)} \mathcal{V}(s).$$

Proof. Let $m = n - p$, let $k_1, \dots, k_m$ be the integers in $[0, n)$ that are not among the E_i , and let $P(X) = \prod_j (X - s_j)$, a monic polynomial of degree p . Consider two $n \times n$ matrices. The rows of Ω are the linear functionals $f \mapsto f(s_j)$ ($j < p$) and $f \mapsto [X^{k_l}]f$ ($l \leq m$), evaluated on the monomials X^a , $a < n$. The columns of Γ are the coefficient vectors of the polynomials X^b ($b < p$) and $X^i P$ ($i < m$), all of degree less than n .

- Ordering the columns of Ω as $E_1, \dots, E_p, k_1, \dots, k_m$ makes it block triangular, with diagonal blocks $[s_j^{E_i}]$ and the identity. So $|\det \Omega| = |\det[s_j^{E_i}]|$.
- In the natural order of the monomials, Γ is block triangular with diagonal blocks the identity and the matrix $([X^{p+l}]X^i P)_{l,i}$, which is unitriangular because P is monic of degree p . So $|\det \Gamma| = 1$.
- The entries of $\Omega\Gamma$ are the values of the functionals on the polynomials. Since $P(s_j) = 0$, the block of evaluations on the $X^i P$ vanishes, so $\Omega\Gamma$ is block triangular with diagonal blocks the Vandermonde matrix $[s_j^b]_{j,b < p}$ and $Q = ([X^{k_l}]X^i P)_{l,i}$.

Hence $|\det[s_j^{E_i}]| = \mathcal{V}(s) |\det Q|$. A determinant is bounded by the product of the ℓ^1 norms of its columns, and each column of Q consists of distinct coefficients of $X^i P$. Its ℓ^1 norm is therefore at most the sum of the absolute values of the coefficients of P , which is $\prod_j (1 + s_j) \leq 2^p$. $\square$

Remark 5.7. The bound is close to optimal. The ratio $\det[s_j^{E_i}]/V(s)$ is a Schur polynomial with nonnegative coefficients, and its value at $s = (1, \dots, 1)$, for a single missing exponent k , is $\binom{n-1}{k}$, of size 2^{n-1} up to a polynomial factor.

5.5. The integrand. Use OpenAI's coordinate $t = 2x/(1+x^2)$, $x \in (-1, 1)$ [7, (64)], write x_i for the coordinates of the free nodes t_i and $y_j \in (-1, 0)$ for those of the fixed nodes τ'_j , and $z = x \parallel y \in (-1, 1)^n$. For a list z of n distinct nonzero numbers in $(-1, 1)$ OpenAI's column factorization [7, §5.1, before (65)] reads

$$(5.2) \quad \det[A_r(t_c)]_{r,c} = \mathcal{R}_n(z) \prod_c t_c^{C-1} (1 - t_c)^h,$$

where $\mathcal{R}_n$ is OpenAI's sheet determinant. OpenAI proves two bounds for it [7, Props. 5.2, 5.3], with $K_0 = 10e^{12}$:

$$(5.3) \quad |\mathcal{R}_n(z)| \leq (1 + K_0 n)^n \mathcal{V}(1/z) \prod_c |z_c|^g \quad \text{if } \sum_c \frac{1 - z_c^2}{1 + z_c^2} \leq D_*,$$

$$(5.4) \quad |\mathcal{R}_n(z)| \leq (3/2)^n n^{n/2} 2^{-\binom{n}{2}} \prod_c (1 + z_c^2)^{\frac{n-1}{2}} |z_c|^{g-(n-1)} \quad \text{always.}$$

Following OpenAI, $\kappa = 2$ refers to (5.3) and $\kappa = 1$ to (5.4). Let $w(x) = |t|^{C-1} (1-t)^h = 2^{C-1} |x|^{C-1} (1-x)^{2h} (1+x^2)^{-(C-1+h)}$.

The p -node majorants and their energy. For $x \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero entries, put $t_i = 2x_i/(1 + x_i^2)$ and

$$\begin{aligned}\mathcal{J}^{(p)}(x, s) &= \prod_i w(x_i) \cdot \frac{\mathcal{V}(t)\mathcal{V}(s)^2}{\prod_{i,j}(1 - t_i s_j)} \prod_j s_j^b (1 - s_j)^q, \\ \mathcal{I}_2^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \mathcal{V}(1/x) \prod_i |x_i|^{g-(n-p)}, \\ \mathcal{I}_1^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \prod_i (1 + x_i^2)^{\frac{n-1}{2}} |x_i|^{g-(n-1)}.\end{aligned}$$

For $p = n$ these are OpenAI's majorants $\mathcal{I}_{2,n}$ and $2^{\binom{n}{2}} \mathcal{I}_{1,n}$ [7, (74)]. Let OpenAI's one-point fields be [7, (76)]

$$\begin{aligned}W_\kappa(x) &= \frac{19}{48} \log |x| + \frac{1}{12} \log(1 - x) - \left(\frac{\kappa}{2} + \frac{17}{48}\right) \log(1 + x^2), \\ W_s(s) &= \frac{7}{48} \log s + \frac{1}{12} \log(1 - s), \quad D(x) = \frac{1}{6} - \frac{2x^2}{1+x^2},\end{aligned}$$

and define the normalized energy

$$\begin{aligned}E_\kappa^{(p)}(x, s) &= \left(\frac{C}{n} + \frac{\kappa - 2}{2} - \frac{\kappa + 1}{2n}\right) \log 2 + \frac{1}{n} \sum_j W_s(s_j) \\ &\quad + \frac{1}{n} \sum_i \left(W_\kappa(x_i) + \frac{\kappa + 2}{2n} \log(1 + x_i^2)\right) \\ &\quad + \frac{1}{n^2} \left(\kappa \sum_{i < j} \log |x_i - x_j| + \sum_{i < j} \log(1 - x_i x_j)\right) \\ &\quad + \frac{2}{n^2} \sum_{i < j} \log |s_i - s_j| - \frac{1}{n^2} \sum_{i,j} \log(1 - 2x_i s_j + x_i^2),\end{aligned}$$

all sums over indices $< p$; note the normalization by n , not p . For $p = n$ this is the normalized logarithm $n^{-2} \log \mathcal{I}_{\kappa,n} - \frac{1}{2} \log 2$ of OpenAI's energy argument. Expanding the coordinate change $t = 2x/(1 + x^2)$ gives, for every $p \leq n$,

$$(5.5) \quad \log \mathcal{I}_\kappa^{(p)}(x, s) = n^2 \left(E_\kappa^{(p)}(x, s) + \frac{1}{2} \log 2\right) + e_\kappa(p) \log 2,$$

with $e_1(p) = p(C - 1) + \binom{p}{2} - n(C - 1)$ and $e_2(p) = e_1(p) - \binom{n}{2}$.

Proposition 5.8 (Pointwise bound). *In the setting of (5.1), let $K_n = \max\{(1 + K_0 n)^n, (3/2)^n n^{n/2}\}$, and let $\kappa = 2$ if $\sum_c \frac{1 - z_c^2}{1 + z_c^2} \leq D_*$ and $\kappa = 1$ otherwise. Then the integrand in (5.1) is at most*

$$K_n 2^{nm} (2^{2h})^m \exp\left(n^2 (E_\kappa^{(p)}(x, s) + \frac{1}{2} \log 2)\right).$$

Moreover, if $\kappa = 1$ then $\sum_{i < p} \frac{1 - x_i^2}{1 + x_i^2} > D_* - m$.

Proof. By (5.2), Cauchy's evaluation

$$\det \left[\frac{1}{1 - t_i s_j} \right] = \frac{V(t)V(s)}{\prod_{i,j} (1 - t_i s_j)}$$

[7, (63)] and Lemma 5.6, the integrand is at most

$$2^{nm} |\mathcal{R}_n(z)| \prod_j w(y_j) \cdot \mathcal{J}^{(p)}(x, s).$$

For a fixed node $-1 < y < 0$ we use $|y| \leq 1$, $(1 - y)^{2h} \leq 2^{2h}$ and $1 + y^2 \geq 1$, together with $C - 1 + g - (n - 1) = A = 19N$ (OpenAI's A) and $(n - 1)/2 < C - 1 + h$:

$$(5.6) \quad w(y) (1 + y^2)^{\frac{n-1}{2}} |y|^{g-(n-1)} \leq 2^{C-1} 2^{2h}, \quad w(y) |y|^{g-(n-1)} \leq 2^{C-1} 2^{2h}.$$

We use $\binom{n}{2} = \binom{p}{2} + pm + \binom{m}{2}$ repeatedly.

Case $\kappa = 1$. By (5.4), the first inequality in (5.6) and (5.5), the integrand is at most

$$K_n 2^{nm} (2^{2h})^m \exp(n^2 (E_1^{(p)} + \frac{1}{2} \log 2)) \cdot 2^{-\binom{n}{2} + m(C-1) + e_1(p)},$$

and $-\binom{n}{2} + m(C - 1) + e_1(p) = \binom{p}{2} - \binom{n}{2} \leq 0$. The case condition fails for z and each fixed node contributes at most 1 to its sum, which gives the last assertion.

Case $\kappa = 2$. By (5.3) we must bound $\mathcal{V}(1/z) \prod_c |z_c|^g$. Split $\mathcal{V}(1/z) = \mathcal{V}(1/x) \prod_{i,j} |1/y_j - 1/x_i| \mathcal{V}(1/y)$. Since $|1/y - 1/x| = |x - y|/|xy| \leq 2/|xy|$ and $\mathcal{V}(1/y) = \mathcal{V}(y) \prod_j |y_j|^{-(m-1)} \leq \prod_j |y_j|^{-(m-1)}$,

$$\mathcal{V}(1/z) \prod_c |z_c|^g \leq 2^{pm} \mathcal{V}(1/x) \prod_i |x_i|^{g-m} \prod_j |y_j|^{g-(n-1)}.$$

With the second inequality in (5.6) and (5.5), the integrand is at most $K_n 2^{nm} (2^{2h})^m \exp(n^2 (E_2^{(p)} + \frac{1}{2} \log 2)) \cdot 2^{pm + m(C-1) + e_2(p)}$, and $pm + m(C - 1) + e_2(p) = -\binom{m}{2} \leq 0$. $\square$

Remark 5.9. Each fixed node brings a factor 2^{C-1} from $w(y)$. It is exactly compensated by the difference between the n -node and p -node normalizations of the energy, so what remains is $2^{2h} = 4^{2N}$ per fixed node, coming from $(1 - y)^{2h}$.

5.6. The energy bound for $p \leq n$ nodes. OpenAI's certificate [7, §7] supplies, for $\kappa = 1, 2$, a parameter $\lambda_\kappa \geq 0$ ($\lambda_2 = 0$, $\lambda_1 = 2.47405979$), trial sequences $u^{(\kappa)}, v^{(\kappa)}$ (OpenAI's p, v), and rational upper bounds

$$(5.7) \quad \sup_{-1 \leq x < 1, x \neq 0} \{W_\kappa + \lambda_\kappa D - 2\kappa T(u^{(\kappa)}, \cdot) - S(v^{(\kappa)}, \cdot)\} \leq B_x^\kappa, \\ \sup_{0 < s < 1} \{W_s + 2T(v^{(\kappa)}, \cdot)\} \leq B_s^\kappa,$$

in the notation of [7, (77)–(78)]. For $\kappa = 1$ we use OpenAI's certificate, which gives $B_x^1 = -1.324376$ and $B_s^1 = -1.427976$. For $\kappa = 2$ we use the

trial sequences of Proposition 5.11 in place of OpenAI's, with

$$B_x^2 = -0.98572217, \quad B_s^2 = -1.60942387.$$

With these, the dual value

$$U_\kappa = -\frac{11}{16} \log 2 + \kappa \|u^{(\kappa)}\|_*^2 + \frac{1}{2} \|v^{(\kappa)}\|_*^2 + B_x^\kappa + B_s^\kappa$$

satisfies $U_1 < -2.296789875$ [7, Prop. 7.1] and $U_2 < -2.29180925$ (Proposition 5.11). Put

$$c_\kappa = (\kappa + 2) \log 2 + |B_x^\kappa| + |B_s^\kappa| + \lambda_\kappa/6,$$

so that $c_1 = 5.24413\dots$ and $c_2 = 5.36773\dots$, and $c^* = 53678/10000 > \max(c_1, c_2)$.

Proposition 5.10 (Energy bound with missing points). *For every $d > 0$ there is N_0 such that for all $N \geq N_0$, all $p \leq n$ (with $m = n - p$) and all $x \in (-1, 1)^p$, $s \in (0, 1)^p$ with distinct nonzero coordinates,*

$$E_\kappa^{(p)}(x, s) \leq U_\kappa + d + \frac{m}{n} c_\kappa,$$

for $\kappa = 2$ unconditionally and for $\kappa = 1$ under $\sum_{i < p} \frac{1-x_i^2}{1+x_i^2} > D_* - m$.

Proof. OpenAI's proof of [7, Prop. 6.1] regularizes the logarithmic interactions by matched damping, controls the diagonal and endpoint corrections, and completes two squares with the trial sequences. All of these steps are estimates for sums over the points, normalized by n , and they remain valid, with the same error terms, uniformly for any number $p \leq n$ of points. The quadratic terms are bounded by $\kappa \|u\|_*^2 + \frac{1}{2} \|v\|_*^2$ as before. Only three places change.

- (1) The constant term. For $p = n$ the powers of 2 combine to $-\frac{11}{16} \log 2$. For $p < n$ the same computation gives $-\frac{11}{16} \log 2 + (\frac{\kappa}{2} + 1)(1 - p^2/n^2) \log 2$, and $1 - p^2/n^2 \leq 2m/n$.
- (2) The one-point terms. They are $\frac{1}{n} \sum_{i < p} X(x_i) \leq \frac{p}{n} B_x = B_x + \frac{m}{n} |B_x|$, since $B_x < 0$, and likewise for the column field with B_s .
- (3) The case field ($\kappa = 1$ only). OpenAI uses $\lambda \frac{1}{n} \sum_i D(x_i) \geq -\lambda/n$ when (5.3)'s condition fails. Since $D(x) = \frac{1-x^2}{1+x^2} - \frac{5}{6}$ and $D_* = \frac{5}{6}n - 1$, the shifted condition gives $\sum_{i < p} D(x_i) > -1 - \frac{m}{6}$, which costs $\lambda(1 + m/6)/n$.

The damping parameter is chosen in terms of d alone, and the remaining errors are $O_d(1/n)$, uniformly in p and the configuration. $\square$

5.7. A new certificate for the second case of the energy dual. For a finite list $c_1, \dots, c_K$ of reals and finitely many pairs (z_j, r_j) of complex numbers with $|z_j| < 1$, consider the trial sequence

$$(5.8) \quad u_k = c_k + \operatorname{Re} \sum_j r_j z_j^k \quad (k \geq 1; \ c_k = 0 \text{ for } k > K).$$

This is the form of OpenAI's trial sequences [7, §7].

Proposition 5.11. *There are trial sequences u, v of the form (5.8), with $K = 40$, coefficients c_k in $10^{-8}\mathbb{Z}$, twelve pairs (z_j, r_j) for u and fifteen for v , all z_j and r_j in $\mathbb{Q} + \mathbb{Q}i$ (they are listed in the file `Dual/Data.lean` of the formalization), such that (5.7) holds for $\kappa = 2$ with $B_x^2 = -0.98572217$ and $B_s^2 = -1.60942387$, and $\|u\|_*^2 \leq 0.066723974$, $\|v\|_*^2 \leq 1.292855037$. Hence*

$$U_2 \leq -\frac{11}{16} \log 2 + 2 \cdot 0.066723974 + \frac{1}{2} \cdot 1.292855037 \\ - 0.98572217 - 1.60942387 < -2.29180925,$$

and Proposition 2.1(b) holds with $U = 2.29180925 - \frac{1}{2} \log 2$ in place of U_{OAI} .

The sequences were found numerically, by maximizing the dual value $-U_2$ over sequences of the form (5.8) with prescribed bases z_j (OpenAI's, together with ± 0.9 , ± 0.97 , 0.96 , ± 0.99 and a few others) by a cutting-plane method, and then rounded. At the optimum the two fields of (5.7) nearly equioscillate, with about fifty and thirty local maxima.

Proof. All computations below are in exact rational arithmetic and are carried out by Lean's kernel. We follow the structure of OpenAI's certificate, but make each step a decidable check of general lemmas.

Closed forms. For $|x| \leq 1$, OpenAI's formulas give

$$T(u, x) = \sum_{k \leq K} \frac{c_k T_k(x)}{k} - \frac{1}{2} \operatorname{Re} \sum_j r_j \log(1 - 2xz_j + z_j^2)$$

and $S(u, x) = \sum_{k \leq K} c_k x^k / k - \operatorname{Re} \sum_j r_j \log(1 - xz_j)$. Expanding the squares and using $\sum_{k \geq 1} w^k / k = -\log(1 - w)$,

$$\|u\|_*^2 = \sum_{k \leq K} \frac{c_k^2 + 2c_k t_k}{k} - \frac{1}{2} \operatorname{Re} \sum_{j,l} (r_j r_l \log(1 - z_j z_l) + r_j \bar{r}_l \log(1 - z_j \bar{z}_l)),$$

$$t_k = \operatorname{Re} \sum_j r_j z_j^k.$$

So the norms and the values of the fields at rational points are rational numbers plus finite sums of terms $\operatorname{Re}((c+di) \log w)$ with c, d, w rational. Each such term is approximated to 10^{-12} by OpenAI's rational approximations of the logarithm and the arctangent [7, §7]; the reduction of w to the range of these approximations is computed and checked rationally.

Stationary points. The derivative of each field F is a sum of rational functions with rational coefficients, hence $F' = A/Q$ with explicit $A, Q \in \mathbb{Q}[x]$ (of degrees 78 and 39 for the row field, 60 and 21 for the column field), where Q does not vanish on the domain. The domain is cut into 154 (respectively 94) intervals. On 104 (respectively 63) of them A has no root: after the substitution $x = a + (b-a)/(1+y)$, all coefficients of $(1+y)^{\deg A} A(x)$ have the same sign, so that A has no root in (a, b) (the case of Descartes's rule of signs without sign changes). The remaining 50 (respectively 31) intervals have length $2 \cdot 10^{-7}$ and contain the roots of A ; a breakpoint between two intervals of the first kind is not a root of A .

Brackets. On a bracket $[l, l+w]$ write $A(l+wt) = \sum a_i t^i$ and $Q(l+wt) = \sum q_i t^i$. For $0 \leq t \leq 1$, $|F'| \leq \sum |a_i|/(|q_0| - \sum_{i \geq 1} |q_i|)$; since A has a root in the bracket, a_0 and a_1 are small, and this gives $|F(x) - F(l)| \leq 10^{-8}$ on the bracket. The value $F(l)$ is computed as above.

Conclusion. The fields tend to $-\infty$ at 0 and 1 (the row field at $0^\pm$ and 1^-), so their suprema are attained at stationary points or at $x = -1$, and every stationary point lies in a bracket. Comparing the certified values with B_x^2 and B_s^2 proves (5.7). The bound for U_2 follows, with $\log 2 > 0.6931471803$. OpenAI's proof of [7, Prop. 7.1] applies to any trial sequences in its two cases and gives $\limsup_N \log |\det \mathcal{A}_N|/n^2 \leq \max(U_1, U_2) + \frac{1}{2} \log 2$, and $U_1 < -2.296789875$. $\square$

The kernel checks take about half an hour of computing time.

Remark 5.12. OpenAI's own certificate for $\kappa = 2$ records its bounds to five digits, as $B_x^2 \leq -0.98399 + 2.4 \cdot 10^{-5}$, $B_s^2 \leq -1.60890 + 2.4 \cdot 10^{-5}$ and a norm term at most 0.77844, while its computations show that the largest bracket values of its two fields are $-0.98403338\dots$ and $-1.60894641\dots$ and its norm term is at most 0.778423472. Read to seven digits, OpenAI's certificate already gives $U_2 < -2.2910468$, and hence the exponent 34 000 by the argument of Section 6. This is also formalized (**SharpBarrier**).

5.8. Completion of the proof.

Proof of Theorem 5.1. Fix $d > 0$ and let N be large. By (5.1), Proposition 5.8, Proposition 5.10 and $U_\kappa < -2.29180925 = -U - \frac{1}{2} \log 2$, every f with $m = m(f)$ satisfies

$$|\det M_f| \leq 2^n K_n 2^{nm} (2^{2h})^m \exp(n^2(-U + d) + nm c^*) = M c^m,$$

with

$$M = 2^n K_n e^{n^2(-U+d)}, \quad c = 2^n 2^{2h} e^{nc^*}.$$

(The degenerate configurations, such as two fixed nodes being equal or a vanishing coordinate, have measure zero or make the determinant vanish.) By Lemma 5.2, $|\beta_{a,k}| \leq B = 8J^2 K^N$. Now

$$\begin{aligned} \log(nJBc) &= \log(8nJ^3) + (n + 2h) \log 2 + N \log K + nc^* \\ &= 52N \log 2 + N \log K + 48Nc^* + \log(8 \cdot 48 \cdot 65^3 N^4), \end{aligned}$$

and $52 \log 2 + \log K + 48c^* = 404.1708\dots < 404.25$; so $nJBc \leq e^{404.25N}$ for $N \geq 2000$. If $|\delta| \leq e^{-404.25N}$ then $J|\delta|Bc \leq 1/n$ and $(1 + J|\delta|Bc)^n - 1 \leq e - 1 < 2$. Lemma 5.4 gives

$$|\Delta_N(x) - \Delta_N(G)| \leq 2M = 2^{n+1} K_n e^{n^2(-U+d)}.$$

Since $\log K_n = O(n \log n)$, taking $d = \varepsilon/2$ proves the theorem. $\square$

Remark 5.13 (The value of ρ). Per unit of N , the cost $\log(Bc)$ of one perturbed column splits as follows: the quadrature weights contribute $\log K \approx 110.47$ (the maximum bounded in the proof of Lemma 5.2 is $e^{110.394\dots}$, so a

finer check would save another 0.08); the generalized Vandermonde bound, $48 \log 2 \approx 33.3$; the fixed nodes, $4 \log 2 \approx 2.8$; and the energy lost by the missing point, $48c^* \approx 257.7$.

6. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Let $\gamma_0 = U - A$ and $\eta = 2304 \cdot 9.996945 \cdot 10^{-4}$, so that $0 < \eta < 2304\gamma_0$ by (2.4), and let $\varepsilon = (\gamma_0 - \eta/2304)/3 > 0$. For large primes p and $|x - G| \leq e^{-404.25p}$, Proposition 5.11 and Theorem 5.1 give $|\Delta_p(x)| \leq 2e^{n^2(-U+\varepsilon/2)}$. Also $\log(2(44p)!) \leq \frac{\varepsilon}{2}n^2$ for large p , and $\log D_p \leq (A + \varepsilon)n^2$ by Proposition 3.2. Hence

$$|Q_p(x)| \leq e^{-n^2(\gamma_0-2\varepsilon)} \leq e^{-n^2(\gamma_0-3\varepsilon)} = e^{-n^2\eta/2304} = e^{-\eta p^2},$$

since $n^2 = 2304p^2$. By Corollary 3.4 the hypotheses of Proposition 4.1 hold with $\xi = G$, $d = 44$, this η and $\rho = 404.25$, and

$$\frac{d}{\eta} = \frac{44}{2304 \cdot 9.996945 \cdot 10^{-4}} = 19.10305 \dots \leq R = 19.1031.$$

With the prime number theorem, $c = 10001/10000$ and $\varepsilon = c/R = 10001/191031$, the exponent is

$$\left(1 + \frac{10001}{191031}\right) \cdot 404.25 \cdot R = 404.25(R + c) = 8126.71 \dots \leq 8127.$$

With Chebyshev's bounds it is at most $3 \cdot 404.25 \cdot R = 23167.28 \dots \leq 23168$. $\square$

Remark 6.1 (Limits of the method). The exponent is essentially $\rho(d/(2304\gamma_0) + c) \approx 404.25 \cdot 20.1 \approx 8.1 \cdot 10^3$. Further improvements must come from ρ , that is, from the contributions listed in the remark at the end of Section 5, or from a larger gap γ_0 . Numerically, enlarging the space of trial sequences in Proposition 5.11 changes U_2 only in the sixth decimal, so a substantially larger γ_0 would require a different dual or a different determinant. The energy $48c^*$ lost per missing point is the largest part of ρ ; we have not tried to optimize it.

7. FORMALIZATION

All results of this paper are formalized in Lean 4 [6] with Mathlib [5], on top of OpenAI's Lean formalization of the irrationality of G [7]. The development is available at

<https://github.com/slitsey/catalan-irrationality-measure>

and consists of about 8900 lines of hand-written proofs in twenty-nine files, together with nineteen generated files (about 1400 lines) holding the data and the kernel checks of Proposition 5.11 and the change of constants in OpenAI's certificate described in the remark after Proposition 5.11. It uses OpenAI's definitions directly: Catalan's constant is `catalan`, defined as the series $\sum_j (-1)^j / (2j+1)^2$; the matrix (2.2) and its rational form are OpenAI's

`determinant` and `determinantRat`. The main statements, in the namespace `OAI.InternalCatalan`, are:

- `catalan_irrationalityMeasure_8127`: for all sufficiently large $b \in \mathbb{N}$ and all $a \in \mathbb{Z}$, $b^{-8127} < |G - a/b|$ (Theorem 1.1);
- `catalan_irrationalityMeasure_23168`: the same with exponent 23 168;
- `eventually_realDeterminant_abs_upper_dual`: Proposition 5.11;
- `lemma6_perturbation`: Theorem 5.1, with $\rho = 404.25$;
- `catalan_irrationalityMeasure` and `catalan_irrationalityMeasure_pnt`: the general exponents of Proposition 4.1.

There is no `sorry`, and the command `#print axioms` reports only the three standard axioms `propext`, `Classical.choice` and `Quot.sound`. The prime number theorem enters through OpenAI’s workspace, which already uses it for Proposition 2.1(c).

File	Content
DeltaPoly, Degree	$\Delta_N(X)$; Lemma 3.1
Denominators, IntegerValued, Integrality	Proposition 3.2; Lemma 3.3; Corollary 3.4
Nonvanishing	Proposition 2.1(d) for $p \nmid \text{den } z$
PrimeWindow, Criterion, PNTWindow, Main	Proposition 4.1 and the reduction to Theorem 5.1
Quadrature, ChebyshevNodes, QuadCircle	Lemma 5.2
DetExpansion	Lemma 5.4
MixedAndreief	Lemma 5.5
GenVandermonde, GenVandermondeSharp	Lemma 5.6
PinMajorant	Proposition 5.8 and (5.5)
PinDampedDual, PinSupDual	Proposition 5.10
Cert/ListPoly, Cert/LogApprox, Cert/FieldEval, Cert/FieldDeriv, Cert/Cover, Cert/Norm	the general lemmas behind the checks of Proposition 5.11
Dual/*, DualTwo	the certificate data, the kernel checks, Proposition 5.11
SharpBarrier Lemma6	the remark after Proposition 5.11 nodes, certified constants, Theorem 5.1, Theorem 1.1

The proofs were checked with Lean v4.34.1 and Mathlib at commit d13f23b, the versions pinned by OpenAI’s repository at commit fd4aeeb. The Palomar registry of formalized mathematics requires a newer Lean version and Lean’s module system. For it, the repository also contains the parts of OpenAI’s formalization that the proof uses, converted to modules; its README describes the details.

DISCLOSURE OF AI USE

The entire work reported here is due to the AI system Claude Opus 5.5 (Anthropic): the mathematics, including the perturbation estimate and all proofs; the Lean formalization; the numerical computations and the certificate generator; the literature search; and the text of this paper. Claude worked in a series of conversations at the request of the author, who posed the problem, directed the work, and is responsible for this submission. The

underlying irrationality proof and its formalization are OpenAI's [7]. The formal verification in Lean is intended as an independent check of every step of the argument above.

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