

AN IRRATIONALITY MEASURE FOR CATALAN'S CONSTANT

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ABSTRACT. Building on OpenAI's recent proof that Catalan's constant $G = \sum_{j \geq 0} (-1)^j / (2j + 1)^2$ is irrational, we prove that

$$\left| G - \frac{a}{b} \right| > b^{-61\,300}$$

for all integers a and all sufficiently large integers b . Thus the irrationality exponent of G is at most 61 300; without the prime number theorem in the final step the same argument gives 182 500. The previous best bound, due to Jha, is 10^8 . The new ingredient is a perturbation estimate for OpenAI's determinant, viewed as a polynomial $\Delta_N(X)$ in the constant: on the interval $|x - G| \leq e^{-412N}$ the polynomial moves by no more than OpenAI's bound for $|\Delta_N(G)|$. It is proved by a quadrature formula for the coefficient of the constant, an Andréief identity with fixed columns, a generalized Vandermonde bound, and OpenAI's energy method rerun for configurations with fewer points. Every step, including the final theorem, is formalized in Lean 4 on top of OpenAI's formalization, without `sorry` and with only the standard axioms.

1. INTRODUCTION

OpenAI [7] recently proved that Catalan's constant

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = 0.91596\,55941\dots$$

is irrational, and formalized the proof in Lean. (An earlier claimed proof [9] was found to contain a gap [10].) The argument is by contradiction: under the hypothesis $G \in \mathbb{Q}$ an explicit determinant Δ_N of size $n = 48N$ is a nonzero rational number whose denominator is controlled prime by prime, so that $\log |\Delta_N| \geq -(A + o(1))n^2$; on the other hand a real-variable estimate gives $\log |\Delta_N| \leq -(U - o(1))n^2$ with $U > A$. The margin is small: $U - A \approx 1.3 \cdot 10^{-4}$. As OpenAI remarks, the argument proves qualitative irrationality; an irrationality measure requires additional control of rational approximations.

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Recall that the *irrationality exponent* $\mu(\xi)$ of a real number ξ is the infimum of the μ such that $|\xi - a/b| > b^{-\mu}$ for all integers a and all sufficiently large b .

Theorem 1.1. *For every integer a and every sufficiently large integer b ,*

$$\left| G - \frac{a}{b} \right| > b^{-61\,300}.$$

In particular $\mu(G) \leq 61\,300$. If in the last step the prime number theorem is replaced by Chebyshev's bounds, the same argument gives $|G - a/b| > b^{-182\,500}$.

Theorem 1.1, together with every result in this paper, has been verified in the Lean 4 proof assistant on top of OpenAI's formalization, with no unproved steps and only the standard axioms; see Section 7. The formalization is available at <https://github.com/slitsey/catalan-irrationality-measure>. The use of AI in this work is disclosed at the end of the paper.

1.1. Previous results. The first finite irrationality measure for G was obtained by Jha [4], who proved $\mu(G) \leq 10^8$, also with a Lean formalization built on OpenAI's. The present work was carried out independently; we found Jha's preprint in a literature search after our proof was complete. Both arguments have the shape that OpenAI's proof suggests: one replaces the determinant by a polynomial in the constant, clears denominators, and needs a perturbation estimate near G . They differ in the perturbation estimate. Jha bounds the complementary minors of OpenAI's matrix in every dimension, by an energy estimate with node padding, with a codimension loss of the form $\exp(1000n(n-m))$. Here the perturbation of the matrix is written, by a quadrature formula, as a combination of point evaluations of OpenAI's row functions, and each perturbed column costs a factor $e^{\rho N}$ with $\rho = 412$ (Theorem 5.1). Our exponent is smaller than Jha's by a factor of about 1600, and the difference comes mainly from this step. For other recent work on rational approximations to G see Eskandari [2]; for earlier work on the arithmetic of G and related values see [8, 11]; for an effective approach to linear independence of related L -values see Calegari, Dimitrov and Tang [1].

1.2. Outline of the proof. OpenAI's matrix has entries in $\mathbb{Q} + \mathbb{Q}G$ (Proposition 2.1(a)). Replacing G by a variable gives a polynomial $\Delta_N(X) \in \mathbb{Q}[X]$, and we proceed in four steps.

- (i) *Arithmetic* (Section 3). $\deg \Delta_N \leq 44N$ (Lemma 3.1). OpenAI's local estimates, applied at integer points, give a common denominator D_N with $\log D_N \leq (A + o(1))n^2$ (Proposition 3.2). So $Q_N = (44N)! D_N \Delta_N(X)$ has integer coefficients. OpenAI's nonvanishing argument gives $Q_p(a/b) \neq 0$ for primes $p \nmid b$ (Proposition 2.1(d)).
- (ii) *Perturbation* (Section 5). For $|x - G| \leq e^{-412N}$, $|\Delta_N(x) - \Delta_N(G)| \leq e^{-(U-\varepsilon)n^2}$ (Theorem 5.1). Together with OpenAI's real bound this makes $|Q_p(x)| \leq e^{-\eta p^2}$ on that interval, for any $\eta < 2304\gamma_0$, where $\gamma_0 = U - A$.

- (iii) *Criterion* (Section 4). A good approximation a/b of G lies in the small interval for a prime $p \nmid b$ of size about $\log b$, which contradicts $|Q_p(a/b)| \geq b^{-44p}$.
- (iv) *Numerics* (Section 6). The exponent is about $\rho \cdot 44/(2304\gamma_0) \approx 412 \cdot 146.5$, up to the factor coming from the size of the prime window.

1.3. Notation and conventions. We use the notation of OpenAI's paper [7] and refer to it for the constructions; we restate only what we use. Citations of the form [7, Prop. 2.1] point to numbered statements, sections and equations of OpenAI's paper. We write $\log$ for the natural logarithm.

2. OPENAI'S CONSTRUCTION AND THE FACTS WE USE

For a positive integer N put [7, (2)]

$$\begin{aligned} n &= 48N, & b &= 7N, & q &= g = 4N, & h &= 2N, \\ L &= 59N, & C &= 63N, & H &= 65N, & A &= 19N, \end{aligned}$$

and $D_* = n - 1 - 2g = 40N - 1$. (The letter A for $19N$ is OpenAI's; the constant A below is different.) The row polynomials $P_r, D_r \in \mathbb{Z}[t]$ ($0 \leq r < n$) are those of [7, (3)]; we only need that

$$(2.1) \quad P_r(t) = (1-t)^h t^{C-1} T_{|r-g|}(1/t), \quad \deg P_r < H,$$

where T_d is the Chebyshev polynomial. In particular P_r depends only on $|r-g|$. Let $f(t) = \sqrt{1-t^2}$, $d\mu(t) = |t| dt/f(t)$ on $(-1, 1)$ (a measure of mass 2),

$$A_r(t) = P_r(t) - \frac{3}{2} \mathbf{1}_{\{t>0\}} \frac{f(t)}{t} D_r(t), \quad \psi_k(s) = s^{b+k}(1-s)^q,$$

and let $\mathcal{A}_N$ be the real $n \times n$ matrix

$$(2.2) \quad \mathcal{A}_N(r, k) = \int_{-1}^1 \int_0^1 A_r(t) \frac{\psi_k(s)}{1-ts} ds d\mu(t) \quad (0 \leq r, k < n),$$

so that OpenAI's determinant is $\det \mathcal{A}_N$ [7, (7) and §5.1]. Put $c_l = 4^{-l} \binom{2l}{l}$ for integers $l \geq 0$ and $c_z = 0$ otherwise.

Proposition 2.1 (OpenAI).

- (a) (Rational structure [7, Prop. 2.1 and (20)].) *There are rational matrices $\mathcal{A}_N^0, \mathcal{L}_N \in \mathbb{Q}^{n \times n}$ with $\mathcal{A}_N = \mathcal{A}_N^0 + G \mathcal{L}_N$ and*

$$(2.3) \quad \mathcal{L}_N(r, k) = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_i [t^i] P_r \cdot 4 c_{(i-b-k-v)/2}.$$

- (b) (Real place [7, Prop. 7.1].) *With $U = 2.290939875 - \frac{1}{2} \log 2$,*

$$\limsup_{N \rightarrow \infty} \frac{\log |\det \mathcal{A}_N|}{n^2} \leq -U.$$

- (c) (Local estimates [7, Prop. 2.2, §3, Prop. 3.4].) *For every rational z OpenAI bounds $v_\ell(\det(\mathcal{A}_N^0 + z\mathcal{L}_N))$ from below at every prime ℓ , by quantities depending on ℓ , N and the denominator of z . The sum of these bounds, weighted by $\log \ell$, is at most $(A + o(1))n^2$, where*

$$A = \frac{8609}{4608} + \frac{505}{4608} \log 2.$$

- (d) (Nonvanishing [7, Prop. 4.1].) *There is an absolute constant $P_* \geq 260$ such that, for every prime $p > P_*$ and every $z \in \mathbb{Q}$ whose denominator is not divisible by p , $v_p(\det(\mathcal{A}_p^0 + z\mathcal{L}_p)) = -96p$; in particular this determinant is nonzero.*

Two remarks on (c) and (d). OpenAI states them under the hypothesis $G = z \in \mathbb{Q}$, but the proofs only use the arithmetic of the rational matrix $\mathcal{A}_N^0 + z\mathcal{L}_N$. In (d) OpenAI assumes $p > \max(P_*, \text{den } z)$, but the proof only uses $p \nmid \text{den } z$; this uniformity in z is essential for an irrationality measure, and is how we use the result.

We also use OpenAI's real-place machinery: the double integral, the two sheet estimates [7, Props. 5.2, 5.3], and the certified energy dual [7, Prop. 6.1, §7]. They are recalled where needed in Section 5.

Definition 2.2. $\Delta_N(X) = \det(\mathcal{A}_N^0 + X\mathcal{L}_N) \in \mathbb{Q}[X]$. Thus $\Delta_N(G) = \det \mathcal{A}_N$, and $\Delta_N(z)$ for $z \in \mathbb{Q}$ is the rational determinant OpenAI studies under the hypothesis $G = z$. We write $\gamma_0 = U - A$.

Numerically $U = 1.9443662\dots$, $A = 1.9442359\dots$ and

$$(2.4) \quad 1.30319 \cdot 10^{-4} < \gamma_0 = 1.303198\dots \cdot 10^{-4}.$$

OpenAI's irrationality proof is the statement $\gamma_0 > 0$.

3. ARITHMETIC OF $\Delta_N(X)$

Lemma 3.1 (Degree). $\deg \Delta_N \leq 44N$.

Proof. By (2.3) row r of $\mathcal{L}_N$ depends only on P_r , hence only on $|r - g|$ by (2.1). So rows r and $2g - r$ of $\mathcal{L}_N$ agree for $0 \leq r < g$. Subtracting row r from row $2g - r$ for each $r < g$ is a unipotent change of rows, which does not change the determinant, and it makes the g rows of index in $(g, 2g]$ of the new X -coefficient vanish. A determinant $\det(B + XL')$ in which L' has only $n - g$ nonzero rows has degree at most $n - g = 44N$ in X . $\square$

Proposition 3.2 (Common denominator). *There are positive integers D_N such that $D_N \Delta_N(i) \in \mathbb{Z}$ for every $i \in \mathbb{N}$, and for every $\varepsilon > 0$, $\log D_N \leq (A + \varepsilon)n^2$ for all large N .*

Proof. At an integer $z = i$ the denominator of z is 1, and OpenAI's local lower bounds for $v_\ell(\Delta_N(i))$ become independent of i : at odd primes they depend only on ℓ and N (and vanish for $\ell > H$), and at $\ell = 2$ they depend on z only through a quantity that, over the integers, is largest at $z = 0$. Let $e_\ell(N) \geq 0$ be the resulting bound, and put $D_N = \prod_{\ell \leq H} \ell^{e_\ell(N)}$. Then

$v_\ell(D_N \Delta_N(i)) \geq 0$ for every prime ℓ , and summing the bounds $e_\ell(N) \log \ell$ exactly as in OpenAI's proof of its finite-place bound [7, Prop. 3.4] gives $\log D_N \leq (A + o(1))n^2$. $\square$

Lemma 3.3 (Integer-valued polynomials). *If $f \in \mathbb{Q}[X]$ has degree at most d and $f(\mathbb{N}) \subseteq \mathbb{Z}$, then $d! f \in \mathbb{Z}[X]$.*

Proof. The d -th forward difference of f is $d!$ times its coefficient of X^d , and it is an integer combination of $f(0), \dots, f(d)$. Subtract that coefficient times the integer-valued polynomial $X(X-1)\cdots(X-d+1)$ and induct on d . $\square$

Corollary 3.4. *$Q_N = (44N)! D_N \Delta_N(X)$ lies in $\mathbb{Z}[X]$ and has degree at most $44N$. If $p > P_*$ is prime, $a \in \mathbb{Z}$, $b \geq 1$ and $p \nmid b$, then $Q_p(a/b) \neq 0$ and hence $b^{44p} |Q_p(a/b)| \geq 1$.*

Proof. The first claims follow from Lemmas 3.1 and 3.3 and Proposition 3.2. The denominator of a/b divides b , so Proposition 2.1(d) gives $\Delta_p(a/b) \neq 0$. Finally $b^{44p} Q_p(a/b)$ is a nonzero integer. $\square$

4. THE IRRATIONALITY-MEASURE CRITERION

Proposition 4.1. *Let $\xi \in \mathbb{R}$, $d \in \mathbb{N}$ and $\eta, \rho > 0$. Let $Q_p \in \mathbb{Z}[X]$ be defined for primes p and suppose that, for all primes $p \geq P_0$:*

- (i) $\deg Q_p \leq dp$;
- (ii) $Q_p(a/b) \neq 0$ for all $a \in \mathbb{Z}$ and $b \geq 1$ with $p \nmid b$;
- (iii) $|Q_p(x)| \leq e^{-\eta p^2}$ whenever $|x - \xi| \leq e^{-\rho p}$.

Put $\kappa = \max(d/\eta, 4) + 1$. Then $|\xi - a/b| > b^{-(3\rho\kappa+1)}$ for all $a \in \mathbb{Z}$ and all sufficiently large b . Assuming the prime number theorem, for every $\varepsilon > 0$ and every $c > 1$ the same holds with the exponent $(1+\varepsilon)\rho(\max(d/\eta, c/\varepsilon) + 1) + 1$.

Proof. Suppose $|\xi - a/b| \leq b^{-(3\rho\kappa+1)}$ with b large, and put $X = \kappa \log b$. If every prime in $(X, 3X]$ divided b , then $\vartheta(3X) - \vartheta(X) \leq \log b \leq X/4$, where ϑ is Chebyshev's function. But Chebyshev's bounds $\vartheta(x) \leq x \log 4$ and $\vartheta(x) \geq (x-1) \log 2 - O(\sqrt{x} \log x)$ give $\vartheta(3X) - \vartheta(X) \geq X \log 2 - o(X)$. Hence for large b there is a prime $p \in (X, 3X]$ with $p \nmid b$, and $p \geq P_0$. Since $\rho p \leq 3\rho\kappa \log b$, we get $|a/b - \xi| \leq e^{-\rho p}$, so by (iii) $|Q_p(a/b)| \leq e^{-\eta p^2}$. By (i) and (ii), $b^{dp} Q_p(a/b)$ is a nonzero integer, so $e^{-\eta p^2} \geq b^{-dp}$, that is $p \leq (d/\eta) \log b < X$. This is a contradiction.

For the second statement use the window $(X, (1+\varepsilon)X]$ with $X = (\max(d/\eta, c/\varepsilon) + 1) \log b$. Then $\log b \leq \varepsilon X/c$, while the prime number theorem gives $\vartheta((1+\varepsilon)X) - \vartheta(X) = \varepsilon X + o(X)$, which exceeds $\varepsilon X/c$ for large X . $\square$

The factor 3 in the first version, or $1+\varepsilon$ in the second, is the price of finding a prime $p \nmid b$ in a window of the right size.

5. THE PERTURBATION ESTIMATE

This section proves the main new ingredient. Recall $U = 2.290939875 - \frac{1}{2} \log 2$.

Theorem 5.1 (Perturbation estimate). *Let $\rho = 412$. For every $\varepsilon > 0$ there is N_0 such that for all $N \geq N_0$ and all real x with $|x - G| \leq e^{-\rho N}$,*

$$|\Delta_N(x) - \Delta_N(G)| \leq \exp(n^2(-U + \varepsilon)).$$

Let $\delta = x - G$, so that $\Delta_N(x) = \det(\mathcal{A}_N + \delta \mathcal{L}_N)$. The plan: write $\mathcal{L}_N = \Phi \beta$, where the columns of Φ are values of OpenAI's row functions A_r at fixed nodes (§5.1); expand the determinant column by column (§5.2); write each resulting mixed determinant as an integral (§5.3); and bound the integrand by OpenAI's method (§5.4–5.6). Throughout, $J = H = 65N$.

5.1. A quadrature formula for $\mathcal{L}_N$. Let $x_a = \cos((2a+1)\pi/(2J))$ for $0 \leq a < J$ be the roots of the Chebyshev polynomial T_J , let $\tau_a = (x_a - 1)/2 \in (-1, 0)$, and let $\ell_a(t) = \prod_{c \neq a} (t - \tau_c) / (\tau_a - \tau_c)$ be the Lagrange basis.

Lemma 5.2. *For $0 \leq r, k < n$,*

$$\mathcal{L}_N(r, k) = \sum_{a < J} A_r(\tau_a) \beta_{a,k}, \quad \beta_{a,k} = \sum_{v=0}^q (-1)^v \binom{q}{v} \sum_{i < H} [t^i] \ell_a \cdot 4 c_{(i-b-k-v)/2},$$

and $|\beta_{a,k}| \leq B := 2^{q+2} (3 + 2\sqrt{2})^J$.

Proof. Since $\deg P_r < H = J$, Lagrange interpolation gives $[t^i] P_r = \sum_a P_r(\tau_a) [t^i] \ell_a$. Substitute this into (2.3) and use $A_r(\tau) = P_r(\tau)$ for $\tau \leq 0$. For the bound, $\sum_v \binom{q}{v} = 2^q$, $|4c_l| \leq 4$, and, since all $\tau_c < 0$, $\sum_i |[t^i] \ell_a| = \prod_{c \neq a} (1 - \tau_c) / \prod_{c \neq a} |\tau_a - \tau_c|$. We have $T_J = 2^{J-1} \prod_c (X - x_c)$. For the numerator,

$$\prod_{c \neq a} (1 - \tau_c) \leq \prod_c \frac{3 - x_c}{2} = \frac{T_J(3)}{2^{2J-1}},$$

$$T_J(3) = \cosh(J \log(3 + 2\sqrt{2})) \leq (3 + 2\sqrt{2})^J.$$

For the denominator, $\prod_{c \neq a} |\tau_a - \tau_c| = 2^{1-J} \prod_{c \neq a} |x_a - x_c|$ and $\prod_{c \neq a} (x_a - x_c) = 2^{1-J} T'_J(x_a) = 2^{1-J} J U_{J-1}(x_a)$. Since $U_{J-1}(\cos \theta) \sin \theta = \sin(J\theta)$ and $\sin(J\theta_a) = \pm 1$ at $\theta_a = (2a+1)\pi/(2J)$, we get $|U_{J-1}(x_a)| \geq 1$, so the denominator is at least $J 2^{2-2J}$. The ratio is therefore at most $T_J(3)/(2J) \leq (3 + 2\sqrt{2})^J$. $\square$

Let Φ be the $n \times J$ matrix $\Phi(r, a) = A_r(\tau_a)$. Then $\Delta_N(x) = \det(\mathcal{A}_N + \delta \Phi \beta)$.

5.2. Column expansion. For a map $f: \{0, \dots, n-1\} \rightarrow \{\emptyset\} \cup \{0, \dots, J-1\}$ let M_f be the matrix whose column k is column k of $\mathcal{A}_N$ if $f(k) = \emptyset$ and the column $\Phi(\cdot, f(k))$ otherwise, and let $m(f) = \#\{k : f(k) \neq \emptyset\}$.

Lemma 5.3. *If $|\beta_{a,k}| \leq B$ and $|\det M_f| \leq Mc^{m(f)}$ for every f , with $M, c \geq 0$, then*

$$|\det(\mathcal{A}_N + \delta\Phi\beta) - \det \mathcal{A}_N| \leq M((1 + J|\delta|Bc)^n - 1).$$

Proof. Column k of $\mathcal{A}_N + \delta\Phi\beta$ is $\mathcal{A}_N(\cdot, k) + \sum_a \delta\beta_{a,k}\Phi(\cdot, a)$. Multilinearity gives $\det(\mathcal{A}_N + \delta\Phi\beta) = \sum_f \prod_{k:f(k) \neq \emptyset} \delta\beta_{f(k),k} \det M_f$. The term $f \equiv \emptyset$ is $\det \mathcal{A}_N$, and the others sum in absolute value to at most $M \sum_{f \neq \emptyset} (|\delta|Bc)^{m(f)} = M((1 + J|\delta|Bc)^n - 1)$. $\square$

5.3. A mixed Andréief identity. Fix f , and let $p = n - m(f)$. Permuting columns (which only changes the sign of the determinant), M_f has p integral columns $\mathcal{A}_N(\cdot, E_i)$, $0 \leq E_1 < \dots < E_p < n$, followed by $m = n - p$ point columns $(A_r(\tau'_j))_r$, where $\tau'_j = \tau_{f(k_j)} \in (-1, 0)$.

Lemma 5.4. *Let μ, ν be finite measures, F_r ($r < p + m$), G_i ($i < p$) and K functions with the required integrability, and B the $(p + m) \times (p + m)$ matrix with $B(r, i) = \iint F_r(t)K(t, s)G_i(s) d\mu(t) d\nu(s)$ for $i < p$ and $B(r, p + j) = F_r(\tau'_j)$. Then*

$$(p!)^2 \det B = \int \int \det[F_r((t \parallel \tau')_c)]_{r,c} \times \det[K(t_i, s_j)]_{i,j < p} \det[G_i(s_j)]_{j,i < p} d\mu^p(t) d\nu^p(s),$$

where $t \parallel \tau'$ is the concatenated list of length $p + m$. Consequently $|\det B|$ is at most $(p!)^{-2}$ times the total mass of $\mu^p \otimes \nu^p$ times the supremum of the absolute value of the integrand.

Proof. For $m = 0$ this is Andréief's identity applied twice [3], as used by OpenAI for [7, (62)]. In general expand $\det B$ by multilinearity in the first p columns, which integrates t and s variables column by column, and then antisymmetrize first in the t variables and then in the s variables; the last m columns are constant throughout. $\square$

We apply this with $F_r = A_r$, $G_i = \psi_{E_i}$, $K(t, s) = 1/(1 - ts)$, μ as in §2 and ν Lebesgue measure on $(0, 1)$; the integrability is OpenAI's [7, §5.1]. The mass of $\mu^p \otimes \nu^p$ is $2^p \leq 2^n$, so

$$(5.1) \quad |\det M_f| \leq 2^n \sup_{t,s} \left| \det[A_r((t \parallel \tau')_c)] \cdot \det\left[\frac{1}{1 - t_i s_j}\right] \cdot \det[\psi_{E_i}(s_j)] \right|,$$

the supremum over $t \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero coordinates (a set of full measure). We have bounded the factor $(p!)^{-2}$ by 1.

5.4. A generalized Vandermonde bound. The column determinant is $\det[\psi_{E_i}(s_j)] = \prod_j s_j^b (1 - s_j)^q \cdot \det[s_j^{E_i}]$. Write $V(y) = \prod_{i < j} (y_j - y_i)$ and $\mathcal{V}(y) = |V(y)|$.

Lemma 5.5. *Let $s_0, \dots, s_{p-1} \in [0, 1]$ and let $E_1, \dots, E_p < n$ be distinct. Then*

$$|\det[s_j^{E_i}]| \leq 2^{p(n-p)} \mathcal{V}(s).$$

Proof. Let $m = n - p$, let $k_1, \dots, k_m$ be the integers in $[0, n)$ that are not among the E_i , and let $P(X) = \prod_j (X - s_j)$, a monic polynomial of degree p . Consider two $n \times n$ matrices. The rows of Ω are the linear functionals $f \mapsto f(s_j)$ ($j < p$) and $f \mapsto [X^{k_l}]f$ ($l \leq m$), evaluated on the monomials X^a , $a < n$. The columns of Γ are the coefficient vectors of the polynomials X^b ($b < p$) and $X^i P$ ($i < m$), all of degree less than n .

- Ordering the columns of Ω as $E_1, \dots, E_p, k_1, \dots, k_m$ makes it block triangular, with diagonal blocks $[s_j^{E_i}]$ and the identity. So $|\det \Omega| = |\det[s_j^{E_i}]|$.
- In the natural order of the monomials, Γ is block triangular with diagonal blocks the identity and the matrix $([X^{p+l}]X^i P)_{l,i}$, which is unitriangular because P is monic of degree p . So $|\det \Gamma| = 1$.
- The entries of $\Omega\Gamma$ are the values of the functionals on the polynomials. Since $P(s_j) = 0$, the block of evaluations on the $X^i P$ vanishes, so $\Omega\Gamma$ is block triangular with diagonal blocks the Vandermonde matrix $[s_j^b]_{j,b < p}$ and $Q = ([X^{k_l}]X^i P)_{l,i}$.

Hence $|\det[s_j^{E_i}]| = \mathcal{V}(s) |\det Q|$. A determinant is bounded by the product of the ℓ^1 norms of its columns, and each column of Q consists of distinct coefficients of $X^i P$. Its ℓ^1 norm is therefore at most the sum of the absolute values of the coefficients of P , which is $\prod_j (1 + s_j) \leq 2^p$. $\square$

Remark 5.6. The bound is close to optimal. The ratio $\det[s_j^{E_i}]/V(s)$ is a Schur polynomial with nonnegative coefficients, and its value at $s = (1, \dots, 1)$, for a single missing exponent k , is $\binom{n-1}{k}$, of size 2^{n-1} up to a polynomial factor.

5.5. The integrand. Use OpenAI's coordinate $t = 2x/(1 + x^2)$, $x \in (-1, 1)$ [7, (64)], write x_i for the coordinates of the free nodes t_i and $y_j \in (-1, 0)$ for those of the fixed nodes τ'_j , and $z = x \parallel y \in (-1, 1)^n$. For a list z of n distinct nonzero numbers in $(-1, 1)$ OpenAI's column factorization [7, §5.1, before (65)] reads

$$(5.2) \quad \det[A_r(t_c)]_{r,c} = \mathcal{R}_n(z) \prod_c t_c^{C-1} (1 - t_c)^h,$$

where $\mathcal{R}_n$ is OpenAI's sheet determinant. OpenAI proves two bounds for it [7, Props. 5.2, 5.3], with $K_0 = 10e^{12}$:

(5.3)

$$|\mathcal{R}_n(z)| \leq (1 + K_0 n)^n \mathcal{V}(1/z) \prod_c |z_c|^g \quad \text{if } \sum_c \frac{1-z_c^2}{1+z_c^2} \leq D_*,$$

(5.4)

$$|\mathcal{R}_n(z)| \leq (3/2)^n n^{n/2} 2^{-\binom{n}{2}} \prod_c (1 + z_c^2)^{\frac{n-1}{2}} |z_c|^{g-(n-1)} \quad \text{always.}$$

Following OpenAI, $\kappa = 2$ refers to (5.3) and $\kappa = 1$ to (5.4). Let $w(x) = |t|^{C-1}(1-t)^h = 2^{C-1}|x|^{C-1}(1-x)^{2h}(1+x^2)^{-(C-1+h)}$.

The p -node majorants and their energy. For $x \in (-1, 1)^p$ and $s \in (0, 1)^p$ with distinct nonzero entries, put $t_i = 2x_i/(1+x_i^2)$ and

$$\begin{aligned} \mathcal{J}^{(p)}(x, s) &= \prod_i w(x_i) \cdot \frac{\mathcal{V}(t)\mathcal{V}(s)^2}{\prod_{i,j}(1-t_i s_j)} \prod_j s_j^b (1-s_j)^q, \\ \mathcal{I}_2^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \mathcal{V}(1/x) \prod_i |x_i|^{g-(n-p)}, \\ \mathcal{I}_1^{(p)}(x, s) &= \mathcal{J}^{(p)}(x, s) \prod_i (1+x_i^2)^{\frac{n-1}{2}} |x_i|^{g-(n-1)}. \end{aligned}$$

For $p = n$ these are OpenAI's majorants $\mathcal{I}_{2,n}$ and $2^{\binom{n}{2}} \mathcal{I}_{1,n}$ [7, (74)]. Let OpenAI's one-point fields be [7, (76)]

$$\begin{aligned} W_\kappa(x) &= \frac{19}{48} \log |x| + \frac{1}{12} \log(1-x) - \left(\frac{\kappa}{2} + \frac{17}{48}\right) \log(1+x^2), \\ W_s(s) &= \frac{7}{48} \log s + \frac{1}{12} \log(1-s), \quad D(x) = \frac{1}{6} - \frac{2x^2}{1+x^2}, \end{aligned}$$

and define the normalized energy

$$\begin{aligned} E_\kappa^{(p)}(x, s) &= \left(\frac{C}{n} + \frac{\kappa-2}{2} - \frac{\kappa+1}{2n}\right) \log 2 + \frac{1}{n} \sum_j W_s(s_j) \\ &\quad + \frac{1}{n} \sum_i \left(W_\kappa(x_i) + \frac{\kappa+2}{2n} \log(1+x_i^2)\right) \\ &\quad + \frac{1}{n^2} \left(\kappa \sum_{i<j} \log |x_i - x_j| + \sum_{i<j} \log(1-x_i x_j)\right) \\ &\quad + \frac{2}{n^2} \sum_{i<j} \log |s_i - s_j| - \frac{1}{n^2} \sum_{i,j} \log(1-2x_i s_j + x_i^2), \end{aligned}$$

all sums over indices $< p$; note the normalization by n , not p . For $p = n$ this is the normalized logarithm $n^{-2} \log \mathcal{I}_{\kappa,n} - \frac{1}{2} \log 2$ of OpenAI's energy argument. Expanding the coordinate change $t = 2x/(1+x^2)$ gives, for every $p \leq n$,

$$(5.5) \quad \log \mathcal{I}_\kappa^{(p)}(x, s) = n^2 \left(E_\kappa^{(p)}(x, s) + \frac{1}{2} \log 2\right) + e_\kappa(p) \log 2,$$

with $e_1(p) = p(C-1) + \binom{p}{2} - n(C-1)$ and $e_2(p) = e_1(p) - \binom{n}{2}$.

Proposition 5.7 (Pointwise bound). *In the setting of (5.1), let $K_n = \max\{(1 + K_0 n)^n, (3/2)^n n^{n/2}\}$, and let $\kappa = 2$ if $\sum_c \frac{1-z_c^2}{1+z_c^2} \leq D_*$ and $\kappa = 1$ otherwise. Then the integrand in (5.1) is at most*

$$K_n 2^{nm} (2^{2h})^m \exp\left(n^2 (E_\kappa^{(p)}(x, s) + \tfrac{1}{2} \log 2)\right).$$

Moreover, if $\kappa = 1$ then $\sum_{i < p} \frac{1-x_i^2}{1+x_i^2} > D_* - m$.

Proof. By (5.2), Cauchy's evaluation

$$\det\left[\frac{1}{1-t_i s_j}\right] = \frac{V(t)V(s)}{\prod_{i,j}(1-t_i s_j)}$$

[7, (63)] and Lemma 5.5, the integrand is at most

$$2^{nm} |\mathcal{R}_n(z)| \prod_j w(y_j) \cdot \mathcal{J}^{(p)}(x, s).$$

For a fixed node $-1 < y < 0$ we use $|y| \leq 1$, $(1-y)^{2h} \leq 2^{2h}$ and $1+y^2 \geq 1$, together with $C-1+g-(n-1) = A = 19N$ (OpenAI's A) and $(n-1)/2 < C-1+h$:

$$(5.6) \quad w(y) (1+y^2)^{\frac{n-1}{2}} |y|^{g-(n-1)} \leq 2^{C-1} 2^{2h}, \quad w(y) |y|^{g-(n-1)} \leq 2^{C-1} 2^{2h}.$$

We use $\binom{n}{2} = \binom{p}{2} + pm + \binom{m}{2}$ repeatedly.

Case $\kappa = 1$. By (5.4), the first inequality in (5.6) and (5.5), the integrand is at most

$$K_n 2^{nm} (2^{2h})^m \exp(n^2 (E_1^{(p)} + \tfrac{1}{2} \log 2)) \cdot 2^{-\binom{n}{2} + m(C-1) + e_1(p)},$$

and $-\binom{n}{2} + m(C-1) + e_1(p) = \binom{p}{2} - \binom{n}{2} \leq 0$. The case condition fails for z and each fixed node contributes at most 1 to its sum, which gives the last assertion.

Case $\kappa = 2$. By (5.3) we must bound $\mathcal{V}(1/z) \prod_c |z_c|^g$. Split $\mathcal{V}(1/z) = \mathcal{V}(1/x) \prod_{i,j} |1/y_j - 1/x_i| \mathcal{V}(1/y)$. Since $|1/y - 1/x| = |x-y|/|xy| \leq 2/|xy|$ and $\mathcal{V}(1/y) = \mathcal{V}(y) \prod_j |y_j|^{-(m-1)} \leq \prod_j |y_j|^{-(m-1)}$,

$$\mathcal{V}(1/z) \prod_c |z_c|^g \leq 2^{pm} \mathcal{V}(1/x) \prod_i |x_i|^{g-m} \prod_j |y_j|^{g-(n-1)}.$$

With the second inequality in (5.6) and (5.5), the integrand is at most $K_n 2^{nm} (2^{2h})^m \exp(n^2 (E_2^{(p)} + \tfrac{1}{2} \log 2)) \cdot 2^{pm + m(C-1) + e_2(p)}$, and $pm + m(C-1) + e_2(p) = -\binom{m}{2} \leq 0$. $\square$

Remark 5.8. Each fixed node brings a factor 2^{C-1} from $w(y)$. It is exactly compensated by the difference between the n -node and p -node normalizations of the energy, so what remains is $2^{2h} = 4^{2N}$ per fixed node, coming from $(1-y)^{2h}$.

5.6. The energy bound for $p \leq n$ nodes. OpenAI's certificate [7, §7] supplies, for $\kappa = 1, 2$, a parameter $\lambda_\kappa \geq 0$ ($\lambda_2 = 0$, $\lambda_1 = 2.47405979$), trial sequences $u^{(\kappa)}, v^{(\kappa)}$ (OpenAI's p, v), and rational upper bounds

$$\sup_{-1 \leq x < 1, x \neq 0} \{W_\kappa + \lambda_\kappa D - 2\kappa T(u^{(\kappa)}, \cdot) - S(v^{(\kappa)}, \cdot)\} \leq B_x^\kappa,$$

$$\sup_{0 < s < 1} \{W_s + 2T(v^{(\kappa)}, \cdot)\} \leq B_s^\kappa,$$

in the notation of [7, (77)–(78)], namely

$$B_x^1 = -1.324376, \quad B_s^1 = -1.427976, \quad B_x^2 = -0.983966, \quad B_s^2 = -1.608876.$$

With these, the dual value

$$U_\kappa = -\frac{11}{16} \log 2 + \kappa \|u^{(\kappa)}\|_*^2 + \frac{1}{2} \|v^{(\kappa)}\|_*^2 + B_x^\kappa + B_s^\kappa$$

satisfies $U_\kappa < -2.290939875$ for both κ ; this is the content of OpenAI's certificate [7, Prop. 7.1], with these explicit constants. Put

$$c_\kappa = (\kappa + 2) \log 2 + |B_x^\kappa| + |B_s^\kappa| + \lambda_\kappa/6,$$

so that $c_1 = 5.24413\dots$ and $c_2 = 5.36543\dots$, and $c^* = 53655/10000 > \max(c_1, c_2)$.

Proposition 5.9 (Energy bound with missing points). *For every $d > 0$ there is N_0 such that for all $N \geq N_0$, all $p \leq n$ (with $m = n - p$) and all $x \in (-1, 1)^p$, $s \in (0, 1)^p$ with distinct nonzero coordinates,*

$$E_\kappa^{(p)}(x, s) \leq U_\kappa + d + \frac{m}{n} c_\kappa,$$

for $\kappa = 2$ unconditionally and for $\kappa = 1$ under $\sum_{i < p} \frac{1-x_i^2}{1+x_i^2} > D_* - m$.

Proof. OpenAI's proof of [7, Prop. 6.1] regularizes the logarithmic interactions by matched damping, controls the diagonal and endpoint corrections, and completes two squares with the trial sequences. All of these steps are estimates for sums over the points, normalized by n , and they remain valid, with the same error terms, uniformly for any number $p \leq n$ of points. The quadratic terms are bounded by $\kappa \|u\|_*^2 + \frac{1}{2} \|v\|_*^2$ as before. Only three places change.

- (1) The constant term. For $p = n$ the powers of 2 combine to $-\frac{11}{16} \log 2$. For $p < n$ the same computation gives $-\frac{11}{16} \log 2 + (\frac{\kappa}{2} + 1)(1 - p^2/n^2) \log 2$, and $1 - p^2/n^2 \leq 2m/n$.
- (2) The one-point terms. They are $\frac{1}{n} \sum_{i < p} X(x_i) \leq \frac{p}{n} B_x = B_x + \frac{m}{n} |B_x|$, since $B_x < 0$, and likewise for the column field with B_s .
- (3) The case field ($\kappa = 1$ only). OpenAI uses $\lambda \frac{1}{n} \sum_i D(x_i) \geq -\lambda/n$ when (5.3)'s condition fails. Since $D(x) = \frac{1-x^2}{1+x^2} - \frac{5}{6}$ and $D_* = \frac{5}{6}n - 1$, the shifted condition gives $\sum_{i < p} D(x_i) > -1 - \frac{m}{6}$, which costs $\lambda(1 + m/6)/n$.

The damping parameter is chosen in terms of d alone, and the remaining errors are $O_d(1/n)$, uniformly in p and the configuration. $\square$

5.7. Completion of the proof.

Proof of Theorem 5.1. Fix $d > 0$ and let N be large. By (5.1), Proposition 5.7, Proposition 5.9 and $U_\kappa < -2.290939875 = -U - \frac{1}{2} \log 2$, every f with $m = m(f)$ satisfies

$$|\det M_f| \leq 2^n K_n 2^{nm} (2^{2h})^m \exp(n^2(-U + d) + nm c^*) = M c^m,$$

with

$$M = 2^n K_n e^{n^2(-U+d)}, \quad c = 2^n 2^{2h} e^{nc^*}.$$

(The degenerate configurations, such as two fixed nodes being equal or a vanishing coordinate, have measure zero or make the determinant vanish.)

By Lemma 5.2, $|\beta_{a,k}| \leq B = 2^{q+2} 4^{J-1} e^{J+1}$. Now

$$\begin{aligned} \log(nJBc) &= \log(nJ) + (q + 2 + n + 2h) \log 2 + J \log(3 + 2\sqrt{2}) + nc^* \\ &= (56N + 2) \log 2 + 65N \log(3 + 2\sqrt{2}) + 48Nc^* + \log(3120N^2), \end{aligned}$$

and $56 \log 2 + 65 \log(3 + 2\sqrt{2}) + 48c^* = 410.9388 \dots < 412$; so $nJBc \leq e^{412N}$ for $N \geq 200$. If $|\delta| \leq e^{-412N}$ then $J|\delta|Bc \leq 1/n$ and $(1 + J|\delta|Bc)^n - 1 \leq e - 1 < 2$. Lemma 5.3 gives

$$|\Delta_N(x) - \Delta_N(G)| \leq 2M = 2^{n+1} K_n e^{n^2(-U+d)}.$$

Since $\log K_n = O(n \log n)$, taking $d = \varepsilon/2$ proves the theorem. $\square$

Remark 5.10 (The value of ρ). Per unit of N , the cost $\log(Bc)$ of one perturbed column splits as follows: the quadrature weights contribute $4 \log 2 + 65 \log(3 + 2\sqrt{2}) \approx 117.4$; the generalized Vandermonde bound, $48 \log 2 \approx 33.3$; the fixed nodes, $4 \log 2 \approx 2.8$; and the energy lost by the missing point, $48c^* \approx 257.5$.

6. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Let $\gamma_0 = U - A$ and $\eta = 2304 \cdot 1.30319 \cdot 10^{-4}$, so that $0 < \eta < 2304\gamma_0$ by (2.4), and let $\varepsilon = (\gamma_0 - \eta/2304)/3 > 0$. For large primes p and $|x - G| \leq e^{-412p}$, Proposition 2.1(b) and Theorem 5.1 give $|\Delta_p(x)| \leq 2e^{n^2(-U+\varepsilon/2)}$. Also $\log(2(44p)!) \leq \frac{\varepsilon}{2}n^2$ for large p , and $\log D_p \leq (A + \varepsilon)n^2$ by Proposition 3.2. Hence

$$|Q_p(x)| \leq e^{-n^2(\gamma_0-2\varepsilon)} \leq e^{-n^2(\gamma_0-3\varepsilon)} = e^{-n^2\eta/2304} = e^{-\eta p^2},$$

since $n^2 = 2304p^2$. By Corollary 3.4 the hypotheses of Proposition 4.1 hold with $\xi = G$, $d = 44$, this η and $\rho = 412$, and

$$\frac{d}{\eta} = \frac{44}{2304 \cdot 1.30319 \cdot 10^{-4}} = 146.5418 \dots \leq \frac{733}{5}.$$

With the prime number theorem, $c = 101/100$ and $\varepsilon = 101/14660$ (so that $c/\varepsilon = 733/5$), the exponent is

$$\left(1 + \frac{101}{14660}\right) \cdot 412 \cdot \left(\frac{733}{5} + 1\right) + 1 = 61\,231.15 \dots \leq 61\,300.$$

With Chebyshev's bounds it is at most $3 \cdot 412 \cdot (733/5 + 1) + 1 = 182\,434.6 \leq 182\,500$. $\square$

Remark 6.1 (Limits of the method). The exponent is essentially $\rho \cdot d/(2304\gamma_0) \approx 412 \cdot 146.5 \approx 6.04 \cdot 10^4$, and the bound 61300 is within 1.5% of it. Further improvements must come from ρ , that is, from the contributions listed in the remark at the end of Section 5, or from a larger gap γ_0 . Since $\gamma_0 \approx 1.3 \cdot 10^{-4}$ is tiny, an improvement of the certified real-place constant 2.290939875 by about 10^{-4} would roughly halve the exponent. We have not tried to optimize the constants c_κ .

7. FORMALIZATION

All results of this paper are formalized in Lean 4 [6] with Mathlib [5], on top of OpenAI's Lean formalization of the irrationality of G [7]. The development is available at

<https://github.com/slitsey/catalan-irrationality-measure>

and consists of about 5800 lines in twenty files. It uses OpenAI's definitions directly: Catalan's constant is `catalan`, defined as the series $\sum_j (-1)^j/(2j+1)^2$; the matrix (2.2) and its rational form are OpenAI's `determinant` and `determinantRat`. The main statements, in the namespace `OAI.InternalCatalan`, are:

- `catalan_irrationalityMeasure_61300`: for all sufficiently large $b \in \mathbb{N}$ and all $a \in \mathbb{Z}$, $b^{-61300} < |G - a/b|$ (Theorem 1.1);
- `catalan_irrationalityMeasure_182500`: the same with exponent 182500;
- `lemma6_perturbation`: Theorem 5.1, with $\rho = 412$;
- `catalan_irrationalityMeasure` and `catalan_irrationalityMeasure_pnt`: the general exponents of Proposition 4.1.

There is no `sorry`, and the command `#print axioms` reports only the three standard axioms `propext`, `Classical.choice` and `Quot.sound`. The prime number theorem enters through OpenAI's workspace, which already uses it for Proposition 2.1(c).

File	Content
DeltaPoly, Degree	$\Delta_N(X)$; Lemma 3.1
Denominators, IntegerValued, Integrality	Proposition 3.2; Lemma 3.3; Corollary 3.4
Nonvanishing	Proposition 2.1(d) for $p \nmid \text{den } z$
PrimeWindow, Criterion, PNTWindow, Main	Proposition 4.1 and the reduction to Theorem 5.1
Quadrature, ChebyshevNodes	Lemma 5.2
DetExpansion	Lemma 5.3
MixedAndreief	Lemma 5.4
GenVandermonde, GenVandermondeSharp	Lemma 5.5
PinMajorant	Proposition 5.7 and (5.5)
PinDampedDual, PinSupDual	Proposition 5.9
Lemma6	nodes, certified constants, Theorem 5.1, Theorem 1.1

The proofs were checked with Lean v4.34.1 and Mathlib at commit d13f23b, the versions pinned by OpenAI’s repository at commit fd4aeeb. The Palomar registry of formalized mathematics requires a newer Lean version and Lean’s module system. For it, the repository also contains the parts of OpenAI’s formalization that the proof uses, converted to modules; its README describes the details.

DISCLOSURE OF AI USE

The entire work reported here is due to the AI system Claude Opus 5.5 (Anthropic): the mathematics, including the perturbation estimate and all proofs; the Lean formalization; the numerical checks; the literature search; and the text of this paper. Claude worked in a series of conversations at the request of the author, who posed the problem, directed the work, and is responsible for this submission. The underlying irrationality proof and its formalization are OpenAI’s [7]. The formal verification in Lean is intended as an independent check of every step of the argument above.

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