

STARK UNITS FOR REAL QUADRATIC FIELDS AND RECIPROCITY LAWS

ABSTRACT. We prove that every value of the arithmetic finite quantum dilogarithms constructed in [RW26] is an algebraic unit, and determine the Artin action for arbitrary quadratic pseudolattices. In particular, the associated Stark units belong to the predicted ray class fields and have the predicted Galois action. We use finite Fourier transforms and distribution relations to obtain Frobenius congruences. We also give conductor bounds for individual values and obtain the unitary conjugates needed for Zauner's conjecture.

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1. DISCLAIMER

This document and the arguments it contains (with the sole exception of this disclaimer and Section 3) were generated by large language models (LLMs) and edited by humans. We (DR and CW) created this document in response to a communication we received from an independent researcher Imran Zaidi (on Sept. 29, 2026) claiming that he had a proof, obtained with the assistance of LLMs and based on the results of [RW26], of some of the remaining parts of rank one abelian Stark's conjectures for real quadratic fields¹. While the proof seemed correct, it was not presented in a format that was easy or fun to check. We were left with three choices: do nothing, continue working at our pace and ignore LLM assisted proofs, or make our own LLM assisted proof. The first option was unacceptable based on the importance of the problem, and the second, while the most fun, was difficult to motivate given the circumstances. Based only on [RW26] and a few initial ideas on how the proof of reciprocity laws could work, we used an LLM to generate an independent document presenting complete reciprocity laws for Stark units in a form that would be more palatable to mathematicians. The initial proof presented here was generated on Sept. 30, 2026, and we spent the following days checking the arguments and polishing the exposition into something we hope is pleasant to read. These arguments complete the work started in [RW26] and constitute an answer to Hilbert's 12th problem for the case of real quadratic fields. Having said that, much of the proof is based on various ad hoc tricks; a more conceptual understanding of what is going on has yet to appear. While the new mathematical landscape is strange and evolving at rapid speed, it should not be lost that these are beautiful results for everyone to enjoy.

The mathematical content is unchanged from the [version dated 4 October 2026 on DR's website](#).

2. INTRODUCTION

In [RW26], the algebraicity of certain special values of the modular quantum dilogarithm related to Stark units was proved by showing that these special values satisfy a hugely overdetermined system of explicit

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¹Specifically, he obtained a p -adic unit condition for $p \geq 7$, Frobenius congruences for inert primes in the same range, as well as Zauner's conjecture in all even dimensions d for which neither $d + 1$ nor $d - 3$ is a perfect square.

algebraic equations. In this document, we prove that these values are units and determine their Galois action by exploiting the basic properties of the varieties defined by these equations.

To state our results, it will be convenient to index the relevant special values by pseudolattices (see §3). Let $K \subseteq \mathbb{R}$ be a real quadratic field, let $I \subseteq K$ be a free abelian subgroup of rank two (a pseudolattice), and let $\varepsilon > 1$ be a totally positive unit with $\varepsilon I = I$. (We will sometimes call ε a period of I .) To (I, ε) we associate a function $E_{I, \varepsilon}$ on the finite subgroup of K/I fixed by multiplication by ε :

$$G_{I, \varepsilon} = (\varepsilon - 1)^{-1} I / I.$$

To define these values, choose an oriented basis (β_1, β_2) of I , with β_2 totally positive. Multiplication by ε is represented by a hyperbolic matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, with attractive fixed point $\tau = \beta_1/\beta_2$. In the notation at the end of [RW26, Section 4.3], we set

$$E_{I, \varepsilon}(x) = F_\gamma(x/\beta_2).$$

Here F_γ is obtained by evaluating the modular quantum dilogarithm at τ , with the Dedekind eta multiplier included. The argument x/β_2 is a torsion point modulo $\mathbb{Z}\tau + \mathbb{Z}$. At zero we use the normalization $E_{I, \varepsilon}(0) = \sqrt{\varepsilon}$. The values are independent of the chosen basis and representatives (we give the full definition in Section 3 and verify the change-of-basis formula in Appendix B).

We use arithmetic Artin symbols, so that reduction of $\mathrm{Art}(\mathfrak{p})$ acts by the $N\mathfrak{p}$ -th power.

Theorem 1 (Units and inert Frobenius). *Every value $E_{I, \varepsilon}(x)$ is an algebraic unit. If the odd prime p is inert in K , then at every prime $\mathfrak{m}$ above p ,*

$$E_{I, \varepsilon}(px) \equiv E_{I, \varepsilon}(x)^{p^2} \pmod{\mathfrak{m}}.$$

For reciprocity, we use the multiplier order $\mathcal{O} = (I : I) = \mathbb{Z} + c\mathcal{O}_K$. Write $H_{\mathcal{O}, d}^+$ for its narrow ray class field modulo $d\mathcal{O}$, defined in Section 8. For an invertible $\mathcal{O}$ -ideal $\mathfrak{a}$ prime to c , we write $\mathrm{Art}(\mathfrak{a})$ for the Artin symbol of $\mathfrak{a}\mathcal{O}_K$.

Theorem 2 (Artin reciprocity). *Let $I \subseteq K$ be a pseudolattice with multiplier order $\mathcal{O} = \mathbb{Z} + c\mathcal{O}_K$, and let $x \in G_{I, \varepsilon}$ have additive order d . Then*

$$\frac{E_{I, \varepsilon}(x)}{E_{I, \varepsilon}(0)} \in H_{\mathcal{O}, d}^+, \quad \mathrm{Art}(\mathfrak{a}) \left(\frac{E_{I, \varepsilon}(x)}{E_{I, \varepsilon}(0)} \right) = \frac{E_{\mathfrak{a}^{-1}I, \varepsilon}(x)}{E_{I, \varepsilon}(0)}$$

for every integral invertible $\mathcal{O}$ -ideal $\mathfrak{a}$ prime to cd . In particular, if $p \nmid cd$ splits in K , then at every prime $\mathfrak{m}$ above a prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ of norm p ,

$$\frac{E_{\mathfrak{p}^{-1}I, \varepsilon}(x)}{E_{I, \varepsilon}(0)} \equiv \left(\frac{E_{I, \varepsilon}(x)}{E_{I, \varepsilon}(0)} \right)^p \pmod{\mathfrak{m}}.$$

If H_n^+ denotes the narrow ray class field of $n\mathcal{O}_K$, then

$$H_d^+ \subseteq H_{\mathcal{O}, d}^+ \subseteq H_{cd}^+.$$

When I is an $\mathcal{O}_K$ -ideal, we have $c = 1$ and recover the modulus $d\mathcal{O}_K$.

We divide by $E_{I, \varepsilon}(0) = \sqrt{\varepsilon}$ to remove a possible quadratic sign. The value $E_{I, \varepsilon}(0)$ is $\sqrt{\varepsilon}$ independently of I , but applying an Artin automorphism over K may change its sign. Equivalently, in a common abelian extension containing the values, we have

$$\mathrm{Art}(\mathfrak{a})(E_{I, \varepsilon}(x)) = \frac{\mathrm{Art}(\mathfrak{a})(\sqrt{\varepsilon})}{\sqrt{\varepsilon}} E_{\mathfrak{a}^{-1}I, \varepsilon}(x)$$

for integral invertible $\mathcal{O}$ -ideals prime to cd and to the conductor of that extension. If $\sqrt{\varepsilon} \in K$, the sign is one, so we may omit the normalization in the reciprocity law.

For Stark's conjectures [Sta76, Sta80], let ϵ_b denote the positive Stark number indexed by $b \in \mathrm{Cl}_{\mathfrak{f}\infty_2}(K)$, where $\mathfrak{f}$ is a proper nonzero ideal and ∞_1 is the selected real embedding. We recall its definition and its relation to E in Section 9.

Theorem 3 (Stark reciprocity). *Let $H = H_{\mathfrak{f}\infty_2}$. Then*

$$\epsilon_b \in \mathcal{O}_H^\times, \quad \mathrm{Art}(\mathfrak{a})(\epsilon_b) = \epsilon_{b[\mathfrak{a}]} \quad ((\mathfrak{a}, \mathfrak{f}) = 1).$$

Every conjugate extending ∞_1 is positive, and every conjugate extending ∞_2 has absolute value one.

We determine the field generated by these units in Proposition 6. In Subsection 9.2, we recall how to obtain unitary conjugates from reciprocity laws and apply them to Zauner's conjecture.

The key ingredient in the proofs is the periodicity of the valuation function $x \mapsto v(E(x))$ under translation by $G[p]$ when $G[p] \cong \mathbb{Z}/p\mathbb{Z}$ and v is a place above p (see Theorem 4, whose proof uses the pentagon relation for E). Together with the distribution relations, this translation invariance implies that all values are units. At an inert prime, distribution over cyclic subgroups gives the Frobenius congruence. At a split rational

prime p , the same argument gives one of two possible congruences, corresponding to the two prime ideals of K above p . Chebotarev's theorem then determines the Artin action up to the same ambiguity, which we resolve using homotheties with different signs at the two real places. For this application of Chebotarev's theorem, congruences at all but finitely many split rational primes suffice.

The required results from [RW26] are the algebraicity theorem, the functional equations for finite quantum dilogarithms, and the theorem on units at 2 (Theorems 1–2 and Section 5.2). We also use basic results from class field theory [Mil20, Neu99].

3. FINITE QUANTUM DILOGARITHMS ON PSEUDOLATTICES

We now give the pseudolattice formulation in detail and collect the identities used in the proofs. Let $K \subseteq \mathbb{R}$ be a real quadratic field, and write x' for the conjugate of $x \in K$. For a subset $X \subseteq K$, let X_+ denote its totally positive elements.

Following Manin [Man04], we call an additive subgroup $I \subseteq K$ a *pseudolattice* if it is free of rank two. Then $\mathbb{Q}I = K$, and I is a fractional ideal of its multiplier order $\mathcal{O}_I = \{a \in K : aI \subseteq I\}$. We fix a norm-one unit $\varepsilon > 1$ with $\varepsilon I = I$; such a unit always exists.

A basis (β_1, β_2) of I is *oriented* if $\beta_1\beta_2' - \beta_1'\beta_2 > 0$. Choose such a basis with β_2 totally positive, and put $\tau = \beta_1/\beta_2$. Write

$$\varepsilon(\beta_1, \beta_2) = (a\beta_1 + b\beta_2, c\beta_1 + d\beta_2), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then $\gamma\tau = \tau$ and $c\tau + d = \varepsilon$.

Definition 1. The finite quantum dilogarithm $E_{I,\varepsilon}$ is a function on

$$G_{I,\varepsilon} = \{x \in K/I : (\varepsilon - 1)x = 0\} = (\varepsilon - 1)^{-1}I/I.$$

For $x \in G_{I,\varepsilon}$, choose a representative in K and write $(\varepsilon^{-1} - 1)x = u\beta_1 + v\beta_2$, with $u, v \in \mathbb{Z}$. We define

$$E_{I,\varepsilon}(x) = \begin{cases} \mu_\gamma \Phi_{\gamma,u,0}(x/\beta_2; \tau), & x \neq 0, \\ \sqrt{\varepsilon}, & x = 0. \end{cases}$$

Here $\Phi_{\gamma,u,0}$ is the modular quantum dilogarithm with definition as in [RW26, Section 2.2], and $\mu_\gamma = \mu_{I,\varepsilon}$ is the Dedekind eta multiplier.

The group has order $N = \mathrm{Tr}(\varepsilon) - 2$, since $\mathrm{Nm}(\varepsilon - 1) = 2 - \mathrm{Tr}(\varepsilon)$. In Appendix B, we verify that $E_{I,\varepsilon}(x)$ is independent of the representative and of the oriented basis with totally positive second vector.

For $c \in K_+$, we may use the basis $(c\beta_1, c\beta_2)$ on cI . The matrix and the normalized arguments are unchanged, so

$$(1) \quad E_{cI,\varepsilon}(cx) = E_{I,\varepsilon}(x).$$

We next define the Gaussian and bicharacter used in the functional equations. Let $\omega_I : I \times I \rightarrow \mathbb{Z}$ be the alternating form with $\omega_I(\beta_1, \beta_2) = 1$ for an oriented basis. We use the same notation for its $\mathbb{Q}$ -bilinear extension to $K \times K$. We also define $s_I : I \rightarrow \{\pm 1\}$ by

$$s_I(t) = \begin{cases} 1, & t \in 2I, \\ -1, & t \notin 2I. \end{cases}$$

Following the notation of [RW26], write $\mathbf{e}(z) = \exp(2\pi iz)$, and define

$$(2) \quad \langle x \rangle = \langle x \rangle_{I,\varepsilon} = s_I((\varepsilon - 1)x) \mathbf{e}\left(\frac{1}{2}\omega_I(x, \varepsilon x)\right).$$

This is the formula in [RW26, Equation (3)], written without coordinates. Using $s_I(x + y)s_I(x)s_I(y) = (-1)^{\omega_I(x,y)}$ for $x, y \in I$, we see that $\langle x \rangle_{I,\varepsilon}$ is independent of the representative of $x \in G_{I,\varepsilon}$. We have $\langle -x \rangle = \langle x \rangle$, and the associated bicharacter is

$$(3) \quad \langle x; y \rangle = \frac{\langle x + y \rangle}{\langle x \rangle \langle y \rangle} = \mathbf{e}(\omega_I(x - \varepsilon x, y)).$$

To check nondegeneracy, note that $\omega_I(x - \varepsilon x, y) = \omega_I(\varepsilon x, (\varepsilon - 1)y)$. As y runs through $(\varepsilon - 1)^{-1}I$, the second argument on the right runs through I . Since ω_I is unimodular on I , if $\langle x; \cdot \rangle$ is a trivial character, we must have $\varepsilon x \in I$, and hence $x \in I$. Thus we obtain a metric group in the sense of [RW26, Section 4.1].

On $G = G_{I,\varepsilon}$, we use the normalized Fourier transform

$$\widehat{f}(y) = \frac{1}{\sqrt{|G|}} \sum_{x \in G} f(x) \langle x; -y \rangle.$$

Write $\delta(x) = 1$ if $x = 0$, and $\delta(x) = 0$ otherwise. We recall the functional equations from [RW26, Theorems 1–2, Proposition 10].

Proposition 1. *Let $E = E_{I,\varepsilon}$ and $G = G_{I,\varepsilon}$ be as above, with $|G| = N$. The values of E are algebraic and satisfy*

$$(4) \quad E(x)E(-x) = \langle x \rangle^{-1} + \sqrt{N}E(0)\delta(x),$$

$$(5) \quad \widehat{E}(x) = \lambda \langle x \rangle E(x).$$

Here $\lambda = \mu_\gamma$ is a twenty-fourth root of unity, and $\lambda^3 = N^{-1/2} \sum_{t \in G} \langle t \rangle^{-1}$.

We will also use the product form of the pentagon relation from [RW26, Equation (36)]:

$$(6) \quad \frac{1}{\sqrt{N}} \sum_{t \in G} E(t) \langle t \rangle E(x-t) \langle t; y \rangle = \langle x+y \rangle E(-x-y) E(x) E(y) - N E(0) \delta(x) \delta(y).$$

For conjugation and powers of the period, we use [RW26, Corollary 1 and Proposition 8]. We view all groups G_{I,ε^r} as subgroups of K/I ; in particular, $G_{I,\varepsilon} \subseteq G_{I,\varepsilon^r}$.

Proposition 2. *Let (I, ε) be as above. Then for all $r \geq 1$ we have*

$$(7) \quad \overline{E_{I,\varepsilon}(x)} = E_{I,\varepsilon}(x) \langle x \rangle, \quad x \in G_{I,\varepsilon},$$

$$(8) \quad E_{I,\varepsilon^r}(x) = E_{I,\varepsilon}(x)^r, \quad x \in G_{I,\varepsilon},$$

$$(9) \quad E_{I,\varepsilon^r}(x) = E_{I,\varepsilon^r}(\varepsilon x), \quad x \in G_{I,\varepsilon^r}.$$

Here by $\bar{z}$ we denote the complex conjugate of z . When I or ε is fixed, we often omit the corresponding subscript in E . We will often make use of the identity $\sqrt{N} = E(0) - E(0)^{-1}$.

The key observation is that distribution relations connect the values of $E_{I,\varepsilon}$ across different pseudolattices I . Note that for a pair of pseudolattices $I \subseteq J$ we have a canonical projection $\pi: K/I \rightarrow K/J$ with finite kernel J/I .

Proposition 3. (i) *Let $I \subseteq J$ be a pair of pseudolattices stabilized by ε and set $d = |J/I|$. Then for any $x \in G_{J,\varepsilon}$ we have*

$$(10) \quad \frac{\mu_{I,\varepsilon}^d}{\mu_{J,\varepsilon}} E_{J,\varepsilon}(x) = \prod_y E_{I,\varepsilon^{r_y}}(y),$$

where the product is over representatives y of ε -orbits in $\pi^{-1}(x)$, one per orbit, and r_y is the least positive integer such that $\varepsilon^{r_y} y = y$ in K/I .

(ii) *Let I be a pseudolattice stabilized by ε and let H be a subgroup of $G_{I,\varepsilon}$. Define $J = \{z \in K : z + I \in H\}$. Then $I \subseteq J \subseteq \frac{1}{\varepsilon-1}I$, $H = J/I$, J is stabilized by ε , and for any $x \in G_{I,\varepsilon}$ we have*

$$(11) \quad \frac{\mu_{I,\varepsilon}^{|H|}}{\mu_{J,\varepsilon}} E_{J,\varepsilon}(\pi(x)) = \prod_{u \in H} E_{I,\varepsilon}(x + u).$$

We prove this in Appendix A. Note that dividing (11) by its specialization at $x = 0$ gives

$$(12) \quad \frac{E_{J,\varepsilon}(\pi(x))}{E_{J,\varepsilon}(0)} = \prod_{u \in H} \frac{E_{I,\varepsilon}(x + u)}{E_{I,\varepsilon}(u)}.$$

4. LOCAL VALUATION ESTIMATES

Fix an odd prime p and a valuation v of $\overline{\mathbb{Q}}$, normalized by $v(p) = 1$. Let $\mathfrak{m}$ and k be the maximal ideal and residue field of the valuation ring of v . Throughout this section, integrality and congruences are with respect to v . We will use Fourier integrality to control reduction on cosets of subgroups of order p and p^2 . In the following lemma, we prove the three properties needed below, using the integral q -Pascal calculation for Fourier matrices from [KW04, Lemma 3.5 and Corollary 3.12]. We will denote the function sending $x \mapsto f(x)^{-1}$ simply by f^{-1} when there should be no confusion.

Lemma 1. *Let G be a finite abelian group with a nondegenerate bicharacter $\langle \cdot; \cdot \rangle$, and suppose its p -primary subgroup is $H = (\mathbb{Z}/p^s\mathbb{Z})^d$, where $s \geq 1$ and $d \in \{1, 2\}$. Let $f: G \rightarrow \overline{\mathbb{Q}}$ and its normalized Fourier transform*

$$\widehat{f}(y) = \frac{1}{\sqrt{|G|}} \sum_{x \in G} f(x) \langle x; -y \rangle$$

be integral, meaning that $v(f(x)) \geq 0$ and $v(\widehat{f}(y)) \geq 0$ for all $x, y \in G$.

- (i) If $d = 1$, the reduction of f on each $H[p]$ -coset is either zero everywhere or nonzero at more than half the points.
- (ii) Suppose $d = 1$, all values of f are units at v , and $\widehat{f^{-1}}$ is integral at v . Then f is constant modulo $\mathfrak{m}$ on each $H[p]$ -coset.
- (iii) Suppose $d = 2$, all values of f are units at v , and $\widehat{f^{-1}}$ is integral at v . If, on an $H[p]$ -coset, the reduction of f modulo $\mathfrak{m}$ has the form $A(u_1)B(u_2)$ in affine coordinates, then at least one of A, B is constant.

Proof. Write $G = H \oplus C$, where $|C|$ and p are coprime. The two subgroups are orthogonal, since a value of the bicharacter on $H \times C$ has order dividing both a power of p and $|C|$. For $c \in C$, put $f_c(h) = f(h + c)$ on H . Character orthogonality on C gives

$$\widehat{f}_c(a) = \frac{1}{\sqrt{|C|}} \sum_{b \in C} \langle c; b \rangle \widehat{f}(a + b), \quad a \in H.$$

Indeed, after expanding $\widehat{f}$, the sum over b is $\sum_{b \in C} \langle c - c'; b \rangle = |C| \delta(c - c')$. The coefficients in this identity are units at v , so $\widehat{f}_c$ is integral. The same argument applies to f^{-1} in parts (ii) and (iii). It therefore suffices to work on H . For $d = 1, 2$ write $x = (x_1, \dots, x_d) \in H$, using integer representatives $0 \leq x_i < p^s$ for the coordinates. Choose a primitive p^s -th root ζ and change frequency coordinates so that the Fourier kernel on H is $\zeta^{-x \cdot y}$.

We will show that, on each $H[p]$ -coset, the sum of the reduction of f against every polynomial of total degree less than $d(p-1)/2$ is zero in k . We first obtain a congruence on H using the Gaussian binomial coefficients

$$\begin{bmatrix} i \\ j \end{bmatrix}_\zeta = \frac{(\zeta; \zeta)_i}{(\zeta; \zeta)_j (\zeta; \zeta)_{i-j}} = \frac{(\zeta^i; \zeta^{-1})_j}{(\zeta; \zeta)_j}, \quad 0 \leq i, j < p^s,$$

where $(t; q)_j = \prod_{r=0}^{j-1} (1 - q^r t)$. These coefficients are integral and reduce to $\binom{i}{j}$, since $\zeta \equiv 1 \pmod{\mathfrak{m}}$ and

$$\begin{bmatrix} i+1 \\ j \end{bmatrix}_q = \begin{bmatrix} i \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} i \\ j \end{bmatrix}_q.$$

Note that the coefficients vanish when $i < j$.

Fix j_i , with $0 \leq j_i < p^s$, and expand

$$\prod_{i=1}^d (X_i; \zeta^{-1})_{j_i} = \sum_a c_a X_1^{a_1} \cdots X_d^{a_d}, \quad c_a \in \mathbb{Z}[\zeta],$$

where $0 \leq a_\nu \leq j_\nu$. The coefficients c_a are integral at v . Substituting $X_\nu = \zeta^{x_\nu}$, multiplying by $f(x)$, and summing over $x \in H$, we obtain

$$\left(\prod_{\nu=1}^d (\zeta; \zeta)_{j_\nu} \right) \sum_{x \in H} f(x) \prod_{i=1}^d \begin{bmatrix} x_i \\ j_i \end{bmatrix}_\zeta = \sum_a c_a \sum_{x \in H} f(x) \zeta^{a \cdot x} = \sqrt{|H|} \sum_a c_a \widehat{f}(-a).$$

Each $\widehat{f}(-a)$ is integral by hypothesis, so the right-hand side has valuation at least $ds/2$. The factors $(\zeta; \zeta)_{j_i}$ are nonzero, and hence

$$(13) \quad v \left(\sum_{x \in H} f(x) \prod_{i=1}^d \begin{bmatrix} x_i \\ j_i \end{bmatrix}_\zeta \right) \geq \frac{ds}{2} - \sum_{i=1}^d v((\zeta; \zeta)_{j_i}).$$

The valuations $v((\zeta; \zeta)_j)$ strictly increase with j , since every factor $(1 - \zeta^r)$ has a positive valuation. Moreover

$$\sum_{r=1}^{p^s-1} v(1 - \zeta^r) = s, \quad v(1 - \zeta^r) = v(1 - \zeta^{p^s-r}).$$

Pairing the factors therefore gives

$$v((\zeta; \zeta)_j) + v((\zeta; \zeta)_{p^s-1-j}) = s.$$

For $d = 1$, the lower bound in (13) is positive when $j_1 < (p^s - 1)/2$. For $d = 2$, it is positive when $j_2 < p^s - 1 - j_1$, by the same identity and strict monotonicity. Reducing the Gaussian binomial coefficients therefore gives

$$\sum_{x \in H} f(x) \prod_{i=1}^d \begin{bmatrix} x_i \\ j_i \end{bmatrix} \equiv 0 \pmod{\mathfrak{m}} \quad \text{when} \quad \sum_{i=1}^d j_i < \frac{d(p^s - 1)}{2}.$$

We now separate the cosets of $H[p]$. Write $x_i = b_i + p^{s-1}u_i$ and $j_i = c_i + p^{s-1}r_i$, where $0 \leq b_i, c_i < p^{s-1}$ and $0 \leq u_i, r_i < p$. The vector b specifies the $H[p]$ -coset; for a fixed b , the vector $u \in \mathbb{F}_p^d$ gives coordinates for

that coset. Fix r with $\sum r_i < d(p-1)/2$. Since $d \in \{1, 2\}$ and the r_i are integers, even the largest choices $c_i = p^{s-1} - 1$ give

$$\sum_{i=1}^d j_i \leq d(p^{s-1} - 1) + p^{s-1} \left(\frac{d(p-1)}{2} - 1 \right) < \frac{d(p^s - 1)}{2}.$$

Thus the preceding congruence applies for every choice of c . In characteristic p , we have $(1+T)^{b+p^{s-1}u} = (1+T)^b(1+T^{p^{s-1}})^u$, so we have partial Lucas congruences

$$\binom{b+p^{s-1}u}{c+p^{s-1}r} \equiv \binom{b}{c} \binom{u}{r} \pmod{p}.$$

For every c , the preceding congruence therefore reads

$$\sum_{0 \leq b_1, \dots, b_d < p^{s-1}} \left(\prod_{i=1}^d \binom{b_i}{c_i} \right) \left(\sum_{u \in \mathbb{F}_p^d} f(b+p^{s-1}u) \prod_{i=1}^d \binom{u_i}{r_i} \right) \equiv 0 \pmod{\mathfrak{m}}.$$

Treating this as a linear system in the inner sums, the associated matrix is a tensor product of triangular Pascal matrices $(\binom{b}{c})_{0 \leq b, c < p^{s-1}}$, each with diagonal one. Inverting this matrix modulo $\mathfrak{m}$, we find that every inner sum belongs to $\mathfrak{m}$. Finally, $\binom{u_i}{r_i}$ has u -degree r_i and leading coefficient $1/r_i!$, which is nonzero because $r_i < p$. An invertible triangular change of basis therefore replaces these weights by monomials, giving

$$(14) \quad \sum_{u \in \mathbb{F}_p^d} f(b+p^{s-1}u) \prod_{i=1}^d u_i^{r_i} \equiv 0 \pmod{\mathfrak{m}} \quad \text{when} \quad \sum r_i < \frac{d(p-1)}{2}.$$

By linearity, the reduction of f on each coset has zero sum in k against every polynomial over k of total degree less than $d(p-1)/2$. An affine change of coordinates preserves this degree bound, so the assertion holds in any affine coordinates on the coset.

To deduce the three assertions, represent each one-variable residue function by its polynomial of degree at most $p-1$. If such a polynomial A is nonzero, then

$$\sum_{t \in \mathbb{F}_p} t^{p-1-\deg A} A(t) = -\text{lc}(A),$$

where $\text{lc}(A)$ is its leading coefficient. For $d=1$, let A represent the reduction of f on a coset. If $\deg A > (p-1)/2$, the monomial $t^{p-1-\deg A}$ has degree less than $(p-1)/2$, contradicting (14). Therefore, $\deg A \leq (p-1)/2$. A nonzero polynomial of this degree has fewer than $p/2$ roots, which proves (i). Under the additional assumptions of (ii), the representatives of f and f^{-1} both have this degree bound. Their product minus one has degree at most $p-1$ and vanishes at all p points; it is therefore identically zero, so both representatives are constant.

For $d=2$, apply the same test in both variables. If $\deg A + \deg B > p-1$, the monomial $u_1^{p-1-\deg A} u_2^{p-1-\deg B}$ has total degree less than $p-1$, but its sum against $A(u_1)B(u_2)$ is nonzero, a contradiction. Applying this also to the reciprocal gives

$$\deg A + \deg B \leq p-1, \quad \deg A^{-1} + \deg B^{-1} \leq p-1.$$

Here A^{-1} denotes the polynomial representing the reciprocal function. A nonconstant A satisfies $\deg A + \deg A^{-1} \geq p$, since otherwise their product would be identically one as a polynomial. The same holds for B , so both cannot be nonconstant. This proves (iii). $\square$

We now apply the lemma to a finite quantum dilogarithm $E: G \rightarrow \overline{\mathbb{Q}}$. By reflection, all its values are nonzero and $v(E(-t)) = -v(E(t))$; here $E(0) = \sqrt{\varepsilon}$ is a unit. For $x \neq 0$, write

$$R_x(t) = \frac{E(x+t)}{E(t)}.$$

Reflection gives $R_x(t) = \langle t \rangle E(x+t)E(-t) - \sqrt{N}E(x)\delta(t)$. Taking its Fourier transform, reversing the summation variable, and using the 5-term equation (6), we obtain $\widehat{R}_x(t) + E(x) = \langle x+t \rangle E(-x-t)E(x)E(t)$. Applying reflection once more, with the exceptional term at $t = -x$, gives

$$(15) \quad \widehat{R}_x(t) = E(x)(R_x(t)^{-1} - 1) + \sqrt{N}\langle x \rangle^{-1}\delta(t+x).$$

Taking the Fourier transform of this identity gives

$$E(\widehat{x})/\widehat{R}_x(t) = R_x(-t) + E(x)\sqrt{N}\delta(t) - \langle x \rangle^{-1}\langle x; t \rangle.$$

Theorem 4 (Cyclic translation). *Suppose the p -primary subgroup of G is nontrivial and cyclic. Then for every $h \in G$ of order p , all the ratios $R_h(t)$ are units at v , and*

$$R_h(t) \equiv 1 \pmod{\mathfrak{m}} \quad (t \in G).$$

In particular, $v(E(h)) = 0$.

Proof. Let $U = G[p]$, and put $w(t) = v(E(t))$. We first show that w is constant on each U -coset. If not, let $D > 0$ be the largest difference between two values of w on a common U -coset, and choose $0 \neq h \in U$ for which $\max_t v(R_h(t)) = D$. Since $w(-h) = -w(h)$, we have $2|w(h)| \leq D$. Choose an algebraic number z with

$$v(z) = \max \left\{ -\min_t v(R_h(t)), D - w(h) \right\}.$$

Then zR_h and $zE(h)/R_h$ are integral, and at least one of these two functions takes a unit value. The bound on $w(h)$ gives $v(z) \geq D - w(h) \geq D/2 > 0$, and $v(zE(h)) \geq D$. The two Fourier identities give

$$\begin{aligned} \widehat{zR_h}(t) &= \frac{zE(h)}{R_h(t)} - zE(h) + z\sqrt{N}\langle h \rangle^{-1}\delta(t+h), \\ z\widehat{E(h)/R_h}(t) &= zR_h(-t) + zE(h)\sqrt{N}\delta(t) - z\langle h \rangle^{-1}\langle h; t \rangle. \end{aligned}$$

Every term is integral, so we can apply part (i) of Lemma 1 to both scaled functions.

If $zE(h)/R_h$ takes a unit value, then $v(zE(h)) = D$, and the unit values occur exactly at the points of

$$S = \{t \in G : v(R_h(t)) = D\}.$$

By part (i) of Lemma 1, some U -coset C satisfies $|S \cap C| > p/2$. Hence $S \cap C$ contains two points $t, t+h$, giving

$$w(t+2h) - w(t) = v(R_h(t+h)) + v(R_h(t)) = 2D,$$

a contradiction. Otherwise zR_h takes a unit value and $v(z) > D - w(h) \geq D/2$. Applying the same argument at the points where $v(R_h) = -v(z)$, we find consecutive points with $w(t) - w(t+2h) = 2v(z) > D$, again a contradiction. Thus w is constant on U -cosets, and in particular $w(h) = w(0) = 0$.

Now R_h and its reciprocal are units with integral Fourier transforms, so by Lemma 1(ii) their reductions modulo $\mathfrak{m}$ are constant on each U -coset. For each $t \in G$, we have

$$\prod_{j=0}^{p-1} R_h(t+jh) = \frac{E(t+ph)}{E(t)} = 1.$$

Thus the common residue ρ on this coset satisfies $\rho^p = 1$, and hence $\rho = 1$ in characteristic p . $\square$

5. ALGEBRAIC UNITS

Fix $\varepsilon > 1$, and consider all ε -stable pseudolattices I . Put $N = \text{Tr}(\varepsilon) - 2$. Fix an odd prime p and a valuation $v \mid p$. For each lattice, regard $w_I(x) = v(E_{I,\varepsilon}(x))$ as a real-valued function and take its ordinary complex Fourier coefficients

$$\widehat{w}_I(\theta) = \frac{1}{\sqrt{N}} \sum_{x \in G_{I,\varepsilon}} w_I(x) \theta(x)^{-1}.$$

We will show that all the coefficients $\widehat{w}_I(\theta)$ vanish. Using the distribution relation, we write each coefficient as an average of coefficients on larger lattices. A nonzero coefficient of largest absolute value must then remain unchanged along these inclusions. We will choose the lattices so that Theorem 4 forces the last coefficient to be zero.

There are only finitely many coefficients $\widehat{w}_I(\theta)$ to consider. Indeed, choose a positive integer c with $c\mathcal{O}_K \subseteq \mathbb{Z}[\varepsilon]$. Then $c\mathcal{O}_K I \subseteq I \subseteq \mathcal{O}_K I$. Up to totally positive homothety, the fractional ideal $\mathcal{O}_K I$ has only finitely many possibilities, by finiteness of the narrow ideal class group. For each choice, there are only finitely many intermediate lattices between $c\mathcal{O}_K I$ and $\mathcal{O}_K I$. Thus there are only finitely many ε -stable pseudolattices up to totally positive homothety. Since E is unchanged under these homotheties, only finitely many valuations and Fourier coefficients occur.

Lemma 2. *Suppose $I \subseteq J$ and $(\varepsilon - 1)J \subseteq I$. Let $\varphi : G_{I,\varepsilon} \rightarrow G_{J,\varepsilon}$ be the natural map, and put $D = [J : I]$. Then for every character θ of $G_{I,\varepsilon}$ that is trivial on J/I ,*

$$\widehat{w}_I(\theta) = \frac{1}{D} \sum_{\theta' \circ \varphi = \theta} \widehat{w}_J(\theta').$$

Proof. The map φ has kernel J/I , and $\varphi(G_{I,\varepsilon})$ has index D in $G_{J,\varepsilon}$. Thus θ descends to a character of $\varphi(G_{I,\varepsilon})$. The descended character has exactly D extensions to $G_{J,\varepsilon}$, namely the characters θ' satisfying $\theta' \circ \varphi = \theta$. Taking valuations in (11), whose eta factors are units, gives

$$w_J(\varphi(x)) = \sum_{h \in J/I} w_I(x+h).$$

Character orthogonality gives, for every $y \in G_{J,\varepsilon}$,

$$\sum_{\theta' \circ \varphi = \theta} \theta'(y)^{-1} = \sum_{\substack{x \in G_{I,\varepsilon} \\ \varphi(x)=y}} \theta(x)^{-1}.$$

Indeed, both sides vanish outside the image of φ ; if $y = \varphi(x)$, both equal $D\theta(x)^{-1}$, since each fibre has D points and θ is trivial on $\ker \varphi$. Consequently, the distribution identity gives

$$\begin{aligned} \sum_{\theta' \circ \varphi = \theta} \widehat{w}_J(\theta') &= \frac{1}{\sqrt{N}} \sum_{x \in G_{I,\varepsilon}} w_J(\varphi(x)) \theta(x)^{-1} \\ &= \frac{1}{\sqrt{N}} \sum_{h \in J/I} \sum_{x \in G_{I,\varepsilon}} w_I(x+h) \theta(x)^{-1} \\ &= D\widehat{w}_I(\theta). \end{aligned}$$

The last equality uses $\theta(h) = 1$ for $h \in J/I$. Dividing by D proves the formula. $\square$

Lemma 3. *Suppose $p \mid N$. For every character θ of $G_{I,\varepsilon}$ there exist a finite chain $I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r$ of inclusions as in Lemma 2 and characters θ_i of $G_{I_i,\varepsilon}$, with $\theta_0 = \theta$, such that*

$$\theta_{i+1} \circ \varphi_i = \theta_i, \quad \varphi_i : G_{I_i,\varepsilon} \longrightarrow G_{I_{i+1},\varepsilon},$$

where φ_i is the natural map. The final group $G_{I_r,\varepsilon}[p^\infty]$ is cyclic, and θ_r is faithful on it.

Proof. Let $M = I^\vee = \{a \in K : \text{Tr}(aI) \subseteq \mathbb{Z}\}$ be the trace dual. Under the trace pairing, $a \in M$ represents the character $x \mapsto \mathbf{e}(\text{Tr}(ax))$, identifying $\widehat{G}_{I,\varepsilon}$ with $M/(\varepsilon-1)M$. Choose a nonzero representative $a \in M$ of the given character; for the trivial character, take any nonzero $a \in (\varepsilon-1)M$. We define

$$M_i = a\mathbb{Z}[\varepsilon] + (\varepsilon-1)^i M, \quad I_i = M_i^\vee.$$

These are ε -stable pseudolattices, and $M_{i+1} \subseteq M_i$, $(\varepsilon-1)M_i \subseteq M_{i+1}$. Dualizing gives

$$I = I_0 \subseteq I_1 \subseteq \cdots, \quad (\varepsilon-1)I_{i+1} \subseteq I_i.$$

At each step take the character θ_i represented by a . These characters satisfy $\theta_{i+1} \circ \varphi_i = \theta_i$, and since $a \in M_{i+1}$, the character on $G_{I_i,\varepsilon}$ is trivial on I_{i+1}/I_i .

The group $M/a\mathbb{Z}[\varepsilon]$ is finite. Since $(\varepsilon-1)^2 = N\varepsilon$ and $\varepsilon M = M$, we have

$$M_{2j}/a\mathbb{Z}[\varepsilon] = N^j(M/a\mathbb{Z}[\varepsilon]).$$

As $p \mid N$, multiplication by N^j kills the p -primary component of this finite group for all sufficiently large j . Thus $[M_{2j} : a\mathbb{Z}[\varepsilon]]$ is prime to p . Put $q = [M_{2j} : a\mathbb{Z}[\varepsilon]]$, so $p \nmid q$. Inclusion and multiplication by q induce maps

$$\frac{a\mathbb{Z}[\varepsilon]}{(\varepsilon-1)a\mathbb{Z}[\varepsilon]} \xrightleftharpoons[q]{\iota} \frac{M_{2j}}{(\varepsilon-1)M_{2j}}.$$

Both compositions are multiplication by q , which is invertible on a finite p -group. Hence ι induces an isomorphism on the p -primary components. In the basis $a, (\varepsilon-1)a$, the relation $(\varepsilon-1)^2 = N\varepsilon$ gives

$$\frac{a\mathbb{Z}[\varepsilon]}{(\varepsilon-1)a\mathbb{Z}[\varepsilon]} \simeq \mathbb{Z}/N\mathbb{Z},$$

generated by the image of a . Its p -primary component is therefore cyclic, and the character represented by a restricts to a generator of the character group of $G_{I_{2j},\varepsilon}[p^\infty]$. This proves both cyclicity and faithfulness. $\square$

Theorem 5. *Every value $E_{I,\varepsilon}(x)$ is an algebraic unit.*

Proof. By [RW26, Theorems 1 and 11], these values are algebraic and are units at every place above two. It remains to prove that their valuations vanish at every odd finite place.

First suppose $p \mid N$. Keep ε and v fixed and choose one representative of each totally positive homothety class of ε -stable lattices. Among the characters on these finitely many lattices, choose a coefficient $\widehat{w}_I(\theta)$ having the largest absolute value. By homothety invariance, its absolute value bounds all coefficients on

every ε -stable lattice. Put $A = \widehat{w}_I(\theta)$. Let $(I_0, \theta_0), \dots, (I_r, \theta_r)$ be the lattices and compatible characters supplied by Lemma 3, starting with $(I_0, \theta_0) = (I, \theta)$. Put

$$A_i = \widehat{w}_{I_i}(\theta_i).$$

We prove $A_i = A$ by induction, starting from $A_0 = A$. Let $\varphi_i : G_{I_i, \varepsilon} \rightarrow G_{I_{i+1}, \varepsilon}$ be the natural map, and put

$$\mathcal{X}_i = \{\chi \in \widehat{G_{I_{i+1}, \varepsilon}} : \chi \circ \varphi_i = \theta_i\}, \quad D_i = [I_{i+1} : I_i].$$

Thus $\mathcal{X}_i$ consists of the D_i characters on the next group compatible with θ_i , including θ_{i+1} . If $A_i = A$, we find that Lemma 2 gives

$$A = \frac{1}{D_i} \sum_{\chi \in \mathcal{X}_i} \widehat{w}_{I_{i+1}}(\chi).$$

By maximality, every summand has absolute value at most $|A|$. If $A = 0$, every summand is therefore zero. If $A \neq 0$, equality in the triangle inequality forces every summand to have modulus $|A|$ and the same argument as A . Thus, in both cases, $\widehat{w}_{I_{i+1}}(\chi) = A$ for every $\chi \in \mathcal{X}_i$. Taking $\chi = \theta_{i+1}$ gives $A_{i+1} = A$, completing the induction. On the other hand, $G_{I_r, \varepsilon}[p^\infty]$ is cyclic and θ_r is faithful on it. If U is its subgroup of order p , Theorem 4 says that w_{I_r} is constant on U -cosets, whereas $\theta_r|_U$ is nontrivial. Summing over these cosets in the Fourier transform gives $A_r = 0$, and thus $A = 0$. Maximality then forces all Fourier coefficients to vanish, and Fourier inversion gives $w_J = 0$ for every ε -stable lattice J .

For $p \nmid N$, choose $r \geq 1$ with $\varepsilon^r \equiv 1 \pmod{p\mathcal{O}_K}$. Since $p \mid \text{Tr}(\varepsilon^r) - 2$, we can apply the case just proved for ε^r . For $x \in G_{I, \varepsilon}$, (8) gives

$$0 = v(E_{I, \varepsilon^r}(x)) = r v(E_{I, \varepsilon}(x)),$$

concluding the proof. $\square$

6. FROBENIUS AT INERT PRIMES

Fix an odd prime p inert in K , and a valuation above it. We denote by $\mathfrak{m}$ and k the maximal ideal and residue field throughout this section. Note that we will now use the fact that all values of E are units, which is the content of Theorem 5.

In this and the following sections, we use the standard notation

$$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}, \quad I_{(p)} = \{x/b : x \in I, b \in \mathbb{Z}, p \nmid b\}.$$

If $I \subseteq J$ are lattices, the natural map identifies

$$(J/I)[p^\infty] \simeq J_{(p)}/I_{(p)}.$$

Indeed, allowing these denominators kills the other primary components and leaves the p -primary component unchanged. In particular, $I_{(p)} = J_{(p)}$ if and only if $[J : I]$ is prime to p . For any period ε of I , multiplication by $\varepsilon - 1$ also identifies

$$G_{I, \varepsilon}[p^\infty] \simeq I_{(p)}/(\varepsilon - 1)I_{(p)}.$$

We call I *locally maximal at p* if $p \nmid [\mathcal{O}_K I : I]$. Equivalently, $I_{(p)} = (\mathcal{O}_K I)_{(p)}$, so multiplication by $\mathcal{O}_K$ preserves $I_{(p)}$. We use the same denominator convention for orders and their ideals. We first prove invariance under translations of order p . Passing to a larger lattice with cyclic p -primary group lets us apply Theorem 4. Proposition 3(ii) then transfers the resulting congruences to products along cyclic summands. Using the following lemma, we can then recover invariance of the individual values from these products.

Lemma 4. *Let $H = (\mathbb{Z}/p^s\mathbb{Z})^2$, with $s \geq 1$, and $U = H[p]$. Suppose $g : H \rightarrow k^\times$ is a function such that for every cyclic direct summand $L \subseteq H$ of order p^s the product $\prod_{\ell \in L} g(x + \ell)$ is invariant under U . Then g itself is invariant under U .*

Proof. Write $[t]$ for the basis element of $\mathbb{Z}[H]$ corresponding to $t \in H$. For $a \in U$, we have

$$([a] - [0]) \sum_L \sum_{\ell \in L} [\ell] = p^s([a] - [0]).$$

Here the sum is over all cyclic direct summands of H of order p^s . Let χ be a complex character of H with $\chi(a) \neq 1$. Since every character of order less than p^s is trivial on $H[p]$, the image of χ has order p^s . Thus $\ker \chi$ is a cyclic direct summand of order p^s . The sum of χ over $\ker \chi$ is p^s , whereas the sum over every other cyclic direct summand is zero. For a character trivial on a , both sides evaluate to zero. This proves the identity. Applying it multiplicatively to g , we obtain

$$1 = \prod_L \prod_{\ell \in L} \frac{g(x + a + \ell)}{g(x + \ell)} = \left(\frac{g(x + a)}{g(x)} \right)^{p^s}.$$

Since $k^\times$ has no nontrivial p -power torsion, the ratio is one, as required. $\square$

Lemma 5. *Suppose I is locally maximal at p , and $\eta I = I$, with $\eta > 1$ totally positive and $\eta \equiv 1 \pmod{p\mathcal{O}_K}$. Then*

$$E_{I,\eta}(x+a) \equiv E_{I,\eta}(x) \pmod{\mathfrak{m}} \quad (x \in G_{I,\eta}, a \in p^{-1}I/I).$$

Proof. Write $\eta - 1 = p^s B$, where $s \geq 1$, $B \in \mathcal{O}_K$, and $B \notin p\mathcal{O}_K$. Since p is inert, the image of B in $\mathcal{O}_K/p\mathcal{O}_K = \mathbb{F}_{p^2}$ is nonzero. Expanding $\text{Nm}(1 + p^s B) = 1$, we obtain $\text{Tr}(B) = -p^s \text{Nm}(B)$. The residue of B in $\mathbb{F}_{p^2}$ is therefore nonzero and has trace zero. Since p is odd, this residue cannot lie in $\mathbb{F}_p$.

The inclusion $I \subseteq \mathcal{O}_K I$ induces an isomorphism modulo p , because its index is prime to p . Thus I/pI is a one-dimensional vector space over $\mathbb{F}_{p^2}$, and multiplication by B preserves no $\mathbb{F}_p$ -line. Moreover, B is a unit in $(\mathcal{O}_K)_{(p)}$, so $BI_{(p)} = I_{(p)}$. It follows that the full p -primary subgroup of $G_{I,\eta}$ is $H = p^{-s}I/I$.

For a cyclic direct summand $L \subseteq H$ of order p^s , let J be its inverse image in K , so $J/I = L$. Then $(\eta - 1)J \subseteq I \subseteq J$, and $\text{Nm}(\eta) = 1$ implies $\eta J = J$. Choose $z \in I$ with $L = \langle p^{-s}z \rangle$. Since $z \notin pI$, we may choose $z_2 \in I$ such that their images form a basis of I/pI . They then form a basis of $I_{(p)}$ over $\mathbb{Z}_{(p)}$: the determinant of the change-of-basis matrix is prime to p . Write (B_{ij}) for multiplication by B in this basis. In the basis $p^{-s}z, z_2$ of $J_{(p)}$, multiplication by $\eta - 1$ has matrix

$$\begin{pmatrix} p^s B_{11} & p^{2s} B_{12} \\ B_{21} & p^s B_{22} \end{pmatrix}.$$

The entry B_{21} is a unit in $\mathbb{Z}_{(p)}$, because multiplication by B does not preserve the line through z modulo p . This unit entry lets one relation eliminate a generator from the cokernel of the matrix, leaving a cyclic group. That cokernel is

$$J_{(p)}/(\eta - 1)J_{(p)} \cong G_{J,\eta}[p^\infty].$$

For $a \in H[p]$, its image in $G_{J,\eta}$ has order at most p . Applying Theorem 4 to $E_{J,\eta}$, and then using Proposition 3(ii), we obtain

$$\prod_{\ell \in L} E_{I,\eta}(x + a + \ell) \equiv \prod_{\ell \in L} E_{I,\eta}(x + \ell) \pmod{\mathfrak{m}}.$$

We conclude by applying Lemma 4 to the reduction of E on each coset of H . $\square$

Theorem 6 (Inert Frobenius). *For every odd prime p inert in K , every valuation above p , and every $x \in G_{I,\varepsilon}$,*

$$E_{I,\varepsilon}(px) \equiv E_{I,\varepsilon}(x)^{p^2} \pmod{\mathfrak{m}}.$$

Proof. First suppose I is locally maximal at p . Let d be the multiplicative order of ε modulo p ; it divides $p + 1$, since $\varepsilon^p \equiv \varepsilon^{-1}$ for inert primes. Every nonzero ε -orbit in $p^{-1}I/I$ has length d . For $x \in G_{I,\varepsilon}$, we divide the orbit distribution formula for $I \subseteq p^{-1}I$ by the same formula at zero. Using positive homothety, we obtain

$$\frac{E_{I,\varepsilon}(px)}{E_{I,\varepsilon}(0)} = \frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \prod_{O \subseteq (p^{-1}I/I) \setminus \{0\}} \frac{E_{I,\varepsilon^d}(x + h_O)}{E_{I,\varepsilon^d}(h_O)},$$

where the product is over ε -orbits, with representatives h_O . Applying Lemma 5 with $\eta = \varepsilon^d$ and using (8), we obtain

$$\frac{E_{I,\varepsilon^d}(x + h_O)}{E_{I,\varepsilon^d}(h_O)} \equiv \frac{E_{I,\varepsilon^d}(x)}{E_{I,\varepsilon^d}(0)} = \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^d \pmod{\mathfrak{m}}.$$

There are $(p^2 - 1)/d$ orbits, so

$$\frac{E_{I,\varepsilon}(px)}{E_{I,\varepsilon}(0)} \equiv \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^{p^2} \pmod{\mathfrak{m}}.$$

Also

$$E_{I,\varepsilon}(0)^{p^2-1} = \varepsilon^{(p^2-1)/2} \equiv 1 \pmod{\mathfrak{m}},$$

since $\varepsilon^{p+1} \equiv 1$ and p is odd. This proves the stated formula in the locally maximal case.

To pass to arbitrary lattices, we first deduce invariance under every p -primary translation on a locally maximal one. If $h \in G_{I,\varepsilon}$ and $p^a h = 0$, repeated application of the congruence gives

$$E_{I,\varepsilon}(z + h)^{p^{2a}} \equiv E_{I,\varepsilon}(p^a z) \equiv E_{I,\varepsilon}(z)^{p^{2a}} \pmod{\mathfrak{m}},$$

and the Frobenius is injective on the residue field.

For arbitrary I , choose $a \geq 0$ such that p^a annihilates the p -primary component of $\mathcal{O}_K I/I$, and put

$$J = I \cap p^a \mathcal{O}_K I.$$

Both lattices in the intersection are ε -stable, so J is also stable. Since $p^a I \subseteq J$, the quotient I/J is a p -group. On the other hand,

$$p^a \mathcal{O}_K I/J \simeq p^a (\mathcal{O}_K I/I)$$

has order prime to p , by the choice of a . As $J \subseteq \mathcal{O}_K J \subseteq p^a \mathcal{O}_K I$, the index $[\mathcal{O}_K J : J]$ is prime to p as well. Thus J is locally maximal at p . Put $D = [I : J]$.

The map $\pi : K/J \rightarrow K/I$ has kernel I/J , a p -group, and is therefore an isomorphism on prime-to- p torsion. For $x \in G_{I,\varepsilon}$, let $y \in K/J$ be the unique prime-to- p lift of the prime-to- p component of x . Every lift of x has prime-to- p component y . Uniqueness and compatibility with multiplication by ε give $\varepsilon y = y$, so $y \in G_{J,\varepsilon}$. If z represents an orbit of length r in $\pi^{-1}(x)$, then $z - y \in G_{J,\varepsilon^r}[p^\infty]$. The invariance under p -primary translations proved above applies to J at every period. Applying it at ε^r gives

$$E_{J,\varepsilon^r}(z) \equiv E_{J,\varepsilon^r}(y) = E_{J,\varepsilon}(y)^r \pmod{\mathfrak{m}}.$$

The orbit lengths sum to D . For the fibre over zero, the same argument applies with $y = 0$. Dividing the two orbit distribution formulas, we obtain

$$\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \equiv \left(\frac{E_{J,\varepsilon}(y)}{E_{J,\varepsilon}(0)} \right)^D \pmod{\mathfrak{m}}.$$

The common prime-to- p component over px is py . Applying the congruence for J , we obtain

$$\begin{aligned} \frac{E_{I,\varepsilon}(px)}{E_{I,\varepsilon}(0)} &\equiv \left(\frac{E_{J,\varepsilon}(py)}{E_{J,\varepsilon}(0)} \right)^D \\ &\equiv \left(\frac{E_{J,\varepsilon}(y)}{E_{J,\varepsilon}(0)} \right)^{Dp^2} \equiv \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^{p^2} \pmod{\mathfrak{m}}. \end{aligned}$$

Finally, we remove the normalization using $E_{I,\varepsilon}(0)^{p^2-1} \equiv 1 \pmod{\mathfrak{m}}$, as above. $\square$

7. TRANSLATION INVARIANCE AT SPLIT PRIMES

Let I be a pseudolattice with multiplier order $\mathcal{O} = \{a \in K : aI \subseteq I\}$, of conductor $c = [\mathcal{O}_K : \mathcal{O}]$. Fix an odd split prime $p \nmid c$ and a valuation above it, again writing $\mathfrak{m}$ and k for its maximal ideal and residue field. Write $p\mathcal{O} = \mathfrak{p}\mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are the invertible prime ideals of $\mathcal{O}$ obtained by contracting the two primes of $\mathcal{O}_K$ above p . We use this notation throughout the section. Since $c\mathcal{O}_K I \subseteq I$ and $p \nmid c$, the lattice I is locally maximal at p . Inclusion identifies I/pI with $\mathcal{O}_K I/p\mathcal{O}_K I$. By the Chinese remainder theorem, this is the direct sum of two one-dimensional spaces, on which $\mathcal{O}_K$ acts through its two residue maps to $\mathbb{F}_p$. Under $p^{-1}I/I \simeq I/pI$, these two spaces correspond to

$$U_{\mathfrak{p}} = \mathfrak{p}^{-1}I/I, \quad U_{\mathfrak{q}} = \mathfrak{q}^{-1}I/I.$$

Both subgroups have order p , and

$$p^{-1}I/I = U_{\mathfrak{p}} \oplus U_{\mathfrak{q}}.$$

We will show that the reduction of E is invariant under translations by one of these two subgroups, with the same subgroup for every argument. First, distribution gives a product of two one-variable functions on each $p^{-1}I/I$ -coset, using coordinates from $U_{\mathfrak{p}}$ and $U_{\mathfrak{q}}$. Lemma 1(iii) then makes one factor constant. A final comparison between cosets makes the choice of subgroup independent of the coset.

Lemma 6. *Let $H = (\mathbb{Z}/p^s\mathbb{Z})^2$, with $s \geq 1$, and let a, b be an $\mathbb{F}_p$ -basis of $H[p]$. Suppose $g : H \rightarrow k^\times$ is a function such that, for every cyclic direct summand $L \subseteq H$ of order p^s with*

$$L[p] \notin \{\langle a \rangle, \langle b \rangle\},$$

the function $x \mapsto \prod_{\ell \in L} g(x + \ell)$ is invariant under translations by $H[p]$. Then

$$g(x + a + b)g(x) = g(x + a)g(x + b) \quad (x \in H).$$

Thus, on each coset of $H[p]$, the function g is a product of functions of the separate coordinates.

Proof. Arguing in the group ring as in Lemma 4, we obtain

$$([a] - [0])([b] - [0]) \sum_L \sum_{\ell \in L} [\ell] = p^s([a] - [0])([b] - [0]),$$

where L ranges over the summands in the statement. A complex character χ of H contributes only if $\chi(a) \neq 1$ and $\chi(b) \neq 1$. Such a character has order p^s , and $\ker \chi$ is a cyclic direct summand of order p^s with $(\ker \chi)[p] \notin \{\langle a \rangle, \langle b \rangle\}$. Exactly one summand, $L = \ker \chi$, therefore contributes p^s ; all the others contribute zero. Applying this identity to g and using the absence of p -power torsion in $k^\times$, we obtain the required equality. Iterating this equality in the two coordinates, we find

$$g(x + ra + tb) = \frac{g(x + ra)g(x + tb)}{g(x)},$$

as claimed. $\square$

Theorem 7. *Suppose I is locally maximal at p , and $\eta \equiv 1 \pmod{p\mathcal{O}_K}$ is a totally positive period. Then at least one choice of $\mathfrak{r} \in \{\mathfrak{p}, \mathfrak{q}\}$ satisfies*

$$E_{I,\eta}(x+a) \equiv E_{I,\eta}(x) \pmod{\mathfrak{m}} \quad (x \in G_{I,\eta}, a \in U_{\mathfrak{r}}),$$

where $U_{\mathfrak{p}} = \mathfrak{p}^{-1}I/I$ and $U_{\mathfrak{q}} = \mathfrak{q}^{-1}I/I$ are the two subgroups just described. The choice of $\mathfrak{r}$ is independent of x .

Proof. Since $(\eta - 1)' = -(\eta - 1)/\eta$, the valuations of $\eta - 1$ at the two primes above p agree. Thus we may write $\eta - 1 = p^s B$, where $s \geq 1$, $B \in \mathcal{O}_K$, and both components of B in $\mathcal{O}_K/p\mathcal{O}_K \simeq \mathbb{F}_p \times \mathbb{F}_p$ are nonzero. By the trace identity in the proof of Lemma 5, these components are opposite. These two scalars are distinct because p is odd. Thus the only one-dimensional subspaces of I/pI preserved by multiplication by B are the two summands described above. Since B is a unit in $(\mathcal{O}_K)_{(p)}$, we have $BI_{(p)} = I_{(p)}$. Hence the full p -primary subgroup is $H = p^{-s}I/I$, and

$$H[p] = U_{\mathfrak{p}} \oplus U_{\mathfrak{q}}.$$

Let $L \subseteq H$ be a cyclic direct summand of order p^s with $L[p] \notin \{U_{\mathfrak{p}}, U_{\mathfrak{q}}\}$, and let J be its inverse image in K . Choose $z \in I$ with $L = \langle p^{-s}z \rangle$. Multiplication by B does not preserve the line generated by z modulo p , so the entry B_{21} in the matrix calculation of Lemma 5 is a unit. That calculation shows that $G_{J,\eta}[p^\infty]$ is cyclic. Applying Theorem 4 to $E_{J,\eta}$, and then Proposition 3(ii), we find that $\prod_{\ell \in L} E_{I,\eta}(x + \ell)$ is invariant modulo $\mathfrak{m}$ under translations by $H[p]$. Choose generators $a \in U_{\mathfrak{p}}$ and $b \in U_{\mathfrak{q}}$. By Lemma 6, on each coset $x + H[p]$ the reduction of $E(x + ua + vb)$ is a product of a function of u and a function of v .

Both E and E^{-1} have integral Fourier transforms. For E , this follows from (5). For its inverse, we use reflection and then take the Fourier transform to obtain

$$E(x)^{-1} = \langle x \rangle E(-x) - \sqrt{N} \delta(x), \quad \widehat{E^{-1}}(y) = \lambda^{-1} E(y) - 1.$$

Applying Lemma 1(iii), we see that on each coset the reduction is constant in at least one coordinate.

It remains to show that the same subgroup works on every coset. Otherwise there are two cosets on which the reduction of E is nonconstant, depending only on u on one and only on v on the other. Choose representatives x_1, x_2 in that order and put $h = x_2 - x_1$. Then

$$R_h(x_1 + ua + vb) \equiv \frac{E(x_2 + vb)}{E(x_1 + ua)} \pmod{\mathfrak{m}},$$

with both one-variable factors nonconstant. All values of E are units, so the same holds for R_h and R_h^{-1} . The translation-ratio identity (15) gives integrality of $\widehat{R_h}$. Taking the Fourier transform of that identity and dividing by $E(h)$ gives

$$\widehat{R_h^{-1}}(t) = \frac{R_h(-t)}{E(h)} + \sqrt{N} \delta(t) - \frac{\langle h \rangle^{-1} \langle h; t \rangle}{E(h)},$$

which is also integral. Lemma 1(iii), applied to R_h , now forces one of the two factors above to be constant, a contradiction. Thus the same subgroup among $U_{\mathfrak{p}}, U_{\mathfrak{q}}$ leaves the reduction invariant on all cosets where it is nonconstant. On the remaining cosets both subgroups do so. Consequently, one of these subgroups leaves the reduction of E invariant on all of $G_{I,\eta}$. $\square$

Theorem 8 (Split Frobenius). *Let I be a pseudolattice with multiplier order $\mathcal{O}$ of conductor c . At each fixed valuation above an odd split prime $p \nmid c$, at least one choice of $\mathfrak{r} \in \{\mathfrak{p}, \mathfrak{q}\}$ satisfies*

$$\frac{E_{\mathfrak{r}^{-1}I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \equiv \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^p \pmod{\mathfrak{m}} \quad (x \in G_{I,\varepsilon}).$$

The same prime $\mathfrak{r}$ works for all x .

Proof. The two residues of ε lie in $\mathbb{F}_p^\times$ and are inverse, so they have the same order $d \mid p-1$. Put $\eta = \varepsilon^d$. Choose $\mathfrak{r} \in \{\mathfrak{p}, \mathfrak{q}\}$ such that the reduction of $E_{I,\eta}$ is invariant under translations by $U_{\mathfrak{r}}$, as in Theorem 7, and set $J = \mathfrak{r}^{-1}I$. Zero is fixed by ε , and the nonzero elements of J/I form $(p-1)/d$ orbits of length d . Dividing the orbit distribution formula at $x \in G_{I,\varepsilon}$ by the same distribution formula at zero, we obtain

$$\frac{E_{J,\varepsilon}(x)}{E_{I,\varepsilon}(0)} = \frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \prod_{O \subseteq (J/I) \setminus \{0\}} \frac{E_{I,\eta}(x + h_O)}{E_{I,\eta}(h_O)}.$$

By Theorem 7 and the power identity (8), each quotient is congruent to $(E_{I,\varepsilon}(x)/E_{I,\varepsilon}(0))^d$. The resulting exponent is $1 + d(p-1)/d = p$, as required. $\square$

7.1. A value off the unit circle. To distinguish the roles of $\mathfrak{p}$ and $\mathfrak{q}$ in reciprocity, we need a nonzero $x \in G$ for which $E(x)$ is not a root of unity. We first show that the required value exists whenever $|G|$ is sufficiently large.

Lemma 7. *If $|G| = N \geq 25$, some nonzero $x \in G$ satisfies $|E(x)| \neq 1$.*

Proof. Suppose that $|E(x)| = 1$ for every $x \neq 0$. By the conjugation formula, this would imply that $E(x)^2 = \langle x \rangle^{-1}$ for $x \neq 0$. Using the Fourier relation at zero and the Gauss sum, we would obtain

$$\sum_{x \neq 0} E(x) = E(0)(\lambda\sqrt{N} - 1), \quad \sum_{x \neq 0} E(x)^2 = \lambda^3\sqrt{N} - 1, \quad |\lambda| = 1.$$

The first sum would therefore have absolute value greater than $\sqrt{N}(\sqrt{N} - 1)$, since $E(0) = (\sqrt{N} + \sqrt{N+4})/2 > \sqrt{N}$. Multiply all summands by the same complex number of absolute value one so that their sum is positive real, and apply Cauchy–Schwarz to their real parts. Using $(\operatorname{Re} z)^2 = (1 + \operatorname{Re}(z^2))/2$ for $|z| = 1$, we obtain

$$\left| \sum_{x \neq 0} E(x) \right|^2 \leq \frac{N-1}{2} \left(N-1 + \left| \sum_{x \neq 0} E(x)^2 \right| \right) \leq \frac{(N-1)(N + \sqrt{N})}{2}.$$

For $N \geq 25$ this is smaller than $N(\sqrt{N} - 1)^2$, a contradiction. $\square$

Since $|G_{I, \varepsilon^r}| = \operatorname{Tr}(\varepsilon^r) - 2$ tends to infinity, there is such a value for every sufficiently large power of the period.

8. RECIPROCITY LAW FOR SPLIT PRIMES

Let I be a pseudolattice, with multiplier order $\mathcal{O} = (I : I) = \mathbb{Z} + c\mathcal{O}_K$. Fix the period ε , writing $E_I(x)$ when there is no ambiguity. We use invertible ideals of $\mathcal{O}$ to translate I . If $\mathfrak{a}$ is an integral invertible ideal prime to c , then we claim that $\mathfrak{a}^{-1}I$ has the same multiplier order and period as I , and

$$I \subseteq \mathfrak{a}^{-1}I, \quad [\mathfrak{a}^{-1}I : I] = N\mathfrak{a} := N(\mathfrak{a}\mathcal{O}_K).$$

To verify the index formula, consider one rational prime ℓ at a time. If $\ell \mid c$, then $\mathfrak{a}_{(\ell)} = \mathcal{O}_{(\ell)}$, so the ℓ -primary component of $\mathfrak{a}^{-1}I/I$ is trivial. If $\ell \nmid c$, then $\mathcal{O}_{(\ell)} = (\mathcal{O}_K)_{(\ell)}$, and $I_{(\ell)}$ is a principal fractional ideal of this ring. Indeed, by the Chinese remainder theorem, a fractional $\mathcal{O}_K$ -ideal has an element with its prescribed valuations at all primes above ℓ ; allowing denominators prime to ℓ makes this element a generator. After cancelling such a generator, the ℓ -primary component has the same order as that of $(\mathfrak{a}\mathcal{O}_K)^{-1}/\mathcal{O}_K$. Taking the product over ℓ gives the asserted index. For $\alpha \in K$,

$$\alpha(\mathfrak{a}^{-1}I) \subseteq \mathfrak{a}^{-1}I \iff \alpha I \subseteq I,$$

by multiplying by $\mathfrak{a}$ in one direction and by $\mathfrak{a}^{-1}$ in the other. Thus $\mathfrak{a}^{-1}I$ and I have the same multiplier order.

For an integer $n \geq 1$, let $C_{\mathcal{O}}^+(n)$ be the group of invertible fractional $\mathcal{O}$ -ideals prime to cn , modulo principal ideals $\alpha\mathcal{O}$ satisfying

$$\alpha, \alpha' > 0, \quad \alpha \in \mathcal{O}_{(\ell)}^\times \quad (\ell \mid c), \quad \alpha - 1 \in n\mathcal{O}_{(\ell)} \quad (\ell \mid n).$$

Here, as above, $\mathcal{O}_{(\ell)}$ consists of fractions a/b with $a \in \mathcal{O}$ and $\ell \nmid b$. The unit condition means that both α and α^{-1} admit such expressions.

The unit condition at conductor primes is essential. Extension and contraction identify the integral ideals prime to cn in the two orders, and this correspondence extends multiplicatively to fractional ideals. Under this identification, let $\alpha \in K^\times$ be totally positive and satisfy

$$\alpha - 1 \in cn(\mathcal{O}_K)_{(\ell)} \quad (\ell \mid cn).$$

Since $c(\mathcal{O}_K)_{(\ell)} \subseteq \mathcal{O}_{(\ell)}$, these congruences imply $\alpha - 1 \in n\mathcal{O}_{(\ell)}$ for $\ell \mid n$. At $\ell \mid c$, they also give

$$\alpha \in 1 + c(\mathcal{O}_K)_{(\ell)} \subseteq \mathcal{O}_{(\ell)}^\times.$$

Thus α satisfies the defining conditions for $C_{\mathcal{O}}^+(n)$. Conversely, those conditions imply $\alpha - 1 \in n(\mathcal{O}_K)_{(\ell)}$ for $\ell \mid n$, the usual ray congruences modulo $n\mathcal{O}_K$. By class field theory, we obtain the order ray class field $H_{\mathcal{O}, n}^+$, with

$$H_n^+ \subseteq H_{\mathcal{O}, n}^+ \subseteq H_{cn}^+, \quad \operatorname{Gal}(H_{\mathcal{O}, n}^+/K) = C_{\mathcal{O}}^+(n).$$

Here H_n^+ denotes the usual narrow ray class field of $n\mathcal{O}_K$. For an $\mathcal{O}$ -ideal $\mathfrak{a}$, its Artin symbol means that of $\mathfrak{a}\mathcal{O}_K$. Conjugation of K preserves the defining conditions, so $H_{\mathcal{O}, n}^+/\mathbb{Q}$ is Galois.

We will also use surjectivity of the maps $C_{\mathcal{O}}^+(n') \rightarrow C_{\mathcal{O}}^+(n)$ for $n \mid n'$. Given a class in $C_{\mathcal{O}}^+(n)$, choose an ideal representative $\mathfrak{a}$. By weak approximation, choose a totally positive α satisfying the level- n conditions such that $\alpha\mathfrak{a}$ is prime to the additional primes dividing n' . Then clear any remaining denominators by a positive integer congruent to one modulo cn' . This gives an integral representative prime to cn' in the original class.

Choose m divisible by four and by twice the exponent of $G_{I,\varepsilon}$, put $H = H_{\mathcal{O},m}^+$, and write $\sigma_{\mathfrak{a}}$ for its arithmetic Artin automorphism. This auxiliary modulus accommodates every $x \in G_{I,\varepsilon}$, the Gaussian factors $\langle x \rangle_I$, and $\sqrt{\varepsilon}$ at once. After proving reciprocity at this level, we will descend each normalized value at x to the modulus given by the additive order of x . In the rest of this section, all acting ideals are integral invertible $\mathcal{O}$ -ideals.

Lemma 8. *For $\mathfrak{a}$ prime to cm , the function $E_{\mathfrak{a}^{-1}I}(x)$, with $x \in G_{I,\varepsilon}$, depends only on the ideal class $[\mathfrak{a}]$ in $C_{\mathcal{O}}^+(m)$. Moreover,*

$$\langle x \rangle_{\mathfrak{a}^{-1}I} = \langle x \rangle_I^{N\mathfrak{a}}, \quad \sigma_{\mathfrak{a}}(\langle x \rangle_I) = \langle x \rangle_I^{N\mathfrak{a}}, \quad E_I(0), \langle x \rangle_I \in H.$$

Proof. The inclusion $I \subseteq \mathfrak{a}^{-1}I$ induces an isomorphism between $G_{I,\varepsilon}$ and $G_{\mathfrak{a}^{-1}I,\varepsilon}$, since its index is prime to their common order. We use this isomorphism to identify the two groups. Suppose that $\mathfrak{b} = \alpha\mathfrak{a}$ is another integral representative, with α totally positive and satisfying the finite conditions in the definition of $C_{\mathcal{O}}^+(m)$. By positive homothety (1), $E_{\mathfrak{b}^{-1}I}(x) = E_{\mathfrak{a}^{-1}I}(\alpha x)$. The congruences $\alpha - 1 \in m\mathcal{O}_{(\ell)}$ for $\ell \mid m$ mean that the coordinates of $(\alpha - 1)/m$ in a $\mathbb{Z}$ -basis of $\mathcal{O}$ have denominators prime to m . We may therefore choose an integer b , prime to m , such that $b(\alpha - 1) \in m\mathcal{O}$. Since $mx \in I$, we obtain

$$b(\alpha - 1)x \in I \subseteq \mathfrak{a}^{-1}I.$$

Also, integrality of $\mathfrak{b} = \alpha\mathfrak{a}$ gives

$$\alpha I = \mathfrak{a}^{-1}\mathfrak{b}I \subseteq \mathfrak{a}^{-1}I,$$

and hence $m(\alpha - 1)x \in \mathfrak{a}^{-1}I$. As $(b, m) = 1$, Bézout's identity now gives $\alpha x \equiv x \pmod{\mathfrak{a}^{-1}I}$, and therefore

$$E_{\mathfrak{a}^{-1}I}(\alpha x) = E_{\mathfrak{a}^{-1}I}(x).$$

This also explains why α need not be integral.

Note that $\omega_{\mathfrak{a}^{-1}I} = N\mathfrak{a}\omega_I$. Since $N\mathfrak{a}$ is odd, the inclusion also preserves the sign function modulo two. Substituting this comparison and the preserved sign function into the Gaussian formula gives the first identity. The root of unity $\langle x \rangle_I$ has order dividing twice the exponent of $G_{I,\varepsilon}$, and hence dividing m . The second identity follows from cyclotomic reciprocity in $K(\zeta_m) \subseteq H_m^+ \subseteq H$.

Finally, $E_I(0) = \sqrt{\varepsilon}$. If $\sqrt{\varepsilon} \notin K$, the discriminant of $T^2 - \varepsilon$ is 4ε . Since ε is a unit, $K(\sqrt{\varepsilon})/K$ has finite conductor dividing (4). This extension is real at both real places of K , so $E_I(0) \in H$. $\square$

Proposition 4. *Every $E_I(x)/E_I(0)$ belongs to H . For every $\mathfrak{a}$ prime to cm , one of the following identities holds for all x simultaneously:*

$$\sigma_{\mathfrak{a}}\left(\frac{E_I(x)}{E_I(0)}\right) = \frac{E_{\mathfrak{a}^{-1}I}(x)}{E_I(0)}, \quad \text{or} \quad \sigma_{\mathfrak{a}}\left(\frac{E_I(x)}{E_I(0)}\right) = \frac{E_{(\mathfrak{a}')^{-1}I}(x)}{E_I(0)}.$$

The same assertion holds after replacing I by any ideal translate.

Proof. Take a finite Galois extension $F/\mathbb{Q}$ containing H and the values on all the ideal translates indexed by $C_{\mathcal{O}}^+(m)$. There are only finitely many such values, by Lemma 8. Fix $g \in \text{Gal}(F/K)$. By Chebotarev's density theorem, there are infinitely many rational primes whose Frobenius conjugacy class contains g . Discard the finitely many dividing cm . The remaining primes split in K . Over each such prime, choose a prime $\mathfrak{m}$ of F whose Frobenius is exactly g . If $g|_H = \sigma_{\mathfrak{a}}$, then the contracted $\mathcal{O}$ -ideal $\mathfrak{p} = \mathfrak{m} \cap \mathcal{O}$ represents $[\mathfrak{a}]$, and $\mathfrak{p}'$ represents $[\mathfrak{a}']$.

At every such split prime, I is locally maximal, and Theorem 8 applies with the contracted primes $\mathfrak{p}, \mathfrak{p}'$. Combining it with the Frobenius congruence, we obtain one of the two asserted formulas modulo $\mathfrak{m}$, with g in place of $\sigma_{\mathfrak{a}}$. One alternative holds at infinitely many of these primes, with the same alternative for all x . Fix that alternative. For each x , the difference between its two sides is a fixed algebraic number divisible by infinitely many primes, and hence is zero. Thus the chosen identity holds for every x .

If g fixes H , both ideal classes are trivial, so both candidates are the original function. Thus every element of $\text{Gal}(F/H)$ fixes the normalized values, proving that they belong to H . For general g , we obtain the two asserted possibilities for its restriction $\sigma_{\mathfrak{a}}$. Apply the same argument to each translated lattice. $\square$

Proposition 5. *If $E_I(x)$ is not a root of unity for some $x \neq 0$, then, for every $\mathfrak{a}$ prime to cm and every $x \in G_{I,\varepsilon}$,*

$$\sigma_{\mathfrak{a}}\left(\frac{E_I(x)}{E_I(0)}\right) = \frac{E_{\mathfrak{a}^{-1}I}(x)}{E_I(0)}.$$

Proof. Let s_1, s_2 be the two real sign classes in $C_{\mathcal{O}}^+(m)$. Represent s_i by a principal ideal whose generator satisfies the finite conditions in the definition of $C_{\mathcal{O}}^+(m)$, is negative at the i -th real place, and is positive at the other real place. Such generators exist by weak approximation. Complex conjugation on H is σ_{s_1} , and ideal conjugation interchanges s_1 and s_2 . Put $J = \mathfrak{a}^{-1}I$, and choose an integral generator α representing s_1 . As in the proof of Lemma 8, we have $\alpha x \equiv x \pmod{J}$. By Proposition 8 and reflection, we obtain for $x \neq 0$

$$E_{\alpha^{-1}J}(x) = E_J(-x)^{-1} = \langle x \rangle_J E_J(x).$$

Thus replacing $[\mathfrak{a}]$ by $[\mathfrak{a}]s_1$ replaces $E_J(x)$ by $\langle x \rangle_J E_J(x)$, where $J = \mathfrak{a}^{-1}I$. Replacing $[\mathfrak{a}]$ by $[\mathfrak{a}]s_2$ replaces $E_J(x)$ by $E_J(x)^{-1}$. Both statements concern $x \neq 0$.

Suppose the direct-ideal formula fails for $\mathfrak{a}$ at some x . Proposition 4 then forces the conjugate-ideal formula for $\mathfrak{a}$ at every x . Apply the proposition also to $[\mathfrak{a}]s_1$. Complex conjugation fixes $E_I(0)$, so

$$\sigma_{\mathfrak{a}s_1} \left(\frac{E_I(x)}{E_I(0)} \right) = \langle x \rangle_I^{N\mathfrak{a}} \sigma_{\mathfrak{a}} \left(\frac{E_I(x)}{E_I(0)} \right).$$

The direct-ideal alternative for $[\mathfrak{a}]s_1$ would give $\langle x \rangle_I^{N\mathfrak{a}} E_{\mathfrak{a}^{-1}I}(x)/E_I(0)$ as the right-hand side. Comparing this with the preceding identity would imply the direct-ideal formula for $\mathfrak{a}$, contrary to our assumption. Hence the conjugate-ideal alternative holds for $[\mathfrak{a}]s_1$ at every x . Since $([\mathfrak{a}]s_1)' = [\mathfrak{a}']s_2$, this gives $1/(E_{(\mathfrak{a}')^{-1}I}(x)E_I(0))$ for $x \neq 0$. Combining this with the conjugate-ideal alternative for $\mathfrak{a}$, we obtain

$$\langle x \rangle_I^{N\mathfrak{a}} \frac{E_{(\mathfrak{a}')^{-1}I}(x)}{E_I(0)} = \frac{1}{E_{(\mathfrak{a}')^{-1}I}(x)E_I(0)}.$$

Thus $E_{(\mathfrak{a}')^{-1}I}(x)^2 = \langle x \rangle_I^{-N\mathfrak{a}}$, so every value at $x \neq 0$ on this lattice would be a root of unity. On the other hand, the conjugate-ideal alternative gives $\sigma_{\mathfrak{a}}(E_I(x)) = \pm E_{(\mathfrak{a}')^{-1}I}(x)$, since $\sigma_{\mathfrak{a}}(E_I(0)) = \pm E_I(0)$. This contradicts the hypothesis. At $x = 0$ both normalized values are one. $\square$

Theorem 9. *For every period ε , every $x \in G_{I,\varepsilon}$, and every $\mathfrak{a}$ prime to cm ,*

$$\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \in H, \quad \sigma_{\mathfrak{a}} \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right) = \frac{E_{\mathfrak{a}^{-1}I,\varepsilon}(x)}{E_{I,\varepsilon}(0)}.$$

Proof. Membership in H follows from Proposition 4. Choose two sufficiently large consecutive integers $r, r+1$. By Lemma 7 and Proposition 5, we have the Artin formula at both periods ε^r and ε^{r+1} . The multiplier order remains $\mathcal{O}$ when the period changes. Choose a common multiple m' of m and the auxiliary levels for ε^r and ε^{r+1} . Choose an integral ideal $\mathfrak{b}$, prime to cm' , representing the class of $\mathfrak{a}$ in $C_{\mathcal{O}}^+(m)$. Let $\tilde{\sigma}$ be the Artin automorphism of $\mathfrak{b}$ in $H_{\mathcal{O},m'}^+$. Then $\tilde{\sigma}|_H = \sigma_{\mathfrak{a}}$, and both higher-period laws hold for $\mathfrak{b}$ and $\tilde{\sigma}$. For $x \in G_{I,\varepsilon}$, we have by (8)

$$\frac{E_{I,\varepsilon^r}(x)}{E_{I,\varepsilon^r}(0)} = \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^r,$$

and similarly for $r+1$ and for every translated lattice. The ratio of the two sides of the Artin formula has both its r -th and $(r+1)$ -st powers equal to one, and is therefore one. Representative independence at level m now gives the formula for the original ideal. $\square$

We chose m so that the proof applied simultaneously to all $x \in G_{I,\varepsilon}$. For a fixed x , we can replace this level by the additive order of x .

Corollary 1. *Let $x \in G_{I,\varepsilon}$ have additive order d . Then*

$$\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \in H_{\mathcal{O},d}^+, \quad \sigma_{\mathfrak{a}} \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right) = \frac{E_{\mathfrak{a}^{-1}I,\varepsilon}(x)}{E_{I,\varepsilon}(0)}$$

for every integral invertible $\mathcal{O}$ -ideal $\mathfrak{a}$ prime to cd . Consequently, if $p \nmid cd$ splits in K , let $\mathfrak{p} \subseteq \mathcal{O}$ be the contraction of one of the primes above p . At every prime $\mathfrak{m}$ above $\mathfrak{p}\mathcal{O}_K$,

$$\frac{E_{\mathfrak{p}^{-1}I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \equiv \left(\frac{E_{I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} \right)^p \pmod{\mathfrak{m}}.$$

This includes $p = 2$ when cd is odd. The unnormalized value belongs to $H_{\mathcal{O},d}^+ K(\sqrt{\varepsilon})$. In particular, the normalized value belongs to the usual narrow ray field H_{cd}^+ .

Proof. The assertion is immediate for $x = 0$. For $x \neq 0$, repeat the Bézout argument in Lemma 8, with d in place of m . Indeed, if $\mathfrak{b} = \alpha\mathfrak{a}$ represents the same class at level d , choose an integer b , prime to d , with $b(\alpha - 1) \in d\mathcal{O}$. Then both $b(\alpha - 1)x$ and $d(\alpha - 1)x$ lie in $\mathfrak{a}^{-1}I$, so $(\alpha - 1)x$ lies there as well. Together with positive homothety, this shows that $E_{\mathfrak{a}^{-1}I,\varepsilon}(x)$ depends only on $[\mathfrak{a}] \in C_{\mathcal{O}}^+(d)$. Since the index $[\mathfrak{a}^{-1}I : I]$ is prime to d , the image of x in $K/\mathfrak{a}^{-1}I$ still has order d .

Choose m as above; in particular, $d \mid m$. Put $U = E_{I,\varepsilon}(x)/E_{I,\varepsilon}(0)$. For a class in the kernel of $C_{\mathcal{O}}^+(m) \rightarrow C_{\mathcal{O}}^+(d)$, choose an integral representative $\mathfrak{b}$ prime to cm . The exact Artin law and level- d independence give

$$\sigma_{\mathfrak{b}}(U) = \frac{E_{\mathfrak{b}^{-1}I,\varepsilon}(x)}{E_{I,\varepsilon}(0)} = U.$$

Thus U lies in the fixed field $H_{\mathcal{O},d}^+$. For an ideal prime to cd , choose a representative of its level- d class prime to cm , apply the theorem, and use representative independence again. This proves the asserted Artin law. Reducing it at an unramified prime gives the congruence, since $H_{\mathcal{O},d}^+ \subseteq H_{cd}^+$. Finally, multiply by $E_{I,\varepsilon}(0) = \sqrt{\varepsilon}$ for the assertion about the unnormalized value. $\square$

9. STARK UNITS

Let $\mathfrak{f} \subsetneq \mathcal{O}_K$ be a nonzero ideal, and let ∞_1 be the selected real embedding. Put

$$\mathcal{B} = \text{Cl}_{\mathfrak{f}\infty_2}(K), \quad H = H_{\mathfrak{f}\infty_2}.$$

Let $r \in \mathcal{B}$ be the sign class at ∞_2 , represented by a principal ideal with generator congruent to one modulo $\mathfrak{f}$, positive at ∞_1 , and negative at ∞_2 . Write $\zeta_{\mathcal{B}}(s, b)$ for the partial zeta function of $b \in \mathcal{B}$. Stark's positive number [Sta76], with parameter $m = 1$, is

$$\epsilon_b = \exp(\zeta'_{\mathcal{B}}(0, b) - \zeta'_{\mathcal{B}}(0, br)).$$

In particular, $\epsilon_{br} = \epsilon_b^{-1}$; if $r = 1$, all these numbers equal one. The theorem below proves the unit, reciprocity, and archimedean assertions of Stark's conjecture for this family [Sta76, Sta80]. We do not address Tate's additional square-root refinement [Tat84], which predicts that $H(\sqrt{\epsilon_b})/K$ is abelian.

Let $\varepsilon = \varepsilon_{\mathfrak{f}} > 1$ generate the totally positive units congruent to one modulo $\mathfrak{f}$, and put

$$t = |\ker(\text{Cl}_K^+(\mathfrak{f}) \rightarrow \mathcal{B})| \in \{1, 2\}.$$

For an integral ideal $\mathfrak{a}$ prime to $\mathfrak{f}$ and representing b , the identity of [RW26, Section 2.3 and Proposition 5] takes the form

$$\epsilon_b = |E_{\mathfrak{f}\mathfrak{a}^{-1},\varepsilon}(1)|^{-t}.$$

The class of 1 is fixed by ε , since $\varepsilon - 1 \in \mathfrak{f} \subseteq \mathfrak{f}\mathfrak{a}^{-1}$, and is nonzero: $1 \in \mathfrak{f}\mathfrak{a}^{-1}$ would imply $\mathfrak{a} \subseteq \mathfrak{f}$, contrary to coprimality. The formula for ϵ_b in terms of E holds for every such representative $\mathfrak{a}$; only the absolute value of E is being identified here.

The reciprocity theorem of the preceding section uses integer moduli. The following lemma recovers the ideal modulus $\mathfrak{f}$ using Artin symbols at prime ideals of K above split rational primes.

Lemma 9. *Let H/K be a ray class field with group C . Suppose that $u_b \in K^{\text{ab}}$, $b \in C$, satisfy*

$$\text{Art}(\mathfrak{p})(u_b) = u_{b[\mathfrak{p}]}$$

for all but finitely many prime ideals $\mathfrak{p}$ of K lying above split rational primes. Then $u_b \in H$ and $\text{Art}(a)(u_b) = u_{ba}$ for every $a, b \in C$.

Proof. Put $F = H(u_b : b \in C)$, a finite abelian extension of K . By Chebotarev's density theorem, every $\sigma \in \text{Gal}(F/K)$ occurs as Frobenius at infinitely many prime ideals of K above split rational primes. Indeed, prime ideals above inert rational primes have density zero among the prime ideals of K , and only finitely many rational primes ramify in K . The assumed formula therefore determines $\sigma(u_b)$, for every b , from $\sigma|_H$. In particular, $\text{Gal}(F/H)$ fixes every u_b , proving both assertions. $\square$

Theorem 10 (Stark reciprocity). *One has*

$$\epsilon_b \in \mathcal{O}_H^\times, \quad \text{Art}(a)(\epsilon_b) = \epsilon_{ba} \quad (a, b \in \mathcal{B}).$$

Every conjugate extending ∞_1 is positive, and every conjugate extending ∞_2 has absolute value one. If $r = 1$, all these units equal one.

Proof. The case $r = 1$ is immediate, so assume $r \neq 1$. Choose an integral ideal $\mathfrak{a}$ prime to $\mathfrak{f}$ representing b , and put $I = \mathfrak{f}\mathfrak{a}^{-1}$. The value $E_{I,\varepsilon}(1)$ is an algebraic unit by Theorem 5, and belongs to K^{ab} : its quotient by $\sqrt{\varepsilon}$ belongs there by Theorem 9, while $K(\sqrt{\varepsilon})/K$ has degree at most two.

Suppose first that $t = 2$. The conjugation formula of Proposition 2 gives

$$\epsilon_b = \langle 1 \rangle_{I,\varepsilon}^{-1} E_{I,\varepsilon}(1)^{-2}.$$

Thus ϵ_b is an algebraic unit in K^{ab} . For every prime ideal $\mathfrak{p}$ above a sufficiently large split rational prime, squaring the normalized Artin law removes the possible sign of $\sqrt{\varepsilon}$. Together with the Gaussian covariance in Lemma 8, this gives

$$\text{Art}(\mathfrak{p})(\epsilon_b) = \langle 1 \rangle_{\mathfrak{p}^{-1}I,\varepsilon}^{-1} E_{\mathfrak{p}^{-1}I,\varepsilon}(1)^{-2} = \epsilon_{b[\mathfrak{p}]},$$

since $\mathfrak{ap}$ represents $b[\mathfrak{p}]$.

Suppose now that $t = 1$. The sign class at ∞_1 is then trivial, so there is a unit $\nu \equiv 1 \pmod{\mathfrak{f}}$ with $\nu < -1$ and $\nu' > 0$. Write $\nu^2 = \varepsilon^j$. The integer j is odd: if it were even, then $\nu = \pm \varepsilon^{j/2}$ would have equal signs at the real places. Replacing ν by $\nu \varepsilon^{-(j-1)/2}$, we may take $\nu = -\sqrt{\varepsilon}$, still congruent to one modulo $\mathfrak{f}$. Lemma 11 therefore gives $E_{I,\varepsilon}(1) > 0$, so

$$\epsilon_b = E_{I,\varepsilon}(1)^{-1}.$$

Again ϵ_b is an algebraic unit in K^{ab} . Since $\sqrt{\varepsilon} = -\nu \in K$, the normalized Artin law has no sign ambiguity and gives

$$\text{Art}(\mathfrak{p})(\epsilon_b) = E_{\mathfrak{p}^{-1}I,\varepsilon}(1)^{-1} = \epsilon_{b[\mathfrak{p}]}$$

for every prime ideal $\mathfrak{p}$ above a sufficiently large split rational prime.

In both cases, Lemma 9, applied directly to $(\epsilon_b)_{b \in \mathcal{B}}$, proves membership in H and the full Artin law. All conjugates over ∞_1 are among the positive numbers ϵ_b . At ∞_2 , complex conjugation corresponds to r . Thus, for an embedding τ extending ∞_2 ,

$$\overline{\tau(\epsilon_b)} = \tau(\epsilon_{br}) = \tau(\epsilon_b)^{-1},$$

which proves the absolute-value assertion. $\square$

9.1. The field generated by the units. We now determine the field generated by these units, including the effect of additional Euler factors in the modulus. For a character χ of $\mathcal{B}$, put

$$L_{\mathfrak{f}}(s, \chi) = \sum_{b \in \mathcal{B}} \chi(b) \zeta_{\mathcal{B}}(s, b), \quad D_{\mathfrak{f}} = \bigcap_{\substack{\chi(r) = -1 \\ L'_{\mathfrak{f}}(0, \chi) \neq 0}} \ker \chi,$$

where an empty intersection means $\mathcal{B}$.

Proposition 6. *One has*

$$K(\epsilon_1) = K(\epsilon_b : b \in \mathcal{B}) = H^{D_{\mathfrak{f}}}.$$

If $r \neq 1$ and every χ with $\chi(r) = -1$ satisfies $L'_{\mathfrak{f}}(0, \chi) \neq 0$, this field equals H .

Proof. Taking the Fourier transform of the defining logarithm gives

$$\sum_{b \in \mathcal{B}} \chi(b) \log \epsilon_b = (1 - \chi(r)) L'_{\mathfrak{f}}(0, \chi).$$

By the Artin law, $a \in \mathcal{B}$ fixes all the units if and only if the real function $b \mapsto \log \epsilon_b$ is invariant under $b \mapsto ba$. Fourier inversion makes this equivalent to $\chi(a) = 1$ for every character with nonzero Fourier coefficient. Thus the common stabilizer is $D_{\mathfrak{f}}$, proving $K(\epsilon_b : b \in \mathcal{B}) = H^{D_{\mathfrak{f}}}$. Moreover, $K(\epsilon_1)/K$ is Galois because H/K is abelian, so it contains every conjugate ϵ_b . This proves the remaining equality.

For the last assertion, choose χ_0 with $\chi_0(r) = -1$. An element in the kernels of all characters with $\chi(r) = -1$ also lies in the kernel of every character with $\chi(r) = 1$, since $\chi = (\chi_0 \chi)/\chi_0$ is a quotient of two characters which take the value -1 on r . Thus the common kernel is trivial, so $D_{\mathfrak{f}} = \{1\}$ and $H^{D_{\mathfrak{f}}} = H$. $\square$

9.2. Unitary conjugates and Zauner's conjecture. The reciprocity law gives unitary conjugates of the arithmetic finite quantum dilogarithms.

Proposition 7. *Let $I \subseteq K$ be a rank-two lattice and let $\varepsilon > 1$ be a totally positive unit such that $\varepsilon I = I$. Then every automorphism τ of $\overline{\mathbb{Q}}$ inducing the nontrivial automorphism of K satisfies*

$$|\tau(E_{I,\varepsilon}(x))| = 1 \quad (0 \neq x \in G_{I,\varepsilon}).$$

Proof. Choose the auxiliary ray field H as in Section 8. It contains $\sqrt{\varepsilon}$ by Lemma 8 and all $E(x)/\sqrt{\varepsilon}$ by Theorem 9. Let c be complex conjugation and put $g = \tau^{-1}c\tau$. Since g fixes K , it preserves the ray class field H . The restriction $g|_H$ is the second real sign class. By Theorem 9 and Proposition 8, for $x \neq 0$ we have

$$g\left(\frac{E(x)}{\sqrt{\varepsilon}}\right) = \frac{1}{\sqrt{\varepsilon}E(x)}.$$

Since $K(\sqrt{\varepsilon})$ is totally real, g fixes $\sqrt{\varepsilon}$. Therefore $g(E(x)) = E(x)^{-1}$, and applying τ gives $\overline{\tau(E(x))} = \tau(E(x))^{-1}$. $\square$

By the argument of [AFK25, Theorem 1.46], Proposition 7, together with the identity of [RW26, Theorem 7], implies the existence of M^2 complex equiangular lines in $\mathbb{C}^M$ for every $M \geq 4$. Together with the known constructions in dimensions two and three, this proves the existence assertion of Zauner's conjecture in all dimensions. See also [AFK26] for the higher-rank twisted convolution identity and the construction of ghost r -SICs.

APPENDIX A. DISTRIBUTION FOR ARBITRARY LATTICES

Here we prove Proposition 3. We use the following analytic fact from [RW26, Section 2.2, before Proposition 2]. The function $\Phi_{\gamma,u,v}(z;\tau)$ continues meromorphically to $c\tau + d \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, where (c,d) is the bottom row of γ . Its zeros and poles in z lie in $\mathbb{Z}\tau + \mathbb{Z}$. In particular, if $c\tau + d > 0$ and $z \notin \mathbb{Z}\tau + \mathbb{Z}$, then $\Phi_{\gamma,u,v}(z;\tau)$ is finite and nonzero.

Choose an oriented basis (β_1, β_2) of J with β_2 totally positive. Dividing the lattices and arguments by β_2 , we write

$$J = \mathbb{Z}\tau + \mathbb{Z}, \quad I = \mathbb{Z}(n\tau + k) + \mathbb{Z}m, \quad n, m > 0, \quad k \in \mathbb{Z}.$$

Put $\tau_I = (n\tau + k)/m$, and let $\gamma_J = \begin{pmatrix} a_J & b_J \\ c_J & d_J \end{pmatrix}$ and $\gamma_I = \begin{pmatrix} a_I & b_I \\ c_I & d_I \end{pmatrix}$ represent multiplication by ε on these bases. Then

$$\gamma_I \begin{pmatrix} n & k \\ 0 & m \end{pmatrix} = \begin{pmatrix} n & k \\ 0 & m \end{pmatrix} \gamma_J, \quad c_I \tau_I + d_I = c_J \tau + d_J, \quad \gamma_I \tau_I = \frac{n(\gamma_J \tau) + k}{m}.$$

At the real fixed points, both automorphy factors equal $\varepsilon > 1$, so both fixed points are attractive. In the following product identity we regard τ as a complex variable; the matrix identities above remain valid.

Lemma 10 (Analytic distribution). *For $z \in \mathbb{C}$ and $u \in \mathbb{Z}$, put*

$$A_i = \left\lfloor \frac{u - i}{n} \right\rfloor, \quad B_{ij} = \left\lfloor \frac{-d_J i + c_J j}{n} \right\rfloor.$$

Near the attractive fixed point, we have the meromorphic identity

$$\Phi_{\gamma_J, u, 0}(z; \tau) = \prod_{i=0}^{n-1} \prod_{j=0}^{m-1} \Phi_{\gamma_I, A_i, B_{ij}} \left(\frac{z + i\tau + j}{m}; \frac{n\tau + k}{m} \right).$$

Proof. For $\text{Im } \tau > 0$, we first multiply the numerator factors over j :

$$\begin{aligned} & \prod_{i=0}^{n-1} \prod_{j=0}^{m-1} \left(\mathbf{e} \left(\frac{z + i\tau + j}{m} + A_i \tau_I \right); \mathbf{e}(\tau_I) \right)_{\infty} \\ &= \prod_{i=0}^{n-1} \prod_{h \geq A_i}^{m-1} (1 - \mathbf{e}(z + (i + nh)\tau + kh)) \\ &= \prod_{\ell \geq u} (1 - \mathbf{e}(z + \ell\tau)) = (\mathbf{e}(z + u\tau); \mathbf{e}(\tau))_{\infty}. \end{aligned}$$

For the first equality we used $\prod_{j=0}^{m-1} (1 - \mathbf{e}(j/m)X) = 1 - X^m$. For the second, we omit the integral term kh inside $\mathbf{e}$ and collect the integers $\ell \geq u$ according to $\ell \equiv i \pmod{n}$. If $A_i < 0$, there are only finitely many additional factors before the convergent tail.

For the denominator put $t = \gamma_J \tau$. Then

$$\frac{z + i\tau + j}{c_J \tau + d_J} = \frac{z}{c_J \tau + d_J} + (d_J i - c_J j)t - b_J i + a_J j.$$

The coordinate map $(i, j) \mapsto (d_J i - c_J j, -b_J i + a_J j)$ represents multiplication by ε^{-1} . Since $\varepsilon I = I$, multiplication by ε^{-1} induces a permutation of J/I . Adding $B_{ij}(n, k)$ puts the first coordinate in $\{0, \dots, n-1\}$. On reducing the second coordinate modulo m , we therefore obtain each pair (ℓ, s) in $\{0, \dots, n-1\} \times \{0, \dots, m-1\}$ exactly once. We now apply the numerator calculation with z replaced by $z/(c_J \tau + d_J)$, τ by t , and $u = 0$. All the numerator indices are then zero, so the denominator equals $(\mathbf{e}(z/(c_J \tau + d_J)); \mathbf{e}(t))_{\infty}$, as required. Both automorphy factors are positive at the attractive fixed point, so we may continue the identity there. $\square$

Proof of Proposition 3. Return to the real quadratic fixed point, where $c_J \tau + d_J = \varepsilon$. Choose a representative $x \in K$, taking $x = 0$ when its image in K/J is zero, and write

$$(\varepsilon^{-1} - 1)x \in u\tau + \mathbb{Z}, \quad u \in \mathbb{Z}.$$

Apply the analytic distribution identity with $z = x$. The factors on the right are indexed by

$$y_{ij} = \frac{x + i\tau + j}{m}, \quad 0 \leq i < n, \quad 0 \leq j < m,$$

which represent the lifts of x modulo I , divided by m . We order each orbit by multiplication by ε^{-1} . If (i', j') follows (i, j) , let $h \in \mathbb{Z}$ be the first coordinate of the lattice difference:

$$\varepsilon^{-1}y_{ij} - y_{i'j'} \in h\tau_I + \mathbb{Z}.$$

Comparing coefficients of τ gives $u + d_J i - c_J j = i' + nh$. Taking ceilings, we obtain

$$A_{i'} = \left\lceil \frac{u - i'}{n} \right\rceil = B_{ij} + h.$$

For an orbit with successive representatives $y_0, \dots, y_{r-1}$, let A_j, B_j be the indices of the factor corresponding to y_j . We rewrite this factor with argument $\varepsilon^{-j}y_0$, arranging that its denominator index equals the next factor's numerator index, so that the cocycle formula applies. Set $y_r = y_0$, $A_r = A_0$, and put

$$\varepsilon^{-j}y_0 - y_j \in p_j\tau_I + \mathbb{Z}, \quad p_j \in \mathbb{Z}, \quad p_0 = 0.$$

These integers are unique because τ_I is irrational. Since $\varepsilon^{-1}y_j - y_{j+1} \in h_j\tau_I + \mathbb{Z}$, we have

$$\varepsilon^{-1}(\varepsilon^{-j}y_0 - y_j) \in (p_{j+1} - h_j)\tau_I + \mathbb{Z}.$$

Moving the argument from y_j to $\varepsilon^{-j}y_0$ replaces the numerator index A_j by $A_j - p_j$, and the denominator index B_j by $B_j - (p_{j+1} - h_j)$. Since $B_j + h_j = A_{j+1}$, the latter is $A_{j+1} - p_{j+1}$. Using the translation formulas of [RW26, Section 2.2, before Equation (16)], we obtain directly

$$\Phi_{\gamma_I, A_j, B_j}(y_j; \tau_I) = \Phi_{\gamma_I, A_j - p_j, A_{j+1} - p_{j+1}}(\varepsilon^{-j}y_0; \tau_I).$$

The indices now match. Since $A_r = A_0$, the iterated cocycle formula [RW26, Equation (18)] gives

$$\prod_{j=0}^{r-1} \Phi_{\gamma_I, A_j, B_j}(y_j; \tau_I) = \Phi_{\gamma_I^r, A_0, A_0 - p_r}(y_0; \tau_I).$$

For a nonzero orbit, $y_0 \notin \mathbb{Z}\tau_I + \mathbb{Z}$. The relation $(\varepsilon^{-r} - 1)y_0 \in p_r\tau_I + \mathbb{Z}$ allows us to shift both indices by $p_r - A_0$, using [RW26, Equation (16)], and obtain

$$\Phi_{\gamma_I^r, p_r, 0}(y_0; \tau_I) = \mu_{I, \varepsilon^r}^{-1} E_{I, \varepsilon^r}(my_0).$$

Every canceled factor is nonzero, since a vanishing factor would put y_0 in $\mathbb{Z}\tau_I + \mathbb{Z}$. Here my_0 represents the original orbit in $\pi^{-1}(x)$.

It remains to consider the zero orbit. It occurs when $x = 0$, so $u = 0$, and consists of $(i, j) = (0, 0)$. Here $A_0 = B_{00} = 0$, and we must evaluate $\Phi_{\gamma_I, 0, 0}(0; \tau_I)$ without shifting its indices.

With τ_I temporarily in the upper half-plane, let the first argument tend to zero in the defining product. Applying the eta transformation law, with the convention of [RW26, Section 1, after Equation (2)], we get

$$\begin{aligned} \Phi_{\gamma_I, 0, 0}(0; \tau_I) &= (c_I\tau_I + d_I) \frac{(\mathbf{e}(\tau_I); \mathbf{e}(\tau_I))_\infty}{(\mathbf{e}(\gamma_I\tau_I); \mathbf{e}(\gamma_I\tau_I))_\infty} \\ &= \mu_{\gamma_I}^{-1} \sqrt{c_I\tau_I + d_I} \exp\left(\frac{\pi i}{12}(\gamma_I\tau_I - \tau_I)\right). \end{aligned}$$

We can now continue the right side to the real fixed point, obtaining

$$\Phi_{\gamma_I, 0, 0}(0; \tau_I) = \mu_{I, \varepsilon}^{-1} \varepsilon^{1/2} = \mu_{I, \varepsilon}^{-1} E_{I, \varepsilon}(0).$$

Repeating the calculation with γ_J , we get $\mu_{J, \varepsilon}^{-1} E_{J, \varepsilon}(0)$ on the left side of the product identity at zero.

Finally, the automorphy factor of γ_I^s at τ_I is $\varepsilon^s > 0$. The square roots in the eta cocycle therefore multiply without a sign, giving

$$\mu_{I, \varepsilon^r} = \mu_{I, \varepsilon}^r.$$

Let $\mathcal{R}_x$ contain one representative of each ε -orbit in $\pi^{-1}(x)$, and put $d = |J/I|$. Multiplying over all orbits and using $\sum_{y \in \mathcal{R}_x} r_y = d$, we obtain

$$\mu_{J, \varepsilon}^{-1} E_{J, \varepsilon}(x) = \mu_{I, \varepsilon}^{-d} \prod_{y \in \mathcal{R}_x} E_{I, \varepsilon^{r_y}}(y).$$

This is (10). The product is independent of the choice of orbit representatives, since $E_{I, \varepsilon^{r_y}}$ is invariant under multiplication by ε .

For part (ii), the definition of J gives

$$I \subseteq J \subseteq (\varepsilon - 1)^{-1}I, \quad J/I = H.$$

Multiplication by ε fixes J/I and preserves I , so $\varepsilon J = J$. For $x \in G_{I,\varepsilon}$, the fibre over $\pi(x)$ is $x + H$, and every point of this fibre is fixed by ε . Thus all its orbits have length one. Applying (10) at $\pi(x)$, with $d = |H|$, gives

$$\frac{\mu_{I,\varepsilon}^{|H|}}{\mu_{J,\varepsilon}} E_{J,\varepsilon}(\pi(x)) = \prod_{u \in H} E_{I,\varepsilon}(x + u),$$

which is (11). □

APPENDIX B. HOMOTHETY AND THE TWO REAL EMBEDDINGS

Change of basis. We first verify that the definition of E is independent of the basis and representative. For the homothety calculation below, we allow oriented bases whose second vector is positive only at the selected embedding. Dividing the lattice and arguments by the second basis vector, write the normalized basis as $(\tau_0, 1)$. Let γ represent multiplication by ε . The bottom row (c, d) satisfies $c\tau_0 + d = \varepsilon > 0$, even when $c < 0$, so the continuation statement recalled at the beginning of Appendix A applies.

For $0 \neq x \in G_{I,\varepsilon}$, choose a representative and write $\varepsilon^{-1}x = r_1\tau_0 + r_2$, with $r = (r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Since $r\gamma - r \in \mathbb{Z}^2$, the quotient

$$\frac{(\mathbf{e}(r_1\tau + r_2); \mathbf{e}(\tau))_\infty}{(\mathbf{e}(r_1\gamma\tau + r_2); \mathbf{e}(\gamma\tau))_\infty} = \Phi_{\gamma, r_1 - (r\gamma)_1, 0}((r\gamma)_1\tau + (r\gamma)_2; \tau)$$

continues to τ_0 , with argument x on the right. The first index of Φ above is the first coordinate of $(\varepsilon^{-1} - 1)x$. Thus the continued quotient at τ_0 equals $\mu_\gamma^{-1} E_{I,\varepsilon}(x)$. Replacing r by an integral translate introduces only finite Pochhammer factors. They are nonzero at τ_0 because $x \notin I$, and their quotient at τ and $\gamma\tau$ tends to one. Since these translations change x by all elements of $\varepsilon I = I$, this proves independence of the representative.

Changing the basis by $A \in \mathrm{SL}_2(\mathbb{Z})$ replaces τ, r, γ by $A\tau, rA^{-1}, A\gamma A^{-1}$. By the defining formula for Φ , the quotient above is multiplied by

$$\frac{\Phi_{A,0,0}(r_1\gamma\tau + r_2; \gamma\tau)}{\Phi_{A,0,0}(r_1\tau + r_2; \tau)}.$$

The automorphy factor of A at τ_0 is the ratio of the second vectors of the two bases, so it is positive. The numerator and denominator therefore continue to finite nonzero values there; they are equal because $\gamma\tau_0 = \tau_0$. We also have $\mu_{A\gamma A^{-1}} = \mu_\gamma$ by the eta cocycle: all square roots have positive limiting values. Hence the value of E is unchanged. At zero, independence follows from the prescribed value $E_{I,\varepsilon}(0) = \sqrt{\varepsilon}$.

Proposition 8. *For $\alpha \in K^\times$ and $0 \neq x \in G_{I,\varepsilon}$,*

$$E_{\alpha I, \varepsilon}(\alpha x) = \begin{cases} E_{I, \varepsilon}(x), & \alpha > 0, \alpha' > 0, \\ E_{I, \varepsilon}(-x), & \alpha < 0, \alpha' < 0, \\ E_{I, \varepsilon}(x)^{-1}, & \alpha > 0, \alpha' < 0, \\ E_{I, \varepsilon}(-x)^{-1}, & \alpha < 0, \alpha' > 0. \end{cases}$$

At zero, $E_{\alpha I, \varepsilon}(0) = E_{I, \varepsilon}(0) = \sqrt{\varepsilon}$.

Proof. For $\alpha, \alpha' > 0$, multiplying both basis vectors by α leaves the normalized lattice and argument unchanged, proving the first formula. For $\alpha > 0 > \alpha'$, take an admissible basis (β_1, β_2) of I . Then $(-\alpha\beta_1, \alpha\beta_2)$ is an oriented basis of αI , with second vector positive at the chosen embedding. Normalizing leaves the argument unchanged and reverses the first basis vector. We show that this inverts the value.

Put $\Delta = \mathrm{diag}(-1, 1)$, and suppose $(\varepsilon^{-1} - 1)x = m\tau_0 + n$. In the basis $(-\tau_0, 1)$, the indices are $(-m, 0)$. Using the reflection formula following [RW26, Equation (16)] and then shifting both indices down by one, we get

$$\Phi_{\Delta\gamma\Delta, -m, 0}(x; -\tau_0) = \Phi_{\gamma, m+1, 1}(x; \tau_0)^{-1} = \Phi_{\gamma, m, 0}(x; \tau_0)^{-1}.$$

No canceled factor vanishes, since $x \notin \mathbb{Z}\tau_0 + \mathbb{Z}$. Conjugating the eta transformation law, using $\eta(-\bar{\tau}) = \overline{\eta(\tau)}$, we get $\mu_{\Delta\gamma\Delta} = \mu_\gamma^{-1}$. Thus reversing the first basis vector replaces $E(x)$ by its inverse, which proves the third formula. The other two follow on replacing α and x by $-\alpha$ and $-x$. □

B.1. A unit of negative norm. The case $\nu < -1$ of the following lemma supplies the positivity needed in the $t = 1$ case of Stark reciprocity. Mixed-sign homothety gives, for a nonzero class x fixed by such a ν ,

$$E_{I, \nu^2}(x) = E_{I, \nu^2}(-x)^{-1} = \overline{E_{I, \nu^2}(x)}.$$

The lemma strengthens this reality statement to positivity.

Lemma 11. *Suppose that $\nu I = I$, $\text{Nm}(\nu) = -1$, and $|\nu| > 1$. Then $\mu_{I,\nu^2} = 1$. Moreover, if $0 \neq x \in K/I$ satisfies $\nu x = x$, then*

$$E_{I,\nu^2}(x) = 1 \quad \text{if } \nu > 1, \quad E_{I,\nu^2}(x) > 0 \quad \text{if } \nu < -1.$$

These assertions about the values remain true with ν^2 replaced by any positive power of ν^2 .

Proof. Normalize an oriented basis to $(\tau_0, 1)$, with second vector positive at both embeddings. Let A represent multiplication by $|\nu|$, and write $x = r_1\tau_0 + r_2$, where $r \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Thus $\det A = -1$, and $rA \equiv r \pmod{\mathbb{Z}^2}$ when $\nu > 0$, whereas $rA \equiv -r \pmod{\mathbb{Z}^2}$ when $\nu < 0$. To factor the period ν^2 into two determinant-one transformations, put

$$\Delta = \text{diag}(-1, 1), \quad \gamma = \Delta A, \quad s = \begin{cases} (-r_1, r_2), & \nu > 0, \\ (r_1, -r_2), & \nu < 0. \end{cases}$$

The coordinates s represent x in the basis $(-\tau_0, 1)$ when $\nu > 0$, and $-x$ when $\nu < 0$. The two matrices $\gamma, A\Delta \in \text{SL}_2(\mathbb{Z})$ satisfy

$$\begin{aligned} \gamma\tau_0 &= -\tau_0, & A\Delta(-\tau_0) &= \tau_0, & (A\Delta)\gamma &= A^2, \\ rA\Delta - s &\in \mathbb{Z}^2, & s\gamma - r &\in \mathbb{Z}^2. \end{aligned}$$

The automorphy factor of γ at τ_0 , and that of $A\Delta$ at $-\tau_0$, both equal $|\nu|$. Since $A\Delta = \Delta\gamma\Delta$, we can conjugate the eta transformation law to obtain $\mu_{A\Delta} = \mu_\gamma^{-1}$. The square roots in the eta cocycle for $(A\Delta)\gamma = A^2$ multiply without a sign, and therefore $\mu_{I,\nu^2} = 1$.

Consider

$$F(\tau) = \frac{(\mathbf{e}(r_1\tau + r_2); \mathbf{e}(\tau))_\infty}{(\mathbf{e}(s_1\gamma\tau + s_2); \mathbf{e}(\gamma\tau))_\infty} = \Phi_{\gamma, r_1 - (s\gamma)_1, 0}((s\gamma)_1\tau + (s\gamma)_2; \tau).$$

The indices are integral because $s\gamma - r \in \mathbb{Z}^2$. The first argument of Φ in the displayed expression for F satisfies

$$(s\gamma)_1\tau_0 + (s\gamma)_2 \equiv x \pmod{\mathbb{Z}\tau_0 + \mathbb{Z}}.$$

Since the automorphy factor is positive and $x \notin I$, $F(\tau_0)$ is finite and nonzero. The same argument, with r, s interchanged and γ replaced by $A\Delta$, shows that the second quotient below has a finite nonzero continuation to $-\tau_0$. We now factor the quotient for A^2 into these two quotients:

$$E_{I,\nu^2}(x) = F(\tau_0) \frac{(\mathbf{e}(s_1w + s_2); \mathbf{e}(w))_\infty}{(\mathbf{e}(r_1A\Delta w + r_2); \mathbf{e}(A\Delta w))_\infty} \Big|_{w=-\tau_0}.$$

It remains to evaluate the second factor. We have $A\Delta(-\tau) = -\gamma\tau$. If $\nu > 0$, we use the extended Pochhammer convention of [RW26, Appendix A.1], $(z; q^{-1})_\infty = (zq; q)_\infty^{-1}$ to write the quotient at $w = -\tau$ as

$$F(\tau)^{-1} \frac{1 - \mathbf{e}(r_1\tau + r_2)}{1 - \mathbf{e}(s_1\gamma\tau + s_2)}.$$

At τ_0 , the numerator and denominator of the finite quotient are equal and nonzero, because $s_1(-\tau_0) + s_2 = r_1\tau_0 + r_2 \notin \mathbb{Z}$. It follows that $E_{I,\nu^2}(x) = 1$.

If $\nu < 0$, the second factor at $w = -\tau$ is instead $\overline{F(\bar{\tau})}$. Indeed, using the same convention, we have

$$\overline{(\mathbf{e}(a\bar{\tau} + b); \mathbf{e}(\bar{\tau}))_\infty} = \frac{1}{(\mathbf{e}((1-a)\tau - b); \mathbf{e}(\tau))_\infty}, \quad a, b \in \mathbb{R}.$$

Consequently $E_{I,\nu^2}(x) = |F(\tau_0)|^2 > 0$. Finally, apply (8) to obtain the assertions for positive powers of ν^2 . $\square$

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