

# ARITHMETIC PROGRESSIONS WITH PRESCRIBED $\ell^p$ GAPS

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**ABSTRACT.** For every  $n \geq 3$  and  $p \in [1, \infty) \setminus \{1, 2, \dots, n-1\}$ , we prove that every measurable subset of  $\mathbb{R}^d$  of positive upper Banach density, with  $d \geq 2^{n-2} + 1$ , contains an  $n$ -term arithmetic progression with every sufficiently large prescribed  $\ell^p$  gap.

## 1. THE RESULT AND ITS INGREDIENTS

For a measurable set  $A \subset \mathbb{R}^d$ , define its upper Banach density by

$$\bar{\delta}(A) = \limsup_{R \rightarrow \infty} \sup_{z \in \mathbb{R}^d} R^{-d} |A \cap (z + [0, R]^d)|.$$

**Theorem 1.1.** *Let  $n \geq 3$  be an integer, let  $p \in (1, \infty) \setminus \{2, \dots, n-1\}$ , and let  $d \geq 2^{n-2} + 1$ . Every measurable  $A \subset \mathbb{R}^d$  with  $\bar{\delta}(A) > 0$  admits  $\lambda_0 > 0$  such that, for every  $\lambda \geq \lambda_0$ , there exist  $x, y \in \mathbb{R}^d$  with*

$$x, x + y, \dots, x + (n-1)y \in A, \quad \|y\|_{\ell^p} = \lambda.$$

For  $n \geq 4$  this answers Problem 4 by Durcik and Kovač [7], implicitly asked earlier by Cook, Magyar, and Pramanik [4]. The sufficient dimensions for  $n = 3, 4, 5, 6$  are, respectively, 3, 5, 9, 17; the same proof gives  $2^{n-2} + 1$  for every larger  $n$ . These bounds are independent of  $p$ , although constants may degenerate near the excluded exponents. We do not establish optimality or settle the planar case (i.e., the case  $d = 2$ , which is the smallest dimension in which the problem makes sense).

The distance theorems of Furstenberg, Katznelson, and Weiss [14] and of Falconer and Marstrand [12] initiated the study of prescribed large sizes of configurations in positive-density subsets of Euclidean space. Bourgain [3] treated nondegenerate simplices and also exhibited an obstruction for three-term progressions with prescribed Euclidean gaps. Cook, Magyar, and Pramanik [4] proved the conclusion for every sufficiently large gap for three-term progressions in the  $\ell^p$  metric, for every  $p \in (1, \infty) \setminus \{2\}$  and sufficiently large dimension depending on  $p$ .

For longer progressions, Durcik and Kovač [7] proved that positive-measure subsets of the unit cube determine an interval of  $\ell^p$  gap lengths whose size has a positive lower bound depending only on the density and the fixed parameters. Their Problem 4 asks for every sufficiently large gap in positive-density subsets of the whole space. Theorem 1.1 supplies this conclusion with a dimension bound independent of  $p$ . Related results for corners and extended boxes appear in [8, 6]; the question for longer progressions was already discussed in [8, Section 4].

Put  $k = n - 2$  and  $Q = 2^k$ . Bilinear stopping reduces uniform shell comparison to Fourier convergence of  $Q$ -fold shell intersections. In dimension  $Q+1$ , these intersections are generically curves. Gowers convergence first gives well-defined positive cube measures; recursive disintegration identifies their full mass with ordinary coarea measure on regular configurations. A polynomial continuation argument then transfers one nonzero-torsion witness to almost every such configuration. We construct that witness for every admissible exponent and obtain the required Fourier convergence by localization.

The final density argument uses two external inputs: the continuous Varnavides estimate [7, Proposition 1] and norm variation of multiple ergodic averages [5, Theorem 1.2]. Their

precise forms are stated in Section 9, including the passage from discrete to continuous translations. The latter theorem strengthens Tao's norm-convergence theorem [18] and extends earlier variation estimates for two and three commuting transformations [9, 10].

The use of shell approximations and uniformity norms has precedents in [4, 7]. Here we stop the Cauchy–Schwarz iteration at a bilinear form and use the geometry of shell intersections to control the resulting Fourier supremum. We also use the coarea formula [13], the analytic implicit-function theorem, one-dimensional van der Corput estimates [16, Chapter VIII, Section 1], and dimension theory for locally subanalytic sets [1]. All dimension arguments are made on relatively compact positive-coordinate charts and then extended by countable exhaustion.

## 2. LOCAL SHELLS AND COUNTING FORMS

Until the final dimension-lifting argument, fix

$$k = n - 2 \geq 1, \quad Q = 2^k, \quad d = Q + 1, \quad p \in (1, \infty) \setminus \{2, \dots, k + 1\}.$$

We call such an exponent  $p$  admissible. On the positive orthant, set  $P(y) = \sum_{i=1}^d y_i^p$ . Choose nonnegative  $\chi_i \in C_c^\infty((0, \infty))$ , each positive at  $d^{-1/p}$ , and write  $\chi(y) = \prod_i \chi_i(y_i)$ . Define the finite positive measure

$$d\sigma(y) = \frac{\chi(y)}{|\nabla P(y)|} d\mathcal{H}^{d-1}(y)|_{\{P=1\}}.$$

Its mass is strictly positive. Fix nonnegative probability densities  $\psi \in C_c^\infty((-1, 1))$  and  $\varphi \in C_c^\infty(\mathbb{R}^d)$ , and put

$$\psi_\eta(s) = \eta^{-1} \psi(s/\eta), \quad S_\eta(y) = \chi(y) \psi_\eta(P(y) - 1).$$

The functions are extended by zero away from the positive orthant. Coarea gives  $S_\eta(y) dy \rightharpoonup \sigma$ , with convergence of masses. All shell supports lie in the same compact positive box.

For a finite measure  $\mu$ , let  $\mu_\lambda$  be its pushforward by  $y \mapsto \lambda y$ . For functions we use the corresponding mass-preserving dilation  $g_\lambda(y) = \lambda^{-d} g(y/\lambda)$ . For bounded, compactly supported inputs, define

$$\Lambda_m(\mathbf{f}; \mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{j=0}^{m-1} f_j(x + jy) dx d\mu(y).$$

We write  $\Lambda_m(\mathbf{f}; g)$  when  $d\mu = g(y) dy$ . The forms depend only on the almost-everywhere classes of the inputs: a null-set change contributes zero after integration in  $x$ , by Fubini. The Fourier transform is  $\widehat{g}(\xi) = \int g(y) e^{-2\pi i y \cdot \xi} dy$ . We write  $(q)_s = q(q-1) \cdots (q-s+1)$  for falling factorials, with  $(q)_0 = 1$ . Constants may depend on the fixed parameters and cutoffs.

## 3. BILINEAR STOPPING AND THE SHELL CRITERION

For an integer  $m \geq 2$ , put

$$Q_m = 2^{m-2}, \quad b_m = 1 - \frac{m-1}{Q_m}.$$

For  $h \in \mathbb{R}^d$  define  $\Delta_h g(y) = g(y) \overline{g(y+h)}$ , and let

$$\mathfrak{B}_m(g)^{Q_m} = \int_{(\mathbb{R}^d)^{m-2}} \left\| \widehat{\Delta_{h_{m-2}} \cdots \Delta_{h_1} g} \right\|_\infty dh_1 \cdots dh_{m-2}. \quad (3.1)$$

For  $m = 2$  this means  $\mathfrak{B}_2(g) = \|\widehat{g}\|_\infty$ . All kernels in the next lemma are bounded and compactly supported, so Fubini's theorem applies to the integrals used in its proof. The suprema in the definition are measurable: each Fourier transform is continuous in  $\xi$ , so its supremum equals the supremum over  $\xi \in \mathbb{Q}^d$ .

**Lemma 3.1** (Explicit localized stopping estimate). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded measurable set of measure  $M > 0$ , and let  $K \subset \mathbb{R}^d$  be a compact set with  $H := |K - K| > 0$ . Suppose that  $|f_j| \leq 1$  and  $\text{supp } f_j \subset \Omega$  for  $0 \leq j < m$ , and that  $\text{supp } g \subset \lambda K$  for some  $\lambda > 0$ . Then*

$$\left| \int_{\mathbb{R}^{2d}} \prod_{j=0}^{m-1} f_j(x + jy) g(y) dx dy \right| \leq M H^{b_m} \lambda^{db_m} \mathfrak{B}_m(g). \quad (3.2)$$

For the mass-preserving dilation  $g_\lambda(y) = \lambda^{-d} g(y/\lambda)$  one has

$$\mathfrak{B}_m(g_\lambda) = \lambda^{-db_m} \mathfrak{B}_m(g). \quad (3.3)$$

In particular, if  $\text{supp } g \subset K$ , then

$$|\Lambda_m(\mathbf{f}; g_\lambda)| \leq M H^{b_m} \mathfrak{B}_m(g) \quad (\lambda > 0). \quad (3.4)$$

*Proof.* For  $m = 2$ , write  $\check{g}(y) = g(-y)$ . The inner integral in  $y$  is  $(f_1 * \check{g})(x)$ . Cauchy–Schwarz in  $x$  and Plancherel therefore give

$$|\Lambda_2(f_0, f_1; g)| \leq \|f_0\|_2 \|f_1 * \check{g}\|_2 \leq \|f_0\|_2 \|f_1\|_2 \|\hat{g}\|_\infty \leq M \mathfrak{B}_2(g),$$

which is (3.2), since  $b_2 = 0$ .

Suppose the estimate has been proved for  $m - 1$ , where  $m \geq 3$ . Removing  $f_0$  by Cauchy–Schwarz gives

$$|\Lambda_m(\mathbf{f}; g)|^2 \leq M \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \prod_{j=1}^{m-1} f_j(x + jy) g(y) dy \right|^2 dx.$$

Expand the square, put the second integration variable equal to  $y + h$ , and then make the measure-preserving substitution  $z = x + y$ . With

$$F_{j,h}(z) = f_j(z) \overline{f_j(z + jh)} \quad (1 \leq j \leq m - 1)$$

this yields

$$|\Lambda_m(\mathbf{f}; g)|^2 \leq M \int_{\lambda(K-K)} |\Lambda_{m-1}(F_{1,h}, \dots, F_{m-1,h}; \Delta_h g)| dh. \quad (3.5)$$

Indeed, the integral over  $h$  before taking absolute values equals the nonnegative square integral above; taking the absolute value inside can only increase it. The restriction on  $h$  follows from  $g(y) \overline{g(y + h)} \neq 0$  only when  $h \in \lambda(K - K)$ . Every  $F_{j,h}$  is bounded by one and supported in  $\Omega$ , while  $\Delta_h g$  is supported in  $\lambda K$ . The induction hypothesis applied for each  $h$  therefore bounds the right side of (3.5) by

$$M^2 H^{b_{m-1}} \lambda^{db_{m-1}} \int_{\lambda(K-K)} \mathfrak{B}_{m-1}(\Delta_h g) dh.$$

Hölder's inequality, with exponent  $Q_{m-1}$ , gives

$$\begin{aligned} \int_{\lambda(K-K)} \mathfrak{B}_{m-1}(\Delta_h g) dh &\leq (\lambda^d H)^{1-1/Q_{m-1}} \left( \int_{\mathbb{R}^d} \mathfrak{B}_{m-1}(\Delta_h g)^{Q_{m-1}} dh \right)^{1/Q_{m-1}} \\ &= (\lambda^d H)^{1-1/Q_{m-1}} \mathfrak{B}_m(g)^2. \end{aligned}$$

The last equality follows directly from the definition, the commuting identity  $\Delta_h \Delta_k g = \Delta_k \Delta_h g$ , and  $Q_m = 2Q_{m-1}$ . For  $m = 3$ , the exponent in Hölder's inequality is one, and the same calculation is simply an equality. Finally,

$$b_m = \frac{1}{2} \left( b_{m-1} + 1 - \frac{1}{Q_{m-1}} \right).$$

Taking square roots proves the induction claim, including its stated constant.

For the dilation identity, write  $r = m - 2$  and use

$$\Delta_{\lambda h_r} \cdots \Delta_{\lambda h_1} g_\lambda(y) = \lambda^{-dQ_m} (\Delta_{h_r} \cdots \Delta_{h_1} g)(y/\lambda).$$

Consequently,

$$\Delta_{\lambda h_r} \cdots \widehat{\Delta_{\lambda h_1}} g_\lambda(\xi) = \lambda^{-d(Q_m-1)} \Delta_{h_r} \cdots \widehat{\Delta_{h_1}} g(\lambda \xi).$$

Changing all  $r$  increment variables in the defining integral gives

$$\mathfrak{B}_m(g_\lambda)^{Q_m} = \lambda^{-d(Q_m-1-r)} \mathfrak{B}_m(g)^{Q_m} = \lambda^{-dQ_m b_m} \mathfrak{B}_m(g)^{Q_m}.$$

This proves (3.3); inserting it into (3.2) proves (3.4).  $\square$

*Remark 3.2.* A compact box containing the support may always be used for  $K$ . The precise constants above are included to exhibit every localization factor. No triangle inequality for  $\mathfrak{B}_m$  is required anywhere in this argument.

### 3.1. From shell comparison to convolution comparison.

**Lemma 3.3** (Direct shell-to-convolution comparison). *Fix  $m \geq 2$  and a bounded input-support set  $\Omega$ . Let  $L > 0$  satisfy  $\text{supp } \varphi \subset B(0, L)$ , and set*

$$\Omega^* = \Omega + B(0, (m-1)L).$$

*Suppose that for some  $a_0 \in (0, 1]$  there is a nondecreasing function  $v : (0, a_0) \rightarrow [0, \infty)$ , with  $v(a) \rightarrow 0$  as  $a \rightarrow 0$ , such that*

$$|\Lambda_m(\mathbf{f}; (S_\eta - S_\tau)_\lambda)| \leq v(a) \quad (3.6)$$

*whenever  $0 < \eta < \tau \leq a < a_0$ ,  $0 < \lambda \leq 1$ , and all inputs are bounded by one and supported in  $\Omega^*$ . Then, for  $0 < \varepsilon < \tau < \tau_0$  and  $0 < \lambda \leq 1$ ,*

$$|\Lambda_m(\mathbf{f}; \sigma_\lambda - \sigma_\lambda * \varphi_{\varepsilon\lambda})| \leq 2v(\tau) + C|\Omega| \frac{\varepsilon}{\tau} \quad (3.7)$$

*for all inputs bounded by one and supported in  $\Omega$ . Here  $0 < \tau_0 < a_0$  is sufficiently small, and  $C$  depends only on the fixed defining function and cutoffs. In particular, taking  $\tau = \sqrt{\varepsilon}$  makes the right-hand side tend to zero uniformly in the inputs and in  $\lambda$ .*

*Proof.* First, the coarea formula in finitely many coordinate neighborhoods where a partial derivative of  $P$  is bounded away from zero gives

$$S_\eta(y) dy \rightarrow \sigma$$

weakly as finite measures. Their supports lie in the fixed compact set  $\text{supp } \chi$ , and their masses are uniformly bounded. For compactly supported inputs bounded by one, the correlation

$$F_{\mathbf{f}}(y) = \int_{\mathbb{R}^d} \prod_{j=0}^{m-1} f_j(x + jy) dx$$

is continuous and compactly supported. To see continuity, expand the difference of the two products by telescoping and use

$$|F_{\mathbf{f}}(y+h) - F_{\mathbf{f}}(y)| \leq \sum_{j=1}^{m-1} \|f_j(\cdot + jh) - f_j\|_1 \rightarrow 0.$$

The compact support follows from the first two inputs: a nonzero product requires  $x \in \text{supp } f_0$  and  $x + y \in \text{supp } f_1$ . Thus we may pass  $\eta \rightarrow 0$  in (3.6), for fixed inputs,  $\tau$ , and  $\lambda$ , obtaining

$$|\Lambda_m(\mathbf{f}; \sigma_\lambda - (S_\tau)_\lambda)| \leq v(\tau). \quad (3.8)$$

Although the weak convergence itself is only pointwise in the inputs, the bound obtained after taking the limit is uniform, since the prelimit bound is uniform.

There is also the elementary estimate

$$\|\nabla S_\tau\|_1 \leq C\tau^{-1} \quad (0 < \tau < \tau_0). \quad (3.9)$$

Indeed,

$$\nabla S_\tau = (\nabla \chi) \psi_\tau(P-1) + \chi \psi'_\tau(P-1) \nabla P.$$

The coordinate neighborhoods above give  $|\text{supp } \chi \cap \{|P-1| \leq C_\psi \tau\}| = O(\tau)$ . The two terms have heights  $O(\tau^{-1})$  and  $O(\tau^{-2})$ , respectively, which proves (3.9). By the fundamental theorem of calculus and translation invariance,

$$\|S_\tau(\cdot - z) - S_\tau\|_1 \leq |z| \|\nabla S_\tau\|_1.$$

It follows that

$$\|S_\tau - S_\tau * \varphi_\varepsilon\|_1 \leq \varepsilon \|\nabla S_\tau\|_1 \int |z| |\varphi(z)| dz \leq C\varepsilon/\tau. \quad (3.10)$$

Use the exact decomposition

$$\begin{aligned} \sigma_\lambda - \sigma_\lambda * \varphi_{\varepsilon\lambda} &= [\sigma_\lambda - (S_\tau)_\lambda] \\ &\quad + [(S_\tau)_\lambda - (S_\tau)_\lambda * \varphi_{\varepsilon\lambda}] \\ &\quad + [(S_\tau)_\lambda - \sigma_\lambda] * \varphi_{\varepsilon\lambda}. \end{aligned}$$

The first term contributes at most  $v(\tau)$  by (3.8). Since  $|F_{\mathbf{f}}(y)| \leq |\Omega|$ , the second contributes at most  $C|\Omega|\varepsilon/\tau$  by (3.10) and invariance of the  $L^1$  norm under mass-preserving dilation.

For the third, expand the convolution and change the gap variable. For  $z \in \text{supp } \varphi_{\varepsilon\lambda}$  put  $f_{j,z}(x) = f_j(x + jz)$ . Fubini's theorem gives

$$\begin{aligned} \Lambda_m(\mathbf{f}; [(S_\tau)_\lambda - \sigma_\lambda] * \varphi_{\varepsilon\lambda}) \\ = \int \varphi_{\varepsilon\lambda}(z) \Lambda_m(\mathbf{f}_z; (S_\tau)_\lambda - \sigma_\lambda) dz. \end{aligned}$$

Because  $\varepsilon\lambda \leq 1$ , all  $f_{j,z}$  are supported in  $\Omega^*$  and bounded by one. Hence (3.8) and  $\|\varphi_{\varepsilon\lambda}\|_1 = 1$  bound this contribution by  $v(\tau)$ . Adding the three bounds proves (3.7) and the convergence assertion.  $\square$

**Corollary 3.4** (A sufficient shell criterion). *If*

$$\omega_m(a) := \sup_{0 < \eta < \tau \leq a} \mathfrak{B}_m(S_\eta - S_\tau) \longrightarrow 0 \quad (a \rightarrow 0),$$

*then the exact-to-convolution comparison in (3.7) holds with a modulus tending to zero, uniformly for inputs bounded by one and supported in a fixed bounded set, and for  $0 < \lambda \leq 1$ .*

*Proof.* Choose  $a_0 > 0$  small enough that  $\omega_m(a) < \infty$  for  $a < a_0$ . All shell differences are supported in the fixed compact set  $\text{supp } \chi$ . Lemma 3.1, applied with  $\Omega^*$  in place of  $\Omega$ , proves (3.6) with  $v(a) = C\omega_m(a)$ . Apply Lemma 3.3.  $\square$

#### 4. GOWERS CONVERGENCE OF THE SHELLS

Gowers introduced uniformity norms in his work on arithmetic progressions [15]. We use their Euclidean version; see [11] for the locally compact abelian group setting. For an integer  $k \geq 2$ , the unnormalized Gowers norm on  $\mathbb{R}^d$  is

$$\|f\|_{U^k(\mathbb{R}^d)}^{2^k} = \int_{\mathbb{R}^{d(k+1)}} \prod_{\omega \in \{0,1\}^k} \mathcal{C}^{|\omega|} f(x + \omega \cdot h) dx dh, \quad \omega \cdot h = \sum_{j=1}^k \omega_j h_j. \quad (4.1)$$

Here  $\mathcal{C}$  is complex conjugation. The integral on the right is nonnegative. For example, separating the last increment writes it as the integral, over  $(h_1, \dots, h_{k-1})$ , of

$$\left| \int_{\mathbb{R}^d} \prod_{\omega \in \{0,1\}^{k-1}} \mathcal{C}^{|\omega|} f(x + \omega \cdot h) dx \right|^2.$$

We use the usual Gowers–Cauchy–Schwarz inequality

$$\left| \int \prod_{\omega \in \{0,1\}^k} \mathcal{C}^{|\omega|} f_{\omega}(x + \omega \cdot h) dx dh \right| \leq \prod_{\omega \in \{0,1\}^k} \|f_{\omega}\|_{U^k}. \quad (4.2)$$

The triangle inequality for  $U^k$  follows by expanding the  $2^k$  factors for  $f + g$ , applying (4.2) to each of the  $2^{2^k}$  terms, and summing:

$$\|f + g\|_{U^k}^{2^k} \leq (\|f\|_{U^k} + \|g\|_{U^k})^{2^k}.$$

Only bounded compactly supported functions and limits controlled by these inequalities are used below.

**Lemma 4.1** (The one-dimensional phase estimate). *Let  $k \geq 2$ , let  $b \in C_c^\infty((0, \infty))$ , and suppose*

$$p > 1, \quad p(p-1) \cdots (p-k+1) \neq 0. \quad (4.3)$$

*Then*

$$\|b(x)e^{2\pi i u x^p}\|_{U^k(\mathbb{R})} \leq C_{k,p,b}(1+|u|)^{-1/2^k} (1+\log(2+|u|))^{(k-2)/2^k}. \quad (4.4)$$

*For  $k = 2$ , condition (4.3) holds for every  $p > 1$ , and the logarithmic exponent is zero.*

*Proof.* Choose  $0 < c < C < \infty$  such that  $\text{supp } b \subseteq (c, C)$ . Put  $f_u(x) = b(x)e^{2\pi i u x^p}$ . Its cube product has amplitude and phase

$$B(x, h) = \prod_{\omega \in \{0,1\}^k} \mathcal{C}^{|\omega|} b(x + \omega \cdot h), \quad \Phi(x, h) = \sum_{\omega \in \{0,1\}^k} (-1)^{|\omega|} (x + \omega \cdot h)^p.$$

The phase is only needed where its arguments are positive. On the amplitude support every vertex belongs to  $[c, C]$ . In particular,  $x \in [c, C]$  and  $|h_j| \leq C - c$  for every  $j$ .

Fix  $x$  and  $h' = (h_1, \dots, h_{k-1})$  for which the factors with  $\omega_k = 0$  do not vanish. Integrate first in  $h_k$ . On the interval on which all the remaining vertices belong to  $[c, C]$ , the fundamental theorem of calculus, applied  $k-1$  times, gives

$$\begin{aligned} \partial_{h_k} \Phi(x, h) &= (-1)^k p(p-1) \cdots (p-k+1) \left( \prod_{j=1}^{k-1} h_j \right) A(x, h), \\ A(x, h) &= \int_{[0,1]^{k-1}} \left( x + h_k + \sum_{j=1}^{k-1} t_j h_j \right)^{p-k} dt. \end{aligned} \quad (4.5)$$

Every point in the last integral lies in  $[c, C]$ : it is a convex combination of the vertices with  $\omega_k = 1$ . Consequently  $A$  is bounded above and below by positive constants. Every derivative of  $A$  in  $h_k$  is uniformly bounded, and the same holds for every fixed-order derivative of  $A^{-1}$ . These bounds are independent of  $x, h'$  on the active set.

For  $\prod_{j < k} h_j \neq 0$ , integrate by parts using

$$e^{2\pi i u \Phi} = \frac{1}{2\pi i u \partial_{h_k} \Phi} \partial_{h_k} e^{2\pi i u \Phi}.$$

The amplitude and each of its derivatives are uniformly bounded and supported in an interval of uniformly bounded length. They vanish near the endpoints of the interval where any vertex first reaches  $c$  or  $C$ , so no boundary terms occur. Equation (4.5) shows that each integration by parts contributes one factor  $(|u| \prod_{j < k} |h_j|)^{-1}$ . After  $N$  repetitions,

$$\left| \int_{\mathbb{R}} B(x, h) e^{2\pi i u \Phi(x, h)} dh_k \right| \leq C_N \min \left\{ 1, \left( |u| \prod_{j=1}^{k-1} |h_j| \right)^{-N} \right\}.$$

The bound by a constant is the trivial estimate; it covers zero increments as well. Combining the nonnegativity in (4.1) with Fubini and the triangle inequality therefore yields

$$\|f_u\|_{U^k}^{2^k} \leq C_N \int_{[-(C-c), C-c]^{k-1}} \min \left\{ 1, \left( |u| \prod_{j=1}^{k-1} |h_j| \right)^{-N} \right\} dh'. \quad (4.6)$$

The bounded  $x$  integral has been absorbed into  $C_N$ .

For completeness, if  $r \geq 1$ ,  $N > 1$ , and  $U \geq 2$ , then

$$\int_{[0,1]^r} \min\{1, (Us_1 \cdots s_r)^{-N}\} ds \leq C_{r,N} U^{-1} (1 + \log U)^{r-1}. \quad (4.7)$$

To prove this, set  $s_j = e^{-v_j}$ . Integration over the simplex of points with  $v_1 + \cdots + v_r = v$  turns the left side into

$$\frac{1}{(r-1)!} \int_0^\infty \min\{1, U^{-N} e^{Nv}\} e^{-v} v^{r-1} dv.$$

On  $0 \leq v \leq \log U$  it is at most

$$\frac{U^{-N} (1 + \log U)^{r-1}}{(r-1)!} \int_0^{\log U} e^{(N-1)v} dv \leq C_{r,N} U^{-1} (1 + \log U)^{r-1}.$$

On  $v \geq \log U$ , put  $v = \log U + w$  and use  $(\log U + w)^{r-1} \leq C_r (1 + \log U)^{r-1} (1 + w)^{r-1}$ . The remaining integral of  $e^{-w} (1 + w)^{r-1}$  is finite and gives the same bound. This also proves the case  $r = 1$ .

Apply (4.7) with  $r = k - 1$ , after rescaling and using symmetry in the increments in (4.6). For  $|u|$  sufficiently large this gives

$$\|f_u\|_{U^k}^{2^k} \leq C |u|^{-1} (1 + \log |u|)^{k-2}.$$

The remaining bounded range of  $u$  follows from the trivial cube integral estimate. Taking the  $2^k$ -th root proves (4.4).  $\square$

**Theorem 4.2** (Gowers Cauchy convergence of shells). *Suppose that  $k \geq 2$ ,  $d > 2^k$ , and  $p$  satisfies (4.3). Let*

$$\chi(y) = \prod_{\ell=1}^d b_\ell(y_\ell), \quad b_\ell \in C_c^\infty((0, \infty)), \quad P_p(y) = \sum_{\ell=1}^d y_\ell^p.$$

Fix  $\psi \in C_c^\infty(\mathbb{R})$  with  $\int \psi = 1$ , and put

$$S_\eta(y) = \chi(y) \eta^{-1} \psi((P_p(y) - 1)/\eta).$$

Then

$$\|S_\eta - S_\tau\|_{U^k(\mathbb{R}^d)} \longrightarrow 0 \quad (\eta, \tau \downarrow 0),$$

and the shell  $U^k$  norms are uniformly bounded for  $0 < \eta \leq 1$ .

*Proof.* For  $G_u(y) = \chi(y) e^{2\pi i u P_p(y)}$ , the defining cube integral factors over the  $d$  scalar coordinates. Thus the tensor identity is exact:

$$\|G_u\|_{U^k(\mathbb{R}^d)} = \prod_{\ell=1}^d \|b_\ell(x) e^{2\pi i u x^p}\|_{U^k(\mathbb{R})}.$$

By Lemma 4.1, with  $\gamma = d/2^k$  and  $L = d(k-2)/2^k$ ,

$$\|G_u\|_{U^k} \leq C(1 + |u|)^{-\gamma} (1 + \log(2 + |u|))^L. \quad (4.8)$$

The right side is integrable on  $\mathbb{R}$ , since  $\gamma > 1$ .

Use the scalar Fourier-transform convention  $\widehat{\psi}(u) = \int \psi(s)e^{-2\pi ius} ds$ . Fourier inversion gives the exact formula

$$S_\eta - S_\tau = \int_{\mathbb{R}} e^{-2\pi iu} [\widehat{\psi}(\eta u) - \widehat{\psi}(\tau u)] G_u du.$$

For fixed positive  $\eta, \tau$ , the scalar coefficient is integrable, so this integral converges uniformly in  $y$  and has fixed compact support. The triangle inequality for  $U^k$  yields

$$\|S_\eta - S_\tau\|_{U^k} \leq \int_{\mathbb{R}} |\widehat{\psi}(\eta u) - \widehat{\psi}(\tau u)| \|G_u\|_{U^k} du. \quad (4.9)$$

One may justify the integral triangle inequality here without invoking a completed Gowers space. On a bounded  $u$  interval, Riemann sums converge uniformly in  $y$ , and hence in  $U^k$ : for functions supported in a fixed compact set  $E$ ,

$$\|f\|_{U^k}^{2^k} \leq |E| |E - E|^k \|f\|_\infty^{2^k}.$$

Apply the finite-sum triangle inequality, pass to the Riemann limit, and then let the interval increase to  $\mathbb{R}$ . The integrability of the Fourier coefficient gives uniform convergence of the tails; (4.8) controls the right side.

The Fourier coefficient in (4.9) tends to zero pointwise as  $\eta, \tau \downarrow 0$  and is bounded by  $2\|\psi\|_1$ . Dominated convergence using (4.8) proves the Cauchy assertion. Applying the same integral estimate to  $S_\eta$  itself, with coefficient  $e^{-2\pi iu}\widehat{\psi}(\eta u)$ , gives uniform boundedness.  $\square$

## 5. CUBE MEASURES AND EXCLUSION OF SINGULAR CONCENTRATION

We first record the measure-theoretic consequence of Gowers convergence. Throughout the next lemma, let

$$S_\epsilon(y) = \chi(y)\psi_\epsilon(P(y) - 1), \quad \psi_\epsilon(s) = \epsilon^{-1}\psi(s/\epsilon),$$

where  $\chi \geq 0$  is smooth and compactly supported,  $P$  is smooth on a neighborhood of its support, and  $\psi \geq 0$  is a smooth compactly supported probability density. Write  $E = \text{supp } \chi$ . All the configuration measures below have common compact support: their base vertex belongs to  $E$ , and each increment belongs to  $E - E$ .

**Lemma 5.1** (Mixed cube limits and kernel factorization). *Let  $r \geq 2$ , and suppose  $S_\epsilon$  is Cauchy in  $U^r(\mathbb{R}^d)$ . Then the positive measures*

$$dM_\epsilon(y, h) = \prod_{\omega \in \{0,1\}^r} S_{\epsilon_\omega}(y + \omega \cdot h) dy dh$$

*converge weakly and in total mass to a common finite measure  $M_r$  as  $\max_\omega \epsilon_\omega \rightarrow 0$ , independently of the relative widths. This limit is supported on the exact incidence*

$$P(y + \omega \cdot h) = 1 \quad (\omega \in \{0,1\}^r).$$

*Put  $\ell = (h_1, \dots, h_{r-1})$ , and define*

$$d\nu_{\ell, \epsilon}(x) = \prod_{\omega \in \{0,1\}^{r-1}} S_\epsilon(x + \omega \cdot \ell) dx, \quad m_\epsilon(\ell) = \nu_{\ell, \epsilon}(\mathbb{R}^d).$$

*There is a measurable kernel of finite positive measures  $\nu_\ell$  such that  $m_\epsilon \rightarrow m := \nu_\ell(\mathbb{R}^d)$  in  $L^2(d\ell)$ , the lower-cube measures converge weakly to*

$$L = \int \nu_\ell d\ell,$$

*and*

$$\int \Phi dM_r = \int d\ell \iint \Phi(x, \ell, z - x) d\nu_\ell(x) d\nu_\ell(z) \quad (5.1)$$

*for every continuous compactly supported  $\Phi$ . The kernel is uniquely determined for almost every  $\ell$  by  $L$ . It is supported on the exact lower-cube fiber for almost every  $\ell$ .*

If  $L$  is concentrated where the lower-cube derivative in  $x$  has rank  $2^{r-1}$  and there equals ordinary weighted coarea measure, then  $\nu_\ell$  is the corresponding fiber coarea measure for almost every  $\ell$ . In that case  $M_r$  is concentrated on the regular locus of the full  $2^r$ -level map and equals ordinary weighted coarea measure there.

*Proof.* Let  $C$  bound the shell  $U^r$  norms. Gowers–Cauchy–Schwarz, (4.2), bounds every mixed  $r$ -cube integral by the product of its input  $U^r$  norms. The same bound is available with a smooth compactly supported configuration test  $\Phi$ , at the cost of  $\|\widehat{\Phi}\|_1$ . Indeed, every Fourier character factors into characters of the base and singleton vertices by

$$\xi \cdot y + \sum_{j=1}^r \eta_j \cdot h_j = \left( \xi - \sum_{j=1}^r \eta_j \right) \cdot y + \sum_{j=1}^r \eta_j \cdot (y + h_j).$$

Linear modulation preserves  $U^r$  for  $r \geq 2$ . Absorbing the conjugations at odd vertices into the inputs and using Fourier inversion therefore gives

$$\left| \int \Phi(y, h) \prod_{\omega} f_{\omega}(y + \omega \cdot h) dy dh \right| \leq \|\widehat{\Phi}\|_1 \prod_{\omega} \|f_{\omega}\|_{U^r}.$$

Telescope one input at a time when comparing two systems of widths. Each summand has one  $U^r$ -small difference and uniformly bounded remaining inputs. Thus all smooth tested integrals are Cauchy, with the same limit for all independent widths. The untested inequality also bounds the masses uniformly. Positivity and common compact support give weakly convergent subsequences, while the Cauchy property on smooth tests makes their limit unique. Uniform approximation by smooth tests on the common compact support proves weak convergence on continuous tests. A smooth test equal to one there proves convergence of total masses.

For  $2^{r-1}$  functions define

$$m_{\mathbf{f}}(\ell) = \int \prod_{\omega \in \{0,1\}^{r-1}} f_{\omega}(x + \omega \cdot \ell) dx.$$

Expanding its squared  $L^2$  norm and substituting  $z = x + h_r$  produces an  $r$ -cube integral. Gowers–Cauchy–Schwarz gives

$$\|m_{\mathbf{f}}\|_2 \leq \prod_{\omega \in \{0,1\}^{r-1}} \|f_{\omega}\|_{U^r}. \quad (5.2)$$

Consequently

$$\|m_{\epsilon} - m_{\tau}\|_2 \leq 2^{r-1} C^{2^{r-1}-1} \|S_{\epsilon} - S_{\tau}\|_{U^r} \longrightarrow 0.$$

For a smooth compactly supported  $\phi(x)$ , write

$$m_{\epsilon}^{\phi}(\ell) = \int \phi(x) d\nu_{\ell, \epsilon}(x).$$

Fourier inversion in  $x$ , Minkowski's inequality, and (5.2) yield the analogous tested estimate

$$\left\| \int \phi(x) \prod_{\omega} f_{\omega}(x + \omega \cdot \ell) dx \right\|_2 \leq \|\widehat{\phi}\|_1 \prod_{\omega} \|f_{\omega}\|_{U^r}.$$

Here the Fourier character modulates only the base-vertex input. Telescoping proves that  $m_{\epsilon}^{\phi}$  is Cauchy in  $L^2$  for every such  $\phi$ .

Choose countably many smooth tests whose restrictions are uniformly dense in  $C(E)$ , including a test equal to one on  $E$ . A diagonal subsequence  $\epsilon_j \rightarrow 0$  has almost-everywhere convergence of all their tested masses and the total masses. For almost every  $\ell$  the latter are bounded. Weak compactness of positive measures on  $E$ , followed by uniqueness from the determining tests, therefore gives a finite measure  $\nu_{\ell}$  with

$$\nu_{\ell, \epsilon_j} \longrightarrow \nu_{\ell} \quad \text{weakly for almost every } \ell.$$

Its total mass is the  $L^2$  limit  $m(\ell)$  of  $m_\epsilon$ . The countable determining family also proves measurability of the kernel  $\ell \mapsto \nu_\ell$ .

The lower-cube measures have a unique weak limit. For tests  $\phi(x)a(\ell)$  this follows from the preceding  $L^2$  convergence and the bounded support in  $\ell$ . Finite sums of such product tests are uniformly dense on the common compact support, and the masses are uniformly bounded by (5.2) and Cauchy–Schwarz in  $\ell$ . It follows that the limit is  $L = \int \nu_\ell d\ell$ . Uniqueness of disintegration determines  $\nu_\ell$  almost everywhere.

For equal widths, the substitution  $h_r = z - x$  gives (5.1) with  $\nu_{\ell,\epsilon}$  in place of  $\nu_\ell$ . Along the chosen subsequence the inner integrals converge for almost every  $\ell$ , by weak convergence of products of finite measures on a compact set. Their absolute values are bounded by  $\|\Phi\|_\infty m_\epsilon(\ell)^2$ . Since

$$\|m_\epsilon^2 - m^2\|_1 \leq \|m_\epsilon - m\|_2(\|m_\epsilon\|_2 + \|m\|_2) \longrightarrow 0,$$

these bounds are uniformly integrable. Vitali's theorem proves (5.1). The already established uniqueness of the full-cube weak limit removes any subsequence restriction from that identity.

Both  $M_r$  and  $L$  are supported on their exact incidences. To see this, any point off the exact incidence has a neighborhood on which at least one level differs from one by a fixed positive amount; the associated shell factor vanishes there for all sufficiently small widths. The same conclusion for almost every conditional measure follows from  $L = \int \nu_\ell d\ell$  and Tonelli's theorem. More generally, concentration of  $L$  on any measurable set implies concentration of its conditional measures on the corresponding sections of that set.

On the full-rank part of the lower-cube derivative in  $x$ , ordinary coarea charts disintegrate the lower-cube level measure into its fiber coarea measures with respect to  $d\ell$ . Uniqueness of the kernel proves the stated identification of  $\nu_\ell$  there. In coordinates  $(x, z, \ell)$ , the two groups of  $2^{r-1}$  levels in (5.1) have independent derivative blocks in  $x$  and  $z$ . Thus the full derivative has rank  $2^r$  almost everywhere for the product kernel. Finally, on any relatively compact submersion chart for the full level map, the shell integrals are products of one-dimensional approximate identities against a smooth density. They converge to ordinary weighted coarea measure, even with independent widths. This identifies  $M_r$  on its regular locus and completes the proof.  $\square$

**Proposition 5.2** (Regular mass at the target dimension). *Let  $k \geq 1$ ,  $d = 2^k + 1$ , and*

$$p > 1, \quad p \notin \{2, \dots, k+1\}.$$

*Let  $P(y) = \sum_{i=1}^d y_i^p$  on the positive orthant, and choose a nonnegative smooth product cutoff  $\chi$  supported compactly there. For every  $1 \leq j \leq k$ , the positive mixed  $j$ -cube shell measures have a common finite weak limit  $M_j$ , with convergence of total masses. This measure is supported on the exact incidence*

$$\Gamma_j = \{(y, h) : P(y + \omega \cdot h) = 1 \text{ for every } \omega \in \{0, 1\}^j\},$$

*is concentrated on the regular locus of its full  $2^j$ -level map, and equals ordinary weighted coarea measure there.*

*Proof.* The single shells converge weakly and in mass to

$$d\sigma(y) = \frac{\chi(y)}{|\nabla P(y)|} d\mathcal{H}^{d-1}(y) \quad \text{on } \{P = 1\},$$

because  $\nabla P$  does not vanish on the active positive patch. Under  $z = y + h$ , the mixed one-cube measure becomes  $S_\eta(y) dy \otimes S_\tau(z) dz$ . Therefore  $M_1$  is the image of  $\sigma \otimes \sigma$ , with weak and mass convergence for independent widths. The map  $(y, z) \mapsto (P(y), P(z))$  is a submersion everywhere on the active patch, so  $M_1$  is ordinary coarea measure on the full two-level incidence.

For  $2 \leq j \leq k$ , Theorem 4.2 applies in  $U^j$ , since  $d > 2^j$  and the required falling factorials of  $p$  are nonzero. Lemma 5.1 supplies the mixed weak limits  $M_j$ . If  $k = 1$ , the preceding paragraph already proves the proposition. Otherwise proceed by induction from  $M_1$ .

Suppose  $1 \leq j < k$  and that  $M_j$  is ordinary coarea measure on the regular full incidence. We show that the derivative in  $y$  alone has rank  $2^j$  at  $M_j$ -almost every point. Rank deficiency implies the existence of  $[c] \in \mathbb{P}^{2^j-1}(\mathbb{R})$  such that

$$\sum_{\omega \in \{0,1\}^j} c_\omega (y_i + \sum_{a=1}^j \omega_a h_{ai})^{p-1} = 0 \quad (1 \leq i \leq d). \quad (5.3)$$

For every nonzero  $c$ , the scalar function in one coordinate block of  $j+1$  variables is nonzero analytic on its connected positive cone. Indeed, set its base variable equal to one and differentiate at zero increments in a square-free subset  $S \subseteq [j]$ . An identically zero function would give

$$(p-1)_{|S|} \sum_{\omega: \omega_a=1 \ (a \in S)} c_\omega = 0 \quad (S \subseteq [j]).$$

All displayed falling factorials are nonzero. Starting with  $S = [j]$  and proceeding in decreasing subset size forces every  $c_\omega = 0$ , a contradiction.

Hence, for fixed  $[c]$ , each coordinate zero set in (5.3) has dimension at most  $j$ . The  $d$  coordinate blocks are independent. Including the projective coefficient parameter and projecting shows that the rank-defect set has dimension at most

$$jd + (2^j - 1).$$

These dimension statements can be made on countably many relatively compact positive charts using analytic stratification; projection in the compact projective coefficient variable does not increase dimension. The regular exact incidence has dimension  $(j+1)d - 2^j$ , and

$$jd + (2^j - 1) < (j+1)d - 2^j, \quad \text{since } d = 2^k + 1 > 2^{j+1} - 1.$$

Thus its rank-defect subset is null for ordinary coarea measure, and consequently for  $M_j$ .

Apply Lemma 5.1 at order  $r = j+1$ . The lower-cube weak limit in that lemma is  $M_j$ , by uniqueness of the already established weak limit. We have just verified its concentration on full-rank  $y$ -fibers and its coarea description. The final assertion of the lemma therefore identifies  $M_{j+1}$  as ordinary coarea measure on its regular full incidence, with no mass on its singular complement. This completes the induction.  $\square$

## 6. POLYNOMIAL TRANSFER FROM ONE SIGNED WITNESS

The geometry of exact intersections can have several connected components. The following argument transfers nonzero torsion through a connected space of coordinate ratios instead. Allowing signed weights in the initial witness will be useful for integer exponents.

For a regular curve  $y = Y(t)$  in  $\mathbb{R}^d$ , nonzero *affine torsion* at a point means

$$\det(Y'(t), Y''(t), \dots, Y^{(d)}(t)) \neq 0.$$

This condition is independent of the regular parameterization and is preserved by invertible affine maps. On a regular family of curves, it is an open condition.

**Proposition 6.1** (Single-witness transfer). *Let  $k \geq 1$ ,  $Q = 2^k$ ,  $d = Q + 1$ , and  $p \in (1, \infty) \setminus \{2, \dots, k+1\}$ . Suppose there are nonzero real weights  $c_1, \dots, c_d$ , a positive vector  $y$ , and increments  $h_1, \dots, h_k$  for which every  $y + \omega \cdot h$  is positive and*

$$\sum_{i=1}^d c_i \left( y_i + \sum_{j=1}^k \omega_j h_{ji} \right)^p = 1 \quad (\omega \in \{0,1\}^k).$$

Suppose the fixed-increment intersection through  $y$  is a regular curve with nonzero affine torsion at  $y$ .

Then almost every regular exact configuration on the ordinary positive unit  $\ell^p$  sphere has a regular fixed-increment curve with nonzero affine torsion. Here regularity of the exact configuration means rank  $Q$  for the full level derivative in  $(y, h_1, \dots, h_k)$ , and almost everywhere refers to its ordinary coarea level measure. In particular, every nonempty relatively open part of that regular incidence contains an ordinary positive witness.

*Proof.* Put

$$\mathcal{U}_k = \{r \in \mathbb{R}^k : 1 + \omega \cdot r > 0 \text{ for all } \omega \in \{0, 1\}^k\}, \quad A(r)_{\omega i} = (1 + \omega \cdot r_i)^p.$$

The domain  $\mathcal{U}_k$  is open and convex and contains the origin. Normalize a configuration with the fixed weights  $c_i$  by

$$b_i = c_i y_i^p, \quad r_i = (h_{1i}, \dots, h_{ki})/y_i.$$

Its exact equations become

$$A(r)b = \mathbf{1}. \tag{6.1}$$

On the chamber  $\text{sgn } b_i = \text{sgn } c_i$ , this normalization is a real-analytic diffeomorphism. Its inverse is

$$y_i = (b_i/c_i)^{1/p}, \quad h_{ji} = y_i(r_i)_j.$$

For the ordinary sphere the same assertion holds with all  $c_i = 1$  and all  $b_i > 0$ . Consequently these maps preserve regularity of the universal incidence and its surface-null sets. On any regular level chart, surface measure and coarea level measure have the same null sets, since their density ratio is smooth and strictly positive.

We first show that  $A$  has rank  $Q$  almost everywhere on every regular incidence in question, irrespective of the signs of  $b$ . The feature map

$$a(r) = ((1 + \omega \cdot r)^p)_{\omega \in \{0, 1\}^k}$$

spans  $\mathbb{R}^Q$  as  $r$  ranges over  $\mathcal{U}_k$ . Indeed, if  $\sum_{\omega} u_{\omega}(1 + \omega \cdot r)^p = 0$  identically, then for each  $S \subseteq [k]$ , square-free differentiation at  $r = 0$  gives

$$(p)_{|S|} \sum_{\omega: \omega_j=1 \ (j \in S)} u_{\omega} = 0.$$

All displayed factors are nonzero. Starting with  $S = [k]$  and proceeding in decreasing subset size, Boolean inversion gives  $u_{\omega} = 0$  for every  $\omega$ . Thus the analytic preimage of any proper linear subspace of  $\mathbb{R}^Q$  under  $a$  has dimension at most  $k - 1$ : it lies in the zero set of a nonzero real-analytic linear combination of the features.

On a rank- $s$  stratum of  $A$ ,  $s < Q$ , choose  $s$  independent columns. Their ratios contribute at most  $ks$  dimensions. Once these ratios are fixed, each remaining ratio must map into their proper linear span, and hence contributes at most  $k - 1$  dimensions. The ratio stratum therefore has dimension at most

$$ks + (k - 1)(d - s).$$

For each such ratio tuple the solutions of (6.1), if any, form an affine space of dimension  $d - s$ . It follows that the exact rank-deficient set has dimension at most

$$ks + (k - 1)(d - s) + (d - s) = kd. \tag{6.2}$$

These dimension statements are local: on relatively compact analytic charts, rank conditions and the incidence equations define subanalytic sets, and the dimension bound for a family is the dimension of its parameter set plus the largest fiber dimension. Apply this fact first to the remaining ratios and then to the weights, and take the finite union over the choices of independent columns. A countable exhaustion handles the unrestricted domains. The regular exact incidence has dimension

$$(k + 1)d - Q = kd + 1.$$

Thus (6.2) makes its rank-defect subset surface-null. The argument imposes no sign condition on the weights.

We next express torsion polynomially in the normalized weights. For fixed  $b, r$  let

$$F_\omega(Y; b, r) = \sum_{i=1}^d b_i (Y_i + \omega \cdot r_i)^p,$$

defined near  $Y = \mathbf{1}$ . Let  $T(Y; b, r)$  be the signed cofactor vector of the  $Q \times d$  matrix  $D_Y F$ , so

$$T_i = (-1)^{i+1} \det((D_Y F)_{\hat{i}}), \quad 1 \leq i \leq d,$$

where the subscript  $\hat{i}$  denotes deletion of column  $i$ . Then  $D_Y F T = 0$ . Write  $\mathcal{D} = T \cdot \nabla_Y$ , differentiating with  $b, r$  held fixed. Set

$$\mathcal{P}(b, r) = \det(T, \mathcal{D}T, \dots, \mathcal{D}^Q T)|_{Y=\mathbf{1}}.$$

The vector  $T$  has homogeneous degree  $Q$  in  $b$ . Consequently  $\mathcal{D}^j T$  has homogeneous degree  $Q(j+1)$ , and  $\mathcal{P}$  is a homogeneous polynomial in  $b$  of degree

$$L = Q \sum_{j=1}^d j = \frac{Qd(d+1)}{2}.$$

Its coefficients are real analytic on all of  $\mathcal{U}_k^d$ .

If  $D_Y F$  has rank  $Q$ , then  $T$  is a nonzero tangent field to the one-dimensional level fiber. Its integral curves have successive derivatives  $T, \mathcal{D}T, \dots$ . Reparameterizing a regular curve multiplies its torsion determinant by a nonzero factor. Hence  $\mathcal{P} \neq 0$  is exactly ordinary affine torsion nonvanishing on a regular fiber; it also implies the required rank by itself. At a fixed original configuration, the linear change  $x_i = y_i Y_i$  takes this normalized weighted fiber to the original fiber with weights  $c_i$  and fixed increments  $h_j$ . It preserves regularity and torsion nonvanishing. In particular,  $\mathcal{P}$  is nonzero at the normalized signed witness.

The fixed-increment rank at the witness implies full universal rank, so its exact incidence is a manifold of dimension  $kd+1$  near that point. By (6.2), we may perturb the witness within this manifold, retaining its sign chamber and nonzero torsion, until  $A$  has rank  $Q$ . Choose a  $Q$ -column minor  $\Delta(r)$  that is nonzero there, and let  $t$  be the omitted weight. With  $I$  the chosen columns and  $j$  the omitted column, Cramer's rule gives

$$b_I(r, t) = \frac{\text{adj}(A_I(r))(\mathbf{1} - tA_j(r))}{\Delta(r)}, \quad b_j(r, t) = t.$$

Write the displayed numerator as  $B_I(r, t)$  and put  $B_j(r, t) = t\Delta(r)$ . All components of  $B$  are analytic on  $\mathcal{U}_k^d \times \mathbb{R}$ . Homogeneity now shows that

$$\mathcal{Q}(r, t) := \mathcal{P}(B(r, t), r) = \Delta(r)^L \mathcal{P}(b(r, t), r) \quad (\Delta(r) \neq 0). \quad (6.3)$$

This explicitly supplies an analytic extension to the whole connected domain  $\mathcal{U}_k^d \times \mathbb{R}$ . It is nonzero at the perturbed signed witness. Therefore its zero set has Lebesgue measure zero in that domain.

The fixed minor  $\Delta$  is also a nonzero analytic function on the connected domain  $\mathcal{U}_k^d$ . On every full-rank incidence chart, including a chart parameterized using another minor, all  $kd$  ratios and one omitted weight are free coordinates. Hence  $\{\Delta = 0\}$  is surface-null on every such chart, by the analytic zero-set theorem and Fubini's theorem. Outside that null set, the fixed Cramer parameterization above applies. Equation (6.3) then makes  $\{\mathcal{P} = 0\}$  null there as well. Together with the rank-defect estimate, this proves  $\mathcal{P} \neq 0$  almost everywhere on the regular positive-weight incidence.

Finally, when  $b_i > 0$ , the fixed diagonal map  $Y_i \mapsto b_i^{1/p} Y_i$  converts the normalized weighted equations into ordinary unit-sphere equations, with increments scaled by the same factors. Equivalently, it is the fiber version of the inverse normalization with  $c_i = 1$ . It preserves

regularity and nonzero torsion, and the normalization preserves null sets on the universal incidence. This proves the asserted almost-everywhere conclusion for the ordinary sphere. Every nonempty relatively open subset of its regular incidence has positive local surface measure, giving the final existence assertion. No connectedness of that incidence has been used.  $\square$

## 7. CONSTRUCTION OF THE WITNESSES

### 7.1. Noninteger exponents.

**Lemma 7.1** (Walsh witness). *Let  $k \geq 1$ ,  $Q = 2^k$ , and let  $p > 1$  be noninteger. There is a regular positive exact  $k$ -cube on the  $\ell^p$  sphere in  $\mathbb{R}^{Q+1}$  whose fixed-increment curve has nonzero affine torsion.*

*Proof.* Here is a direct construction. Index the first  $Q$  coordinates by  $\epsilon \in \{-1, 1\}^k$  and the vertices by  $\sigma \in \{-1, 1\}^k$ . Choose  $a > 0$  and  $\delta > 0$  small. At the vertex  $\sigma$  prescribe the first  $Q$  coordinates to be

$$a + \delta \sum_{j=1}^k \epsilon_j \sigma_j.$$

The sum of their  $p$ th powers is independent of  $\sigma$ , by permuting the signs  $\epsilon$ ; denote it by  $C$ . Writing  $\sigma_j = 2\omega_j - 1$ , these are the vertices of an affine cube with base coordinates  $a - \delta \sum_j \epsilon_j$  and increments  $h_{j,\epsilon} = 2\delta\epsilon_j$ .

The derivative matrix in the base coordinates is

$$M_{\sigma,\epsilon} = p \left( a + \delta \sum_j \epsilon_j \sigma_j \right)^{p-1}.$$

The Walsh character of a subset  $S$  is an eigenvector, with eigenvalue

$$\lambda_S = p \sum_{\epsilon \in \{-1,1\}^k} \left( \prod_{j \in S} \epsilon_j \right) \left( a + \delta \sum_j \epsilon_j \right)^{p-1}.$$

For  $s = |S|$  its leading term as  $\delta \rightarrow 0$  is

$$\lambda_S = 2^k p(p-1) \cdots (p-s) a^{p-1-s} \delta^s + O(\delta^{s+1}).$$

All these leading coefficients are nonzero under the stated assumption. Thus  $M$  is invertible for sufficiently small fixed  $\delta > 0$ .

Choose positive numbers  $c_1, \dots, c_k$  whose subset sums  $s_\omega = \sum_j \omega_j c_j$  are pairwise distinct, for example  $c_j = 2^{j-1}$ . Introduce a small positive weight  $\beta$  in a final coordinate, whose base coordinate is  $t > 0$  and whose increments are  $c_j$ . The implicit function theorem solves the  $Q$  equations

$$\sum_{\epsilon} \left( Y_{\epsilon}(t, \beta) + 2\delta \sum_j \omega_j \epsilon_j \right)^p + \beta(t + s_{\omega})^p = C$$

near any fixed  $t_0 > 0$ , with

$$Y_{\epsilon}(t, 0) = a - \delta \sum_j \epsilon_j.$$

The solution is jointly analytic in  $(t, \beta)$  and

$$Y(t, \beta) = Y(t, 0) - \beta M^{-1} f(t) + O(\beta^2), \quad f_{\omega}(t) = (t + s_{\omega})^p,$$

with the vertex rows permuted if necessary. The error is analytic, so it can be differentiated any fixed number of times. For the curve  $\gamma_\beta(t) = (Y(t, \beta), t)$ , expansion along the last coordinate gives

$$\det(\gamma'_\beta, \dots, \gamma_\beta^{(Q+1)}) = \pm(-\beta)^Q (\det M)^{-1} \det(f^{(2)}(t), \dots, f^{(Q+1)}(t)) + O(\beta^{Q+1}).$$

Put  $x_\omega = t + s_\omega$ . The determinant on the right equals, up to the harmless sign from reversing columns,

$$\left( \prod_{m=2}^{Q+1} p(p-1) \cdots (p-m+1) \right) \left( \prod_{\omega} x_\omega^{p-Q-1} \right) \prod_{\omega < \omega'} (x_{\omega'} - x_\omega).$$

It is nonzero. Consequently the curve has nonzero torsion for every sufficiently small  $\beta > 0$ . Its defining gradients are independent because the first  $Q$  columns remain invertible. All its cube vertices are positive.

Finally apply the fixed invertible diagonal map multiplying the last coordinate by  $\beta^{1/p}$ , then scale every coordinate by  $C^{-1/p}$ . This converts the weighted equal-level equations to ordinary unit  $\ell^p$  sphere equations. Nonzero affine torsion and regularity are preserved. This proves the lemma.  $\square$

*Remark 7.2* (A positive regular incidence exists for all admissible exponents). The Walsh eigenvalue calculation needs only  $p(p-1) \cdots (p-k) \neq 0$ , which holds for every admissible  $p$ . For any such  $p$ , add a last coordinate equal to one, with zero increments and unit weight, and scale the common level to one. The first  $Q$  normal columns are still independent. Thus the regular positive exact incidence is nonempty for every exponent used below, including the integers where the displayed leading torsion coefficient vanishes. Every such regular point has a neighborhood of dimension  $kd + 1$ ; the rank-defect bound ensures that this neighborhood contains a positive full-rank Cramer chart.

## 7.2. Integer exponents.

**Lemma 7.3** (Integer exponents). *For every  $k \geq 1$  and every integer  $p \geq k + 2$ , there is a signed witness satisfying Proposition 6.1 in  $d = 2^k + 1$  coordinates.*

*Proof.* Index  $Q-1$  coordinates by the nonempty subsets  $S \subseteq [k]$ , and use two more coordinates denoted by  $u, v$ . Set  $d_S = p - |S| \geq 2$ . Choose positive numbers  $\beta_1, \dots, \beta_k$  so that

$$R_S := \left( \prod_{j \in S} \beta_j \right)^{1/d_S} > 3 \quad (S \neq \emptyset)$$

are pairwise distinct. For an entirely explicit choice, put  $M = 2p+1$ ,  $B = 3^p$ , and  $\beta_j = B^{M^{j-1}}$ . If  $R_S = R_T$ , then

$$d_T \sum_{j \in S} M^{j-1} - d_S \sum_{j \in T} M^{j-1} = 0.$$

The coefficients in this integer relation have absolute value at most  $p$ , and the largest nonzero term exceeds the sum of all smaller terms because  $M - 1 > p$ . Hence all coefficients vanish, forcing  $S = T$ . The lower bound  $R_S > 3$  is immediate. Write  $\beta_S = \prod_{j \in S} \beta_j$ .

Define  $k$  increment-direction vectors by

$$a_S^{(j)} = \mathbf{1}_{j \in S}, \quad a_u^{(j)} = 1, \quad a_v^{(j)} = \beta_j.$$

For a parameter  $\varepsilon > 0$ , give the coordinates the weights

$$b_S = \varepsilon^{-d_S}, \quad b_u = 1, \quad b_v = -1.$$

First consider the homogeneous finite-difference limit equations

$$H_R(y) := \sum_i b_i \left( \prod_{j \in R} a_i^{(j)} \right) y_i^{p-|R|} = 0 \quad (\emptyset \neq R \subseteq [k]), \quad P_b(y) := \sum_i b_i y_i^p = 1. \quad (7.1)$$

Before imposing the last normalization, set  $z_S = \varepsilon X_S$ ,  $z_u = t$ , and  $z_v = 1$ . The nonempty equations become the triangular system

$$X_R^{d_R} + \sum_{S \supsetneq R} \varepsilon^{|S|-|R|} X_S^{d_R} + t^{d_R} - \beta_R = 0.$$

At  $\varepsilon = 0$  its positive solution on  $3/2 < t < 5/2$  is

$$X_R(t, 0) = (\beta_R - t^{d_R})^{1/d_R}.$$

The derivative with respect to the  $X_R$  variables is diagonal and invertible there. The analytic implicit-function theorem therefore supplies positive real-analytic solutions  $X_R(t, \varepsilon)$  near any compact subinterval when  $\varepsilon$  is sufficiently small. Put

$$K_\varepsilon(t) = \left( t^p - 1 + \sum_{S \neq \emptyset} \varepsilon^{|S|} X_S(t, \varepsilon)^p \right)^{1/p}.$$

This is positive on the same interval for sufficiently small  $\varepsilon$ . By homogeneity,  $y = z/K_\varepsilon$  solves (7.1). After the invertible diagonal change of coordinates that divides the subset coordinates by  $\varepsilon$ , the curve is

$$\gamma_\varepsilon(t) = \frac{((X_S(t, \varepsilon))_{S \neq \emptyset}, t, 1)}{K_\varepsilon(t)}.$$

As  $\varepsilon \rightarrow 0$ , it converges in  $C^m$  on compact subintervals for every fixed nonnegative integer  $m$  to

$$\gamma_0(t) = \frac{(((\beta_S - t^{d_S})^{1/d_S})_{S \neq \emptyset}, t, 1)}{(t^p - 1)^{1/p}}.$$

We claim that this limit curve lies in no affine hyperplane. It is enough to prove linear independence, over  $\mathbb{C}$ , of the analytic germs

$$1, \quad t, \quad K_0(t) = (t^p - 1)^{1/p}, \quad X_S(t, 0) = (\beta_S - t^{d_S})^{1/d_S} \quad (S \neq \emptyset).$$

The branch points of  $K_0$  lie on the unit circle. Those of  $X_S$  lie on  $|t| = R_S$ , and these circles are pairwise distinct. Suppose a linear relation of these germs holds. Continue it from the real interval to a neighborhood of a zero of  $\beta_S - t^{d_S}$ , avoiding all the finitely many other branch points, and continue once around a small loop about that zero. This multiplies  $X_S$  by  $e^{2\pi i/d_S} \neq 1$  and fixes all other germs. Subtracting the original relation forces the coefficient of  $X_S$  to vanish. Doing this for every  $S$ , and then around one zero of  $t^p - 1$ , forces the coefficient of  $K_0$  to vanish. The remaining affine polynomial in  $t$  has both coefficients zero. This proves the claim.

For real-analytic functions, linear independence implies that their Wronskian is not identically zero; see [2]. Applied to the constant function and the  $d$  coordinates of  $\gamma_0$ , it follows that

$$\det(\gamma_0'(t_*), \dots, \gamma_0^{(d)}(t_*)) \neq 0$$

for some real  $t_* \in (3/2, 5/2)$ . Here is an elementary justification of the analytic Wronskian fact. In any finite-dimensional space of independent analytic germs, constant linear changes of basis eliminate successive leading Taylor terms and give a basis  $g_j(t) = c_j(t - t_0)^{m_j} + \dots$ , where  $c_j \neq 0$  and the nonnegative integers  $m_j$  are distinct. The leading Wronskian coefficient is the product of the  $c_j$  times the determinant of the falling factorials  $(m_j)_r$ ,  $r = 0, \dots, d$ . That determinant is the Vandermonde product in the distinct  $m_j$  and is nonzero. Hence the Wronskian cannot vanish identically. The Wronskian of 1 and the coordinates of  $\gamma_0$  expands along its constant column to precisely the displayed torsion. The real-analytic identity theorem ensures that a nonzero Wronskian cannot vanish on the entire real interval. The determinant remains nonzero for  $\gamma_\varepsilon$  when  $\varepsilon > 0$  is sufficiently small. Thus the original curve  $y = z/K_\varepsilon$  has nonzero affine torsion.

The nonempty gradients in (7.1) have rank  $Q-1$ : the submatrix using the subset coordinates is triangular in decreasing subset size, with nonzero diagonal entries. Euler's identity gives  $\nabla H_R(y) \cdot y = 0$  at a solution, whereas  $\nabla P_b(y) \cdot y = p$ . Thus all  $Q$  defining gradients are independent, and this is a regular curve.

Finally, fix such a sufficiently small  $\varepsilon > 0$  and deform from the limit equations to genuine translated shells. For nonempty  $R$ , replace  $H_R$  by

$$H_{R,\delta}(y) = \frac{\sum_{T \subseteq R} (-1)^{|R|-|T|} P_b\left(y + \delta \sum_{j \in T} a^{(j)}\right)}{(p)_{|R|} \delta^{|R|}}, \quad (p)_m = p(p-1) \cdots (p-m+1).$$

These are polynomial functions, hence analytic, in  $y$  and  $\delta$  after removing the displayed removable singularity, and  $H_{R,0} = H_R$ . Keep the equation  $P_b(y) = 1$ . Choose a linear functional  $\ell$  with  $\ell(y'(t_*)) \neq 0$  and adjoin the equation  $\ell(y) = s$ . The analytic implicit-function theorem gives a jointly analytic solution  $y = y(s, \delta)$  near  $(s, \delta) = (\ell(y(t_*)), 0)$ . At  $\delta = 0$ , this is a regular reparameterization of the curve above. Thus the nearby regular curves converge in  $C^d$  as  $\delta \rightarrow 0$ , and their torsion remains nonzero at the corresponding point for sufficiently small  $\delta$ . Vanishing of all the finite differences is equivalent, by Boolean inversion, to

$$P_b\left(y + \delta \sum_j \omega_j a^{(j)}\right) = 1 \quad (\omega \in \{0, 1\}^k).$$

All vertex coordinates are positive if  $\delta$  is sufficiently small. This is the required signed witness.  $\square$

**Theorem 7.4** (Witnesses in every progression length). *For every  $k \geq 1$  and every  $p \in (1, \infty) \setminus \{2, \dots, k+1\}$ , the positive unit  $\ell^p$  sphere in  $\mathbb{R}^{2^k+1}$  has a regular exact  $k$ -cube whose fixed-increment intersection has nonzero affine torsion. Moreover, this property holds at almost every regular positive exact configuration, with respect to ordinary coarea level measure.*

*Proof.* If  $p$  is noninteger, Lemma 7.1 gives a positive witness directly. If  $p$  is an integer, admissibility says  $p \geq k+2$ ; Lemma 7.3 and Proposition 6.1 give the almost-everywhere positive conclusion. The regular positive incidence is nonempty by the remark following the Walsh construction, so this also gives an actual positive witness. In both cases the polynomial transfer in Proposition 6.1 gives the asserted almost-everywhere statement.  $\square$

## 8. FOURIER LOCALIZATION AND UNIFORM SHELL COMPARISON

**Lemma 8.1** (Fourier convergence on almost-everywhere good curves). *Let  $k \geq 1$ ,  $Q = 2^k$ , and  $d = Q+1$ . Suppose the positive mixed  $k$ -cube shell measures converge weakly and in mass to  $M$ . Suppose  $M$ -almost every exact configuration has a regular fixed-increment fiber with nonzero affine torsion. Then*

$$\int \sup_{\xi \in \mathbb{R}^d} \left| \int e^{-2\pi i \xi \cdot y} \prod_{\omega \in \{0,1\}^k} (S_\eta - S_\tau)(y + \omega \cdot h) dy \right| dh \longrightarrow 0 \quad (0 < \eta < \tau \downarrow 0).$$

*Proof.* Let  $\gamma > 0$ . By finiteness and inner regularity of  $M$ , choose a smooth  $0 \leq \rho \leq 1$ , supported in finitely many regular nonzero-torsion charts, with  $\int (1 - \rho) dM < \gamma$ . This is possible by first taking a compact subset of the good locus of complement mass below  $\gamma$ , then a smooth cutoff supported in the good open set. A partition of unity decomposes  $\rho$  into smooth amplitudes compactly supported in these charts. Shrink the charts so torsion stays bounded away from zero on their closures and on nearby levels.

In one chart the submersion coordinates are  $y = Y(t, s, h)$ , with  $t \in \mathbb{R}$  and  $s = (P(y + \omega \cdot h) - 1)_\omega \in \mathbb{R}^Q$ . The Fourier transform of its localized level measure is

$$J(s, h, \xi) = \int e^{-2\pi i \xi \cdot Y(t, s, h)} a(t, s, h) dt,$$

where  $a$  is smooth and compactly supported. It is jointly continuous. The vectors  $\partial_t^j Y$ ,  $1 \leq j \leq d$ , span  $\mathbb{R}^d$  uniformly on the compact chart. Therefore, for every unit direction  $\theta$ , at least one  $|\theta \cdot \partial_t^j Y|$  has a fixed positive lower bound. A finite smooth partition in the chart variables and in  $\theta$ , using sufficiently small  $t$ -intervals, lets us fix such a derivative throughout each integration interval. For  $j \geq 2$ , apply the one-dimensional van der Corput estimate [16, Chapter VIII, Section 1]; for  $j = 1$  integrate by parts, using the uniformly bounded second derivative and amplitude derivatives. It follows that

$$|J(s, h, \xi)| \leq C(1 + |\xi|)^{-1/d}$$

uniformly in the relevant nearby levels and increments.

Set  $e_{\eta, \tau} = \psi_\eta - \psi_\tau$ . The localized signed-shell Fourier transform is

$$\int J(s, h, \xi) \prod_{\omega} e_{\eta, \tau}(s_\omega) ds.$$

Each factor has integral zero,  $L^1$  norm at most two, and shrinking support. The displayed decay controls large frequencies uniformly. For bounded frequencies, subtract  $J(0, h, \xi)$  and use uniform continuity and cancellation. The resulting Fourier supremum tends to zero uniformly in  $h$ , whose support is fixed and compact. Thus the integral tends to zero on the part multiplied by  $\rho$ .

For the remaining part, discard the Fourier phase and expand the absolute product. Its integrated Fourier supremum is at most

$$\sum_{\epsilon \in \{\eta, \tau\}^Q} \int (1 - \rho) dM_\epsilon.$$

Weak convergence and common compact support bound its limsup by  $2^Q \int (1 - \rho) dM < 2^Q \gamma$ . Let  $\gamma \downarrow 0$ .  $\square$

**Corollary 8.2** (The single-witness shell criterion). *For admissible  $p$  and  $d = 2^{n-2} + 1$ , the existence of a signed witness as in Proposition 6.1 implies*

$$\mathfrak{B}_n(S_\eta - S_\tau) \longrightarrow 0 \quad (0 < \eta < \tau \downarrow 0).$$

*Consequently the exact-to-convolution comparison holds uniformly for unit-bounded inputs on a fixed bounded set and  $0 < \lambda \leq 1$ .*

*Proof.* Proposition 5.2 identifies the limiting cube measure with ordinary coarea measure on the full regular incidence. Proposition 6.1 makes its fixed-increment fibers regular with nonzero torsion almost everywhere. The preceding lemma is exactly the convergence of the  $Q$ th power of (3.1), since the shells are real. Apply Corollary 3.4 for the final assertion.  $\square$

## 9. A FINITE-BOX BRIDGE USING RADIAL REGULARIZATION

This section gives the passage from a uniform approximation estimate for the exact spherical count to the conclusion for every sufficiently large radius. It applies to a measure supported on an arbitrarily small open cap.

Write  $\rho(y) = \|y\|_{\ell^p}$ ,  $V_p = \{y : \rho(y) < 1\}$ , and  $S_p = \{y : \rho(y) = 1\}$ . Let  $\kappa_p$  be the probability cone measure, normalized by the polar-coordinate formula

$$\int_{\mathbb{R}^d} F(y) dy = dV_p \int_{S_p} \int_0^\infty F(s\omega) s^{d-1} ds d\kappa_p(\omega).$$

Throughout this section let  $\mu = a\kappa_p$  be a probability measure, where  $a$  is nonnegative and continuous and is strictly positive on some relatively open subset of  $S_p$ . A smooth, nonzero, nonnegative cap cutoff satisfies this hypothesis. For a measurable set  $B$  of finite measure define

$$C_\lambda(B) = \int_{S_p} \int_{\mathbb{R}^d} \prod_{i=0}^{n-1} 1_B(x + i\lambda\omega) dx d\mu(\omega).$$

The continuity of translations in  $L^1$  shows that the integrated correlation is continuous in its gap vector; hence the spherical and radial integrals below are well defined.

We use the following uniform approximation hypothesis. For a fixed smooth compactly supported probability density  $\varphi$ , there is a function  $u(\varepsilon) \rightarrow 0$  as  $\varepsilon \downarrow 0$  such that, for every  $R > 0$ , every translate  $Q$  of  $[0, R]^d$ , every measurable  $B \subset Q$ , and every  $0 < \lambda \leq R$ ,

$$\left| C_\lambda(B) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \prod_{i=0}^{n-1} 1_B(x + iy) (\mu_\lambda * \varphi_{\varepsilon\lambda})(y) dy dx \right| \leq R^d u(\varepsilon). \quad (9.1)$$

**Lemma 9.1** (Uniform continuity in relative radius). *Under (9.1), for every  $\tau > 0$  there is an  $\eta \in (0, 1/4)$  such that*

$$|C_{s\lambda}(B) - C_\lambda(B)| \leq \tau R^d \quad (1 - \eta \leq s \leq 1 + \eta)$$

whenever  $B \subset Q$ ,  $Q$  is a cube of side  $R$ , and  $(1 + \eta)\lambda \leq R$ .

*Proof.* Set  $K_\varepsilon = \mu * \varphi_\varepsilon$  and use mass-preserving dilations  $(K_\varepsilon)_s(y) = s^{-d} K_\varepsilon(y/s)$ . Applying (9.1) at  $\lambda$  and  $s\lambda$  and observing that the integrated correlation is bounded by  $|B|$  gives

$$|C_{s\lambda}(B) - C_\lambda(B)| \leq 2R^d u(\varepsilon) + |B| \|(K_\varepsilon)_s - K_\varepsilon\|_1.$$

Choose  $\varepsilon$  so that  $2u(\varepsilon) < \tau/2$ . Continuity of dilation in  $L^1$ , with this  $\varepsilon$  fixed, then gives the assertion, since  $|B| \leq R^d$ .  $\square$

**Lemma 9.2** (A coarse count in an angular cap). *For every  $\delta \in (0, 1]$  there are constants  $c > 0$  and  $L_0 \geq 1$ , depending only on  $n, d, p, \mu, \delta$ , such that*

$$P_\lambda(B) := \int_0^1 ds^{d-1} C_{s\lambda}(B) ds \geq cR^d \quad (9.2)$$

for every cube  $Q$  of side  $R$ , every measurable  $B \subset Q$  with  $|B| \geq \delta R^d$ , and every  $0 < \lambda \leq R/L_0$ .

*Proof.* Polar coordinates identify  $P_\lambda(B)$  with the progression count whose gap kernel is the dilation of

$$K_0(y) = V_p^{-1} a(y/\rho(y)) 1_{\{0 < \rho(y) < 1\}}.$$

Choose a point in the open set where  $a > 0$ . By taking a sufficiently small closed cone around the corresponding ray, there exist an invertible real matrix  $T$  and a number  $\gamma > 0$  such that

$$a(Tv/\rho(Tv)) \geq \gamma \quad \text{for every } v \in [0, \infty)^d \setminus \{0\}. \quad (9.3)$$

Indeed, choose the  $d$  linearly independent columns of  $T$  sufficiently close to a common nonzero vector on the selected ray. Every nonzero nonnegative linear combination, after division by the sum of its coefficients, lies in the convex hull of these columns. Radial projection of that convex hull stays inside the set where  $a \geq \gamma$  if the columns were chosen sufficiently close.

Put

$$M = \max_{v \in [0, 1]^d} \rho(Tv), \quad D = \text{diam}(T[0, 1]^d), \quad \ell = \lambda/M, \quad v_\lambda = \ell^d |\det T|.$$

Tile  $\mathbb{R}^d$  by the half-open parallelepipeds  $P_k = \ell T(k + [0, 1]^d)$ ,  $k \in \mathbb{Z}^d$ . Each has measure  $v_\lambda$  and Euclidean diameter  $D\ell$ . Retain those cells that are contained in  $Q$ , disregarding boundaries

of measure zero. Their union contains the points of  $Q$  at distance greater than  $D\ell$  from its boundary. Consequently the measure discarded from  $Q$  is at most

$$2dD\ell R^{d-1}$$

when  $R \geq 2D\ell$ . Taking, for example,

$$L_0 = \max\{1, 4dD/(M\delta)\}$$

ensures that the discarded measure is at most  $\delta R^d/2$ . The mass of  $B$  in retained cells is therefore at least  $\delta R^d/2$ .

Call a retained cell dense when  $|B \cap P_k| \geq (\delta/4)|P_k|$ . The other retained cells contain at most  $\delta R^d/4$  of  $B$ . Hence the union of the dense cells has measure at least  $\delta R^d/4$ .

Let  $q = q(n, d, \delta/4) > 0$  be the continuous Varnavides lower bound on the unit cube: every  $E \subset [0, 1]^d$  of measure at least  $\delta/4$  satisfies

$$\int_{[0,1]^d} \int_{[0,1]^d} \prod_{i=0}^{n-1} 1_E(u + iv) dv du \geq q,$$

where  $1_E$  is extended by zero. This is [7, Proposition 1], a continuous form of Szemerédi's theorem [17] obtained by the averaging argument of Varnavides [19]. Apply it on a dense cell using  $u \mapsto \ell T(k + u)$ . The resulting contribution, with starting point in that cell and gap in  $\ell T[0, 1]^d$ , is at least  $qv_\lambda^2$ . The Jacobian is  $\ell^d |\det T|$  in each of the starting-point and gap variables.

For almost every  $y \in \ell T[0, 1]^d$  one has  $\rho(y) < \lambda$ , and (9.3) gives  $(K_0)_\lambda(y) \geq \gamma/(V_p \lambda^d)$ . The exceptional boundary is immaterial. Summing the preceding Varnavides contributions is legitimate because their starting-point cells are disjoint and each restricted indicator is bounded by  $1_B$ . It follows that

$$P_\lambda(B) \geq \frac{\gamma q}{V_p \lambda^d} v_\lambda \frac{\delta R^d}{4} = \frac{\gamma q |\det T| \delta}{4 V_p M^d} R^d.$$

This proves (9.2). □

We next record exactly the consequence of ray variation used in the finite-box argument. Set  $m = n - 1$  and

$$M_t^\omega(f_1, \dots, f_m)(x) = \frac{1}{t} \int_0^t \prod_{i=1}^m f_i(x + iu\omega) du.$$

For  $f_i \in L^{2(n-1)}(\mathbb{R}^d)$ ,  $1 \leq i \leq n - 1$ , the continuous ray estimate is

$$\left( \sum_{k=0}^{N-1} \|M_{t_{k+1}}^\omega - M_{t_k}^\omega\|_2^r \right)^{1/r} \leq C_{n,r} \prod_{i=1}^{n-1} \|f_i\|_{2(n-1)} \quad (0 < t_0 < \dots < t_N). \quad (9.4)$$

Here one may take  $r = 2$  when  $n = 3$  and any  $r > 2^{n-2}$  when  $n \geq 4$ . These are the exponents for  $n - 1$  moving factors; the unshifted factor will be paired with the output in  $L^2$ . Here is the complete passage from discrete to continuous time. Fix  $h > 0$  and use the commuting transformations  $T_i x = x + ih\omega$ ,  $1 \leq i \leq n - 1$ , on the  $\sigma$ -finite space  $(\mathbb{R}^d, dx)$ . Theorem 1.2 of [5] gives the same bound for the averages  $N^{-1} \sum_{j=0}^{N-1} \prod_i f_i(\cdot + ijh\omega)$ , uniformly in  $h$  and in the direction. Put  $H_s = \prod_{i=1}^{n-1} f_i(\cdot + is\omega)$ . Telescoping and Hölder give

$$\|H_s - H_t\|_2 \leq \sum_{i=1}^{n-1} \|f_i(\cdot + is\omega) - f_i(\cdot + it\omega)\|_{2(n-1)} \prod_{j \neq i} \|f_j\|_{2(n-1)}.$$

Strong continuity of translations makes  $s \mapsto H_s$  continuous in  $L^2$ , uniformly on compact time intervals. For any finite list  $0 < t_0 < \dots < t_N$ , the integers  $N_j(h) = \lfloor t_j/h \rfloor$  are positive

and strictly increasing for sufficiently small  $h$ . The difference between the discrete average of length  $N_j(h)$  and  $(hN_j(h))^{-1} \int_0^{hN_j(h)} H_s ds$  is bounded in  $L^2$  by

$$\sup\{\|H_s - H_t\|_2 : 0 \leq s, t \leq t_N + 1, |s - t| \leq h\},$$

which tends to zero. Since  $hN_j(h) \rightarrow t_j$ , the integral averages also tend to  $M_{t_j}^\omega$  in  $L^2$ . Apply the discrete variation bound to this finite list and pass to the limit in its finite sum. This proves (9.4) with precisely the stated exponents and constants.

**Lemma 9.3** (Radial profiles on separated scales). *Let  $0 < \alpha < \beta < \infty$ , and let  $w$  be a real function of bounded variation on  $\mathbb{R}$ , supported in  $[\alpha, \beta]$ , with  $\int w(s) ds = 0$ . Define the finite signed measure  $d\nu(s) = -s dw(s)$ , with the derivative taken on all of  $\mathbb{R}$ , including endpoint jumps. If*

$$\beta\lambda_j < \alpha\lambda_{j+1} \quad (1 \leq j < J),$$

*then, for every measurable  $B$  of finite measure,*

$$\left( \sum_{j=1}^J \left| \int w(s) C_{s\lambda_j}(B) ds \right|^r \right)^{1/r} \leq C_{n,r} \|\nu\|_{\text{TV}} |B|. \quad (9.5)$$

*Proof.* Integration by parts gives  $\nu(\mathbb{R}) = \int w = 0$ . Apply Bochner integration by parts to the locally absolutely continuous  $L^2$ -valued function  $s \mapsto sM_{s\lambda}^\omega$ ; its derivative is  $\prod_{i=1}^{n-1} 1_B(\cdot + i\lambda s\omega)$ . It follows that

$$\int w(s) C_{s\lambda}(B) ds = \int_{S_p} \int \langle 1_B, M_{s\lambda}^\omega - M_{\alpha\lambda}^\omega \rangle d\nu(s) d\mu(\omega).$$

For every fixed  $s \in [\alpha, \beta]$ , the intervals  $[\alpha\lambda_j, s\lambda_j]$  are ordered and disjoint. Include all their endpoints in the partition in (9.4); the intervening increments only add nonnegative summands. Zero-length intervals may be omitted. Minkowski's inequality in  $\ell^r$ , followed by the  $L^2$  pairing and (9.4), proves

$$\left( \sum_j \left| \int w(s) C_{s\lambda_j}(B) ds \right|^r \right)^{1/r} \leq C_{n,r} \|\nu\|_{\text{TV}} \|1_B\|_2 \prod_{i=1}^{n-1} \|1_B\|_{2(n-1)}.$$

The product of norms on the right equals  $|B|$ . □

**Proposition 9.4** (Finite-list lemma with exact quantifiers). *Assume (9.1) and (9.4). For every  $\delta \in (0, 1]$  there exist numbers  $\Lambda > 1$ ,  $L \geq 1$ , and an integer  $J_\delta \geq 1$  with the following property. For every list of  $J_\delta$  positive radii satisfying*

$$\lambda_{j+1} > \Lambda\lambda_j \quad (1 \leq j < J_\delta),$$

*every  $R \geq L\lambda_{J_\delta}$ , every translate  $Q$  of  $[0, R]^d$ , and every measurable  $B \subset Q$  with  $|B| \geq \delta R^d$ , at least one of the counts  $C_{\lambda_j}(B)$  is strictly positive. All three constants may depend on  $n, d, p, \mu, \varphi, \delta$ , the uniform modulus in (9.1), and the fixed admissible exponent  $r$ ; they do not depend on the radii,  $Q$ , or  $B$ .*

*Proof.* Take  $c, L_0$  from Lemma 9.2. Choose  $\alpha \in (0, 1/4)$  with  $2\alpha^d < c/4$ , and put

$$w_\alpha(s) = \frac{ds^{d-1}}{1 - \alpha^d} 1_{[\alpha, 1]}(s).$$

This is a probability density, and its  $L^1$  distance from  $ds^{d-1} 1_{(0,1)}(s)$  equals  $2\alpha^d$ . Since  $0 \leq C_t(B) \leq |B| \leq R^d$ , we obtain

$$\int w_\alpha(s) C_{s\lambda}(B) ds \geq 3cR^d/4 \quad (\lambda \leq R/L_0). \quad (9.6)$$

Choose  $\eta \in (0, 1/4)$  from Lemma 9.1 with  $\tau = c/4$ , and choose a smooth nonnegative probability density  $w_\eta$  supported in  $[1 - \eta, 1 + \eta]$ . Then

$$\left| \int w_\eta(s) C_{s\lambda}(B) ds - C_\lambda(B) \right| \leq cR^d/4 \quad ((1 + \eta)\lambda \leq R). \quad (9.7)$$

Set  $\beta = 1 + \eta$ ,  $\Lambda = \beta/\alpha$ ,  $L = \max\{L_0, \beta\}$ , and  $w = w_\eta - w_\alpha$ . The hypotheses of Lemma 9.3 hold, with  $\nu = -s dw$ . Define

$$J_\delta = 1 + \left\lceil \left( \frac{2C_{n,r}\|\nu\|_{\text{TV}}}{c} \right)^r \right\rceil.$$

If all  $C_{\lambda_j}(B)$  vanished, then (9.6) and (9.7) would imply

$$\left| \int w(s) C_{s\lambda_j}(B) ds \right| \geq cR^d/2 \quad \text{for every } j.$$

Equation (9.5) would consequently give  $J_\delta^{1/r} cR^d/2 \leq C_{n,r}\|\nu\|_{\text{TV}} R^d$ , contradicting the definition of  $J_\delta$ .  $\square$

### 9.1. Completion of the proof.

*Proof of Theorem 1.1.* Let  $k = n - 2$  and  $d_0 = 2^k + 1$ . In dimension  $d_0$ , Theorem 7.4 supplies the witness required by Corollary 8.2. Thus the exact-to-convolution comparison has a modulus tending to zero, uniformly on a unit input cube and for  $0 < \lambda \leq 1$ .

Normalize the cap measure by  $\mu = \sigma/\sigma(\mathbb{R}^{d_0})$ . Homogeneity and the cone-measure formula give  $d\sigma = (d_0 V_p/p)\chi d\kappa_p$ , so this  $\mu$  has the continuous nonnegative angular density required above, positive on a nonempty open patch. Translating and rescaling an input cube of side  $R$  multiplies the counting forms by  $R^{d_0}$  and changes  $\lambda$  to  $\lambda/R$ . Hence (9.1) holds in dimension  $d_0$ , and the finite-list proposition applies.

To prove the result in any  $d \geq d_0$ , fix  $0 < \delta < \bar{\delta}(A)$  and take the constants  $J_\delta, \Lambda, L$  of Proposition 9.4 in dimension  $d_0$ . If missing radii were unbounded, choose  $J_\delta$  of them with successive ratios larger than  $\Lambda$ . Then choose a  $d$ -dimensional cube of side  $R \geq L\lambda_{J_\delta}$  whose relative density in  $A$  exceeds  $\delta$ . If  $d > d_0$ , Fubini supplies a measurable  $d_0$ -dimensional coordinate slice of relative density at least  $\delta$ ; if  $d = d_0$ , use the cube itself. The finite-list proposition gives a positive exact count at a listed missing radius, a contradiction. Appending the fixed slice coordinates to the progression points and zero coordinates to the gap preserves membership in  $A$  and the  $\ell^p$  gap length. No fixed slice of positive upper Banach density is required.  $\square$

### AI DISCLOSURE

The manuscript was generated gradually in the following three chats with OpenAI's models:

- ChatGPT-5.6 Sol, August 28 – September 3, 2026,
- ChatGPT-6 Astra, September 5–9, 2026,
- ChatGPT-6 Astra, September 9, 2026.

The chats were mutually dependent via ChatGPT's internal library. They were initially directed towards the papers [7] and [5], and then guided by generic prompts. My intention is to digest and address the proof in a cooperative, non-competitive, time-insensitive environment.

### REFERENCES

- [1] E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Publ. Math. Inst. Hautes Études Sci. **67** (1988), 5–42. doi:10.1007/BF02699126.
- [2] A. Bostan and P. Dumas, *Wronskians and linear independence*, Amer. Math. Monthly **117** (2010), no. 8, 722–727. doi:10.4169/000298910X515785.
- [3] J. Bourgain, *A Szemerédi type theorem for sets of positive density in  $\mathbb{R}^k$* , Israel J. Math. **54** (1986), no. 3, 307–316. doi:10.1007/BF02764959.

- [4] B. Cook, Á. Magyar, and M. Pramanik, *A Roth-type theorem for dense subsets of  $\mathbb{R}^d$* , Bull. Lond. Math. Soc. **49** (2017), no. 4, 676–689. doi:10.1112/blms.12043.
- [5] F. van Doorn, P. Durcik, J. Roos, L. Slavíková, and C. Thiele, *A blueprint for the formalization of norm-variation of multiple ergodic averages for commuting transformations*, preprint, 2026. arXiv:2608.27321.
- [6] P. Durcik and V. Kovač, *Boxes, extended boxes, and sets of positive upper density in the Euclidean space*, Math. Proc. Cambridge Philos. Soc. **171** (2021), no. 3, 481–501. doi:10.1017/S0305004120000316.
- [7] P. Durcik and V. Kovač, *A Szemerédi-type theorem for subsets of the unit cube*, Anal. PDE **15** (2022), no. 2, 507–549. doi:10.2140/apde.2022.15.507.
- [8] P. Durcik, V. Kovač, and L. Rimanić, *On side lengths of corners in positive density subsets of the Euclidean space*, Int. Math. Res. Not. IMRN **2018** (2018), no. 22, 6844–6869. doi:10.1093/imrn/rnx093.
- [9] P. Durcik, V. Kovač, K. A. Škreb, and C. Thiele, *Norm variation of ergodic averages with respect to two commuting transformations*, Ergodic Theory Dynam. Systems **39** (2019), no. 3, 658–688. doi:10.1017/etds.2017.48.
- [10] P. Durcik, L. Slavíková, and C. Thiele, *Norm-variation of triple ergodic averages for commuting transformations*, Anal. PDE **19** (2026), no. 3, 539–586. doi:10.2140/apde.2026.19.539.
- [11] T. Eisner and T. Tao, *Large values of the Gowers–Host–Kra seminorms*, J. Anal. Math. **117** (2012), 133–186. arXiv:1012.3509.
- [12] K. J. Falconer and J. M. Marstrand, *Plane sets with positive density at infinity contain all large distances*, Bull. Lond. Math. Soc. **18** (1986), no. 5, 471–474. doi:10.1112/blms/18.5.471.
- [13] H. Federer, *Geometric Measure Theory*, Grundlehren Math. Wiss. **153**, Springer-Verlag, New York, 1969.
- [14] H. Furstenberg, Y. Katznelson, and B. Weiss, *Ergodic theory and configurations in sets of positive density*, in *Mathematics of Ramsey Theory*, J. Nešetřil and V. Rödl (eds.), Algorithms Combin. **5**, Springer, Berlin, 1990, 184–198. doi:10.1007/978-3-642-72905-8\_13.
- [15] W. T. Gowers, *A new proof of Szemerédi’s theorem*, Geom. Funct. Anal. **11** (2001), no. 3, 465–588. doi:10.1007/s00039-001-0332-9.
- [16] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton University Press, Princeton, NJ, 1993.
- [17] E. Szemerédi, *On sets of integers containing no  $k$  elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245. doi:10.4064/aa-27-1-199-245.
- [18] T. Tao, *Norm convergence of multiple ergodic averages for commuting transformations*, Ergodic Theory Dynam. Systems **28** (2008), no. 2, 657–688. doi:10.1017/S0143385708000011.
- [19] P. Varnavides, *On certain sets of positive density*, J. Lond. Math. Soc. **34** (1959), no. 3, 358–360. doi:10.1112/jlms/s1-34.3.358.

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