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A counterexample to Han's conjecture over arbitrary fields

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Abstract

Based on the constructions of Kong–Liu–Shen and Liu–Shen, we give a counterexample to Han's conjecture over every field.

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The results of this paper were obtained using GPT-6-Astra-Pro in a research run lasting approximately 54 minutes, followed by an independent review and light editing using GPT-6-Astra. My recent work on extending the results associated with OpenAI Problems 198 and 199 prompted me to consider extending the counterexample to Han's conjecture to arbitrary fields.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

For a field k and a unital associative k-algebra A, write

$$\mathrm{HH}_n(A/k) = \mathrm{Tor}_n^{A\otimes_k A^{\mathrm{op}}}(A, A)$$

for Hochschild homology. Han's conjecture asks whether eventual vanishing of these groups for a finite-dimensional algebra forces finite global dimension [4]. Kong–Liu–Shen construct a counterexample over $\mathbb{C}$ by passing from a phantom category to an endomorphism algebra [6]. Liu–Shen give an explicit three-vertex algebra with a nonfull exceptional triple in characteristic zero [7]. We use their ungraded input and construct replacements for its exceptional triple in the four characteristics where direct reduction fails.

Theorem 1.1. *For every field k, there is a finite-dimensional unital associative k-algebra A_k such that*

$$\mathrm{HH}_n(A_k/k) = 0 \quad (n \geq 1), \qquad \mathrm{gldim}\, A_k = \infty.$$

The construction in this paper satisfies

$$\dim_k A_k = \begin{cases} 10787, & \mathrm{char}\, k \in \{2, 43\}, \\ 10786, & \mathrm{char}\, k \notin \{2, 43\}. \end{cases}$$

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Unless indicated otherwise, tensor products and dimensions are over k. We write $\operatorname{mod}\text{-}\Gamma$ for finite-dimensional right Γ-modules, $\operatorname{proj}\text{-}\Gamma$ for its finitely generated projective modules, and $\mathrm{D}^b(\operatorname{mod}\text{-}\Gamma)$ for their bounded derived category. The notation $\operatorname{add} X$ means finite direct sums of direct summands of X, and $\operatorname{pd}_\Gamma X$ denotes projective dimension. We use cohomological grading for complexes: $d : P^q \to P^{q+1}$ and $(P[n])^q = P^{q+n}$ with differential $(-1)^n d_P$. The differential on a degree-n map is $d(f) = d_Q f - (-1)^n f d_P$. A thick subcategory is closed under shifts, cones, and direct summands; $\operatorname{thick}(X_1, \dots, X_r)$ denotes the one generated by the indicated objects. For an ordinary finite-dimensional algebra Γ, $\operatorname{Perf}(\Gamma)$ is the category of bounded complexes of finitely generated projective right modules up to homotopy. We write H^n for degree-n cohomology and $\mathbf{R}\mathrm{Hom}$ for the derived Hom complex. For a k-complex V, $V^* = \operatorname{Hom}_k(V, k)$ is its dual complex. All categories used for Hochschild invariants carry their indicated differential graded (dg) enhancements; the same symbol denotes a full dg enhancement when a dg tensor product or quotient is formed. Dg Morita equivalence means equivalence after passing to categories of perfect dg modules.

A split exceptional object E has $\operatorname{Hom}(E, E[n]) = 0$ for $n \neq 0$ and $\operatorname{End}(E) = k$. A split exceptional collection $(E_1, \dots, E_r)$ additionally satisfies $\operatorname{Hom}(E_j, E_i[n]) = 0$ for $j > i$ and all n. Its right orthogonal consists of objects X with $\operatorname{Hom}(E_i, X[n]) = 0$ for every i, n. A collection is full if it generates the ambient thick subcategory. A dg category is proper if the total cohomology of every morphism complex is finite-dimensional, and smooth if its diagonal bimodule is perfect. A full triangulated subcategory is admissible if its inclusion has both adjoints. The notation $\langle \mathcal{T}_1, \dots, \mathcal{T}_s \rangle$ denotes a semiorthogonal decomposition: the components generate, and morphisms from a later component to any shift of an earlier one vanish. The group K_0 is generated by object classes with relations $[Y] = [X] + [Z]$ for distinguished triangles $X \to Y \to Z \to X[1]$. Hochschild cohomology HH^n is the degree-n derived endomorphism space of the diagonal bimodule; for an ordinary algebra it is $\operatorname{Ext}^n_{A \otimes_k A^{\mathrm{op}}}(A, A)$.

The proof uses a three-vertex algebra R of dimension 25 and three left R-modules E, F, G. Their matrices are listed in Appendix A. Integer Smith forms determine their extension groups in all but four characteristics. Separate inputs over $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7$ cover the remaining cases. An elementary calculation of HH^0 proves that each exceptional triple is nonfull. Its right orthogonal is therefore a nonzero category with vanishing Hochschild homology. Folding projective resolutions and taking an endomorphism ring then produces A_k.

2 The three-vertex algebra and its extension complexes

Let Q have arrows $a_1, \dots, a_4 : 1 \to 2$ and $b_1, \dots, b_5 : 2 \to 3$. Products are composed from right to left, so $b_j a_i$ first traverses a_i. Let V have ordered basis $(b_1 a_1, b_1 a_2, \dots, b_5 a_4)$. For a matrix over k of rank seven $\mathcal{R} = (r_{\ell j i})$ with seven rows and twenty columns, set

$$R = kQ/(\rho_1, \dots, \rho_7), \qquad \rho_\ell = \sum_{j=1}^{5} \sum_{i=1}^{4} r_{\ell j i} b_j a_i.$$

Here $r_{\ell j i}$ occupies column $4(j-1) + i$. Put $W = V/\langle \rho_1, \dots, \rho_7 \rangle$, and write $\mu : V \to W$ for the quotient map. The vertex idempotents are e_1, e_2, e_3.

Lemma 2.1. *The algebra R has dimension 25, semisimple quotient k^3, and a finite projective bimodule resolution of length two. In particular, R and $\Gamma = R^{\mathrm{op}}$ are homologically smooth and have global dimension at most two.*

Proof. The arrows and paths of length two span the radical. There are no longer paths, and $\dim W = 13$, giving $\dim R = 3 + 9 + 13 = 25$. Consider

$$0 \longrightarrow (Re_3 \otimes_k e_1 R)^7 \xrightarrow{d_2} \bigoplus_{a:i\to j} Re_j \otimes_k e_i R \xrightarrow{d_1} \bigoplus_{v} Re_v \otimes_k e_v R \xrightarrow{\text{mult}} R \longrightarrow 0. \tag{1}$$

On the summand indexed by $a : i \to j$, $d_1(x \otimes y)$ has components $xa \otimes y$ at i and $-x \otimes ay$ at j. The image under d_2 of the generator in the ℓth summand has components $\sum_j r_{\ell ji} b_j \otimes e_1$ at a_i and $\sum_i r_{\ell ji} e_3 \otimes a_i$ at b_j. Thus $d_1 d_2 = 0$.

Exactness can be checked after applying $e_u(-)e_v$. The diagonal components are identities. For $(u, v) = (2, 1), (3, 2)$ the complex is the difference map into two copies of the arrow space followed by addition. For $(3, 1)$ it is

$$0 \to \ker \mu \longrightarrow V \oplus V \longrightarrow W \oplus V \oplus W \longrightarrow W \to 0,$$

with maps

$$r \mapsto (r, r), \quad (x, y) \mapsto (\mu x, -x + y, -\mu y), \quad (w_1, z, w_3) \mapsto w_1 + \mu z + w_3.$$

The middle kernel consists of $x = y \in \ker \mu$. If $w_1 + \mu z + w_3 = 0$, choose x with $\mu x = w_1$ and put $y = x + z$ to obtain a preimage. All other components vanish.

Each term in (1) is projective as a bimodule. Forgetting the left action gives a split exact resolution of the projective right module R, so tensoring with any left module preserves exactness. The resulting terms are projective left modules. The analogous argument applies to right modules. This proves the global-dimension assertions and smoothness. □

For a left R-module X, put $X_v = e_v X$ and write $A_i^X : X_1 \to X_2$, $B_j^X : X_2 \to X_3$ for its arrow maps. Such matrices define an R-module precisely when

$$\sum_{j,i} r_{\ell ji} B_j^X A_i^X = 0 \qquad (1 \le \ell \le 7). \tag{2}$$

Left R-modules will also be viewed as right Γ-modules, with $x \cdot r^{\mathrm{op}} = rx$.

Lemma 2.2. *For finite-dimensional left modules X, Y, all their extension groups are computed by*

$$C^0(X, Y) \xrightarrow{\delta^0} C^1(X, Y) \xrightarrow{\delta^1} C^2(X, Y),$$

where

$$\begin{aligned} C^0 &= \bigoplus_{v=1}^{3} \mathrm{Hom}_k(X_v, Y_v), \\ C^1 &= \mathrm{Hom}_k(X_1, Y_2)^4 \oplus \mathrm{Hom}_k(X_2, Y_3)^5, \\ C^2 &= \mathrm{Hom}_k(X_1, Y_3)^7, \end{aligned}$$

For $\phi = (\phi_1, \phi_2, \phi_3) \in C^0$ and $\xi = ((\xi_{a_i})_{i=1}^4, (\xi_{b_j})_{j=1}^5) \in C^1$, the maps are

$$(\delta^0\phi)_{a_i} = A_i^Y \phi_1 - \phi_2 A_i^X, \qquad (\delta^0\phi)_{b_j} = B_j^Y \phi_2 - \phi_3 B_j^X,$$
$$(\delta^1\xi)_\ell = \sum_{j,i} r_{\ell ji} (B_j^Y \xi_{a_i} + \xi_{b_j} A_i^X).$$

There are no extension groups in degrees greater than two.

Proof. Tensor (1) on the right with X and apply $\mathrm{Hom}_R(-,Y)$. Evaluation on $e_v \otimes X_v$ identifies the resulting spaces and gives the displayed maps by precomposition with d_1, d_2. This is the usual unsigned Hom complex of a homological projective resolution. If the resolution is instead placed in cohomological degrees $-2,-1,0$, its dg Hom complex has maps $-\delta^0, \delta^1$; multiplication by $1,-1,-1$ in degrees $0,1,2$ identifies these conventions. Lemma 2.1 supplies exactness and the bound on higher extensions. □

3 Exact calculations in every characteristic

For a representation X, its dimension vector is $\underline{\dim} X = (\dim_k X_1, \dim_k X_2, \dim_k X_3)$. The modules have dimension vectors

$$\underline{\dim} E = (3,2,1), \qquad \underline{\dim} F = (1,1,1), \qquad \underline{\dim} G = (1,2,4).$$

Appendix A specifies five sets of matrices. The integral set is the ungraded input of [7, Sections 2.1–2.2]. Use it over $\mathbb{Q}$ and in characteristics outside $\{2,3,5,7\}$. In characteristic $p \in \{2,3,5,7\}$ use the set labelled p, interpreted in the prime subfield of k.

For an integer matrix M, a Smith form is a diagonal matrix $D = UMV$ with integral U, V of determinant ± 1, whose positive nonzero diagonal entries divide their successors. Its rank in characteristic p is the number of these entries not divisible by p; in characteristic zero it is their number. Thus finitely many exact integer identities specify the ranks in every characteristic.

Proposition 3.1. *For the selected input over any field, $\mathcal{R}$ has rank seven, the module equations (2) hold, and (E,F,G) is split exceptional. In the following table, XY means extensions from X to Y. Their dimensions, listed in degrees $0,1,2$, are*

Input and characteristic	EF	EG	FG
Integral, outside $2,3,5,7,43$	$(0,0,5)$	$(0,0,31)$	$(0,0,7)$
Integral, 43	$(0,0,5)$	$(0,0,31)$	$(0,1,8)$
New, 2	$(0,1,6)$	$(0,0,31)$	$(0,0,7)$
New, $3,5,7$	$(0,0,5)$	$(0,0,31)$	$(0,0,7)$

Proof. Substitution verifies (2). The integer relation matrix has seven unit Smith entries. The remaining integer calculations are shown below. The two ranks are over $\mathbb{Q}$; only nonunit Smith entries are displayed, with $a^{\times m}$ denoting m entries equal to a.

XY	$(\dim C^0, \dim C^1, \dim C^2)$	$(\mathrm{rank}\,\delta^0, \mathrm{rank}\,\delta^1)$	δ^0	δ^1
EE	$(14,34,21)$	$(13,21)$	$2,2,2$	$2,6,6,84,84$
EF	$(6,22,21)$	$(6,16)$	–	$6,6,6,6$
EG	$(11,64,84)$	$(11,53)$	–	$2^{\times 7}, 6^{\times 4}, 30$
FE	$(6,13,7)$	$(6,7)$	2	$3,30$
FF	$(3,9,7)$	$(2,7)$	–	$3,6$
FG	$(7,28,28)$	$(7,21)$	–	$3,3,3,129$
GE	$(11,18,7)$	$(11,7)$	2	$2,2,4$
GF	$(7,14,7)$	$(7,7)$	–	$2,2,2,4$
GG	$(21,48,28)$	$(20,28)$	–	$2^{\times 14}, 4,4,20,20$

The integral certificates contain U, V, D for these eighteen matrices and for $\mathcal{R}$. Checking $UMV = D$ and $\det U, \det V \in \{1,-1\}$ verifies the table. For a matrix M with a rows and b columns and index

sets I, J, let $M[:, J]$ denote its columns indexed by J, and let $M[I, J]$ denote the corresponding row-and-column submatrix. Each finite-field rank certificate of rank h consists of index sets $|I| = |J| = h$ and an $h \times b$ matrix H satisfying $M = M[:, J]H$ and $\det M[I, J] \neq 0$. These identities give both rank inequalities. The new inputs have the following ranks; their complex dimensions are the same as above:

p	EE	EF	EG	FE	FF	FG	GE	GF	GG
2	$13, 21$	$6, 15$	$11, 53$	$6, 7$	$2, 7$	$7, 21$	$11, 7$	$7, 7$	$20, 28$
$3, 5, 7$	$13, 21$	$6, 16$	$11, 53$	$6, 7$	$2, 7$	$7, 21$	$11, 7$	$7, 7$	$20, 28$

Appendix B gives an additional verifier that reconstructs the differentials directly from the five input files and computes the ranks and Smith forms independently. For dimensions (c_0, c_1, c_2) and ranks (r_0, r_1) the extension dimensions are $(c_0 - r_0, c_1 - r_0 - r_1, c_2 - r_1)$. Applying this formula proves the assertions. In characteristic 43 precisely one Smith entry in the forward map $\delta^1(F, G)$ becomes zero. Finally, extending the prime field preserves these finite matrix ranks. □

For the original input the excluded characteristics are necessary: the self-extension dimensions of E are $(4, 8, 5)$ in characteristic 2, $(1, 4, 4)$ in characteristic 3, and $(1, 2, 2)$ in characteristic 7; those of G are $(1, 2, 2)$ in characteristic 5.

4 Nonfullness and the residual category

Put $\mathcal{E} = \operatorname{thick}(E, F, G) \subset \operatorname{Perf}(\Gamma)$. A strictly unital minimal A_∞ model over $S = k^3$ may be chosen for the dg endomorphism algebra of the direct sum of projective representatives; see [5, Section 3.5]. Its underlying graded algebra is $\operatorname{Ext}^*_\Gamma(E \oplus F \oplus G, E \oplus F \oplus G)$. The n-ary operation m_n has degree $2 - n$; minimality means $m_1 = 0$, and m_2 is the extension product. Strict unitality makes m_n vanish on an identity input for $n \geq 3$, and S-linearity forces the inputs to be composable. The complement to the three identity classes consists of the three forward off-diagonal blocks.

There are no three composable inputs from these off-diagonal blocks, so all products m_n for $n \geq 3$ vanish. The only possible product of two off-diagonal inputs goes from E through F to G. By Proposition 3.1, its degree is at least three, whereas the target is concentrated in degree two. Thus this product also vanishes. Consequently $\mathcal{E}$ is dg Morita equivalent to the zero-differential graded algebra

$$B = S \oplus J, \qquad J^2 = 0,$$

where J is the direct sum of the three forward extension spaces, with the left and right S actions determined by their target and source vertices. The minimal-model construction applies over every field: $S \otimes_k S^{\mathrm{op}} \cong k^9$ is semisimple, so the required splittings can be chosen S-bilinearly.

Lemma 4.1. *For this graded algebra,*

$$\dim_k \mathrm{HH}^0(B) \;\geq\; 1 + \dim \operatorname{Ext}^2(E, F)\, \dim \operatorname{Ext}^2(F, G)\, \dim \operatorname{Ext}^2(E, G) > 1.$$

Proof. Since S is separable over k, the relative normalized bar resolution computes Hochschild cohomology. Its cochains of length n are

$$\operatorname{Hom}_{S\text{-}S}(J^{\otimes_S n}, B).$$

For $n = 0$, interpret the source as S. A map of internal degree d raises the sum of input degrees by d, and has total cohomological degree $d + n$. This convention comes from suspending each bar input down by one. Since $J^{\otimes_S 3} = 0$, only lengths $0, 1, 2$ occur.

The maps

$$\mathrm{Ext}^2(F,G)\otimes_k \mathrm{Ext}^2(E,F)\longrightarrow \mathrm{Ext}^2(E,G)$$

have internal degree -2 and total degree zero. Extend such maps by zero on all other input degrees. Their internal differential vanishes and their Hochschild differential has length three, hence is zero.

There are no total-degree -1 cochains. At length zero all diagonal outputs have degree zero. At length one a degree--1 cochain would have internal degree -2; an input has degree one or two, and the same off-diagonal block has no output of degree -1 or zero. At length two the internal degree would be -3; input degrees sum to at most four, and the EG block has no output in degree at most one. Thus none of the displayed cocycles is a boundary. The common scalar identity is an additional independent degree-zero cocycle, and is not a boundary either. □

Proposition 4.2. *The right orthogonal $\mathcal{P}=\mathcal{E}^{\perp}$ is nonzero, smooth and proper, and satisfies $\mathrm{HH}_*(\mathcal{P})=0$ and $K_0(\mathcal{P})=0$.*

Proof. The center of R is k: commuting with the vertex idempotents forces a central element to be diagonal, each diagonal block is k, and commuting with the arrows makes the three scalars equal. Hence $\mathrm{HH}^0(\Gamma)=k$. Fullness of $\mathcal{E}$ would give a dg Morita equivalence with $\mathrm{Perf}(\Gamma)$, contradicting Lemma 4.1.

Each exceptional object generates an admissible subcategory. For the embedding $V\mapsto V\otimes_k E$, the right adjoint is $\mathbf{R}\mathrm{Hom}(E,-)$ and the left adjoint is $\mathbf{R}\mathrm{Hom}(-,E)^*$; these take values in perfect k-complexes because the ambient category is proper. Successive projections give

$$\mathrm{Perf}(\Gamma)=\langle \mathcal{P},E,F,G\rangle.$$

Nonfullness implies $\mathcal{P}\neq 0$. Smoothness and properness pass to admissible subcategories.

The relative normalized Hochschild chains of R over k^3 have no positive-length terms after taking the cyclic tensor: every such term would require a directed cycle. The length-zero term is k^3. Thus $\mathrm{HH}_0(\Gamma)=k^3$ and all other homology groups vanish. Additivity for the displayed semiorthogonal decomposition, with each exceptional component contributing k in degree zero, gives $\mathrm{HH}_*(\mathcal{P})=0$.

The group $K_0(\Gamma)$ is $\mathbb{Z}^3$, identified by dimension vectors. The matrix of the three classes is

$$\begin{pmatrix}3&1&1\\2&1&2\\1&1&4\end{pmatrix},\qquad \det=1.$$

They form an integral basis. The direct-sum decomposition on K_0 therefore gives $K_0(\mathcal{P})=0$. □

5 Folding and the canonical singularity category

In this section and the next, Γ is any finite-dimensional homologically smooth ordinary k-algebra. It has finite global dimension: a bounded finite projective bimodule resolution, tensored with a module, supplies a uniform bound as in Lemma 2.1. Let $(E_1,\dots,E_r)$ be a split exceptional collection with nonzero right orthogonal $\mathcal{P}$ and $\mathrm{HH}_*(\mathcal{P})=0$. Choose bounded finite projective right-module representatives $P_i^{\bullet}$.

Set $\Delta=k[\epsilon]/(\epsilon^2)$ and $\Lambda=\Gamma\otimes_k\Delta$. For a right Γ-module P and a square-zero Γ-linear map $d:P\to P$, write (P,d) for the right Λ-module on which ϵ acts as d. Define

$$N_i=\left(\bigoplus_q P_i^q,d_{P_i}\right),\qquad M=\Lambda\oplus\bigoplus_{i=1}^{r}N_i,\qquad A=\mathrm{End}_{\Lambda}(M).\tag{3}$$

The endomorphism-ring product is composition.

The singularity category is $\mathrm{D}_{\mathrm{sg}}(\Lambda) = \mathrm{D}^b(\mathrm{mod}\text{-}\Lambda)/\operatorname{Perf}(\Lambda)$. Its canonical dg enhancement $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ is the corresponding dg quotient. We take pretriangulated and idempotent completions when forming dg Morita equivalences.

Lemma 5.1. *Let $\mathcal{C} = C^b_{\mathrm{dg}}(\mathrm{proj}\text{-}\Gamma)$ be the dg category of bounded projective complexes with the morphism differential fixed in the introduction, and let $U = k[t, t^{-1}]$ be the associative graded Laurent algebra with $|t| = 1$ and zero differential. There is a dg Morita equivalence*

$$\operatorname{Perf}(\mathcal{C} \otimes_k U) \simeq \mathcal{D}_{\mathrm{sg}}(\Lambda)$$

which carries each specified $P_i^\bullet$ to N_i.

Proof. Introduce a dg category with objects (P, d_P), where P is finite projective over Γ and $d_P^2 = 0$. Its degree-n morphisms from P to Q are $\mathrm{Hom}_\Gamma(P, Q)$, with differential $d_Q f - (-1)^n f d_P$ and ordinary composition. On objects, let Ψ send $P^\bullet$ to $(\bigoplus_q P^q, d_P)$. For integers ℓ, s and a homogeneous map $f : P^\bullet \to Q^\bullet$ of degree ℓ, write $f^q : P^q \to Q^{q+\ell}$ for its component and define

$$\Psi(f \otimes t^s)|_{P^q} = (-1)^{sq} f^q. \tag{4}$$

It has degree $\ell + s$. The tensor product multiplication is

$$(g \otimes t^u)(f \otimes t^s) = (-1)^{u\ell}(gf) \otimes t^{u+s}.$$

The blockwise sign in composing (4) is $(-1)^{sq+u(q+\ell)}$, which is this same sign. For the differential, the second term has exponent $\ell + s + s(q+1) \equiv \ell + sq \pmod 2$; the first has exponent sq. These are precisely the signs obtained by applying Ψ to $(d_Q f - (-1)^\ell f d_P) \otimes t^s$. Thus Ψ is a dg functor. In total degree n, a block $P^q \to Q^{q+\ell}$ corresponds uniquely to $s = n - \ell$. Boundedness gives only finitely many blocks, proving that Ψ is an isomorphism on morphism complexes.

For an object (P, d) of the intermediate category, define a two-sided complex of projective Λ-modules

$$T(P, d)^j = P \otimes_\Gamma \Lambda, \qquad D^j = \epsilon + (-1)^j d \quad (j \in \mathbb{Z}).$$

Here d is extended by $d \otimes 1$, and ϵ means multiplication by ϵ. Write $P \otimes \Delta = P \oplus P\epsilon$. Then, with $s = (-1)^j$,

$$D^j(x, y) = (sdx, x + sdy), \qquad \ker D^j = \{(-sdy, y) : y \in P\} = \operatorname{im} D^{j-1}.$$

To check the dual complex, write a Λ-linear map $P \otimes \Delta \to \Lambda$ as

$$(x, y) \longmapsto \alpha(x) + (\beta(x) + \alpha(y))\epsilon, \qquad \alpha, \beta \in P^* = \mathrm{Hom}_\Gamma(P, \Gamma).$$

Precomposition with D^j takes (α, β) to $(s\alpha d, \alpha + s\beta d)$. The same kernel-image formula applies, with $d^*(\alpha) = \alpha d$; the overall signs from the Hom differential do not change kernels or images. Hence $T(P, d)$ is totally acyclic, meaning that it and its Λ-dual are exact. Its cokernel at D^{-1} is (P, d) via $(x, y) \mapsto x + dy$. Thus these modules are Gorenstein-projective, i.e. cokernels in totally acyclic complexes of finite projectives.

For a degree-n morphism f define

$$T(f)^j = (-1)^{n(n-1)/2 + nj}(f \otimes 1). \tag{5}$$

Writing the sign as $a_{n,j}$, the identities

$$a_{n,j+1} = (-1)^n a_{n,j}, \quad a_{n+1,j} = (-1)^{n+j} a_{n,j}, \quad a_{m,j+n} a_{n,j} = a_{m+n,j}$$

verify the differential and composition rules. In particular, the ϵf terms in $dT(f)$ cancel, and the remaining expression is $a_{n+1,j}(d_Q f - (-1)^n f d_P)$. The exponent $n(n-1)/2$ is an integer for every integer n.

Truncate $T(P,d)$ brutally to degrees at most zero. The result is a projective resolution of (P,d). On morphisms truncate the families (5) in the same way. The failure of truncation to commute with differential or composition is supported in finitely many degrees. In the Vogel dg category morphism complexes are products of component maps modulo the subcomplex of finite-support families; these failures therefore vanish. Chen–Wang's comparison [2, Proposition 4.1], applied to right modules by taking the opposite ring, identifies this bounded-above Vogel model with the canonical dg singularity category and acts identically on its resolution objects.

We verify that the resulting functor is a quasi-isomorphism on morphism complexes. In degree zero the closed maps are the Λ-linear maps between (P,d_P) and (Q,d_Q). Its boundaries are $d_Q h + h d_P$. They factor through a projective by

$$P \xrightarrow{p \mapsto (d_P p, p)} P \otimes \Delta \xrightarrow{h \otimes 1} Q \otimes \Delta \xrightarrow{(x,y) \mapsto x + d_Q y} Q.$$

Conversely, a Λ-linear map $P \to P' \otimes \Delta$ has the form $(h d_P, h)$ for a Γ-linear map $h : P \to P'$. A Λ-linear map $P' \otimes \Delta \to Q$ sends (x,y) to $a(x) + d_Q a(y)$ for a Γ-linear map $a : P' \to Q$. Their composition is $a h d_P + d_Q a h$. Every finite projective Λ-module is a summand of one of this form, so boundaries are exactly the maps factoring through projectives. The canonical embedding of the stable Gorenstein-projective category into $\mathrm{D_{sg}}(\Lambda)$ is fully faithful [2, Theorem 4.11]. In degree zero our functor induces that embedding, because its augmentation sends $T(f)^0$ to f.

For degree n, put $Q_n = (Q, (-1)^n d_Q)$. Degree-n cocycles are exactly degree-zero maps $P \to Q_n$; their boundary subspaces agree up to the scalar $(-1)^n$. The complexes $T(Q_n)$ and $T(Q)[n]$ are isomorphic by multiplying degree j by $(-1)^{nj}$. Under this identification, (5) differs from the degree-zero comparison by the constant $(-1)^{n(n-1)/2}$. Hence it induces an isomorphism in every cohomological degree.

Finally the image contains $(\Gamma, 0)$. A simple Λ-module is a simple Γ-module with zero ϵ action. Its finite Γ-projective resolution, with zero ϵ on each term, is also an exact sequence of Λ-modules, and every term is in $\mathrm{add}(\Gamma, 0)$. These objects therefore generate all simple modules, and by finite-length filtrations and truncations they generate $\mathrm{D}^b(\mathrm{mod}\text{-}\Lambda)$. The image thickly generates the singularity category. This proves the Morita equivalence and its stated action on each $P_i^\bullet$. □

6 The endomorphism ring and its Hochschild homology

Lemma 6.1. *For A in (3), there is a dg Morita equivalence*

$$\mathcal{D}_{\mathrm{sg}}(A) \simeq \mathrm{Perf}(\mathcal{P} \otimes_k U).$$

Proof. Let $e \in A$ project onto Λ, and let $f_i \in A$ project onto N_i for $1 \le i \le r$. Put $C_0 = A/AeA$. The ideal AeA consists of maps factoring through finite projective Λ-modules, since all such modules belong to $\mathrm{add}\,\Lambda$. Thus C_0 is the stable endomorphism algebra of $\bigoplus_i N_i$. Write $\underline{\mathrm{Hom}}_\Lambda$ for module homomorphisms modulo those factoring through finite projective modules. Lemma 5.1 identifies its blocks, as vector spaces, with

$$\underline{\mathrm{Hom}}_\Lambda(N_i, N_j) \cong \bigoplus_{q \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Perf}(\Gamma)}(E_i, E_j[q]).$$

These sums are finite. The exceptional conditions imply that C_0 is directed with diagonal blocks k. More explicitly, $f_jC_0f_i$ vanishes for $i > j$, while $f_jC_0f_j = k$. Thus the radical of the right projective f_jC_0 has composition factors only at vertices $i < j$. Let $S_j = f_jC_0/\operatorname{rad}(f_jC_0)$ be its simple top. Induction on j gives $\operatorname{pd}_{C_0} S_j \leq j-1$: $S_1 = f_1C_0$ is projective, and a composition series of the radical, together with the induction hypothesis, bounds its projective dimension by $j-2$. The short exact sequence defining S_j gives the claim. A composition series of any finite module then gives $\operatorname{gldim} C_0 \leq r-1$.

For $N_i = (P, d)$ there is an exact sequence

$$0 \to (P,d) \xrightarrow{p\mapsto(dp,p)} P \otimes \Delta \xrightarrow{(x,y)\mapsto x-dy} (P,-d) \to 0.$$

The parity operator $p \mapsto (-1)^q p$ for $p \in P_i^q$ identifies $(P,-d)$ with (P,d). Denote the resulting projective middle term by Q_i. Applying $\operatorname{Hom}_\Lambda(M,-)$ gives

$$0 \to \operatorname{Hom}_\Lambda(M,N_i) \to \operatorname{Hom}_\Lambda(M,Q_i) \to \operatorname{Hom}_\Lambda(M,N_i) \to f_iC_0 \to 0, \tag{6}$$

Indeed, every map factoring through a projective lifts along $Q_i \twoheadrightarrow N_i$, so the cokernel is the stable Hom space, with its right A action by precomposition. The first three terms are projective right A-modules. Thus $\operatorname{pd}_A(f_iC_0) \leq 2$. A projective C_0-resolution of any finite C_0-module, combined with these resolutions, gives projective dimension at most $r+1$ over A.

Evaluation gives an isomorphism of left A-modules $M \cong Ae$. For a right A-module X, the identification $X \otimes_A M \cong Xe$ shows that $- \otimes_A M$ is exact. Its kernel consists precisely of the finite C_0-modules: $Xe = 0$ is equivalent to $X(AeA) = 0$. The right-module version of Chen's partial-resolution theorem [1, Proposition 1.1 and Lemmas 2.1–2.3] applies, and this tensor functor induces

$$\mathrm{D}_{\mathrm{sg}}(A) \simeq \mathrm{D}^b(\text{mod-}\Lambda)/\operatorname{thick}(M) \simeq \mathrm{D}_{\mathrm{sg}}(\Lambda)/\operatorname{thick}(N_1,\dots,N_r). \tag{7}$$

This particular equivalence lifts to the dg quotients. Apply $- \otimes_A M$ degreewise to bounded complexes. Exactness preserves acyclic complexes, so it descends to a dg quasi-functor on the bounded derived enhancements. Its composite with the quotient by $\operatorname{thick}_{\mathrm{dg}}(M)$ kills every perfect A-complex, since $A \otimes_A M = M$. The universal property of the dg quotient [3, Theorem 1.6.2] therefore gives a quasi-functor from $\mathcal{D}_{\mathrm{sg}}(A)$ to that quotient. On H^0 it is the equivalence (7). In pretriangulated models the nth cohomology of a morphism complex is the H^0 morphism space into the shift by n, so this lift is quasi-fully faithful and induces a Morita equivalence.

Let $\mathcal{D} \subset \mathcal{C}$ be the full dg subcategory of complexes representing $\operatorname{thick}(E_1,\dots,E_r)$. Lemma 5.1 identifies the specified objects with N_i. The thick subcategory generated by their images is also generated by the images of all of $\mathcal{D}$: extension of scalars $\mathcal{C} \to \operatorname{Perf}(\mathcal{C} \otimes U)$ preserves finite cones and summands. Thus the quotient in (7) is the Morita completion of

$$(\mathcal{C} \otimes U)/(\mathcal{D} \otimes U) \simeq (\mathcal{C}/\mathcal{D}) \otimes U.$$

The last equivalence is [3, Proposition 1.6.3]; U is homotopically flat over the field k. The inclusion of the admissible right orthogonal into $\mathcal{C}/\mathcal{D}$ is fully faithful and essentially surjective on the homotopy category, by the semiorthogonal decomposition. Its dg lift is a Morita equivalence by the same shift argument. Therefore $\mathcal{C}/\mathcal{D}$ is Morita equivalent to $\mathcal{P}$, proving the lemma. □

Lemma 6.2. *For a field extension K/k, write $X_K = X \otimes_k K$ for scalar extension of an algebra, module, or complex (degreewise for complexes). The construction (3) commutes with scalar extension. The extended exceptional collection has a nonzero right orthogonal with vanishing Hochschild homology.*

Proof. Finite projective representatives give base change for all derived Hom spaces, so the collection remains split exceptional. A finite projective bimodule resolution shows that Γ_K remains smooth.

The projection onto the right orthogonal is obtained by successively applying

$$L_E(X) = \mathrm{Cone}(\mathbf{R}\mathrm{Hom}(E, X) \otimes_k E \longrightarrow X)$$

in the order $E_r, E_{r-1}, \ldots, E_1$, where the arrow is evaluation and Cone is its cochain mapping cone. Applying $\mathrm{Hom}(E_j, -)$ shows that a later step with $i < j$ preserves orthogonality to E_j, since $\mathrm{Hom}(E_j, E_i[*]) = 0$. These projections commute with scalar extension. Apply the composite to Γ and denote its value by T. Because Γ generates the ambient category and the projection is exact and preserves summands, T generates $\mathcal{P}$; its scalar extension generates the extended orthogonal.

Choose a bounded finite projective representative of T and let $B_T = \mathrm{End}^\bullet_\Gamma(T)$. Then $\mathrm{End}^\bullet_{\Gamma_K}(T_K) = B_T \otimes_k K$. Since $\mathcal{P} \neq 0$, the identity of T has a nonzero cohomology class, which remains nonzero by faithful flatness. The ordinary direct-sum Hochschild bar complex of B_T commutes with extension of fields, so its homology still vanishes. This proves the assertions about the orthogonal.

Folding visibly commutes with scalar extension. Endomorphisms of a finite-dimensional module are the kernel of a finite system of linear equations expressing compatibility with a basis of the algebra action. Flatness preserves this kernel, giving $\mathrm{End}_\Lambda(M) \otimes_k K \cong \mathrm{End}_{\Lambda_K}(M_K)$. □

Theorem 6.3. *Under the assumptions of Section 5, the ordinary algebra A in (3) satisfies* $\operatorname{gldim} A = \infty$ *and* $\mathrm{HH}_n(A/k) = 0$ *for all* $n \geq 1$.

Proof. Choose a nonzero object $X \in \mathcal{P}$. Its identity class remains nonzero in the t^0 summand of $H^0(\mathrm{Hom}_\mathcal{P}(X, X) \otimes U)$. Thus Lemma 6.1 gives a nonzero singularity category, and A has infinite global dimension.

Using the generator T of Lemma 6.2, compute the homology of $\mathcal{P} \otimes U$ by the dg algebra $B_T \otimes U$. Write $C_*(D)$ for the ordinary direct-sum Hochschild chain complex of a dg algebra D, graded so that its nth homology is $\mathrm{HH}_n(D)$. The shuffle Künneth quasi-isomorphism [8, Theorem 2.8 and Proposition 2.9] gives

$$C_*(B_T \otimes U) \simeq C_*(B_T) \otimes_k C_*(U).$$

The first factor on the right is acyclic. An acyclic complex of vector spaces is contractible, and tensoring its contraction with the identity contracts this total tensor complex. Hence $\mathrm{HH}_n(\mathcal{D}_{\mathrm{sg}}(A)) = 0$ for every integer n.

If k is algebraically closed, Wang–Arunachalam–Keller [9, Corollary 5] give

$$\mathrm{HH}_n(\mathcal{D}_{\mathrm{sg}}(A)) \cong \mathrm{HH}_{n-1}(A/k) \qquad (n \geq 2).$$

Their result applies in every characteristic; a nonbasic algebra is reduced to a basic one by Morita invariance. We conclude that $\mathrm{HH}_m(A/k) = 0$ for every $m \geq 1$.

For general k, let $\overline{k}$ be an algebraic closure. Lemma 6.2 permits the same argument after extension to $\overline{k}$. The ordinary Hochschild bar complex gives

$$\mathrm{HH}_m(A \otimes_k \overline{k}/\overline{k}) \cong \mathrm{HH}_m(A/k) \otimes_k \overline{k}.$$

Faithful flatness then yields the required vanishing over k. □

7 The explicit algebra and its dimension

Return to the selected three-vertex input and $\Gamma = R^{\mathrm{op}}$. For $X \in \{E, F, G\}$ use the resolution of Lemma 2.1, with terms

$$\begin{aligned} P_X^0 &= Re_1 \otimes_k X_1 \ \oplus \ Re_2 \otimes_k X_2 \ \oplus \ Re_3 \otimes_k X_3, \\ P_X^{-1} &= \bigoplus_{i=1}^{4} Re_2 \otimes_k X_1 \ \oplus \ \bigoplus_{j=1}^{5} Re_3 \otimes_k X_2, \\ P_X^{-2} &= \bigoplus_{\ell=1}^{7} Re_3 \otimes_k X_1. \end{aligned} \tag{8}$$

The augmentation sends $x \otimes u$ to xu. On the a_i summand the first differential sends $x \otimes u$ to $xa_i \otimes u - x \otimes A_i^X u$, in the first and second vertex summands respectively. On the b_j summand use $xb_j \otimes u - x \otimes B_j^X u$. On the ℓth relation summand the next differential has a_i component $\sum_j r_{\ell ji} x b_j \otimes u$ and b_j component $\sum_i r_{\ell ji} x \otimes A_i^X u$. These formulas specify all matrix entries independently of any choice in the residual category.

Fold (8) to obtain N_X and set

$$A_k = \mathrm{End}_{\Gamma \otimes_k \Delta}\big((\Gamma \otimes_k \Delta) \oplus N_E \oplus N_F \oplus N_G\big).$$

The program supplied with the article constructs matrices by row-reducing $\mathcal{R}$ in its fixed column order and taking the nonpivot paths as a basis of W. It produces a representation M with vertex dimensions $(7, 55, 294)$, arrow matrices a_i^M, b_j^M, and square-zero operators D_v giving the action of ϵ. Writing $M_d(k)$ for the algebra of $d \times d$ matrices, we can describe A_k as the ordinary matrix algebra

$$\left\{ (f_1, f_2, f_3) \in M_7(k) \oplus M_{55}(k) \oplus M_{294}(k) : \begin{array}{l} f_2 a_i^M = a_i^M f_1 \ (1 \le i \le 4), \\ f_3 b_j^M = b_j^M f_2 \ (1 \le j \le 5), \\ f_v D_v = D_v f_v \ (1 \le v \le 3) \end{array} \right\}.$$

Multiplication is componentwise matrix multiplication.

Proof of Theorem 1.1. Proposition 3.1 and Proposition 4.2 verify the hypotheses of Theorem 6.3, which gives the global-dimension and Hochschild-homology assertions for every field.

It remains to compute the dimension. The underlying Γ-module of M is

$$(Re_1)^7 \oplus (Re_2)^{27} \oplus (Re_3)^{68}.$$

Here a left R-module is identified with a right Γ-module as before. The multiplicities follow by adding (8) for the three modules and two copies of the regular module. Since $\dim Re_1 = 18$, $\dim Re_2 = 6$, $\dim Re_3 = 1$, the vertex dimensions are $(7, 55, 294)$ and $\dim M = 356$. Moreover,

$$\begin{aligned} \dim_k \mathrm{End}_\Gamma(M) = {} & 7^2 + 27^2 + 68^2 + 4 \cdot 7 \cdot 27 \\ & + 5 \cdot 27 \cdot 68 + 13 \cdot 7 \cdot 68 = 21526. \end{aligned}$$

On this endomorphism space define $c(f) = df - fd$ and $s(f) = df + fd$, where d is the ϵ action. Then $A_k = \ker c$. Give each folded summand its original parity, and give the two copies of Γ in Λ opposite parities. The resulting invertible operator σ satisfies $\sigma d = -d\sigma$, whence $c(f\sigma) = s(f)\sigma$.

Thus c, s have the same rank. Since $cs = 0$, and the stable endomorphisms are $\ker c / \operatorname{im} s$ by the factorization argument in Lemma 5.1,

$$\dim A_k = \frac{21526 + \dim C_0}{2}.$$

This is an identity of integer dimensions, also in characteristic two. The three diagonal blocks of C_0 contribute 3, and its forward blocks contribute the sum of all the extension dimensions in Proposition 3.1. Thus $\dim C_0 = 3 + 5 + 31 + 7 = 46$ outside characteristics $2, 43$. In characteristic 2 the EF block has dimension $1 + 6 = 7$, and in characteristic 43 the FG block has dimension $1 + 8 = 9$, giving 48 in both cases. Substitution proves the dimension formula. □

A The five inputs

All indices in the data below start at zero. Each row of `relations` uses the path order $b_1a_1, b_1a_2, \dots, b_5a_4$. Each module record lists its dimension vector, followed by the four matrices for a_i and the five matrices for b_j. Matrices are lists of rows; a single-column matrix is still written as a list of one-entry rows. The integral set is interpreted over $\mathbb{Q}$ or reduced in the indicated characteristic; the other four sets are interpreted modulo their labels.

Integral input

```
relations = [
  [-1, 2, 0, 0, 1, -2, 0, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 0],
  [0, 2, 0, 0, 1, -3, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0],
  [0, 0, -1, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
  [0, 0, 0, -2, 0, 0, 0, 0, -2, 1, 0, 0, 0, 1, 0, 0, 0, 0, -2, 0],
  [0, 0, 0, 1, 0, 0, 0, -1, -2, 1, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0],
  [0, 0, 0, 1, 0, 0, 0, 0, -2, 0, 0, 0, -2, 1, 0, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, -1, 1, 1, 0, 0],
]
E dims = [3, 2, 1]
E a[0] = [[0, 1, 0], [-1, 0, 0]]
E a[1] = [[0, 1, 0], [-2, 0, 0]]
E a[2] = [[0, 0, 0], [0, 0, 0]]
E a[3] = [[-2, 0, 0], [0, 0, 2]]
E b[0] = [[-1, 0]]
E b[1] = [[-1, 0]]
E b[2] = [[0, -1]]
E b[3] = [[0, -2]]
E b[4] = [[0, 0]]
F dims = [1, 1, 1]
F a[0] = [[-2]]
F a[1] = [[-1]]
F a[2] = [[0]]
F a[3] = [[0]]
F b[0] = [[-1]]
F b[1] = [[-2]]
F b[2] = [[0]]
F b[3] = [[0]]
F b[4] = [[0]]
G dims = [1, 2, 4]
G a[0] = [[-2], [0]]
G a[1] = [[0], [0]]
G a[2] = [[0], [-1]]
G a[3] = [[0], [0]]
G b[0] = [[0, 2], [1, 0], [0, 0], [0, 0]]
G b[1] = [[0, 1], [1, 0], [1, 0], [0, 0]]
G b[2] = [[0, 0], [0, 1], [0, 1], [1, 0]]
G b[3] = [[0, 0], [0, -1], [0, -2], [-1, 0]]
G b[4] = [[0, 0], [0, 0], [0, 0], [0, -2]]
```

Characteristic 2

```
relations = [
  [1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1],
  [1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1],
  [1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1],
  [1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0],
  [0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1],
  [1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 0],
  [0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0],
]
E dims = [3, 2, 1]
E a[0] = [[0, 0, 1], [0, 1, 1]]
E a[1] = [[1, 0, 0], [1, 0, 1]]
E a[2] = [[0, 1, 1], [0, 1, 0]]
E a[3] = [[0, 0, 0], [1, 1, 1]]
E b[0] = [[1, 0]]
E b[1] = [[0, 1]]
E b[2] = [[1, 1]]
E b[3] = [[1, 1]]
E b[4] = [[0, 1]]
F dims = [1, 1, 1]
F a[0] = [[1]]
F a[1] = [[0]]
F a[2] = [[1]]
F a[3] = [[0]]
F b[0] = [[1]]
F b[1] = [[0]]
F b[2] = [[1]]
F b[3] = [[1]]
F b[4] = [[1]]
G dims = [1, 2, 4]
G a[0] = [[0], [0]]
G a[1] = [[1], [0]]
G a[2] = [[1], [1]]
G a[3] = [[0], [1]]
G b[0] = [[1, 1], [0, 0], [0, 1], [0, 1]]
G b[1] = [[0, 0], [1, 0], [1, 1], [0, 1]]
G b[2] = [[0, 1], [0, 1], [1, 1], [0, 1]]
G b[3] = [[1, 1], [1, 1], [1, 1], [1, 1]]
G b[4] = [[1, 0], [0, 1], [0, 0], [0, 1]]
```

Characteristic 3

```
relations = [
  [2, 2, 1, 2, 1, 0, 1, 2, 2, 2, 0, 2, 2, 1, 2, 0, 2, 2, 2, 0],
  [1, 2, 0, 2, 2, 1, 2, 1, 1, 0, 2, 2, 0, 2, 1, 0, 2, 2, 1, 0],
  [0, 2, 1, 2, 1, 0, 1, 0, 1, 0, 1, 1, 0, 2, 0, 2, 2, 2, 2, 2],
  [0, 2, 0, 1, 2, 2, 2, 2, 0, 0, 1, 0, 0, 0, 1, 0, 2, 2, 1, 1],
  [1, 1, 2, 2, 1, 1, 2, 1, 1, 0, 2, 0, 2, 1, 0, 2, 2, 1, 0, 1],
  [1, 0, 1, 2, 0, 1, 0, 2, 2, 0, 0, 0, 1, 1, 1, 2, 0, 1, 0, 0],
  [2, 0, 1, 2, 0, 2, 2, 0, 1, 2, 2, 1, 0, 1, 1, 0, 2, 1, 0, 2],
]
E dims = [3, 2, 1]
E a[0] = [[1, 0, 1], [2, 0, 0]]
E a[1] = [[1, 1, 2], [0, 0, 1]]
E a[2] = [[1, 0, 0], [1, 1, 2]]
E a[3] = [[1, 1, 0], [0, 1, 2]]
E b[0] = [[1, 2]]
E b[1] = [[2, 0]]
E b[2] = [[1, 1]]
E b[3] = [[2, 2]]
E b[4] = [[1, 1]]
F dims = [1, 1, 1]
F a[0] = [[0]]
F a[1] = [[1]]
F a[2] = [[1]]
```

```
F a[3] = [[1]]
F b[0] = [[1]]
F b[1] = [[2]]
F b[2] = [[0]]
F b[3] = [[2]]
F b[4] = [[1]]
G dims = [1, 2, 4]
G a[0] = [[2], [2]]
G a[1] = [[1], [1]]
G a[2] = [[1], [2]]
G a[3] = [[0], [1]]
G b[0] = [[2, 2], [1, 0], [0, 1], [2, 0]]
G b[1] = [[2, 2], [2, 1], [2, 1], [0, 0]]
G b[2] = [[0, 2], [1, 1], [2, 2], [0, 2]]
G b[3] = [[1, 2], [2, 0], [1, 0], [2, 1]]
G b[4] = [[1, 1], [1, 2], [1, 1], [0, 1]]
```

Characteristic 5

```
relations = [
  [3, 0, 3, 0, 4, 2, 1, 1, 3, 0, 4, 0, 2, 0, 2, 4, 4, 3, 0, 2],
  [4, 4, 2, 2, 1, 4, 4, 3, 4, 3, 0, 3, 3, 0, 2, 0, 0, 4, 0, 2],
  [0, 0, 3, 2, 4, 4, 3, 2, 4, 1, 2, 2, 2, 1, 3, 4, 1, 3, 0, 0],
  [1, 4, 2, 0, 1, 4, 2, 2, 1, 3, 4, 3, 4, 0, 1, 3, 4, 3, 4, 0],
  [0, 1, 3, 4, 2, 3, 4, 0, 2, 0, 1, 4, 3, 0, 4, 3, 0, 0, 4, 4],
  [1, 4, 4, 0, 0, 2, 0, 0, 2, 2, 0, 2, 4, 1, 1, 4, 2, 2, 0, 2],
  [3, 4, 2, 1, 0, 2, 0, 2, 2, 0, 3, 2, 0, 0, 2, 3, 0, 0, 2, 2],
]
E dims = [3, 2, 1]
E a[0] = [[3, 1, 1], [2, 3, 2]]
E a[1] = [[1, 2, 3], [0, 2, 3]]
E a[2] = [[4, 0, 1], [4, 2, 3]]
E a[3] = [[0, 4, 4], [4, 2, 2]]
E b[0] = [[0, 4]]
E b[1] = [[1, 2]]
E b[2] = [[2, 3]]
E b[3] = [[2, 1]]
E b[4] = [[0, 1]]
F dims = [1, 1, 1]
F a[0] = [[4]]
F a[1] = [[2]]
F a[2] = [[4]]
F a[3] = [[3]]
F b[0] = [[2]]
F b[1] = [[0]]
F b[2] = [[3]]
F b[3] = [[2]]
F b[4] = [[4]]
G dims = [1, 2, 4]
G a[0] = [[0], [2]]
G a[1] = [[4], [0]]
G a[2] = [[4], [4]]
G a[3] = [[2], [2]]
G b[0] = [[1, 0], [2, 1], [4, 3], [3, 4]]
G b[1] = [[4, 1], [4, 2], [2, 3], [1, 2]]
G b[2] = [[3, 4], [1, 0], [2, 1], [1, 1]]
G b[3] = [[4, 4], [2, 0], [2, 0], [2, 0]]
G b[4] = [[4, 2], [1, 0], [0, 1], [4, 3]]
```

Characteristic 7

```
relations = [
  [1, 6, 4, 5, 1, 0, 4, 3, 5, 2, 6, 6, 2, 5, 2, 3, 4, 3, 1, 3],
  [0, 4, 1, 6, 6, 2, 4, 2, 3, 2, 5, 6, 4, 3, 2, 6, 1, 6, 0, 5],
  [4, 5, 5, 4, 2, 1, 4, 6, 2, 6, 0, 2, 1, 4, 5, 6, 3, 0, 0, 5],
  [6, 4, 4, 6, 4, 0, 6, 1, 1, 5, 1, 3, 1, 1, 3, 3, 6, 0, 1, 3],
```

```
  [5, 3, 0, 5, 3, 2, 4, 1, 5, 3, 3, 5, 3, 3, 1, 3, 3, 6, 1, 2],
  [0, 5, 5, 2, 1, 4, 5, 3, 2, 3, 0, 4, 4, 1, 0, 5, 0, 3, 1, 2],
  [0, 3, 6, 2, 1, 1, 5, 6, 2, 0, 4, 0, 0, 2, 5, 4, 1, 1, 4, 3],
]
E dims = [3, 2, 1]
E a[0] = [[2, 1, 5], [3, 2, 6]]
E a[1] = [[6, 4, 4], [5, 0, 6]]
E a[2] = [[2, 5, 4], [2, 6, 1]]
E a[3] = [[1, 3, 3], [2, 6, 2]]
E b[0] = [[4, 1]]
E b[1] = [[5, 1]]
E b[2] = [[6, 3]]
E b[3] = [[5, 4]]
E b[4] = [[2, 4]]
F dims = [1, 1, 1]
F a[0] = [[5]]
F a[1] = [[0]]
F a[2] = [[3]]
F a[3] = [[6]]
F b[0] = [[1]]
F b[1] = [[3]]
F b[2] = [[1]]
F b[3] = [[1]]
F b[4] = [[6]]
G dims = [1, 2, 4]
G a[0] = [[4], [3]]
G a[1] = [[0], [3]]
G a[2] = [[2], [6]]
G a[3] = [[0], [5]]
G b[0] = [[6, 6], [6, 1], [4, 4], [3, 6]]
G b[1] = [[0, 3], [5, 4], [1, 0], [6, 0]]
G b[2] = [[3, 0], [0, 1], [4, 4], [2, 5]]
G b[3] = [[4, 1], [0, 3], [3, 5], [5, 2]]
G b[4] = [[4, 2], [2, 1], [0, 5], [0, 6]]
```

B Exact verification program

The following Python program uses SymPy 1.14 and the five JSON files matching Appendix A. Run `python verify_inputs.py` in the supplementary directory. It reconstructs the maps using column vectorization, recomputes the integer Smith forms and the finite-field ranks, verifies the module equations and complex conditions, and checks the extension table. For a matrix F, column vectorization stacks its columns and obeys $\mathrm{vec}(AFB) = (B^{\mathsf{T}} \otimes A)\,\mathrm{vec}(F)$, where $\otimes$ on matrices is the Kronecker product. The supplementary files also include the integer and minor certificates and their verifier, and `build_counterexample.py`, which implements (8) and checks its exactness and augmentation. The finite computations enter the proof only through Proposition 3.1.

```
"""Exact check of the five inputs; requires SymPy 1.14."""
from pathlib import Path
import json
import sympy as s
from sympy.matrices.normalforms import smith_normal_form
from sympy.polys.domains import ZZ, GF
from sympy.polys.matrices import DomainMatrix
ROOT = Path(__file__).resolve().parent

def require(ok):
    if not ok:
        raise ValueError('verification failed')

def zero(M, p):
    return all(int(v) % p == 0 for v in M) if p else not any(M)

def rank(M, p):
```

```
    if not p:
        return M.rank()
    return DomainMatrix.from_Matrix(M).convert_to(GF(p)).rank()

def smith(M):
    D = smith_normal_form(M, domain=ZZ)
    return [abs(int(D[i, i])) for i in range(min(D.shape))
            if D[i, i]]

def complex_matrices(X, Y, R):
    x, y = X['dims'], Y['dims']
    c0 = [x[i]*y[i] for i in range(3)]
    c1 = [x[0]*y[1]]*4 + [x[1]*y[2]]*5
    o0 = [0] + [sum(c0[:i+1]) for i in range(3)]
    o1 = [0] + [sum(c1[:i+1]) for i in range(9)]
    d0 = s.zeros(sum(c1), sum(c0))
    d1 = s.zeros(7*x[0]*y[2], sum(c1))
    K = s.kronecker_product
    for a in range(9):
        kind, i, j, z = ('a', 0, 1, a) if a < 4 else (
            'b', 1, 2, a-4)
        d0[o1[a]:o1[a+1], o0[i]:o0[i+1]] = K(
            s.eye(x[i]), Y[kind][z])
        d0[o1[a]:o1[a+1], o0[j]:o0[j+1]] = -K(
            X[kind][z].T, s.eye(y[j]))
    z = x[0]*y[2]
    for l in range(7):
        for j in range(5):
            for i in range(4):
                c = R[l, 4*j+i]
                d1[l*z:(l+1)*z, o1[i]:o1[i+1]] += c*K(
                    s.eye(x[0]), Y['b'][j])
                d1[l*z:(l+1)*z, o1[4+j]:o1[5+j]] += c*K(
                    X['a'][i].T, s.eye(y[2]))
    return d0, d1

def expected(a, b, p):
    if a == b:
        return [1, 0, 0]
    if a > b:
        return [0, 0, 0]
    return {'EF': [0, 1, 6] if p == 2 else [0, 0, 5],
            'EG': [0, 0, 31],
            'FG': [0, 1, 8] if p == 43 else [0, 0, 7]}[a+b]

# Nonunit Smith entries, in order; ranks determine the unit entries.
NONUNITS = {
 'EE': ([2]*3, [2,6,6,84,84]),
 'EF': ([], [6]*4), 'EG': ([], [2]*7+[6]*4+[30]),
 'FE': ([2], [3,30]), 'FF': ([], [3,6]),
 'FG': ([], [3,3,3,129]), 'GE': ([2], [2,2,4]),
 'GF': ([], [2,2,2,4]), 'GG': ([], [2]*14+[4,4,20,20])}

for p in [0, 2, 3, 5, 7]:
    name = 'char0_original.json' if p == 0 else f'char{p}.json'
    data = json.loads((ROOT/'inputs'/name).read_text())
    R = s.Matrix(data['relations'])
    require(R.shape == (7, 20) and rank(R, p) == 7)
    if p == 0:
        require(smith(R) == [1]*7)
    modules = data['modules']
    for name, X in modules.items():
        require(X['dims'] == {'E':[3,2,1], 'F':[1,1,1],
                              'G':[1,2,4]}[name])
        for kind in ['a', 'b']:
            X[kind] = list(map(s.Matrix, X[kind]))
        for l in range(7):
            value = s.zeros(X['dims'][2], X['dims'][0])
```

```
            for j in range(5):
                for i in range(4):
                    value += R[l,4*j+i]*X['b'][j]*X['a'][i]
            require(zero(value, p))
    for a, X in modules.items():
        for b, Y in modules.items():
            d0, d1 = complex_matrices(X, Y, R)
            require(zero(d1*d0, p))
            r0, r1 = rank(d0, p), rank(d1, p)
            ext = [d0.cols-r0, d0.rows-r0-r1, d1.rows-r1]
            require(ext == expected(a, b, p))
            if p == 0:
                v0, v1 = smith(d0), smith(d1)
                require(len(v0) == r0 and len(v1) == r1)
                require(([v for v in v0 if v > 1],
                         [v for v in v1 if v > 1]) == NONUNITS[a+b])
                r0, r1 = (sum(v % 43 != 0 for v in vs)
                          for vs in [v0, v1])
                require([d0.cols-r0, d0.rows-r0-r1,
                         d1.rows-r1] == expected(a, b, 43))
            print(p, a+b, ext, flush=True)
print('ALL FIVE INPUTS AND THE INTEGER SMITH TABLE VERIFIED')
```

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