

A VARIATIONAL FORMULA FOR AFFINE SEMICIRCULAR DETERMINANTS

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ABSTRACT. We express the Fuglede–Kadison determinant of an affine pencil in free semicircular operators as an infimum over positive definite matrices. The infimum is attained if and only if the pencil is boundedly invertible, in which case its minimizer is unique. At zero constant term, a scalar rescaling recovers the determinant-capacity identity of Mai and Speicher. For selfadjoint coefficients we compare the formula with a strictly constrained Hilbert–Schmidt variational principle and establish ordered min–max formulas under their respective positivity hypotheses. Strictly positive pencils admit a minimum over a linear matrix inequality. Applications to Brown measure include a uniform determinant lower bound, restrictions on atoms, radial distribution formulas, and a Hessian formula for the regularized density. The text of this manuscript was generated entirely by OpenAI’s large language model Astra.

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1. INTRODUCTION

Disclosure of AI assistance. The mathematical content and text of this manuscript were generated entirely by an autonomous exploration system driven by OpenAI’s large language model Astra under human guidance. No part of the work has been checked and understood by the author. This manuscript records our exploration and its findings. We share it in its current form to invite feedback and collaboration from interested readers.

This work was motivated by a question posed by Roland Speicher in his blog [Spe26]: whether the Fuglede–Kadison determinant formula for centered operator-valued semicircular elements extends to arbitrary mean. It is also an experiment in doing mathematics with AI.

The Fuglede–Kadison determinant extends the absolute determinant of a finite matrix to operators in a finite tracial von Neumann algebra. For a matrix-valued semicircular element with zero mean, Mai and Speicher [MS24, Theorem 1.2] express this determinant through the capacity of a completely positive map determined by the coefficients. A deterministic term changes the

question substantially. Such terms arise in particular when one studies the determinant of $T - zI$ as a function of the spectral parameter z .

In this paper we express the Fuglede–Kadison determinant of an affine semicircular pencil with complex matrix coefficients as an infimum over positive definite matrices. We prove that the infimum is attained exactly when the pencil is invertible, and determine the unique minimizer in that case.

1.1. Setting and the coefficient maps. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with faithful normal tracial state. Throughout, $s_1, \dots, s_n \in \mathcal{M}$ are freely independent standard semicircular elements. Each has spectral density $(2\pi)^{-1}\sqrt{4-t^2}$ on $[-2, 2]$. Freeness means that the trace of a product of centered elements from the generated unital algebras vanishes whenever adjacent factors come from different algebras; see [MS17, Section 1.11 and Chapter 2, Section 2].

For $m \geq 1$, write Tr_m for the ordinary matrix trace, $\text{tr}_m = \text{Tr}_m/m$, $\varphi_m = \text{tr}_m \otimes \tau$, and $E_m = \text{id}_{M_m(\mathbb{C})} \otimes \tau$. Subscripts on Tr and φ will be omitted when the size is fixed. The notation $b > 0$ means that the finite matrix b is positive definite, while $b \geq 0$ means positive semidefinite. The identity matrix of size m is written 1_m , following [MS24, Section 2.2].

Definition 1.1 (Determinant and capacity). For $Y \in M_m(\mathcal{M})$, let $\mu_{|Y|}$ be the spectral probability measure of $|Y| = (Y^*Y)^{1/2}$ under φ_m . We use the Fuglede–Kadison determinant

$$(1.1) \quad \Delta(Y) = \exp \left(\int_{[0, \infty)} \log t \, d\mu_{|Y|}(t) \right), \quad \log 0 = -\infty, \quad \exp(-\infty) = 0.$$

For $b \in M_m(\mathbb{C})$, put $\Delta_m(b) = |\det b|^{1/m}$. For a completely positive map $\eta : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$, its capacity is

$$(1.2) \quad \text{cap}(\eta) = \inf_{b > 0} \frac{\det \eta(b)}{\det b}.$$

These are the conventions of [MS24, Definitions 1.1 and 2.1]; see also [FK52, Gur04]. Complete positivity means that every matrix amplification of the map preserves positivity. In particular, maps of the form $b \mapsto \sum_j a_j b a_j^*$ have this property [MS24, Section 2.2]. The quotient in (1.2) is the homogeneous form of the definition with $\det b = 1$. Capacity also connects this determinant problem with operator scaling and noncommutative rank; see [GGOW20, Sections 1.1–1.3]. The homogeneous function $\log \det \eta(b) - \log \det b$ is studied by geodesically convex optimization in [AZGL⁺18, Section 1.1].

Definition 1.2 (Affine semicircular pencil). For $a_0, a_1, \dots, a_n \in M_m(\mathbb{C})$, an affine semicircular pencil is the operator

$$(1.3) \quad S = a_0 + \sum_{j=1}^n a_j \otimes s_j \in M_m(\mathcal{M}).$$

Here a_0 denotes $a_0 \otimes 1_{\mathcal{M}}$. We use the same convention for other matrices in $M_m(\mathbb{C})$. We associate to the coefficients $a_1, \dots, a_n$ the complex-linear maps

$$(1.4) \quad \eta(b) = \sum_{j=1}^n a_j b a_j^*, \quad \rho(b) = \sum_{j=1}^n a_j b a_j, \quad b \in M_m(\mathbb{C}).$$

This terminology includes nonselfadjoint coefficients, as does the semicircular matrix convention in [MS24, p. 3]. When we require selfadjoint coefficients, we state that assumption separately. The maps η and ρ follow the notation of [MS24, Theorem 1.2 and Lemma 3.5(ii)].

At $a_0 = 0$, the identity of Mai and Speicher is

$$(1.5) \quad \Delta \left(\sum_j a_j \otimes s_j \right) = e^{-1/2} \text{cap}(\eta)^{1/(2m)}.$$

For a nonzero constant term, a_0 and η alone do not determine the determinant. Indeed, let s be standard semicircular and $t > 2$. The scalar pencils $t + s$ and $t + is$ have the same constant term and the same map $\eta(b) = b$, but

$$(1.6) \quad \Delta(t + s) < t < \Delta(t + is).$$

Strict concavity of the logarithm gives $\tau \log(t + s) < \log t$, whereas $\frac{1}{2} \tau \log(t^2 + s^2) > \log t$. Their maps ρ are respectively $b \mapsto b$ and $b \mapsto -b$. The general formula below retains this additional information. Its appearance can also be seen directly in the covariance of the selfadjoint block operator associated with S .

1.2. The general formula and attainment. For each $b > 0$, define

$$(1.7) \quad H_b = 1_m + \eta(b), \quad \mathcal{T}_b(Y) = H_b Y b^{-1} + \rho(Y^*), \quad Y \in M_m(\mathbb{C}).$$

This is a real-linear map on the real vector space of complex matrices. Its invertibility for every $b > 0$ is part of the next theorem.

Theorem 1.3. *Let S be the pencil in Definition 1.2, with arbitrary complex coefficient matrices. For every $b > 0$, the map $\mathcal{T}_b$ in (1.7) is invertible as a real-linear map on $M_m(\mathbb{C})$. Write $Z_b = \mathcal{T}_b^{-1}(a_0)$. Then*

$$(1.8) \quad \Delta(S) = e^{-1/2} \inf_{b>0} \left[\frac{\det(1_m + \eta(b))}{\det b} \right]^{1/(2m)} \exp \left[\frac{1}{2m} \text{Re Tr}(a_0^* Z_b) \right].$$

Define

$$(1.9) \quad \mathcal{V}(b) = \log \det H_b - \log \det b - m + \text{Re Tr}(a_0^* Z_b).$$

The following conditions are equivalent:

- (i) the infimum in (1.8) is attained;
- (ii) $\mathcal{V}$ has a critical point on the positive definite cone;
- (iii) S is invertible in the bounded operator algebra $M_m(\mathcal{M})$.

When these conditions hold, let b_* denote the unique minimizer, which is also the unique critical point of $\mathcal{V}$, and put $Z_{b_*} = \mathcal{T}_{b_*}^{-1}(a_0)$. Then

$$(1.10) \quad b_* = E_m((S^* S)^{-1}), \quad Z_{b_*} = E_m((S^*)^{-1}).$$

Equivalently,

$$\log \Delta(S) = \frac{1}{2m} \inf_{b>0} \mathcal{V}(b).$$

The minimizers are the same as in (1.8). Here a critical point means that the first variation of $\mathcal{V}$ vanishes in every selfadjoint matrix direction.

At zero constant term, $Z_b = 0$ for every trial, and

$$(1.11) \quad \inf_{b>0} \frac{\det(1_m + \eta(b))}{\det b} = \text{cap}(\eta).$$

For the lower bound, use $1_m + \eta(tc) \geq \eta(tc)$ and determinant monotonicity. Conversely, for every fixed $c > 0$, the substitution $b = tc$ gives

$$\frac{\det(1_m + \eta(tc))}{\det(tc)} = \frac{\det(t^{-1} 1_m + \eta(c))}{\det c} \longrightarrow \frac{\det \eta(c)}{\det c}.$$

Taking the infimum over c proves (1.11). Hence our formula recovers (1.5) directly. The proof of Theorem 1.3 uses the semicircular Dyson equation, rather than the centered determinant identity, to establish the general formula.

1.3. Other variational descriptions. Throughout this subsection assume $a_j = a_j^*$ for $0 \leq j \leq n$. We first prove a strict-domain version of the formula proposed in Speicher's public discussion [Spe26]. For $c > 0$, put $\mathcal{C}_c(x) = cxc$ and

$$(1.12) \quad \mathcal{B}_\eta = \{c > 0 : \mathcal{C}_c - \eta > 0 \text{ on } M_m(\mathbb{C})\}.$$

Here strict positivity is with respect to the Hilbert–Schmidt inner product $\langle x, y \rangle = \text{Tr}(x^*y)$.

Proposition 1.4 (Strict Hilbert–Schmidt formula). *For every $c \in \mathcal{B}_\eta$, the map $\mathcal{C}_c + \eta$ is invertible. Put $h_c = (\mathcal{C}_c + \eta)^{-1}(a_0)$. Then*

$$(1.13) \quad \Delta(S) = e^{-1/2} \inf_{c \in \mathcal{B}_\eta} \Delta_m(c) \exp\left(\frac{1}{2m} \text{Tr}(a_0 h_c)\right).$$

Section 4 proves this formula and compares its trial domain and quadratic correction with those of Theorem 1.3.

Strictly positive pencils admit a minimum over a linear matrix inequality.

Definition 1.5 (Feasible matrices). For $b > 0$, put

$$(1.14) \quad L(b) = \begin{pmatrix} a_0 - \eta(b) & 1_m \\ 1_m & b \end{pmatrix}.$$

The feasible and strictly feasible sets are respectively

$$(1.15) \quad \mathcal{B} = \{b > 0 : L(b) \geq 0\}, \quad \mathcal{B}_{\text{str}} = \{b > 0 : L(b) > 0\}.$$

Define

$$(1.16) \quad J(b) = -\log \det b + \text{Tr}(a_0 b) - \frac{1}{2} \text{Tr}(b\eta(b)) - m.$$

Theorem 1.6 (Strictly positive pencils). *The pencil S is strictly positive, meaning $S \geq \gamma 1_m$ for some $\gamma > 0$, if and only if $\mathcal{B}_{\text{str}}$ is nonempty. Under this hypothesis,*

$$(1.17) \quad \Delta(S) = \exp\left(\frac{1}{m} \min_{b \in \mathcal{B}} J(b)\right) = \exp\left(\frac{1}{m} \inf_{b \in \mathcal{B}_{\text{str}}} J(b)\right).$$

The minimum is attained uniquely at $q = E_m(S^{-1})$, which satisfies

$$q^{-1} = a_0 - \eta(q), \quad q \leq b \quad (b \in \mathcal{B}).$$

The infimum on $\mathcal{B}_{\text{str}}$ is not attained.

Section 5 proves the theorem and relates q to the minimizer $b_* = E_m(S^{-2})$ in Theorem 1.3.

Section 6 gives ordered min–max formulas for scalar pencils, commuting coefficients and isotropic covariance, and a separate matrix formula when η is strictly positive as a Hilbert–Schmidt operator.

1.4. Consequences for Brown measure. Let $T = a_0 + \sum_j a_j \otimes s_j$. The Brown measure μ_T is the probability measure whose logarithmic potential is $u(z) = \log \Delta(T - z1_m)$; the precise convention is recalled in Section 7, following [MS17, Chapter 11, Section 9]. Spectral shifts in Theorem 1.3 therefore give u and the spectral criterion

$$z \notin \text{spec}(T) \iff \text{the infimum for } T - z1_m \text{ is attained.}$$

Theorem 1.3 and the nonnegative quadratic correction in (1.9) give the following consequence.

Corollary 1.7 (Capacity bound and absence of atoms). *For every $z \in \mathbb{C}$,*

$$(1.18) \quad \Delta(T - z1_m) \geq e^{-1/2} \text{cap}(\eta)^{1/(2m)}.$$

If $\text{cap}(\eta) > 0$, then μ_T has no atoms.

The proof is given in Section 7.2, where Proposition 7.2 also shows that every atom of μ_T lies in $\text{spec}(a_0)$ without a capacity assumption.

When both a_0 and ρ vanish, Theorem 1.3 determines the radial distribution and the operator spectrum. This includes linear combinations of free circular elements with matrix coefficients; see Example 7.3. Write $r(\eta)$ for the spectral radius of η , the maximum modulus of its eigenvalues.

Proposition 1.8 (Radial Brown measure). *Assume $a_0 = 0$ and $\rho = 0$. The logarithmic potential has the form $u(z) = f(|z|^2)$, where f is finite, continuously differentiable, nondecreasing and concave on $(0, \infty)$. Its radial distribution is*

$$(1.19) \quad F(s) := \mu_T(\{|z|^2 \leq s\}) = 2sf'(s), \quad s > 0.$$

The measure μ_T is absolutely continuous with respect to planar area on $\mathbb{C} \setminus \{0\}$. If $\text{cap}(\eta) > 0$, it is absolutely continuous on all of $\mathbb{C}$. Moreover,

$$(1.20) \quad \text{spec}(T) = \{z \in \mathbb{C} : |z|^2 \leq r(\eta)\}.$$

Section 7.3 proves this proposition and gives the distribution in terms of regularized minimizers, together with bounds for the density. For general coefficients, Section 7.5 computes the Laplacian of the regularized logarithmic potential through the Hessian at the minimizer.

1.5. Proof mechanism and organization. The proof of the general formula starts with a regularized variational formula. The function being minimized has compact sublevel sets in the positive definite cone. A positive-map argument shows that every critical point has the signs required by the selfadjoint block resolvent. Dyson uniqueness then identifies the global minimizer. Rescaling the trial variable removes the regularizer. For the attainment assertion, a critical point yields a strict spectral-radius estimate and an analytic continuation of the matrix Cauchy transform through zero. This proof is independent of the later Hilbert–Schmidt comparison and of the applications to Brown measure.

Section 2 establishes the common Dyson and spectral tools. Section 3 proves Theorem 1.3. Section 4 treats imaginary constant terms and the strict Hilbert–Schmidt comparison. Section 5 gives the order-theoretic formula for strictly positive pencils, and Section 6 establishes the ordered min–max principles. Section 7 develops the applications to spectra and Brown measures. Appendix A contains the global comparison argument used in Section 4.

2. PRELIMINARIES AND THE DYSON EQUATION

We use the determinant and trace conventions introduced in Section 1. In particular, all operator determinants use the normalized trace of the relevant matrix size. We use multiplicativity and the block-diagonal determinant rule as in [MS24, Proposition 2.3].

2.1. The matrix Cauchy transform. Consider first any selfadjoint pencil

$$K = A \otimes 1 + \sum_j B_j \otimes s_j \in M_d(\mathcal{M}), \quad A = A^*, \quad B_j = B_j^* \in M_d(\mathbb{C}),$$

Put $\Theta(b) = \sum_j B_j b B_j$ for $b \in M_d(\mathbb{C})$.

Definition 2.1 (Matrix Cauchy transform). The matrix-valued Cauchy transform of K is

$$G(w) = E_d[(w1_d - K)^{-1}], \quad \text{Im } w > 0.$$

We use the resolvent convention of [MS17, Chapter 9, (9.5)]. The operator-valued semicircular moment rule gives the *Dyson equation*

$$(2.1) \quad G(w)^{-1} = w1_d - A - \Theta(G(w)), \quad \text{Im } G(w) < 0.$$

Here $\text{Im } G = (G - G^*)/(2i)$, and the inequality is negative definiteness. The source input is [MS17, Chapter 9, Definition 10, Theorem 11 and Proposition 13]. The moment recursion is

$$Q_0 = 1_d, \quad Q_k = AQ_{k-1} + \sum_{r+s=k-2} \Theta(Q_r)Q_s, \quad Q_k = E_d[K^k].$$

The empty sum is zero. Inserting this recursion into the convergent large- $|w|$ resolvent series proves (2.1) there; analytic continuation proves it throughout the upper half-plane.

The imaginary-part condition selects exactly one solution at each w . With $W = iG$ and $V = -i(w1_d - A)$, the equation takes the positive-real-part form $VW + \Theta(W)W = 1_d$ studied in [HRFS07, Theorem 2.1 and Section 3] by holomorphic contraction. We include a direct proof because its estimate also gives the invertibility needed when differentiating the Dyson equation. Suppose G_1, G_2 are solutions and put $V_j = -\text{Im } G_j > 0$, $v = \text{Im } w$. Taking imaginary parts of the inverse equation gives

$$(2.2) \quad V_j = G_j(v1_d + \Theta(V_j))G_j^* = G_j^*(v1_d + \Theta(V_j))G_j.$$

For $D = G_1 - G_2$ we have $D = G_1\Theta(D)G_2$. After setting

$$W = V_1^{-1/2}DV_2^{-1/2}, \quad L_i = V_1^{-1/2}G_1B_iV_1^{1/2}, \quad R_i = V_2^{1/2}B_iG_2V_2^{-1/2},$$

this becomes $W = \sum_i L_i W R_i$. Equation (2.2) implies $\sum_i L_i L_i^* < 1_d$ and $\sum_i R_i^* R_i < 1_d$. The row and column operator norm estimate therefore gives

$$\|W\| \leq \left\| \sum_i L_i L_i^* \right\|^{1/2} \left\| \sum_i R_i^* R_i \right\|^{1/2} \|W\| < \|W\|$$

unless $W = 0$. This proves uniqueness. We call this selected solution the *resolvent solution*; no choice among real algebraic roots is implicit in the notation. Taking $G_1 = G_2 = G$ in the same estimate shows that $X \mapsto G\Theta(X)G$ is a strict contraction in the norm $\|V^{-1/2}XV^{-1/2}\|$, where $V = -\text{Im } G$. Consequently $X \mapsto X - G\Theta(X)G$ is invertible. We use uniqueness to identify the variational critical points in Sections 3 and 6, and this invertibility for the derivatives in Section 7.5.

2.2. Integrating the Cauchy transform. The covariance is symmetric for the bilinear trace pairing: $\text{Tr}(\Theta(X)Y) = \text{Tr}(X\Theta(Y))$. Choose the analytic logarithm of $\det G(w)$ compatible with $G(w) \sim w^{-1}1_d$ at infinity. Then

$$(2.3) \quad F(w) = -\frac{1}{d} \log \det G(w) + \frac{1}{2d} \text{Tr}(\Theta(G(w))G(w)) = \varphi_d(\text{Log}(w1_d - K)).$$

To prove this identity, differentiate (2.1) and take the trace to obtain

$$\text{Tr } G = -\text{Tr}(G^{-1}G') + \text{Tr}(\Theta(G)G').$$

Thus both sides of (2.3) have derivative $\text{tr}_d G$. Both are $\log w + o(1)$ as $w = it$ tends to infinity, which fixes the additive constant. Taking real parts gives

$$(2.4) \quad \log \Delta(K - w1_d) = -\frac{1}{d} \log |\det G(w)| + \frac{1}{2d} \text{Re } \text{Tr}(\Theta(G(w))G(w)).$$

The identity extends to a real spectral gap by continuity. It fixes the value at a resolvent solution; an extremum over free trial matrices requires a separate argument. Related finite log-determinant expressions, evaluated at solutions of deterministic resolvent equations, occur in the random-matrix deterministic equivalents of [HLN07, Theorem 4.1]. The shared method is integration of the resolvent; the minimization over arbitrary positive matrix trials is established in Section 3.

2.3. Selfadjoint blocks and regularization. For the general pencil S define its hermitization by

$$(2.5) \quad \mathcal{H} = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_0 \\ a_0^* & 0 \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & a_j \\ a_j^* & 0 \end{pmatrix}.$$

This is a selfadjoint pencil of size $2m$; see [MS24, Section 3] and [MS17, Chapter 11, Section 9] for this block construction. Put $\Theta(b) = \sum_j B_j b B_j$ and

$$\eta^*(b) = \sum_j a_j^* b a_j, \quad \rho^*(b) = \sum_j a_j^* b a_j^*.$$

These are the Hilbert–Schmidt adjoints of η and ρ for $\langle b, c \rangle = \text{Tr}(b^* c)$. We follow the notation of [MS24, Lemma 3.5(ii)]. Direct multiplication gives

$$(2.6) \quad \Theta \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \eta(b_{22}) & \rho(b_{21}) \\ \rho^*(b_{12}) & \eta^*(b_{11}) \end{pmatrix}.$$

This displays the role of ρ anticipated in the introduction.

For $\varepsilon > 0$, set

$$(2.7) \quad \log \Delta_\varepsilon(S) = \frac{1}{2} \varphi_m \log(S^* S + \varepsilon^2 1_m).$$

The operators $S^* S$ and $S S^*$ have the same tracial spectral distribution, including at zero, by polar decomposition and traciality. Since $\mathcal{H}^2 = \text{diag}(S S^*, S^* S)$, the resolvent $G_\varepsilon = E_{2m}((i\varepsilon 1_{2m} - \mathcal{H})^{-1})$ satisfies

$$(2.8) \quad \log \Delta_\varepsilon(S) = -\frac{1}{2m} \log |\det G_\varepsilon| + \frac{1}{4m} \text{Re Tr}(\Theta(G_\varepsilon) G_\varepsilon).$$

This follows from (2.4). Monotone spectral integration gives $\log \Delta_\varepsilon(S) \downarrow \log \Delta(S)$, also when the limit is $-\infty$.

2.4. Positive maps and spectral gaps. We record two tools for identifying spectral gaps from the matrix Cauchy transform. For a linear map on a finite-dimensional space, write $r(\Psi)$ for its spectral radius, the largest modulus of its eigenvalues. If $\Psi(b) = \sum_j c_j b c_j^*$ on $M_m(\mathbb{C})$, then the row operator $[c_1 \ \cdots \ c_n]$ gives

$$\|\Psi\| = \|\Psi(1_m)\|.$$

Indeed it gives the upper bound, and evaluation at 1_m gives equality. If $W > 0$ and $W - \Psi(W) > 0$, conjugating the map by $b \mapsto W^{1/2} b W^{1/2}$ makes its value at the identity strictly less than the identity. Its spectral radius is therefore less than one. In finite dimension this implies convergence of $(\text{id} - \Psi)^{-1} = \sum_{k \geq 0} \Psi^k$, which preserves positivity. This is the finite-dimensional positive-map form of the strict Stein criterion; see [Pop03, Lemma 5.5 and Theorem 5.9] for its relation to similarity to a strict contraction.

Next let $K = K^* \in M_d(\mathcal{M})$, with scalar spectral measure ν_K , and let $g(w) = E_d((w 1_d - K)^{-1})$. For real $a < b$,

$$(2.9) \quad -\lim_{v \downarrow 0} \frac{1}{\pi} \int_a^b \text{Im tr}_d g(x + iv) dx = \nu_K((a, b)) + \frac{1}{2} \nu_K(\{a, b\}),$$

$$(2.10) \quad -\lim_{v \downarrow 0} v \text{Im tr}_d g(x + iv) = \nu_K(\{x\}).$$

To verify the first formula, integrate the kernel $v/((x-t)^2 + v^2)$ over $x \in (a, b)$ and then over ν_K . The normalized inner integral is bounded by one and tends to the indicator of (a, b) with half weights at its endpoints. The second formula follows in the same way from $v^2/((x-t)^2 + v^2)$. Dominated convergence proves both assertions.

Faithfulness implies $\text{supp } \nu_K = \text{spec}(K)$: every nonzero spectral projection has positive trace. Consequently, if the matrix Cauchy transform g continues analytically across a real interval and has

selfadjoint values there, that interval is a spectral gap. On its compact subintervals the imaginary part tends uniformly to zero, so (2.9) gives zero mass. The converse follows from the operator resolvent. This criterion concerns continuation of the matrix Cauchy transform, not an arbitrary solution of the matrix equation at a real parameter.

3. PROOF OF THE GENERAL FORMULA AND ATTAINMENT

We retain the notation of Theorem 1.3 and the block covariance in (2.6). All traces in this section are unnormalized matrix traces. Adjoints of maps are taken for the Hilbert–Schmidt inner product unless stated otherwise.

3.1. A coercive regularization. For $\varepsilon > 0$ and $c > 0$, put

$$C = \varepsilon^2 1_m + \eta(c), \quad \mathcal{L}_{\varepsilon,c}(Y) = CYc^{-1} + \rho(Y^*).$$

On the real vector space $M_m(\mathbb{C})$ use the inner product $\langle X, Y \rangle_{\mathbb{R}} = \operatorname{Re} \operatorname{Tr}(X^*Y)$. The map $\mathcal{L}_{\varepsilon,c}$ is selfadjoint for this inner product, and its quadratic form is

$$(3.1) \quad B_{\varepsilon,c}(Y) = \operatorname{Tr}(Y^*CYc^{-1}) + \operatorname{Re} \operatorname{Tr}(Y^*\rho(Y^*)).$$

For the second summand, symmetry follows from cyclicity of the trace: $\operatorname{Re} \operatorname{Tr}(X^*a_jY^*a_j) = \operatorname{Re} \operatorname{Tr}(Y^*a_jX^*a_j)$. To check positivity, set $D_j = c^{-1/2}Y^*a_jc^{1/2}$. Then

$$|\operatorname{Tr}(D_j^2)| \leq \operatorname{Tr}(D_j^*D_j) = \operatorname{Tr}(Y^*a_jca_j^*Yc^{-1}).$$

The inequality is the Hilbert–Schmidt Cauchy–Schwarz inequality applied to D_j^* and D_j . Consequently,

$$(3.2) \quad B_{\varepsilon,c}(Y) \geq \varepsilon^2 \operatorname{Tr}(Y^*Yc^{-1}) > 0 \quad (Y \neq 0).$$

Thus $\mathcal{L}_{\varepsilon,c}$ is invertible as a real-linear map on $M_m(\mathbb{C})$. Its unique solution $Y = Y_{\varepsilon,c}$ at right-hand side a_0 depends smoothly on c . In particular, $\varepsilon = 1$, $c = b$ proves the asserted invertibility of $\mathcal{T}_b$.

Define

$$(3.3) \quad \mathcal{F}_{\varepsilon}(c) = \log \det C - \log \det c + \operatorname{Tr} c - m + \operatorname{Re} \operatorname{Tr}(a_0^*Y_{\varepsilon,c}).$$

Completing the square in the real inner product gives

$$(3.4) \quad \operatorname{Re} \operatorname{Tr}(a_0^*Y_{\varepsilon,c}) = \max_Y \{2 \operatorname{Re} \operatorname{Tr}(a_0^*Y) - B_{\varepsilon,c}(Y)\} \geq 0.$$

Hence

$$\mathcal{F}_{\varepsilon}(c) \geq m \log \varepsilon^2 + \operatorname{Tr} c - \log \det c - m.$$

Since $t - \log t \rightarrow +\infty$ as $t \downarrow 0$ or $t \rightarrow \infty$, every sublevel set is contained in $\{\alpha 1_m \leq c \leq \beta 1_m\}$ for some $0 < \alpha \leq \beta < \infty$. Continuity then shows that $\mathcal{F}_{\varepsilon}$ attains a global minimum in the positive cone.

3.2. Critical points and the resolvent solution. We now identify every critical point of $\mathcal{F}_{\varepsilon}$. In this subsection $Y = Y_{\varepsilon,c}$. Differentiating the maximum in (3.4) requires only the explicit c -derivative: the derivative with respect to Y vanishes at its maximizer. Put

$$(3.5) \quad P = C^{-1} - Yc^{-1}Y^*.$$

This matrix is initially only known to be selfadjoint. Differentiating (3.3) in every selfadjoint direction gives

$$(3.6) \quad c^{-1} = 1_m + \eta^*(P) + c^{-1}Y^*CYc^{-1}.$$

Indeed, the terms involving the derivative of C combine to $\operatorname{Tr}(\eta^*(P)dc)$; the derivative of c^{-1} in the quadratic form contributes $\operatorname{Tr}(c^{-1}Y^*CYc^{-1}dc)$.

We next prove that P is positive definite. Set

$$U = C^{1/2}Yc^{-1/2}, \quad K = 1_m - U^*U, \quad W = 1_m - UU^*,$$

and define the completely positive map

$$\Phi(x) = C^{-1/2}\eta(c^{1/2}xc^{1/2})C^{-1/2}.$$

Then $P = C^{-1/2}WC^{-1/2}$, and (3.6) is equivalent to

$$(3.7) \quad K = c + \Phi^*(W), \quad \Phi(1_m) = 1_m - \varepsilon^2 C^{-1} < 1_m.$$

Choose a unitary extension V of the polar factor of the square matrix U , so that $U = V(U^*U)^{1/2}$ and $W = VKV^*$. For $\Lambda(x) = \Phi^*(VxV^*)$ we have $\Lambda^*(1_m) = V^*\Phi(1_m)V < 1_m$. The positive-map estimate in Section 2.4 implies that the spectral radius of Λ , equal to that of its adjoint, is less than one. Thus

$$K = (\text{id} - \Lambda)^{-1}(c) = \sum_{k \geq 0} \Lambda^k(c) > 0.$$

It follows that $W > 0$ and $P > 0$ as required.

Consider the invertible selfadjoint block matrix

$$\mathfrak{q} = \begin{pmatrix} P & Y \\ Y^* & -c \end{pmatrix}.$$

Its Schur complement is $P + Yc^{-1}Y^* = C^{-1}$, so

$$(3.8) \quad \mathfrak{q}^{-1} = \begin{pmatrix} C & CYc^{-1} \\ c^{-1}Y^*C & -c^{-1} + c^{-1}Y^*CYc^{-1} \end{pmatrix} = \begin{pmatrix} \varepsilon^2 1_m & a_0 \\ a_0^* & -1_m \end{pmatrix} - \Theta(\mathfrak{q}).$$

The upper right equality uses $CYc^{-1} + \rho(Y^*) = a_0$, and the lower diagonal equality is (3.6). Now set

$$(3.9) \quad \mathcal{R} = \begin{pmatrix} i\varepsilon P & Y \\ Y^* & ic/\varepsilon \end{pmatrix}.$$

Multiplying (3.8) by the corresponding diagonal scalars, or directly inverting (3.9), gives

$$\mathcal{R}^{-1} = A - \Theta(\mathcal{R}) - i\varepsilon 1_{2m}.$$

Since $\text{Im } \mathcal{R} > 0$, the matrix $-\mathcal{R}$ is the unique solution of (2.1) at $i\varepsilon$. Thus $\mathcal{R} = E_{2m}((\mathcal{H} - i\varepsilon 1_{2m})^{-1})$. The operator block inverse yields

$$(3.10) \quad \begin{aligned} P_\varepsilon &= E_m((SS^* + \varepsilon^2 1_m)^{-1}), \\ Y_\varepsilon &= E_m((SS^* + \varepsilon^2 1_m)^{-1}S), \\ c_\varepsilon &= \varepsilon^2 E_m((S^*S + \varepsilon^2 1_m)^{-1}). \end{aligned}$$

Every critical point therefore has the same $c = c_\varepsilon$. Since a minimum exists, c_ε is the unique minimizer and the unique critical point of $\mathcal{F}_\varepsilon$.

3.3. The minimum value and removal of regularization. All quantities in the next calculation are evaluated at (3.10). Write $\gamma_\rho = \text{Re } \text{Tr}(Y^*\rho(Y^*))$. The Schur complement gives $|\det \mathcal{R}| = \det c / \det C$, while block multiplication gives

$$\frac{1}{2} \text{Re } \text{Tr}(\Theta(\mathcal{R})\mathcal{R}) = -\text{Tr}(P\eta(c)) + \gamma_\rho.$$

Applying (2.8) to $G_\varepsilon = -\mathcal{R}$ therefore gives

$$(3.11) \quad 2m \log \Delta_\varepsilon(S) = \log \det C - \log \det c - \text{Tr}(P\eta(c)) + \gamma_\rho.$$

On the other hand, $\mathcal{L}_{\varepsilon,c}(Y) = a_0$ and the trace of (3.6) after multiplication by c give

$$\text{Re } \text{Tr}(a_0^*Y) = \text{Tr}(Y^*CYc^{-1}) + \gamma_\rho, \quad m = \text{Tr } c + \text{Tr}(P\eta(c)) + \text{Tr}(Y^*CYc^{-1}).$$

Substitution in (3.11) proves

$$(3.12) \quad 2m \log \Delta_\varepsilon(S) = \min_{c>0} \mathcal{F}_\varepsilon(c).$$

The substitution $c = \varepsilon^2 b$ is a bijection of the positive cone. It gives $C = \varepsilon^2 H_b$ and $\mathcal{L}_{\varepsilon,c} = \mathcal{T}_b$, hence

$$(3.13) \quad 2m \log \Delta_\varepsilon(S) = \min_{b>0} \{\mathcal{V}(b) + \varepsilon^2 \operatorname{Tr} b\}.$$

Let $\alpha = \inf_{b>0} \mathcal{V}(b) \in [-\infty, \infty)$. The right-hand side is at least α . For each fixed $b > 0$ it is at most $\mathcal{V}(b) + \varepsilon^2 \operatorname{Tr} b$, so its upper limit as $\varepsilon \downarrow 0$ is at most $\mathcal{V}(b)$. Taking the infimum over fixed trials proves convergence to α , also when $\alpha = -\infty$. Since the left-hand side decreases to $2m \log \Delta(S)$, we obtain $2m \log \Delta(S) = \inf_{b>0} \mathcal{V}(b)$. Exponentiating proves (1.8).

3.4. A critical point forces a spectral gap. Suppose now that $b > 0$ is a critical point of $\mathcal{V}$. Write $H = 1_m + \eta(b)$, $Y = Z_b$, and $P = H^{-1} - Yb^{-1}Y^*$. The same differentiation as above, with the term $\operatorname{Tr} c$ absent, gives

$$(3.14) \quad b^{-1} = \eta^*(P) + b^{-1}Y^*HYb^{-1}.$$

Set $U = H^{1/2}Yb^{-1/2}$ and use the same K, W, V, Φ as in (3.7), with $C = H$ and $c = b$. This time $K = \Phi^*(W) = \Lambda(K)$ and $\Phi(1_m) = 1_m - H^{-1} < 1_m$. Thus $K = 0$. Consequently U is unitary, $P = 0$, and Y is invertible. We have proved

$$(3.15) \quad b = Y^*(1_m + \eta(b))Y, \quad a_0 = Y^{-*} + \rho(Y^*),$$

where $Y^{-*} = (Y^*)^{-1}$.

These identities also give a spectral-radius bound on the full block space. Define the completely positive maps

$$\mathcal{A}(x) = Y^*\eta(x)Y, \quad \mathcal{B}(x) = Y\eta^*(x)Y^*.$$

The first identity in (3.15) says $b - \mathcal{A}(b) = Y^*Y > 0$, so the spectral radius of $\mathcal{A}$ is less than one. The map $\mathcal{B}$ is similar to $\mathcal{A}^* : x \mapsto \eta^*(YxY^*)$, by conjugation with the invertible map $x \mapsto YxY^*$. It too has spectral radius less than one. Hence

$$P_0 = (\operatorname{id} - \mathcal{B})^{-1}(YY^*) > 0, \quad P_0 - \mathcal{B}(P_0) = YY^*.$$

Put

$$(3.16) \quad G_0 = - \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix}, \quad D = \begin{pmatrix} P_0 & 0 \\ 0 & b \end{pmatrix} > 0, \quad \mathcal{S}(x) = G_0\Theta(x)G_0.$$

The second identity in (3.15) and its adjoint give $G_0^{-1} = -A - \Theta(G_0)$. Moreover,

$$(3.17) \quad D - \mathcal{S}(D) = G_0^2 = \operatorname{diag}(YY^*, Y^*Y) > 0.$$

The positive-map estimate therefore gives $r(\mathcal{S}) < 1$ on all of $M_{2m}(\mathbb{C})$.

We explain why the selfadjoint solution G_0 is the value at zero of the analytically continued matrix Cauchy transform. For invertible G near G_0 set

$$\mathcal{E}(w, G) = G^{-1} - w1_{2m} + A + \Theta(G).$$

Its derivative with respect to G at $(0, G_0)$ is the invertible complex-linear map

$$L(X) = -G_0^{-1}(\operatorname{id} - \mathcal{S})(X)G_0^{-1}.$$

This produces a unique local analytic solution $g(w)$ with $g(0) = G_0$. Indeed, on a sufficiently small closed ball about G_0 the map $G \mapsto G - L^{-1}\mathcal{E}(w, G)$ has derivative in G of norm at most $1/2$. For $|w|$ small it maps that ball into itself, because its value at the center tends to G_0 . Its iterates converge uniformly to the unique fixed point, and locally uniform convergence of the holomorphic iterates gives analyticity in w .

Adjoint symmetry and local uniqueness give $g(\bar{w}) = g(w)^*$. Differentiating the equation at zero and using (3.17) yields

$$(3.18) \quad g'(0) = -(\text{id} - S)^{-1}(G_0^2) = -D < 0.$$

Thus $\text{Im } g(i\varepsilon) = -\varepsilon D + O(\varepsilon^2) < 0$ for small $\varepsilon > 0$. Dyson uniqueness identifies $g(i\varepsilon)$ with the matrix Cauchy transform of $\mathcal{H}$. The identity theorem extends this equality to the upper half of a small disk. Since g is selfadjoint on the real diameter, the spectral-gap criterion in Section 2.4 shows that $\mathcal{H}$ is boundedly invertible. As $\mathcal{H}^2 = \text{diag}(SS^*, S^*S)$, this is equivalent to bounded invertibility of S .

3.5. Attainment and the unique minimizer. Conversely, suppose that S is boundedly invertible. The minimizing trial in (3.13) is

$$b_\varepsilon = E_m((S^*S + \varepsilon^2 1_m)^{-1}).$$

Norm continuity of inversion in a spectral gap gives

$$b_\varepsilon \longrightarrow b_* = E_m((S^*S)^{-1}) > 0, \quad Y_\varepsilon \longrightarrow E_m((S^*)^{-1}).$$

The matrices b_ε stay bounded, so $\varepsilon^2 \text{Tr } b_\varepsilon \rightarrow 0$. Passing to the limit in (3.13) and using continuity of $\mathcal{V}$ shows that b_* attains its infimum. Continuity of $\mathcal{T}_b^{-1}(a_0)$ gives $Z_{b_*} = E_m((S^*)^{-1})$.

Every attained minimum lies in the open positive cone and is a critical point of the smooth real function $\mathcal{V}$. This also applies to the expression minimized in (1.8), which equals $\exp(\mathcal{V}(b)/(2m))$ for every $b > 0$. The preceding subsection has shown that every critical point forces invertibility. It also determines that critical point uniquely: once S is invertible, the continued matrix Cauchy transform satisfies

$$G_0 = -E_{2m}(\mathcal{H}^{-1}), \quad g'(0) = -E_{2m}(\mathcal{H}^{-2}).$$

The upper right block of the first identity determines $Y = E_m((S^*)^{-1})$, and the lower diagonal block of (3.18) determines $b = E_m((S^*S)^{-1})$. This proves all equivalences and the uniqueness assertion in Theorem 1.3.

Remark 3.1 (Capacity and attainment). Attainment of homogeneous capacity is related to exact operator scaling [Gur04, Theorem 4.7], and is a different question from attainment in Theorem 1.3. For $m = 1$, $a_0 = 0$ and $\eta(b) = b$, the homogeneous quotient $\eta(b)/b$ is identically one, whereas $(1 + \eta(b))/b = 1 + 1/b$ reaches its infimum only as $b \rightarrow \infty$. The corresponding standard semicircular element is not invertible.

4. IMAGINARY CONSTANT TERMS AND THE STRICT HILBERT–SCHMIDT FORMULA

Throughout this section $a_j = a_j^*$ for $1 \leq j \leq n$. We write

$$T = \sum_j a_j \otimes s_j = T^*, \quad \eta(x) = \sum_j a_j x a_j, \quad S = a_0 + T,$$

For the formula for $S = a_0 + T$, we also assume $a_0 = a_0^*$. The same positive-cone minimization will give a formula for $T - iD$, with $D > 0$, and supply the global comparison needed for a strict Hilbert–Schmidt formula for S .

4.1. A positive-cone minimum. For $D > 0$ and $b > 0$, define

$$(4.1) \quad J_D(b) = -\log \det b + \text{Tr}(Db) + \frac{1}{2} \text{Tr}(b\eta(b)).$$

All finite traces here are unnormalized.

Lemma 4.1. *The function J_D has a unique minimizer on the positive definite cone. This minimizer is its only critical point and satisfies*

$$(4.2) \quad b^{-1} = D + \eta(b).$$

Proof. If $D \geq \delta 1_m$, then

$$J_D(b) \geq \sum_{k=1}^m (-\log \lambda_k + \delta \lambda_k),$$

where $\lambda_k > 0$ are the eigenvalues of b . Indeed, $\text{Tr}(ba_jba_j) = \text{Tr}((b^{1/2}a_jb^{1/2})^2) \geq 0$. Each scalar summand is bounded below and tends to infinity at both ends of $(0, \infty)$. Every sublevel set is therefore contained in a compact subset of the positive definite cone, and a minimum exists.

To see uniqueness, join two positive matrices by $b(t) = b_0^{1/2}e^{tx}b_0^{1/2}$, where $x = x^*$. In a basis diagonalizing x , write its eigenvalues as x_k , and set $d = b_0^{1/2}Db_0^{1/2} > 0$, $c_j = b_0^{1/2}a_jb_0^{1/2} = c_j^*$. Then

$$\text{Tr}(Db(t)) = \sum_k d_{kk}e^{tx_k}, \quad \text{Tr}(b(t)a_jb(t)a_j) = \sum_{k,l} |(c_j)_{kl}|^2 e^{t(x_k+x_l)}.$$

The second expressions are convex in t , the first is strictly convex if $x \neq 0$, and $-\log \det b(t)$ is affine. Thus J_D is strictly convex along every nonconstant such path. Finally, symmetry of η for the trace pairing gives

$$dJ_D(b)[x] = \text{Tr}((-b^{-1} + D + \eta(b))x), \quad x = x^*.$$

This proves the critical-point equation. A critical point minimizes along each of these paths, which join it to every positive matrix, so it is the unique global minimizer. $\square$

The lemma also applies to any completely positive map that is selfadjoint on Hilbert–Schmidt space. To verify this assertion, start with a Kraus representation $\Psi(x) = \sum_j k_j x k_j^*$ and write $k_j = u_j + iv_j$ with u_j, v_j selfadjoint. Since $\Psi = \Psi^*$,

$$\Psi(x) = \frac{1}{2} \sum_j (k_j x k_j^* + k_j^* x k_j) = \sum_j (u_j x u_j + v_j x v_j).$$

Thus Ψ has the selfadjoint coefficient representation used in the proof. Complete positivity here does not mean positivity as a Hilbert–Schmidt operator: for $a_1 = \text{diag}(1, -1)$, the map $x \mapsto a_1 x a_1$ sends every selfadjoint off-diagonal matrix x to $-x$. The path argument is needed precisely because ordinary linear convexity of J_D can fail.

Remark 4.2 (Positive matrix equations and geodesic convexity). Equation (4.2) is a positive matrix equation of the form studied by Helton, Rashidi Far and Speicher [HRFS07, Theorem 2.1]; their result gives existence, uniqueness and a convergent fixed-point iteration. Under Hilbert–Schmidt selfadjointness of η , Lemma 4.1 also identifies its solution as a minimizer. The paths used in the proof are the positive-definite geodesics in [SH15, Section 2, equation (2.2)]. Thus the proof establishes strict geodesic convexity of J_D , meaning strict convexity along each nonconstant such path.

Proposition 4.3 (Imaginary constant term). *For every $D > 0$,*

$$(4.3) \quad \Delta(T - iD) = \exp \left(\frac{\min_{b>0} J_D(b) - m}{m} \right).$$

The minimizer is unique and satisfies (4.2).

Proof. First let $D = 1_m$. The expected resolvent $G = E_m((i1_m - T)^{-1})$ is purely imaginary, because T and $-T$ have the same matrix-valued distribution. Its imaginary part is strictly negative, so $G = -ib$ with $b > 0$. The Dyson equation gives $b^{-1} = 1_m + \eta(b)$, and (2.4) gives

$$\log \Delta(T - i1_m) = -\frac{1}{m} \log \det b - \frac{1}{2m} \text{Tr}(b\eta(b)).$$

The trace identity $m = \text{Tr} b + \text{Tr}(b\eta(b))$ identifies this value with $(J_{1_m}(b) - m)/m$. Lemma 4.1 identifies b as the unique minimizer.

For general D , put $T' = D^{-1/2}TD^{-1/2}$ and $\eta'(x) = \sum_j (D^{-1/2}a_j D^{-1/2})x(D^{-1/2}a_j D^{-1/2})$. The congruence $T - iD = D^{1/2}(T' - i1_m)D^{1/2}$ and determinant multiplicativity give

$$\log \Delta(T - iD) = \frac{1}{m} \log \det D + \log \Delta(T' - i1_m).$$

The positive-cone bijection $c = D^{1/2}bD^{1/2}$ satisfies $J_{1_m}'(c) = J_D^\eta(b) - \log \det D$. This reduces the assertion to the case already proved. $\square$

Letting $D = \varepsilon 1_m$ tend to zero in Proposition 4.3 gives a formula for T . Put $J_0(b) = -\log \det b + \frac{1}{2} \operatorname{Tr}(b\eta(b))$. Then

$$(4.4) \quad \Delta(T) = \exp\left(\frac{\inf_{b>0} J_0(b) - m}{m}\right), \quad \operatorname{cap}(\eta) = \exp\left(2 \inf_{b>0} J_0(b) - m\right).$$

The convention is $\exp(-\infty) = 0$, and no attainment is asserted. Indeed, $J_{\varepsilon 1_m}(b) = J_0(b) + \varepsilon \operatorname{Tr} b$, and testing each fixed b shows that its infima decrease to $\inf J_0$. On the operator side, $\log \Delta(T - i\varepsilon 1_m) = \frac{1}{2} \varphi_m \log(T^2 + \varepsilon^2 1_m)$ decreases to $\log \Delta(T)$. This proves the first equality. The second follows from the zero-mean case of Theorem 1.3, which recovers the capacity identity in [MS24, Theorem 1.2].

Scaling $b = tc$, with $\det c = 1$, also gives

$$(4.5) \quad \operatorname{cap}(\eta) = \inf_{b>0} \frac{(\operatorname{Tr}(b\eta(b))/m)^m}{(\det b)^2}.$$

For $\beta = \operatorname{Tr}(c\eta(c)) > 0$, the minimum of $-m \log t + t^2 \beta/2$ is $m/2 + (m/2) \log(\beta/m)$; if $\beta = 0$, its infimum is $-\infty$. This proves (4.5).

Remark 4.4 (Capacity and common congruences). For selfadjoint coefficients, (4.5) can be written as

$$m \operatorname{cap}(\eta)^{1/m} = \inf_{\substack{c>0 \\ \det c=1}} \sum_j \|c^{1/2} a_j c^{1/2}\|_{\text{HS}}^2, \quad \|x\|_{\text{HS}}^2 = \operatorname{Tr}(x^* x).$$

This expresses capacity through an infimum over common positive congruences of the coefficient matrices. It is related to the capacity and norm minimizations in operator scaling [AZGL⁺18], where independent left and right changes of basis are allowed. Here selfadjointness permits the same positive factor on both sides; the displayed identity follows directly from (4.5).

4.2. The strict Hilbert–Schmidt formula. We now prove Proposition 1.4. The domain $\mathcal{B}_\eta$ in (1.12) is nonempty, since large scalar multiples of 1_m belong to it.

Remark 4.5 (The strict domain). Speicher’s public discussion [Spe26] presents an affine determinant proposal attributed there to ChatGPT, without an inspectable proof. Its displayed nonstrict domain can leave the inverse in the proposed expression undefined. For example, take $a_1 = \operatorname{diag}(1, -1)$ and $c = 1_2$. Then $\mathcal{C}_c - \eta \geq 0$, but $\mathcal{C}_c + \eta$ vanishes on off-diagonal matrices. Proposition 1.4 uses strict positivity of $\mathcal{C}_c - \eta$ as a Hilbert–Schmidt operator, which guarantees that the inverse exists. The attribution to [Spe26] concerns the candidate expression; the argument below proves its strict-domain version.

To compare the expressions in (1.8) and (1.13), we first parameterize $\mathcal{B}_\eta$ by positive definite matrices.

Lemma 4.6 (Positive coordinates for the strict domain). *For $b > 0$, put $H_b = 1_m + \eta(b)$ and*

$$(4.6) \quad c_b = b^{-1/2} (b^{1/2} H_b b^{1/2})^{1/2} b^{-1/2}.$$

Then $b \mapsto c_b$ is a bijection from the positive definite cone onto $\mathcal{B}_\eta$, with inverse $b_c = (\mathcal{C}_c - \eta)^{-1}(1_m)$. Moreover,

$$(4.7) \quad c_b b c_b = 1_m + \eta(b), \quad 2 \log \det c_b = \log \det H_b - \log \det b.$$

Proof. For $c > 0$, set $\Psi = \mathcal{C}_{c^{-1/2}} \eta \mathcal{C}_{c^{-1/2}}$. It is completely positive and Hilbert–Schmidt selfadjoint. If $c \in \mathcal{B}_\eta$, congruence gives $\text{id} - \Psi > 0$. For a Hermitian eigenmatrix x of Ψ , positivity gives

$$|\text{Tr}(x\Psi(x))| \leq \text{Tr}(|x|\Psi(|x|)) \leq \lambda_{\max}(\Psi) \text{Tr}(x^2).$$

The first inequality follows by expanding $x = x_+ - x_-$; the diagonal and mixed traces are all nonnegative. Every eigenvalue has a nonzero Hermitian eigenmatrix, since Ψ preserves adjoints. Consequently $\|\Psi\|_{2 \rightarrow 2} = \lambda_{\max}(\Psi) < 1$. Both $\text{id} - \Psi$ and $\text{id} + \Psi$ are therefore positive definite. The Neumann series for $(\text{id} - \Psi)^{-1}$ is completely positive, and its zeroth term shows that

$$b_c = c^{-1/2}(\text{id} - \Psi)^{-1}(c^{-1})c^{-1/2} > 0.$$

This also proves the invertibility of $\mathcal{C}_c + \eta$ claimed in Proposition 1.4.

Conversely, if $cbc - \eta(b) = 1_m$ with $b > 0$, then $p = c^{1/2}bc^{1/2}$ satisfies $p - \Psi(p) = c^{-1} > 0$. The positive-map estimate of Section 2.4 gives $r(\Psi) < 1$. Selfadjointness gives $\text{id} - \Psi > 0$, hence $c \in \mathcal{B}_\eta$. Finally, $cbc = H_b$ has the unique positive solution (4.6), by uniqueness of the positive square root of $b^{1/2}H_b b^{1/2}$. This proves both compositions and the determinant identity. $\square$

The matrix in (4.6) is the geometric mean $c_b = b^{-1} \# H_b$: for $A, B > 0$, $A \# B$ is the unique positive solution of $XA^{-1}X = B$; see [BJL19, Section 2]. This identifies the explicit positive solution, while the positive-map argument above proves the bijection with the strict domain.

Write $h_b = (\mathcal{C}_{c_b} + \eta)^{-1}(a_0)$ and retain $Z_b = \mathcal{T}_b^{-1}(a_0)$ from Theorem 1.3; here $\rho = \eta$. Define

$$(4.8) \quad q_{\text{HS}}(b) = \text{Tr}(a_0 h_b), \quad q_{\text{gen}}(b) = \text{Re Tr}(a_0 Z_b).$$

Define

$$(4.9) \quad \begin{aligned} \Phi_{\text{HS}}(c) &= \frac{2 \log \det c - m + \text{Tr}(a_0(\mathcal{C}_c + \eta)^{-1}(a_0))}{2m}, \\ \Phi_{\text{gen}}(b) &= \frac{\mathcal{V}(b)}{2m}. \end{aligned}$$

Proposition 4.7 (Comparison of the corrections). *For every $b > 0$,*

$$(4.10) \quad 0 \leq q_{\text{HS}}(b) \leq q_{\text{gen}}(b), \quad \Phi_{\text{HS}}(c_b) \leq \Phi_{\text{gen}}(b).$$

The inequality can be strict, even when $\eta = 0$.

Proof. Fix b and put $c = c_b$, $p = c^{1/2}bc^{1/2}$, $a = c^{-1/2}a_0c^{-1/2}$, $\Psi = \mathcal{C}_{c^{-1/2}} \eta \mathcal{C}_{c^{-1/2}}$. The transformed equations for $w = c^{1/2}h_b c^{1/2}$ and $X = c^{1/2}Z_b c^{1/2}$ are

$$(4.11) \quad (\text{id} + \Psi)w = a, \quad pXp^{-1} + \Psi(X^*) = a.$$

The quadratic corrections are $\langle a, w \rangle_{\mathbb{R}}$ and $\langle a, X \rangle_{\mathbb{R}}$, for $\langle x, y \rangle_{\mathbb{R}} = \text{Re Tr}(x^*y)$.

On this real Hilbert space let $J(x) = x^*$ and $K(x) = pxp^{-1}$. The operator K is positive definite and selfadjoint, J is a selfadjoint involution, and $KJK = J$. Splitting into Hermitian and skew-Hermitian matrices gives

$$K = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi_+ & 0 \\ 0 & \Psi_- \end{pmatrix}.$$

Here adjoints and inequalities are for the real inner product. The identity $KJK = J$ yields $A^2 - BB^* = \text{id}$, $AB = BD$ and $D^2 - B^*B = \text{id}$. Thus $A, D \geq \text{id}$ and

$$(4.12) \quad B(D + \text{id})^{-1}B^* = A - \text{id}.$$

The operator $K + \Psi J$ is positive definite. Indeed it is a congruence of $\mathcal{T}_b$, whose real quadratic form is bounded below by $\text{Tr}(Y^*Yb^{-1})$ in the proof of Theorem 1.3. Its Schur complement is therefore

$$F = A + \Psi_+ - B(D - \Psi_-)^{-1}B^* > 0.$$

Since $\|\Psi\|_{2 \rightarrow 2} < 1$, we have $0 < D - \Psi_- < D + \text{id}$. Inverting and using (4.12) gives $F \leq \text{id} + \Psi_+$. The right-hand side a in (4.11) is Hermitian, so block inversion gives

$$q_{\text{gen}}(b) = \langle a, F^{-1}a \rangle_{\mathbb{R}} \geq \langle a, (\text{id} + \Psi_+)^{-1}a \rangle_{\mathbb{R}} = q_{\text{HS}}(b) \geq 0.$$

Together with (4.7), this proves the claim. $\square$

The remaining ingredient gives $\log \Delta(S) \leq \Phi_{\text{HS}}(c)$ for every $c \in \mathcal{B}_\eta$. We state the estimate here; its proof in Appendix A uses the logarithmic primitive and Lemma 4.1.

Lemma 4.8 (Global comparison). *Let Ψ be completely positive and Hilbert–Schmidt selfadjoint, with $\theta = \|\Psi\|_{2 \rightarrow 2} < 1$, and let T_Ψ be a centered selfadjoint matrix-valued semicircular element with covariance Ψ . For every $a = a^*$,*

$$(4.13) \quad \log \Delta(a + T_\Psi) \leq -\frac{1}{2} + \frac{1}{2m} \text{Tr}(a(\text{id} + \Psi)^{-1}(a)).$$

The assertion includes determinant zero.

Proof of Proposition 1.4. For each $c \in \mathcal{B}_\eta$, apply Lemma 4.8 to $a = c^{-1/2}a_0c^{-1/2}$ and $\Psi = \mathcal{C}_{c^{-1/2}\eta\mathcal{C}_{c^{-1/2}}}$. Determinant multiplicativity and trace cyclicity give

$$\log \Delta(S) \leq \Phi_{\text{HS}}(c).$$

For the converse, Lemma 4.6, Proposition 4.7 and Theorem 1.3 give

$$(4.14) \quad \inf_{c \in \mathcal{B}_\eta} \Phi_{\text{HS}}(c) = \inf_{b > 0} \Phi_{\text{HS}}(c_b) \leq \inf_{b > 0} \Phi_{\text{gen}}(b) = \log \Delta(S).$$

These inequalities are valid in the extended real line. Exponentiating proves (1.13). $\square$

Remark 4.9 (Comparison under the change of variables). Lemma 4.6 identifies the determinant factors under its bijection of trial domains, but the inequality between the quadratic corrections in Proposition 4.7 can be strict. For example, take $\eta = 0$,

$$a_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \text{diag}(1, 4), \quad c_b = \text{diag}(1, 1/2).$$

Then $Z_b = a_0b$, $h_b = c_b^{-1}a_0c_b^{-1}$, and $q_{\text{gen}}(b) = 5$ whereas $q_{\text{HS}}(b) = 4$. Thus $\Phi_{\text{HS}}(c_b)$ and $\Phi_{\text{gen}}(b)$ need not agree. More generally, when $\eta = 0$,

$$\begin{aligned} q_{\text{gen}}(b) - q_{\text{HS}}(b) &= \text{Tr}(ba_0^2) - \text{Tr}(b^{1/2}a_0b^{1/2}a_0) \\ &= \frac{1}{2} \| [b^{1/2}, a_0] \|_{\text{HS}}^2 = \text{Tr}(b) I_{\text{WY}}(b / \text{Tr}(b), a_0). \end{aligned}$$

Here $I_{\text{WY}}(d, a) = -\frac{1}{2} \text{Tr}([d^{1/2}, a]^2)$ is the Wigner–Yanase skew information for $d \geq 0$, $\text{Tr} d = 1$ and $a = a^*$; see [Han08]. Hence this difference vanishes exactly when b and a_0 commute. For every η in the present setting, $\Phi_{\text{HS}}(c_b)$ and $\Phi_{\text{gen}}(b)$ coincide pointwise when $a_0 = 0$. For scalar selfadjoint data $\eta(b) = \sigma b$, both h_b and Z_b are $a_0/(b^{-1} + 2\sigma)$ and $c_b^2 = \sigma + b^{-1}$, so the entire scalar formulas also coincide pointwise.

For general selfadjoint coefficient matrices, (4.14) provides only one direction. Lemma 4.8 supplies the other; the strict Hilbert–Schmidt formula is therefore not merely a substitution into the general formula.

Finally, arbitrary complex coefficients can be treated by applying Proposition 1.4 to the selfadjoint pencil $\mathcal{H} = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$. With the normalized operator trace in each size, the two determinants satisfy $\Delta(\mathcal{H}) = \Delta(S)$. This yields a strict-domain formula in size $2m$. The coordinate comparison above then also uses $2m \times 2m$ trial matrices, whereas Theorem 1.3 uses $m \times m$ trial matrices.

5. A VARIATIONAL FORMULA FOR STRICTLY POSITIVE AFFINE PENCILS

We prove Theorem 1.6 and relate its minimizer to the one in Theorem 1.3.

Throughout this section all $a_j = a_j^*$, including a_0 , so that $S = S^*$ and $\eta(b) = \sum_j a_j b a_j$. Recall $L(b)$, $\mathcal{B}$ and $\mathcal{B}_{\text{str}}$ from Definition 1.5, and put

$$(5.1) \quad F(b) = a_0 - b^{-1} - \eta(b), \quad b > 0.$$

Eliminating the lower diagonal block of $L(b)$ gives $L(b) \geq 0$ if and only if $F(b) \geq 0$, and likewise for strict positivity. This is the Schur complement calculation; it follows by multiplying $L(b)$ by invertible triangular block matrices to obtain $\text{diag}(F(b), b)$. In particular, the condition $L(b) \geq 0$ is a linear matrix inequality: its matrix entries depend affinely on the selfadjoint variable b . Both feasible sets are convex and are specified without solving a Dyson equation.

Remark 5.1 (Semicircular spectral edges and semidefinite programming). Kunisky's semidefinite formulation of Lehner's semicircular norm formula [Kun26, Theorem 1.9 and Section 4.1], applied to $-S$, gives its upper spectral edge as

$$\sup \text{spec}(-S) = \inf \left\{ t \in \mathbb{R} : \text{there is } b > 0 \text{ with } \begin{pmatrix} t1_m + a_0 - \eta(b) & 1_m \\ 1_m & b \end{pmatrix} \geq 0 \right\}.$$

The block at $t = 0$ is $L(b)$. Its Schur complement is $t1_m + F(b)$, so a negative feasible t exists exactly when $F(b) > 0$ for some $b > 0$. Thus the strict-positivity criterion in Theorem 1.6 agrees with this spectral-edge formula. The determinant minimum and the least feasible matrix are obtained by the argument below.

The function J in (1.16) is the ordinary log determinant of the positive matrix

$$(5.2) \quad P(b) = b^{-1/2} \exp \left(b^{1/2} (a_0 - \tfrac{1}{2} \eta(b)) b^{1/2} - 1_m \right) b^{-1/2} > 0.$$

Indeed, $\log \det P(b) = J(b)$, so $\Delta_m(P(b)) = \exp(J(b)/m)$.

Proof of Theorem 1.6. Positivity from coefficients. Use the full Fock space $\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{k \geq 1} (\mathbb{C}^n)^{\otimes k}$. The left creation operators $\ell_j \in \mathcal{B}(\mathcal{F})$ prepend the j th basis vector and satisfy $\ell_i^* \ell_j = \delta_{ij} I$. The operators $\ell_j + \ell_j^*$ have the joint semicircular distribution of the s_j ; see [Miy22, Section 2.1, pp. 3–5]. The vacuum state on their generated von Neumann algebra is faithful and tracial. Thus equality of these joint distributions gives the same matrix spectral laws and positivity tests as in $M_m(\mathcal{M})$.

Given $b > 0$, put $d_j = b^{1/2} a_j b^{1/2}$ and $C = \sum_j d_j \otimes \ell_j \in \mathcal{B}(\mathbb{C}^m \otimes \mathcal{F})$. Direct multiplication gives the square completion

$$(5.3) \quad b^{1/2} S b^{1/2} = (I + C)^*(I + C) + b^{1/2} F(b) b^{1/2}.$$

If $F(b) \geq \delta 1_m$ with $\delta > 0$, this proves $S \geq \delta 1_m$. Conversely, suppose $S \geq \gamma 1_m$. For $0 \leq t < \gamma$ set $q(t) = E_m((S - t1_m)^{-1}) > 0$. The selfadjoint Dyson equation, continued from the upper half-plane through this spectral gap, gives

$$(5.4) \quad q(t)^{-1} = a_0 - t1_m - \eta(q(t)), \quad F(q(t)) = t1_m.$$

Hence $q(t) \in \mathcal{B}_{\text{str}}$ for $t > 0$ and $q = q(0)$ belongs to $\mathcal{B}$.

The expectation of S^{-1} is the least feasible matrix. Fix any $b \in \mathcal{B}$ and start the iteration

$$(5.5) \quad q_0 = 0, \quad q_{k+1} = (a_0 - \eta(q_k))^{-1}.$$

For $0 \leq X \leq b$, positivity of η gives $a_0 - \eta(X) \geq a_0 - \eta(b) \geq b^{-1} > 0$. The iteration is therefore defined, order preserving and bounded above by b , so $0 < q_1 \leq q_2 \leq \dots \leq b$. Its limit is the expectation of S^{-1} , as can be seen without selecting an algebraic root. Compress the Fock realization of S to words of length at most N . Each compression is at least γI . Separate the vacuum from the subtrees classified by the last letter of a word. Every subtree carries the same pencil truncated at depth $N - 1$; eliminating these blocks shows that the vacuum block of the

inverse is exactly q_{N+1} in (5.5). The compressions, extended by γI on the orthogonal complements, converge strongly to S and have uniformly bounded inverses. Their inverses converge strongly to S^{-1} , by the resolvent identity. Taking vacuum matrix coefficients identifies the limit as q . Thus

$$(5.6) \quad q \leq b \quad (b \in \mathcal{B}).$$

Global minimality and strictness. The first variation of J is

$$dJ(b)[Y] = \text{Tr}(F(b)Y), \quad Y = Y^*.$$

For $b \in \mathcal{B}$ write $Y = b - q \geq 0$ and $b_t = q + tY$. The function F is operator concave on positive matrices: its second derivative along this segment is $-2b_t^{-1}Yb_t^{-1}Yb_t^{-1} \leq 0$. Consequently $F(b_t) \geq (1-t)F(q) + tF(b) \geq 0$, and

$$(5.7) \quad \frac{d}{dt}J(b_t) = \text{Tr}(F(b_t)Y) \geq 0.$$

This proves $J(b) \geq J(q)$.

For uniqueness it suffices to prove strictness near q . Let $q' = \frac{d}{dt}q(t)\big|_{t=0} = E_m(S^{-2}) > 0$. Differentiation of (5.4) gives $q' = q^2 + q\eta(q')q$. Set $c_j = q^{1/2}a_jq^{1/2}$ and $\Phi(x) = \sum_j c_jxc_j$. With $H = q^{-1/2}q'q^{-1/2} > 0$ this reads $H - \Phi(H) = q > 0$. The positive-map estimate in Section 2.4 shows that the spectral radius of Φ is less than one. The map Φ is also selfadjoint for the Hilbert-Schmidt inner product $\langle X, Y \rangle = \text{Tr}(X^*Y)$. It follows that the Hessian of J at q , in the direction $Y = q^{1/2}Zq^{1/2} \neq 0$, is

$$d^2J(q)[Y, Y] = \text{Tr}(Z^2) - \text{Tr}(Z\Phi(Z)) > 0.$$

Since $dJ(q) = 0$, this gives $J(q + tY) > J(q)$ for small positive t . Together with (5.7), it proves uniqueness.

Finally, (2.4) at zero, where $G(0) = -q$, gives

$$\log \Delta(S) = \frac{1}{m} \left(-\log \det q + \frac{1}{2} \text{Tr}(q\eta(q)) \right) = J(q)/m.$$

The last equality uses the trace of (5.4) at $t = 0$. Since $q(t) \rightarrow q$ and $q(t) \in \mathcal{B}_{\text{str}}$ for $t > 0$, the strict-domain infimum is the same. It cannot be attained there because the unique minimizer on $\mathcal{B}$ has $F(q) = 0$. $\square$

The feasible set is convex. The proof of minimality uses its least element and the nonnegative directional derivative of J along segments starting there; it does not require convexity of J .

The two variational formulas minimize different functions and have different optimizers. Under the present assumption that S is strictly positive, Theorem 1.3 gives $b_* = E_m(S^{-2})$ and $Z_{b_*} = q$, whereas Theorem 1.6 minimizes at q itself. The differentiated Dyson equation above relates them by

$$(5.8) \quad b_* = q^2 + q\eta(b_*)q.$$

Thus the strict-positivity formula supplies an order description of the expectation of S^{-1} and a feasibility criterion in addition to the general determinant formula.

Replacing S by $-S$ treats strictly negative pencils. Nonstrict feasibility alone does not provide the positive gap used here. For example, the scalar pencil $2 + s$ has the feasible trial $b = 1$, but zero belongs to its spectrum, so Theorem 1.6 does not apply. The general infimum in Theorem 1.3 continues to apply to this boundary case.

6. ORDERED MIN-MAX PRINCIPLES

The preceding formulas minimize over positive definite matrices. For certain selfadjoint pencils we can instead maximize first over a positive semidefinite matrix and then minimize over a selfadjoint matrix. The order is essential. We begin with scalar and isotropic covariance, where the maximum over the nonnegative variable is explicit, and then treat a different class in which the covariance is strictly positive as a Hilbert–Schmidt operator. Throughout this section $a_j = a_j^*$ for $0 \leq j \leq n$.

6.1. A shifted semicircle and commuting coefficients. For $\sigma > 0$ and $a \in \mathbb{R}$, put

$$(6.1) \quad \ell_{a,\sigma}(p, q) = \frac{(a-p)^2 - q^2}{2\sigma} + \frac{1}{2} \log(p^2 + q^2), \quad p \in \mathbb{R}, \quad q \geq 0,$$

with $\log 0 = -\infty$.

Proposition 6.1. *For a standard semicircular element s ,*

$$(6.2) \quad \log \Delta_\tau(a + \sqrt{\sigma} s) = \min_{p \in \mathbb{R}} \max_{q \geq 0} \ell_{a,\sigma}(p, q).$$

For every p , the maximizer in q is unique. The function $p \mapsto \max_{q \geq 0} \ell_{a,\sigma}(p, q)$ has a unique minimizer. The optimizers are

$$(6.3) \quad p_* = \begin{cases} a/2, & |a| \leq 2\sqrt{\sigma}, \\ (a + \operatorname{sgn}(a)\sqrt{a^2 - 4\sigma})/2, & |a| > 2\sqrt{\sigma}, \end{cases} \quad q_*^2 = (\sigma - p_*^2)_+.$$

Here $r_+ = \max\{r, 0\}$ for $r \in \mathbb{R}$.

Proof. For fixed p , differentiation in $r = q^2$ gives

$$\frac{\partial}{\partial r} \left(-\frac{r}{2\sigma} + \frac{1}{2} \log(p^2 + r) \right) = -\frac{1}{2\sigma} + \frac{1}{2(p^2 + r)}.$$

This derivative decreases strictly wherever the expression is finite. The unique maximizing $q(p)$ therefore satisfies $q(p)^2 = (\sigma - p^2)_+$, and the maximum as a function of p equals

$$(6.4) \quad f_{a,\sigma}(p) = \frac{(a-p)^2}{2\sigma} + \begin{cases} \frac{p^2}{2\sigma} - \frac{1}{2} + \frac{1}{2} \log \sigma, & |p| \leq \sqrt{\sigma}, \\ \log |p|, & |p| \geq \sqrt{\sigma}. \end{cases}$$

Its value and first derivative agree at both joining points. Its second derivative is $2/\sigma$ in the inner interval and $1/\sigma - 1/p^2 > 0$ in the two open outer intervals. Thus it is continuously differentiable, strictly convex, and tends to $+\infty$ as $|p| \rightarrow \infty$. Solving $f'_{a,\sigma}(p) = 0$ gives (6.3).

The scalar Dyson equation and (2.4) give

$$\log \Delta_\tau(a + \sqrt{\sigma} s) = \begin{cases} \frac{1}{2} \log \sigma - \frac{1}{2} + \frac{a^2}{4\sigma}, & |a| \leq 2\sqrt{\sigma}, \\ -\log |g| + \frac{\sigma}{2} g^2, & |a| > 2\sqrt{\sigma}, \end{cases}$$

where in the second line g is the root of $\sigma g^2 - ag + 1 = 0$ with smaller absolute value. Indeed the boundary value of the Cauchy transform in the first regime has modulus $\sigma^{-1/2}$; substitution in the same primitive gives its displayed value. Boundary passage is justified by the semicircular spectral integral, including the endpoints. In the second regime $g = 1/p_*$ and $a - p_* = \sigma g$. Substitution in (6.4) now proves (6.2) in both regimes. $\square$

The pair (p_*, q_*) need not be a global saddle point of $\ell_{a,\sigma}$. If $|a| > 2\sqrt{\sigma}$, then $q_* = 0$, whereas $\ell_{a,\sigma}(p, 0) \rightarrow -\infty$ as $p \rightarrow 0$. Hence p_* does not minimize the slice with $q = q_*$. The convexity used in the proof is the convexity after maximizing over q ; it does not authorize an exchange of the two extrema.

Suppose next that all the selfadjoint coefficients of $S = a_0 + \sum_{j=1}^n a_j \otimes s_j$ commute. Simultaneous unitary diagonalization gives real entries a_{jk} , $1 \leq k \leq m$. Put $a_k = a_{0k}$ and $\sigma_k = \sum_{j=1}^n a_{jk}^2$. The k th diagonal entry has a semicircular distribution with mean a_k and variance σ_k . Therefore, if all $\sigma_k > 0$,

$$(6.5) \quad \log \Delta_\varphi(S) = \frac{1}{m} \min_{p \in \mathbb{R}^m} \max_{q \in [0, \infty)^m} \sum_{k=1}^m \ell_{a_k, \sigma_k}(p_k, q_k).$$

This follows from Proposition 6.1, the block-diagonal determinant rule and separability of the successive extrema. When $\sigma_k = 0$, replace its summand by the deterministic value $\log |a_k|$ and omit its variables. In particular a zero deterministic entry makes the logarithmic determinant $-\infty$.

6.2. Isotropic covariance and a scalar spectral coordinate. Assume now that

$$(6.6) \quad \eta(b) = \sum_{j=1}^n a_j b a_j = \sigma \operatorname{tr}_m(b) 1_m, \quad \sigma > 0,$$

and let $a_0 = a_0^*$ be arbitrary. Write $\nu = m^{-1} \sum_{k=1}^m \delta_{\lambda_k}$ for its eigenvalue probability measure, with algebraic multiplicities, and define

$$\ell_{x, \nu, \sigma}(p, q) = \frac{(x - p)^2 - q^2}{2\sigma} + \frac{1}{2} \int_{\mathbb{R}} \log((p - t)^2 + q^2) d\nu(t).$$

The logarithmic term is also $m^{-1} \log |\det((p + iq)1_m - a_0)|$. The spectral coordinate and density below are the finite-atomic case of Biane's description of free convolution with a semicircular distribution [Bia97, Section 3, Lemma 4, Proposition 2 and Corollary 3]. We give the proof in this setting together with the ordered variational expression for the logarithmic determinant. For comparison, [BCG25, Theorem 1.1 and (1.6)] gives a branch-restricted variational formula for scalar free additive convolutions at real points below the spectrum. That work permits more general input measures; the ordered expression below uses the finite-atomic measure ν and holds for every real x .

Proposition 6.2. *Under (6.6), for every $x \in \mathbb{R}$,*

$$(6.7) \quad \log \Delta_\varphi(S - x 1_m) = \min_{p \in \mathbb{R}} \max_{q \geq 0} \ell_{x, \nu, \sigma}(p, q).$$

Both successive extrema are attained, and $p \mapsto \max_{q \geq 0} \ell_{x, \nu, \sigma}(p, q)$ has a unique minimizer.

More precisely, set

$$I(p, q) = \int_{\mathbb{R}} \frac{d\nu(t)}{(p - t)^2 + q^2},$$

interpreting a pole at $q = 0$ as $+\infty$. For fixed p , the maximizing $q(p)$ is zero if $I(p, 0) \leq 1/\sigma$ and is otherwise the unique positive solution of $I(p, q) = 1/\sigma$. The function

$$(6.8) \quad X(p) = p + \sigma \int_{\mathbb{R}} \frac{p - t}{(p - t)^2 + q(p)^2} d\nu(t)$$

is a strictly increasing homeomorphism of $\mathbb{R}$, and the minimizing p is $X^{-1}(x)$. The entire spectral measure of S has the continuous density

$$(6.9) \quad \frac{d\mu_S}{dx}(x) = \frac{q(X^{-1}(x))}{\pi\sigma}.$$

Proof. Differentiation in q^2 gives the stated maximizing condition. The function $I(p, q)$ decreases strictly in $q > 0$, so the maximizer $q(p)$ is unique and continuous: the positive branch can approach zero only at a non-pole point where $I(p, 0) = 1/\sigma$, while an atom of mass w at p forces $q(p)^2 \geq \sigma w$. The set where $q(p) > 0$ is a bounded finite union of intervals. Its endpoints solve the rational

equation $I(p, 0) = 1/\sigma$ away from the finitely many atoms, and that equation has finitely many roots.

On an interval with $q = q(p) > 0$, let

$$B = \int_{\mathbb{R}} \frac{d\nu(t)}{((p-t)^2 + q^2)^2}, \quad C = \int_{\mathbb{R}} \frac{p-t}{((p-t)^2 + q^2)^2} d\nu(t).$$

Implicit differentiation gives $q' = -C/(qB)$ and hence

$$(6.10) \quad X'(p) = 2\sigma \left(q^2 B + \frac{C^2}{B} \right) > 0.$$

On an open interval where $q = 0$, one has $X'(p) = 1 - \sigma I(p, 0) > 0$ except possibly at the finitely many solutions of $I(p, 0) = 1/\sigma$. Continuity across the endpoints proves that X is strictly increasing. Moreover $X(p) = p + O(1/p)$ at infinity, so it maps $\mathbb{R}$ onto itself.

The derivative of $p \mapsto \ell_{x,\nu,\sigma}(p, q(p))$ is $(X(p) - x)/\sigma$. This follows on each of the intervals just considered; the derivatives match at their endpoints because q is continuous and no endpoint is a pole. Consequently this function is continuously differentiable, strictly convex and coercive, with minimum at $p = X^{-1}(x)$.

It remains to identify the value. For $z = x + i\varepsilon$, $\varepsilon > 0$, the Dyson equation reduces to

$$(6.11) \quad G(z) = (w(z)1_m - a_0)^{-1}, \quad g(z) = \text{tr}_m G(z) = \int_{\mathbb{R}} \frac{d\nu(t)}{w(z) - t}, \quad w(z) = z - \sigma g(z).$$

Write $w = p + iq$, with $q > 0$. Imaginary parts give

$$\varepsilon = q(1 - \sigma I(p, q)), \quad I(p, q) < 1/\sigma.$$

Thus $|g|^2 \leq I(p, q) < 1/\sigma$, $|w - z| \leq \sqrt{\sigma}$, and $|w - \lambda_k| \geq \sqrt{\sigma/m}$ for every eigenvalue. As $\varepsilon \downarrow 0$, the points w stay bounded and uniformly away from all poles. Every subsequential limit satisfies the maximizing condition for $q(p)$ and the real equation $X(p) = x$. The preceding uniqueness proves convergence to $X^{-1}(x) + iq(X^{-1}(x))$.

The primitive (2.4) becomes

$$(6.12) \quad \log \Delta_\varphi(S - z1_m) = \int_{\mathbb{R}} \log |w(z) - t| d\nu(t) + \frac{\sigma}{2} \text{Re}(g(z)^2).$$

In the boundary limit $g = (x - p - iq)/\sigma$, and the right side is exactly $\ell_{x,\nu,\sigma}(p, q)$ at $p = X^{-1}(x)$ and $q = q(p)$. Pole separation makes the limit finite. On the operator side, selfadjoint spectral calculus gives the same limit as the ordinary logarithmic determinant. This proves (6.7).

The boundary values of g are continuous, and $|g(x + i\varepsilon)| \leq 1/\sqrt{\sigma}$ uniformly. Stieltjes inversion, in the convention of (2.9), gives (6.9): dominated convergence applies on each finite interval, and (2.10) excludes atoms. The interval formula identifies the full measure, so no singular continuous part remains. Faithfulness identifies its support with the operator spectrum. $\square$

6.3. Strict positivity on Hilbert–Schmidt space. On $M_m(\mathbb{C})$ use the Hilbert–Schmidt inner product $\langle x, y \rangle_2 = \text{Tr}(x^*y)$. Assume that the selfadjoint covariance η is strictly positive for this inner product:

$$(6.13) \quad \text{Tr}(x^*\eta(x)) > 0 \quad (x \neq 0).$$

Its inverse $K = \eta^{-1}$ is a positive definite selfadjoint operator on Hilbert–Schmidt space.

Remark 6.3 (Isotropic covariance and Hilbert–Schmidt positivity). For $m > 1$, the isotropic covariance in (6.6) vanishes on every traceless matrix. It is therefore not invertible as an operator on matrix space and does not satisfy (6.13). Thus Theorem 6.4 does not include the isotropic case of Proposition 6.2 when $m > 1$. On the other hand, this isotropic map satisfies the order-flatness

condition of [AEK19, Assumption A1], since its value on $b \geq 0$ is exactly $\sigma \operatorname{tr}_m(b)1_m$. Order flatness therefore does not imply strict positivity on Hilbert–Schmidt space.

For $u = u^*$ and $v \geq 0$, define

$$(6.14) \quad \mathcal{H}_{a_0}(u, v) = \frac{1}{2} \operatorname{Tr}((a_0 - u)K(a_0 - u)) - \frac{1}{2} \operatorname{Tr}(vK(v)) + \log |\det(u + iv)|,$$

with value $-\infty$ when $u + iv$ is singular.

Theorem 6.4. *Let $S = a_0 + \sum_j a_j \otimes s_j$, with all $a_j = a_j^*$, including a_0 , and assume (6.13). Then*

$$(6.15) \quad \log \Delta_\varphi(S) = \frac{1}{m} \min_{u=u^*} \max_{v \geq 0} \mathcal{H}_{a_0}(u, v).$$

Both successive extrema are attained and the value is finite. The function $u \mapsto \max_{v \geq 0} \mathcal{H}_{a_0}(u, v)$ is convex and tends to $+\infty$ as $\|u\|_2 \rightarrow \infty$.

Proof. We first add a positive imaginary regularizer. Put $D = \eta(1_m) > 0$. Indeed a vector in the kernel of $\sum_j a_j^2$ would be killed by every a_j , so its rank-one projection would be in the kernel of η , contradicting (6.13). For $\varepsilon > 0$, set

$$(6.16) \quad \begin{aligned} \mathcal{H}_{a_0, \varepsilon}(u, v) &= \frac{1}{2} \operatorname{Tr}((a_0 - u)K(a_0 - u)) + \log |\det(u + iv)| \\ &\quad - \frac{1}{2} \operatorname{Tr}((v - \varepsilon D)K(v - \varepsilon D)). \end{aligned}$$

Since $K(D) = 1_m$, its difference from (6.14) is $\varepsilon \operatorname{Tr} v - \frac{1}{2} \varepsilon^2 \operatorname{Tr} D$.

For fixed u , strict positivity of K makes the negative quadratic term in v dominate the logarithmic term as $\|v\|_2 \rightarrow \infty$. Upper semicontinuity at singular matrices, together with the finite trial $v = 1_m$, gives a global maximum over $v \geq 0$. Its maximizers are uniformly bounded for u in a compact set and $0 \leq \varepsilon \leq 1$.

Every maximizing v is in fact positive definite if $\varepsilon > 0$. Its determinant is nonzero; write $r = (u + iv)^{-1} = p - iq$, where $p = p^*$ and $q \geq 0$. The gradient in v is $-K(v) + \varepsilon 1_m + q$. Testing positive directions and both radial directions at a maximum gives

$$C = K(v) - \varepsilon 1_m - q \geq 0, \quad \operatorname{Tr}(Cv) = 0.$$

Applying the positive map η yields $v = \eta(q) + \varepsilon D + \eta(C) \geq \varepsilon D > 0$. The maximum is therefore interior and satisfies

$$(6.17) \quad v = \eta(q) + \varepsilon D.$$

To prove uniqueness of this maximizer, associate to every interior stationary point the matrices and covariance

$$\mathcal{G} = \begin{pmatrix} iq & p \\ p & iq \end{pmatrix}, \quad \Xi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \eta(w) & 0 \\ 0 & \eta(x) \end{pmatrix}.$$

The map Ξ is completely positive and has selfadjoint Kraus matrices: for each a_j use

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & a_j \\ a_j & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & ia_j \\ -ia_j & 0 \end{pmatrix}.$$

The scalar Hadamard unitary diagonalizes the two-by-two block pattern and gives

$$(6.18) \quad \mathcal{G}^{-1} = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} - i\varepsilon \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} - \Xi(\mathcal{G}), \quad \operatorname{Im} \mathcal{G} > 0.$$

Congruence by $\operatorname{diag}(D^{-1/2}, D^{-1/2})$, and adjunction if needed to use the upper half-plane convention, reduce this equation to the uniqueness argument of Section 2. Thus $\mathcal{G}$, and hence the maximizing $v = v_\varepsilon(u)$, is unique. The auxiliary equation proves uniqueness; it does not restrict the freely chosen trial variables in (6.15). The equation for $v_\varepsilon(u)$ is also the case of [Bel15, Proposition

3.1 and Remark 3.2] with the source variables $X = 0$ and $q = \varepsilon D$. That result describes the imaginary part of an operator-valued subordination function as a function of its real part. Here $-\operatorname{Im}(u + iv)^{-1} = (v + uv^{-1}u)^{-1}$.

We next prove convexity after maximizing over v . At $v = v_\varepsilon(u)$, define operators on the real Hilbert space of Hermitian matrices by

$$\mathcal{S}_r(x) = \frac{1}{2}(rxr^* + r^*xr), \quad \mathcal{A}_r(x) = \operatorname{Re}(rxr), \quad \mathcal{C}_r(x) = \operatorname{Im}(rxr).$$

Here real and imaginary parts mean Hermitian parts. The map $\mathcal{S}_r$ is completely positive and selfadjoint for the trace pairing, and $r^*vr = rvr^* = q$. Therefore

$$\eta\mathcal{S}_r(v) = \eta(q) = v - \varepsilon D < v.$$

Conjugating the positive map $\eta\mathcal{S}_r$ by $x \mapsto v^{1/2}xv^{1/2}$ makes its value at 1_m strictly less than 1_m , so its spectral radius is less than one. The Hilbert–Schmidt selfadjoint operator $\eta^{1/2}\mathcal{S}_r\eta^{1/2}$ is similar to it. Its spectrum lies in $(-1, 1)$, which proves $K - \mathcal{S}_r > 0$ as a quadratic form. The square root $\eta^{1/2}$ here is the positive square root on Hilbert–Schmidt space; no positivity as a matrix map is used. Since

$$(\mathcal{S}_r - \mathcal{A}_r)(x) = 2qxq$$

and $x \mapsto qxq$ is positive semidefinite for the trace pairing, $B = K - \mathcal{A}_r$ is positive definite. Direct differentiation of (6.16) gives

$$\mathcal{H}_{uu} = B, \quad \mathcal{H}_{vv} = -B, \quad \mathcal{H}_{uv} = \mathcal{C}_r.$$

The map $\mathcal{C}_r$ is selfadjoint for the real trace pairing by cyclicity. The implicit-function theorem makes $v_\varepsilon(u)$ smooth, and the reduced Hessian is

$$(6.19) \quad D_u^2[\max_{v \geq 0} \mathcal{H}_{a_0, \varepsilon}(u, v)] = B + \mathcal{C}_r B^{-1} \mathcal{C}_r > 0.$$

This is the Schur complement obtained by eliminating the maximizing variable; compare [BV04, Appendix A.5.5]. Thus $u \mapsto \max_{v \geq 0} \mathcal{H}_{a_0, \varepsilon}(u, v)$ is strictly convex. Testing $v = 1_m$ also bounds it below by $\frac{1}{2} \operatorname{Tr}((a_0 - u)K(a_0 - u)) - C_0$, with C_0 independent of $0 \leq \varepsilon \leq 1$, because $|\det(u + i1_m)| \geq 1$. Its minimizer in u exists and is unique.

At this minimizing u and $v = v_\varepsilon(u)$, stationarity in u gives $u = a_0 - \eta(p)$. Together with (6.17), it gives

$$r^{-1} = a_0 + i\varepsilon D - \eta(r), \quad \operatorname{Im} r < 0.$$

The resolvent uniqueness argument identifies $r = E_m((S + i\varepsilon D)^{-1})$. The primitive (2.4), after congruence by $D^{-1/2}$, gives

$$(6.20) \quad m \log \Delta_\varphi(S + i\varepsilon D) = -\log |\det r| + \frac{1}{2} \operatorname{Re} \operatorname{Tr}(r\eta(r)).$$

The right side is exactly $\mathcal{H}_{a_0, \varepsilon}(u, v)$ at these matrices: substitute $a_0 - u = \eta(p)$ and $v - \varepsilon D = \eta(q)$. This identifies the global ordered extremum, not merely its stationary value.

Finally, the uniform bounds on the maximizing matrices v and the explicit difference $\mathcal{H}_{a_0, \varepsilon} - \mathcal{H}_{a_0}$ give local uniform convergence of $u \mapsto \max_{v \geq 0} \mathcal{H}_{a_0, \varepsilon}(u, v)$ as $\varepsilon \downarrow 0$. The common coercive lower bound keeps the minimizing matrices u in a compact set. The minima therefore converge to a finite, attained minimum of $u \mapsto \max_{v \geq 0} \mathcal{H}_{a_0}(u, v)$. On the operator side set $\tilde{S} = D^{-1/2}SD^{-1/2} = \tilde{S}^*$. Then

$$\log \Delta_\varphi(S + i\varepsilon D) = \frac{1}{m} \log \det D + \frac{1}{2} \varphi \log(\tilde{S}^2 + \varepsilon^2 1_m) \longrightarrow \log \Delta_\varphi(S)$$

by selfadjoint spectral calculus and determinant multiplicativity. This proves (6.15) and all its attainment claims. $\square$

In the centered case $a_0 = 0$, the function $u \mapsto \max_{v \geq 0} \mathcal{H}_0(u, v)$ is even, so convexity places a minimum at $u = 0$. Consequently Theorem 6.4 gives

$$(6.21) \quad \begin{aligned} & \max_{v > 0} \left\{ \log \det v - \frac{1}{2} \operatorname{Tr}(vK(v)) \right\} \\ &= \min_{b > 0} \left\{ -\log \det b + \frac{1}{2} \operatorname{Tr}(b\eta(b)) \right\} - m. \end{aligned}$$

Both optimizers are unique under (6.13). Indeed the function minimized on the right is strictly convex and coercive, with stationary equation $\eta(b) = b^{-1}$. The function maximized on the left is strictly concave and tends to $-\infty$ at its boundary and at infinity; its stationary equation is $K(v) = v^{-1}$. The substitution $v = b^{-1}$ identifies the two optimizers and the corresponding values in (6.21). By the centered case of Theorem 1.3, this common value is $\frac{1}{2} \log \operatorname{cap}(\eta) - \frac{m}{2}$. Here capacity is positive and the equation $\eta(b) = b^{-1}$ has a positive definite solution; the main theorem also covers covariances without these properties.

6.4. Why the covariance hypothesis matters. Complete positivity alone does not suffice for the formula involving η^{-1} in (6.15). Let

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta(b) = ZbZ, \quad a_0 = tX, \quad t > 0.$$

This covariance is completely positive, invertible and Hilbert–Schmidt selfadjoint, but $K = \eta$ satisfies $K(X) = -X$. At $u = 0$, the function $v \mapsto \mathcal{H}_{a_0}(0, v)$ is

$$-t^2 + \log \det v - \frac{1}{2} \operatorname{Tr}(vZvZ).$$

For $v > 0$ with diagonal entries α, δ , $\operatorname{Tr}(vZvZ) = (\alpha - \delta)^2 + 2 \det v$. The inequality $\log r - r \leq -1$ for $r > 0$ shows that the maximum over $v \geq 0$ is $-t^2 - 1$, achieved at $v = 1_2$. Singular v gives value $-\infty$. The right-hand side of (6.15) is therefore at most $-(1 + t^2)/2$. In contrast, the actual pencil $S = tX + Z \otimes s$ satisfies

$$S^2 = (t^2 + s^2)1_2, \quad \log \Delta_\varphi(S) = \frac{1}{2} \tau \log(t^2 + s^2) > -\frac{1}{2}.$$

The final inequality follows by comparison with $\tau \log(s^2) = -1$. Thus deleting (6.13) changes the variational value, not just the proof of convexity. Like the scalar formula, the matrix identity maximizes in v first and then minimizes in u ; it does not assert a global saddle point or an interchange of extrema.

Remark 6.5. There is also an ordered form of the strict Hilbert–Schmidt formula which does not require an inverse covariance. For $b > 0$, let c_b be the positive definite matrix defined in (4.6). Since $\mathcal{C}_{c_b} + \eta$ is positive definite on Hermitian matrices, quadratic conjugacy, as in [BV04, Example 3.22], gives

$$\frac{1}{2} \operatorname{Tr}(a_0(\mathcal{C}_{c_b} + \eta)^{-1}(a_0)) = \max_{h=h^*} \left\{ \operatorname{Tr}(a_0 h) - \frac{1}{2} \operatorname{Tr}(hc_b hc_b + h\eta(h)) \right\}.$$

The unique maximizer is $h = (\mathcal{C}_{c_b} + \eta)^{-1}(a_0)$. Inserting this identity in Proposition 1.4 and using (4.7) yields

$$(6.22) \quad \begin{aligned} \log \Delta_\varphi(S) = & -\frac{1}{2} + \inf_{b > 0} \max_{h=h^*} \left\{ \frac{1}{2m} \log \frac{\det(1_m + \eta(b))}{\det b} + \operatorname{tr}_m(a_0 h) \right. \\ & \left. - \frac{1}{2} \operatorname{tr}_m(hc_b hc_b + h\eta(h)) \right\}. \end{aligned}$$

This identity holds for all selfadjoint coefficients and does not use η^{-1} . Unlike (6.15), it does not assert attainment of the infimum over b .

7. SPECTRAL APPLICATIONS

Throughout this section let

$$T = a_0 + \sum_{j=1}^n a_j \otimes s_j \in M_m(\mathcal{M}), \quad S_z = T - z1_m, \quad a_z = a_0 - z1_m,$$

with arbitrary complex coefficient matrices. We keep the maps η and ρ from (1.4). Applying Theorem 1.3 to S_z gives the logarithmic potential of μ_T and characterizes $\text{spec}(T)$.

7.1. Shifted potentials and spectral interpretation. For $b > 0$, write $Z_b(z) = \mathcal{T}_b^{-1}(a_z)$ and

$$\mathcal{V}_z(b) = \log \det H_b - \log \det b - m + \text{Re Tr}(a_z^* Z_b(z)), \quad H_b = 1_m + \eta(b).$$

Theorem 1.3 gives

$$(7.1) \quad u(z) := \log \Delta(S_z) = \frac{1}{2m} \inf_{b>0} \mathcal{V}_z(b),$$

with the value $-\infty$ permitted. It also gives

$$(7.2) \quad z \notin \text{spec}(T) \iff \mathcal{V}_z \text{ has a critical point} \iff \inf_{b>0} \mathcal{V}_z(b) \text{ is attained.}$$

At a point of the resolvent set, the unique minimizer is $b_z = E_m((S_z^* S_z)^{-1})$. The criterion concerns bounded invertibility; finiteness of $u(z)$ alone does not imply it.

We use the Wirtinger derivatives $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, where $z = x + iy$.

Definition 7.1 (Brown measure). The Brown measure of T with respect to φ_m is the probability measure

$$(7.3) \quad \mu_T = \frac{1}{2\pi} \nabla_z^2 u, \quad \nabla_z^2 = 4\partial_z \partial_{\bar{z}},$$

where the Laplacian is taken in the sense of distributions.

Here u is subharmonic and locally integrable, and $\text{supp } \mu_T \subseteq \text{spec}(T)$. With the normalization at infinity fixed by the determinant, it is exactly the logarithmic potential

$$(7.4) \quad u(z) = \int_{\mathbb{C}} \log |z - w| d\mu_T(w).$$

These properties of the Brown measure are recalled in [MS17, Chapter 11, Theorem 5, Definition 6 and (11.6)–(11.8)]. We use them to interpret the variational formula; no equality between $\text{supp } \mu_T$ and $\text{spec}(T)$ is assumed.

For $\varepsilon > 0$, set

$$(7.5) \quad u_\varepsilon(z) = \frac{1}{2} \varphi_m \log(S_z^* S_z + \varepsilon^2 1_m).$$

The rescaled minimum (3.13) reads

$$(7.6) \quad u_\varepsilon(z) = \frac{1}{2m} \min_{b>0} F_\varepsilon(z, b), \quad F_\varepsilon(z, b) = \mathcal{V}_z(b) + \varepsilon^2 \text{Tr } b.$$

Its unique minimizer $b_\varepsilon(z)$ and the matrix $Z_\varepsilon(z) = \mathcal{T}_{b_\varepsilon(z)}^{-1}(a_z)$ are

$$(7.7) \quad \begin{aligned} b_\varepsilon(z) &= E_m((S_z^* S_z + \varepsilon^2 1_m)^{-1}), \\ Z_\varepsilon(z) &= Z_{b_\varepsilon(z)}(z) = E_m((S_z S_z^* + \varepsilon^2 1_m)^{-1} S_z). \end{aligned}$$

We will use these matrices both to recover radial masses and to compute a regularized density. The regularized Cauchy transform and its relation to hermitization are developed in [BSS18, arXiv version, Sections 2.1.6 and 2.2]. Here (7.6) expresses the same regularized potential through a positive-matrix minimum.

7.2. Determinant lower bounds and atoms of the Brown measure. We first prove Corollary 1.7, then locate the possible atoms without assuming positive capacity. For $a \in M_m(\mathbb{C})$ and $b > 0$, put $q_a(b) = \operatorname{Re} \operatorname{Tr}(a^* \mathcal{T}_b^{-1}(a))$.

Proof of Corollary 1.7. Use the real inner product $\langle X, Y \rangle_{\mathbb{R}} = \operatorname{Re} \operatorname{Tr}(X^* Y)$. The positivity established in (3.2), at $\varepsilon = 1$, implies

$$q_a(b) = \langle a, \mathcal{T}_b^{-1}(a) \rangle_{\mathbb{R}} \geq 0.$$

Put $T_0 = \sum_j a_j \otimes s_j$. Taking infima in Theorem 1.3 gives, for every $a \in M_m(\mathbb{C})$,

$$(7.8) \quad \Delta(a + T_0) \geq \Delta(T_0) = e^{-1/2} \operatorname{cap}(\eta)^{1/(2m)}.$$

Taking $a = a_0 - z1_m$ proves (1.18). At positive capacity, this gives the uniform bound

$$u(z) \geq -\frac{1}{2} + \frac{1}{2m} \log \operatorname{cap}(\eta) > -\infty.$$

An atom at z would make (7.4) equal $-\infty$ there, since its positive part is bounded on the compact support. This proves the assertion about atoms. $\square$

Proposition 7.2 (Possible atom locations). *For every $z \in \mathbb{C}$,*

$$(7.9) \quad \det(a_0 - z1_m) \neq 0 \implies \Delta(T - z1_m) > 0.$$

In particular every atom of μ_T belongs to $\operatorname{spec}(a_0)$.

Proof. Put $T_0 = \sum_j a_j \otimes s_j$, and let u_0 be the logarithmic potential of μ_{T_0} . Applying Theorem 1.3 to $T_0 - re^{i\theta}1_m$ and using the real-linearity of $\mathcal{T}_b^{-1}$, we obtain

$$(7.10) \quad u_0(re^{i\theta}) = \frac{1}{2m} \inf_{b>0} \{A(b) + r^2 q_{-e^{i\theta}1_m}(b)\}, \quad A(b) = \log \det H_b - \log \det b - m.$$

Since $q_{-e^{i\theta}1_m}(b) \geq 0$ by (3.2), the potential is nondecreasing in $r \geq 0$ for each fixed θ . It cannot take the value $-\infty$ at a positive radius. Indeed, for any bounded real interval I and any $w \in \mathbb{C}$,

$$\int_I \log^- |re^{i\theta} - w| dr \leq \int_{-1}^1 -\log |t| dt = 2, \quad \log^- t = \max\{-\log t, 0\}.$$

The inequality follows from $|re^{i\theta} - w| \geq |r - \operatorname{Re}(e^{-i\theta}w)|$. The positive logarithmic part is uniformly bounded when w ranges over a compact set. Fubini's theorem therefore shows that a compactly supported logarithmic potential is integrable on every finite line segment. If $u_0(r_0 e^{i\theta}) = -\infty$ for $r_0 > 0$, monotonicity would make it $-\infty$ on $[r_0/2, r_0]e^{i\theta}$, a contradiction. Hence $\Delta(T_0 - z1_m) > 0$ for $z \neq 0$.

Now suppose $A = a_0 - z1_m$ is invertible and write

$$T - z1_m = A(1_m + C), \quad C = \sum_j A^{-1} a_j \otimes s_j.$$

The centered conclusion applied to C at the shift -1 gives $\log \Delta(1_m + C) > -\infty$. If $c > 0$ is the smallest singular value of A , then, with $B = 1_m + C$, $(AB)^*(AB) \geq c^2 B^* B$. Add $\varepsilon^2 1_m$, apply the operator-monotone logarithm and the trace, and let $\varepsilon \downarrow 0$. The result is

$$\log \Delta(T - z1_m) \geq \log c + \log \Delta(1_m + C) > -\infty.$$

This proves (7.9) and the last assertion. $\square$

The atom locations in Proposition 7.2 need not all carry positive mass. Nor does atomlessness imply absolute continuity with respect to planar area. For example, if $T = s$ is a standard selfadjoint semicircular element, then $\eta = \operatorname{id}$ and $\operatorname{cap}(\eta) = 1$, while μ_T is supported on $\mathbb{R}$.

A second distinction concerns regularization at a fixed point. Spectral calculus gives

$$(7.11) \quad \lim_{\varepsilon \downarrow 0} \varepsilon^2 \operatorname{tr}_m b_\varepsilon(z) = \varphi_m(1_{\{0\}}(S_z^* S_z)).$$

This is the trace of the kernel projection of S_z , which need not equal $\mu_T(\{z\})$. More generally, the former is at most the latter; see [ACSY24, Section 2.1, p. 12] for this distinction between point-spectrum weights and atoms of the Brown measure. For the deterministic matrix $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the right side at zero is $1/2$, whereas $\mu_T = \delta_0$: indeed $\log \Delta(T - z1_2) = \log |z|$. Thus divergence of $b_\varepsilon(z)$ alone does not determine $\mu_T(\{z\})$.

7.3. Centered pencils with vanishing ρ . Assume in this subsection that $a_0 = 0$ and $\rho = 0$ as a map on $M_m(\mathbb{C})$.

Example 7.3 (Matrix coefficients and free circular elements). Let $s_1, \dots, s_{2r}$ be free standard semicircular elements and set

$$c_k = \frac{s_{2k-1} + i s_{2k}}{\sqrt{2}}, \quad 1 \leq k \leq r.$$

Then $c_1, \dots, c_r$ are free standard circular elements. For matrices $d_1, \dots, d_r \in M_m(\mathbb{C})$, the operator

$$T = \sum_{k=1}^r d_k \otimes c_k$$

is a centered semicircular pencil with coefficients

$$a_{2k-1} = \frac{d_k}{\sqrt{2}}, \quad a_{2k} = \frac{i d_k}{\sqrt{2}}.$$

For every $x \in M_m(\mathbb{C})$, these pairs give

$$\rho(x) = \frac{1}{2} \sum_{k=1}^r (d_k x d_k + (i d_k) x (i d_k)) = 0, \quad \eta(x) = \sum_{k=1}^r d_k x d_k^*.$$

Thus Proposition 1.8 applies to T with this map η .

For a centered semicircular pencil, all $M_m(\mathbb{C})$ -valued cumulants of (T, T^*) except those of order two vanish. The condition $\rho = 0$ also gives $E_m(TxT) = E_m(T^*xT^*) = 0$. Its only possibly nonzero cumulants are therefore

$$E_m(TxT^*) = \eta(x), \quad E_m(T^*xT) = \eta^*(x), \quad x \in M_m(\mathbb{C}).$$

By [DG24, Theorem 2.9(b)], T is $M_m(\mathbb{C})$ -valued R -diagonal. Since all cumulants of order greater than two vanish, [DG24, Definition 2.10] identifies T as an $M_m(\mathbb{C})$ -valued circular element. These cumulants are unchanged when T is replaced by $e^{i\theta}T$, so its $M_m(\mathbb{C})$ -valued $*$ -distribution is rotation invariant. We now apply Theorem 1.3 to $T - z1_m$ to prove Proposition 1.8 and the following formulas for the logarithmic potential and density of μ_T .

Proposition 7.4 (Radial density and minimizers). *The function f in Proposition 1.8 is given by*

$$(7.12) \quad f(s) = \frac{1}{2m} \inf_{b > 0} \left\{ \log \det H_b - \log \det b - m + s \operatorname{Tr}(H_b^{-1} b) \right\}.$$

The radial distribution F in (1.19) is locally Lipschitz on $(0, \infty)$, and

$$(7.13) \quad \mu_T = \alpha \delta_0 + \beta(z) d^2 z, \quad \beta(z) = \frac{1}{\pi} F'(|z|^2), \quad 0 \leq \beta(z) \leq \frac{F(|z|^2)}{\pi |z|^2} \leq \frac{1}{\pi |z|^2} \quad \text{a.e. } z \neq 0,$$

where $\alpha = \lim_{s \downarrow 0} F(s)$. For every $s > 0$,

$$(7.14) \quad F(s) = \frac{s}{m} \lim_{\varepsilon \downarrow 0} \operatorname{Tr}((1_m + \eta(b_\varepsilon(s)))^{-1} b_\varepsilon(s)),$$

where the minimizer in (7.7) depends only on $s = |z|^2$.

Proof of Propositions 1.8 and 7.4. Since $\rho = 0$, the solution of $\mathcal{T}_b(Z_b(z)) = -z1_m$ is $Z_b(z) = -zH_b^{-1}b$. Substitution in (7.1) proves (7.12). Its slope at each fixed trial is positive, so f is nondecreasing and concave. Radiality of the potential gives rotation invariance of its distributional Laplacian. Let ν be the image of μ_T under $z \mapsto |z|^2$. Averaging the logarithmic kernel over angles yields

$$(7.15) \quad f(s) = \frac{1}{2} \int_{[0,\infty)} \log \max\{s, t\} d\nu(t), \quad s > 0.$$

The scalar identity used here is $(2\pi)^{-1} \int_0^{2\pi} \log |r - te^{i\theta}| d\theta = \log \max\{r, t\}$; it follows by expanding the logarithm after factoring out the larger radius, and by an integrable limit when the radii agree. Compact support bounds the positive part, while the identity bounds the integrated negative part. Thus the interchange of integrals is valid and the result is finite.

The left and right derivatives of (7.15) are

$$f'_-(s) = \frac{\nu([0, s))}{2s}, \quad f'_+(s) = \frac{\nu([0, s])}{2s}.$$

The measure inequality gives $f'_-(s) \leq f'_+(s)$, while concavity gives the opposite inequality. Hence $\nu(\{s\}) = 0$ for every $s > 0$, $f \in C^1((0, \infty))$, and (1.19) follows. Also $F(s)/s = 2f'(s)$ is nonincreasing. For $0 < s_1 < s_2$ this implies

$$0 \leq F(s_2) - F(s_1) \leq (s_2 - s_1) \frac{F(s_1)}{s_1} \leq \frac{s_2 - s_1}{s_1}.$$

Thus F is locally Lipschitz, with $0 \leq F'(s) \leq F(s)/s$ almost everywhere. Polar coordinates, $d^2z = \frac{1}{2} ds d\theta$, now give (7.13). Every possible singular part is concentrated at zero.

The function $F_\varepsilon(z, b)$ in (7.6) depends on z only through $s = |z|^2$. Write $u_\varepsilon(z) = f_\varepsilon(s)$. Uniqueness of the minimizer makes b_ε independent of the argument of z . This minimizer is smooth for $s > 0$, either from its resolvent expression along $z = \sqrt{s}$ or by differentiation of the stationary equation. Differentiating the minimum value therefore gives

$$f'_\varepsilon(s) = \frac{1}{2m} \text{Tr}(H_{b_\varepsilon(s)}^{-1} b_\varepsilon(s)).$$

The functions f_ε are concave and decrease to f . For a fixed positive s , bound their derivatives between their left and right secant slopes. First send ε to zero, then send the secant increment to zero. Since f is differentiable, $f'_\varepsilon(s) \rightarrow f'(s)$, proving (7.14).

For the spectrum, the critical identities (3.15) with $a_z = -z1_m$ and $\rho = 0$ force $z \neq 0$, $Y = -1_m/\bar{z}$, and

$$(7.16) \quad (s \text{ id} - \eta)b = 1_m, \quad b > 0, \quad s = |z|^2.$$

If $s > r(\eta)$, the positive convergent series $b = \sum_{k \geq 0} s^{-k-1} \eta^k (1_m)$ solves this equation. With the displayed Y it satisfies $\mathcal{T}_b(Y) = -z1_m$, has $P = H_b^{-1} - Yb^{-1}Y^* = 0$, and satisfies (3.14). Theorem 1.3 therefore puts z in the resolvent set. Conversely, any positive solution of (7.16) makes

$$x \mapsto s^{-1}b^{-1/2}\eta(b^{1/2}xb^{1/2})b^{-1/2}$$

completely positive with value $1_m - s^{-1}b^{-1} < 1_m$ at the identity. Its norm, and hence its spectral radius, is less than one. Similarity gives $r(\eta) < s$. At zero the first critical identity would read $0 = Y^{-*}$, which is impossible. This proves (1.20). Finally Corollary 1.7 removes the origin atom when capacity is positive. $\square$

The spectral disk in (1.20) is the full operator spectrum. The argument above does not show that $\text{supp } \mu_T$ fills that disk. The estimate on $F(s)/s$ concerns the average mass per unit squared radius; it does not assert that the pointwise density is nonincreasing. At zero the density need not have a finite bound.

Under the stronger two-sided flatness assumption $c \text{tr}_m(x)1_m \leq \eta(x) \leq C \text{tr}_m(x)1_m$ for all $x \geq 0$, with $C > c > 0$, Alt and Krüger prove analyticity and positive upper and lower bounds for the density inside the disk [AK20, Proposition 2.9 and Theorem 2.5]. The argument above allows degenerate maps and yields the weaker regularity stated in Propositions 1.8 and 7.4.

Remark 7.5 (Comparison with scalar R -diagonal elements). For a scalar R -diagonal element, represented in $*$ -distribution as uh with u Haar unitary, $h \geq 0$ nonconstant and u, h free, Haagerup and Larsen describe the radial distribution through the S -transform of the law of h^2 [HL00, author preprint, Theorem 4.4]. The formula above instead uses the matrix covariance map and its variational minimizers. The condition $\rho = 0$ gives circularity over $M_m(\mathbb{C})$ in Example 7.3; it does not by itself give scalar R -diagonality with respect to φ_m . Thus the scalar S -transform formula requires an additional hypothesis in the matrix-coefficient setting.

7.4. An invertibility criterion and enclosed mass. The attainment criterion can be expressed using an invertible matrix and a spectral-radius bound. For $g \in M_m(\mathbb{C})$ invertible, put

$$\Psi_g(x) = g\eta(x)g^*.$$

Proposition 7.6. *A point z lies outside $\text{spec}(T)$ if and only if there is an invertible $g \in M_m(\mathbb{C})$ such that*

$$(7.17) \quad a_z = g^{-1} + \rho(g), \quad r(\Psi_g) < 1.$$

This matrix is unique and equals $g_z = E_m(S_z^{-1})$. The unique minimizer is

$$(7.18) \quad b_z = (\text{id} - \Psi_{g_z})^{-1}(g_z g_z^*) = \sum_{k \geq 0} \Psi_{g_z}^k(g_z g_z^*) > 0.$$

The strict spectral-radius condition is equivalent to the existence of a positive matrix b with $b - \Psi_g(b) > 0$.

Proof. If $z \notin \text{spec}(T)$, Theorem 1.3 gives an attained minimum. The identities (3.15) then give (7.17) with $g = Y^*$ and $b - g\eta(b)g^* = gg^* > 0$. Conjugating by $b^{1/2}$ turns Ψ_g into a completely positive map with value strictly less than 1_m at the identity, so its spectral radius is less than one.

Conversely, suppose (7.17) holds and define b by the positive series in (7.18). Then $b = gH_b g^*$. Put $Y = g^*$. We have

$$H_b Y b^{-1} = g^{-1}, \quad H_b Y b^{-1} + \rho(Y^*) = a_z,$$

so $Y = \mathcal{T}_b^{-1}(a_z)$. Also $H_b^{-1} - Y b^{-1} Y^* = 0$ and $Y^* H_b Y = b$. These two identities give (3.14). Theorem 1.3 therefore gives $z \notin \text{spec}(T)$, $g = E_m(S_z^{-1})$ and $b = b_z$. It also proves uniqueness.

The implication from $b - \Psi_g(b) > 0$ to $r(\Psi_g) < 1$ has already been proved. For the reverse implication, $r(\Psi_g) < 1$ makes $b = \sum_{k \geq 0} \Psi_g^k(1_m) > 0$ a solution of $b - \Psi_g(b) = 1_m$. $\square$

The equation $a_z = g^{-1} + \rho(g)$ alone need not determine $E_m(S_z^{-1})$. The additional condition $r(\Psi_g) < 1$ selects that matrix; equivalently, one can find $b > 0$ with $b - \Psi_g(b) > 0$.

Corollary 7.7. *Let Γ be a positively oriented piecewise smooth simple closed curve in the resolvent set of T , and let Ω be its bounded interior. Then*

$$(7.19) \quad \mu_T(\Omega) = -\frac{1}{2\pi i} \int_{\Gamma} \text{tr}_m(g_z) dz.$$

If $\rho = 0$, the full spectrum and the exterior potential are

$$(7.20) \quad \text{spec}(T) = \text{spec}(a_0) \cup \{z \notin \text{spec}(a_0) : r(x \mapsto a_z^{-1}\eta(x)a_z^{-*}) \geq 1\},$$

$$(7.21) \quad u(z) = \log \Delta_m(a_z) = \frac{1}{m} \log |\det a_z|, \quad z \notin \text{spec}(T).$$

In this case

$$(7.22) \quad \mu_T(\Omega) = \frac{1}{m} \#\{\text{eigenvalues of } a_0 \text{ in } \Omega, \text{ counted algebraically}\}.$$

Proof. By Proposition 7.6, $g_z = E_m(S_z^{-1})$ off the spectrum. Differentiation of the trace logarithm gives

$$2\partial_z u(z) = -\varphi_m(S_z^{-1}) = -\text{tr}_m(g_z).$$

Differentiating (7.4) there instead gives $2\partial_z u(z) = \int (z - w)^{-1} d\mu_T(w)$. The curve has positive distance from $\text{supp } \mu_T$, so integration and the scalar Cauchy formula prove (7.19).

If $\rho = 0$, (7.17) forces $g_z = a_z^{-1}$. The equation has no solution when a_z is singular. When a_z is invertible, $r(\Psi_{g_z}) < 1$ holds exactly under the strict inequality complementary to (7.20). At a resolvent point, $b_z = a_z^{-1}H_{b_z}a_z^{-*}$ and $Z_{b_z} = a_z^{-*}$. Thus $\det H_{b_z}/\det b_z = |\det a_z|^2$, while the quadratic correction equals $\text{Re Tr}(a_z^*a_z^{-*}) = m$. Substitution proves (7.21) in every resolvent component. Finally $-g_z = (z1_m - a_0)^{-1}$ in (7.19), and the finite-dimensional argument principle proves (7.22). $\square$

In particular, the mass of any region separated by such a contour is a multiple of $1/m$ when $\rho = 0$. The contour hypothesis is part of this statement; it does not give an unconditional formula for the mass of each atom of μ_T .

Remark 7.8 (Diagonal variance profiles). Assume $\rho = 0$, $a_0 = \text{diag}(\alpha_1, \dots, \alpha_m)$ and $\eta(x) = \text{diag}(\sum_j s_{ij}x_{jj})$, with $s_{ij} \geq 0$. Put $S = (s_{ij})$ and $D_z = \text{diag}(|\alpha_i - z|^{-2})$ for $z \notin \text{spec}(a_0)$. The map $x \mapsto a_z^{-1}\eta(x)a_z^{-*}$ vanishes on matrices with zero diagonal and restricts to $D_z S$ on diagonal matrices. Hence its spectral radius equals $r(D_z S)$. In this case (7.20) reduces to the variance-profile test for a deformed circular element over a commutative base. Alt and Krüger use the same threshold $r(SD_z) = 1$ in [AK26, arXiv version, Proposition 5.16]; the two products are similar. Under their block-primitivity assumption A1 and data regularity assumption A2, they also identify the support of the Brown measure and prove boundedness and interior analyticity of the density [AK26, arXiv version, Theorem 2.2]. Assumption A2 is automatic for a finite base, whereas A1 remains an additional condition. The elementary reduction above requires neither condition and concerns the operator spectrum.

Remark 7.9 (Spectral subspaces). When the ambient algebra is a II_1 -factor, Haagerup and Schultz associate to every Borel set B an invariant spectral subspace whose projection has trace $\mu_T(B)$ [HS09, Theorem 1.1]. For a region separated by a resolvent contour, the subspace is the range of the Riesz idempotent $e_\Omega = (2\pi i)^{-1} \int_\Gamma (z1_m - T)^{-1} dz$; compare [HS09, Introduction, (1.3)]. Thus (7.19) computes the mass of this spectral block. Under $\rho = 0$, the further eigenvalue-count identity follows from the finite-dimensional argument principle.

7.5. Density from the variational optimizer. To differentiate the minimum value, we first prove that the Hessian at the minimizer is positive definite, using a positive matrix regularizer. All gradients and Hessians with respect to b are taken on the real Hilbert space of Hermitian matrices with pairing $\langle k, l \rangle = \text{Tr}(kl)$.

Lemma 7.10. *Fix an affine pencil S with constant term a , and let $\mathcal{V}_a$ be the function in (1.9) with $a_0 = a$. For every $D \in M_m(\mathbb{C})$ with $D > 0$,*

$$(7.23) \quad \frac{1}{2} \varphi_m \log(S^* S + D) = \frac{1}{2m} \min_{b > 0} \{\mathcal{V}_a(b) + \text{Tr}(Db)\}.$$

Its unique minimizer is $b_D = E_m((S^*S + D)^{-1})$. The real Hessian $\mathfrak{H}_D = D_b^2 \mathcal{V}_a(b_D)$ is positive definite, and its inverse is the completely positive map

$$(7.24) \quad \mathfrak{H}_D^{-1}(k) = E_m((S^*S + D)^{-1}k(S^*S + D)^{-1}).$$

Proof. Apply (3.13), at $\varepsilon = 1$, to $SD^{-1/2}$. Its coefficient maps are

$$\eta_D(x) = \eta(D^{-1/2}xD^{-1/2}), \quad \rho_D(x) = \sum_j a_j D^{-1/2}x a_j D^{-1/2}.$$

Use the trial bijection $b' = D^{1/2}bD^{1/2}$. If $Z = \mathcal{T}_b^{-1}(a)$, the corresponding solution for $SD^{-1/2}$ is $Z' = ZD^{1/2}$: its equation is the old equation multiplied on the right by $D^{-1/2}$. The quadratic correction $\text{Re Tr}(a^*Z)$ is unchanged, the logarithm of the determinant ratio decreases by $\log \det D$, and the linear term is $\text{Tr}(Db)$. The same determinant term appears on the operator side, since

$$(SD^{-1/2})^*(SD^{-1/2}) + 1_m = D^{-1/2}(S^*S + D)D^{-1/2}.$$

The trace-log congruence identity follows without commutation: for positive invertible Q , differentiate $\varphi_m \log(D^{-t/2}QD^{-t/2})$. Cyclicity of the trace makes its derivative $-\text{tr}_m \log D$; integrating from $t = 0$ to $t = 1$ gives the required subtraction of $m^{-1} \log \det D$. This proves (7.23). Transforming the known unique minimizer gives b_D as stated.

Put $Q_D = S^*S + D$. Stationarity and its derivative in a Hermitian direction k give

$$\nabla \mathcal{V}_a(b_D) + D = 0, \quad D_D b_D[k] = -E_m(Q_D^{-1}kQ_D^{-1}) =: -\mathcal{A}_D(k), \quad \mathfrak{H}_D \mathcal{A}_D = \text{id}.$$

The map $\mathcal{A}_D$ is completely positive and selfadjoint for the trace pairing. Moreover

$$\langle k, \mathcal{A}_D(k) \rangle = (\text{Tr} \otimes \tau)((Q_D^{-1/2}kQ_D^{-1/2})^2) > 0 \quad (k = k^* \neq 0).$$

Faithfulness and invertibility of Q_D give strictness. Thus $\mathcal{A}_D$ is positive definite on the finite-dimensional real space, so $\mathfrak{H}_D = \mathcal{A}_D^{-1} > 0$ and (7.24) follows. $\square$

The same formula at $D = t1_m \downarrow 0$ shows that, whenever S is boundedly invertible, the Hessian at its unregularized minimizer is positive definite with inverse $k \mapsto E_m((S^*S)^{-1}k(S^*S)^{-1})$. This is a statement at the minimizer. It does not imply global convexity of $\mathcal{V}_a$. At positive regularizer one obtains the quadratic-form bounds

$$\lambda_{\min}(D)^2 \text{id} \leq \mathfrak{H}_D \leq (\|S\|^2 + \|D\|)^2 \text{id}.$$

Indeed the operator $Q_D^{-1/2}kQ_D^{-1/2}$ in the proof has Hilbert–Schmidt norm between $\|Q_D\|^{-1}\|k\|_2$ and $\lambda_{\min}(D)^{-1}\|k\|_2$. The lower bound can degenerate as D tends to zero.

Proposition 7.11. *For each $\varepsilon > 0$, the regularized logarithmic potential satisfies*

$$(7.25) \quad \partial_z u_\varepsilon(z) = -\frac{1}{2m} \text{Tr } Z_\varepsilon(z)^*.$$

Evaluate all derivatives of F_ε at $(z, b_\varepsilon(z))$, and set

$$\mathfrak{H} = D_b^2 F_\varepsilon > 0, \quad h_x = \partial_x \nabla_b F_\varepsilon, \quad h_y = \partial_y \nabla_b F_\varepsilon.$$

Set $\beta_\varepsilon = (2\pi)^{-1} \nabla_z^2 u_\varepsilon$. Then

$$(7.26) \quad \begin{aligned} \beta_\varepsilon(z) = & \frac{1}{4\pi m} (2\langle 1_m, \mathcal{T}_b^{-1}(1_m) \rangle_{\mathbb{R}} + 2\langle i1_m, \mathcal{T}_b^{-1}(i1_m) \rangle_{\mathbb{R}} \\ & - \langle h_x, \mathfrak{H}^{-1}h_x \rangle - \langle h_y, \mathfrak{H}^{-1}h_y \rangle), \quad b = b_\varepsilon(z). \end{aligned}$$

It is positive, and $\beta_\varepsilon(z) d^2 z$ converges weakly to μ_T as $\varepsilon \downarrow 0$.

Proof. For fixed positive ε , the resolvent formulas (7.7) are real analytic in x, y . At the minimizer the derivative with respect to b vanishes, so differentiation of the minimum value uses only the explicit parameter derivative. The quadratic correction $\operatorname{Re} \operatorname{Tr}(a^* Z_b)$ has first variation $2\langle da, Z_b \rangle_{\mathbb{R}}$. Since $\partial_x a_z = -1_m$ and $\partial_y a_z = -i1_m$, it follows that

$$\partial_x u_\varepsilon = -\frac{1}{m} \operatorname{Re} \operatorname{Tr} Z_\varepsilon, \quad \partial_y u_\varepsilon = -\frac{1}{m} \operatorname{Im} \operatorname{Tr} Z_\varepsilon.$$

This is (7.25).

Lemma 7.10, with $D = \varepsilon^2 1_m$, proves $\mathfrak{H} > 0$. Differentiating stationarity gives $\partial_x b_\varepsilon = -\mathfrak{H}^{-1} h_x$ and $\partial_y b_\varepsilon = -\mathfrak{H}^{-1} h_y$. Hence the real Hessian of the optimized value is the Schur complement

$$(7.27) \quad \partial_{jk} u_\varepsilon = \frac{1}{2m} (F_{jk} - \langle h_j, \mathfrak{H}^{-1} h_k \rangle), \quad j, k \in \{x, y\}.$$

At fixed b , $F_\varepsilon(z, b)$ is a real quadratic function of x, y , so $F_{xx} = 2\langle 1_m, \mathcal{T}_b^{-1}(1_m) \rangle_{\mathbb{R}}$ and $F_{yy} = 2\langle i1_m, \mathcal{T}_b^{-1}(i1_m) \rangle_{\mathbb{R}}$. Taking the normalized Laplacian proves (7.26).

For the sign, put $Q = S_z^* S_z + \varepsilon^2 1_m$ and $P = S_z S_z^* + \varepsilon^2 1_m$. Direct trace differentiation and $S_z Q^{-1} S_z^* = 1_m - \varepsilon^2 P^{-1}$ give

$$\partial_z \partial_z u_\varepsilon = \frac{1}{2} \varphi_m(Q^{-1} - Q^{-1} S_z Q^{-1} S_z^*) = \frac{\varepsilon^2}{2} \varphi_m(Q^{-1} P^{-1}).$$

Thus

$$(7.28) \quad \beta_\varepsilon(z) = \frac{\varepsilon^2}{\pi} \varphi_m(Q^{-1} P^{-1}) > 0.$$

The trace equals that of the positive invertible operator $Q^{-1/2} P^{-1} Q^{-1/2}$; commutation is unnecessary. The weak convergence follows from [MS17, Chapter 11, Section 9, (11.14)–(11.15)], with the normalization in (7.3). $\square$

Remark 7.12 (Free circular addition and uniform density bounds). Suppose x is an operator in a finite tracial von Neumann algebra and c_t is a circular element of variance $t > 0$, scalar $*$ -free from x . Zhong's density formula for $x + c_t$ on its region where the subordination parameter is positive combines a two-resolvent term with the derivative of an implicitly determined parameter [Zho26, arXiv version, Theorem 4.2]. This gives a useful comparison with the optimizer differentiation above. Belinschi, Zhi Yin and Zhong prove, in this setting, that the regularized densities are uniformly bounded by $1/(\pi t)$ and that the limiting Brown measure is absolutely continuous with the same bound [BYZ24, arXiv version, Theorem 7.10 and Lemma 7.11]. Such a bound supplies the control needed when the regularizer tends to zero. The scalar freeness and nonzero circular variance in these results are additional assumptions; the general matrix-coefficient formula (7.26) does not supply them.

Formula (7.26) uses the positive minimizer, the inverse of $\mathcal{T}_b$, and two linear systems with the positive definite coefficient operator $\mathfrak{H}$ on Hermitian matrices. It provides another way to compute the same derivatives as the hermitized Dyson equation. To make the comparison explicit, set

$$A_z = \begin{pmatrix} 0 & a_z \\ a_z^* & 0 \end{pmatrix}, \quad G^{-1} = i\varepsilon 1_{2m} - A_z - \Theta(G), \quad \operatorname{Im} G < 0.$$

With $E_{21} = \begin{pmatrix} 0 & 0 \\ 1_m & 0 \end{pmatrix}$, differentiation gives the uniquely solvable equation

$$(7.29) \quad X - G\Theta(X)G = -GE_{21}G,$$

and

$$(7.30) \quad \partial_z u_\varepsilon = \frac{1}{2} \operatorname{tr}_m G_{21}, \quad \beta_\varepsilon = \frac{1}{\pi} \operatorname{tr}_m X_{21}.$$

Indeed $G_{21} = -E_m(Q^{-1}S_z^*) = -Z_\varepsilon^*$ and $X = \partial_z G$; invertibility of the linearized equation follows from the weighted contraction after (2.2). Neither formulation gives a uniform zero-regularizer inverse bound. Weak convergence permits atoms and curve-supported mass and does not justify pointwise convergence of the densities for arbitrary coefficients.

Remark 7.13 (Linear stability of the Dyson equation). Writing $\mathcal{C}_G(X) = GXG$, the operator in (7.29) is $\text{id} - \mathcal{C}_G \Theta$. Under the convention $M = -G$, it is the stability operator studied in [AEK19, Section 4.2, Proposition 4.4 and Theorem 2.6]. Their quantitative estimates use a flatness assumption, and Theorem 2.6 also assumes spatial decay. Our block covariance does not satisfy the strictly positive lower bound in the flatness assumption: for $P > 0$, (2.6) gives

$$\Theta \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \eta^*(P) \end{pmatrix}.$$

Here invertibility at positive ε follows from the weighted estimate in Section 2. At the real parameter used in Section 3, the strict inequality (3.17) instead gives the invertible linearization needed for analytic continuation and the spectral gap.

7.6. Recoveries and examples. If a_0, a_j are selfadjoint, $T = T^*$ and the smaller matrix Cauchy transform satisfies

$$g(w) = E_m((w1_m - T)^{-1}), \quad g(w)^{-1} = w1_m - a_0 - \eta(g(w)), \quad \text{Im } g(w) < 0 \quad (\text{Im } w > 0).$$

Its normalized trace recovers interval masses and atoms by (2.9) and (2.10); the potential is given by (2.4). For a selfadjoint operator the Brown measure equals its scalar spectral measure on the real line [MS17, Chapter 11, p. 270], and faithfulness identifies its support with the spectrum. The gap criterion in Section 2.4 therefore applies directly.

For isotropic covariance $\eta(b) = \sigma \text{tr}_m(b)1_m$, Proposition 6.2 reduces the calculation to two real variables. The density of the spectral measure of T is $q(X^{-1}(x))/(\pi\sigma)$, as stated in (6.9). This one-dimensional density and the planar densities in Proposition 7.4 refer to their respective hypotheses.

For a scalar circular variable of variance $v > 0$, put $\eta(b) = vb$ and $\rho = 0$. In (7.12), the substitution $t = v + b^{-1} > v$ gives

$$f(s) = \frac{1}{2} \inf_{t > v} \{\log t - 1 + s/t\} = \begin{cases} \frac{1}{2}(\log v - 1 + s/v), & 0 \leq s \leq v, \\ \frac{1}{2} \log s, & s \geq v. \end{cases}$$

For $s \leq v$ the infimum is the limit at $t = v$; for $s > v$ it is attained at $t = s$. Thus $F(s) = \min\{s/v, 1\}$ and $\beta(z) = (\pi v)^{-1} 1_{\{|z|^2 < v\}}$, the normalized uniform law on the disk of radius $\sqrt{v}$. If instead $T = \text{diag}(c, 0)$ with c standard circular, the normalized potential is the mean of the two diagonal potentials. It gives half the circular law and half an atom at zero, illustrating the possibility allowed by (7.13). More generally, invariant block decompositions give weighted sums of the Brown measures of the diagonal blocks; in a II_1 -factor this is [HS09, Proposition 6.5], with the weights given by the traces of the block projections.

Finally, for the scalar deterministic pencil $T = \lambda$,

$$F_\varepsilon(z, b) = -\log b - 1 + (|\lambda - z|^2 + \varepsilon^2)b, \quad b_\varepsilon = (|\lambda - z|^2 + \varepsilon^2)^{-1}.$$

Its Hessian is b_ε^{-2} and its mixed derivatives are $h_x = 2(x - \text{Re } \lambda)$, $h_y = 2(y - \text{Im } \lambda)$. Substitution into (7.26) yields

$$\beta_\varepsilon(z) = \frac{\varepsilon^2}{\pi(|\lambda - z|^2 + \varepsilon^2)^2}.$$

Polar integration gives total mass one, and the mass outside every fixed neighborhood of λ tends to zero. Hence the limiting measure is δ_λ , in agreement with the determinant potential. This also verifies the signs and both real-direction contributions in the Hessian formula.

APPENDIX A. PROOF OF THE GLOBAL HILBERT–SCHMIDT COMPARISON

We prove Lemma 4.8, retaining a positive regularizer until the last step. All traces on finite matrix algebras and their corners are unnormalized. Set $K = \text{id} + \Psi$ and

$$\ell_\varepsilon(a) = \frac{1}{2} \varphi_m \log((a + T_\Psi)^2 + \varepsilon^2 1_m), \quad F_\varepsilon(a) = \ell_\varepsilon(a) - \frac{1}{2m} \text{Tr}(aK^{-1}(a)), \quad a = a^*.$$

We will establish the uniform bound

$$(A.1) \quad F_\varepsilon(a) \leq -\frac{1}{2} + \frac{\varepsilon + \varepsilon^2/2}{1 - \theta}, \quad \theta = \|\Psi\|_{2 \rightarrow 2} < 1.$$

A.1. An inverse comparison. If $A = A^*$ has $\|A\| \leq \theta < 1$ on a finite-dimensional Hilbert space $\mathcal{H}$ and $J = J^* = J^{-1}$, then

$$(A.2) \quad L = (J - A)^{-1} \leq (\text{id} - A)^{-1}, \quad \|L\| \leq (1 - \theta)^{-1}.$$

The norm bound follows by factoring $J - A = J(\text{id} - JA)$ and using the Neumann series. To prove the order bound, put $D_0 = \text{id} - A > 0$ and $P = (\text{id} - J)/2$, an orthogonal projection. When $P \neq 0$, block elimination, or direct multiplication, gives

$$(D_0 - 2P)^{-1} = D_0^{-1} + 2D_0^{-1}P(\text{id}_{P\mathcal{H}} - 2PD_0^{-1}P)^{-1}PD_0^{-1}.$$

Because $D_0^{-1} \geq (1 + \theta)^{-1} \text{id} > \frac{1}{2} \text{id}$, the middle inverse is strictly negative on $P\mathcal{H}$. The correction is negative semidefinite, which proves (A.2). For $P = 0$ the correction is absent.

There is an additional positivity-preservation fact when A is a completely positive map and $J(x) = uxu$ with $u = u^* = u^{-1}$:

$$(A.3) \quad L = \sum_{k \geq 0} (JA)^k J$$

is completely positive. Each summand is completely positive and the series converges in norm. This does not assert that L is positive as a Hilbert–Schmidt operator; the order comparison in (A.2) is a separate assertion.

A.2. A maximum over the constant term. For fixed $\varepsilon > 0$, the function F_ε attains its global maximum on the real vector space of Hermitian matrices. Indeed, $K^{-1} \geq (1 + \theta)^{-1} \text{id}$, whereas

$$\ell_\varepsilon(a) \leq \frac{1}{2} \log((\|a\| + \|T_\Psi\|)^2 + \varepsilon^2).$$

The negative quadratic term consequently dominates as $\|a\|_2 \rightarrow \infty$. Continuity gives a maximum.

Fix a maximizing a , and write

$$R = E_m((a + T_\Psi - i\varepsilon 1_m)^{-1}) = H + iQ, \quad H = H^*, \quad Q > 0.$$

Differentiation under the trace gives $d\ell_\varepsilon(a)[h] = m^{-1} \text{Tr}(Hh)$ for $h = h^*$. For example, differentiate $(a + T_\Psi)^2 + \varepsilon^2 1_m$ and use the traced logarithmic derivative and cyclicity. Stationarity therefore gives $H = K^{-1}(a)$. The Dyson equation becomes

$$(A.4) \quad R^{-1} = a - \Psi(R) - i\varepsilon 1_m = H - iB, \quad B = \Psi(Q) + \varepsilon 1_m > 0.$$

The logarithmic primitive (2.4) gives

$$\ell_\varepsilon(a) = -\frac{1}{m} \log |\det R| + \frac{1}{2m} \text{Tr}(H\Psi(H) - Q\Psi(Q)).$$

Taking real traces in $RR^{-1} = 1_m$ yields $\text{Tr} H^2 + \text{Tr}(Q\Psi(Q)) + \varepsilon \text{Tr} Q = m$. Thus

$$(A.5) \quad F_\varepsilon(a) = -\frac{1}{2} - \frac{1}{m} \log |\det R| + \frac{\varepsilon}{2m} \text{Tr} Q.$$

We next control the last two terms, including when H has a kernel.

A.3. The support and kernel of the real part. Multiplying (A.4) in both orders gives $[Q, B] + i[Q + B, H] = 0$. The first summand is skewadjoint and the second is selfadjoint, so each vanishes. The matrix $QH - HB$ is then selfadjoint. Separating the two parts of $H^2 + QB + i(QH - HB) = 1_m$ gives

$$(A.6) \quad QB = BQ = 1_m - H^2, \quad QH = HB, \quad HQ = BH.$$

Let e be the orthogonal projection onto the range of H , let $f = 1_m - e$, and put $d = \text{rank } e$, $r = \text{rank } f$. The matrices Q, B, R commute with these projections, since Q and B commute with H^2 .

On the e -corner the matrix H_e is invertible. Set $u = \text{sign}(H_e)$, the selfadjoint unitary with eigenvalues the signs of the eigenvalues of H_e . Since Q_e commutes with $|H_e|$, (A.6) implies $B_e = uQ_e u$. Consequently $R_e^{-1} = uR_e^* u$, so $|\det R_e| = 1$. On the f -corner, $H_f = 0$, and the same identities give $R_f = iQ_f$, $B_f = Q_f^{-1}$. Therefore

$$(A.7) \quad \log |\det R| = \log \det Q_f.$$

When a corner is absent, its determinant is one and its trace is zero.

For $v, w \in \{e, f\}$ write $\Psi_{vw}(x) = v\Psi(w x w)v$ for the maps between the corner matrix algebras $M_e = eM_m(\mathbb{C})e$ and $M_f = fM_m(\mathbb{C})f$. On their direct sum the compressed map is selfadjoint, has norm at most θ , and has mutually adjoint off-diagonal blocks. Each block preserves complete positivity.

Assume first that $e \neq 0$, and let $A = \Psi_{ee}$, $J(x) = u x u$, $L = (J - A)^{-1}$. By (A.2)–(A.3), L is selfadjoint and completely positive, with $\|L\| \leq (1 - \theta)^{-1}$ and $L \leq (\text{id} - A)^{-1}$. The e -part of $B = \Psi(Q) + \varepsilon 1_m$ gives

$$(A.8) \quad Q_e = L\Psi_{ef}(Q_f) + \varepsilon L(e).$$

On the f -corner define

$$(A.9) \quad \Phi = \Psi_{ff} + \Psi_{fe} L \Psi_{ef}, \quad D = f + \Psi_{fe}(L(e)).$$

Then Φ is completely positive and Hilbert–Schmidt selfadjoint, and $D \geq f > 0$ on this corner. Moreover $\text{id}_f - \Phi > 0$ as a Hilbert–Schmidt operator. Indeed, the positive block operator $\text{id} - \Psi|_{M_e \oplus M_f}$ has positive Schur complement

$$\text{id}_f - \Psi_{ff} - \Psi_{fe}(\text{id}_e - A)^{-1} \Psi_{ef} > 0.$$

Replacing $(\text{id}_e - A)^{-1}$ by the smaller operator L preserves this inequality. In particular,

$$(A.10) \quad \text{Tr } \Phi(f) < r \quad \text{if } r > 0.$$

The remaining Dyson equation and the trace of (A.8) now read

$$(A.11) \quad Q_f^{-1} = \Phi(Q_f) + \varepsilon D, \quad \text{Tr } Q = \text{Tr}(DQ_f) + \varepsilon \text{Tr } L(e).$$

The second equality uses selfadjointness of L and $\Psi_{ef}^* = \Psi_{fe}$. If $e = 0$, take $\Phi = \Psi$, $D = 1_m$ and omit all terms containing L ; the same conclusions hold.

A.4. The positive-cone estimate. Suppose $r > 0$. Lemma 4.1, applied on the f -corner with covariance Φ and positive matrix parameter εD , shows that Q_f minimizes

$$-\log \det x + \varepsilon \text{Tr}(Dx) + \frac{1}{2} \text{Tr}(x\Phi(x)), \quad x > 0 \text{ on } f\mathbb{C}^m.$$

Comparison with $x = f$, together with the trace of (A.11) multiplied by Q_f , yields

$$\log \det Q_f \geq \frac{\varepsilon}{2} \text{Tr}(DQ_f) - \varepsilon \text{Tr } D + \frac{1}{2}(r - \text{Tr } \Phi(f)).$$

Substitute this into (A.5) and use (A.7)–(A.11). The terms containing $\text{Tr}(DQ_f)$ cancel, and the last term is nonnegative by (A.10). We obtain

$$(A.12) \quad F_\varepsilon(a) \leq -\frac{1}{2} + \frac{\varepsilon}{m} \text{Tr } D + \frac{\varepsilon^2}{2m} \text{Tr } L(e).$$

If $f = 0$, (A.7) and (A.8) give the same bound with the D -term omitted.

These error terms are uniform over the maximizing matrices a and the support projections of the associated matrices H . Since the Hilbert–Schmidt norm of a rank- d identity is $\sqrt{d}$, complete positivity and the norm bounds give

$$0 \leq \text{Tr } L(e) \leq \frac{d}{1-\theta}, \quad \text{Tr } D \leq r + \frac{\theta\sqrt{dr}}{1-\theta} \leq \frac{m}{1-\theta}.$$

Thus (A.12) implies (A.1) at every maximum and hence at every Hermitian constant term a . Finally, spectral integration gives $\ell_\varepsilon(a) \downarrow \log \Delta(a + T_\Psi)$ as $\varepsilon \downarrow 0$. This proves (4.13), including a limit of $-\infty$.

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