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REPRESENTATION EMBEDDINGS AND THE THIRD BRAUER–THRALL CONJECTURE

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Abstract. Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k. We prove that for every infinite cardinal κ there is an indecomposable A-module M with $\dim_k M = \kappa$. Over algebraically closed fields, this settles Simson's conjecture, also called the third Brauer–Thrall conjecture, that every representation-infinite finite-dimensional algebra has indecomposable modules of arbitrarily large cardinality. To construct these modules, we extend Bongartz's representation embedding from finite-dimensional Kronecker modules without a simple injective summand to such modules of arbitrary dimension. The extended functor commutes with direct sums and filtered colimits and preserves the dimension of infinite-dimensional modules. Applying it to the indecomposable Kronecker modules of every infinite dimension constructed by Göbel and Simson yields the required A-modules.

Contents

1. Introduction 1
1.1. Representation embeddings on modules of arbitrary dimension 2
1.2. Indecomposable modules of prescribed infinite dimension 3
1.3. Outline of the proof 3
1.4. Related work 4
2. Kronecker modules and nilpotent ideals 5
2.1. Kronecker modules without simple injective summands 5
2.2. Nilpotent kernels and indecomposability 7
3. Bongartz's representation embeddings 7
3.1. Reduction to minimal representation-infinite algebras 7
3.2. Quotients of projective modules by submodules of the socle 8
3.3. Galois coverings and push-down functors 9
3.4. Push-down along a closed walk 11
3.5. Existence of the embeddings 12
4. Homomorphisms between the embedded modules 13
4.1. Homomorphisms between quotients by submodules of the socle 13
4.2. Vertex evaluations and block matrices 14
4.3. Endomorphisms of push-downs along a covering 15
4.4. Endomorphisms of push-downs along a closed walk 16
4.5. A uniform nilpotence bound 18
5. The embedding theorem and large families 19
Appendix A. Corner quotients of a special biserial algebra 22
References 23

1. Introduction

Let A be a finite-dimensional algebra over a field k. The algebra A is called *representation-finite* if there are only finitely many isomorphism classes of finite-dimensional indecomposable

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A-modules, and *representation-infinite* otherwise. The Brauer–Thrall conjectures concern the indecomposable modules of a representation-infinite algebra; see Ringel [20] for their origin. The first Brauer–Thrall conjecture asserts that a representation-infinite algebra has finite-dimensional indecomposable modules of arbitrarily large dimension. Roiter [24] proved it for finite-dimensional algebras over any field, and Auslander [2] proved the analogue for Artin rings, with length in place of dimension. The second Brauer–Thrall conjecture asserts that, over an infinite field, a representation-infinite algebra has infinitely many positive integers d for which there are infinitely many pairwise non-isomorphic indecomposable modules of dimension d. Over algebraically closed fields of arbitrary characteristic, it was proved by Bautista [4], using Bongartz's standardness theorem [7]; Fischbacher [12] and Bretscher and Todorov [8] subsequently gave alternative proofs (see Bongartz [6, §5.1] for the history). In [6, Theorem 6], Bongartz also proved the second conjecture over algebraically closed fields by constructing representation embeddings from the category of finite-dimensional Kronecker modules without a simple injective summand into the category $\operatorname{mod} A$ of finite-dimensional A-modules. These functors transfer indecomposable Kronecker modules to indecomposable A-modules and are the starting point of this paper.

Both of these conjectures concern finite-dimensional modules. For modules of arbitrary size, Ringel and Tachikawa [23, Corollary 4.4] proved that every module over a representation-finite Artin algebra is a direct sum of finitely generated indecomposable modules (see also Auslander [3, pp. 1–2]), and Auslander [3, Theorem A] proved conversely that if every indecomposable module over an Artin algebra is finitely generated, then the algebra is representation-finite. Thus every representation-infinite finite-dimensional algebra has infinite-dimensional indecomposable modules, and one may ask how large they can be. As an infinite-dimensional counterpart of the first Brauer–Thrall conjecture, Simson [27, Conjecture 1_∞] conjectured that every representation-infinite finite-dimensional algebra over a field has indecomposable modules of arbitrarily large cardinality, meaning that for every infinite cardinal κ there exists an indecomposable module M with $|M| \geq \kappa$. Following Iovanov [18, §6.1], who calls this a Brauer–Thrall 3 question, we refer to it as the *third Brauer–Thrall conjecture*. Since any infinite-dimensional vector space M over k satisfies $|M| = \max(|k|, \dim_k M)$, the condition that $|M|$ can be made arbitrarily large is equivalent to the existence, for every infinite cardinal κ, of an indecomposable A-module M with $\dim_k M \geq \kappa$.

In this paper, we settle this conjecture over algebraically closed fields by proving that every representation-infinite finite-dimensional algebra over an algebraically closed field has an indecomposable module of dimension equal to each prescribed infinite cardinal. Our main result (Theorem A) constructs representation embeddings on the subcategory of Kronecker modules of arbitrary dimension without a simple injective summand. Applying these embeddings to the indecomposable Kronecker modules of every infinite dimension constructed by Göbel and Simson [14] gives Corollary 1.1, which also counts the resulting modules and bounds the nilpotence exponent of their endomorphism radicals.

1.1. **Representation embeddings on modules of arbitrary dimension.** Let K_2 be the Kronecker quiver $1 \rightrightarrows 2$. We consider its representations of arbitrary dimension, identified with left kK_2-modules. Write such a representation as $(V, W; a, b)$, with V at vertex 1, W at vertex 2 and arrow maps $a, b : V \to W$. Let $S_+ = (k, 0; 0, 0)$ and $S_- = (0, k; 0, 0)$ be its simple injective and simple projective representations, respectively. Let $\mathcal{D}$ be the full subcategory of $\operatorname{Mod} kK_2$ consisting of the modules T without a direct summand isomorphic to S_+, and put $\mathcal{D}_{\mathrm{fin}} = \mathcal{D} \cap \operatorname{mod} kK_2$. Since S_+ is simple and injective, $T \in \mathcal{D}$ if and only if $\operatorname{Hom}_{K_2}(S_+, T) = 0$; the category $\mathcal{D}$ is closed under submodules, direct sums and filtered colimits (Lemma 2.2(2)), and a short sequence in $\mathcal{D}$ is called exact when it is short exact in $\operatorname{Mod} kK_2$. The finite-dimensional subcategory $\mathcal{D}_{\mathrm{fin}}$ is the common domain of Bongartz's embeddings: one of his constructions, applied to all Kronecker modules, sends S_+ to zero and therefore cannot reflect isomorphism classes on that larger category.

Following Bongartz [6, §2.1], we call a k-linear functor $G : \mathcal{D}_{\mathrm{fin}} \to \operatorname{mod} A$ or $G : \mathcal{D} \to \operatorname{Mod} A$ a *representation embedding* if it preserves short exact sequences and indecomposability and reflects isomorphism classes, in the sense that $G(T) \cong G(U)$ implies $T \cong U$. Our main result is the following.

Theorem A. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k. There is a representation embedding*

$$F : \mathcal{D} \longrightarrow \operatorname{Mod} A$$

with the following properties.

(1) *F sends finite-dimensional modules to finite-dimensional modules.*
(2) *F commutes with arbitrary direct sums and filtered colimits.*
(3) *If $T \in \mathcal{D}$ has infinite dimension, then $\dim_k F(T) = \dim_k T$.*

Theorem 5.2 proves a stronger form for all the choices of Bongartz's construction described in Section 1.3, and also shows that the resulting functors are faithful.

1.2. **Indecomposable modules of prescribed infinite dimension.** For every infinite cardinal κ, Göbel and Simson [14, Theorem 1.2] construct a family of Kronecker modules T_β ($\beta \subseteq \kappa$) of dimension κ, with neither simple module as a direct summand, satisfying

$$\operatorname{Hom}_{K_2}(T_\beta, T_\gamma) \cong \begin{cases} k, & \beta \subseteq \gamma, \\ 0, & \beta \nsubseteq \gamma, \end{cases} \qquad \operatorname{End}_{K_2}(T_\beta) \cong k$$

(see Theorem 5.6). For our embeddings F, we construct algebra surjections $\operatorname{End}_A(F(T_\beta)) \to \operatorname{End}_{K_2}(T_\beta) \cong k$ that are left inverses to the maps induced by F. Their kernels are nilpotent, so the endomorphism algebras $\operatorname{End}_A(F(T_\beta))$ are local with residue field k. Using Bongartz's bound [6, Theorem 7] on the rank over $k[t]$ of a free bimodule defining an embedding of finite-dimensional $k[t]$-modules, we can choose F so that the nilpotence exponent of these kernels is at most $\max(30, 4\dim_k A)$ for every κ and every $\beta \subseteq \kappa$ (Proposition 5.5). Combining Theorem A with the family of Göbel and Simson then yields the following result.

Corollary 1.1 (= Corollary 5.7)**.** *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k. There is an integer $1 \le n \le \max(30, 4\dim_k A)$ such that, for every infinite cardinal κ, there are 2^κ pairwise non-isomorphic indecomposable A-modules M satisfying*

$$\dim_k M = \kappa, \qquad \operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \cong k, \qquad (\operatorname{rad}\operatorname{End}_A(M))^n = 0.$$

If $\kappa \ge |k|$, this is the largest possible number of isomorphism classes in dimension κ.

1.3. **Outline of the proof.** We first reduce to a basic minimal representation-infinite algebra, that is, a basic representation-infinite algebra whose proper quotients are representation-finite. For such an algebra, Bongartz [6, Theorem 6 and its proof] supplies at least one of three constructions of a representation embedding on $\mathcal{D}_{\mathrm{fin}}$ (Theorem 3.14): a quotient of projective modules, a push-down along a Galois covering, or a push-down along a closed walk. Because these constructions are defined using tensor products with finite-dimensional spaces, quotients, and finite direct sums of vertex spaces, their formulas extend directly to modules of arbitrary dimension; we establish their exactness and colimit properties in Section 3.

To control homomorphisms between the resulting modules, let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ consisting of the Kronecker modules with neither simple module as a direct summand. For each of the three functors F, we construct k-linear maps

$$\rho_{T,U} : \operatorname{Hom}_A(F(T), F(U)) \longrightarrow \operatorname{Hom}_{K_2}(T, U), \qquad T, U \in \mathcal{C},$$

preserving composition and satisfying $\rho_{T,U}(F(f)) = f$, so that each $\rho_{T,T}$ is a surjective algebra homomorphism onto $\operatorname{End}_{K_2}(T)$. For each T, the kernel of $\rho_{T,T}$ is a nilpotent ideal of $\operatorname{End}_A(F(T))$, with exponent bounded independently of T. Consequently, when $T \in \mathcal{C}$ is indecomposable, $\operatorname{End}_A(F(T))$ contains no nontrivial idempotent and $F(T)$ is indecomposable; likewise, applying $\rho_{T,U}$ and $\rho_{U,T}$ to an isomorphism $F(T) \xrightarrow{\sim} F(U)$ and its inverse recovers an isomorphism $T \xrightarrow{\sim} U$. Finally, to extend reflection of isomorphism classes from $\mathcal{C}$ to all of $\mathcal{D}$, we split off the simple projective summands S_- and cancel their images under F.

1.4. **Related work.** Classical constructions of infinite-dimensional indecomposable modules include generic, Prüfer and adic modules; see Crawley-Boevey [10, §§1.3–1.5, 3.1]. These modules do not by themselves attain every infinite dimension; for instance, the Prüfer modules over the Kronecker algebra all have countable dimension. For a fully k-wild algebra A, meaning that there exists a fully faithful exact k-linear functor from finite-dimensional $k\langle x, y\rangle$-modules to finite-dimensional A-modules, Simson [27, Proposition 2.2] constructs a fully faithful exact k-linear functor $\operatorname{Mod} k\langle x, y\rangle \to \operatorname{Mod} A$ and obtains indecomposable modules of arbitrarily large infinite dimension. Over an algebraically closed field, Simson [26, Theorem 2.8] also constructs indecomposable modules of arbitrarily large infinite dimension for four classes of representation-infinite algebras: loop-finite algebras, strongly simply connected algebras, algebras with radical square zero, and group algebras of finite groups. By contrast, Corollary 1.1 provides indecomposable modules of every infinite dimension over any representation-infinite finite-dimensional algebra over an algebraically closed field.

In [27, Theorem 3.1], Simson states that every representation-infinite special biserial algebra over an arbitrary field k has indecomposable modules of arbitrarily large cardinality. However, Case 4 of that proof relies on [27, Proposition 3.7], which asserts that every such algebra R has a corner eRe with a quotient isomorphic either to $k\langle x, y\rangle/(x^2, xy, yx, y^2)$ or to the path algebra of an acyclic orientation of a cycle with at least two vertices. Example A.1 in Appendix A refutes this assertion over every field, leaving a gap in Case 4. Over algebraically closed fields, Corollary 1.1 establishes the existence of indecomposable modules of arbitrarily large cardinality for all representation-infinite special biserial algebras; over other fields, that conclusion is not established by Simson's proof.

For a representation-infinite finite-dimensional algebra R over a field K with $(\operatorname{rad} R)^2 = 0$ and $R/\operatorname{rad} R \cong K^m$ for some $m \geq 1$, Simson [27, Corollary 3.6] constructs R-modules U of arbitrarily large cardinality satisfying

$$\operatorname{End}_R(U)/\operatorname{rad}\operatorname{End}_R(U) \cong K, \qquad (\operatorname{rad}\operatorname{End}_R(U))^2 = 0,$$

so that $\operatorname{End}_R(U)$ is local, although the dimension of U is not prescribed. Over an algebraically closed field, Corollary 1.1 bounds the nilpotence exponent of the endomorphism radical uniformly for families of indecomposable A-modules in every infinite dimension.

Ringel [22, Corollary on p. 407] showed that for a connected tame hereditary algebra R, the category of all R-modules contains a full exact subcategory equivalent to the category of all modules over a wild hereditary algebra; thus tame hereditary algebras are wild with respect to modules of arbitrary dimension. For representation-infinite algebras A over an algebraically closed field k that are loop-finite or strongly simply connected, Simson [25, Theorem 1.2] constructs a fully faithful exact functor from the category of all Kronecker modules to $\operatorname{Mod} A$ sending finite-dimensional modules to finite-dimensional modules. In contrast to these full embeddings for special classes, Theorem A applies to every representation-infinite algebra over an algebraically closed field, but its functor on $\mathcal{D}$ need not be full, as Example 4.2 shows.

Iovanov [18, §6.1] asks whether a given exact functor between categories of all comodules whose restriction to finite-dimensional comodules is a representation embedding must itself be a representation embedding. For modules over finite-dimensional algebras, Theorem 5.2 establishes a version with domain $\mathcal{D}$: for the functors constructed here, being a representation embedding on $\mathcal{D}_{\mathrm{fin}}$ implies being a representation embedding on $\mathcal{D}$.

Gregory and Prest [17, Theorem 4.2] study tensor representation embeddings $G : \operatorname{Mod} S \to \operatorname{Mod} R$ induced by finite-dimensional bimodules between finite-dimensional algebras, and construct a functor $H : \operatorname{Mod} R \to \operatorname{Mod} S$ satisfying $H(G(X)) \cong X$ for every S-module X. Their construction assumes that for finite-dimensional S-modules X and Y, the space $\operatorname{Hom}_R(G(X), G(Y))$ decomposes as the direct sum of the image of $\operatorname{Hom}_S(X, Y)$ and an ideal of radical morphisms factoring through finite direct sums of a fixed finite-dimensional R-module. By contrast, our maps $\rho_{T,U}$ are defined only between objects in the image of the embedding rather than as a functor on all of $\operatorname{Mod} A$, and in place of this factorization condition we require only a uniform nilpotence bound on the endomorphism kernels $\ker \rho_{T,T}$ (Definition 2.3).

Over perfect fields, Bautista, Pérez and Salmerón [5, Theorem 1.4] construct a representation embedding into $\operatorname{mod} A$ for every representation-infinite finite-dimensional algebra A, with domain the category of modules over a principal ideal domain that have finite dimension over k. While this provides a finite-dimensional representation embedding over a wider class of fields, our argument relies on the specific quotient, covering and walk constructions of Theorem 3.14, which are established over algebraically closed fields.

Organization. In Section 2, we describe how to split off the simple summands of a Kronecker module and show that retractions on Hom spaces with nilpotent kernels preserve indecomposability and reflect isomorphisms. In Section 3, we reduce to basic minimal representation-infinite algebras, define the three functors in arbitrary dimension, and state Theorem 3.14. In Section 4, we construct retractions on Hom spaces and bound the nilpotence exponents of their kernels using the Loewy length in the quotient case and the fibre sizes in the covering and walk cases. In Section 5, we cancel the simple projective summands, transfer the embedding to the original algebra, and prove the uniform numerical bound and Corollary 1.1. Appendix A gives the counterexample discussed in Section 1.4.

Conventions and notation. Throughout, the letter k denotes a field, assumed to be algebraically closed where stated, and we assume the axiom of choice. All algebras and categories are k-linear, and modules are left modules. We write $\operatorname{Mod} A$ for the category of all A-modules and $\operatorname{mod} A$ for the full subcategory of finite-dimensional A-modules. We write $\dim_k$ for dimension over k, $\operatorname{rad} E$ for the Jacobson radical of an algebra E, and $X^{(I)}$ for the direct sum of copies of X indexed by a set I. Tensor products without a subscript are over k; for an A-module P and a vector space V, the left A-action on $P \otimes V$ is given by $a(p \otimes v) = ap \otimes v$.

For arrows $\alpha : i \to j$ and $\beta : j \to l$ in a quiver, the product $\beta\alpha$ traverses α first, so that left modules over a path algebra identify with covariant quiver representations. We write Hom_{K_2} and End_{K_2} for Hom and End over kK_2. A module over a k-linear category is a covariant k-linear functor to vector spaces. For a representation X of a quiver, or a module X over a k-linear category, we put $\operatorname{supp} X = \{i : X(i) \neq 0\}$.

Formalization. Theorem A and the full statement of Corollary 1.1 have been formalized in Lean 4 using Mathlib. The source is available at `https://github.com/haruhisa-enomoto/third-brauer-thrall-conjecture`. Rather than assuming the cited results of Bongartz and Göbel–Simson as additional axioms, the development proves the existence of the quotients, pairs of modules on coverings, and walks needed in Theorem 3.14 (using alternative arguments at the two steps noted in Remark 3.15), establishes the numerical bound of Proposition 5.5 directly without the bounded-bimodule assertion of Lemma 5.4, and proves Theorem 5.6 for arbitrary infinite fields, which suffices for our application over algebraically closed fields. In the walk construction it uses a distinguished sink in place of a source; Lemma 3.9 and Appendix A are not formalized.

Use of AI. The research was done under the author's direction using GPT-6-Astra, which found and developed the proof presented here and wrote the earlier drafts of the manuscript. The present text was revised with Claude Opus 5.5. The Lean formalization was carried out mainly by Claude Opus 5.5. GPT-6-Astra provided mathematical advice for steps that Claude Opus 5.5 could not complete and performed the final audit. The author directed the research and revisions and is responsible for the mathematical content and the final manuscript.

2. Kronecker modules and nilpotent ideals

Throughout this section, k is any field and A is any finite-dimensional k-algebra. We describe the simple summands of arbitrary Kronecker modules and record how retractions on Hom spaces with nilpotent kernels preserve indecomposability and reflect isomorphisms.

2.1. Kronecker modules without simple injective summands.

We first describe $\mathcal{D}$ and $\mathcal{C}$ in terms of arrow maps. A Kronecker module is written $T = (V, W; a, b)$, with $a, b : V \to W$; a morphism $(g, h) : T \to (V', W'; a', b')$ satisfies $ha = a'g$ and $hb = b'g$. For any Kronecker module

$T = (V, W; a, b)$, the submodule $(0, W; 0, 0) \cong S_- \otimes W$ and the quotient $(V, 0; 0, 0) \cong S_+ \otimes V$ fit into the canonical short exact sequence

$$0 \longrightarrow S_- \otimes W \longrightarrow T \longrightarrow S_+ \otimes V \longrightarrow 0. \tag{2.1}$$

Moreover, $\mathrm{Hom}_{K_2}(S_+, T)$ identifies with $\ker a \cap \ker b$: choosing a complement V_1 to $\ker a \cap \ker b$ in V gives a decomposition $T \cong (V_1, W; a|_{V_1}, b|_{V_1}) \oplus (\ker a \cap \ker b, 0; 0, 0)$, where the second summand is a direct sum of copies of S_+. Hence

$$T \in \mathcal{D} \quad \Longleftrightarrow \quad \ker a \cap \ker b = 0. \tag{2.2}$$

The following definition describes the subcategory $\mathcal{C}$ announced in Section 1.3 in terms of the arrow maps.

Definition 2.1. Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ consisting of the modules $T = (V, W; a, b)$ with $a(V) + b(V) = W$. Write $\mathcal{C}_{\mathrm{fin}} = \mathcal{C} \cap \mathrm{mod}\, kK_2$.

For $T \in \mathcal{D}$, choosing a complement to $a(V) + b(V)$ in W gives a decomposition

$$T \cong T_0 \oplus S_-^{(I)}, \qquad T_0 \in \mathcal{C}. \tag{2.3}$$

Here $T_0 = (V, a(V) + b(V); a, b)$ and I indexes a basis of the chosen complement. Conversely, if $T = T' \oplus S_-$ with $T' = (V', W'; a', b')$, then $W = W' \oplus k$ and $a(V) + b(V) = a'(V') + b'(V') \subseteq W'$, so $a(V) + b(V) \neq W$. Thus $\mathcal{C}$ consists precisely of the Kronecker modules having neither S_+ nor S_- as a direct summand. In particular, every infinite-dimensional indecomposable Kronecker module belongs to $\mathcal{C}$.

A morphism in $\mathcal{D}$ need not be determined by its component at vertex 1 alone, since every endomorphism of S_- vanishes at vertex 1. For modules in $\mathcal{C}$, however, evaluation at either vertex is injective. We record this fact together with the closure properties and finite-dimensional submodules of these categories.

Lemma 2.2. *The subcategories $\mathcal{D}$ and $\mathcal{C}$ have the following properties.*

1. *Both vertex evaluations $(g, h) \mapsto g$ and $(g, h) \mapsto h$ are injective on $\mathrm{Hom}_{K_2}(T, U)$ for $T, U \in \mathcal{C}$.*
2. *$\mathcal{D}$ is closed under submodules, direct sums and filtered colimits, and $\mathcal{C}$ is closed under finite direct sums.*
3. *Every object of $\mathcal{D}$ is the union of its finite-dimensional submodules. For $T = (V, W; a, b) \in \mathcal{C}$, the submodules*

$$T_{V_0} = (V_0, a(V_0) + b(V_0); a, b), \qquad V_0 \subseteq V, \quad \dim_k V_0 < \infty,$$

are cofinal among its finite-dimensional submodules and belong to $\mathcal{C}$.

Proof. (1) Let $(g, h) : T \to U$ be a morphism between $T = (V, W; a, b)$ and $U = (V', W'; a', b')$ in $\mathcal{C}$. If $g = 0$, then $ha = a'g = 0$ and $hb = b'g = 0$, so $h = 0$ because $a(V) + b(V) = W$. Conversely, if $h = 0$, then $a'g = ha = 0$ and $b'g = hb = 0$, so $\mathrm{im}\, g \subseteq \ker a' \cap \ker b' = 0$ by (2.2) and hence $g = 0$.

(2) A Kronecker module $T = (V, W; a, b)$ satisfies (2.2) if and only if the map $\binom{a}{b} : V \to W \oplus W$ is injective. This condition is inherited by submodules and preserved by direct sums and filtered colimits, since filtered colimits of vector spaces are exact. The additional condition $a(V) + b(V) = W$ in Definition 2.1 is equivalent to the surjectivity of $(a\ b) : V \oplus V \to W$, which is preserved by finite direct sums.

(3) Since the algebra kK_2 is finite-dimensional, every finitely generated submodule is finite-dimensional, so every object of $\mathcal{D}$ is the union of its finite-dimensional submodules. For $T = (V, W; a, b) \in \mathcal{C}$, each submodule T_{V_0} belongs to $\mathcal{C}$ by definition. Given any finite-dimensional submodule $U = (V_1, W_1; a, b) \subseteq T$, any finite spanning set of W_1 lies in $a(V_0) + b(V_0)$ for a sufficiently large finite-dimensional subspace $V_0 \subseteq V$; enlarging V_0 so that $V_1 \subseteq V_0$ gives $U \subseteq T_{V_0}$, which proves cofinality. □

2.2. **Nilpotent kernels and indecomposability.** To show that a functor Ψ preserves indecomposability and reflects isomorphism classes, we will recover morphisms between Kronecker modules from morphisms between their images under Ψ. Ardizzoni and Menini [1, Definition 1.1] call a functor *heavily separable* if it admits natural retractions on Hom spaces preserving composition. We use k-linear retractions of this kind whose endomorphism kernels satisfy a uniform nilpotence bound.

Definition 2.3. Let Ψ be a k-linear functor from a full subcategory of $\operatorname{Mod} kK_2$ to $\operatorname{Mod} A$, and let $\mathcal{X}$ be a full subcategory of its domain. We call k-linear maps

$$\rho_{X,Y} : \operatorname{Hom}_A(\Psi(X), \Psi(Y)) \longrightarrow \operatorname{Hom}_{K_2}(X, Y), \qquad X, Y \in \mathcal{X},$$

retractions for Ψ on $\mathcal{X}$ if they preserve composition and satisfy $\rho_{X,Y}(\Psi(u)) = u$ for every morphism $u : X \to Y$. Thus each $\rho_{X,Y}$ is a retraction of the linear map $\operatorname{Hom}_{K_2}(X, Y) \to \operatorname{Hom}_A(\Psi(X), \Psi(Y))$ induced by Ψ. For an integer $n \geq 1$, these retractions have *nilpotence bound n* if

$$(\ker \rho_{X,X})^n = 0 \qquad \text{for every } X \in \mathcal{X}. \tag{2.4}$$

Taking $u = 1_X$ shows that $\rho_{X,X}(1_{\Psi(X)}) = 1_X$, so each $\rho_{X,X}$ is a unital k-algebra homomorphism and $\ker \rho_{X,X}$ is a two-sided ideal of $\operatorname{End}_A(\Psi(X))$. Moreover, composition with maps in the image of Ψ shows that $\rho_{X,Y}$ is natural in both variables.

An infinite-dimensional indecomposable module need not have a local endomorphism algebra: for instance, $(k[t], k[t]; 1, t)$ has endomorphism algebra $k[t]$. Instead, we detect indecomposability through the absence of nontrivial idempotents.

Lemma 2.4. *Let $\Psi : \mathcal{C} \to \operatorname{Mod} A$ be a k-linear functor admitting retractions with nilpotence bound $n \geq 1$. Then Ψ preserves indecomposability and reflects isomorphism classes.*

Proof. Let $T \in \mathcal{C}$ be indecomposable. Since $\rho_{T,T}(\Psi(1_T)) = 1_T \neq 0$, we have $\Psi(T) \neq 0$. Let $e \in \operatorname{End}_A(\Psi(T))$ be an idempotent. Because $\rho_{T,T}$ is a unital algebra homomorphism and T is indecomposable, the idempotent $\rho_{T,T}(e) \in \operatorname{End}_{K_2}(T)$ is either 0 or 1_T. Consequently, either e or $1_{\Psi(T)} - e$ lies in $\ker \rho_{T,T}$ and is therefore a nilpotent idempotent and hence zero. Thus $\Psi(T)$ is indecomposable. Finally, if $T, U \in \mathcal{C}$ and $f : \Psi(T) \to \Psi(U)$ is an isomorphism, then $\rho_{T,U}(f) : T \to U$ and $\rho_{U,T}(f^{-1}) : U \to T$ are mutually inverse morphisms in $\mathcal{C}$, so $T \cong U$. □

3. Bongartz's representation embeddings

In Section 3.1, k is any field and A is a representation-infinite finite-dimensional k-algebra. In the remaining subsections, k is algebraically closed and A denotes a connected basic minimal representation-infinite finite-dimensional k-algebra. We define the three functors used in Bongartz's representation embedding theorem [6, Theorem 6] on modules of arbitrary dimension. Theorem 3.14 supplies the required algebras, modules and walks, and Section 4 proves the nilpotence needed to preserve indecomposability.

3.1. **Reduction to minimal representation-infinite algebras.** Recall that an algebra is called *minimal representation-infinite* if it is representation-infinite and all of its proper quotients are representation-finite. The following lemma records the functor used in the reduction of [6, §5.2] to transfer embeddings back to A in arbitrary dimension.

Lemma 3.1. *Let A be a representation-infinite finite-dimensional algebra over a field k. Choose a full idempotent e (so $AeA = A$) such that $A_0 = eAe$ is basic, and a minimal representation-infinite quotient B of A_0. Then B is basic and connected, $\dim_k B \leq \dim_k A$, and there is an exact fully faithful functor $\Phi : \operatorname{Mod} B \to \operatorname{Mod} A$ preserving finite-dimensional modules, arbitrary direct sums, and filtered colimits. For every infinite-dimensional B-module X, $\dim_k \Phi(X) = \dim_k X$.*

Proof. Since A and $A_0 = eAe$ are Morita equivalent, A_0 is representation-infinite, and any quotient B of A_0 of least dimension among its representation-infinite quotients is minimal representation-infinite with $\dim_k B \leq \dim_k A_0 \leq \dim_k A$. Any minimal representation-infinite quotient B of A_0 is basic because A_0 is basic, and connected because otherwise one of its connected factors would be a proper representation-infinite quotient. Let $\Phi : \operatorname{Mod} B \to \operatorname{Mod} A$ be inflation from B to A_0

followed by the Morita equivalence $Ae \otimes_{A_0} -$. Both inflation and $Ae \otimes_{A_0} -$ are exact and fully faithful, preserve finite-dimensional modules, and commute with arbitrary direct sums and filtered colimits. Since the inverse Morita functor is $e(-)$, every B-module X satisfies

$$\dim_k X = \dim_k e\Phi(X) \le \dim_k \Phi(X) \le (\dim_k Ae) \dim_k X,$$

and hence $\dim_k \Phi(X) = \dim_k X$ whenever $\dim_k X$ is infinite. □

In the remainder of this section, we write A for the reduced algebra B of Lemma 3.1, as stipulated at the start of this section.

3.2. **Quotients of projective modules by submodules of the socle.** In the first construction, a Kronecker module is sent to a quotient of a direct sum of copies of one indecomposable projective module over an algebra Λ, with relations defined by the two arrow maps.

Setting 3.2. Throughout this subsection, let Λ be a basic finite-dimensional algebra over the algebraically closed field k, and put $J = \operatorname{rad}\Lambda$. Fix a complete set $\{e_z\}$ of primitive orthogonal idempotents of Λ. Choose two idempotents e_x, e_y from this set (which may coincide) and linearly independent elements $s, t \in e_x J e_y$ satisfying $Js = Jt = sJ = tJ = 0$.

With the socle elements s, t of Setting 3.2, the functor of [6, §3.1] takes the following form, as used in the proof of [6, Corollary 2].

Construction 3.3 ([6, §3.1])**.** For $T = (V, W; a, b) \in \mathcal{D}$, put

$$F_1(T) = (\Lambda e_y \otimes W)/\{s \otimes a(v) + t \otimes b(v) : v \in V\}. \tag{3.1}$$

On a morphism (g, h), let $F_1(g, h)$ be the map induced by $1 \otimes h$.

To see that the relation space $\{s \otimes a(v) + t \otimes b(v) : v \in V\}$ is a Λ-submodule of $\Lambda e_y \otimes W$, note that it is annihilated by J and satisfies $e_z s = \delta_{zx} s$ and $e_z t = \delta_{zx} t$ for each primitive idempotent e_z; since $\Lambda/J \cong \prod_z k$, it is stable under Λ and lies in the socle of $\Lambda e_y \otimes W$. For any morphism $(g, h) : T \to (V', W'; a', b')$, the relations $ha = a'g$ and $hb = b'g$ imply that $1 \otimes h$ maps the relation submodule for T into that for $(V', W'; a', b')$, so $F_1(g, h)$ is well defined. Moreover, since $s, t \in J$, the relation submodule is contained in $J(\Lambda e_y \otimes W)$, so the top of $F_1(T)$ is identified with $S_y \otimes W$, where $S_y = \Lambda e_y / J e_y$; we use this identification of the induced map on tops in Proposition 4.1.

Lemma 3.4. *The functor $F_1 : \mathcal{D} \to \operatorname{Mod}\Lambda$ is exact and faithful, commutes with arbitrary direct sums and filtered colimits, and preserves finite-dimensional modules. For infinite-dimensional $T \in \mathcal{D}$, $\dim_k F_1(T) = \dim_k T$.*

Proof. For $T = (V, W; a, b) \in \mathcal{D}$, define $j_T : V \to \Lambda e_y \otimes W$ by $j_T(v) = s \otimes a(v) + t \otimes b(v)$. Since s and t are linearly independent and $T \in \mathcal{D}$, we have $\ker j_T = \ker a \cap \ker b = 0$ by (2.2), so j_T is injective. Viewing $V \cong S_x \otimes V$ as a semisimple Λ-module via the simple module $S_x = \Lambda e_x / J e_x$, we may regard j_T as a Λ-module homomorphism with cokernel $F_1(T)$ (so the formula (3.1) defines a right exact functor on $\operatorname{Mod} kK_2$). For a short exact sequence $0 \to T' \to T \to T'' \to 0$ in $\mathcal{D}$, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_{T'}} & & \downarrow{\scriptstyle j_T} & & \downarrow{\scriptstyle j_{T''}} & & \\ 0 & \longrightarrow & \Lambda e_y \otimes W' & \longrightarrow & \Lambda e_y \otimes W & \longrightarrow & \Lambda e_y \otimes W'' & \longrightarrow & 0 \end{array}$$

has exact rows and injective vertical maps, so the induced sequence of cokernels $0 \to F_1(T') \to F_1(T) \to F_1(T'') \to 0$ is exact. Thus F_1 is exact on $\mathcal{D}$. If $(g, h) : T \to U$ is a morphism in $\mathcal{D}$ with $F_1(g, h) = 0$, then its induced map $1_{S_y} \otimes h$ on tops vanishes, so $h = 0$ and hence $g = 0$ by (2.2) for U; thus F_1 is faithful. Since tensor products and cokernels commute with arbitrary direct sums and filtered colimits, so does F_1. Finally, F_1 preserves finite-dimensional modules, and the inequalities

$$\dim_k W \le \dim_k F_1(T) \le (\dim_k \Lambda e_y) \dim_k W$$

together with the embedding $\binom{a}{b} : V \hookrightarrow W \oplus W$ show that $\dim_k F_1(T) = \dim_k W = \dim_k T$ whenever $T \in \mathcal{D}$ is infinite-dimensional. □

3.3. **Galois coverings and push-down functors.** Fix a complete set $\{e_q\}$ of primitive orthogonal idempotents of the basic algebra A, indexed by a set of vertices q, and let $\mathcal{A}$ be the k-linear category with these vertices as objects and morphism spaces $\mathcal{A}(q,q') = e_{q'}Ae_q$; we identify A-modules with $\mathcal{A}$-modules. In the second construction, a larger category R carries an action of a group G that is free on objects, and a functor $\pi : R \to \mathcal{A}$ identifies each G-orbit of objects with a single object of $\mathcal{A}$. Kronecker modules are first embedded fully faithfully into modules over R supported on finitely many objects, and the spaces above each object of $\mathcal{A}$ are then summed.

Definition 3.5 ([13, 3.1])**.** A small k-linear category R is *locally bounded* if distinct objects are nonisomorphic, each endomorphism algebra is local, all morphism spaces are finite-dimensional, and for every object x the set $\{y : R(x,y) \neq 0 \text{ or } R(y,x) \neq 0\}$ is finite. A *Galois covering with group* G is a k-linear functor $\pi : R \to \mathcal{A}$ from a locally bounded category R, surjective on objects, together with an action of G by k-linear automorphisms, such that the action on objects is free, $\pi g = \pi$, each fibre is a G-orbit, and both maps

$$\bigoplus_{\pi y = q} R(x,y) \longrightarrow \mathcal{A}(\pi x, q), \qquad \bigoplus_{\pi y = q} R(y,x) \longrightarrow \mathcal{A}(q, \pi x) \tag{3.2}$$

induced by π are isomorphisms for every x, q.

For the rest of this subsection, fix a Galois covering $\pi : R \to \mathcal{A}$ with group G as in Definition 3.5. We allow arbitrary vertex spaces, but require the support to be finite. Its size will bound the nilpotence exponent in Section 4.

Construction 3.6. For an R-module X of finite support, set

$$(\pi_* X)(q) = \bigoplus_{\pi x = q} X(x).$$

Given a morphism $u : q \to q'$ in $\mathcal{A}$ and an object x with $\pi x = q$, the first isomorphism in (3.2) allows us to write $u = \sum_{\pi y = q'} \pi(u_{y,x})$ with unique $u_{y,x} \in R(x,y)$. We define $(\pi_* X)(u) : (\pi_* X)(q) \to (\pi_* X)(q')$ to be the linear map whose (y,x)-block is $X(u_{y,x}) : X(x) \to X(y)$, and on morphisms of R-modules we define π_* by taking the direct sum of the vertex maps. This defines the *push-down functor* π_*.

By the uniqueness in (3.2), these block matrices respect composition.

We consider modules supported on a finite full subcategory $\mathcal{S} \subset R$ satisfying

$$R(x,j) \circ R(i,x) = 0 \qquad (i,j \in \mathcal{S},\ x \notin \mathcal{S}). \tag{3.3}$$

Write $\Gamma = \bigoplus_{i,j \in \mathcal{S}} R(i,j)$, with multiplication induced by composition in R and zero on noncomposable factors, so that Γ-modules identify with modules over $\mathcal{S}$. Condition (3.3) ensures that extending a Γ-module by zero outside $\mathcal{S}$ yields an R-module, and that extension by zero is an exact fully faithful functor from Γ-modules to R-modules.

We use the construction by extensions of Ringel [21, 1.5, Lemma]. Its form for modules of arbitrary dimension is also used by Göbel and Simson [15, Proposition 3.1] and Simson [25, proof of Theorem 1.2].

Construction 3.7. Choose finite-dimensional Γ-modules M_-, M_+ with disjoint supports and $\operatorname{End}_\Gamma(M_-) = \operatorname{End}_\Gamma(M_+) = k$, and two linearly independent classes in $\operatorname{Ext}^1_\Gamma(M_+, M_-)$. Fix a bound quiver presentation of Γ. Represent each class by an extension with vertex spaces $M_-(i) \oplus M_+(i)$, and write $z_j(\alpha) : M_+(i) \to M_-(l)$ for the off-diagonal arrow entries of the j-th extension, for $\alpha : i \to l$ and $j = 1, 2$. For any Kronecker module $T = (V, W; a, b)$, define $L_2(T)$ by

$$L_2(T)(i) = (M_-(i) \otimes W) \oplus (M_+(i) \otimes V), \qquad L_2(T)(\alpha) = \begin{pmatrix} M_-(\alpha) \otimes 1_W & z_1(\alpha) \otimes a + z_2(\alpha) \otimes b \\ 0 & M_+(\alpha) \otimes 1_V \end{pmatrix}.$$

For a morphism $(g,h) : T \to T'$, we define $L_2(g,h)$ at each vertex $i \in \mathcal{S}$ by the diagonal blocks $1_{M_-(i)} \otimes h$ and $1_{M_+(i)} \otimes g$. Viewing $L_2(T)$ as an R-module via extension by zero outside $\mathcal{S}$, we put $F_2 = \pi_* L_2 : \operatorname{Mod} kK_2 \to \operatorname{Mod} A$.

To check that $L_2(T)$ satisfies the defining relations of Γ, observe that in any product of the upper triangular matrices $L_2(T)(\alpha)$, each term in the off-diagonal block contains a single off-diagonal factor. For any relation of Γ, the off-diagonal block of the evaluated relation is therefore the sum of the corresponding blocks for the two chosen extensions, tensored with a and b respectively. Both vanish because those extensions are Γ-modules, and the diagonal blocks satisfy the relations because M_- and M_+ do, so $L_2(T)$ is a Γ-module in arbitrary dimension.

Lemma 3.8. *The functor L_2 is exact and fully faithful. The functor F_2 is exact and faithful. Both commute with direct sums and filtered colimits and preserve finite-dimensional modules. For infinite-dimensional T, $\dim_k F_2(T) = \dim_k T$.*

Proof. Let $T = (V, W; a, b)$ and $T' = (V', W'; a', b')$ be Kronecker modules. Since M_- and M_+ have disjoint supports, at each vertex $i \in \mathcal{S}$ at most one of $M_-(i)$ and $M_+(i)$ is nonzero. Any Γ-module homomorphism $L_2(T) \to L_2(T')$ therefore decomposes into maps $M_- \otimes W \to M_- \otimes W'$ and $M_+ \otimes V \to M_+ \otimes V'$, and the diagonal blocks of the arrow equations show that these two components are Γ-module homomorphisms. For a finite-dimensional Γ-module M with $\mathrm{End}_\Gamma(M) = k$, every homomorphism $M \otimes U \to M \otimes U'$ has the form $1_M \otimes h$ for a unique linear map $h : U \to U'$: indeed, $M \otimes U$ is a direct sum of copies of M, and since M is finitely generated, each copy maps into a finite direct sum of copies of M via scalar endomorphisms of M, which define the map h. Thus the two components of our morphism are $1_{M_-} \otimes h$ and $1_{M_+} \otimes g$ for unique linear maps $h : W \to W'$ and $g : V \to V'$. The remaining arrow equations are

$$z_1(\alpha) \otimes (ha - a'g) + z_2(\alpha) \otimes (hb - b'g) = 0 \quad \text{for every arrow } \alpha.$$

Since the two families z_1, z_2 represent linearly independent extension classes, they are linearly independent in the finite-dimensional space

$$\bigoplus_{\alpha : i \to l} \mathrm{Hom}_k(M_+(i), M_-(l)).$$

Applying linear functionals that separate z_1 and z_2 yields $ha = a'g$ and $hb = b'g$, which proves that L_2 is fully faithful. Since L_2 is defined at each vertex by tensoring with finite-dimensional spaces and taking direct sums, it is exact, preserves finite-dimensional modules, and commutes with arbitrary direct sums and filtered colimits. Extending by zero to R and applying π_* preserve these properties, and π_* is exact and faithful because it sums vertex spaces and maps over each fibre. Finally,

$$\dim_k F_2(T) = (\dim_k M_-) \dim_k W + (\dim_k M_+) \dim_k V,$$

and since $\dim_k M_-$ and $\dim_k M_+$ are positive integers, $\dim_k F_2(T) = \dim_k T$ whenever T is infinite-dimensional. □

In the covering case of Theorem 3.14, Bongartz supplies a functor on finite-dimensional modules. The next lemma identifies that functor with the finite restriction of L_2, so that Construction 3.7 extends the specified functor to arbitrary dimension.

Lemma 3.9. *Let $E : \mathrm{mod}\, kK_2 \to \mathrm{mod}\, \Gamma$ be an exact fully faithful k-linear functor such that $E(S_-)$ and $E(S_+)$ have disjoint supports. Put $M_- = E(S_-)$ and $M_+ = E(S_+)$. Represent the images under E of the extension classes with middle terms $(k, k; 1, 0)$ and $(k, k; 0, 1)$ by extensions, and let z_1, z_2 be their off-diagonal arrow entries as in Construction 3.7. Then these data satisfy the hypotheses of that construction, and E is naturally isomorphic to the restriction to $\mathrm{mod}\, kK_2$ of the resulting functor L_2.*

Proof. Since E is fully faithful, we have $\mathrm{End}_\Gamma(M_-) = \mathrm{End}_\Gamma(M_+) = k$. The k-linear map $\mathrm{Ext}^1_{K_2}(S_+, S_-) \to \mathrm{Ext}^1_\Gamma(M_+, M_-)$ induced by the exact k-linear functor E is injective, since if the image of an extension splits in $\mathrm{mod}\, \Gamma$, full faithfulness lifts its splitting to $\mathrm{mod}\, kK_2$. Thus the images of the two basis extensions with middle terms $(k, k; 1, 0)$ and $(k, k; 0, 1)$ are linearly independent. For any finite-dimensional Kronecker module $T = (V, W; a, b)$, applying E to (2.1) yields an exact sequence $0 \to M_- \otimes W \to E(T) \to M_+ \otimes V \to 0$. Because M_- and M_+ have disjoint supports, at each vertex $i \in \mathcal{S}$ at most one of $M_-(i)$ and $M_+(i)$ is nonzero. The exact sequence therefore gives a canonical identification $E(T)(i) \cong (M_-(i) \otimes W) \oplus (M_+(i) \otimes V) = L_2(T)(i)$ at every

vertex $i \in \mathcal{S}$, and every vertex linear map $M_+(i) \otimes V \to M_-(i) \otimes W$ vanishes. Consequently, there are no nonzero coboundaries on the vertex spaces, so an extension class in $\operatorname{Ext}^1_\Gamma(M_+ \otimes V, M_- \otimes W)$ determines unique off-diagonal arrow maps $M_+(i) \otimes V \to M_-(l) \otimes W$. Since the extension class of (2.1) has coefficients a and b with respect to the two basis extensions, under this identification of vertex spaces the off-diagonal arrow maps of $E(T)$ are $z_1(\alpha) \otimes a + z_2(\alpha) \otimes b$. Since these identifications commute with the maps induced by any Kronecker morphism (g, h), they define a natural isomorphism $E \cong L_2|_{\operatorname{mod} kK_2}$. □

3.4. **Push-down along a closed walk.** The third construction likewise sums vertex spaces over fibres: a closed walk in a quiver Q determines a cycle quiver together with a map from that cycle to Q, and we push representations down along this map.

Definition 3.10. A walk in a quiver Q is a composable sequence of arrows and formal inverse arrows, written in traversal order; it is *closed* if it ends at its starting vertex, and *reduced* if adjacent letters are never mutual inverses. A nonempty closed walk is *cyclically reduced* if this also holds at the join of its last and first letters, and *primitive* if it is not a proper power of a shorter closed walk. For a closed walk $w = c_1^{\varepsilon_1} \cdots c_r^{\varepsilon_r}$ passing through vertices $q_0, q_1, \ldots, q_r = q_0$, we form a cycle quiver D_w with vertices $d_0, \ldots, d_{r-1}$ (indices read modulo r, often written simply as i for d_i), where the i-th arrow goes from d_{i-1} to d_i if $\varepsilon_i = 1$ and in the reverse direction if $\varepsilon_i = -1$. The quiver map $p : D_w \to Q$ sends d_i to q_i and the i-th arrow to c_i; we call $p(e)$ the label of an arrow e of D_w.

When w is cyclically reduced,

$$p \text{ is injective on the arrows of } D_w \text{ with a fixed source, and on those with a fixed target,} \tag{3.4}$$

since two outgoing arrows with the same label, or two incoming arrows with the same label, would correspond to adjacent mutually inverse letters in w (possibly at the join).

Setting 3.11. For the rest of this subsection, fix a presentation $A = kQ/I$ as a bound quiver algebra and a primitive cyclically reduced closed walk w whose cycle D_w has a source u and $r \geq 2$ vertices. Write $p : D_w \to Q$ for the map of Definition 3.10, and fix an ordering of the two arrows out of u.

Construction 3.12 ([6, proof of Theorem 7])**.** For $T = (V, W; a, b) \in \operatorname{Mod} kK_2$, let $L_3(T)$ be the representation of D_w with V at u, W at all other vertices of D_w, the maps a, b on the ordered arrows out of u, and the identity map 1_W on the remaining arrows; a morphism (g, h) acts as g at u and as h at all other vertices. For a representation X of D_w and a vertex q of Q, set

$$(p_*X)(q) = \bigoplus_{p(i)=q} X(i).$$

For an arrow $c : q \to q'$ of Q, we define $(p_*X)(c)$ so that its block from $X(i)$ to $X(j)$ is $X(e)$ if there is an arrow $e : i \to j$ in D_w with label c, and zero otherwise; such an arrow is unique by (3.4), so each block is well defined. On morphisms, p_* takes the direct sum of the vertex maps. The composite $F_3 = p_*L_3$ is defined on $\operatorname{Mod} kK_2$ and takes values in representations of Q.

Theorem 3.14(3) supplies a walk for which the images of all finite-dimensional Kronecker modules are annihilated by I. The next lemma extends this property to arbitrary modules.

Lemma 3.13. *In Setting 3.11, the functor L_3 is exact and fully faithful, and $L_3(S_+)$ and $L_3(S_-)$ are nonzero with disjoint supports. Suppose that I annihilates $F_3(T)$ for every finite-dimensional Kronecker module T. Then $F_3 : \operatorname{Mod} kK_2 \to \operatorname{Mod} A$ is exact and faithful, commutes with arbitrary direct sums and filtered colimits, and preserves finite-dimensional modules. For infinite-dimensional T, $\dim_k F_3(T) = \dim_k T$.*

Proof. Because the identity arrows of $L_3(T)$ connect all vertices of D_w other than u, any morphism $L_3(T) \to L_3(V', W'; a', b')$ must have the same component h at all those vertices, and the remaining two commutativity conditions at the arrows out of u are precisely $ha = a'g$ and $hb = b'g$. Thus L_3 is fully faithful; it is also exact because each vertex space is either V or W. Moreover,

$L_3(S_+)$ and $L_3(S_-)$ are nonzero and have disjoint supports $\{u\}$ and $(D_w)_0 \setminus \{u\}$, just as for L_2 in Construction 3.7.

Since the push-down p_* takes finite direct sums of vertex spaces and maps, it is exact and faithful, and both L_3 and p_* commute with arbitrary direct sums and filtered colimits and preserve finite-dimensional modules. Because every Kronecker module is the union of its finite-dimensional submodules and $F_3 = p_*L_3$ commutes with filtered colimits, the ideal I annihilates $F_3(T)$ for every $T \in \operatorname{Mod} kK_2$, so F_3 takes values in $\operatorname{Mod} A$ and is exact and faithful. Finally, $\dim_k F_3(T) = \dim_k V + (r-1)\dim_k W = \dim_k T$ whenever T is infinite-dimensional. □

3.5. **Existence of the embeddings.** Over the algebraically closed field k, Bongartz [6, Theorem 6 and its proof] shows that at least one of the three preceding constructions yields a representation embedding on $\mathcal{D}_{\mathrm{fin}}$. While [6, Theorem 6] states the existence of an embedding, the free covering group, finite full subcategory and primitive closed walk in the alternatives below come from the proof of [6, Theorem 7] in [6, §5.4]. The subcategory $\mathcal{S}$ chosen there has the property that a nonzero composition between its objects cannot pass through an object outside $\mathcal{S}$; this gives (3.3). The disjoint supports in alternative (2) follow from [6, Lemma 5], as we explain after the theorem.

Theorem 3.14 ([6, Theorem 6 and its proof])**.** *Let A be a connected basic minimal representation-infinite algebra over an algebraically closed field. Then at least one of the following holds.*

1. *A has elements s, t as in Setting 3.2, with $\Lambda = A$, such that F_1 from Construction 3.3 is a representation embedding $\mathcal{D}_{\mathrm{fin}} \to \operatorname{mod} A$* [6, Theorem 1 and Corollary 2].
2. *There are a Galois covering $\pi : R \to \mathcal{A}$ whose group G is a free group, a finite full subcategory $\mathcal{S}$ of R satisfying (3.3), and an exact fully faithful k-linear functor $E : \operatorname{mod} kK_2 \to \operatorname{mod}\Gamma$ such that $E(S_+)$ and $E(S_-)$ are nonzero with disjoint supports and $\pi_*E|_{\mathcal{D}_{\mathrm{fin}}}$, with E extended by zero from $\mathcal{S}$ to R, is a representation embedding $\mathcal{D}_{\mathrm{fin}} \to \operatorname{mod} A$* [6, Theorem 3, Lemma 5, and proof of Theorem 7].
3. *There are a presentation $A = kQ/I$ and a walk as in Setting 3.11 such that F_3 from Construction 3.12 sends finite-dimensional Kronecker modules to $\operatorname{mod} A$ and $F_3|_{\mathcal{D}_{\mathrm{fin}}}$ is a representation embedding $\mathcal{D}_{\mathrm{fin}} \to \operatorname{mod} A$* [6, proof of Theorem 7].

In alternative (1), the construction in [6, Corollary 2] initially uses a quotient Λ of A. Since the modules $(k, k; \lambda, 1)$ $(\lambda \in k)$ are pairwise non-isomorphic indecomposables in $\mathcal{D}_{\mathrm{fin}}$, the existence of a representation embedding $\mathcal{D}_{\mathrm{fin}} \to \operatorname{mod}\Lambda$ implies that Λ is representation-infinite; by the minimality of A, we have $\Lambda = A$.

For alternative (2), the proof of [6, Theorem 7] chooses $\mathcal{S}$ so that Γ is tame concealed and verifies the representation-embedding property after push-down. By [6, Lemma 5 and its proof], there is a source or sink x and, in the sink case, an exact sequence of Γ-modules $0 \to P_x \to N_x \to U_x \to 0$, where P_x is simple projective, $\dim_k N_x(x) = 1$, and

$$\operatorname{End}_\Gamma(P_x) = \operatorname{End}_\Gamma(U_x) = k, \qquad \operatorname{Hom}_\Gamma(P_x, U_x) = \operatorname{Hom}_\Gamma(U_x, P_x) = 0, \qquad \dim_k \operatorname{Ext}^1_\Gamma(U_x, P_x) = 2.$$

Since P_x is supported at x and $\dim_k N_x(x) = 1$, exactness at x gives $U_x(x) = 0$, so $M_- = P_x$ and $M_+ = U_x$ have disjoint supports. When x is a source, the lemma is applied to Γ^{op}: P_x, N_x, U_x are then Γ^{op}-modules, and we take the Γ-modules $M_- = U_x^*$ and $M_+ = P_x^*$, where $(-)^* = \operatorname{Hom}_k(-, k)$. Duality preserves disjointness and identifies $\operatorname{Ext}^1_\Gamma(P_x^*, U_x^*)$ with $\operatorname{Ext}^1_{\Gamma^{\mathrm{op}}}(U_x, P_x)$. The functor E used in [6, proof of Theorem 7] is the finite-dimensional restriction of the extension functor of Construction 3.7 for this pair (M_-, M_+), so $E(S_-) = M_-$ and $E(S_+) = M_+$. Any choice of basis of the two-dimensional space $\operatorname{Ext}^1_\Gamma(M_+, M_-)$ gives the same representation-embedding property after push-down, since a change of basis replaces the two Kronecker arrow maps by invertible linear combinations and thus acts by an autoequivalence of $\operatorname{mod} kK_2$ preserving $\mathcal{D}_{\mathrm{fin}}$.

For each available alternative ν of Theorem 3.14, fix its data and write

$$F_\nu : \mathcal{D} \longrightarrow \operatorname{Mod} A \tag{3.5}$$

for the functor given by that construction, with F_2, F_3 restricted to $\mathcal{D}$. In alternative (2), Lemma 3.9 supplies the Construction 3.7 data from the specified exact fully faithful k-linear

functor E, so the finite restriction of F_2 is naturally isomorphic to $\pi_* E|_{\mathcal{D}_{\mathrm{fin}}}$. Put $F_{\nu,\mathrm{fin}} = F_\nu|_{\mathcal{D}_{\mathrm{fin}}}$. Since every $T \in \mathcal{D}$ is the union of its finite-dimensional submodules,

$$F_\nu(T) \cong \varinjlim_{\substack{U \subseteq T \\ \dim_k U < \infty}} F_{\nu,\mathrm{fin}}(U), \tag{3.6}$$

naturally in $T \in \mathcal{D}$. This identifies F_ν with the extension of $F_{\nu,\mathrm{fin}}$ by filtered colimits, uniquely up to natural isomorphism among extensions preserving filtered colimits. The same holds after applying Φ from Lemma 3.1; we use it in Step 1 of the proof of Theorem 5.2.

Remark 3.15. The Lean formalization gives alternative arguments for the pair of modules with disjoint supports in Theorem 3.14(2) and the closed walk in Theorem 3.14(3), replacing respectively [6, Lemma 5] and the use of the classification of large sincere algebras in [6, proof of Theorem 7].

4. Homomorphisms between the embedded modules

Throughout this section, k is algebraically closed and A denotes a connected basic minimal representation-infinite finite-dimensional k-algebra. Working with Kronecker modules of arbitrary dimension, whose images under the covering and walk constructions still have finite support on R and D_w, we construct retractions on $\mathcal{C}$ for each of the functors F_ν of (3.5), with a nilpotence bound determined by the Loewy length of A or the maximum fibre size of the support.

4.1. **Homomorphisms between quotients by submodules of the socle.** We work in Setting 3.2 and Construction 3.3. As noted after Construction 3.3, the top of $F_1(T)$ is $S_y \otimes W$, and multiplying lifts of this top by s and t defines a linear map whose kernel is precisely $\{(a(v), b(v)) : v \in V\}$ (see (4.1) below). In the proof of [6, Theorem 1], Bongartz uses this observation to recover Kronecker morphisms from the induced maps on tops; we formulate this recovery as a system of retractions.

Proposition 4.1. *There are retractions*

$$\rho_{T,U} : \mathrm{Hom}_\Lambda(F_1(T), F_1(U)) \longrightarrow \mathrm{Hom}_{K_2}(T, U), \qquad T, U \in \mathcal{D},$$

for F_1 on $\mathcal{D}$. If $J^n = 0$, these retractions have nilpotence bound n.

Proof. For any Λ-module M, put $\mathrm{top}_y(M) = e_y(M/JM)$. Since $sJ = tJ = 0$, the map

$$d_M : \mathrm{top}_y(M) \oplus \mathrm{top}_y(M) \longrightarrow e_x M, \qquad (w_1, w_2) \longmapsto sm_1 + tm_2,$$

where $m_i \in e_y M$ lifts w_i, is independent of the choice of lifts. For $T = (V, W; a, b) \in \mathcal{D}$, under the identification $\mathrm{top}_y(F_1(T)) = W$, the map $d_{F_1(T)}$ is the composite of the injection $W \oplus W \to \Lambda e_y \otimes W$, $(w_1, w_2) \mapsto s \otimes w_1 + t \otimes w_2$, with the quotient map onto $F_1(T)$. Its kernel is therefore

$$\ker d_{F_1(T)} = \{(a(v), b(v)) : v \in V\}. \tag{4.1}$$

Now write $U = (V', W'; a', b')$. Given $f : F_1(T) \to F_1(U)$, let $h : W \to W'$ be the map induced by f on top_y. By the Λ-linearity of f, the square

$$\begin{array}{ccc} W \oplus W & \xrightarrow{d_{F_1(T)}} & e_x F_1(T) \\ {\scriptstyle h \oplus h}\downarrow & & \downarrow{\scriptstyle f} \\ W' \oplus W' & \xrightarrow[d_{F_1(U)}]{} & e_x F_1(U) \end{array}$$

commutes. For each $v \in V$, we have $d_{F_1(T)}(a(v), b(v)) = 0$ by (4.1) for T, so the commutativity of the diagram gives $d_{F_1(U)}(ha(v), hb(v)) = 0$. By (4.1) for U and the condition $\ker a' \cap \ker b' = 0$, there is a unique linear map $g : V \to V'$ satisfying $ha = a'g$ and $hb = b'g$. We set $\rho_{T,U}(f) = (g, h)$. If $\rho_{U,T''}(f') = (g', h')$ for a composable morphism $f' : F_1(U) \to F_1(T'')$, then $f'f$ induces $h'h$ on top_y, and $g'g$ satisfies the two defining equations for $h'h$, so uniqueness yields $\rho_{T,T''}(f'f) = (g'g, h'h) = \rho_{U,T''}(f')\rho_{T,U}(f)$. The same uniqueness argument shows that $\rho_{T,U}$ is k-linear, and since $F_1(g, h)$ is induced by $1 \otimes h$, we have $\rho_{T,U}(F_1(g, h)) = (g, h)$. Finally, since $F_1(T)$ is generated by $e_y F_1(T)$, we have $\rho_{T,U}(f) = 0$ if and only if $h = 0$, which holds if and only if $f(F_1(T)) \subseteq JF_1(U)$.

Every such homomorphism satisfies $f(J^iF_1(T)) \subseteq J^{i+1}F_1(U)$ for all $i \geq 0$, so any n of them compose to zero when $J^n = 0$. □

Example 4.2. The embeddings of Theorem 5.2 need not be full, and the retractions of Proposition 4.1 can have nonzero kernels. Let $\Lambda = k\langle\xi,\eta\rangle/(\xi^2, \xi\eta, \eta\xi, \eta^2)$ and take $s = \xi$, $t = \eta$ in Setting 3.2. This is a basic minimal representation-infinite algebra: its quotients $\Lambda/(\xi - \lambda\eta)$, $\lambda \in k$, are pairwise non-isomorphic indecomposable Λ-modules, whereas every proper nonzero algebra quotient is isomorphic to k or $k[t]/(t^2)$ and is representation-finite. Taking $A = \Lambda$ and the identity reduction in Theorem 5.2 gives $F = F_1$. We have $F_1(S_-) = \Lambda$, and the retraction $\mathrm{End}_\Lambda(\Lambda) \to \mathrm{End}_{K_2}(S_-) = k$ is the canonical projection $\Lambda^{\mathrm{op}} \to k$, whose kernel is nonzero. Thus F_1 is not full.

4.2. **Vertex evaluations and block matrices.** We write $L = L_2$ or $L = L_3$ for the functors of Constructions 3.7 and 3.12, respectively, and retain this notation for the rest of this section. In the covering and walk constructions, we recover an endomorphism of $L(T)$ from the diagonal blocks of an endomorphism of its push-down. The next two lemmas establish the injectivity of vertex evaluations on $\mathrm{End}(L(T))$ and a nilpotence bound for the kernel of taking the diagonal part. Finite direct sums in $\mathcal{C}$ will then give retractions on all Hom spaces in Section 4.5.

Lemma 4.3. *Let L be either functor just described and put $\Sigma = \operatorname{supp} L(S_+) \cup \operatorname{supp} L(S_-)$. For nonzero $T \in \mathcal{C}$, we have $\operatorname{supp} L(T) = \Sigma$, and evaluation $\mathrm{End}(L(T)) \to \mathrm{End}_k(L(T)(x))$ is injective at every $x \in \Sigma$; here endomorphisms are taken over R for the functor of Construction 3.7 and over D_w for that of Construction 3.12.*

Proof. Write $T = (V, W; a, b)$. Applying the exact functor L to (2.1) yields a short exact sequence. Since $L(S_+)$ and $L(S_-)$ have disjoint supports, this sequence identifies $L(T)(x)$ naturally with $L(S_+)(x) \otimes V$ for $x \in \operatorname{supp} L(S_+)$ and with $L(S_-)(x) \otimes W$ for $x \in \operatorname{supp} L(S_-)$. Since $0 \neq T \in \mathcal{C}$, both V and W are nonzero, so $\operatorname{supp} L(T) = \Sigma$. Because L is fully faithful, every endomorphism of $L(T)$ has the form $L(g, h)$ for some $(g, h) \in \mathrm{End}_{K_2}(T)$, and its component at $x \in \Sigma$ is $1 \otimes g$ or $1 \otimes h$. If this component vanishes, then $g = 0$ or $h = 0$, and hence $(g, h) = 0$ by Lemma 2.2(1). □

After summing vertex spaces over each fibre, morphisms between push-downs become block matrices. Fernández-Alonso, Gavito and Pérez-Terrazas [11] recover a morphism from a diagonal block of such a matrix in their construction for truncated polynomial algebras; in Propositions 4.5 and 4.8 below, we recover endomorphisms from the diagonal part of such block matrices. The next lemma uses injectivity of the vertex evaluations to establish a nilpotence bound depending only on the maximum fibre size. In our applications, Ω is the support of a module on the covering or the vertex set of D_w, the partition $\mathcal{P}$ consists of the fibres over the vertices of A, and $\mathcal{E}$ is the endomorphism algebra of the push-down.

Lemma 4.4. *Let Ω be a finite nonempty set with a partition $\mathcal{P}$, and let $M_i \neq 0$ be vector spaces for $i \in \Omega$. Let $\mathcal{E}$ be a unital subalgebra of $\mathrm{End}_k(\bigoplus_{i\in\Omega} M_i)$ preserving each subspace $\bigoplus_{i\in\Omega'} M_i$, $\Omega' \in \mathcal{P}$. For a matrix $f = (f_{j,i})$ with blocks $f_{j,i} : M_i \to M_j$, so that columns are indexed by sources, let $\delta(f)$ be the diagonal part of f, retaining only the blocks $f_{i,i}$. Assume:*

(1) *$\delta(\mathcal{E}) \subseteq \mathcal{E}$;*
(2) *for $\mathcal{E}_0 = \delta(\mathcal{E})$, every coordinate map $\mathcal{E}_0 \to \mathrm{End}_k(M_i)$ is injective;*
(3) *$N = \ker(\delta|_{\mathcal{E}})$ is spanned by matrices $Z \in N$ such that*
   (a) *each column of Z has at most one nonzero block;*
   (b) *at least one column of Z is zero.*

Then $\delta : \mathcal{E} \to \mathcal{E}_0$ is a split algebra surjection, N is a two-sided ideal, and

$$N^m = 0, \qquad m = \max_{\Omega'\in\mathcal{P}} |\Omega'|.$$

Proof. By hypothesis (1), the diagonal matrices in $\mathcal{E}$ form a unital subalgebra $\mathcal{E}_0 = \delta(\mathcal{E})$, and $\mathcal{E} = \mathcal{E}_0 \oplus N$ as vector spaces. Let $\mathcal{Z}$ be the set of matrices $Z \in N$ satisfying conditions (a) and (b) of hypothesis (3), so that $\mathcal{Z}$ spans N. Any product of matrices in $\mathcal{Z}$ again has at most one nonzero block in each column, and a nonempty product Z of matrices in $\mathcal{Z}$ inherits a zero column

from its rightmost factor. If column i of Z is zero, then $Z_{i,i} = 0$; since the coordinate map at i is injective on $\mathcal{E}_0$ by hypothesis (2), it follows that $\delta(Z) = 0$. In particular, $N^2 \subseteq N$. For $r \in \mathcal{E}_0$ and $f \in \mathcal{E}$, diagonal multiplication satisfies $\delta(rf) = r\delta(f)$ and $\delta(fr) = \delta(f)r$, so $\mathcal{E}_0 N + N\mathcal{E}_0 \subseteq N$. Thus N is a two-sided ideal of $\mathcal{E}$, and $\delta : \mathcal{E} \to \mathcal{E}_0$ is an algebra surjection split by the inclusion $\mathcal{E}_0 \hookrightarrow \mathcal{E}$.

Now suppose that $Z_m \cdots Z_1$ is a product of m matrices in $\mathcal{Z}$ having a nonzero block in column i_0. By condition (a), each factor Z_t has at most one nonzero block in each column, so there is a unique sequence of indices $i_0, i_1, \ldots, i_m \in \Omega$ such that $(Z_t)_{i_t, i_{t-1}} \neq 0$ for all $1 \leq t \leq m$, and the nonzero block of $Z_m \cdots Z_1$ in column i_0 is the composite $(Z_m)_{i_m, i_{m-1}} \cdots (Z_1)_{i_1, i_0}$. Since every factor preserves the subspaces indexed by the parts of $\mathcal{P}$, all $m+1$ indices $i_0, \ldots, i_m$ belong to a single part of $\mathcal{P}$, which has at most m elements. By the pigeonhole principle, there exist $0 \leq p < q \leq m$ with $i_p = i_q$. For the subproduct $Z' = Z_q \cdots Z_{p+1}$, column i_p likewise has the single nonzero block

$$Z'_{i_q, i_p} = (Z_q)_{i_q, i_{q-1}} \cdots (Z_{p+1})_{i_{p+1}, i_p},$$

which must be nonzero because the full composite $(Z_m)_{i_m, i_{m-1}} \cdots (Z_1)_{i_1, i_0}$ is nonzero. Since $i_p = i_q$, this is a nonzero diagonal block of Z', contradicting $\delta(Z') = 0$ from the first paragraph. Since $\mathcal{Z}$ spans N, we conclude that $N^m = 0$. □

The nilpotence argument in Lemma 4.4 uses the same idea as the proof of [16, Theorem 3.1], where Gordon and Green show that a $\mathbb{Z}$-graded Artin algebra whose subalgebra of degree zero is local is itself local: there a repeated partial sum of degrees in a long product produces a subproduct of degree zero, just as a repeated index produces a diagonal block here.

4.3. **Endomorphisms of push-downs along a covering.** Let $\pi : R \to \mathcal{A}$ be a Galois covering with group G as in Definition 3.5, with push-down π_* as in Construction 3.6. For objects x, y in the same fibre of π, there is a unique $\gamma \in G$ with $y = \gamma x$, and we group the blocks of an endomorphism according to this group element, following the decomposition of morphisms between push-downs in Gabriel [13, §3.2 and proof of Theorem 3.6(c)]. The covering groups in Theorem 3.14(2) are free and therefore torsion-free, as required by the next proposition.

Proposition 4.5. *Suppose G is torsion-free. Let X be an R-module with finite nonempty support Σ, and suppose every evaluation $\mathrm{End}_R(X) \to \mathrm{End}_k(X(x))$, $x \in \Sigma$, is injective. The map sending f to the family of blocks $f_{x,x} : X(x) \to X(x)$ is a split algebra surjection*

$$\mathrm{End}_A(\pi_* X) \longrightarrow \mathrm{End}_R(X)$$

whose kernel has m-th power zero, where $m = \max_q |\Sigma \cap \pi^{-1}(q)|$.

Proof. We decompose endomorphisms according to G, and then apply Lemma 4.4 to the components indexed by $\gamma \neq 1$.

Step 1: Components of an endomorphism. Every $f \in \mathrm{End}_A(\pi_* X)$ preserves the fibre spaces $(\pi_* X)(q)$, since it commutes with the vertex idempotents of A. For $\gamma \in G$, let $f^{[\gamma]}$ be the matrix obtained by retaining only the blocks $f_{\gamma x, x}$. We show that each $f^{[\gamma]}$ belongs to $\mathrm{End}_A(\pi_* X)$. Let $\psi : x \to y$ be a morphism in R. For each $\eta \in G$, the decomposition of $\pi(\psi)$ with source ηx in Construction 3.6 has the single term $\eta(\psi) : \eta x \to \eta y$, by uniqueness in (3.2); thus $(\pi_* X)(\pi(\psi))$ has blocks $X(\eta(\psi)) : X(\eta x) \to X(\eta y)$ for $\eta \in G$, and zero blocks elsewhere. Comparing the blocks from $X(x)$ to $X(\gamma y)$ in $f \circ (\pi_* X)(\pi(\psi)) = (\pi_* X)(\pi(\psi)) \circ f$, and using the fact that G acts freely on objects so that only $\eta = 1$ contributes on the left and only $\eta = \gamma$ on the right, we obtain

$$f_{\gamma y, y} X(\psi) = X(\gamma(\psi)) f_{\gamma x, x}. \tag{4.2}$$

Applying (4.2) to every G-translate of ψ shows that $f^{[\gamma]}$ commutes with $(\pi_* X)(\pi(\psi))$. Since the images of morphisms of R span the morphism spaces of $\mathcal{A}$, it follows that $f^{[\gamma]} \in \mathrm{End}_A(\pi_* X)$. Because Σ is finite, $f^{[\gamma]}$ is nonzero for only finitely many $\gamma \in G$. For $\gamma = 1$, equation (4.2) is precisely the condition for an endomorphism of X, so the diagonal part $f^{[1]}$ is the image of a unique element of $\mathrm{End}_R(X)$ under π_*.

Step 2: Nilpotence of the kernel. For $\gamma \neq 1$, the matrix $f^{[\gamma]}$ has at most one nonzero block in each row and column and has zero diagonal blocks. Moreover, $\gamma\Sigma \neq \Sigma$: otherwise a positive

power of γ would fix an element of the finite nonempty set Σ, contradicting the freeness of the action and the torsion-freeness of G. Hence there exists $x \in \Sigma$ with $\gamma x \notin \Sigma$, so the column of $f^{[\gamma]}$ indexed by x is zero. We now apply Lemma 4.4 to $\mathcal{E} = \mathrm{End}_A(\pi_* X)$, partitioning Σ by the fibres of π. Hypothesis (1) holds because $\delta(f) = f^{[1]} \in \mathcal{E}$; hypothesis (2) is the assumed injectivity of the vertex evaluations; and hypothesis (3) holds because $f - \delta(f) = \sum_{\gamma \neq 1} f^{[\gamma]}$. Lemma 4.4 therefore gives the result. □

4.4. **Endomorphisms of push-downs along a closed walk.** We work in Setting 3.11 with the push-down functor p_* of Construction 3.12. For an endomorphism of $p_* X$, the matrix blocks are indexed by pairs of vertices lying in the same fibre of p, and the morphism relations link blocks whose indices are connected by arrows of D_w carrying the same label. We encode these index pairs and relations in the matching quiver H defined below, following the idea used by Crawley-Boevey [9, Theorem in §2] to describe homomorphisms between tree modules and by Krause [19, Theorem, p. 191] between tree and band modules via pairs of label-preserving maps from a common quiver. Here the vertex spaces may be infinite-dimensional and the arrow maps a, b out of u may be noninvertible. We give the block argument in this setting to obtain a nilpotence bound independent of their dimensions.

Definition 4.6. The *matching quiver* H has vertices (j, i) with $p(j) = p(i)$, ordered as target and source of a matrix block. For each pair of arrows $\alpha : j \to j'$ and $\beta : i \to i'$ of D_w with $p(\alpha) = p(\beta)$, it has an arrow

$$(\alpha, \beta) : (j, i) \longrightarrow (j', i').$$

Let Δ be the subquiver consisting of the diagonal vertices (i, i) and the arrows (e, e), where e runs over the arrows of D_w. The coordinate projections $H \to D_w$ send (j, i) to j and i, respectively, and (α, β) to α and β. By a *component* of H we mean a connected component of its underlying unoriented graph.

For an endomorphism $f : p_* X \to p_* X$, a vertex (j, i) of H indexes the block $f_{j,i} : X(i) \to X(j)$, and a matching arrow $(\alpha, \beta) : (j, i) \to (j', i')$ encodes the relation

$$f_{j',i'} X(\beta) = X(\alpha) f_{j,i}, \tag{4.3}$$

which we derive in the proof of Proposition 4.8. For instance, for the walk $c(c')^{-1}$ on two distinct loops at a single vertex, D_w is the Kronecker quiver, the diagonal vertices form a Kronecker component Δ of H, and the two off-diagonal vertices are isolated. For the walk $w = \ell_1 \ell_2 \ell_1 \ell_3^{-1}$ on three distinct loops at a single vertex, the cycle D_w and one off-diagonal component of H are

$$\begin{array}{ccc} d_0 & \xrightarrow{\ell_1} & d_1 \\ {\scriptstyle \ell_3}\downarrow & & \downarrow{\scriptstyle \ell_2} \\ d_3 & \xleftarrow[\ell_1]{} & d_2 \end{array} \qquad\qquad (d_2, d_0) \longrightarrow (d_3, d_1).$$

The two coordinate projections of this component pick out the two distinct arrows of D_w labelled ℓ_1, and each projection misses two vertices of D_w; accordingly, any block matrix whose blocks vanish off the vertices of this component has zero columns at the vertices missed by its second projection. The next lemma establishes this property for all off-diagonal components of H, verifying hypothesis (3) of Lemma 4.4 for Proposition 4.8.

Lemma 4.7. *For a primitive cyclically reduced closed walk whose cycle D_w has a source, Δ is a whole component of H. Both coordinate projections are injective on the vertices of each component. On a component other than Δ, neither projection is surjective onto $(D_w)_0$.*

Proof. We first prove that the coordinate projections are injective on each component and that Δ is a component, then show that any component with a surjective projection induces an automorphism of the cycle D_w, and finally use the primitivity of w to rule out such components other than Δ.
Step 1: Injectivity and the diagonal component. Put $r = |(D_w)_0|$. At any vertex (j, i) of H, an arrow of D_w incident to j determines at most one arrow incident to i with the same label

and the same orientation (incoming or outgoing), and conversely, by (3.4). Thus both coordinate projections $H \to D_w$ are injective on the incident edges at each vertex, and every vertex of H has degree at most two. Any path in H that never traverses the same edge twice in succession therefore projects, in each coordinate, to a walk in the cycle D_w that never backtracks along the edge just traversed and hence moves in one fixed direction around the cycle. If a shortest path in H between two vertices had at least r edges, its first r edges would project to one complete turn in each coordinate, returning to the starting vertex of H and contradicting minimality; thus every shortest path in H has fewer than r edges.

Now suppose that two vertices in the same component of H share a coordinate. The projection of a shortest path between them to that coordinate moves in one direction around D_w and begins and ends at the same vertex, so its length is a multiple of r; since its length is less than r, it must be zero. This proves that both coordinate projections are injective on the vertices of each component. Finally, at a diagonal vertex (i, i), the two arrows in any incident edge of H have the same label and the same source or target i, so they coincide by (3.4); thus every edge of H incident to Δ lies in Δ, and Δ is a whole component.

Step 2: A surjective projection gives a cycle automorphism. When $r = 2$, since D_w has a source u, both arrows of D_w go from u to the other vertex, so every off-diagonal vertex of H is isolated and cannot project onto $(D_w)_0$. Assume now that $r \geq 3$, and let C be a component of H on which one of the coordinate projections is surjective. By Step 1, C has at most r vertices, so it has exactly r vertices and both coordinate projections are bijective on the vertices of C. Since C is connected and has degree at most two, it is a path or a cycle. It therefore contains a spanning path of length $r - 1$, whose two projections are spanning paths of the cycle D_w. Let σ be the permutation of $(D_w)_0$ defined by requiring $(\sigma(i), i)$ to lie in C for every $i \in (D_w)_0$. Then σ maps the second projection of the spanning path edge by edge onto the first projection, and therefore maps the two endpoints of the second projection to those of the first. Since each projection omits a single edge of D_w, namely the edge joining its two endpoints, σ also maps the omitted edge of the second projection to that of the first, and hence extends to an automorphism of the underlying unoriented cycle of D_w.

Step 3: Excluding a component other than Δ. If σ fixes a vertex i, then $(i, i) \in C$, so C meets Δ and therefore equals Δ by Step 1. Suppose for a contradiction that $C \neq \Delta$, so that σ has no fixed vertex. Then σ is either a nonidentity rotation or a reflection without fixed vertices. A reflection without fixed vertices requires r to be even; it maps two opposite edges of the cycle to themselves while exchanging their endpoints. Since the spanning path of C omits only one edge, at least one of these two edges, say $e : i \to j$, belongs to the second projection of the spanning path. Here $\sigma(i) = j$ and $\sigma(j) = i$. By the definition of a matching arrow, the edge of C above e would pair it with an arrow $\sigma(i) \to \sigma(j)$, that is, an arrow $j \to i$. Since $r \geq 3$, the cycle D_w has only one arrow between i and j, so this is impossible. Thus σ is a nonidentity rotation.

Fix a cyclic orientation of D_w, which σ preserves, and assign to each edge of D_w its arrow label if its orientation agrees with that cyclic orientation, and the formal inverse label otherwise. Along the spanning path of C, paired arrows have the same label and orientation by Definition 4.6, so σ preserves their signed labels. Consequently, the two projections of the spanning path carry the same multiset of signed labels. Since adding the single omitted edge to either projection yields all r edges of D_w, the two omitted edges must also carry the same signed label; since they correspond under σ by Step 2, the rotation σ preserves the signed labels of all r edges of D_w. If σ shifts vertices along the cycle by τ steps ($1 \leq \tau < r$), then the cyclic word of signed labels is periodic with period $\ell = \gcd(r, \tau) < r$, and is therefore the (r/ℓ)-th power of its first ℓ letters. That shorter walk is closed in Q, since the equal signed arrows at positions 1 and $\ell + 1$ have the same initial vertex. Read from d_0, the full signed word is either w or w^{-1} according to the chosen cyclic orientation, so w is a proper power of a shorter closed walk, contradicting the primitivity of w. □

We now apply Lemma 4.7 to the endomorphism algebra of a push-down p_*X.

Proposition 4.8. *Let X be a representation of D_w with nonzero space at every vertex. Suppose p_*X is an A-module and every evaluation $\mathrm{End}_{D_w}(X) \to \mathrm{End}_k(X(i))$ is injective. The map sending*

f to the family of blocks $f_{i,i} : X(i) \to X(i)$ is a split algebra surjection

$$\mathrm{End}_A(p_*X) \longrightarrow \mathrm{End}_{D_w}(X)$$

whose kernel has m-th power zero, where $m = \max_q |p^{-1}(q)|$.

Proof. We separate the morphism equations according to the components of H, and then apply Lemma 4.4 to the components other than Δ.

Step 1: Decomposition by components of H. Since the relations for a morphism of A-modules are the same as those for the underlying representations of Q, we may compute with the arrows of Q. Let f be an endomorphism of p_*X and let $c : q \to q'$ be an arrow of Q. For vertices $i \in p^{-1}(q)$ and $j' \in p^{-1}(q')$, the block from $X(i)$ to $X(j')$ of the commutativity relation $f \circ (p_*X)(c) = (p_*X)(c) \circ f$ has the form (4.3), where $\beta : i \to i'$ is the unique arrow of D_w labelled c starting at i and $\alpha : j \to j'$ is the unique arrow of D_w labelled c ending at j', whenever such arrows exist (with the corresponding term omitted if no such arrow exists). When both α and β exist, the two blocks in (4.3) are indexed by the endpoints (j, i) and (j', i') of the matching arrow (α, β) in H. When only β exists, the relation reduces to $f_{j',i'}X(\beta) = 0$, and here $j' \neq i'$ because otherwise β itself would be an arrow labelled c ending at j'. Likewise, when only α exists, the relation reduces to $X(\alpha)f_{j,i} = 0$ with $j \neq i$, since otherwise α would be an arrow labelled c starting at i. In both of these cases, the relation constrains a single off-diagonal block. Consequently, every commutativity relation involves only blocks indexed by vertices in a single component of H, so retaining the blocks supported on any single component of H yields an endomorphism of p_*X. In particular, the part supported on Δ is precisely the image of $\mathrm{End}_{D_w}(X)$ under p_*.

Step 2: Nilpotence of the kernel. We apply Lemma 4.4 to $\mathcal{E} = \mathrm{End}_A(p_*X)$, partitioning the vertices of D_w by the fibres of p; $\mathcal{E}$ preserves the fibre subspaces $(p_*X)(q)$. Hypothesis (1) holds because the diagonal part of f is the part supported on Δ, which belongs to $\mathcal{E}$ by Step 1; hypothesis (2) is the assumed injectivity of the vertex evaluations; and hypothesis (3) holds because $\ker(\delta|_{\mathcal{E}})$ is spanned by the matrices obtained by restricting endomorphisms to single components of H other than Δ. Each such matrix has at most one nonzero block in each row and column because both coordinate projections are injective on that component; it has a zero column because the second projection is not surjective (Lemma 4.7). □

4.5. **A uniform nilpotence bound.** Since $\mathcal{C}$ is closed under finite direct sums (Lemma 2.2(2)), Lemma 4.3 and Propositions 4.5 and 4.8 apply to finite direct sums in $\mathcal{C}$. Passing to matrix entries between direct summands then yields retractions on all Hom spaces in $\mathcal{C}$ whose nilpotence bound depends only on the fixed supports of $L(S_+)$ and $L(S_-)$, and is therefore independent of the modules as required by (2.4).

Proposition 4.9. *For each available alternative ν of Theorem 3.14, the functor F_ν of (3.5) admits retractions on $\mathcal{C}$ with nilpotence bound*

$$n = \begin{cases} \min\{j \geq 1 : (\operatorname{rad} A)^j = 0\}, & \text{in alternative (1)}, \\ \max_q |\Sigma \cap \pi^{-1}(q)|, & \text{in alternative (2)}, \\ \max_q |p^{-1}(q)|, & \text{in alternative (3)}, \end{cases}$$

where $\Sigma = \operatorname{supp} L_2(S_+) \cup \operatorname{supp} L_2(S_-)$ in (2).

Proof. For alternative (1) of Theorem 3.14, this follows from Proposition 4.1. For alternatives (2) and (3), fix $\nu \in \{2, 3\}$; we first obtain retractions on endomorphism algebras and then use finite direct sums to extend them to all Hom spaces.

Step 1: Endomorphism algebras. Put $\Sigma = \operatorname{supp} L(S_+) \cup \operatorname{supp} L(S_-)$, and let $T \in \mathcal{C}$ be nonzero. In alternative (2), G is torsion-free and $L(T)$ has support Σ with injective vertex evaluations by Lemma 4.3, so Proposition 4.5 applies to $L(T)$. In alternative (3), $L_3(T)$ has full support $\Sigma = (D_w)_0$ and injective vertex evaluations by Lemma 4.3, and $p_*L_3(T)$ is an A-module by Theorem 3.14(3) and Lemma 3.13, so Proposition 4.8 applies. In either case, taking the diagonal part and using the full faithfulness of L gives a split algebra surjection

$$\mathrm{End}_A(F_\nu(T)) \longrightarrow \mathrm{End}(L(T)) \cong \mathrm{End}_{K_2}(T) \tag{4.4}$$

whose kernel has m-th power zero, where m is the maximum fibre size in the fixed support Σ.

Step 2: Hom spaces and the nilpotence bound. Let $T, U \in \mathcal{C}$. If $T = 0$ or $U = 0$, then both Hom spaces vanish and we set $\rho_{T,U} = 0$. Assume now that T and U are nonzero. Given $f : F_\nu(T) \to F_\nu(U)$, we regard f as the endomorphism of $F_\nu(T \oplus U) = F_\nu(T) \oplus F_\nu(U)$ whose (U,T)-component is f and whose other three components are zero; we then apply (4.4) for $T \oplus U$ to this endomorphism and define $\rho_{T,U}(f) : T \to U$ to be the (U,T)-component of the resulting endomorphism of $T \oplus U$. Explicitly, at each vertex $i \in \Sigma$, the (U,T)-component of the (i,i)-block retained by (4.4) is simply the (i,i)-block $f_{i,i} : L(T)(i) \to L(U)(i)$ of f. Because the inclusions and projections for a finite direct sum act separately at each vertex, the resulting family of vertex maps $(f_{i,i})_{i\in\Sigma} : L(T) \to L(U)$ is independent of the ambient finite direct sum in $\mathcal{C}$ used to compute it, and $\rho_{T,U}(f)$ is its unique preimage under the fully faithful functor L. In particular, $\rho_{T,U}$ is k-linear, and since the (i,i)-blocks of $F_\nu(g,h)$ are the vertex maps of $L(g,h)$, we have $\rho_{T,U}(F_\nu(g,h)) = (g,h)$. Moreover, for a third nonzero object $T'' \in \mathcal{C}$ and a morphism $f' : F_\nu(U) \to F_\nu(T'')$, applying the algebra surjection (4.4) to $T \oplus U \oplus T''$ shows that $\rho_{T,T''}(f'f) = \rho_{U,T''}(f')\rho_{T,U}(f)$. Finally, for $U = T$, this construction yields the endomorphism retraction of Step 1, whose kernel has m-th power zero; hence these retractions have nilpotence bound $n = m$. □

5. The embedding theorem and large families

Throughout this section, k is algebraically closed and A is an arbitrary representation-infinite finite-dimensional k-algebra. We finish the embedding theorem by cancelling the simple projective summands and transferring the functors from the reduced algebra. We then bound the nilpotence exponent and apply the embedding to Göbel–Simson families.

The cancellation uses the following fact about a module with local endomorphism ring and nilpotent radical. Maps factoring through direct sums of copies of this module form a nilpotent ideal on modules having no summand isomorphic to it.

Lemma 5.1. *Let P be a nonzero A-module with local endomorphism ring and choose $\ell \geq 1$ with $(\operatorname{rad}\operatorname{End}_A(P))^\ell = 0$. On the full subcategory of $\operatorname{Mod} A$ consisting of modules with no direct summand isomorphic to P, let $\mathcal{J}(X,Y)$ consist of maps $X \to Y$ factoring through an arbitrary direct sum of copies of P. Then $\mathcal{J}$ is an ideal and $\mathcal{J}^{\ell+1} = 0$.*

Proof. Closure under addition and composition with arbitrary morphisms is immediate from the definition of $\mathcal{J}$. For any module X with no direct summand isomorphic to P, every composite $P \to X \to P$ is a nonunit of $\operatorname{End}_A(P)$, since a unit would split a copy of P off X. Since $\operatorname{End}_A(P)$ is local, every such composite belongs to $\operatorname{rad}\operatorname{End}_A(P)$.

Now consider $\ell+1$ composable morphisms in $\mathcal{J}$, written as $X_{i-1} \xrightarrow{u_i} P^{(I_i)} \xrightarrow{v_i} X_i$ $(1 \leq i \leq \ell+1)$. Their composite is

$$v_{\ell+1}(u_{\ell+1}v_\ell)\cdots(u_2v_1)u_1.$$

For each $1 \leq i \leq \ell$, put $w_i = u_{i+1}v_i : P^{(I_i)} \to P^{(I_{i+1})}$; by the first paragraph, every matrix component $(w_i)_{s,r} : P \to P$ $(r \in I_i,\ s \in I_{i+1})$ belongs to $\operatorname{rad}\operatorname{End}_A(P)$. Fix $s_1 \in I_1$ and an element $x \in P$ in the s_1-th summand of $P^{(I_1)}$. Since each w_i sends an element of a summand of $P^{(I_i)}$ to an element of $P^{(I_{i+1})}$ with only finitely many nonzero coordinates, the $s_{\ell+1}$-th coordinate of $w_\ell \cdots w_1(x)$ in $P^{(I_{\ell+1})}$ is the finite sum

$$\sum_{s_2\in I_2,\ldots,s_\ell\in I_\ell} (w_\ell)_{s_{\ell+1},s_\ell}\cdots(w_1)_{s_2,s_1}(x)$$

for every $s_{\ell+1} \in I_{\ell+1}$. Each product $(w_\ell)_{s_{\ell+1},s_\ell}\cdots(w_1)_{s_2,s_1}$ lies in $(\operatorname{rad}\operatorname{End}_A(P))^\ell = 0$, so every term of the sum vanishes. Since elements supported in single summands span $P^{(I_1)}$, we have $w_\ell \cdots w_1 = 0$, and hence $v_{\ell+1}w_\ell \cdots w_1u_1 = 0$. □

We can now prove Theorem A for every choice in Theorem 3.14.

Theorem 5.2. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k. Let A_0, B and Φ be as in Lemma 3.1, and let $F_B : \mathcal{D} \to \operatorname{Mod} B$ be any of the*

functors F_ν in (3.5), formed from data in Theorem 3.14 for B. Then $F = \Phi F_B : \mathcal{D} \to \operatorname{Mod} A$ is a faithful representation embedding with the following properties.

(1) *F sends finite-dimensional modules to finite-dimensional modules.*
(2) *F commutes with arbitrary direct sums and filtered colimits.*
(3) *If $T \in \mathcal{D}$ has infinite dimension, then $\dim_k F(T) = \dim_k T$.*

Moreover, F admits retractions on $\mathcal{C}$ with the same nilpotence bound as F_B in Proposition 4.9.

Proof. Lemmas 3.4, 3.8, 3.13 and 3.1 show that F is exact and faithful. We first check the three stated properties.

(1) The same lemmas show that F_B and Φ preserve finite-dimensional modules.

(2) Those lemmas also show that F_B and Φ commute with arbitrary direct sums and filtered colimits.

(3) By Lemmas 3.4, 3.8 and 3.13, $\dim_k F_B(T) = \dim_k T$ for infinite-dimensional T. Lemma 3.1 gives $\dim_k F(T) = \dim_k F_B(T)$.

For $T, U \in \mathcal{C}$, compose the inverse of the Hom isomorphism induced by Φ with the retraction for F_B:

$$\operatorname{Hom}_A(F(T), F(U)) \xrightarrow{\sim} \operatorname{Hom}_B(F_B(T), F_B(U)) \xrightarrow{\rho^B_{T,U}} \operatorname{Hom}_{K_2}(T, U).$$

These maps preserve composition and have the bound in Proposition 4.9. By Lemma 2.4, $F|_{\mathcal{C}}$ preserves indecomposability and reflects isomorphism classes. Put $F_{\mathrm{fin}} = F|_{\mathcal{D}_{\mathrm{fin}}}$. The finite restriction of F_B is a representation embedding by Theorem 3.14 and, in alternative (2), Lemma 3.9. Since Φ is exact and fully faithful by Lemma 3.1, F_{fin} is also a representation embedding. Thus $P = F(S_-)$ is finite-dimensional and indecomposable, so $\operatorname{End}_A(P)$ is local with nilpotent radical. We complete the proof on $\mathcal{D}$ in three steps.

Step 1: Excluding P as a summand. Let $T_0 \in \mathcal{C}$, and suppose that $P \to F(T_0)$ is a split monomorphism with retraction $p : F(T_0) \to P$. By part (2) and (3.6), $F(T_0)$ is the filtered colimit of the modules $F(U)$ for finite-dimensional $U \subseteq T_0$. Since P is finitely presented and the finite-dimensional submodules $T_1 \subseteq T_0$ belonging to $\mathcal{C}_{\mathrm{fin}}$ are cofinal (Lemma 2.2(3)), the map $P \to F(T_0)$ factors through $F(T_1)$ for some $T_1 \in \mathcal{C}_{\mathrm{fin}}$. Composing the canonical map $F(T_1) \to F(T_0)$ with p shows that P is isomorphic to a direct summand of $F(T_1)$. The indecomposable summands of T_1 belong to $\mathcal{C}_{\mathrm{fin}}$, since they have neither simple summand, and their images under F are indecomposable. The Krull–Schmidt theorem therefore gives an indecomposable summand T' of T_1 with $F(T') \cong P = F(S_-)$. Since F_{fin} reflects isomorphism classes, $T' \cong S_-$, contradicting the fact that $T_1 \in \mathcal{C}$ has no summand isomorphic to S_-.

Step 2: Reflection of isomorphism classes. Let $T, U \in \mathcal{D}$ satisfy $F(T) \cong F(U)$. By (2.3), we may write $T = T_0 \oplus S_-^{(I)}$ and $U = U_0 \oplus S_-^{(K)}$ with $T_0, U_0 \in \mathcal{C}$. Put $X = F(T_0)$ and $Y = F(U_0)$. Since F commutes with direct sums, an isomorphism $F(T) \xrightarrow{\sim} F(U)$ and its inverse take the block form

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : X \oplus P^{(I)} \longrightarrow Y \oplus P^{(K)}, \qquad f^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

so that

$$g_{11} f_{11} = 1_X - g_{12} f_{21}, \qquad f_{11} g_{11} = 1_Y - f_{12} g_{21}. \tag{5.1}$$

The composites $g_{12}f_{21}$ and $f_{12}g_{21}$ factor through direct sums of copies of P. Since X and Y have no direct summand isomorphic to P by Step 1, Lemma 5.1 implies that $g_{12}f_{21}$ and $f_{12}g_{21}$ are nilpotent. Consequently, both right sides of (5.1) are invertible (with inverses given by finite geometric series), so $f_{11} : X \to Y$ is an isomorphism, and Lemma 2.4 gives $T_0 \cong U_0$. Moreover, multiplying f on the left and right by invertible triangular matrices yields

$$\begin{pmatrix} 1 & 0 \\ -f_{21} f_{11}^{-1} & 1 \end{pmatrix} f \begin{pmatrix} 1 & -f_{11}^{-1} f_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} - f_{21} f_{11}^{-1} f_{12} \end{pmatrix},$$

which shows that $P^{(I)} \cong P^{(K)}$. Taking dimensions over k gives $(\dim_k P)|I| = (\dim_k P)|K|$; since $0 < \dim_k P < \infty$, we obtain $|I| = |K|$ and hence $T \cong U$.

Step 3: Preservation of indecomposability. By (2.3), every indecomposable object of $\mathcal{D}$ either belongs to $\mathcal{C}$ or is isomorphic to S_-. In the first case, F preserves indecomposability by Lemma 2.4; in the second, $F(S_-) \cong P$ is indecomposable because F_{fin} preserves indecomposability. □

Remark 5.3. In the proof of Theorem 5.2, the assumption that F_{fin} is a representation embedding can be replaced by the indecomposability of $F(S_-)$ and the condition $F(S_-) \ncong F(U)$ for every $U \in \mathcal{C}_{\mathrm{fin}}$, with the other properties of F and its retractions retained. These are the two finite-dimensional facts used in the Lean proof of Theorem 5.2. They also suffice for the proof given here: the retractions imply that $F|_{\mathcal{C}}$ preserves indecomposability, so the Krull–Schmidt argument in Step 1 would give $U \in \mathcal{C}_{\mathrm{fin}}$ with $F(U) \cong F(S_-)$, contrary to the stated condition. Step 3 uses only the indecomposability of $F(S_-)$.

The rank bound in [6, Theorem 7] will bound the nilpotence exponent. We record it together with the choices of construction made in that proof. Write $\iota : \operatorname{mod} k[t] \to \mathcal{D}_{\mathrm{fin}}$ for the functor $\iota(U) = (U, U; t, 1)$, acting by (f, f) on a $k[t]$-module homomorphism f.

Lemma 5.4 ([6, §5.2, Theorem 7 and §5.4])**.** *Let B be a connected basic minimal representation-infinite algebra over an algebraically closed field k, and put $d = \dim_k B$.*

(1) *If the lattice of two-sided ideals of B is not distributive, then alternative (1) of Theorem 3.14 applies.*
(2) *If this lattice is distributive, there is a choice of F_B in alternative (2) or (3) of Theorem 3.14 and a bimodule ${}_B\mathcal{B}_{k[t]}$, free as a right $k[t]$-module of rank $1 \le \mu \le \max(30, 4d)$, such that*

$$(F_B|_{\mathcal{D}_{\mathrm{fin}}}) \circ \iota \cong \mathcal{B} \otimes_{k[t]} - \quad \text{as functors } \operatorname{mod} k[t] \longrightarrow \operatorname{mod} B.$$

Part (1) is the final paragraph of [6, §5.2], applying [6, Corollary 2]. The tensor description and the upper bound in part (2) are the assertion of [6, Theorem 7]. The proof in [6, §5.4] obtains this tensor functor from one of the covering or walk constructions in Theorem 3.14. The lower bound $\mu \ge 1$ follows because the composite is a representation embedding. Evaluating the tensor functor at $k[t]/(t)$ bounds the support sizes in Proposition 4.9 and hence gives the next result.

Proposition 5.5. *There is a choice of the embedding F in Theorem 5.2 and an integer $1 \le n \le \max(30, 4 \dim_k A)$ such that F admits retractions on $\mathcal{C}$ with nilpotence bound n.*

Proof. Choose $A_0 = eAe$ and B as in Lemma 3.1, and put $d = \dim_k B \le \dim_k A$. If the lattice of two-sided ideals of B is not distributive, Lemma 5.4(1) gives alternative (1). Here the quotient construction is over B itself, whose Loewy length is at most d, so the assertion follows from Proposition 4.9.

If the lattice of two-sided ideals of B is distributive, choose F_B and the bimodule $\mathcal{B}$, free of rank $\mu \le \max(30, 4d)$ over $k[t]$, as in Lemma 5.4(2). For $T_* = (k, k; 0, 1) = \iota(k[t]/(t))$, we have $F_B(T_*) \cong \mathcal{B} \otimes_{k[t]} k[t]/(t)$ and hence $\dim_k F_B(T_*) = \mu$. In alternative (2), every vertex of $\Sigma = \operatorname{supp} M_- \cup \operatorname{supp} M_+$ contributes at least one dimension to $F_B(T_*)$, so the maximum fibre size is at most $|\Sigma| \le \mu$. In alternative (3), all r vertex spaces of $L_3(T_*)$ are one-dimensional, so $r = \mu$ and the maximum fibre size is at most μ. By Proposition 4.9 and Theorem 5.2, the resulting functor $F = \Phi F_B$ admits retractions on $\mathcal{C}$ with nilpotence bound $n \le \mu \le \max(30, 4 \dim_k A)$. □

We use the following specialization of the rigid families of Göbel and Simson. It records the cardinality and Hom spaces needed for the application.

Theorem 5.6 ([14, Theorem 1.2(i),(iii) and Theorem 3.3])**.** *For every field k and every infinite cardinal κ, there are Kronecker modules $T_\beta = (V_\beta, W_\beta; a_\beta, b_\beta)$, indexed by $\beta \subseteq \kappa$, such that*

$$\dim_k V_\beta = \dim_k W_\beta = \kappa, \qquad a_\beta, b_\beta \text{ are injective}, \qquad a_\beta(V_\beta) + b_\beta(V_\beta) = W_\beta,$$

and

$$\operatorname{Hom}_{K_2}(T_\beta, T_\gamma) \cong \begin{cases} k, & \beta \subseteq \gamma, \\ 0, & \beta \not\subseteq \gamma, \end{cases} \qquad \operatorname{End}_{K_2}(T_\beta) \cong k.$$

The Hom isomorphisms are k-linear and the endomorphism isomorphisms are isomorphisms of k-algebras.

The displayed arrow conditions show that $T_\beta \in \mathcal{C}$.

Applying our chosen embedding gives the announced families.

Corollary 5.7. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k. There is an integer $1 \le n \le \max(30, 4\dim_k A)$ such that, for every infinite cardinal κ, there are 2^κ pairwise non-isomorphic indecomposable A-modules M satisfying*

$$\dim_k M = \kappa, \qquad \operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \cong k, \qquad (\operatorname{rad}\operatorname{End}_A(M))^n = 0.$$

If $\kappa \ge |k|$, this is the largest possible number of isomorphism classes in dimension κ.

Proof. Fix F and n as in Proposition 5.5, and put $M_\beta = F(T_\beta)$ for the family in Theorem 5.6. Each T_β is indecomposable because $\operatorname{End}_{K_2}(T_\beta) \cong k$, and for $\beta \neq \gamma$ at least one of $\operatorname{Hom}_{K_2}(T_\beta, T_\gamma)$ and $\operatorname{Hom}_{K_2}(T_\gamma, T_\beta)$ vanishes, so the modules T_β are pairwise non-isomorphic. Theorem 5.2 therefore gives 2^κ pairwise non-isomorphic indecomposable A-modules M_β of dimension κ. For each β, the retraction ρ_{T_β, T_β} is a surjective k-algebra homomorphism

$$\operatorname{End}_A(M_\beta) \longrightarrow \operatorname{End}_{K_2}(T_\beta) \cong k$$

whose kernel N_β satisfies $N_\beta^n = 0$. Every nilpotent ideal is contained in the Jacobson radical, and the quotient k has zero radical, so $N_\beta = \operatorname{rad}\operatorname{End}_A(M_\beta)$.

Finally, suppose that $\kappa \ge |k|$. Any k-vector space of dimension κ has cardinality κ and therefore admits at most $\kappa^\kappa = 2^\kappa$ linear endomorphisms. Since an A-module structure on this space is determined by the actions of a finite basis of A, there are at most $(2^\kappa)^{\dim_k A} = 2^\kappa$ such module structures, and hence at most 2^κ isomorphism classes of A-modules of dimension κ. □

Note that in Corollary 5.7, the conclusion $\operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \cong k$ cannot in general be replaced by $\operatorname{End}_A(M) \cong k$. Indeed, over any finite-dimensional local algebra A with $A/\operatorname{rad} A \cong k$ (for example, the representation-infinite local algebra $k\langle x, y\rangle/(x, y)^2$), every nonsimple indecomposable A-module M satisfies $(\operatorname{rad} A)M \neq 0$, since otherwise M would be semisimple. Because $\operatorname{rad} A$ is nilpotent, both the top $M/(\operatorname{rad} A)M$ and the socle of $(\operatorname{rad} A)M$ are nonzero; composing a nonzero projection $p : M \to k$ factoring through $M/(\operatorname{rad} A)M$ with an injection $i : k \to M$ whose image lies in $(\operatorname{rad} A)M$ gives a nonzero endomorphism $ip \in \operatorname{End}_A(M)$ with $(ip)^2 = 0$.

Appendix A. Corner quotients of a special biserial algebra

Throughout this appendix, k is an arbitrary field and all algebras are finite-dimensional k-algebras. We examine the assertion about corner quotients used in Case 4 of the proof of [27, Theorem 3.1, pp. 128–129]. Proposition 3.7 of [27] asserts that every representation-infinite special biserial algebra R over a field k admits an idempotent e and a surjective k-algebra homomorphism $eRe \to S$, where S is either $k\langle x, y\rangle/(x^2, xy, yx, y^2)$ or the path algebra of an acyclic orientation of a cycle with at least two vertices; here eRe is the *corner* of R determined by e. Case 4 applies this assertion to an arbitrary representation-infinite special biserial algebra and then transfers modules from S to R through eRe. Example A.1 below shows that over any field k, the required corner quotient need not exist. Because the algebra R in Example A.1 has radical square zero, Case 1 of the same proof already establishes the conclusion of [27, Theorem 3.1] for R; thus the example refutes Proposition 3.7, invoked in Case 4, rather than the conclusion of [27, Theorem 3.1].

Recall that a finite-dimensional algebra kQ/I with I admissible is *special biserial* if each vertex has at most two incoming and two outgoing arrows and, for every arrow, at most one preceding arrow and at most one following arrow have a product outside I [27, p. 124, (SP1)–(SP2)]. The following algebra satisfies these conditions over any field k but has no corner admitting either quotient above.

Example A.1. Let k be any field. Let Q have vertices $1, 2$, loops α at 1 and δ at 2, and arrows $\beta : 1 \to 2$ and $\gamma : 2 \to 1$. Put

$$R = kQ/(\text{all paths of length two}).$$

Then R is a six-dimensional representation-infinite special biserial algebra. For every idempotent $e \in R$, the corner eRe has no quotient isomorphic to

$$k\langle x, y\rangle/(x^2, xy, yx, y^2)$$

or to the path algebra of an acyclic orientation of a cycle with at least two vertices (for two vertices, this is the Kronecker quiver).

Proof. Let e_i be the trivial path at vertex i for $i = 1, 2$; the elements $e_1, e_2, \alpha, \beta, \gamma, \delta$ form a basis of R. Since each vertex has two incoming and two outgoing arrows and all products of two arrows vanish in R, the algebra R is special biserial. For each integer $m \geq 1$, let C_m be the nilpotent Jordan block of size m, and define a representation M_m of Q by placing $k^m \oplus k^m$ at each vertex and setting

$$M_m(\alpha) = M_m(\delta) = M_m(\beta) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}, \qquad M_m(\gamma) = \begin{pmatrix} 0 & 0 \\ C_m & 0 \end{pmatrix}.$$

Since all products of two arrow maps vanish, M_m is an R-module. Any endomorphism of M_m commutes with the actions of α and δ, so its vertex matrices take the form

$$E_i = \begin{pmatrix} X_i & 0 \\ Y_i & X_i \end{pmatrix} \quad (i = 1, 2).$$

Commutativity with β gives $X_1 = X_2 =: X$, and commutativity with γ gives $XC_m = C_m X$. Because C_m has a cyclic vector, its commutant in $\mathrm{Mat}_m(k)$ is $k[C_m] \cong k[t]/(t^m)$. The algebra homomorphism $\mathrm{End}_R(M_m) \to k[C_m]$ sending (E_1, E_2) to X is surjective with square-zero kernel; since $k[C_m]$ is local, $\mathrm{End}_R(M_m)$ is also local, and hence M_m is indecomposable. Since $\dim_k M_m = 4m$ is unbounded, R is representation-infinite over every field k.

Put $J = \mathrm{rad}\, R$, so that $J^2 = 0$. Every idempotent $e \in R$ is conjugate to one of $0, e_1, e_2, 1$: writing $e = f + r$ with $f \in \{0, e_1, e_2, 1\}$ and $r \in J$, the relation $e^2 = e$ gives $r = fr(1 - f) + (1 - f)rf$, so $z = (1 - f)rf - fr(1 - f)$ satisfies $z^2 = 0$ and

$$(1 + z)f(1 - z) = f + r = e.$$

Up to isomorphism, the nonzero corners of R are therefore $e_1Re_1 \cong e_2Re_2 \cong k[t]/(t^2)$ and R itself. The two-dimensional corners $k[t]/(t^2)$ cannot map onto any algebra of dimension at least three. Under any algebra homomorphism from R to the local three-dimensional algebra $k\langle x, y\rangle/(x^2, xy, yx, y^2)$, the orthogonal idempotents e_1, e_2 must map to $1, 0$ in some order; consequently, β, γ and the loop at the vertex whose idempotent maps to 0 lie in the kernel, so the image is spanned by 1 and the remaining loop and no such homomorphism can be surjective.

Finally, every quotient of a corner of R has at most two isomorphism classes of simple modules, which rules out path algebras of cycle quivers with three or more vertices. The only remaining candidate is the Kronecker algebra kK_2. Any algebra surjection $R \to kK_2$ would map e_1, e_2 to the two primitive orthogonal idempotents of kK_2 in some order; however, both e_1Re_2 and e_2Re_1 have dimension one, whereas the arrow space of kK_2 has dimension two, so no such surjection exists. □

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