

REPRESENTATION EMBEDDINGS AND THE THIRD BRAUER–THRALL CONJECTURE

HARUHISA ENOMOTO

ABSTRACT. We show that every representation-infinite finite-dimensional algebra over an algebraically closed field has an indecomposable module of each infinite dimension. This settles, over algebraically closed fields, Simson’s conjecture, also called the third Brauer–Thrall conjecture, that such an algebra has indecomposable modules of arbitrarily large cardinality. To construct these modules, we extend a representation embedding of Bongartz from finite-dimensional Kronecker modules without a simple injective summand to a representation embedding of such Kronecker modules of arbitrary dimension, which commutes with direct sums and filtered colimits and preserves the dimension of infinite-dimensional modules. Applying it to the indecomposable Kronecker modules of every infinite dimension constructed by Göbel and Simson gives the required modules.

CONTENTS

1. Introduction	2
1.1. Extending representation embeddings to arbitrary modules	2
1.2. Indecomposable modules of prescribed dimension	3
1.3. Related work	3
1.4. Outline of the proof	4
2. Extending representation embeddings by filtered colimits	5
2.1. Kronecker modules without simple injective summands	5
2.2. Exactness and dimension of the extension	6
2.3. Nilpotent kernels and indecomposability	7
2.4. Cancelling simple projective summands	8
3. Bongartz’s representation embeddings	9
3.1. Reduction to minimal representation-infinite algebras	9
3.2. Quotients of projective modules by submodules of the socle	10
3.3. Galois coverings and push-down functors	10
3.4. Push-down along a closed walk	11
3.5. Existence of the embeddings	11
4. Homomorphisms between the embedded modules	12
4.1. Homomorphisms between quotients by submodules of the socle	12
4.2. Vertex evaluations and block matrices	13
4.3. Endomorphisms of push-downs along a covering	14
4.4. Endomorphisms of push-downs along a closed walk	14
4.5. A uniform nilpotence bound	16
5. Proof of Theorem A and nilpotent endomorphism radicals	17
References	18

Date: October 10, 2026.

2020 Mathematics Subject Classification. 16G60, 16G20, 16D70.

Key words and phrases. representation embedding, large indecomposable module, Brauer–Thrall conjecture, Kronecker module, Galois covering.

1. INTRODUCTION

Let A be a finite-dimensional algebra over a field k . The algebra A is called *representation-finite* if there are only finitely many isomorphism classes of finite-dimensional indecomposable A -modules, and *representation-infinite* otherwise. The Brauer–Thrall conjectures predict how the indecomposable modules of a representation-infinite algebra grow; Ringel [15] describes their origin. The first Brauer–Thrall conjecture asserts that a representation-infinite algebra has finite-dimensional indecomposable modules of arbitrarily large dimension. Roiter [19] proved it for finite-dimensional algebras over any field, and Auslander [2] gave another proof, for artin rings. The second Brauer–Thrall conjecture asserts that, over an infinite field, a representation-infinite algebra has infinitely many dimensions d in each of which there are infinitely many pairwise non-isomorphic indecomposable modules of dimension d . Over algebraically closed fields it was proved in the 1980s; Bongartz [5, §5.1] recalls this history and gives a proof [5, Theorem 6] by representation embeddings, which are functors from Kronecker modules to A -modules that preserve short exact sequences and indecomposability and reflect isomorphism classes. These embeddings are the starting point of this paper.

Both conjectures concern finite-dimensional modules. Auslander [3, Theorem A] showed that an artin algebra is representation-finite if and only if all its indecomposable modules are finitely generated, the forward implication being due to Ringel and Tachikawa [18]. Thus a representation-infinite algebra has infinite-dimensional indecomposable modules, and one may ask how large they can be. Simson [22, Conjecture 1_∞] conjectured, as a version of the first Brauer–Thrall conjecture, that every representation-infinite finite-dimensional algebra over a field has indecomposable modules of arbitrarily large cardinality: for every infinite cardinal κ there should be an indecomposable module M with $|M| \geq \kappa$. Iovanov [13, §6.1] proposes calling it a Brauer–Thrall 3 question, whence the name third Brauer–Thrall conjecture. An infinite-dimensional module M over k satisfies $|M| = \max(|k|, \dim_k M)$. Applying the conjecture to a cardinal greater than $\max(\kappa, |k|)$ shows that it is equivalent to asking for an indecomposable M with $\dim_k M \geq \kappa$ for every infinite cardinal κ .

We settle the conjecture over algebraically closed fields, in the stronger form that every infinite dimension occurs: if k is algebraically closed and A is representation-infinite, then A has an indecomposable module of dimension κ for every infinite cardinal κ (Corollary 1.1). To construct these modules, we extend a representation embedding of Bongartz from finite-dimensional Kronecker modules to Kronecker modules of arbitrary dimension, and apply the extension to Kronecker modules of Göbel and Simson [9] whose endomorphism algebra is k .

1.1. Extending representation embeddings to arbitrary modules. Let K_2 be the Kronecker quiver $1 \rightrightarrows 2$. We identify its representations with left kK_2 -modules and write a representation as $(V, W; a, b)$, with V at vertex 1, W at vertex 2 and arrow maps $a, b : V \rightarrow W$. Let $S_+ = (k, 0; 0, 0)$ and $S_- = (0, k; 0, 0)$ be its simple injective and simple projective representations, respectively. Let $\mathcal{D}$ be the full subcategory of $\text{Mod } kK_2$ consisting of the modules T without a direct summand isomorphic to S_+ . Since S_+ is simple and injective, $T \in \mathcal{D}$ if and only if $\text{Hom}_{K_2}(S_+, T) = 0$. Hence $\mathcal{D}$ is closed under submodules and, since S_+ is finitely presented, under direct sums and filtered colimits. A short sequence in $\mathcal{D}$ is called exact when it is short exact in $\text{Mod } kK_2$. Put $\mathcal{D}_{\text{fin}} = \mathcal{D} \cap \text{mod } kK_2$.

Following [5, §2.1], a *representation embedding* is a k -linear functor $G : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ which preserves short exact sequences and indecomposability and reflects isomorphism classes: $G(T) \cong G(U)$ implies $T \cong U$. The same definition applies to functors $\mathcal{D} \rightarrow \text{Mod } A$. Our main result is the following.

Theorem A. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k . There is a representation embedding*

$$F : \mathcal{D} \longrightarrow \text{Mod } A$$

with the following properties.

- (1) *F sends finite-dimensional modules to finite-dimensional modules.*

- (2) F commutes with arbitrary direct sums and filtered colimits.
- (3) If $T \in \mathcal{D}$ has infinite dimension, then $\dim_k F(T) = \dim_k T$.

We obtain F by extending a representation embedding of Bongartz [5, Theorem 6] by a filtered colimit. His proof passes from A to a minimal representation-infinite algebra B , that is, one whose proper quotients are all representation-finite, and shows that one of three explicit constructions over B gives a representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } B$: a quotient of a direct sum of copies of one projective module by relations formed from the arrow maps, or a push-down of representations along a Galois covering, or along the map to the quiver of B given by a closed walk (Section 3). Composing with a functor $\text{mod } B \rightarrow \text{mod } A$ given by inflation and a Morita equivalence (Lemma 3.1) gives a representation embedding $F_{\text{fin}} : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$. Set

$$F(T) := \varinjlim_{\substack{U \subseteq T \\ \dim_k U < \infty}} F_{\text{fin}}(U), \quad T \in \mathcal{D}. \quad (1.1)$$

Every $T \in \mathcal{D}$ is the union of its finite-dimensional submodules, and they lie in $\mathcal{D}_{\text{fin}}$. Since a morphism maps finite-dimensional submodules into finite-dimensional submodules, (1.1) defines a functor $F : \mathcal{D} \rightarrow \text{Mod } A$ extending F_{fin} . Theorem 5.1 shows that F has the properties stated in Theorem A for each F_{fin} obtained in this way.

1.2. Indecomposable modules of prescribed dimension. Applying Theorem A to the Kronecker modules constructed by Göbel and Simson [9] gives indecomposable modules of every infinite dimension.

Corollary 1.1. *Let A be a finite-dimensional algebra over an algebraically closed field k , and assume that A is representation-infinite. For every infinite cardinal κ there is an indecomposable A -module of dimension κ .*

Proof. Over the algebraically closed field k , Göbel and Simson [9, Corollary 1.3] construct, for every infinite cardinal κ , a Kronecker module T with $\dim_k T = \kappa$ and $\text{End}_{K_2}(T) = k$. It is indecomposable and infinite-dimensional, so it has no simple direct summand. In particular $T \in \mathcal{D}$. For a representation embedding F as in Theorem A, the module $F(T)$ is indecomposable, and it has dimension κ by Theorem A(3). $\square$

1.3. Related work. Simson [21, Theorem 2.8] gives indecomposable modules of arbitrarily large infinite dimension for four classes of representation-infinite algebras over an algebraically closed field: loop-finite algebras, strongly simply connected algebras, algebras with radical square zero, and group algebras of finite groups. For representation-infinite special biserial algebras over arbitrary fields, Simson [22, Theorem 3.1] states that there are indecomposable modules of arbitrarily large cardinality. Corollary 1.1 attains every infinite dimension, for every representation-infinite algebra over an algebraically closed field; Simson’s special biserial statement also concerns fields that are not algebraically closed.

Simson [22, Theorem 3.1 and Corollary 3.6] also controls endomorphism algebras. For a representation-infinite finite-dimensional algebra R over a field K such that either $(\text{rad } R)^2 = 0$ and $R/\text{rad } R$ is a product of copies of K , or R is local with $R/\text{rad } R \cong K$, or R is the group algebra of a finite group over such a local algebra, he shows that for every K -algebra E there are an R -module U and a two-sided ideal I of $\text{End}_R(U)$ with $I^2 = 0$ and $\text{End}_R(U)/I \cong E$. Taking $E = K$, the ideal I is the radical of $\text{End}_R(U)$, so U has a local endomorphism algebra whose radical has square zero; the dimension of U is not prescribed. Corollary 5.3 below shows that the modules of Corollary 1.1 can be chosen with local endomorphism algebras whose radicals are nilpotent with an exponent that depends on F_{fin} but not on the dimension.

Ringel [17, Corollary on p. 407] showed that for a connected tame hereditary algebra R , the category of all R -modules contains a full exact subcategory equivalent to the category of all modules over a certain wild hereditary algebra; thus tame hereditary algebras are wild with respect to modules of arbitrary dimension. For representation-infinite algebras A over k that are loop-finite or strongly simply connected, Simson [20, Theorem 1.2] constructs a fully faithful exact functor from the category of all Kronecker modules to $\text{Mod } A$ which sends finite-dimensional modules to

finite-dimensional modules; its proof uses Ringel's full exact embedding of all Kronecker modules defined by a suitable pair of bricks [16, 1.5, Lemma]. These embeddings are full, but they are available only for special classes of algebras. Theorem A applies to every representation-infinite algebra over an algebraically closed field; its functor, the extension (1.1) of Bongartz's embeddings to $\mathcal{D}$, is not asserted to be full.

Iovanov [13, §6.1] asks whether an exact functor between categories of comodules over coalgebras is a representation embedding whenever its restriction to finite-dimensional comodules is one. The modules over a finite-dimensional algebra are the comodules over its dual coalgebra. The analogous question for Kronecker modules asks whether an exact functor $\mathcal{D} \rightarrow \text{Mod } A$ whose restriction to $\mathcal{D}_{\text{fin}}$ is a representation embedding is itself one. Theorem 5.1 answers this question affirmatively for the functors (1.1).

Gregory and Prest [11] recover modules of arbitrary dimension from their images under certain representation embeddings. Let R and S be finite-dimensional algebras and X an R - S -bimodule such that $\tilde{G} = X \otimes_S - : \text{Mod } S \rightarrow \text{Mod } R$ is exact and restricts to a representation embedding $G : \text{mod } S \rightarrow \text{mod } R$. If G is finitely controlled, a condition in [11] on the maps between modules $G(M)$, then [11, Theorem 4.2] gives a functor $I : \text{Mod } R \rightarrow \text{Mod } S$ commuting with direct products and filtered colimits, with $I(\tilde{G}(M)) \cong M$ for every S -module M . Hence $\tilde{G}$ reflects isomorphism classes, but this alone does not show that $\tilde{G}$ preserves indecomposability.

Over perfect fields, Bautista, Pérez and Salmerón [4, Theorem 1.4] construct a representation embedding into $\text{mod } A$ for every representation-infinite finite-dimensional algebra A . Its domain consists of the finitely generated modules over a bounded principal ideal domain, that is, a principal ideal domain in which every nonzero one-sided ideal contains a nonzero two-sided ideal. Like [5, Theorem 6], this concerns finitely generated modules.

Assuming the generalized continuum hypothesis, Guil Asensio, Cortés Izurdiaga and Torrecillas [12, Corollary 5.12] construct large modules over every ring that is not left pure semisimple: for some infinite cardinals $\lambda \leq \kappa$, a module generated by κ elements in which every subset of fewer than κ elements lies in a direct summand that is a direct sum of modules generated by fewer than λ elements, although the module itself is not such a direct sum; in particular, these modules are decomposable.

1.4. Outline of the proof. Let us explain the idea of the proof. Exactness of F and the dimension formula in Theorem A(3) follow from properties of filtered colimits and from faithfulness of F_{fin} (Lemma 3.11 and Proposition 2.3). The main difficulty is indecomposability. A finite-dimensional module is indecomposable if and only if its endomorphism algebra is local. An infinite-dimensional indecomposable module need not have a local endomorphism algebra: the indecomposable Kronecker module with spaces $k[t], k[t]$ and arrow maps $1, t$ has endomorphism algebra $k[t]$. We therefore use the characterization of indecomposability by the absence of nontrivial idempotents.

Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ consisting of the Kronecker modules with neither simple module as a direct summand. For Bongartz's functors we construct k -linear maps

$$\rho_{T,U} : \text{Hom}_A(F(T), F(U)) \longrightarrow \text{Hom}_{K_2}(T, U), \quad T, U \in \mathcal{C},$$

preserving composition and satisfying $\rho_{T,U}(F(f)) = f$, together with an integer $n \geq 1$ such that every composite of n composable maps lying in kernels of these maps is zero. For each T , the kernel of $\rho_{T,T}$ is a nilpotent ideal of $\text{End}_A(F(T))$, and $\rho_{T,T}$ is surjective. Since idempotents lift modulo a nilpotent ideal, $\text{End}_A(F(T))$ has a nontrivial idempotent if and only if $\text{End}_{K_2}(T)$ does, so F preserves indecomposability on $\mathcal{C}$. Applying ρ to an isomorphism and its inverse shows that F reflects isomorphism classes on $\mathcal{C}$. We construct these maps for finite-dimensional modules over the minimal representation-infinite algebra in Section 4, pass to arbitrary modules by filtered colimits in Section 2, and transport them to A in Section 5. Finally, every module in $\mathcal{D}$ is the direct sum of a module in $\mathcal{C}$ and copies of S_- (see (2.2)). To reflect isomorphism classes on all of $\mathcal{D}$, we cancel the corresponding copies of $F(S_-)$ using a nilpotence property of maps that factor through direct sums of copies of $F(S_-)$ (Lemma 2.6).

Organization. Section 2 proves Theorem 2.7, a sufficient condition for extending a faithful representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ by filtered colimits, including the cancellation of arbitrary

multiplicities of S_- . Section 3 reduces A to a connected basic minimal representation-infinite algebra and describes Bongartz's three constructions; Theorem 3.10 states his theorem that one of them gives a representation embedding, and Lemma 3.11 shows that these functors are faithful. Section 4 constructs the maps ρ required by Theorem 2.7 for finite-dimensional modules. Section 5 proves Theorem 5.1, which implies Theorem A, together with Proposition 5.2, which transports the maps ρ from the minimal representation-infinite algebra to A , and Corollary 5.3, which shows that the modules of Corollary 1.1 can be chosen with local endomorphism algebras whose radicals are nilpotent with an exponent independent of the dimension.

Conventions and notation. All algebras and categories are k -linear, and modules are left modules. The letter k denotes a field; algebraic closure is assumed where stated. All tensor products are over k . For an A -module P and a vector space V , the action on $P \otimes V$ is $a(p \otimes v) = ap \otimes v$. We use $\text{Mod } A$ for all modules and $\text{mod } A$ for finite-dimensional modules. For arrows $\alpha : i \rightarrow j$ and $\beta : j \rightarrow l$, the product $\beta\alpha$ traverses α first. Thus left modules over a path algebra are covariant quiver representations. Write $\dim_k$ for dimension over k and $\text{rad } E$ for the Jacobson radical of an algebra E . We write Hom_{K_2} and End_{K_2} for Hom and End over kK_2 , and $X^{(I)}$ for the direct sum of copies of X indexed by a set I . A module over a k -linear category is a covariant k -linear functor to vector spaces. For a representation X of a quiver, or a module X over a k -linear category, put $\text{supp } X = \{i : X(i) \neq 0\}$. We assume the axiom of choice.

2. EXTENDING REPRESENTATION EMBEDDINGS BY FILTERED COLIMITS

We first give a sufficient condition for extending a faithful representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ by filtered colimits while preserving indecomposability and reflecting all isomorphism classes. The condition requires maps from the Hom spaces between images of finite-dimensional modules back to Kronecker Hom spaces that are left inverse to the maps induced by the functor, together with a common nilpotence bound for their kernels. Throughout this section, k is any field and A is any finite-dimensional k -algebra.

2.1. Kronecker modules without simple injective summands. We describe $\mathcal{D}$ and $\mathcal{C}$ in terms of arrow maps. A Kronecker module is written $T = (V, W; a, b)$, with $a, b : V \rightarrow W$; a morphism $(g, h) : T \rightarrow (V', W'; a', b')$ satisfies $ha = a'g$ and $hb = b'g$. Choosing a complement V_1 to $\ker a \cap \ker b$ in V gives a decomposition $T \cong (V_1, W; a|_{V_1}, b|_{V_1}) \oplus (\ker a \cap \ker b, 0)$. The second summand is a direct sum of copies of S_+ , and $\text{Hom}_{K_2}(S_+, T)$ is identified with $\ker a \cap \ker b$. Hence

$$T \in \mathcal{D} \iff \ker a \cap \ker b = 0. \quad (2.1)$$

The following definition describes the subcategory $\mathcal{C}$ of Section 1.4 by arrow maps; the equivalence of the two descriptions is shown after it.

Definition 2.1. Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ consisting of the modules $T = (V, W; a, b)$ with $a(V) + b(V) = W$. Write $\mathcal{C}_{\text{fin}} = \mathcal{C} \cap \text{mod } kK_2$.

For $T \in \mathcal{D}$, choosing a complement to $a(V) + b(V)$ in W gives a decomposition

$$T \cong T_0 \oplus S_-^{(I)}, \quad T_0 \in \mathcal{C}. \quad (2.2)$$

Conversely, if $T = T' \oplus S_-$ with $T' = (V', W'; a', b')$, then $W = W' \oplus k$ and $a(V) + b(V) = a'(V') + b'(V') \subseteq W'$, so $a(V) + b(V) \neq W$. Thus $\mathcal{C}$ consists exactly of the Kronecker modules with neither simple summand. In particular every infinite-dimensional indecomposable Kronecker module is in $\mathcal{C}$.

Lemma 2.2. *The subcategories $\mathcal{D}$ and $\mathcal{C}$ have the following properties.*

- (1) Both vertex maps $(g, h) \mapsto g$ and $(g, h) \mapsto h$ are injective on $\text{Hom}_{K_2}(T, U)$ for $T, U \in \mathcal{C}$.
- (2) $\mathcal{D}$ is closed under submodules, direct sums and filtered colimits, and $\mathcal{C}$ is closed under finite direct sums.
- (3) Every object of $\mathcal{D}$ is the union of its finite-dimensional submodules. For $T = (V, W; a, b) \in \mathcal{C}$, the submodules

$$T_{V_0} = (V_0, a(V_0) + b(V_0); a, b), \quad V_0 \subseteq V, \quad \dim_k V_0 < \infty,$$

are cofinal among its finite-dimensional submodules and belong to $\mathcal{C}$.

Proof. (1): If $g = 0$, then $ha = hb = 0$, so $h = 0$ because $a(V) + b(V) = W$. If $h = 0$, then $\text{im } g \subseteq \ker a' \cap \ker b' = 0$.

(2): The condition defining $\mathcal{D}$ is injectivity of $\begin{pmatrix} a \\ b \end{pmatrix} : V \rightarrow W \oplus W$. It is inherited by submodules, and it is preserved by direct sums and filtered colimits, since filtered colimits of vector spaces are exact. The additional condition defining $\mathcal{C}$ is surjectivity of $(a \ b) : V \oplus V \rightarrow W$, which is preserved by finite direct sums.

(3): The algebra kK_2 is finite-dimensional, so every finitely generated submodule is finite-dimensional. For $T \in \mathcal{C}$, any finite collection of vectors in W lies in $a(V_0) + b(V_0)$ for a sufficiently large finite-dimensional $V_0 \subseteq V$. Enlarging V_0 to contain specified vectors in V proves cofinality. $\square$

2.2. Exactness and dimension of the extension. We first show that the extension of a faithful exact functor by filtered colimits is again faithful and exact, commutes with direct sums and filtered colimits, and satisfies $\dim_k F(T) = \dim_k T$ for infinite-dimensional T .

Proposition 2.3. *Let $F_{\text{fin}} : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ be a faithful exact k -linear functor. Then the colimit (1.1) defines a faithful exact k -linear functor $F : \mathcal{D} \rightarrow \text{Mod } A$ extending F_{fin} . It commutes with arbitrary direct sums and filtered colimits, and $\dim_k F(T) = \dim_k T$ for every infinite-dimensional $T \in \mathcal{D}$.*

Proof. For $f : T \rightarrow T'$, a finite-dimensional submodule $U \subseteq T$ maps into the finite-dimensional submodule $f(U) \subseteq T'$. Applying F_{fin} and taking colimits defines $F(f)$. For finite-dimensional T , the index set has the largest element T , so $F(T) \cong F_{\text{fin}}(T)$ naturally.

Next we show that F commutes with filtered colimits. Let $T = \varinjlim_i T_i$ be a filtered colimit in $\mathcal{D}$, with transition maps $\varphi_{ij} : T_i \rightarrow T_j$ and canonical maps $\varphi_i : T_i \rightarrow T$; we show that the canonical map $c : \varinjlim_i F(T_i) \rightarrow F(T)$ is bijective. Finite-dimensional kK_2 -modules and finite-dimensional A -modules are finitely presented. Thus a map from a finite-dimensional module into a filtered colimit, of Kronecker modules or of A -modules, factors through one of the modules of the filtered system, and two such maps that agree in the colimit already agree in a later module of the system. Every finite-dimensional submodule $U \subseteq T$ is finitely generated, so $U = \varphi_i(U_i)$ for some i and some finite-dimensional $U_i \subseteq T_i$. The kernel of $U_i \rightarrow U$ is a finite-dimensional module in $\mathcal{D}$, so $F_{\text{fin}}(U_i) \rightarrow F_{\text{fin}}(U)$ is surjective by exactness; hence c is surjective. For injectivity, we find, for an element in the kernel of c , an index j and a finite-dimensional submodule of T_j that φ_j maps isomorphically onto a finite-dimensional submodule of T in which the element already vanishes. Let $x \in F(T_i)$ satisfy $c(x) = 0$, and represent x in $F_{\text{fin}}(Z_i)$ for a finite-dimensional $Z_i \subseteq T_i$. Its image in $F_{\text{fin}}(\varphi_i(Z_i))$ vanishes in $F_{\text{fin}}(U')$ for some finite-dimensional U' with $\varphi_i(Z_i) \subseteq U' \subseteq T$. Lifting generators of U' , we find an index $l \geq i$ and a finite-dimensional $U_l \subseteq T_l$ containing $\varphi_{il}(Z_i)$ with $\varphi_l(U_l) = U'$. The kernel of $U_l \rightarrow U'$ is finite-dimensional and maps to zero in some T_j with $j \geq l$. Then $U_j = \varphi_{lj}(U_l)$ contains $\varphi_{ij}(Z_i)$, and φ_j maps U_j isomorphically onto U' , since the kernel of $U_l \rightarrow U'$ maps to zero in T_j . Since $\varphi_j \varphi_{ij} = \varphi_i$, the isomorphism $F_{\text{fin}}(U_j) \cong F_{\text{fin}}(U')$ induced by φ_j carries the image of x in $F_{\text{fin}}(U_j)$ to its image in $F_{\text{fin}}(U')$, which is zero. Thus x maps to zero in $F(T_j)$, and c is injective. Since F_{fin} is additive, F preserves finite sums; expressing an arbitrary sum as the filtered colimit of its finite subsums proves preservation of all direct sums.

Let $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ be exact in $\mathcal{D}$. For each finite-dimensional submodule $U \subseteq T$, there is an exact sequence

$$0 \rightarrow U \cap T' \rightarrow U \rightarrow \text{im}(U \rightarrow T'') \rightarrow 0$$

in $\mathcal{D}_{\text{fin}}$, by submodule closure. The original sequence is the filtered colimit of these sequences. Exactness of F_{fin} and of filtered colimits, together with the commutation of F with filtered colimits just proved, gives exactness of F . To prove faithfulness, let $f : X \rightarrow Y$ satisfy $F(f) = 0$. Restrict f to a finite-dimensional $U \subseteq X$ and factor it through a finite-dimensional $U' \subseteq Y$. The resulting map $F_{\text{fin}}(U) \rightarrow F(Y)$ is zero. Finite presentation of $F_{\text{fin}}(U)$ shows that $F_{\text{fin}}(U) \rightarrow F_{\text{fin}}(U'')$ is zero for some finite-dimensional U'' with $U' \subseteq U'' \subseteq Y$. Faithfulness of F_{fin} then gives $f|_U = 0$. Hence $f = 0$.

Put $P = F_{\text{fin}}(S_-)$. It is finite-dimensional, and nonzero because F_{fin} is faithful and $1_{S_-} \neq 0$. For $T = (V, W; a, b) \in \mathcal{D}$, evaluation at vertex 2 identifies $\text{Hom}_{K_2}(S_-, T)$ with W , and faithfulness gives an injection

$$W = \text{Hom}_{K_2}(S_-, T) \longrightarrow \text{Hom}_A(P, F(T)).$$

Consequently $\dim_k W \leq (\dim_k P) \dim_k F(T)$. By (2.1), the map $V \rightarrow W \oplus W$, $v \mapsto (a(v), b(v))$, is injective, so $\dim_k V \leq 2 \dim_k W$, so $\dim_k T = \dim_k W$ if T is infinite-dimensional. For the reverse inequality, the regular module kK_2 belongs to $\mathcal{D}$, as do its direct sums and their submodules. An epimorphism $kK_2^{(I)} \rightarrow T$ with $|I| \leq \dim_k T$ therefore has its kernel in $\mathcal{D}$. Exactness and preservation of sums give

$$\dim_k F(T) \leq (\dim_k F_{\text{fin}}(kK_2)) \dim_k T.$$

These two inequalities imply $\dim_k F(T) = \dim_k T$ for infinite-dimensional T . $\square$

2.3. Nilpotent kernels and indecomposability. We now formalize the maps ρ of Section 1.4.

Definition 2.4. Let Ψ be a k -linear functor from a full subcategory of $\text{Mod } kK_2$ to $\text{Mod } A$, and let $\mathcal{X}$ be a full subcategory of its domain. We call k -linear maps

$$\rho_{X,Y} : \text{Hom}(\Psi(X), \Psi(Y)) \longrightarrow \text{Hom}_{K_2}(X, Y), \quad X, Y \in \mathcal{X},$$

retractions for Ψ on $\mathcal{X}$ if they preserve composition and satisfy $\rho_{X,Y}(\Psi(u)) = u$ for every morphism $u : X \rightarrow Y$. Thus each $\rho_{X,Y}$ is a retraction of the linear map $\text{Hom}_{K_2}(X, Y) \rightarrow \text{Hom}(\Psi(X), \Psi(Y))$ induced by Ψ , and these retractions are compatible with composition.

Taking $u = 1_X$ shows that retractions preserve identities. Since $\rho_{X,Y}(\Psi(v) \circ f \circ \Psi(u)) = v \circ \rho_{X',Y'}(f) \circ u$ for $X, X', Y, Y' \in \mathcal{X}$ and morphisms $u : X \rightarrow X'$, $f : \Psi(X') \rightarrow \Psi(Y')$ and $v : Y' \rightarrow Y$, retractions are natural in both variables. Ardizzoni and Menini [1, Definition 1.1] call a functor heavily separable if there are maps from the Hom spaces between images to the original Hom spaces which are natural in both variables, left inverse to the functor, and compatible with composition. Thus retractions for Ψ on $\mathcal{X}$ are exactly k -linear such maps for the restriction of Ψ to $\mathcal{X}$. We show that retractions for F_{fin} on $\mathcal{C}_{\text{fin}}$ whose kernels satisfy a common nilpotence bound extend, by filtered colimits, to retractions for F on $\mathcal{C}$ with the same bound.

Proposition 2.5. *Under the hypotheses of Proposition 2.3, let*

$$\rho_{U,U'}^{\text{fin}} : \text{Hom}_A(F_{\text{fin}}(U), F_{\text{fin}}(U')) \longrightarrow \text{Hom}_{K_2}(U, U'), \quad U, U' \in \mathcal{C}_{\text{fin}},$$

be retractions for F_{fin} on $\mathcal{C}_{\text{fin}}$. Assume that there is an integer $n \geq 1$ such that every composite of n composable maps lying in kernels of the maps ρ^{fin} is zero. Then there are retractions

$$\rho_{X,Y} : \text{Hom}_A(F(X), F(Y)) \longrightarrow \text{Hom}_{K_2}(X, Y), \quad X, Y \in \mathcal{C},$$

for F on $\mathcal{C}$ with the same nilpotence bound. In particular, $F|_{\mathcal{C}}$ preserves indecomposability and reflects isomorphism classes.

Proof. Use the cofinal finite-dimensional submodules in $\mathcal{C}$ from Lemma 2.2(3). Finite presentation of these submodules U and of the modules $F_{\text{fin}}(U)$ gives

$$\text{Hom}_A(F(X), F(Y)) \cong \varinjlim_{U \subseteq X} \varinjlim_{U' \subseteq Y} \text{Hom}_A(F_{\text{fin}}(U), F_{\text{fin}}(U')), \quad (2.3)$$

$$\text{Hom}_{K_2}(X, Y) \cong \varinjlim_{U \subseteq X} \varinjlim_{U' \subseteq Y} \text{Hom}_{K_2}(U, U'). \quad (2.4)$$

All submodules indexing these limits are finite-dimensional and in $\mathcal{C}$. The maps ρ^{fin} are natural in both variables, as noted after Definition 2.4, so they are compatible with the transition maps of both systems. Hence, through the isomorphisms (2.3) and (2.4), the maps ρ^{fin} induce a map $\rho : \text{Hom}_A(F(X), F(Y)) \rightarrow \text{Hom}_{K_2}(X, Y)$, split by F . To check composition, restrict to $F_{\text{fin}}(U)$, factor the first map through $F_{\text{fin}}(U')$ for a finite-dimensional submodule U' of the middle module, and factor the second map on $F_{\text{fin}}(U')$ through $F_{\text{fin}}(U'')$ for a finite-dimensional submodule U'' of the target. Compatibility of ρ^{fin} with this composite gives the required identity.

Let $f : F(X) \rightarrow F(Y)$ satisfy $\rho(f) = 0$, and represent its restriction to $F_{\text{fin}}(U)$ by $\varphi : F_{\text{fin}}(U) \rightarrow F_{\text{fin}}(U')$. By construction, the composite $U \xrightarrow{\rho^{\text{fin}}(\varphi)} U' \hookrightarrow Y$ is $\rho(f)|_U = 0$. Since $U' \hookrightarrow Y$ is injective, $\rho^{\text{fin}}(\varphi) = 0$. For n composable maps lying in kernels of the maps ρ , start from any $F_{\text{fin}}(U)$ and factor the maps successively through modules $F_{\text{fin}}(U')$ with U' finite-dimensional. Each representative lies in a kernel of ρ^{fin} , so their composite is zero. Taking the colimit over U proves the same bound n .

For an indecomposable $X \in \mathcal{C}$, let $e \in \text{End}_A(F(X))$ be idempotent. Then $\rho(e)$ is 0 or 1, so e or $1 - e$ belongs to the nilpotent kernel. A nilpotent idempotent is zero. Faithfulness gives $F(X) \neq 0$, hence $F(X)$ is indecomposable. Finally, applying ρ to an isomorphism and its inverse gives inverse Kronecker morphisms. $\square$

2.4. Cancelling simple projective summands. To pass from $\mathcal{C}$ to $\mathcal{D}$, we must recover the multiplicity of S_- in (2.2). For a representation embedding F_{fin} , the module $F_{\text{fin}}(S_-)$ is finite-dimensional and indecomposable. The following lemma controls maps factoring through arbitrary direct sums of copies of such a module.

Lemma 2.6. *Let P be a finite-dimensional indecomposable A -module and choose $\ell \geq 1$ with $(\text{rad End}_A(P))^\ell = 0$. On the full subcategory of $\text{Mod } A$ consisting of modules with no direct summand isomorphic to P , let $\mathcal{J}(X, Y)$ consist of maps $X \rightarrow Y$ factoring through an arbitrary direct sum of copies of P . Then $\mathcal{J}$ is an ideal and $\mathcal{J}^{\ell+1} = 0$.*

Proof. Closure under addition and composition with arbitrary morphisms is immediate from the factorization description. For a module X in the stated subcategory, every composite $P \rightarrow X \rightarrow P$ is a nonunit of $\text{End}_A(P)$: a unit would split a copy of P off X . Since P is finite-dimensional and indecomposable, $\text{End}_A(P)$ is local, so this composite lies in its radical.

Write $\ell + 1$ composable maps in $\mathcal{J}$ as $X_{i-1} \xrightarrow{u_i} P^{(I_i)} \xrightarrow{v_i} X_i$. Their product contains the ℓ middle maps $u_{i+1}v_i$ between sums of copies of P . Write such a map $P^{(I)} \rightarrow P^{(J)}$ as a matrix whose column indexed by an element of I lists the components in $\text{End}_A(P)$ of its restriction to the corresponding copy of P . Each entry lies in $\text{rad End}_A(P)$ by the preceding paragraph, and each column has only finitely many nonzero entries because P is finitely generated. Matrix multiplication therefore expresses each entry of the product as a finite sum of products of ℓ radical elements, hence as zero. The full composite vanishes. $\square$

Combining this lemma with Propositions 2.3 and 2.5 gives the following sufficient condition for the extension to be a representation embedding.

Theorem 2.7. *Let $F_{\text{fin}} : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ be a faithful representation embedding. Suppose that there are retractions for F_{fin} on $\mathcal{C}_{\text{fin}}$ and an integer $n \geq 1$ such that every composite of n composable maps lying in kernels of these retractions is zero. Then the extension F of F_{fin} defined by (1.1) is a representation embedding on $\mathcal{D}$. It commutes with arbitrary sums and filtered colimits, satisfies $\dim_k F(T) = \dim_k T$ for every infinite-dimensional $T \in \mathcal{D}$, and has retractions on $\mathcal{C}$ with the same nilpotence bound.*

Proof. Propositions 2.3 and 2.5 give all assertions except preservation of indecomposability and reflection of isomorphism classes on $\mathcal{D}$. Put $P = F_{\text{fin}}(S_-)$; it is finite-dimensional and indecomposable. For $T_0 \in \mathcal{C}$, the module $F(T_0)$ has no summand isomorphic to P . Indeed, a split inclusion $P \rightarrow F(T_0)$ factors through $F_{\text{fin}}(T_1)$ for some finite-dimensional $T_1 \subseteq T_0$ in $\mathcal{C}$, since P is finitely presented and such submodules are cofinal among the finite-dimensional submodules of T_0 (Lemma 2.2(3)). Composing the canonical map $F_{\text{fin}}(T_1) \rightarrow F(T_0)$ with a projection $F(T_0) \rightarrow P$ that splits this inclusion splits P off $F_{\text{fin}}(T_1)$. Decompose T_1 into finitely many indecomposables. Their images are indecomposable, so the Krull–Schmidt theorem and reflection of isomorphism classes by F_{fin} force one of them to be isomorphic to S_- . This contradicts $T_1 \in \mathcal{C}$.

Let $T, U \in \mathcal{D}$ and suppose $F(T) \cong F(U)$. By (2.2) we may write $T = T_0 \oplus S_-^{(I)}$ and $U = U_0 \oplus S_-^{(J)}$ with $T_0, U_0 \in \mathcal{C}$. Put $X = F(T_0)$ and $Y = F(U_0)$. Since F commutes with direct sums, we may

write an isomorphism and its inverse as

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : X \oplus P^{(I)} \longrightarrow Y \oplus P^{(J)}, \quad f^{-1} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then

$$\alpha'\alpha = 1_X - \beta'\gamma, \quad \alpha\alpha' = 1_Y - \beta\gamma'. \quad (2.5)$$

The maps $\beta'\gamma$ and $\beta\gamma'$ factor through direct sums of copies of P . By the preceding paragraph, X and Y have no direct summand isomorphic to P , so Lemma 2.6 shows that $\beta'\gamma$ and $\beta\gamma'$ are nilpotent. Hence both right sides of (2.5) are invertible, with inverses given by finite geometric series, and $\alpha : X \rightarrow Y$ is an isomorphism. Proposition 2.5 gives $T_0 \cong U_0$. Multiplying f on the left and right by invertible triangular matrices gives

$$\begin{pmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{pmatrix} f \begin{pmatrix} 1 & -\alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{pmatrix}.$$

Hence $P^{(I)} \cong P^{(J)}$. Positive finite dimension of P implies $|I| = |J|$, proving $T \cong U$.

It remains to prove preservation of indecomposability. By (2.2), every indecomposable object of $\mathcal{D}$ either belongs to $\mathcal{C}$ or is isomorphic to S_- . Proposition 2.5 treats the first case. In the second, $F(S_-) \cong P$ is indecomposable because F_{fin} is a representation embedding. $\square$

3. BONGARTZ'S REPRESENTATION EMBEDDINGS

We will apply Theorem 2.7 to the representation embeddings $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ constructed by Bongartz [5]. His proof first reduces A to a connected basic minimal representation-infinite algebra (Section 3.1). For such an algebra it uses one of three constructions, each defined from auxiliary data attached to that algebra: a quotient algebra Λ of it with two suitable radical elements, used to form quotients of direct sums of copies of an indecomposable projective Λ -module (Section 3.2), a Galois covering of it (Section 3.3), or a closed walk in a quiver presenting it (Section 3.4). Theorem 3.10, due to Bongartz, states that for suitable data at least one of them is a representation embedding. Section 4 then constructs retractions with a common nilpotence bound for the resulting functors.

3.1. Reduction to minimal representation-infinite algebras. Recall that an algebra is called *minimal representation-infinite* if it is representation-infinite and all of its proper quotients are representation-finite. Bongartz [5, §5.2] first passes to a basic algebra Morita equivalent to A and then to a minimal representation-infinite quotient of that basic algebra. The following lemma extends this reduction to modules of arbitrary dimension.

Lemma 3.1. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field. Let A_0 be a basic algebra Morita equivalent to A . Let B be a quotient of A_0 that is minimal representation-infinite; such a quotient exists. Then B is basic and connected, and there is an exact fully faithful functor $\Phi : \text{Mod } B \rightarrow \text{Mod } A$ which preserves direct sums, filtered colimits, finite-dimensional modules and the dimension of every infinite-dimensional module.*

Proof. Representation type is invariant under Morita equivalence, so A_0 is representation-infinite. A representation-infinite quotient of A_0 of least dimension is therefore minimal representation-infinite. As a quotient of a basic algebra, B is basic. It is connected, since otherwise one of its connected factors would be a proper representation-infinite quotient. Compose inflation along $A_0 \rightarrow B$ with a k -linear Morita equivalence $\text{Mod } A_0 \rightarrow \text{Mod } A$ to obtain Φ . Both functors have the stated properties except possibly the assertion about dimension. Inflation preserves the underlying vector space. The Morita equivalence and its inverse are given by tensoring with finite-dimensional bimodules, so each bounds dimension by a fixed finite multiple. They therefore preserve the dimension of every infinite-dimensional module. $\square$

By Lemma 3.1, we may pass from A to a connected basic minimal representation-infinite algebra. From now on until Section 5, we assume that A is such an algebra over an algebraically closed field k .

3.2. Quotients of projective modules by submodules of the socle. The first construction turns the two Kronecker maps into relations in a sum of copies of one projective module over an algebra Λ . In Theorem 3.10(1), Λ is a quotient of A , and Λ -modules become A -modules by inflation.

Setting 3.2. Let Λ be a basic finite-dimensional algebra, put $J = \text{rad } \Lambda$, choose primitive idempotents e_x, e_y from a complete set of primitive orthogonal idempotents of Λ , indexed by x, y , and choose linearly independent $s, t \in e_x J e_y$ with $Js = Jt = sJ = tJ = 0$. The idempotents may coincide.

Under these assumptions, the construction of [5, §3.1], specialized as in the proof of [5, Corollary 2], is the following functor.

Construction 3.3 ([5, §3.1]). For $T = (V, W; a, b) \in \mathcal{D}_{\text{fin}}$, put

$$F_1(T) = (\Lambda e_y \otimes W) / \{s \otimes a(v) + t \otimes b(v) : v \in V\}. \quad (3.1)$$

On a morphism (g, h) , let $F_1(g, h)$ be induced by $1 \otimes h$.

The relation space is a Λ -submodule of $\Lambda e_y \otimes W$: it is annihilated by J , and Λ/J acts on s and t through its coordinate at x , hence by scalars. Being annihilated by J , it lies in the socle. For a morphism $(g, h) : T \rightarrow (V', W'; a', b')$, the equations $ha = a'g$ and $hb = b'g$ show that $1 \otimes h$ maps this space into the corresponding space for $(V', W'; a', b')$. Since $s, t \in J$, the relation submodule lies in $J(\Lambda e_y \otimes W)$, so the top of $F_1(T)$ is $S_y \otimes W$, where $S_y = \Lambda e_y / J e_y$.

3.3. Galois coverings and push-down functors. Fix a complete set of primitive orthogonal idempotents of the basic algebra A , called its vertices. Let $\mathcal{A}$ be the k -linear category with these vertices as objects and morphism spaces $\mathcal{A}(i, j) = e_j A e_i$; we identify A -modules with $\mathcal{A}$ -modules. In the second construction, a larger category R carries an action of a group G that is free on objects, and a functor $\pi : R \rightarrow \mathcal{A}$ identifies each G -orbit of objects with one object of $\mathcal{A}$. Finite-dimensional Kronecker modules are first embedded fully faithfully into modules over R supported on finitely many objects. The spaces above each object of $\mathcal{A}$ are then summed.

Definition 3.4 ([8, 3.1]). A small k -linear category R is *locally bounded* if distinct objects are nonisomorphic, each endomorphism algebra is local, all morphism spaces are finite-dimensional, and each object has nonzero morphism spaces to or from only finitely many objects. The dimension of an R -module X is $\sum_x \dim_k X(x)$.

A *Galois covering with group G* is a k -linear functor $\pi : R \rightarrow \mathcal{A}$ from a locally bounded category R , surjective on objects, together with an action of G by k -linear automorphisms, such that the action on objects is free, $\pi g = \pi$, each fibre is a G -orbit, and both maps

$$\bigoplus_{\pi y = b} R(x, y) \longrightarrow \mathcal{A}(\pi x, b), \quad \bigoplus_{\pi y = b} R(y, x) \longrightarrow \mathcal{A}(b, \pi x) \quad (3.2)$$

induced by π are isomorphisms for every x, b .

Construction 3.5. For a finite-dimensional R -module X , define

$$(\pi_* X)(b) = \bigoplus_{\pi x = b} X(x).$$

Let $u : b \rightarrow c$ be a morphism of $\mathcal{A}$ and x an object with $\pi x = b$. The first isomorphism in (3.2) writes $u = \sum_{\pi y = c} \pi(u_{y, x})$ with unique $u_{y, x} \in R(y, x)$. Give $(\pi_* X)(u)$ the block $X(u_{y, x}) : X(x) \rightarrow X(y)$. On morphisms take the direct sum of the vertex maps. This defines the *push-down functor* π_* .

Uniqueness in (3.2) shows that these matrices respect composition.

We use modules supported on a finite full subcategory $\mathcal{S} \subset R$ satisfying

$$\text{every composite } i \rightarrow x \rightarrow j \text{ in } R \text{ with } i, j \in \mathcal{S}, x \notin \mathcal{S} \text{ is zero.} \quad (3.3)$$

Write $\Gamma = \bigoplus_{i, j \in \mathcal{S}} R(i, j)$, with multiplication by composition and zero on noncomposable factors; Γ -modules are the same as modules over $\mathcal{S}$. Condition (3.3) ensures that a Γ -module extended by

zero is an R -module; extension by zero is then an exact fully faithful functor from Γ -modules to R -modules.

Construction 3.6. Suppose there is an exact fully faithful k -linear functor $L : \text{mod } kK_2 \rightarrow \text{mod } \Gamma$ such that $L(S_+)$ and $L(S_-)$ are nonzero and have disjoint supports. Extend L by zero to R -modules, retaining the notation L , and put $F_2 = \pi_* L$.

3.4. Push-down along a closed walk. The third construction sums spaces at different positions of a closed walk in a quiver presenting A .

Definition 3.7. A walk in a quiver Q is a composable sequence of arrows and formal inverse arrows, written in traversal order; it is *closed* if it ends at its starting vertex. It is *reduced* if adjacent letters are never mutual inverses. A nonempty closed walk is *cyclically reduced* if this also holds at the join of its last and first letters, and *primitive* if it is not a proper power of a shorter closed walk. For $w = c_1^{\varepsilon_1} \cdots c_r^{\varepsilon_r}$ through $q_0, q_1, \dots, q_r = q_0$, form a cycle quiver D_w with vertices $d_0, \dots, d_{r-1}$, indices read modulo r ; we often write i for the vertex d_i . Its i -th arrow goes from d_{i-1} to d_i if $\varepsilon_i = 1$, and in the reverse direction otherwise. The map $p : D_w \rightarrow Q$ sends d_i to q_i and its i -th arrow to c_i . We call $p(e)$ the label of the arrow e .

For a cyclically reduced walk,

$$p \text{ is injective on the arrows of } D_w \text{ with a fixed source, and on those with a fixed target.} \quad (3.4)$$

Indeed, two outgoing arrows with the same label, or two incoming arrows with the same label, would give adjacent mutually inverse letters, including at the join. Property (3.4) makes the block formula in Construction 3.9 below well defined.

Setting 3.8. Present A as a bound quiver algebra kQ/I , and let w be a primitive cyclically reduced closed walk whose cycle D_w has a source u and $r \geq 2$ vertices. Fix an ordering of the two arrows out of u .

Construction 3.9 ([5, proof of Theorem 7]). For $T = (V, W; a, b) \in \text{mod } kK_2$, put V at u , W at all other vertices of D_w , a, b on the ordered arrows out of u , and identity maps on the remaining arrows. Denote this representation by $L(T)$. A morphism (g, h) acts as g at u and as h elsewhere. For a representation X of D_w and a vertex q of Q , set

$$(p_* X)(q) = \bigoplus_{p(i)=q} X(i).$$

For an arrow c of Q , the map $(p_* X)(c)$ has block $X(e)$ from $X(i)$ to $X(j)$ if $e : i \rightarrow j$ has label c , and zero otherwise. On morphisms take the direct sum of the vertex maps. Put $F_3 = p_* L$, a functor to representations of Q .

The identity arrows connect all vertices other than u , so their morphism equations force one common map h there. The other two equations are exactly $ha = a'g$ and $hb = b'g$. Thus L is fully faithful; its vertex formula also proves exactness. The representations $L(S_+)$ and $L(S_-)$ are nonzero, with the disjoint supports $\{u\}$ and the set of the other vertices.

3.5. Existence of the embeddings. The following theorem of Bongartz states that, for suitable data, one of the three constructions gives a representation embedding. We then show that these functors are faithful.

Theorem 3.10 ([5, Theorem 6 and its proof]). *Let A be a connected basic minimal representation-infinite algebra over an algebraically closed field. Then at least one of the following holds.*

- (1) *A has a quotient Λ with elements s, t as in Setting 3.2 such that F_1 from Construction 3.3, followed by inflation, is a representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ [5, Theorem 1 and Corollary 2].*
- (2) *There are a Galois covering $\pi : R \rightarrow A$ whose group G is a free group, a finite full subcategory $\mathcal{S}$ of R satisfying (3.3), and an exact fully faithful functor $L : \text{mod } kK_2 \rightarrow \text{mod } \Gamma$ such that $L(S_+)$ and $L(S_-)$ are nonzero with disjoint supports and $F_2|_{\mathcal{D}_{\text{fin}}}$ from Construction 3.6 is a representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ [5, Theorem 3, Lemma 5, and proof of Theorem 7].*

- (3) There are a presentation $A = kQ/I$ and a walk as in Setting 3.8 such that F_3 from Construction 3.9 takes values in $\text{mod } A$ and $F_3|_{\mathcal{D}_{\text{fin}}}$ is a representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ [5, proof of Theorem 7].

By a functor of Theorem 3.10 we mean F_1 followed by inflation, $F_2|_{\mathcal{D}_{\text{fin}}}$ or $F_3|_{\mathcal{D}_{\text{fin}}}$, built from data for which the corresponding alternative of the theorem holds.

Lemma 3.11. *Each of the functors $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ in Theorem 3.10 is faithful.*

Proof. (1): Inflation is faithful. Let $(g, h) : T \rightarrow U$ be a morphism of $\mathcal{D}_{\text{fin}}$ with $F_1(g, h) = 0$, and write $U = (V', W'; a', b')$. The map induced by $F_1(g, h)$ on the tops $S_y \otimes W \rightarrow S_y \otimes W'$ is $1 \otimes h$, so $h = 0$. Then $a'g = ha = 0$ and $b'g = hb = 0$, so $g = 0$ by (2.1) for U .

(2) and (3): The functor L , extended by zero in (2), is fully faithful. On morphisms, π_* and p_* take direct sums of the vertex maps, so they are faithful. Hence $F_2 = \pi_*L$ and $F_3 = p_*L$ are faithful. $\square$

Fix one of the functors of Theorem 3.10 for this algebra A and call it F_{fin} . By Lemma 3.11, it is a faithful representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$. In Section 5, where A again denotes the original algebra, this functor is written $F_{B, \text{fin}}$.

4. HOMOMORPHISMS BETWEEN THE EMBEDDED MODULES

We now verify the remaining hypotheses of Theorem 2.7 for F_{fin} : retractions on $\mathcal{C}_{\text{fin}}$ with a common nilpotence bound. For the quotient construction, the map induced on tops determines a Kronecker morphism. For coverings and closed walks, we recover it from the diagonal blocks of push-down morphisms. All modules in this section are finite-dimensional.

4.1. Homomorphisms between quotients by submodules of the socle. Use Setting 3.2. Inflation along the quotient map $A \rightarrow \Lambda$ is fully faithful, so we may compute Hom spaces over Λ . As noted after Construction 3.3, the top of $F_1(T)$ is $S_y \otimes W$. Multiplication by s and t on lifts of this top defines a map whose kernel is exactly $\{(a(v), b(v)) : v \in V\}$; see (4.1) below. In the proof of [5, Theorem 1], Bongartz recovers Kronecker morphisms from the top in this way. We record the recovery as retractions.

Proposition 4.1. *There are retractions*

$$\rho_{T,U} : \text{Hom}_{\Lambda}(F_1(T), F_1(U)) \longrightarrow \text{Hom}_{K_2}(T, U), \quad T, U \in \mathcal{D}_{\text{fin}},$$

for F_1 on $\mathcal{D}_{\text{fin}}$. If $J^n = 0$, every composite of n maps lying in the kernels of these retractions is zero.

Proof. For a Λ -module M , put $\text{top}_y(M) = e_y(M/JM)$. Since $sJ = tJ = 0$, the formula

$$d_M : \text{top}_y(M) \oplus \text{top}_y(M) \longrightarrow e_x M, \quad (w_1, w_2) \longmapsto sm_1 + tm_2,$$

where $m_i \in e_y M$ lifts w_i , is independent of the lifts. For $T = (V, W; a, b)$, identify $\text{top}_y(F_1(T)) = W$. The injection $j : W \oplus W \rightarrow \Lambda e_y \otimes W$, $(w_1, w_2) \mapsto s \otimes w_1 + t \otimes w_2$, followed by the quotient map, is $d_{F_1(T)}$. Its kernel is therefore

$$\ker d_{F_1(T)} = \{(a(v), b(v)) : v \in V\}. \quad (4.1)$$

Write $U = (V', W'; a', b')$. For $f : F_1(T) \rightarrow F_1(U)$, let $h : W \rightarrow W'$ be the map induced on the tops. Then

$$\begin{array}{ccc} W \oplus W & \xrightarrow{d_{F_1(T)}} & e_x F_1(T) \\ h \oplus h \downarrow & & \downarrow f \\ W' \oplus W' & \xrightarrow{d_{F_1(U)}} & e_x F_1(U) \end{array}$$

commutes by Λ -linearity. For $v \in V$, (4.1) for T gives $d_{F_1(T)}(a(v), b(v)) = 0$, so the diagram gives $d_{F_1(U)}(ha(v), hb(v)) = 0$. Equation (4.1) for U and the condition $\ker a' \cap \ker b' = 0$ give a unique $g : V \rightarrow V'$ with $ha = a'g$ and $hb = b'g$. Set $\rho(f) = (g, h)$. Maps on tops and uniqueness

of g prove linearity and compatibility with composition. The map $F_1(g, h)$ is induced by $1 \otimes h$, so $\rho F_1 = 1$. Finally, $\rho(f) = 0$ exactly when $f(F_1(T)) \subseteq JF_1(U)$. Such maps lower the radical filtration: $f(J^i F_1(T)) \subseteq J^{i+1} F_1(U)$. Any n of them compose to zero if $J^n = 0$. $\square$

4.2. Vertex evaluations and block matrices. For the other two constructions, let L be the functor of Construction 3.6, extended by zero, or of Construction 3.9. The next lemma uses the disjoint supports of $L(S_+)$ and $L(S_-)$ to show that an endomorphism of $L(T)$ is determined by its value at any vertex of the support.

Lemma 4.2. *Let L be either functor just described and put $\Sigma = \text{supp } L(S_+) \cup \text{supp } L(S_-)$. For nonzero $T \in \mathcal{C}_{\text{fin}}$, we have $\text{supp } L(T) = \Sigma$, and evaluation $\text{End}(L(T)) \rightarrow \text{End}_k(L(T)(x))$ is injective at every $x \in \Sigma$; here endomorphisms are taken over R for the functor of Construction 3.6 and over D_w for that of Construction 3.9.*

Proof. Write $T = (V, W; a, b)$. Applying L to

$$0 \longrightarrow S_- \otimes W \longrightarrow T \longrightarrow S_+ \otimes V \longrightarrow 0$$

gives an exact sequence. Since $L(S_+)$ and $L(S_-)$ have disjoint supports, it identifies $L(T)(x)$ naturally with $L(S_+)(x) \otimes V$ on $\text{supp } L(S_+)$ and with $L(S_-)(x) \otimes W$ on $\text{supp } L(S_-)$. Both V, W are nonzero when $0 \neq T \in \mathcal{C}$, so the support is Σ . By full faithfulness, every endomorphism is $L(g, h)$. Its value at x is $1 \otimes g$ or $1 \otimes h$. Vanishing implies $g = 0$ or $h = 0$, and then Lemma 2.2(1) gives $(g, h) = 0$. $\square$

After summing vertex spaces in a fibre, morphisms become block matrices. In the proof of [7, Theorem 5.5], Fernández-Alonso, Gavito and Pérez-Terrazas recover a morphism from a diagonal block of such a matrix, for a functor from modules over a polynomial ring in one variable to modules over a truncated polynomial algebra in two variables; Propositions 4.4 and 4.7 below recover endomorphisms from the diagonal part of such matrices. The following lemma turns injectivity of the vertex evaluations into a nilpotence bound depending only on the size of the largest fibre. In the applications, Ω is the support of a module over the covering or the vertex set of D_w , the parts of $\mathcal{P}$ are the fibres, and E is the endomorphism algebra of the push-down.

Lemma 4.3. *Let Ω be a finite nonempty set with a partition $\mathcal{P}$, and let $M_i \neq 0$ be vector spaces for $i \in \Omega$. Let E be a unital subalgebra of $\text{End}_k(\bigoplus_{i \in \Omega} M_i)$ preserving each subspace $\bigoplus_{i \in \Omega'} M_i$, $\Omega' \in \mathcal{P}$. For a matrix $f = (f_{j,i})$ with blocks $f_{j,i} : M_i \rightarrow M_j$, so that columns are indexed by sources, let $\delta(f)$ be the diagonal part of f , retaining only the blocks $f_{i,i}$. Assume:*

- (1) $\delta(E) \subseteq E$;
- (2) for $E_0 = \delta(E)$, every coordinate map $E_0 \rightarrow \text{End}_k(M_i)$ is injective;
- (3) $N = \ker(\delta|_E)$ is spanned by elements of N having at most one nonzero block in each row and column, and having some column all of whose blocks are zero.

Then $\delta : E \rightarrow E_0$ is a split algebra surjection, N is a two-sided ideal, and

$$N^m = 0, \quad m = \max_{\Omega' \in \mathcal{P}} |\Omega'|.$$

Proof. The diagonal elements of E form a unital subalgebra, and hypothesis (1) identifies this subalgebra with E_0 . Call the elements of N described in hypothesis (3) spanning matrices. Products of spanning matrices still have at most one nonzero block in each row and column. A nonempty product Z has a zero column inherited from its rightmost factor. If this is column i , then $Z_{i,i} = 0$. The coordinate map at i is injective on E_0 by hypothesis (2), so $\delta(Z) = 0$. In particular $N^2 \subseteq N$. For $r \in E_0$, diagonal multiplication gives $\delta(rf) = r\delta(f)$ and $\delta(fr) = \delta(f)r$. Thus $E_0 N + N E_0 \subseteq N$. Since $E = E_0 \oplus N$ as vector spaces, N is an ideal and δ is an algebra homomorphism split by inclusion.

Suppose $Z_m \cdots Z_1$ is a product of spanning matrices with a nonzero block. At most one sequence of intermediate indices contributes to that block; write it as $i_0, i_1, \dots, i_m$. Every factor preserves the subspaces indexed by the parts of $\mathcal{P}$, so all $m+1$ indices lie in a single part, which has at most m elements. Choose $\mu < \nu$ with $i_\mu = i_\nu$. The composite from M_{i_μ} to M_{i_ν} along this sequence is nonzero: if it were zero, the full composite from M_{i_0} to M_{i_m} would be zero. It is the (i_ν, i_μ) block

of $Z_\nu \cdots Z_{\mu+1}$, and $i_\mu = i_\nu$ makes it a diagonal block. This contradicts the preceding paragraph, which applies to every nonempty product of the spanning matrices. Expanding products of linear combinations gives $N^m = 0$. $\square$

The second paragraph of this proof follows the proof of [10, Theorem 3.1], where Gordon and Green show that a $\mathbb{Z}$ -graded Artin algebra whose subalgebra of degree zero is local is itself local. There a repeated partial sum of degrees in a long product gives a subproduct of degree zero, as a repeated index gives a diagonal block here.

4.3. Endomorphisms of push-downs along a covering. Let $\pi : R \rightarrow \mathcal{A}$ be a Galois covering as in Definition 3.4. For objects x, y in one fibre, there is a unique $\gamma \in G$ with $y = \gamma x$. We group matrix blocks according to this element. For finite-dimensional R -modules X and Y , Gabriel [8, §3.2 and proof of Theorem 3.6(c)] identifies $\text{Hom}_A(\pi_* X, \pi_* Y)$ with the direct sum of the spaces $\text{Hom}_R(X, Y \circ \gamma)$, $\gamma \in G$; the grouping is the matrix form of this decomposition.

Proposition 4.4. *Suppose G is torsion-free. Let X be a finite-dimensional R -module with nonempty support Σ , and suppose every evaluation $\text{End}_R(X) \rightarrow \text{End}_k(X(x))$, $x \in \Sigma$, is injective. Taking the diagonal part gives a split algebra surjection*

$$\text{End}_A(\pi_* X) \longrightarrow \text{End}_R(X)$$

whose kernel has m -th power zero, where $m = \max_b |\Sigma \cap \pi^{-1}(b)|$.

Proof. For $f \in \text{End}_A(\pi_* X)$, retain the blocks $f_{\gamma x, x}$ to obtain $f^{[\gamma]}$. These components are again morphisms. Indeed, let $\psi : x \rightarrow y$ be a morphism of R . For each $\eta \in G$, the decomposition of $\pi(\psi)$ with source ηx in Construction 3.5 has the single term $\eta(\psi) : \eta x \rightarrow \eta y$, by uniqueness in (3.2). Thus $\pi_* X(\pi(\psi))$ has the blocks $X(\eta(\psi))$ from $X(\eta x)$ to $X(\eta y)$, $\eta \in G$, and no others. Compare the blocks from $X(x)$ to $X(\gamma y)$ in $f \circ \pi_* X(\pi(\psi)) = \pi_* X(\pi(\psi)) \circ f$. Since G acts freely on objects, only $\eta = 1$ contributes on the left and only $\eta = \gamma$ on the right, so

$$f_{\gamma y, y} X(\psi) = X(\gamma(\psi)) f_{\gamma x, x}. \quad (4.2)$$

Applying (4.2) also to every translate of ψ shows that $f^{[\gamma]}$ commutes with $\pi_* X(\pi(\psi))$. The images of morphisms of R span the morphism spaces of $\mathcal{A}$, so $f^{[\gamma]}$ is a morphism. Only finitely many γ occur, since Σ is finite. For $\gamma = 1$, equation (4.2) is the equation for an endomorphism of X . Hence the diagonal part is precisely the image of $\text{End}_R(X)$ under π_* .

For $\gamma \neq 1$, the component $f^{[\gamma]}$ has at most one nonzero block in each row and column and has no diagonal block. Moreover $\gamma \Sigma \neq \Sigma$: otherwise a positive power of γ would fix an object of the finite nonempty set Σ , contrary to freeness of the action and torsion-freeness of G . Thus some $x \in \Sigma$ has $\gamma x \notin \Sigma$, giving a zero column. Apply Lemma 4.3 to $E = \text{End}_A(\pi_* X)$, partitioning Σ by the fibres of π ; E preserves the corresponding subspaces because A -linear maps preserve each vertex space $(\pi_* X)(b)$. Hypothesis (1) holds because the diagonal part of f is the morphism $f^{[1]}$; hypothesis (2) is the injectivity of the evaluations; and hypothesis (3) holds because $f - \delta(f) = \sum_{\gamma \neq 1} f^{[\gamma]}$ with each $f^{[\gamma]}$ of the form just described. The lemma gives the assertion. $\square$

4.4. Endomorphisms of push-downs along a closed walk. Use the walk and map $p : D_w \rightarrow Q$ from Setting 3.8. For an endomorphism of $p_* X$, the blocks are indexed by pairs of positions in one fibre of p , and the morphism equations relate blocks whose positions are joined by equally labelled arrows. These pairs and relations form the matching quiver H defined below, and we first describe its components. Crawley-Boevey [6, Theorem in §2] describes homomorphisms between tree modules, and Krause [14, Theorem, p. 191] between tree and band modules, by pairs of label-preserving maps from a common quiver; the two coordinate projections of a component of the matching quiver form such a pair.

Definition 4.5. The matching quiver H has vertices (j, i) with $p(j) = p(i)$. For each pair of arrows $\alpha : j \rightarrow j'$ and $\beta : i \rightarrow i'$ of D_w with $p(\alpha) = p(\beta)$, it has an arrow

$$(\alpha, \beta) : (j, i) \longrightarrow (j', i').$$

Let Δ be the subquiver consisting of the diagonal vertices (i, i) and the arrows (e, e) , where e runs over the arrows of D_w . Components mean connected components of the underlying unoriented graph.

For an endomorphism $f : p_*X \rightarrow p_*X$, a vertex (j, i) indexes the block from $X(i)$ to $X(j)$. The matching arrow $(\alpha, \beta) : (j, i) \rightarrow (j', i')$ records the equation

$$f_{j', i'} X(\beta) = X(\alpha) f_{j, i}, \quad (4.3)$$

derived in the proof of Proposition 4.7. For example, for the walk $c(c')^{-1}$ on two distinct loops at one vertex, D_w is the Kronecker quiver. The diagonal pairs form one Kronecker component of H , and the two off-diagonal pairs are isolated. The following lemma describes the components of H in general. In the proof of Proposition 4.7, it gives hypothesis (3) of Lemma 4.3.

Lemma 4.6. *For a primitive cyclically reduced closed walk whose cycle D_w has a source, Δ is a whole component of H . Both coordinate projections are injective on each component's vertices. On a component other than Δ , neither projection is surjective onto $(D_w)_0$.*

Proof. We first prove that both projections are injective on each component and that Δ is a component. A component other than Δ with a surjective projection would then give a nonidentity rotation of the cycle D_w compatible with the arrow labels, and we show that this contradicts primitivity of w .

Put $r = |(D_w)_0|$. At a vertex (j, i) of H , an arrow of D_w at j determines the matching arrow at i , and conversely, by (3.4); thus both projections $H \rightarrow D_w$ are injective on the edges at each vertex. Thus every vertex of H has degree at most two. Consider a path in H that never traverses an edge twice in succession. By (3.4), it projects, in each coordinate, to a walk in the cycle D_w that never returns along the edge just traversed; such a walk moves in one fixed direction around the cycle. A shortest path between two vertices never traverses an edge twice in succession, and it has fewer than r edges: after r steps both coordinates return to their starting values, so a longer path would revisit a vertex. If two vertices of a component have the same coordinate, the projection of their shortest connecting path to that coordinate moves in one direction and returns to its start, so its length is divisible by r . That length is zero, proving injectivity of both projections. At a diagonal vertex, (3.4) forces every incident matching arrow to be diagonal. Hence Δ is a whole component.

It remains to exclude a surjective projection from any other component. For $r = 2$, the two arrows go from the source to the sink; an off-diagonal pair is isolated and cannot project onto both vertices. Assume $r \geq 3$. A component with a surjective projection has exactly r vertices by injectivity, and both projections are bijective on vertices. It contains a spanning path, whose two projections are spanning paths of the cycle. Let σ be the bijection of $(D_w)_0$ with $(\sigma(i), i)$ in the component for every i . It maps the second projection of the spanning path, edge by edge, onto the first projection, and hence maps the endpoints of the one to the endpoints of the other. Each projection omits exactly one edge of the cycle, and that edge joins its two endpoints. Hence σ maps the omitted edge of the second projection to the omitted edge of the first, and so σ extends to an automorphism of the underlying cycle.

If σ fixes a vertex, the component meets Δ and equals it. Otherwise it is a nonidentity rotation or a reflection without a fixed vertex. Such a reflection maps two edges to themselves with their endpoints exchanged, and at least one of them, say e joining i and j , lies in the spanning path. The edge of the component whose second coordinate is e would then be a matching arrow pairing e with an arrow of D_w from σ of its source to σ of its target, that is, an arrow in the opposite direction between i and j . For $r \geq 3$ the only arrow joining i and j is e , so this is impossible. Thus σ is a nonidentity rotation.

Relative to a fixed direction around the cycle, assign to each edge its arrow label if its orientation agrees with that direction, and the inverse label otherwise. The rotation preserves these signed labels on all edges of the spanning path, so the two projections of the spanning path carry the same multiset of signed labels. Adding its omitted edge to either projection gives all edges of the cycle, so the two omitted edges carry equal signed labels. They correspond under the rotation, as shown above, so the rotation preserves the signed labels of all r edges. If the rotation shifts

positions by τ , the signed word is periodic with period $\ell = \gcd(r, \tau) < r$. It is therefore the r/ℓ -th power of its first ℓ letters. That shorter walk is closed: the equal signed arrows in positions 1 and $\ell + 1$ have the same initial vertex. Periodicity does not depend on the starting position, and read from d_0 the signed word is w or the inverse of w , according to the chosen direction. Hence w is a proper power of a shorter closed walk, contrary to primitivity. $\square$

We apply the component description to the endomorphism equations. The next proposition assumes that p_*X is an A -module; for $X = L(T)$ with the walk of Theorem 3.10(3), this holds by that theorem.

Proposition 4.7. *Let X be a finite-dimensional representation of D_w with nonzero space at every vertex. Suppose p_*X is an A -module and every evaluation $\text{End}_{D_w}(X) \rightarrow \text{End}_k(X(i))$ is injective. Taking the diagonal part gives a split algebra surjection*

$$\text{End}_A(p_*X) \longrightarrow \text{End}_{D_w}(X)$$

whose kernel has m -th power zero, where $m = \max_q |p^{-1}(q)|$.

Proof. The equations for a morphism of A -modules are the same as those for the underlying Q -representations, so we may compute with the arrows of Q . Let f be an endomorphism of p_*X and $c : q \rightarrow q'$ an arrow of Q . For positions $i \in p^{-1}(q)$ and $j' \in p^{-1}(q')$, the block from $X(i)$ to $X(j')$ of the equation $f \circ (p_*X)(c) = (p_*X)(c) \circ f$ is (4.3). Here $\beta : i \rightarrow i'$ is the unique arrow labelled c starting at i , and $\alpha : j \rightarrow j'$ is the unique arrow labelled c ending at j' , if they exist; a missing term is zero. When both occur, the blocks are indexed by the endpoints of the matching arrow (α, β) . When only β exists, the equation reads $f_{j',i'}X(\beta) = 0$, and $j' \neq i'$, since otherwise β itself would be an arrow labelled c ending at j' . Similarly, when only α exists, the equation reads $X(\alpha)f_{j,i} = 0$, and $j \neq i$, since otherwise α would be an arrow labelled c starting at i . In both cases the equation constrains a single off-diagonal block. Thus each equation involves only blocks indexed by vertices of one component of H , and retaining the blocks in any single component of H preserves all equations, including the ones with only one term. The diagonal component gives precisely $p_*\text{End}_{D_w}(X)$.

Apply Lemma 4.3 to $E = \text{End}_A(p_*X)$, partitioning the vertices of D_w by the fibres of p ; E preserves the corresponding subspaces $(p_*X)(q)$. Hypothesis (1) holds because the diagonal part of f is the part supported on Δ , which is a morphism by the preceding paragraph. Hypothesis (2) is the injectivity of the evaluations. For hypothesis (3), the kernel of the map taking the diagonal part is spanned by matrices supported on single components other than Δ . By Lemma 4.6, such a matrix has at most one nonzero block in each row and column, and it has a zero column because its second projection is not surjective. $\square$

4.5. A uniform nilpotence bound. Since $\mathcal{C}_{\text{fin}}$ is closed under finite direct sums (Lemma 2.2(2)), Lemma 4.2 and Propositions 4.4 and 4.7 apply to direct sums such as $T \oplus U$. Taking matrix entries between summands gives retractions on $\mathcal{C}_{\text{fin}}$.

Proposition 4.8. *Each of the functors in Theorem 3.10 admits retractions on $\mathcal{C}_{\text{fin}}$, and there is an integer $n \geq 1$ such that every composite of n composable maps lying in kernels of these retractions is zero.*

Proof. Alternative (1) is Proposition 4.1, since inflation preserves Hom spaces. In (2) and (3), use Lemma 4.2 with fixed support $\Sigma = \text{supp } L(S_+) \cup \text{supp } L(S_-)$. If T or U is zero, both Hom spaces are zero and we put $\rho_{T,U} = 0$. Let $T \in \mathcal{C}_{\text{fin}}$ be nonzero. In (2), Proposition 4.4 applies to $L(T)$: G is torsion-free, and $L(T)$ has support Σ with injective evaluations by Lemma 4.2. In (3), Σ is the whole vertex set of D_w , so $L(T)$ is nonzero at every vertex; its evaluations are injective by Lemma 4.2, and $p_*L(T)$ is an A -module by Theorem 3.10(3). Thus Proposition 4.7 applies. Taking the diagonal part and then applying full faithfulness of L gives

$$\text{End}_A(F_{\text{fin}}(T)) \longrightarrow \text{End}(L(T)) \cong \text{End}_{K_2}(T),$$

a split algebra surjection. Its kernel has m -th power zero, where m is the largest fibre size in the fixed support Σ .

For nonzero T, U and $f : F_{\text{fin}}(T) \rightarrow F_{\text{fin}}(U)$, regard f as the endomorphism of $F_{\text{fin}}(T \oplus U) = F_{\text{fin}}(T) \oplus F_{\text{fin}}(U)$ whose only nonzero component is f , apply this surjection to it, and let $\rho_{T,U}(f)$ be the component from T to U of the resulting endomorphism of $T \oplus U$. The surjection keeps the diagonal blocks $f_{i,i}$, indexed by the positions i in the fibres, and then applies full faithfulness of L . Both operations commute with taking components with respect to direct sums, so $\rho_{T,U}$ does not depend on the containing direct sum, and $\rho_{F_{\text{fin}}} = 1$. Applying the surjection to $T \oplus U \oplus Z$, for a third object Z of $\mathcal{C}_{\text{fin}}$, proves compatibility with composition. For a composite of m composable maps lying in kernels of the maps ρ , put all its objects in one finite direct sum and extend the maps by zero on the other summands. The support is still Σ , so the same bound m makes their product zero. Thus the proposition holds with $n = m$. $\square$

5. PROOF OF THEOREM A AND NILPOTENT ENDOMORPHISM RADICALS

By Lemma 3.11 and Proposition 4.8, each functor of Theorem 3.10 satisfies the hypotheses of Theorem 2.7. We apply that theorem and transfer the result through the reduction of Lemma 3.1. The representation embeddings constructed in the proof of [5, Theorem 6] are of this form: that proof passes to algebras A_0 and B as in Lemma 3.1 (see [5, §5.2]) and uses one of the functors of Theorem 3.10 for B . The following theorem implies Theorem A: by Proposition 2.3 its functor F extends F_{fin} , so F sends finite-dimensional modules to finite-dimensional modules.

Theorem 5.1. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field k . Let A_0 , B and Φ be as in Lemma 3.1, let $F_{B,\text{fin}} : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } B$ be one of the functors of Theorem 3.10 for B , and put $F_{\text{fin}} = \Phi F_{B,\text{fin}}$. Then $F_{\text{fin}} : \mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$ is a faithful representation embedding, and the functor*

$$F : \mathcal{D} \longrightarrow \text{Mod } A$$

defined by (1.1) is a representation embedding with the following properties.

- (1) *F commutes with arbitrary direct sums and filtered colimits.*
- (2) *If $T \in \mathcal{D}$ has infinite dimension, then $\dim_k F(T) = \dim_k T$.*

Proof. Sections 3 and 4 apply to B in place of A . By Theorem 3.10, Lemma 3.11 and Proposition 4.8, the functor $F_{B,\text{fin}}$ is a faithful representation embedding with retractions on $\mathcal{C}_{\text{fin}}$ and a common nilpotence bound. By Theorem 2.7, its extension $F_B : \mathcal{D} \rightarrow \text{Mod } B$ defined by (1.1) is a representation embedding that commutes with arbitrary direct sums and filtered colimits and satisfies $\dim_k F_B(T) = \dim_k T$ for every infinite-dimensional $T \in \mathcal{D}$.

The functor Φ is exact and fully faithful, and it preserves finite-dimensional modules. Full faithfulness preserves endomorphism algebras, hence indecomposability, and reflects isomorphism classes. Thus $F_{\text{fin}} = \Phi F_{B,\text{fin}}$ is a faithful representation embedding $\mathcal{D}_{\text{fin}} \rightarrow \text{mod } A$. Since Φ commutes with filtered colimits, ΦF_B is naturally isomorphic to the extension F of F_{fin} defined by (1.1). Since Φ is exact and fully faithful, ΦF_B preserves short exact sequences and indecomposability and reflects isomorphism classes, as F_B does; so F is a representation embedding, and Lemma 3.1 gives Theorem 5.1(1) and (2). $\square$

In this proof, Theorem 2.7 also provides retractions for F_B on $\mathcal{C}$ with a common nilpotence bound. The next proposition transports them to retractions for F on $\mathcal{C}$.

Proposition 5.2. *Let F be the functor of Theorem 5.1. There are an integer $n \geq 1$ and retractions*

$$\rho_{T,U} : \text{Hom}_A(F(T), F(U)) \longrightarrow \text{Hom}_{K_2}(T, U), \quad T, U \in \mathcal{C},$$

for F on $\mathcal{C}$. Their kernels form an ideal of the full subcategory on the modules $F(T)$, $T \in \mathcal{C}$, whose n -th power is zero.

Proof. For the algebra B of Theorem 5.1, Theorem 2.7, applied in the proof of Theorem 5.1, gives the retractions for F_B on $\mathcal{C}$ and the bound. Transport them to $F \cong \Phi F_B$ through the Hom isomorphisms of the fully faithful functor Φ . These isomorphisms preserve composition and hence the same nilpotence exponent. The kernels form an ideal because ρ is compatible with composition: if $\rho(\varphi) = 0$, then $\rho(\psi'\varphi\psi) = \rho(\psi')\rho(\varphi)\rho(\psi) = 0$. $\square$

An infinite-dimensional indecomposable module need not have a local endomorphism algebra (Section 1.4). Applying Proposition 5.2 to the Kronecker modules used in the proof of Corollary 1.1 gives local endomorphism algebras with radicals of bounded nilpotence index.

Corollary 5.3. *Under the hypotheses of Theorem 5.1, there is an integer $n \geq 1$ such that for every infinite cardinal κ there is an A -module M satisfying*

$$\dim_k M = \kappa, \quad \text{End}_A(M)/\text{rad End}_A(M) \cong k, \quad (\text{rad End}_A(M))^n = 0.$$

In particular M is indecomposable.

Proof. Choose the Kronecker module T with $\dim_k T = \kappa$ and $\text{End}_{K_2}(T) = k$ used in the proof of Corollary 1.1. Since T is infinite-dimensional and indecomposable, $T \in \mathcal{C}$, as noted after (2.2). Put $M = F(T)$; Theorem 5.1(2) gives $\dim_k M = \kappa$. Let $N = \ker \rho_{T,T}$. With n as in Proposition 5.2, that proposition gives $N^n = 0$. The map $\rho_{T,T}$ is an algebra homomorphism, surjective because $\rho_{T,T}F = 1$, so $\text{End}_A(M)/N \cong \text{End}_{K_2}(T) = k$. Every nilpotent ideal lies in the Jacobson radical: for $x \in N$ and $y \in \text{End}_A(M)$, the element $1 - yx$ is invertible by a finite geometric series. Conversely the quotient k has zero radical, so $\text{rad End}_A(M) \subseteq N$. Thus $N = \text{rad End}_A(M)$. $\square$

Use of AI. The research was done under the author's direction using GPT-6-Astra, which found and developed the proof presented here and wrote the earlier drafts of the manuscript. The present text was revised with Claude Opus 5.5. The author directed the research and revisions and is responsible for the mathematical content and the final manuscript.

REFERENCES

- [1] A. Ardizzoni and C. Menini, *Heavily separable functors*, J. Algebra **543** (2020), 170–197.
- [2] M. Auslander, *Representation theory of Artin algebras II*, Comm. Algebra **1** (1974), 269–310.
- [3] M. Auslander, *Large modules over Artin algebras*, in: Algebra, Topology, and Category Theory (A. Heller and M. Tierney, eds.), Academic Press, New York, 1976, 1–17.
- [4] R. Bautista, E. Pérez and L. Salmerón, *A representation embedding for algebras of infinite type*, J. Pure Appl. Algebra **229** (2025), no. 6, 107955.
- [5] K. Bongartz, *Representation embeddings and the second Brauer–Thrall conjecture*, [arXiv:1611.02017v5](https://arxiv.org/abs/1611.02017v5), 2023.
- [6] W. W. Crawley-Boevey, *Maps between representations of zero-relation algebras*, J. Algebra **126** (1989), 259–263.
- [7] R. Fernández-Alonso, S. Gavito and J. E. Pérez-Terrazas, *On the connection between the representation type of an algebra and its lattice of preradicals*, Comm. Algebra **46** (2018), 176–190.
- [8] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Representations of algebras (Puebla, 1980), Lecture Notes in Math. **903**, Springer, Berlin, 1981, pp. 68–105.
- [9] R. Göbel and D. Simson, *Rigid families and endomorphism algebras of Kronecker modules*, Israel J. Math. **110** (1999), 293–315.
- [10] R. Gordon and E. L. Green, *Graded Artin algebras*, J. Algebra **76** (1982), 111–137.
- [11] L. Gregory and M. Prest, *Representation embeddings, interpretation functors and controlled wild algebras*, J. Lond. Math. Soc. (2) **94** (2016), 747–766.
- [12] P. A. Guil Asensio, M. Cortés Izurdiaga and B. Torrecillas, *Accessible subcategories of modules and pathological objects*, Forum Math. **22** (2010), 485–507.
- [13] M. C. Iovanov, *On the infinite tame-wild dichotomy conjecture and related problems*, [arXiv:1803.00173v2](https://arxiv.org/abs/1803.00173v2), 2018.
- [14] H. Krause, *Maps between tree and band modules*, J. Algebra **137** (1991), 186–194.
- [15] C. M. Ringel, *Report on the Brauer–Thrall conjectures: Rojter’s theorem and the theorem of Nazarova and Rojter*, in: Representation Theory I, Lecture Notes in Math. **831**, Springer, Berlin, 1980, 104–136.
- [16] C. M. Ringel, *Representations of K -species and bimodules*, J. Algebra **41** (1976), 269–302.
- [17] C. M. Ringel, *Infinite dimensional representations of finite dimensional hereditary algebras*, Symposia Math. **23** (1979), 321–412.
- [18] C. M. Ringel and H. Tachikawa, *QF-3 rings*, J. Reine Angew. Math. **272** (1975), 49–72.
- [19] A. V. Roiter, *Unbounded dimensionality of indecomposable representations of an algebra with an infinite number of indecomposable representations*, Math. USSR-Izv. **2** (1968), 1223–1230; Russian original: Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1275–1282.
- [20] D. Simson, *An endomorphism algebra realization problem and Kronecker embeddings for algebras of infinite representation type*, J. Pure Appl. Algebra **172** (2002), 293–303.
- [21] D. Simson, *On large indecomposable modules, endo-wild representation type and right pure semisimple rings*, Algebra Discrete Math. **2003**, no. 2, 93–118.
- [22] D. Simson, *On Corner type Endo-Wild algebras*, J. Pure Appl. Algebra **202** (2005), 118–132.

PARAKEET INC., JAPAN

Email address: `haruhisa.enomoto.math@gmail.com`