

SUPEREXPONENTIALLY DECAYING SEQUENCES IN THE ERDŐS SIMILARITY PROBLEM

GENERATED BY VJEKOSLAV KOVAČ USING CHATGPT-6 ASTRA

ABSTRACT. We prove a non-universality criterion for the Erdős similarity conjecture that applies beyond exponential (i.e., geometric) decay. In particular, it covers decreasing positive null sequences satisfying $\log(1/a_n) = o(n \log \log n)$ and $\log(a_n/a_{n+1}) \geq e^{-n^\alpha}$ eventually for some $0 < \alpha < 1$.

1. THE RESULT

An infinite set $A \subset \mathbb{R}$ is *measure universal* if every Lebesgue-measurable set of positive measure contains $x + sA$ for some $x \in \mathbb{R}$ and $s \neq 0$. The Erdős similarity conjecture [5] asserts that no infinite set is measure universal. We write $m(E)$ for Lebesgue measure and use natural logarithms.

Our main result concerns a countable union of finite configurations.

Theorem 1. *Let*

$$A = \bigcup_{j \geq 1} \{c_j, c_j + b_{j,1}, \dots, c_j + b_{j,m_j}\}$$

be bounded, where $m_j \in \mathbb{N}$, $c_j \in \mathbb{R}$, and $b_{j,1} > \dots > b_{j,m_j} > 0$. Suppose that numbers $\varepsilon_j \in (0, 1]$ satisfy

$$\log \frac{b_{j,i}}{b_{j,i+1}} \geq \varepsilon_j \quad (1 \leq i < m_j).$$

Put

$$\ell_j = \log \frac{e}{\varepsilon_j}, \quad B_j = \frac{m_j}{\ell_j}, \quad L_j = \log \frac{b_{j,1}}{b_{j,m_j}}.$$

Assume, as $j \rightarrow \infty$, that

$$B_j \rightarrow \infty, \quad L_j = o(m_j \log \log B_j). \tag{1.1}$$

Then, for every $\eta \in (0, 1)$, there is a compact $E \subset [0, 1]$ with $m(E) > 1 - \eta$ containing no $x + sA$, $x \in \mathbb{R}$, $s \neq 0$.

The logarithmic conditions are invariant under translation and positive rescaling of each configuration. The first limit is equivalent to $\ell_j = o(m_j)$. Reflection gives the analogous statement for configurations on the other side of their anchors.

The first consequence gives an explicit condition allowing superexponential decay.

Corollary 2. *Let $a_1 > a_2 > \dots > 0$ tend to zero and suppose*

$$\log \frac{1}{a_n} = o(n \log \log n).$$

If, for some $\beta \in (0, 1)$, eventually

$$d_n = \log \frac{a_n}{a_{n+1}} \geq \exp \left(-n \exp \left(-(\log n)^\beta \right) \right),$$

then $\{a_n : n \geq 1\}$ has the conclusion of Theorem 1. In particular, it suffices that $d_n \geq e^{-n^\alpha}$ eventually for some $\alpha \in (0, 1)$. No monotonicity of the d_n is required.

The next consequence collects two conditions involving consecutive ratios; the first is the result of Jiang and Xi [12].

Corollary 3. *Let $a_n > 0$. Then $\{a_n : n \geq 1\}$ has the conclusion of Theorem 1 under either of the following assumptions:*

- (i) $a_n \rightarrow 0$ and $\liminf_{n \rightarrow \infty} a_{n+1}/a_n > 0$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ and $\log \frac{a_n}{a_{n+1}} = o(\log \log n)$.

For example, Corollary 3(ii) covers

$$\begin{aligned} a_n &= \exp(-n(\log \log n)^\theta), & 0 < \theta < 1, \\ a_n &= \exp(-n \log \log n), \\ a_n &= \exp\left(-\frac{n \log \log n}{\log \log \log n}\right), \end{aligned} \tag{1.2}$$

where the formulas are used for sufficiently large n . All three examples decay superexponentially and have consecutive ratios tending to zero.

Both corollaries are deduced in Section 7.

For background, see the surveys of Svetic [20] and Jung, Lai, and Moorooogen [13]. Falconer [6] and Eigen [4] treated sequences with $a_{n+1}/a_n \rightarrow 1$. Humke and Laczkovich [9] gave a finite combinatorial characterization. The probabilistic criteria of Kolountzakis [14] and Chlebík [3, Theorem 15] use finite subsets whose logarithmic normalized minimum-gap cost is sublinear in their cardinality.

The classical relative-gap criterion does not cover even a tail with ratios bounded above by a fixed $q < 1$: the normalized minimum gap of any m selected points is at most $q^{m-2}/(1-q)$. The geometric-sequence draft [18] and Jiang and Xi's extension [12] to null sequences with $\liminf_n a_{n+1}/a_n > 0$ also exclude (1.2). To our knowledge, positive-measure affine avoidance for the examples in (1.2) has not previously been established.

Bourgain [2] treated sums of three infinite sets; Mora Cuellar, Iosevich, Kulkarni, Rojas Aravena, and Yavicoli [17] treated two-fold sums and differences with a geometric summand or a decreasing positive null summand whose adjacent ratios are bounded away from zero. Such sumset results do not imply nonuniversality of the summands.

Recent progress for uncountable sets includes results for thin Cantor sets [15, 11], sets of positive logarithmic dimension above suitable thresholds [19], and sets supporting Rajchman measures [10]. These do not imply the corresponding assertion for a contained countable sequence.

Related variants concern bi-Lipschitz embeddings [7], unbounded sequences and sets of large measure in every unit interval [16, 8], and dilations bounded away from zero [1]. Their conclusions differ from positive-measure avoidance of every nontrivial affine copy.

Our proof modifies the finite-tree construction of [18]. Either a short block admits the classical random-grid construction, or every such block decreases enough to stabilize the tree. We then average over shifted index windows and a random stopping vertex and add the missed anchors to the hitting set.

2. FINITE CONFIGURATIONS AND PERIODIC HITTING SETS

An affine change of A does not change the family of its nontrivial affine copies. We may therefore assume throughout the proof that $\text{diam } A \leq 1/8$. All positive offsets of configurations in A are then at most $1/8$.

For every $p \in (0, 1)$, we will find a measurable 1-periodic H with $m(H \cap [0, 1)) < 2p$ that meets every $x + tA$, $x \in \mathbb{R}$, $t \in [1, 2]$. We call copies with $t \in [1, 2]$ *normalized copies*. It suffices to hit every normalized copy of one finite configuration in A . Translating that configuration to its anchor does not affect this assertion, since the translation parameter x is arbitrary.

The following elementary construction will handle short blocks.

Lemma 4. *Let $Y_j \subset A$ be finite sets with $r_j + 1$ points, diameter R_j , and minimum distance s_j between distinct points. If $r_j \rightarrow \infty$ and*

$$\log(R_j/s_j) = o(r_j),$$

then for every $p \in (0, 1)$ there is a measurable 1-periodic H with $m(H \cap [0, 1)) < 2p$ meeting every $x + tA$, $x \in \mathbb{R}$, $t \in [1, 2]$.

Proof. Fix p and one configuration, omitting its subscript. Subtract its minimum to obtain $0 = u_0 < u_1 < \dots < u_r = R$. Set $q = 2^{\lceil \log_2(4/s) \rceil}$, so $4/s \leq q < 8/s$. Independently color the q cells of the grid $q^{-1}\mathbb{Z}$ in $[0, 1)$ with probability p , extend the coloring 1-periodically, and let B be the union of the colored half-open cells. Then $\mathbb{E}m(B \cap [0, 1)) = p$.

For fixed x and $t \in [1, 2]$, the r positive test points $x + tu_i$ occupy distinct cells: their mutual distances are at least s , and they lie in an interval of length at most $2R \leq 1/4$. Their failure probability is $(1 - p)^r \leq e^{-pr}$. As t varies, the whole vector of cell addresses changes only when one test crosses a grid boundary. Including the endpoints, at most

$$2 + r + q \sum_{i=1}^r u_i \leq 12rR/s$$

representatives suffice. The cells include their left endpoints, so the cell-address vectors are right-continuous in t .

Define the open periodic set

$$Z = \{x : \exists t \in [1, 2], x + tu_i \notin \overline{B} \text{ for all } 1 \leq i \leq r\}.$$

Taking the closure of B adds only finitely many endpoints per period. A union bound and integration give

$$\mathbb{E}m(Z \cap [0, 1)) \leq 12r(R/s)e^{-pr} \rightarrow 0.$$

Some realization therefore has $m(\overline{B} \cap [0, 1)) + m(Z \cap [0, 1)) < 2p$. The set $H = \overline{B} \cup Z$ hits every $x + t\{0, u_1, \dots, u_r\}$: if the positive tests all miss $\overline{B}$, their anchor x belongs to Z . It consequently hits every normalized copy of A . $\square$

3. THE SHORT-BLOCK DICHOTOMY

We now prove Theorem 1. Fix $p \in (0, 1)$ and let the family index tend to infinity. Suppress the family index and write

$$b_1 > \dots > b_m > 0, \quad \ell = \log(e/\varepsilon), \quad K = m/\ell, \quad L = \log(b_1/b_m).$$

Choose the integers

$$M = \left\lfloor \frac{\log \log K}{2 \log 4} \right\rfloor, \quad D = 4^M, \quad q = (4M)^D, \quad g = \left\lfloor \frac{m}{2q} \right\rfloor, \quad N = gq.$$

We use only sufficiently large configurations, so $M, g \geq 2$. Since $K \rightarrow \infty$,

$$M \asymp \log \log K, \quad \log q = O(\sqrt{\log K} \log \log \log K) = o(\log K), \quad N \sim m/2.$$

In particular,

$$\ell + \log m = o(g), \quad \frac{L}{MN} \rightarrow 0. \tag{3.1}$$

Indeed, $q = K^{o(1)}$ and $m = \ell K$ give $\ell/g \sim 2q/K \rightarrow 0$ and $g = m^{1-o(1)}$. The last limit follows from (1.1).

Put

$$T = 4(\ell + \log q) = o(g). \tag{3.2}$$

Suppose first that, for infinitely many configurations, some block $b_v, \dots, b_{v+g}$ has logarithmic span at most T . Its diameter is at most b_v , and its minimum ordinary gap is at least $(1 - e^{-\varepsilon})b_{v+g} \geq (\varepsilon/2)b_{v+g}$. The logarithm of its diameter divided by its minimum gap is at most

$$\log \frac{b_v}{(\varepsilon/2)b_{v+g}} \leq T + \ell + \log 2 = o(g).$$

These are finite subsets of the original configurations in A , after putting back their anchors. Lemma 4 gives the required periodic hitting sets.

We may henceforth pass to a subfamily for which no such block exists. Thus

$$\log \frac{b_v}{b_{v+g}} > T \quad (1 \leq v \leq m - g). \quad (3.3)$$

This block decrease will stabilize the tree.

4. WINDOWS, AVERAGING, AND GRIDS

Take the complete ordered M -ary tree of depth D , with leaves of height 0, and set $\ell_h = g(4M)^h$ for $0 \leq h \leq D$. Assign its root the integer interval $[1, N]$, where $N = \ell_D$. A height- h vertex has an interval of length ℓ_h . Divide this into M consecutive blocks of length $4w$, where $w = \ell_{h-1}$. In each block, use the first $2w$ indices as the window of the edge to that child, leave the next w unused, and assign the final w indices to the child:

$$4w = \underbrace{2w}_{\text{edge window}} + \underbrace{w}_{\text{gap}} + \underbrace{w}_{\text{child interval}}.$$

Leaf intervals are unused. Every vertex interval ends with g unused indices. Consequently consecutive windows have at least g unused indices between them. Their order is preorder: an edge precedes its child's whole subtree, which precedes the next sibling edge.

Write $W_e = [u_e, v_e] \cap \mathbb{N}$, $\mathcal{I} = \bigcup_e W_e$, and $r_e = |W_e|$. If v_e^* is the largest endpoint among the window and its descendant windows, the construction gives

$$r_e = 2\ell_{h-1}, \quad v_e^* - u_e + 1 \leq 2r_e$$

for an edge from a height- h vertex.

Choose J uniformly from $\{0, \dots, N - 1\}$ and shift all indices by J . Since $2N \leq m$, every selected index belongs to the finite configuration. Set

$$A_e(J) = \log \frac{b_{J+u_e}}{b_{J+v_e^*}}.$$

In the average over J , each consecutive logarithmic gap appears at most $2r_e$ times. Their total sum is L , so

$$\mathbb{E}_J A_e(J) \leq \frac{2r_e L}{N}.$$

Call an edge good if $A_e(J) \leq pMr_e/8$, and an internal vertex good if at least $M/2$ of its outgoing edges are good. At a fixed vertex, its outgoing windows have a common length r . If it is bad, the sum of its outgoing $A_e(J)$ exceeds $pM^2r/16$. Markov's inequality therefore gives

$$\mathbb{P}_J(U \text{ is bad}) \leq \frac{32L}{pMN}. \quad (4.1)$$

Set

$$c_\varepsilon = \frac{8}{\varepsilon}, \quad \log c_\varepsilon \leq \ell + \log 8.$$

Conditional on $J = j$, abbreviate $a_n = b_{j+n}$ and use the grids

$$\Gamma_k = n_k^{-1}\mathbb{Z}, \quad n_k = 2^{\lceil \log_2(c_\varepsilon/a_k) \rceil}, \quad c_\varepsilon/a_k \leq n_k < 2c_\varepsilon/a_k.$$

Cells include their left endpoints and exclude their right endpoints. The grids are nested because the n_k are nondecreasing powers of two.

Call x stable for j if, for every window except the first, with v' the endpoint of its predecessor,

$$(x, x + 2a_{u_e}] \cap \Gamma_{v'} = \emptyset.$$

For a fixed window, the bad centers in $[0, 1)$ have measure at most

$$2n_{v'}a_{u_e} \leq 4c_\varepsilon \frac{a_{u_e}}{a_{v'}} \leq 4c_\varepsilon e^{-T}.$$

The last inequality follows from (3.3), since $u_e - v' \geq g + 1$. There are at most $2M^D$ windows. The stable set S_j thus satisfies, uniformly in j ,

$$m((\mathbb{R} \setminus S_j) \cap [0, 1)) \leq 8c_\varepsilon M^D e^{-T} \rightarrow 0.$$

Indeed, $D \log M \leq \log q$ and (3.2) give $\log(8c_\varepsilon M^D e^{-T}) \leq O(1) - 3\ell - 3 \log q \rightarrow -\infty$. At a stable center, translation by ta_n , $n \in W_e$, $t \in [1, 2]$, preserves the cell of every grid attached to an edge preceding e . It preserves the cell of $\Gamma_{v'}$ because $ta_n \leq 2a_{u_e}$, and hence all earlier cells by nesting.

At every center, the points x and $x + ta_n$, $n \in W_e$, occupy distinct cells of Γ_{v_e} and all finer grids. Their mutual distances are at least $(1 - e^{-\varepsilon})a_{v_e}$, whereas the cell width is at most a_{v_e}/c_ε . They lie in an interval of length at most $2a_1 \leq 1/4$, so their circular distances equal their ordinary distances.

5. A RANDOM STOPPING TREE

All cell values below are chosen on $[0, 1)$ and extended 1-periodically. After choosing $J = j$, give each edge e an independent random 1-periodic selector S_e , constant on the cells of Γ_{v_e} , with independent fair cell values. Give each nonroot vertex V an independent color function T_V on its incoming grid, with independent Bernoulli- p cell values. At the root use a single Bernoulli- p constant. Conditional on J , all cell values chosen within one period and the root color are mutually independent; the joint probability space is finite.

Starting at the root, route a point z to the first child whose incoming selector equals 1 at z . Stop if all M selectors vanish or if a leaf is reached. If $V(z)$ is the stopping vertex, define

$$B = \{z : T_{V(z)}(z) = 1\}.$$

Every realization is a finite union of half-open cells per period. Conditioning on J and all selectors fixes the color address read at z ; the color value at that address still has conditional mean p . Hence

$$\mathbb{E}m(B \cap [0, 1)) = p. \tag{5.1}$$

For each fixed x, j , the center values $S_e(x)$ are independent fair bits. Thus the law of $V(x)$ is independent of both x and j . The probability of reaching a leaf without an internal stop is

$$(1 - 2^{-M})^D \leq e^{-2^M}.$$

Write w_U for the stopping probability at an internal vertex U . Since $\sum_U w_U \leq 1$ and the weights do not depend on J , (4.1) gives

$$\mathbb{P}(V(x) \text{ is internal and bad for } J) = \sum_U w_U \mathbb{P}_J(U \text{ is bad}) \leq \frac{32L}{pMN}.$$

This averaging is performed before imposing stability.

Fix $J = j$ and $x \in S_j$, and reveal $S_e(x)$ for every edge. Condition on an assignment ξ for which the stopping vertex U is internal and good. Its outgoing window length r and the set $G \subset \{1, \dots, M\}$ of indices of its good outgoing edges are now fixed, with $|G| \geq M/2$. Write $e_i = (U, i)$ for the edge to child U_i . All outgoing selectors vanish at the center. Other selector entries retain their independent laws because their addresses were fixed before this conditioning.

Let $V_i(z)$ be the stopping vertex reached by starting at child U_i , and set

$$Q_i(z) = S_{e_i}(z)T_{V_i(z)}(z)$$

for $i \in G$. For $n \in W_{e_i}$, stability preserves the center's decisions at ancestors of U and its zero selectors at children $1, \dots, i-1$. All those edges precede e_i , whether or not they are good. Thus

$$Q_i(x + ta_n) = 1 \implies x + ta_n \in B. \quad (5.2)$$

For fixed t , the $|G|r$ test selectors are independent fair bits under this conditioning. Within each table they occupy distinct cells and avoid its exposed center cell; different edges have independent tables. Now condition on all selectors. The tests read colors at distinct vertex-cell pairs. Different child subtrees are disjoint; within one subtree, tests stopping at the same vertex remain separated because its grid refines the incoming grid of that subtree. The colors read are independent Bernoulli- p variables. If k selectors succeed, the failure probability is $(1-p)^k$. Averaging over the selectors gives

$$\mathbb{P}(Q_i(x + ta_n) = 0 \text{ for all } i \in G, n \in W_{e_i} \mid J = j, \xi) = (1-p/2)^{|G|r} \leq e^{-pMr/4}. \quad (5.3)$$

Each Q_i is constant on the cells of $\Gamma_{v_{e_i}^*}$, since it uses only its edge and child subtree. Goodness bounds the number of crossings of that grid by a single test path by

$$1 + a_n n_{v_{e_i}^*} \leq 1 + 2c_\varepsilon e^{A_{e_i}(j)} \leq 1 + 2c_\varepsilon e^{pMr/8}.$$

Collect the crossing times and the endpoints 1, 2. At most $C_\varepsilon Mre^{pMr/8}$ representatives are needed, where $C_\varepsilon = 4(1+c_\varepsilon)$. They depend only on the geometry and the exposed assignment, not on the unexposed random entries. Right-continuity of the cell-address vectors ensures that these representatives cover every scale. Equations (5.2)–(5.3) and a union bound give

$$\mathbb{P}(\exists t \in [1, 2] \forall n \in \mathcal{I} : x + ta_n \notin B \mid J = j, \xi) \leq C_\varepsilon Mre^{-pMr/8} \leq 2C_\varepsilon Mge^{-pMg/4} \rightarrow 0.$$

Here $r \geq 2g$, and $r \mapsto re^{-pMr/8}$ is decreasing on this range for large configurations. Moreover $\log(C_\varepsilon Mg) = O(\ell + \log m + \log M) = o(g)$ by (3.1). The estimate is uniform over the good stopping vertices.

6. REPAIR AND ALL DILATIONS

For each joint outcome, let

$$Z = \{x : \exists t \in [1, 2] \forall n \in \mathcal{I}, x + tb_{J+n} \notin \overline{B}\}.$$

The set Z is open and periodic: it is a union, over t , of finite intersections of open translates of $\mathbb{R} \setminus \overline{B}$. Taking the closure of B adds finitely many endpoints per period, so $m(\overline{B} \cap [0, 1)) = m(B \cap [0, 1))$.

A missed center is unstable, reaches a leaf without stopping, stops at a bad vertex, or fails all its good-vertex tests at some scale. Integrating the preceding bounds yields

$$\mathbb{E}m(Z \cap [0, 1)) \leq 8c_\varepsilon M^D e^{-T} + e^{-2M} + \frac{32L}{pMN} + 2C_\varepsilon Mge^{-pMg/4} \rightarrow 0.$$

Stability is integrated separately, so no independence of J and $x \in S_J$ is assumed. Together with (5.1), this gives a realization with $m(\overline{B} \cap [0, 1)) + m(Z \cap [0, 1)) < 2p$. Set $H = \overline{B} \cup Z$. If a normalized copy of $\{0, b_1, \dots, b_m\}$ misses the closed colored part at every selected positive offset, its anchor lies in Z . Thus H hits every such copy. Restoring the anchor c shows that it hits every normalized copy of A .

The dense-block case in Section 3 gives the same conclusion by Lemma 4. Thus, for every $p \in (0, 1)$, a periodic hitting set H with $m(H \cap [0, 1)) < 2p$ has been constructed.

Choose $p_k > 0$, $k \in \mathbb{Z}$, with $4 \sum_k p_k \max(1, 2^k) < \eta$ and obtain corresponding sets H_k . Put

$$C = \bigcup_{k \in \mathbb{Z}} (2^k H_k \cup (-2^k H_k)).$$

Periodicity gives

$$m(C \cap [0, 1]) \leq 4 \sum_k p_k \max(1, 2^k) < \eta.$$

For positive s , write $s = 2^k t$ with $t \in [1, 2)$ and apply the normalized hitting property at $2^{-k}x$. Negative dilations are handled by reflection. Hence C meets every nontrivial affine copy of A . By inner regularity choose a compact $E \subset [0, 1] \setminus C$ with $m(E) > 1 - \eta$. This set contains no nontrivial affine copy of A , proving Theorem 1.

7. DEDUCTIONS FROM THE MAIN THEOREM

We use the elementary observation that a compact set avoiding every nontrivial affine copy of $Y \subset \overline{X}$ also avoids every nontrivial affine copy of X : closedness would otherwise force it to contain the corresponding copy of $\overline{X}$ and hence of Y .

Proof of Corollary 2. The function $n \exp(-(\log n)^\beta)$ is eventually increasing. For all large n , the first n points have every logarithmic gap at least

$$\varepsilon_n = \exp\left(-n \exp(-(\log n)^\beta)\right);$$

the finitely many initial positive gaps also exceed this quantity eventually. Thus

$$\ell_n = 1 + n \exp(-(\log n)^\beta), \quad \frac{n}{\ell_n} \sim e^{(\log n)^\beta}, \quad \log \log(n/\ell_n) \sim \beta \log \log n.$$

Their logarithmic span is $\log(a_1/a_n) = o(n \log \log n)$. Apply Theorem 1, using all sufficiently large j , with $c_j = 0$, $m_j = j$, and $b_{j,i} = a_i$ for $1 \leq i \leq j$. The resulting countable set is $\{0\} \cup \{a_n : n \geq 1\}$, so the observation above gives the conclusion for $\{a_n : n \geq 1\}$. The special case follows because, for any fixed $\alpha, \beta \in (0, 1)$, $n^\alpha \leq n \exp(-(\log n)^\beta)$ eventually. $\square$

Proof of Corollary 3. In case (i), choose $c \in (0, 1)$ with $a_{n+1} \geq ca_n$ eventually. Starting sufficiently far out, select the first subsequent term at most e^{-1} times the preceding selected term. If the selected terms are b_j , first passage gives

$$ce^{-1}b_j < b_{j+1} \leq e^{-1}b_j.$$

Their logarithmic gaps lie between 1 and $1 + \log(1/c)$. The first m selected points therefore have $\varepsilon = 1$, $\ell = 1$, and $L = O(m)$. Apply Theorem 1 to $\{0\} \cup \{b_j : j \geq 1\}$ and use the observation above.

In case (ii), the sequence is eventually strictly decreasing to zero, and $d_n = \log(a_n/a_{n+1})$ is eventually bounded below by a positive constant. Telescoping gives

$$\log \frac{1}{a_n} = \log \frac{1}{a_1} + \sum_{k=1}^{n-1} d_k = o(n \log \log n).$$

Apply Corollary 2 to a tail. Avoiding that tail also avoids the whole sequence. $\square$

For the examples in (1.2), write $a_n = e^{-nf(n)}$. In each case, $f(n) \rightarrow \infty$, $f(n) = o(\log \log n)$, and $xf'(x) = o(f(x))$. The mean value theorem gives $\log(a_n/a_{n+1}) \sim f(n)$, verifying Corollary 3(ii).

The finite-pattern formulation also includes the classical translation-invariant criterion of [3]. Let X be bounded, and let $Y_j \subset \overline{X}$ have $m_j + 1$ points, diameter R_j , and minimum distance s_j between distinct points. Suppose that $m_j \rightarrow \infty$ and $H_j = \log(R_j/s_j) = o(m_j)$. Anchor each Y_j at its minimum. The positive offsets have logarithmic span at most H_j and consecutive logarithmic gaps at least s_j/R_j : indeed, $\log(u/v) \geq (u-v)/u$ for $u > v > 0$. Taking $\varepsilon_j = s_j/R_j$ gives $\ell_j = 1 + H_j = o(m_j)$ and verifies (1.1). Apply Theorem 1 to $\bigcup_j Y_j$ and use the same observation to obtain the conclusion for X .

AI DISCLOSURE

The manuscript was generated by OpenAI's ChatGPT-6 Astra on October 9, 2026, in this chat, guided by a sequence of (somewhat generic) prompts. My intention is to digest and address the proof in a cooperative, non-competitive, time-insensitive environment.

REFERENCES

- [1] A. Bërdëllima, *Almost Erdős sets*, Proc. Edinb. Math. Soc. **69** (2026), no. 3, 932–940. doi:10.1017/S0013091526101370.
- [2] J. Bourgain, *Construction of sets of positive measure not containing an affine image of a given infinite structure*, Israel J. Math. **60** (1987), no. 3, 333–344. doi:10.1007/BF02780397.
- [3] M. Chlebík, *On the Erdős similarity problem*, preprint (2015), arXiv:1512.05607.
- [4] S. J. Eigen, *Putting convergent sequences into measurable sets*, Studia Sci. Math. Hungar. **20** (1985), nos. 1–4, 411–412.
- [5] P. Erdős, *Problems*, Math. Balkanica **4** (1974), 203–204, Problem 4.33.7*.
- [6] K. J. Falconer, *On a problem of Erdős on sequences and measurable sets*, Proc. Amer. Math. Soc. **90** (1984), no. 1, 77–78. doi:10.2307/2044671.
- [7] D.-J. Feng, C.-K. Lai, and Y. Xiong, *Erdős similarity problem via bi-Lipschitz embedding*, Int. Math. Res. Not. IMRN **2024**, no. 17, 12327–12342. doi:10.1093/imrn/rnae167.
- [8] X. Gao, Y. Moorooogen, and C. H. Yip, *On an Erdős similarity problem in the large*, Bull. London Math. Soc. **57** (2025), no. 6, 1801–1818. doi:10.1112/blms.70062.
- [9] P. D. Humke and M. Laczkovich, *A visit to the Erdős problem*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 819–822. doi:10.1090/S0002-9939-98-04167-7.
- [10] A. Iosevich, N. Kulkarni, N. Mora Cuéllar, I. Rojas Aravena, and A. Yavicoli, *The Erdős similarity conjecture and Rajchman measures*, preprint (2026), arXiv:2609.04456.
- [11] A. Iosevich and A. Yavicoli, *Falconer lattice sets and the Erdős similarity problem*, preprint (2026), arXiv:2604.01493.
- [12] K. Jiang and L. Xi, *The Erdős similarity problem for null sequences with adjacent ratios bounded below*, preprint, October 8, 2026, arXiv:2610.11449v1.
- [13] Y. Jung, C.-K. Lai, and Y. Moorooogen, *Fifty years of the Erdős similarity conjecture*, Res. Math. Sci. **12** (2025), Article 9. doi:10.1007/s40687-025-00495-w. Correction: *ibid.* **12** (2025), Article 50, doi:10.1007/s40687-025-00537-3.
- [14] M. N. Kolountzakis, *Infinite patterns that can be avoided by measure*, Bull. London Math. Soc. **29** (1997), no. 4, 415–424. doi:10.1112/S0024609397003056.
- [15] M. N. Kolountzakis, *Sets of full measure avoiding Cantor sets*, Bull. Hellenic Math. Soc. **67** (2023), 1–11. arXiv:2209.10823.
- [16] M. N. Kolountzakis and E. Papageorgiou, *Large sets containing no copies of a given infinite sequence*, Anal. PDE **18** (2025), no. 1, 93–108. doi:10.2140/apde.2025.18.93.
- [17] N. Mora Cuéllar, A. Iosevich, N. Kulkarni, I. Rojas Aravena, and A. Yavicoli, *The Erdős similarity conjecture for two-fold sumsets with a geometric summand*, preprint (2026), arXiv:2607.03584v2.
- [18] OpenAI, *The geometric case of the Erdős similarity conjecture*, research draft, October 5, 2026. Source draft.
- [19] P. Shmerkin and A. Yavicoli, *Full measure universality for Cantor sets*, Adv. Math. **495** (2026), Article 110978. doi:10.1016/j.aim.2026.110978. arXiv:2503.21079.
- [20] R. E. Svetic, *The Erdős similarity problem: a survey*, Real Anal. Exchange **26** (2000/01), no. 2, 525–539.