

The Haagerup VOA at central charge eight

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Abstract

We give an explicit construction of a vertex operator algebra U of central charge eight inside $V_{E_6 \oplus A_2}$, and prove the ribbon equivalence $\text{Rep}(U) \simeq Z(\mathcal{H}_3)$, where $\mathcal{H}_3$ is a Haagerup fusion category. The algebra is generated as a vertex algebra by six affine generators and three specified weight-two vectors. The argument combines exact lattice-vector calculations with an all-weight strong-generation induction, a finite presentation and semisimple descent for a full corner of Zhu's algebra, and the actual multiplication of two holomorphic E_8 extensions. The categorical identification uses splitting maps and a comparison with the Evans–Gannon reconstruction, not modular data alone.

Computational certificates C01–C21 and verification software

https://github.com/Doctorfm98/Haagerup_VOA

Download the complete certificate ZIP from the repository's Releases page and open `CERTIFICATES.md` after extraction. The detailed access instructions are in the reading guide.

1 Introduction

It is well known that a strongly rational vertex operator algebra (VOA) V gives rise to a modular tensor category $\text{Rep}(V)$. A fundamental reconstruction problem asks for the opposite direction: which modular tensor categories arise from strongly rational VOAs? Significant work has been done toward answering this question. Evans and Gannon proved the first general reconstruction theorem of this kind for twisted Drinfeld doubles: every twisted Drinfeld double of a finite group can be realized by a completely rational conformal net, and for solvable groups the realization can be taken to be a strongly rational VOA [1]. More recently, the author, together with Gannon, developed a broader reconstruction theory for orbifolds of pointed VOAs, including the case in which the automorphism group fixes all simple modules [2].

One particular modular tensor category which has resisted capture is the double of the Haagerup fusion category. The Haagerup category occupies a special place in the subject as one of the first genuinely exotic fusion categories: it emerged from the exceptional small-index subfactor discovered by Haagerup and does not arise from the usual group-theoretic or quantum-group constructions [5, 6, 11]. Its Drinfeld center was computed and studied from several directions, beginning with the subfactor and endomorphism constructions of Izumi and continuing through the work of Evans, Gannon, Grossman, Snyder, and others [7, 10, 11]. The problem of realizing this modular tensor category by a strongly rational VOA has remained open.

In this paper, with the assistance of AI, we show that the answer is yes: the double of the Haagerup fusion category can be realized by a strongly rational VOA. More precisely, we construct

a VOA

$$U \subset V_{E_6 \oplus A_2} \subset V_{E_8}$$

and prove a ribbon equivalence

$$\text{Rep}(U) \simeq Z(\mathcal{H}_3).$$

The appearance of $V_{E_6 \oplus A_2}$ and V_{E_8} is not accidental. Evans and Gannon had already shown that the Haagerup modular data places extremely strong restrictions on any possible conformal realization and, at central charge eight, leads directly to the $E_6 \oplus A_2$ lattice VOA and its E_8 extensions [10, Sections 1 and 2.4]. Their analysis therefore gives a remarkably precise target for the construction carried out here.

The basic reason AI could be useful at this point is that the existing mathematical work already constrained the search so severely. Rather than searching blindly for a VOA, the computation could work inside a particular lattice VOA, with a particular character target and a short list of affine and coset possibilities. AI helped propose the generating vectors and calculations, but an AI assertion of success is not itself a mathematical argument. The decisive task is to make each exact calculation inspectable, and then explain why the finite calculations imply the required structural and categorical statements. The companion certificates and the arguments below are organized with that distinction in mind.

Before discussing the AI contributions and verification process, we first recall the work of the VOA and fusion-category communities which made this problem possible to solve at all. In particular, Terry Gannon’s work on the Haagerup modular data and on the constraints that any possible conformal realization would have to satisfy has played an especially important role in bringing the problem to its present form [10, 24]. Much of the search carried out here is possible only because this earlier work narrows dramatically what a Haagerup VOA could possibly look like.

More personally, the author’s understanding of this problem has been deeply influenced by Gannon’s work and by many conversations with him. The calculation carried out in this paper should therefore be understood as the final step of a much longer mathematical story, rather than as an isolated computation: the preceding work of Gannon and others narrowed the space of possibilities far enough that a sufficiently large exact computation could finally finish the problem.

Given both the unusual way in which this result was obtained and the long history of the problem, the author strongly encourages feedback from the community on how the AI-assisted aspects of the work are presented, as well as on the historical account below. Any corrections, missing references, or relevant prior work would be greatly appreciated.

1.1 Historical background

The history of the Haagerup categories begins with Jones’s index theory. For an inclusion $N \subset M$ of II_1 factors, the Jones index $[M : N]$ measures the size of the inclusion, and Jones’s fundamental work showed that the possible values below 4 are rigidly quantized [4]. This immediately made the region just above 4 a natural testing ground: if new algebraic structures existed beyond those coming from finite groups, Lie theory, or quantum groups, one might hope to see them first at small index.

This is exactly what happened. In his analysis of possible principal graphs just above index 4, Haagerup found a remarkable candidate at

$$\frac{5 + \sqrt{13}}{2} = 4.302 \dots$$

He announced the existence of the corresponding subfactor in 1993. Asaeda and Haagerup subsequently gave the first published proof of its existence, together with the construction of the distinct Asaeda–Haagerup subfactor of index $(5 + \sqrt{17})/2$ [5]. At the time these examples were particularly striking precisely because they did not arise from any of the standard sources then known: classical groups, quantum groups, or the standard constructions of rational VOAs. The index $(5 + \sqrt{13})/2$ example became known as the *Haagerup subfactor*. It was the first indication that finite-depth subfactors could carry genuinely new algebraic structure rather than merely repackage familiar symmetry.

The subsequent classification of subfactors of index at most five put this discovery into a much broader picture. Outside the standard infinite families, only a small collection of exceptional examples survive in this range, with the Haagerup, Asaeda–Haagerup, and extended Haagerup subfactors occupying a distinguished place. Jones–Morrison–Snyder give an account of this classification and its history in [6]. Thus the Haagerup example is not simply one isolated unusual subfactor: it is one of the basic exceptional objects produced by the small-index classification program.

A sequence of later constructions exposed different parts of the structure hidden in the original subfactor. Izumi gave an endomorphism realization using Cuntz algebras and analyzed the corresponding Longo–Rehren inclusion, the operator-algebraic analogue of taking a Drinfeld center. In particular, he obtained the dual principal graphs and the modular S - and T -matrices of the double [7]. This was an important step toward the present problem: the Haagerup object was no longer visible only through a principal graph, but through modular data of exactly the kind that one expects from a strongly rational VOA. Izumi’s work also placed the Haagerup example into what is now understood as the larger Haagerup–Izumi family.

Peters later gave a completely different construction using planar algebras [8]. In this description the Haagerup planar algebra appears as a singly generated subfactor planar algebra inside a graph planar algebra, giving in particular a new proof of existence. This was followed by the work of Grossman and Snyder, who determined the quantum subgroups and the complete Morita picture of the Haagerup fusion categories [11]. They showed that the Morita class contains exactly three fusion categories, which we denote

$$\mathcal{H}_1, \quad \mathcal{H}_2, \quad \mathcal{H}_3.$$

The category $\mathcal{H}_3$, of rank six, will be our reference category. Since the Drinfeld center is invariant under Morita equivalence, all three categories determine the same braided fusion category up to equivalence. We therefore write

$$Z(\mathcal{H}_3)$$

for the modular tensor category which we seek to realize by a VOA. This Morita flexibility will also be useful later when comparing the categorical data produced by the construction.

The decisive connection with vertex operator algebras was made by Evans and Gannon. Their work [10] transformed the question from the vague problem of finding “some VOA with Haagerup representation theory” into a surprisingly concrete reconstruction problem. Starting from the Haagerup modular data, they compared it with more familiar modular data, analyzed the associated vector-valued modular functions, and determined the possible character vectors. They showed, in particular, that at central charge $c = 8$ there are exactly two possible character vectors compatible with the Haagerup modular data.

More importantly for the present paper, Evans and Gannon identified the ambient VOA in which a Haagerup realization should occur. One of the central-charge-eight character vectors is naturally related to the lattice VOA

$$V_{E_6 \oplus A_2}.$$

The conformal embedding

$$L_1(\mathfrak{sl}_3) \otimes L_1(\mathfrak{e}_6) \subset L_1(\mathfrak{e}_8)$$

places this VOA inside V_{E_8} . Their analysis of modular invariants, the relation with the Drinfeld double of S_3 , and the corresponding branching data led them to an extremely concrete prediction: the desired Haagerup VOA should occur as a conformal subalgebra of $V_{E_6 \oplus A_2}$. Indeed, at the end of their analysis they identify the remaining task as constructing the Haagerup VOA inside $V_{E_6 \oplus A_2}$ [10, Section 2.4].

The present paper can in a very literal sense be viewed as completing this program. The ambient algebra, the central charge, the expected characters, and much of the necessary branching information were already forced by Evans and Gannon. What remained was the difficult part that modular data alone cannot supply: to write down the subalgebra itself, prove that the proposed vectors actually generate the required VOA, prove the necessary finiteness and rationality properties, construct its modules, and finally identify its full braided representation category with $Z(\mathcal{H}_3)$. In this sense the large calculation carried out below does not search blindly for a Haagerup VOA. It begins only after the preceding theory has narrowed the problem to a remarkably rigid configuration.

This point is especially important for understanding the role of Terry Gannon’s work in the present paper. Together with David Evans, Gannon supplied the modular and conformal framework which makes the construction possible in the first place: the target modular data, the character analysis, the $c = 8$ constraint, the connection with $V_{E_6 \oplus A_2}$, and the route through V_{E_8} [10]. Their later endomorphism work gives further reconstruction tools for Haagerup–Izumi categories and their doubles [27], while Gannon’s extension and branching framework is used directly in the identification carried out below [24]. Thus the construction in this paper should not be thought of as having discovered the Haagerup VOA from an unconstrained search. The preceding work, and especially the program developed by Evans and Gannon, had already determined much of what the answer was forced to look like. The problem was to turn that prediction into an actual vertex operator algebra and then prove that it has the predicted category of representations.

The Haagerup–Izumi program has continued to develop. Most recently, Gannon, Schopieray, and Yadav constructed Haagerup–Izumi categories for every odd cyclic order and proved the Evans–Gannon conjectural description of the modular data of their centers [9]. This substantially enlarges the family surrounding the original Haagerup example and makes the question of conformal realization still more natural. At the same time, recent work of Albert, Honda, Kaidi, and Zheng has returned directly to the E_8 picture, proposing a further construction of Haagerup symmetry based on E_8 [3]. These developments give a second, rather different, route back to the same remarkable appearance of E_8 .

Finally, the passage from an explicit candidate subalgebra to the categorical statement

$$\mathrm{Rep}(U) \simeq Z(\mathcal{H}_3)$$

depends on a large amount of foundational work in vertex operator algebras and tensor categories. The lattice construction of Frenkel–Lepowsky–Meurman supplies the ambient vertex operators [12]. Results of Dong–Li–Mason and Dong–Lin provide the affine-extension and unitary tools needed to control the subalgebra and its extensions [18, 29, 14]. Zhu’s algebra turns the classification of ordinary modules into a finite algebraic problem at lowest conformal weight [17], while the C_2 -Poisson algebra and its structure, in the form used here, provide the corresponding finiteness mechanism [16]. McRae’s theorem is then used in passing from these structural properties to rationality [20].

On the categorical side, Huang–Kirillov–Lepowsky identify suitable VOA extensions with commutative algebra objects in braided tensor categories [19]. Davydov–Müger–Nikshych–Ostrik explain how Lagrangian algebra objects recover Drinfeld centers [25], and the pivotal and positivity results of Etingof–Nikshych–Ostrik supply the fusion-category structure used in making the final identification [22]. These theorems are the bridges which allow explicit vertex-operator calculations to end in a statement about the full ribbon category of modules.

Thus, by the time one arrives at the present work, a remarkable amount is already known. The subfactor community constructed the Haagerup categories and determined their Morita theory. Izumi computed the structure of their double. Evans and Gannon determined the modular target and, at $c = 8$, essentially identified where the missing VOA had to live. The VOA and tensor-category literature supplied the machinery needed to pass from an explicit subalgebra to a theorem about its representation category. The new task is therefore very concrete, although far from automatic: to construct the predicted subalgebra inside $V_{E_6 \oplus A_2}$, prove the required structural properties, and verify that the resulting category is precisely $Z(\mathcal{H}_3)$. That is the task undertaken below.

1.2 The result

Theorem 1.1. *Let $U \subset V_{E_6 \oplus A_2}$ be the conformal vertex subalgebra of Definition 2.1. Then U is simple, unitary, self-contragredient, and strongly rational, with $U = \bigoplus_{n \geq 0} U_n$, $U_0 = \mathbb{C}\mathbf{1}$, and central charge 8. Its weight-one Lie algebra is $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, at levels 1 and 39. It has twelve irreducible ordinary modules, and there is a $\mathbb{C}$ -linear ribbon equivalence*

$$\mathrm{Rep}(U) \simeq Z(\mathcal{H}_3).$$

1.3 AI contributions and the author’s role

Generative AI was used extensively through author-directed conversations with GPT Pro 6, Astra 6, and Sol 6.1. The systems proposed the explicit candidate and generating lattice vectors, produced much of the code and exact linear algebra, and drafted substantial parts of the initial mathematical arguments and exposition. The supporting programs, witnesses, and verification instructions document the finite calculations.

The author formulated the research objective, supplied and refined the mathematical constraints, and actively guided the development of the proof strategy. In particular, the author recognized that the holomorphic E_8 extensions should give rise to Lagrangian algebra objects and used this insight to guide the categorical identification. Throughout the project, the author contributed mathematical ideas, critically examined and refined AI-generated arguments, identified gaps requiring further justification, and directed independent and adversarial verification efforts. The author also undertook extensive revisions of the mathematical notation, organization, and exposition to make the arguments accessible to human mathematicians. A substantial part of the author’s ongoing work consists of reconstructing, refining, and clarifying the AI-generated proofs and calculations into rigorous, transparent arguments that can be independently understood and verified by human readers.

The author takes responsibility for the mathematical claims and final presentation. Appendix R states the exact inputs, algorithms, witnesses, and scope of the finite computations. The global mathematical deductions are given in the text. A second lattice-product implementation replays specified calculations using an independent product engine, while reusing generation words and

linear-algebra choices.

1.4 Proof strategy

The proof has five logically distinct stages. We give the main mechanism of each stage and indicate the explanatory appendix methods and their computational certificates. The reading guide (p. 15) records the established input for each stage, including the associator and ribbon comparisons, and collects the principal notation.

1. An explicit conformal subalgebra and its positive form

Put

$$W = V_{E_6 \oplus A_2}, \quad \mathcal{A} = L_1(\mathfrak{sl}_2) \otimes L_{39}(\mathfrak{sl}_2).$$

The level-one factor comes from a chosen root A_1 in E_6 . The other factor is obtained by diagonally embedding the principal $\mathfrak{sl}_2$ triples in the orthogonal $A_2 \oplus A_5$ system; the levels add to $4 + 35 = 39$. In the weight-two coset $\text{Com}(\mathcal{A}, W)$, an explicitly fixed four-dimensional basis $b_0, \dots, b_3$ gives commuting Virasoro vectors

$$\begin{aligned} \omega_C = \omega_W - \omega_{\mathcal{A}} &= \frac{8b_0 + 9b_1 + 2b_2 + 2b_3}{41}, \\ e &= \frac{2b_0 + 6b_1}{31} + \frac{2-r}{93}b_2 + \frac{2-2r}{93}b_3, \quad f = \omega_C - e, \quad r = \sqrt{13} > 0. \end{aligned}$$

They have central charges $70/31$ and $2400/1271$. Let u_1, u_2, u_3 be the specified exponential basis of the weight-two affine highest-weight space of type $(1, 13)$. We take

$$v = \frac{-r+3q}{20}u_1 + \frac{r+3q}{20}u_2 + u_3, \quad q = \sqrt{-3} = i\sqrt{3},$$

and define

$$U = \langle \mathcal{A}, e, f, v \rangle_{\text{VA}} \subset W. \tag{1.1}$$

The six affine generators, together with e, f, v , are *vertex-algebra* generators, or in other words only weakly generate U ; finite *strong* generation requires a further proof. As $\omega_{\mathcal{A}} + e + f = \omega_W$, the inclusion is conformal and $c(U) = 8$. The fixed lattice roots, cocycle, bases, and exact product calculations define the candidate without an appeal to its conjectured representation theory. The lattice enumeration, coset-space solve, products, and seed calculation are explained in Appendices B–E. **[CALC C01, p. 130]**

An explicit transported conjugation of the lattice VOA preserves U , including a direct calculation expressing the conjugate of v inside U . Restricting the positive invariant Hermitian form, and using the resulting invariant bilinear form and a minimal-weight ideal argument, gives unitarity, self-contragredience, and simplicity. The inherited grading is nonnegative and integral, with $U_0 = \mathbb{C}\mathbf{1}$. Appendix G gives the form and extremal-lowering calculations. **[CALC C02, p. 130]**

2. Strong generation and C_2 -cofiniteness

The nine defining vectors generate U under arbitrary vertex modes. To prove finite strong generation, we must find a finite set from which every vector of U can be expressed using only derivatives and normal products. Exact lattice calculations first construct spaces of dimensions 120 and 956 in

weights two and three. In weight two, the derivatives and normal products of the six affine currents span a 22-dimensional space, leaving 98 additional representatives. In weight three, derivatives and normal products of the lower-weight vectors span a 634-dimensional space, leaving 322 additional representatives. To bound further additions, we study zeroth products modulo derivatives and normal products, while controlling positive-mode actions separately. In these quotients, skew symmetry and the Jacobi identity make zeroth products behave as Lie brackets. Using $\mathfrak{g}$ -equivariance, the resulting relation calculations show that at most three additional representatives are needed in weight four, and that the next zeroth products introduce none in weight five. Positive-mode recovery and a separate triangular induction then show that the chosen vectors strongly generate U in *all* weights. Thus the strong generating set contains at most 6, 98, 322, and 3 vectors in weights one, two, three, and four, respectively. The finite calculations supply the input to the induction; they do not alone establish strong generation. Appendix H explains the bases and finite maps, while Subsections H.4, H.7, and H.8 give the Jacobi relations, positive-mode recovery, and all-weight induction. [CALC C03, p. 130] [CALC C04, p. 130] [CALC C05, p. 130] [CALC C06, p. 131]

The strong generators make

$$R_U = U/C_2(U)$$

a finitely generated commutative Poisson algebra. Its nilradical is a Poisson ideal in characteristic zero. Integrability puts the classes of the weight-one generators in this ideal. Their zero-mode action then puts the classes of all nontrivial irreducible $\mathfrak{g}$ -modules in the ideal. Exact square identities deal with the remaining coset invariant vectors e, f ; the higher strong generators arise by the indicated mode operations and likewise have nilpotent classes. Once every positive-weight algebra generator of R_U is nilpotent, R_U is finite-dimensional, proving C_2 -cofiniteness. Subsection H.9 constructs the square identities and explains their exact residual checks. [CALC C07, p. 131]

3. Semisimplicity of the Zhu algebra $A(U)$ through Morita equivalence and semisimple descent

A central difficulty is to prove U is strongly rationality without an explicit list of its irreducible U -modules or directly showing that Zhu's algebra $A(U)$ is semisimple. To that end, the key algebraic simplification is a simple current extension of $\mathcal{A}$:

$$\mathcal{A}^+ = \mathcal{A} \oplus J_{(1,39)} \subset U,$$

whose 20 simple modules supply explicit lowest-weight matrix blocks T_t . We then decompose $A(\mathcal{A}^+)$ into matrix blocks corresponding to the lowest-conformal-weight spaces of its simple modules. All but one of these spaces are irreducible $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ -modules, determined by a single highest weight. The exceptional space is

$$T_{20} = V(0, 20) \oplus V(1, 19),$$

whose two summands have the same lowest conformal weight. It contributes a single matrix block $M_{61}(\mathbb{C})$, since $21 + 40 = 61$. $T_{20} = V(0, 20) \oplus V(1, 19)$, of dimension $21 + 40 = 61$. Here 20 is the affine label j ; it becomes quiver vertex $v = 10$, since $j = 2v$. We select the highest line in its $(0, 20)$ constituent for the rank-one corner; the full affine matrix algebra connects both constituents (Section 8). The inclusion preserves translation and conformal weights on $\mathcal{A}^+$, and hence induces a unital map $A(\mathcal{A}^+) \rightarrow A(U)$ (Proposition 7.2). If $E_{ij}^{(t)}$ are the images of matrix units, let

$$e_Z = \sum_t E_{11}^{(t)}, \quad D = e_Z A(U) e_Z.$$

The matrix-unit identity

$$1 = \sum_{t,i} E_{i1}^{(t)} e_Z E_{1i}^{(t)}$$

shows that the idempotent is full, even if certain blocks are killed by the map. Hence D is Morita equivalent to the *whole* Zhu algebra $A(U)$. Appendix J, especially Subsection J.2, gives the affine block calculation and the selected idempotents. **[CALC C10, p. 131]**

The strong-generation reduction yields 22 Zhu generators: the classes of six weight-one generators and sixteen highest-weight vectors $[z_0], \dots, [z_{15}]$. Decomposing the latter into normalized matrix coefficients between the T_t gives the arrows of a finite quiver. We obtain explicit relations from kernels of three $\mathfrak{g}$ -equivariant maps of lattice vectors: 66 families of symmetric normal products of weight-two vectors, 408 weight-three families involving translations and zeroth products, and 446 mixed weight-four families. These are 920 *families of VOA identities*, not 920 scalar quiver relations: projecting a family to its matrix coefficients can produce multiple path relations. Zhu's projection is performed with its translation and lower-mode terms retained. The resulting relations, along with conformal centrality and exact eliminations, define an upper algebra S_K over $K = \mathbb{Q}(\sqrt{13}, \sqrt{-3})$ with a surjection

$$S_K \otimes_K \mathbb{C} \twoheadrightarrow D.$$

It is enough to show that this upper algebra is semisimple; a complete determination of the kernel is not needed for this step. Subsections K.1–K.3 construct and convert the complete lattice-vector kernels; Subsection K.4 explains the typed vertex-unit witnesses and quadratic saturation. **[CALC C11, p. 132] [CALC C12, p. 132] [CALC C13, p. 132]**

For this, the relations are put into an integral form S_R over the local discrete valuation ring

$$R = \mathbb{Z}[\theta]_{(10009, \theta - 2914)}, \quad \theta = \sqrt{13} + \sqrt{-3}, \quad S_R \otimes_R K = S_K.$$

Integral shortening of every three-arrow path expresses it in a span of 1,081 specified paths of length at most two. This proves that S_R is *finite over R before reduction modulo the prime*. The special fiber over $\mathbb{F}_{10009}$ has dimension 546 and an exact split semisimple matrix decomposition with block sizes

$$1, 1, 1, 1, 2, 3, 4, 6, 9, 10, 10, 14.$$

The nondegenerate regular trace pairing gives a separate semisimplicity test. Removing R -torsion before applying the integral semisimple-descent argument shows that S_K is absolutely semisimple, with at most twelve absolute simple blocks. In particular, *flatness of S_R itself is not a prerequisite* for this deduction. The quotient D is therefore semisimple, and fullness transfers this to $A(U)$. Applying McRae's rationality criterion after establishing simplicity, self-contragredience, the nonnegative integral grading with $U_0 = \mathbb{C}\mathbf{1}$, C_2 -cofiniteness, and Zhu semisimplicity proves strong rationality [20, Corollary 5.19]. Subsections K.5.1 and K.5.2 explain coefficient regularity and the integral spanning bound; Subsections K.5.3 and K.5.4 give confluence and splitting. Subsection R.1 fixes the coefficient arithmetic. **[CALC C14, p. 132] [CALC C15, p. 132] [CALC C17, p. 133]**

4. Exhaustion of the twelve simple modules

We first construct six actual modules from lattice coset modules and verify their simplicity, distinctness and self-duality using top-space intertwiners and an invariant lattice pairing. They will eventually help show that every block in a finite upper bound survives, but they are *not* used to infer Zhu semisimplicity. Appendix I constructs the lowest-weight lattice vectors, exact matrices, and invariant pairing. **[CALC C08, p. 131] [CALC C09, p. 131]**

The six explicit lattice modules have the following lowest weights and top dimensions:

	U	B	P	Q	X_+	X_-
h	0	1	$\frac{1}{3}$	$\frac{2}{3}$	1	1
dim top	1	80	3	27	81	81.

The upper algebra gives at most twelve simple modules. A conformal trace identity for S_K , computed modulo commutators, restricts the six remaining lowest weights to

$$\frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}, \frac{11}{13}, \frac{15}{13}.$$

Lattice branching and positivity force at least one additional actual simple module. Only *after* strong rationality is established do we apply modular Galois symmetry: the square-Galois action on the relevant T -eigenvalues is transitive on these six possibilities. One new module therefore forces all six. This both exhausts the list and proves that every proposed block survives, giving 12 simple modules, $\dim D = 546$, and $\dim A(U) = 36,625$. This is cyclotomic symmetry of the actual modular category, not conjugation of the coefficient field K . Subsection K.6 constructs the trace witness; Subsection 11.5 gives the square-Galois action and its theorem hypotheses.[21] **[CALC C16, p. 133] [CALC C17, p. 133]**

5. The category, including its associator and ribbon structure

The discriminant group of $E_6 \oplus A_2$ is $G \cong \mathbb{Z}_3^2$, with quadratic form

$$q(e, a) = \frac{2e^2 + a^2}{3} \pmod{\mathbb{Z}}.$$

Its two isotropic lines determine holomorphic E_8 lattice extensions. Their *actual* lattice multiplication maps produce connected commutative separable algebra objects

$$A_{\pm} = U \oplus B \oplus 2X_{\pm} \quad \text{in } \mathcal{C} = \text{Rep}(U).$$

Their local-module categories are trivial; hence $A_{\pm}$ are Lagrangian. The center theorem of Davydov–Müger–Nikshych–Ostrik gives braided equivalences

$$\mathcal{C} \simeq_{\text{br}} Z(\mathcal{D}_{\pm}), \quad \mathcal{D}_{\pm} = \mathcal{C}_{A_{\pm}}.$$

The extension determines actual module-category tensor products, not merely the branching of characters.[19, 25]

The category $\mathcal{D}_+$ has six simple objects. Three are invertible, forming a $\mathbb{Z}_3$ subgroup, and the other three form an orbit under it. The complementary isotropic line of the lattice discriminant form establishes that the associator *restricted to this pointed $\mathbb{Z}_3$ subcategory* is cohomologically trivial. We make no claim that the associator of the whole fusion category is trivial. A seven-dimensional finite Fourier calculation, together with branching and positivity, determines the dimension of the other orbit and the fusion rules

$$\alpha^3 = \mathbf{1}, \quad \alpha\rho = \rho\alpha^2, \quad \rho^2 = \mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho.$$

The global-dimension computation and the fusion rules use the actual modular structure and are recorded separately from the associator comparison. Subsections O.1 and O.2 explain the finite class-algebra and Fourier construction and the Verlinde checks. **[CALC C18, p. 133] [CALC C19, p. 133]**

These fusion rules do *not* by themselves identify a tensor category. The final step chooses splitting inclusions and projections for the four summands of ρ^2 , compares their compositions inside categorical Hom spaces, and normalizes the resulting coefficients to the Evans–Gannon reconstruction equations. The comparison is expanded to all tensor words to prove full faithfulness of the resulting tensor functor. Together with the classification and the positive pivotal structure, this identifies $\mathcal{D}_+$ with $\mathcal{H}_2$ or $\mathcal{H}_3$. Those two categories are Morita equivalent, so their centers agree. The chosen braiding orientation and positive spherical structures then upgrade the identification to the claimed *ribbon* equivalence. [27, 11] Appendix M develops the typed coefficient comparison and full equivalence on words; Appendices N and P explain the normalized actions and finite solution check. Appendix Q gives the ribbon comparison. **[CALC C20, p. 133] [CALC C21, p. 133]**

The associated braid representations and hypergeometric functions provided motivation for the search and possible consistency tests; they are not being substituted here for the VOA calculations or the categorical equivalence, and no independent derivation of those functions from U is claimed in this proof outline.

1.5 How to follow a calculation

A small bold sans-serif marker such as **[CALC C01, p. 130]** identifies the finite input behind the nearby assertion. Read the indicated explanatory appendix subsection first: it describes the mathematical input, the finite map or equations, how the computation is formed, and what its output proves. The marker then opens the corresponding entry in Appendix R, which gives the precise witness files, reader guide, and verification command. Supplied witnesses and reports generated by replay are distinguished in those guides. The printed page number serves the same purpose on paper.

For example, the coset-square identities used in Section 5 are constructed in Subsection H.9; C07 supplies their exact coefficients and the command that reconstructs the identity and checks every residual. Thus the reading route is mathematical assertion, appendix method, then certificate. The reading guide (p. 15) records the wider proof dependencies and the principal notation.

The substantive deductions remain in the text: in particular, the all-weight induction, integral semisimple descent, module exhaustion, associator normalization, and ribbon comparison are mathematical arguments. A marker on an intermediate input identifies the finite assertion needed to use that argument. Major conclusions inherit the inputs explicitly marked in their proofs. The companion `README.md` gives the installation steps and commands in copyable form.

Background needed at each stage. Readers can approach the proof in parts. Section 2 and Appendix B fix the lattice VOA and its mode calculus. Sections 4–5 introduce the strong-generation spaces and the C_2 Poisson quotient, with the closure arguments in Appendix H. The Zhu operations are introduced in Definition 6.1; affine matrix blocks and their relations are developed in Sections 7–9 and Appendices J–K. The local DVR and integral descent are explained in Subsection R.1 and Lemma 9.6. For the categorical part, Sections 12 and 13 define the algebra objects and comparisons being used; Appendices M–Q supply the detailed coefficient, modular, and ribbon arguments. Precise external theorem locators are given at their applications.

1.6 Conventions and finite calculations

We use $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, $T = L_{-1}$, and $o(a) = a_{(\text{wt}(a)-1)}$ for homogeneous a . For a Heisenberg vector h , write $h(n) = (h(-1)\mathbf{1})_{(n)}$. All gradings are by conformal

weight. Strong rationality means simplicity, self-contragredience, rationality, C_2 -cofiniteness, and a nonnegative integral grading with weight-zero space $\mathbb{C}\mathbf{1}$. A $\mathfrak{g}$ -module label (r, j) denotes highest weights, with finite dimension $(r + 1)(j + 1)$; certificate dimension labels are converted to this convention. The reading guide (p. 15) collects these symbols and their changes of meaning. Local tables specify matrix indices, path directions, and the types of the maps at each step.

The affine algebra is $\mathcal{A}$, its extension is $\mathcal{A}^+$, the 80-dimensional-top module is B , and the Zhu corner is D . We reserve $\mathcal{D}$ for a fusion category. Genus-one matrices use $T = e^{-2\pi ic/24} \text{diag}(\vartheta)$ and positive vacuum normalization for S .

Finite spaces are defined over K and scalar-extended to $\mathbb{C}$ for comparison with U_n . A nonzero modular minor proves a rank lower bound; quotient upper bounds also require valid relations over K . The Zhu argument proves an integral spanning bound before using the special fiber. The appendices supply coordinates, full expansions, and finite witnesses (Appendix R).

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Reading guide and notation

Where to find all 21 certificates. The companion computational package is hosted at https://github.com/Doctorfm98/Haagerup_VOA. To obtain it, open https://github.com/Doctorfm98/Haagerup_VOA/releases, expand the release’s *Assets*, and download the author-uploaded ZIP containing `CERTIFICATES.md`, `certificates/`, and `shared/`. The automatically generated GitHub *Source code* archives are not the certificate download. Extract the complete ZIP and open its root `CERTIFICATES.md`: this index links to every guide, C01–C21, under `certificates/`. Each guide identifies its input data, programs, witnesses, and verification command; the shared data and programs are under `shared/`. The root `README.md` gives installation and reproduction instructions. Throughout this manuscript, computational file paths and commands are relative to the root of that extracted companion package. The Hexagon submission contains the manuscript’s LaTeX sources; the computational package is downloaded separately from GitHub. The clickable **CALC** markers lead to the detailed registry in Subsection R.2, which covers all 21 certificates.

The main argument and its computational evidence have different tasks. The main text explains why each finite assertion is enough for the next mathematical step. The indicated appendix explains how its vectors, maps, or relation matrices are constructed. A **CALC** entry then identifies the witness, the reader guide, and the command for exact verification. For example, the square identities used in Section 5 are constructed in Subsection H.9; C07 provides their coefficients and replay. The file is the last stage of that reading route, rather than a substitute for its mathematical explanation.

Step	Established input	Result and detailed explanation
Definition and positive form	Fixed lattice vectors and their conjugation	Proposition 3.1; Appendices B–G
Strong generation	Finite closure and valid Jacobi relations	Theorem 4.4; Appendix H
C_2 -cofiniteness	Strong generation, affine integrability, and two square witnesses	Theorem 5.2; Subsection H.9
Semisimple Zhu algebra	An integral upper presentation, a spanning bound, and a split special fiber	Section 9; Appendix K
Module exhaustion	Six actual lattice modules, trace profiles, branching, and Galois symmetry	Section 11; Appendices I, K
Fusion and associator	Actual lattice multiplication, modular dimensions, and typed categorical comparisons	Section 12; Appendices M, N
Ribbon equivalence	Positive pivotal structure and compatible braiding	Section 13; Appendix Q

A few distinctions to keep in view. The spaces S_i initially denote constructed finite spaces, not a claim that every vector of U_i has been found. The twenty affine blocks index the matrix coordinates; they are not twenty ordinary U -modules. An upper algebra maps onto the actual Zhu corner, and equality is proved only after module exhaustion. Likewise the 1081 integral paths form a spanning set, whereas the 546 normal paths form a basis in the special fiber. Numerical matrices of the affine generators describe fixed intertwining patterns; their unknown coefficients remain independent path variables.

Notation	Meaning in the proof
$W = V_L, U$	Ambient lattice VOA and its explicitly generated conformal subVOA
$H, \mathfrak{g}$	Six-dimensional space of weight-one generators and its zero-mode Lie algebra
T, L_n	Translation L_{-1} and Virasoro operators; modular T is identified explicitly when used
S_i, N_i, D_i	Constructed weight spaces, lower-weight normal spans, and coefficient representatives modulo those spans
$A(U), D$	Zhu algebra and the selected full corner $e_Z A(U) e_Z$
e_t, e_Z	One rank-one affine matrix idempotent per block and their sum
K, R, S_R, S_K	Exact coefficient field, local DVR, integral upper algebra, and its generic fiber
$\mathcal{C}, \mathcal{D}$	The modular category $\text{Rep}(U)$ and its category of modules over the lattice algebra
R_0, K_7	An intermediate categorical algebra and the seven-dimensional modular subspace, not the local ring or coefficient field
$[a]$	A quotient class: a C_2 class only in Section 5, and a Zhu class in the Zhu-algebra part

Throughout, $\mathfrak{g}$ acts by the zero modes of the six weight-one generators. A type (r, j) denotes a finite-dimensional irreducible $\mathfrak{g}$ -module, and its lowering vectors are obtained by applying $(F_s)_{(0)}$ and $(F_t)_{(0)}$ to a highest-weight vector. For an ordinary module, its *lowest space* or *top* is the eigenspace of L_0 with least eigenvalue. A lattice module's lowest space is spanned by the exponentials of the minimum-norm vectors in its lattice coset. The subsequent local tables specify matrix entries, path directions, coefficient encodings, and induction functors.

Part I

The Construction

2 The lattice subVOA U

The construction takes place inside

$$W := V_L, \quad L = E_6 \oplus A_2.$$

The affine part consists of a root $\mathfrak{sl}_2$ and a diagonal principal $\mathfrak{sl}_2$ in $A_2 \oplus A_5$. The coset contributes two Virasoro vectors. An additional weight-two vector has specified affine highest weight and coset eigenvalues. All products are inherited from the lattice VOA [12, Ch. 8]. We fix the lattice coordinates, cocycle, and roots in Appendix B; these choices are part of the definition. The finite calculations used below are listed in Appendix R.

2.1 The commuting affine generators

For the fixed root s , define

$$E_s = e^s, \quad F_s = -e^{-s}, \quad H_s = s(-1)\mathbf{1}. \quad (2.1)$$

Put $t = \sum_i c_i \rho_i$, where the c_i are the principal-triple coefficients fixed in Appendix B.

$$E_t = \sum_i e^{\rho_i}, \quad F_t = -\sum_i c_i e^{-\rho_i}, \quad H_t = t(-1)\mathbf{1}. \quad (2.2)$$

The coefficients 2, 2 and 5, 8, 9, 8, 5 are the familiar $i(m-i)$ coefficients of the principal $\mathfrak{sl}_2$ triples in A_2 and A_5 . This explains the otherwise unusual level 39: the diagonal triple has level

$$\sum_i c_i = (2+2) + (5+8+9+8+5) = 4+35 = 39.$$

Indeed, $(t, \rho_i) = 2$, distinct positive simple roots have no root difference, and the cocycle gives $(e^{\rho_i})_{(1)}(-e^{-\rho_i}) = \mathbf{1}$. Thus $(H_t)_{(0)}E_t = 2E_t$, $(E_t)_{(0)}F_t = H_t$, and $(E_t)_{(1)}F_t = 39\mathbf{1}$. Orthogonality to s gives commutation with the level-one triple. Their nonzero brackets and pairings, and the identification of their generated affine VOA, are explained in Appendix B. **[CALC C01, p. 130]** Let

$$\mathcal{A} := \langle E_s, F_s, H_s, E_t, F_t, H_t \rangle_{\text{VA}} \cong L_1(\mathfrak{sl}_2) \otimes L_{39}(\mathfrak{sl}_2). \quad (2.3)$$

Its Sugawara conformal vector $\omega_{\mathcal{A}}$ is specified in (B.7), and its central charge is

$$c_{\mathcal{A}} = 1 + \frac{3 \cdot 39}{41} = \frac{158}{41}.$$

Here $\mathcal{A}$ is a vertex subalgebra with its own Sugawara conformal vector; that vector differs from ω_W . Set

$$\omega_C := \omega_W - \omega_{\mathcal{A}}, \quad C_W := \text{Com}(\mathcal{A}, W). \quad (2.4)$$

The coset central charge is $170/41$.

Concretely, since the affine generators generate $\mathcal{A}$, a vector belongs to C_W exactly when each affine generator h satisfies $h_{(n)}x = 0$ for $n \geq 0$. In weight two this is a finite system: the output has weight $2-n$, so only $n = 0, 1, 2$ can contribute. Appendix D builds that system before exhibiting its four-dimensional nullspace.

2.2 The chosen coset plane

The space $(C_W)_2$, graded by ω_W , has dimension four. Fix the ordered basis b_0, b_1, b_2, b_3 given by the complete coefficient tables in Appendix C. In particular, the cocycle transport in that appendix is part of the definition of these vectors. Their products, bilinear pairings, and a full ambient dimension check appear in Appendix D. **[CALC C01, p. 130]** In this basis,

$$\omega_C = \frac{8b_0 + 9b_1 + 2b_2 + 2b_3}{41}. \quad (2.5)$$

Fix $r = \sqrt{13} > 0$, and choose

$$e = \frac{2}{31}b_0 + \frac{6}{31}b_1 + \frac{2-r}{93}b_2 + \frac{2-2r}{93}b_3, \quad f = \omega_C - e. \quad (2.6)$$

The singular products are given in (D.1). Hence e, f are commuting Virasoro vectors with

$$c_e = \frac{70}{31}, \quad c_f = \frac{2400}{1271}, \quad c_e + c_f = \frac{170}{41}.$$

Their first-mode algebra is the semisimple plane $\mathbb{C}e \oplus \mathbb{C}f$, whose unit is $\omega_C/2$. **[CALC C01, p. 130]**

Here the product is $a * b = a_{(1)}b$. The displayed singular identities give $e * e = 2e$, $f * f = 2f$, and $e * f = 0$, so $e/2$ and $f/2$ are orthogonal idempotents. Their sum $\omega_C/2$ acts as the identity on this plane. We choose this particular plane by the explicit axis formula, rather than asserting that every plane in $(C_W)_2$ has this property. The multiplication table in Appendix D is what makes the choice checkable.

2.3 The additional weight-two seed

The fixed lattice vectors in (E.1) have Gram matrix with diagonal four and off-diagonal two. The vectors $u_i = e^{\beta_i}$ form the entire weight-two affine highest-weight space of type $(1, 13)$, where the pair denotes highest weights. The zero-mode highest-weight conditions and all positive affine-mode conditions are checked in Appendix E. **[CALC C01, p. 130]**

Fix $q = \sqrt{-3} = i\sqrt{3}$, and choose the line represented by

$$v = \frac{-r + 3q}{20}u_1 + \frac{r + 3q}{20}u_2 + u_3. \quad (2.7)$$

For exact arithmetic let $\theta = r + q$. Then

$$\theta^4 - 20\theta^2 + 256 = 0, \quad K = \mathbb{Q}(\theta) = \mathbb{Q}(r, q). \quad (2.8)$$

For arithmetic we use $v' = ((-r + 3q)/20)^{-1}v$, with coefficient of u_1 equal to one. Its K -coordinates are given in (E.2). The seed's coset eigenvalues are

$$e_{(1)}v = \frac{13}{31}v, \quad f_{(1)}v = \frac{180}{1271}v. \quad (2.9)$$

The four sign choices for r, q are recorded in Appendix F.

The Virasoro eigenvalues select a two-dimensional space in this three-dimensional multiplicity space. They do not specify a unique seed. The displayed coefficients of v choose the line used here. The eigenvalue calculation is in Appendix E. **[CALC C01, p. 130]**

The dimension three counts highest-weight vectors. Each generates an irreducible $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ -module of dimension $(1+1)(13+1) = 28$. A line in the multiplicity space therefore selects one copy of this 28-dimensional module.

Vector	Weight	Normalization	Role
v	2	Equation (2.7)	Defining seed
v'	2	Coefficient of u_1 is one	Arithmetic representative of the same line
z_9	2	Appendix A	Certificate-normalized highest representative

In fact $z_9 = v' = ((-r + 3q)/20)^{-1}v$, as the identical coefficients in (A.18) show. A nonzero rescaling of the seed does not change the vertex subalgebra it generates.

2.4 Definition and conformal structure

Definition 2.1. Let U be the vertex subalgebra of W generated by the nine explicit vectors

$$\boxed{E_s, F_s, H_s, E_t, F_t, H_t, e, f, v.} \quad (2.10)$$

Equivalently, $U = \langle \mathcal{A}, e, f, v \rangle_{\text{VA}} \subset W$. Here generation means the smallest subspace containing $\mathbf{1}$ and these vectors, stable under translation and all inherited vertex modes. It does not assert strong generation by these nine vectors.

Defining vectors	Weight	Purpose
$E_s, F_s, H_s, E_t, F_t, H_t$	1	Two commuting affine factors
e, f	2	The chosen coset plane and the complementary conformal vector
v	2	A highest-weight vector with specified affine type and coset eigenvalues

These nine vectors define U using all modes. The finite list of strong generators and the proof that it controls all weights come later. At this stage we have a positively graded conformal subalgebra; its positive form and simplicity are established in the next section.

The vector $\omega_{\mathcal{A}}$ belongs to the vertex subalgebra generated by the six affine generators, because (B.7) expresses it using their negative modes. Therefore

$$\omega_W = \omega_{\mathcal{A}} + e + f \in U.$$

Restricting the lattice conformal structure makes $U \subset W$ a conformal subVOA of central charge eight. Its vacuum, translation, and Jacobi identities are inherited from W . It has finite-dimensional nonnegative integral weight spaces and $U_0 = \mathbb{C}\mathbf{1}$, since it is a graded subspace of W .

Finiteness, rationality, and the module category will be proved below.

3 The positive form, simplicity, and duality

A conformal subalgebra of a unitary VOA need not be stable under its unitary conjugation. Here the needed conjugation can be written down. The point of the calculation is to show that it preserves the seed line's generated algebra.

Proposition 3.1. *The defined U is a simple unitary VOA with $U = \bigoplus_{n \geq 0} U_n$ and $U_0 = \mathbb{C}\mathbf{1}$, $U \cong U'$ as a U -module, and $L_1 U_1 = 0$.*

Proof. The grading and vacuum space were established in Section 2. Also $L_1 W_1 = 0$, hence $L_1 U_1 = 0$.

The standard conjugation exchanges positive and negative root exponentials with unit coefficients. Our diagonal principal triple has the unequal coefficients c_i in its lowering affine generator. A lattice character corrects exactly these coefficients: we require $\chi(s) = 1$ and $\chi(\rho_i) = c_i$. Transporting the form and conjugation together preserves positivity and invariance, and reduces stability of U to the nine generator images below.

Define a positive lattice character χ by

$$(\chi(\alpha_0), \dots, \chi(\alpha_7)) = \left(2, 2, 1, 5, \frac{1}{24}, 8, 9, 8\right). \quad (3.1)$$

Let ϕ_0 be the standard lattice conjugation, and set $g(e^\lambda) = \chi(\lambda)^{-1/2} e^\lambda$, with g fixing the Heisenberg vectors. This is a VOA automorphism. Conjugating ϕ_0 by g gives the antilinear involution

$$\phi(h_1(-n_1) \cdots h_\ell(-n_\ell) e^\lambda) = (-1)^\ell \chi(\lambda) h_1(-n_1) \cdots h_\ell(-n_\ell) e^{-\lambda}, \quad (3.2)$$

for real h_i , with complex conjugation of coefficients. The cocycle of (B.2) satisfies $\varepsilon(-a, -b) = \varepsilon(a, b)$ and $\varepsilon(\lambda, -\lambda) = (-1)^{(\lambda, \lambda)/2}$, so this is the standard lattice unitary normalization. Transporting the standard positive form gives

$$H_\chi(x, y) = H_0(g^{-1}x, g^{-1}y).$$

Here Hermitian forms are linear in the first argument.

The generator images are

$$\begin{aligned} \phi(E_s) &= -F_s, & \phi(F_s) &= -E_s, & \phi(H_s) &= -H_s, \\ \phi(E_t) &= -F_t, & \phi(F_t) &= -E_t, & \phi(H_t) &= -H_t, \\ \phi(e) &= e, & \phi(f) &= f, & \phi(v) &= -\frac{(F_s)_{(0)}(F_t)_{(0)}^{13}v}{13!}. \end{aligned} \quad (3.3)$$

The affine generator images follow from $\chi(s) = 1$ and $\chi(\rho_i) = c_i$. The two axes are fixed term by term. The seed identity is the explicit highest-to-lowest-weight calculation in Appendix G. **[CALC C02, p. 130]** All nine images belong to U . Since ϕ preserves vertex modes, translation, and the vacuum, it therefore preserves the generated algebra U . Its positive invariant form restricts to a unitary structure on U [14, Proposition 2.6 and Theorem 4.12].

Nondegeneracy and self-duality. Let B_W be the normalized invariant lattice bilinear form. The character automorphism preserves it, and

$$H_\chi(x, y) = B_W(x, \phi y).$$

Hence $B_U = B_W|_{U \times U}$ is nondegenerate: for $x \neq 0$, $B_U(x, \phi x) = H_\chi(x, x) > 0$. Different weights are orthogonal, so this is nondegenerate in every finite-dimensional grade. Invariance makes $x \mapsto B_U(x, -)$ an isomorphism to the restricted contragredient $U' = \bigoplus_n U_n^*$.

Simplicity. An ideal is stable under the diagonal operator L_0 and therefore graded: the projection of any finite sum of weight vectors onto a single weight is a polynomial in L_0 on that sum. A proper ideal cannot contain the vacuum, so a proper nonzero ideal I has $I_0 = 0$. Choose $0 \neq x$ of its least weight $n > 0$; then $L_1 x = 0$. Indeed $L_1 x \in I_{n-1} = 0$. In the invariant-form identity the factor e^{zL_1} consequently fixes x , while $(-z^{-2})^{L_0}$ contributes $(-1)^n z^{-2n}$. Its vacuum coefficient is therefore the scalar displayed next. Invariance and skew symmetry give

$$\phi(x)_{(2n-1)} x = x_{(2n-1)} \phi(x) = (-1)^n H_\chi(x, x) \mathbf{1} \neq 0.$$

The left-hand side belongs to I_0 , a contradiction. Thus U is simple. $\square$

Lemma 3.2. *Every lattice module $V_{L+\lambda}$ restricts to a completely reducible ordinary U -module. After C_2 -cofiniteness has been proved, this is a finite direct sum.*

Proof. We work with an ordinary restriction: the inherited L_0 is diagonal, bounded below, and has finite-dimensional eigenspaces. No rationality of U is being assumed here. First positivity splits off cyclic irreducible submodules; later C_2 -cofiniteness bounds the number and multiplicities of those summands.

The transported lattice forms are positive and invariant for ϕ . The adjoint formula makes the graded orthogonal complement of any submodule invariant. Complete reducibility follows as in [14, Lemma 2.5]: split a lowest space into irreducible summands for its adjoint-closed zero-mode algebra, generate cyclic modules, and split them off orthogonally. A cyclic module so obtained is irreducible, since a proper submodule would have an invariant orthogonal complement containing its lowest-space generator.

Once C_2 -cofiniteness is known, $A(U)$ is finite-dimensional. There are finitely many possible irreducible ordinary modules, by Zhu's correspondence. Each has a fixed lowest weight, and its multiplicity is bounded by the finite dimension of that grade in the lattice module. The sum is therefore finite. $\square$

Part II

Strong Generation and C_2 Cofiniteness

4 Strong generation: two finite obstructions

Strong generation asks whether every vector is a linear combination of normal products of derivatives of finitely many generating vectors. The definition of U only gives generation by arbitrary modes. The difference is precisely what must be controlled here.

The useful observation is that zeroth products behave like Lie brackets after derivatives and normal products have been removed. Indeed, on positive-weight vectors a zeroth mode preserves their span:

$$a_{(0)}Tb = T(a_{(0)}b), \quad a_{(0)}(p_{(-1)}q) = (a_{(0)}p)_{(-1)}q + p_{(-1)}(a_{(0)}q).$$

Skew symmetry becomes antisymmetry in this quotient, and the zeroth-mode commutator becomes Jacobi. Thus the possible new generators are controlled by relations among iterated brackets.

We organize the low-weight vectors into finite-dimensional irreducible $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ -modules. The calculation leaves a set of representatives in weight three, at most three new representatives in weight four, and none in weight five. An all-weight lemma then proves that these representatives strongly generate U .

4.1 The finite input

All calculations use the field K of (2.8). For K -subspaces $P, Q \subset U$, write

$$P_{(n)}Q := \text{span}_K\{p_{(n)}q : p \in P, q \in Q\}.$$

We keep a vector or subspace index separate from its mode index; for example, $(a_i)_{(0)}$ and $(S_2)_{(n)}Q$. Let $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, acting by zero modes. Let H be the six-dimensional space of weight-one generators, and let $S_2 \subset U_2$ be the 120-dimensional space obtained from the defining weight-two vectors, their zero-mode lowering vectors, and their first-mode products. Write $V_{r,j}$ for the finite-dimensional irreducible $\mathfrak{g}$ -module of highest weight (r, j) . The $\mathfrak{g}$ -module decomposition of S_2 and its ordered basis are recorded in Appendix H. **[CALC C03, p. 130]** The defining construction includes TH , $H_{(-1)}H$, and ω_W . Set

$$N_3 = TS_2 + H_{(-1)}S_2, \quad S_3 = N_3 + (S_2)_{(0)}S_2.$$

The saved exact basis of S_3 has 956 elements. **[CALC C04, p. 130]** At this point S_2, S_3 are constructed subspaces of U ; their equality with the full grades will follow from strong generation.

The notation distinguishes construction from conclusion. S_i is a finite space actually generated in weight i ; N_i is the part already made by translation and normal products from lower weights. D_3 is a chosen complement to N_3 , while D_4 below is a coefficient space whose lifts span the quotient and need not remain independent.

Space	Weight	Size	Meaning
H	1	6	Affine generators
S_2	2	120	Constructed $\mathfrak{g}$ -stable space
B_2	2	22	Affine derivatives and normal products
E	2	97	Complement to $B_2 + K\omega_W$
N_3	3	634	$TS_2 + H_{(-1)}S_2$
D_3	3	322	Chosen representatives of S_3/N_3
S_3	3	956	$N_3 + (S_2)_{(0)}S_2$
N_4, S_4	4	Defined below	Normal span and one further bracket
D_4	4	3	At most three new lifts
N_5	5	Defined below	Normal span in which all next brackets lie

Proposition 4.1 (Finite mode closure). *With $S_0 = K\mathbf{1}$, $S_1 = H$, and $S_i = 0$ for $i < 0$, the constructed spaces satisfy*

$$H_{(n)}S_i \subset S_{i-n} \quad (n \geq 0), \quad (S_2)_{(n)}S_i \subset S_{i+1-n} \quad (n \geq 1), \quad i = 2, 3.$$

In particular, $L_1S_i \subset S_{i-1}$ because $\omega_W \in S_2$.

Finite calculation. The lattice products reduce with zero residual in the stated bases. The input and basis conventions are fixed in Subsections H.1–H.2. Zero-mode covariance gives $[h_{(0)}, a_{(n)}] = (h_{(0)}a)_{(n)}$; applying it repeatedly transports the highest-input checks to products involving all zero-mode lowering vectors. Thus the verification covers the lowering inputs as well as the highest-weight vectors. The bases are rebuilt from their lattice generation words and each output is reduced in complete lattice-monomial coordinates. [CALC C03, p. 130] [CALC C04, p. 130] The scalar outputs and all larger modes follow from the weight grading. $\square$

4.2 What can be new in weights four and five?

Define the normal spans

$$\begin{aligned} N_4 &= TS_3 + H_{(-1)}S_3 + (S_2)_{(-1)}S_2, & S_4 &= N_4 + (S_2)_{(0)}S_3, \\ N_5 &= TS_4 + H_{(-1)}S_4 + (S_2)_{(-1)}S_3. \end{aligned} \tag{4.1}$$

In each weight, N_i contains what is already built from lower weights by derivatives and normal products. The extra term in S_4 allows another bracket. The key assertion is that one more bracket adds nothing new:

$$(S_2)_{(0)}S_4 \subset N_5. \tag{4.2}$$

We prove this by writing down the Jacobi relations among these brackets and checking that they exhaust the possible new terms.

Let B_2 be the 22-dimensional affine weight-two space. Choose the 97-dimensional $\mathfrak{g}$ -stable complement E recorded in Appendix H, so that

$$S_2 = B_2 + K\omega_W + E.$$

Use the splitting $S_3 = N_3 \oplus D_3$ proved in Appendix H, with $\dim N_3 = 634$ and $\dim D_3 = 322$. [CALC C05, p. 130] Let $\pi : S_3 \rightarrow D_3$ be the corresponding projection and put

$$\beta(a, b) = \pi \left(\frac{a_{(0)}b - b_{(0)}a}{2} \right), \quad \beta : \Lambda^2 E \longrightarrow D_3. \tag{4.3}$$

Skew symmetry and the finite mode closure give $a_{(0)}b - \beta(a, b) \in N_3$. Thus D_3 represents the quotient by derivatives and affine generator normal products.

Proposition 4.2 (The two obstruction calculations). *The map β is onto. There is a three-dimensional coefficient space D_4 and a lift $\rho : D_4 \rightarrow S_4$ such that*

$$S_4 = N_4 + \rho(D_4).$$

The further zeroth products $E_{(0)}\rho(D_4)$ lie in N_5 . Consequently (4.2) holds.

Proof. The selected bracket minor proves that β is onto. [CALC C06, p. 131] Normal-input terms contribute only to N_4, N_5 , by (H.1). Thus possible new generators are represented by $E \otimes D_3$ in weight four, and by $E \otimes D_4$ in weight five.

In weight four, Jacobi gives the cyclic relations (H.2). Their checked rank leaves three representatives. Exact elimination gives the lift ρ and the reduction (H.3). [CALC C06, p. 131]

In weight five, take a relation $\sum_i a_i \wedge b_i \in \ker \beta$. The vector $\sum_i (a_i)_{(0)} b_i$ is then normal. Bracketing it with $x \in D_3$ gives zero modulo N_5 . The zeroth-mode commutator turns this into the rows (H.4); their checked full column rank kills every remaining representative. [CALC C06, p. 131]

These are relations over K before reduction modulo the prime. Their nonzero minors therefore give the required quotient upper bounds. Appendix H.4 gives the projected maps and the complete argument. $\square$

Calculation	Columns	Rank	Conclusion
Bracket map	4656	322	Surjects onto D_3
Quartic valid relations	31234	31231	At most three new weight-four lifts
Quintic valid relations	291	291	No new weight-five lift

A nonzero minor gives a rank lower bound. Here the rows are first proved to be relations, so that lower bound gives the complementary upper bound on the quotient of possible new generators. Subsection H.5 explains the unit minors and column coverage. In particular, the three weight-four free columns provide representatives; no assertion of their independence in U is needed.

4.3 Why the calculation controls every higher weight

The calculation has closed the chosen vectors under singular actions by $H + S_2$. To conclude, we must control actions by the newly added vectors as well. Each of them is a bracket of a weight-two vector with a lower-weight expression. This is the triangular form used below. We use the standard normal-product and mode identities [15, Section 2].

There is one additional closure step between the zeroth-mode calculation and the lemma. For $a \in S_2, x \in S_4$,

$$2a_{(1)}x = L_1(a_{(0)}x) - a_{(0)}L_1x - (L_1a)_{(0)}x.$$

Its terms lie in S_4 , using $a_{(0)}x \in N_5$ and $L_1N_5 \subset S_4$. Further L_1 commutators recover mode two, but the coefficient of mode three is zero at that step. The L_2 identity recovers mode three, and then L_1 recovers mode four. Subsection H.7 gives each term and output space. This is why the lemma has closure under every singular $H + S_2$ action, rather than just the zeroth actions.

Lemma 4.3 (Triangular strong generation). *Let $A \subset G$ be finite homogeneous positive-weight sets in a nonnegatively graded vertex algebra, with vacuum space $K\mathbf{1}$. Let $\mathcal{N}(G)$ span all parenthesized normal products of arbitrary derivatives of members of G , together with the vacuum. Suppose the following conditions hold.*

1. $a_{(n)}g \in \mathcal{N}(G)$ for $a \in A$, $g \in G$, $n \geq 0$;
2. Every $g \in G \setminus A$ of weight d is a normal expression in smaller-weight generators plus terms $a_{(0)}b$, where $a \in A$, $\text{wt}(a) \geq 2$, and b is a normal expression of weight $d - \text{wt}(a) + 1 < d$ in smaller-weight generators.

Then $\mathcal{N}(G)$ is a vertex subalgebra strongly generated by G .

Proof. Write $N = \mathcal{N}(G)$. The zeroth modes of A preserve N : they commute with T , are derivations of normal products, and preserve the generators by hypothesis 1.

If singular closure failed, choose homogeneous normal expressions x, y with $x_{(n)}y \notin N$, $n \geq 0$, and smallest ordered pair $(\text{wt}(x) + \text{wt}(y), \text{wt}(x))$.

This is a well-founded lexicographic order on positive integers: we first decrease total input weight, and only when it is unchanged decrease the weight of the first input. For instance, a singular-product correction $(x_{(j)}p)_{(n-1-j)}q$ from a normal second input has total outer input weight $\text{wt}(x) + \text{wt}(p) + \text{wt}(q) - j - 1$, strictly less than the original total. The remaining term is a normal product of already controlled expressions.

Delete vacuum factors. The derivative and normal-product identities reduce both inputs to generators: every singular-product correction has smaller total input weight. Appendix H.8 gives the expansions. A first input in A is now covered by hypothesis 1.

For another generator, hypothesis 2 leaves a bracket $a_{(0)}b$ with $\text{wt}(b) < \text{wt}(a_{(0)}b)$. Its action is

$$(a_{(0)}b)_{(n)}y = a_{(0)}(b_{(n)}y) - b_{(n)}(a_{(0)}y).$$

The nested product $b_{(n)}y$ in the first term has smaller total input weight, and the outer $a_{(0)}$ preserves N . In the second term, total input weight is unchanged but the first input is the smaller b . Both terms contradict minimality. Thus N is a vertex subalgebra.

The conformal-weight filtration, with differentiation of degree zero, then rewrites its normal expressions into the usual right-nested form. Appendix H.8 gives this final step too. $\square$

Apply the lemma with A a basis of $H + S_2$, and add a basis of D_3 and the three lifts $\rho(D_4)$ to obtain G . The definitions of S_3, S_4 give the triangular expressions. Proposition 4.1 and (4.2) control zeroth products. The Virasoro commutators in Appendix H recover all positive S_2 -modes on S_4 ; affine modes are already controlled by the definitions. Thus hypothesis 1 holds. All these vectors lie in U , and A contains the defining generators, so $\mathcal{N}(G) = U$.

Theorem 4.4. *The VOA U is strongly generated in weights at most four, by at most 429 generators: 6 of weight one, 98 of weight two, 322 of weight three, and 3 of weight four. Moreover, $U_1 = H$, $U_2 = S_2$, and $U_3 = S_3$.*

Proof. The preceding application proves strong generation. [CALC C06, p. 131] Omit the 22 affine weight-two vectors, which are derivatives and normal products of affine generators. The count is $6 + (120 - 22) + 322 + 3 = 429$: affine generators, the remaining weight-two representatives, the weight-three complement, and the three weight-four lifts. These are upper bounds on a strong generating list, not a minimality assertion. Since their normal expressions of weights one, two, and three lie in the constructed H, S_2, S_3 , those spaces are the full grades. $\square$

In particular, the vacuum character begins

$$\text{ch}_U(\tau) = q_\tau^{-1/3}(1 + 6q_\tau + 120q_\tau^2 + 956q_\tau^3 + O(q_\tau^4)), \quad q_\tau = e^{2\pi i\tau}.$$

These dimensions agree with the central-charge-eight character possibility in [10]; the strong-generation argument establishes them directly from the defining vectors.

5 C_2 -cofiniteness: the classical quotient is supported at a point

Here $C_2(U) = \text{span}_{\mathbb{C}}\{a_{(-2)}b : a, b \in U\}$. Translation satisfies $Ta = a_{(-2)}\mathbf{1}$, so every derivative lies in this subspace. Unlike Zhu's later star product, the product of classes below is commutative and records only normal multiplication.

The C_2 quotient is the commutative Poisson algebra

$$R_U = U/C_2(U), \quad [a][b] = [a_{(-1)}b], \quad \{[a], [b]\} = [a_{(0)}b].$$

Derivatives vanish in this quotient. Strong generation has already made R_U finitely generated. We need only show that its positive-weight generators are nilpotent, rather than compute the whole quotient. After affine integrability removes the weight-one generators and the nontrivial irreducible $\mathfrak{g}$ -summands, the only new input is two square identities for the coset invariant vectors. We will prove that the reduced algebra is $\mathbb{C}$: the associated variety is a single point.

The nilradical $\mathfrak{n}$ is a Poisson ideal. This follows from an elementary fact: every derivation D in characteristic zero preserves nilpotence. If $x^N = 0$, the Leibniz expansion gives

$$0 = D^N(x^N) = N!(Dx)^N + xy$$

for some y . Every term except the one differentiating each factor once contains an undifferentiated x . Hence $(Dx)^{N^2} = 0$. Apply this with the derivations $D = \{a, -\}$ [16, Corollary 2.4.2].

Lemma 5.1. *The classes of H, S_2, S_3, S_4 lie in $\mathfrak{n}$.*

Proof. Pass to the reduced Poisson algebra $\overline{R} = R_U/\mathfrak{n}$, and write $\overline{a}$ for the image of a vector. We show that every positive-weight strong generator has zero image.

For an affine raising generator at level k , integrability gives $E_{(-1)}^{k+1}\mathbf{1} = 0$, so $\overline{E} = 0$. The Weyl action gives $\overline{F} = 0$, and $\{\overline{E}, \overline{F}\} = \overline{H}$ gives the Cartan affine generator. Thus every affine generator vanishes in $\overline{R}$.

Let $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. For a nontrivial finite-dimensional irreducible $\mathfrak{g}$ -module M , the submodule $\mathfrak{g}M$ is nonzero, so $M = \mathfrak{g}M$. The same equality holds for finite direct sums of these modules. Every vector in such a sum is therefore a sum of zero-mode images of the weight-one generators. In $\overline{R}$, this action is the Poisson bracket with their images, which are zero. Thus all these vectors vanish in $\overline{R}$, as do derivatives and products of the weight-one generators. The only weight-two images still to consider are the two coset invariant vectors $\overline{e}, \overline{f}$.

The axis certificate gives complete lattice-vector equalities

$$e_{(-1)}e, f_{(-1)}f \in TS_3 + H_{(-1)}S_3 + \mathcal{I}, \tag{5.1}$$

where $\mathcal{I}$ is spanned by invariant normal products of pairs of nontrivial irreducible $\mathfrak{g}$ -modules in S_2 . Every term on the right vanishes in $\overline{R}$. Consequently $\overline{e}^2 = \overline{f}^2 = 0$. The algebra is reduced, so $\overline{e} = \overline{f} = 0$. These two identities suffice; the certificate also checks $e_{(-1)}f$ in the same span. Subsection H.9 explains the construction and exact reduction of the 310 candidate columns. The actual coefficient witnesses and the independent replay are linked by C07. [CALC C07, p. 131]

Finally, S_3 and S_4 are built from H and S_2 by translation, normal products, and zeroth products. These become zero, multiplication, and Poisson brackets in $\overline{R}$. Their images therefore vanish too. This proves the lemma and $\overline{R} = \mathbb{C}$. $\square$

The preceding argument proves nilpotence of the classes; it does not require e or f themselves to belong to $C_2(U)$, nor one uniform nilpotence exponent. Finitely many strong generators permit finitely many individual exponents, which is exactly what the next proof uses.

Theorem 5.2. *The VOA U is C_2 -cofinite.*

Proof. By Theorem 4.4, the classes $x_1, \dots, x_m$ of a finite strong generating set generate R_U as a commutative algebra [16, Corollary 2.6.2]. By Lemma 5.1 [CALC C07, p. 131], choose N_i with $x_i^{N_i} = 0$. There is a surjection

$$\mathbb{C}[X_1, \dots, X_m] / (X_1^{N_1}, \dots, X_m^{N_m}) \longrightarrow R_U.$$

The source is spanned by the finitely many monomials with $0 \leq a_i < N_i$. Hence R_U is finite-dimensional. $\square$

Part III

Semisimplicity of the Zhu algebra

We prove semisimplicity of $A(U)$ through four reductions.

1. Choose 22 Zhu generators adapted to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.
2. Use the affine extension by an invertible module $\mathcal{A} \subset \mathcal{A}^+ \subset U$ to obtain a finite block decomposition.
3. Take the highest-weight corner $D = e_Z A(U) e_Z$, where e_Z is the image in $A(U)$ of the chosen affine idempotent. This full corner is Morita equivalent to $A(U)$.
4. Express the corner generators as normalized highest-weight matrices, convert the VOA relations into path-algebra relations, and prove that the resulting presentation is semisimple.

6 Zhu algebra generators and reduction

6.1 Definition and identities

For completeness, we recall the definition of the Zhu algebra and establish a set of 22 generators of $A(U)$: the six affine generator classes corresponding to $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, together with the classes of sixteen $\mathfrak{g}$ -highest-weight vectors in S_2 .

Definition 6.1 (Zhu algebra). For a homogeneous vector $a \in U$ of conformal weight d and $b \in U$, define

$$a * b = \sum_{j=0}^d \binom{d}{j} a_{(j-1)} b, \quad a \circ b = \sum_{j=0}^d \binom{d}{j} a_{(j-2)} b, \quad (6.1)$$

and extend these operations bilinearly. Let $O(U)$ be the linear span of all $a \circ b$. The *Zhu algebra* is

$$A(U) = U/O(U),$$

with the associative product induced by $*$ and identity [1]. We write $[a]$ for the class of a in $A(U)$. The construction and associativity are established in [17, Definition 2.1.2 and Theorem 2.1.1].

The notation $[a]$ in this part always means a Zhu class. The circle relations remove precisely the vectors acting trivially on lowest spaces, and the star product models composition there. Its lower-mode terms distinguish it from the commutative C_2 product.

The Zhu algebra records the action of the operators $o_M(a)$ on lowest spaces. On the lowest space M_h of an ordinary module M , a homogeneous vector a of weight d acts through

$$o_M(a) = a_{(d-1)}|_{M_h}. \quad (6.2)$$

These operators satisfy

$$o_M(a * b) = o_M(a)o_M(b), \quad o_M(O(U)) = 0,$$

and hence define an $A(U)$ -module structure on M_h [17, Theorem 2.1.2]. Zhu's correspondence says that simple lowest spaces determine simple generated modules up to isomorphism [20, Section 2.2]. This correspondence does not require rationality of U .

General Zhu identities. Let $T = L_{-1}$ denote the translation operator. For homogeneous vectors $u, v \in U$, with $d = \text{wt}(u)$, the definition of the Zhu product gives

$$[u] * [v] = \sum_{j \geq 0} \binom{d}{j} [u_{(j-1)}v]; \quad (6.3)$$

see [17, Definition 2.1.2, equation (2.1.1)]. The translation and commutator identities are

$$[Tu] = -d[u], \quad (6.4)$$

$$[u] * [v] - [v] * [u] = \sum_{j \geq 0} \binom{d-1}{j} [u_{(j)}v]. \quad (6.5)$$

Equation (6.4) is [17, Lemma 2.1.1], which states that $Tu + L_0u \in O(U)$. Equation (6.5) is the mode expansion of [17, Lemma 2.1.3, equation (2.1.4)]. All sums are finite by truncation.

Specialization to affine generators and weight-two vectors. Let $x \in U$ be homogeneous. The following specializations are the formulas used in the later relation conversion:

Operation	Class in $A(U)$
Translation	$[Tx] = -\text{wt}(x)[x]$
Affine generator normal product	$[h_{(-1)}x] = [x] * [h]$
Weight-two zeroth product	$[a_{(0)}x] = [a] * [x] - [x] * [a] - [a_{(1)}x]$
Weight-two normal product	$[a_{(-1)}x] = [a] * [x] - 2[a_{(0)}x] - [a_{(1)}x]$

The last terms are genuine corrections in the same algebra; they are retained in Subsection K.1.

For an affine generator $h \in H$, we have $\text{wt}(h) = 1$. Equations (6.3) and (6.5) therefore become

$$[h] * [x] = [h_{(-1)}x] + [h_{(0)}x], \quad [h] * [x] - [x] * [h] = [h_{(0)}x].$$

Consequently,

$$[h_{(-1)}x] = [h] * [x] - [h_{(0)}x], \quad (6.6)$$

$$[h_{(0)}x] = [h] * [x] - [x] * [h]. \quad (6.7)$$

In particular, affine generator zeroth products become commutators in $A(U)$.

For $a \in S_2$, we have $\text{wt}(a) = 2$. The same general identities now give

$$[a] * [x] = [a_{(-1)}x] + 2[a_{(0)}x] + [a_{(1)}x],$$

$$[a] * [x] - [x] * [a] = [a_{(0)}x] + [a_{(1)}x].$$

Rearranging yields the two formulas needed below:

$$[a_{(0)}x] = [a] * [x] - [x] * [a] - [a_{(1)}x], \quad (6.8)$$

$$[a_{(-1)}x] = [a] * [x] - 2[a_{(0)}x] - [a_{(1)}x]. \quad (6.9)$$

6.2 Why weight one and two suffice

Let $z_0, \dots, z_{15}$ be the ordered highest-weight vectors of S_2 used in calculation C03. The vectors $z_0, \dots, z_{15}$ are listed explicitly, with the certificate ordering and normalizations, in Appendix A. For example,

$$z_0 = TE_s, \quad z_1 = (H_s)_{(-1)}H_s, \quad z_5 = TE_t,$$

and the certificate-normalized seed is

$$z_9 = e^{\beta_1} + \left(\frac{11}{10} - \frac{3\theta^2}{40}\right)e^{\beta_2} - \left(\frac{\theta^3}{32} + \frac{3\theta}{8}\right)e^{\beta_3}.$$

The certificate constructs an ordered basis of S_2 by exact lattice mode products and normalization at coordinate pivots over K . It then obtains the z_a by exact row reduction of the simultaneous equations $(E_s)_{(0)}z = (E_t)_{(0)}z = 0$ in each Cartan-weight subspace; the precise normalization procedure is given in Appendix A.4. Their zero-mode lowering vectors span S_2 . **[CALC C03, p. 130]**

The sixteen labels count highest-weight representatives. Applying the zero-mode lowering operators spans their irreducible $\mathfrak{g}$ -modules. Together with the six affine generators they give an associative generating list, not a minimality claim. In particular this count is different from the 429 strong generators: the next proof expresses the Zhu classes of the weight-three and weight-four strong generators using weight-one and weight-two classes.

Proposition 6.2 (Twenty-two Zhu generators). *The Zhu algebra $A(U)$ is generated by the six affine generator classes*

$$[E_s], [F_s], [H_s], [E_t], [F_t], [H_t]$$

and the sixteen highest-weight classes

$$[z_0], \dots, [z_{15}].$$

Equivalently,

$$A(U) = \langle [E_s], [F_s], [H_s], [E_t], [F_t], [H_t], [z_0], \dots, [z_{15}] \rangle_{\text{alg}}. \quad (6.10)$$

Proof. Let $\mathcal{B} \subset A(U)$ be the unital subalgebra generated by the classes of H and S_2 . Recall the defining formulas for the next two spaces:

$$\begin{aligned} S_3 &= TS_2 + H_{(-1)}S_2 + (S_2)_{(0)}S_2, \\ S_4 &= TS_3 + H_{(-1)}S_3 + (S_2)_{(-1)}S_2 + (S_2)_{(0)}S_3. \end{aligned} \quad (6.11)$$

The second formula is the definition in (4.1), with N_4 expanded. The finite mode closure of Proposition 4.1 gives, in particular,

$$H_{(0)}S_i \subset S_i, \quad (S_2)_{(1)}S_i \subset S_i, \quad i = 2, 3. \quad (6.12)$$

First consider S_3 . Equation (6.4) puts the classes of TS_2 in $\mathcal{B}$. Equation (6.6), together with $H_{(0)}S_2 \subset S_2$, does the same for $H_{(-1)}S_2$. Finally, (6.8) expresses the class of $a_{(0)}b$, for $a, b \in S_2$, as a commutator of classes in $\mathcal{B}$ minus $[a_{(1)}b]$. The correction lies in $\mathcal{B}$ because $a_{(1)}b \in S_2$. The first line of (6.11) therefore shows that $[S_3] \subset \mathcal{B}$.

Now consider S_4 . The same identities, using $[S_3] \subset \mathcal{B}$ and (6.12), put the classes of TS_3 , $H_{(-1)}S_3$, and $(S_2)_{(0)}S_3$ in $\mathcal{B}$. For the remaining summand, (6.9) expresses $[a_{(-1)}b]$, with $a, b \in S_2$, in terms of $[a] * [b]$, $[a_{(0)}b]$, and $[a_{(1)}b]$. All three terms already belong to $\mathcal{B}$. Hence $[S_4] \subset \mathcal{B}$.

By Theorem 4.4, U has a strong generating set contained in $H + S_2 + S_3 + S_4$. The classes of a strong generating set generate the Zhu algebra.

To see this directly, replace a derivative using $[Ta] = -\text{wt}(a)[a]$, and replace a normal product by the star product minus its nonnegative-mode terms. Those correction vectors have lower total conformal weight. Induction on that weight therefore expresses every normal word as a polynomial in the strong generator classes. Since all these classes lie in $\mathcal{B}$, we obtain $\mathcal{B} = A(U)$.

It remains to reduce the classes of S_2 to the sixteen highest-weight classes. Repeated application of (6.7) gives

$$[(F_s)_{(0)}^p (F_t)_{(0)}^q z_i] = \text{ad}([F_s])^p \text{ad}([F_t])^q [z_i],$$

where $\text{ad}(u)(v) = u * v - v * u$. The zero-mode lowering vectors of the z_i span S_2 , so every class in $[S_2]$ belongs to the algebra generated by the six affine generator classes and $[z_0], \dots, [z_{15}]$. These 22 classes therefore generate $A(U)$. $\square$

Consequently, irreducibility and homomorphisms of lowest spaces can be checked against a fixed set of 22 matrices.

7 An affine extension makes the Zhu problem smaller

The 22 generators still act on large irreducible $\mathfrak{g}$ -modules. A rational affine extension supplies matrix units that remove this repeated $\mathfrak{g}$ -module structure. Keeping one highest line in each source block produces a full Morita corner: it retains every simple module and every possible radical, while greatly reducing the matrix sizes.

7.1 Finding the invertible affine module inside U

Put

$$w = (s + t)/2 = (1, 1, 3, 5, 5, 9, 7, 4), \quad (w, w) = 20.$$

Write $v' = u_1 + \gamma_2 u_2 + \gamma_3 u_3$, with the nonzero coefficients of (E.2). The three seed lattice vectors have pairwise inner product two, and $\beta_1 + \beta_2 + \beta_3 = w + s$. In $(e^a)_{(n)} e^b$, the required Heisenberg degree is $-n - 1 - (a, b)$. The mode -3 therefore keeps only distinct seed pairs, with no Heisenberg-mode factors. The next mode, -5 , keeps only the third distinct lattice vector. This leaves the six orders of the same triple. Finally $(F_s)_{(2)}$ removes s , again without Heisenberg-mode factors. The cocycle gives the same final sign, $+1$, in all six terms, so

$$(F_s)_{(2)}(v'_{(-5)}(v'_{(-3)}v')) = 6\gamma_2\gamma_3 e^w \neq 0.$$

Thus $e^w \in U$.

The intermediate conformal weights are 6, 12, and 10: for homogeneous vectors, $\text{wt}(a_{(n)}b) = \text{wt}(a) + \text{wt}(b) - n - 1$. Thus the final vector has exactly the weight $(w, w)/2$ of the pure exponential. The Heisenberg-degree calculation explains why no unseen Heisenberg terms accompany it; Subsection J.1 checks every surviving order. Appendix J records the cocycle check and the original exact coefficient. **[CALC C10, p. 131]**

The Cartan eigenvalues show that e^w is affine highest of type $(1, 39)$. Its affine Sugawara weight is

$$h_{\mathcal{A}}(r, j) = \frac{r(r+2)}{12} + \frac{j(j+2)}{164}, \quad h_{\mathcal{A}}(1, 39) = 10 = \text{wt}(e^w).$$

Thus $L_0^C e^w = 0$. **[CALC C10, p. 131]**

Proposition 7.1. *For $C = \text{Com}(\mathcal{A}, U)$, the double commutant is the actual extension by an invertible module*

$$\text{Com}(C, U) = \mathcal{A}^+ = \mathcal{A} \oplus J_{(1,39)}. \quad (7.1)$$

It is rational and regular.

Proof. First, L_0^C has nonnegative eigenvalues. On the multiplicity space of an affine highest weight, L_0^C equals the total L_0 minus the fixed affine lowest weight, so its spectrum is discrete and bounded below. The affine modes commute with L_0^C . A vector x of least L_0^C -eigenvalue is killed by L_1^C , and $\|L_{-1}^C x\|^2 = 2\langle L_0^C x, x \rangle$ excludes a negative least eigenvalue. On the zero eigenspace, the same identity gives $L_{-1}^C x = 0$. For $c \in C$, all sufficiently large nonnegative modes act by zero on x ; descending with $[L_{-1}^C, c_{(n)}] = -nc_{(n-1)}$ shows that all nonnegative modes act by zero on x . Conversely, the commutant condition for ω_C forces $L_0^C x = 0$. Hence the zero eigenspace is exactly $\text{Com}(C, U)$.

It remains to identify the affine summands in this zero eigenspace. There are two separate restrictions: the affine lowest weight must be an integer, and a highest lattice vector at that weight must realize the required Cartan eigenvalues with no excess norm.

Its affine highest weights must have integral $h_{\mathcal{A}}(r, j)$, with $r = 0, 1$ and $0 \leq j \leq 39$. Reduction modulo 41 leaves $j = 0, 39$; checking r leaves only $(0, 0)$ and $(1, 39)$. The vacuum multiplicity is one. A lattice vector with Cartan eigenvalues $(1, 39)$ has orthogonal projection w onto $\text{span}\{s, t\}$. Its norm is at least 20, with equality only for w . At total weight ten the remaining Heisenberg degree is zero. Thus the $(1, 39)$ highest-weight multiplicity is also one, supplied by (J.1).

The module $J_{(1,39)}$ is an order-two invertible module of integral weight ten. For rationality use [18, Theorem 4.5 and Proposition 5.1]. The base $\mathcal{A}$ is rational, the group of summands has order two, and the nontrivial summand is the tensor product of the two affine invertible modules obtained by the Cartan shifts $H_s/2, H_t/2$. Their zero modes are semisimple with integral eigenvalues on $\mathcal{A}$, and their tensor product has integral lowest weight ten, as already checked. The actual inherited multiplication provides the VOA extension required by that theorem. Finally $\mathcal{A}$ is regular and

has the same Sugawara conformal vector as $\mathcal{A}^+$. Hence [13, Proposition 3.4] upgrades the rational extension to a regular one. These are properties of the affine extension, before rationality of U is known. $\square$

There is a conformal-vector issue to check before using its matrix algebra inside $A(U)$. The inclusion does not send the affine Sugawara vector to ω_U . What matters for the Zhu formulas, however, is the weight and translation of each vector in the included algebra. The double-commutant condition makes their coset contributions zero, as the following proof shows.

Proposition 7.2 (The homomorphism of Zhu algebras). *The inclusion $\iota: \mathcal{A}^+ \hookrightarrow U$ induces a unital homomorphism of associative algebras*

$$\Phi: A(\mathcal{A}^+) \longrightarrow A(U), \quad [a]_{\mathcal{A}^+} \longmapsto [a]_U.$$

Proof. The coset conformal vector

$$\omega_C = \omega_U - \omega_{\mathcal{A}}$$

belongs to C . For $a \in \mathcal{A}^+ = \text{Com}(C, U)$, the commutant condition gives

$$c_{(n)}a = 0 \quad (c \in C, n \geq 0).$$

Applying this to $c = \omega_C$, and writing $L_n^C = (\omega_C)_{(n+1)}$, gives

$$L_{-1}^C a = 0, \quad L_0^C a = 0.$$

Consequently,

$$L_0^U a = L_0^{\mathcal{A}^+} a, \quad T_U a = T_{\mathcal{A}^+} a.$$

Thus the conformal grading and translation on $\mathcal{A}^+$ agree with those inherited from U .

The vertex operators on $\mathcal{A}^+$ are also inherited from U . Hence, for homogeneous $a \in \mathcal{A}^+$ of common conformal weight d , and $b \in \mathcal{A}^+$, the defining Zhu products satisfy

$$\begin{aligned} a *_{\mathcal{A}^+} b &= \sum_{j=0}^d \binom{d}{j} a_{(j-1)} b = a *_{\mathcal{U}} b, \\ a \circ_{\mathcal{A}^+} b &= \sum_{j=0}^d \binom{d}{j} a_{(j-2)} b = a \circ_{\mathcal{U}} b. \end{aligned}$$

Since $O(\mathcal{A}^+)$ is spanned by these circle products,

$$\iota(O(\mathcal{A}^+)) \subseteq O(U).$$

The inclusion therefore descends to the stated map

$$\Phi: \mathcal{A}^+/O(\mathcal{A}^+) \longrightarrow U/O(U).$$

The equality of star products proves that Φ preserves multiplication, and the common vacuum gives $\Phi([1]_{\mathcal{A}^+}) = [1]_U$. $\square$

7.2 The $\mathfrak{g}$ -module structure of $A(\mathcal{A}^+)$

Definition 7.3 (Module notation). These are modules for the regular affine extension. They supply a coordinate system for the subsequent Zhu calculation, not yet a list of ordinary U -modules.

Let

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2,$$

with the two ideals corresponding, respectively, to the affine factors of levels 1 and 39.

Let $V(m)$ denote the simple finite-dimensional $\mathfrak{sl}_2$ -module of highest weight m , and set

$$V(r, j) := V(r) \otimes V(j), \quad \dim V(r, j) = (r+1)(j+1).$$

The two ideals of $\mathfrak{g}$ act on the respective tensor factors.

Let $L_k(m)$ denote the simple $L_k(\mathfrak{sl}_2)$ -module with finite highest weight m , where $0 \leq m \leq k$. Put

$$\mathcal{L}_{r,j} := L_1(r) \otimes L_{39}(j), \quad r \in \{0, 1\}, \quad 0 \leq j \leq 39.$$

These are the simple $\mathcal{A}$ -modules. Their least L_0 -eigenvalues and corresponding eigenspaces are

$$h_{\mathcal{A}}(r, j) = \frac{r(r+2)}{12} + \frac{j(j+2)}{164}, \quad (\mathcal{L}_{r,j})_{h_{\mathcal{A}}(r,j)} \cong V(r, j).$$

Write $\boxtimes_{\mathcal{A}}$ for the tensor product in the category of ordinary $\mathcal{A}$ -modules.

Lemma 7.4 (Simple modules of the extension). *The module $\mathcal{L}_{r,j}$ induces an ordinary $\mathcal{A}^+$ -module precisely when $r+j$ is even. Its induced module is then simple.*

A complete set of pairwise nonisomorphic simple ordinary $\mathcal{A}^+$ -modules is

$$M_t := \mathcal{A}^+ \boxtimes_{\mathcal{A}} \mathcal{L}_{0,t}, \quad t \in I := \{0, 2, \dots, 38\}.$$

Their restrictions satisfy

$$M_t|_{\mathcal{A}} \cong \mathcal{L}_{0,t} \oplus \mathcal{L}_{1,39-t}. \tag{7.2}$$

Thus the twenty modules correspond to the orbits $\{(0, t), (1, 39-t)\}$.

Proof. The module $J := \mathcal{L}_{1,39}$ is invertible of order two, and

$$J \boxtimes_{\mathcal{A}} \mathcal{L}_{r,j} \cong \mathcal{L}_{1-r,39-j}.$$

Its least L_0 -eigenvalue is 10. The induction criterion for this extension [18, 19] therefore reduces to

$$h_{\mathcal{A}}(1-r, 39-j) - h_{\mathcal{A}}(r, j) \in \mathbb{Z}.$$

The difference is

$$10 - \frac{r+j}{2},$$

so the criterion is equivalent to $r+j$ being even.

No simple affine module is fixed by tensoring with J , since $r \neq 1-r$ for $r \in \{0, 1\}$. The extension induction theorem consequently gives one simple ordinary module for each orbit satisfying this criterion, and every simple ordinary extension module arises in this way. Each orbit has a unique representative $(0, t)$ with t even, giving the stated list and restriction formula. $\square$

Definition 7.5 (Lowest spaces and block labels). For $t \in I$, define

$$h_t = \min\{h_{\mathcal{A}}(0, t), h_{\mathcal{A}}(1, 39 - t)\}, \quad T_t := (M_t)_{h_t}, \quad \mathcal{B}_t := \text{End}_{\mathbb{C}}(T_t).$$

The index t labels the source block $\mathcal{B}_t$.

Lemma 7.6 (Decomposition of the lowest spaces). *The zero modes of the six weight-one generators give the following $\mathfrak{g}$ -module decomposition:*

$$T_t \cong \begin{cases} V(0, t), & t = 0, 2, \dots, 18, \\ V(0, 20) \oplus V(1, 19), & t = 20, \\ V(1, 39 - t), & t = 22, 24, \dots, 38. \end{cases}$$

Proof. The least eigenvalues of the two affine summands in (7.2) differ by

$$h_{\mathcal{A}}(1, 39 - t) - h_{\mathcal{A}}(0, t) = 10 - \frac{t}{2}.$$

The first summand therefore has smaller least eigenvalue when $t < 20$, and the second when $t > 20$. Their least eigenvalues coincide only when $t = 20$.

Vectors of positive grade in each affine summand have strictly greater eigenvalue than the least eigenvalue of their summand. Hence the least eigenspace of M_t is the least eigenspace of the smaller summand, or the direct sum of both least eigenspaces in the equality case. $\square$

The following table records the $\mathfrak{g}$ -module structures used in the next section.

Table 1: The twenty source blocks and their simple modules restricted to $\mathfrak{g}$.

t	T_t as a $\mathfrak{g}$ -module	$\mathcal{B}_t$
0	$V(0, 0)$	$\mathbb{C}$
2	$V(0, 2)$	$M_3(\mathbb{C})$
4	$V(0, 4)$	$M_5(\mathbb{C})$
6	$V(0, 6)$	$M_7(\mathbb{C})$
8	$V(0, 8)$	$M_9(\mathbb{C})$
10	$V(0, 10)$	$M_{11}(\mathbb{C})$
12	$V(0, 12)$	$M_{13}(\mathbb{C})$
14	$V(0, 14)$	$M_{15}(\mathbb{C})$
16	$V(0, 16)$	$M_{17}(\mathbb{C})$
18	$V(0, 18)$	$M_{19}(\mathbb{C})$
20	$V(0, 20) \oplus V(1, 19)$	$M_{61}(\mathbb{C})$
22	$V(1, 17)$	$M_{36}(\mathbb{C})$
24	$V(1, 15)$	$M_{32}(\mathbb{C})$
26	$V(1, 13)$	$M_{28}(\mathbb{C})$
28	$V(1, 11)$	$M_{24}(\mathbb{C})$
30	$V(1, 9)$	$M_{20}(\mathbb{C})$
32	$V(1, 7)$	$M_{16}(\mathbb{C})$
34	$V(1, 5)$	$M_{12}(\mathbb{C})$

t	T_t as a $\mathfrak{g}$ -module	$\mathcal{B}_t$
36	$V(1, 3)$	$M_8(\mathbb{C})$
38	$V(1, 1)$	$M_4(\mathbb{C})$

Remark 7.7 (The exceptional block). Only the simple Zhu module T_{20} becomes reducible upon restriction to $\mathfrak{g}$. Its decomposition is

$$T_{20} \cong V(0, 20) \oplus V(1, 19),$$

with both summands occurring once. Their least L_0 -eigenvalues agree:

$$h_{\mathcal{A}}(0, 20) = h_{\mathcal{A}}(1, 19) = \frac{110}{41}.$$

Their dimensions are 21 and 40, respectively.

Consequently, the exceptional source block is the single simple algebra

$$\mathcal{B}_{20} = \text{End}_{\mathbb{C}}(V(0, 20) \oplus V(1, 19)) \cong M_{61}(\mathbb{C}),$$

including all linear maps between the two $\mathfrak{g}$ -summands.

Knowing the $\mathfrak{g}$ -summands of T_t allows us to describe $A(U)$ as an associative algebra.

Reading the exceptional block. At $t = 20$ the two affine constituents have the same lowest weight. Their sum is one simple $\mathcal{A}^+$ -top of dimension $21 + 40 = 61$, so its Zhu action is the full 61 by 61 matrix algebra, including maps between the constituents. A rank-one line in its $(0, 20)$ constituent consequently gives a full corner of this entire block. It is not necessary to choose a second vertex for $(1, 19)$. Appendix J describes the projector selecting that line; the matrix-unit argument in the next section proves fullness.

8 Matrix coordinates and normalized numerical products

The affine extension supplies matrix units in $A(U)$. They separate a known numerical matrix factor from the remaining coefficients in the full corner D . This section fixes that separation, the normalization of the highest-weight matrices, and the numerical products used later in the relation calculation. The next section replaces the coefficients by formal arrows and extracts the path relations.

8.1 The full corner and its matrix coordinates

For the quiver notation in this section, the vertices are indexed by $t = 0, \dots, 19$: vertex t denotes the affine summand previously indexed by $2t$. Thus the exceptional affine label 20 becomes vertex 10.

The affine Zhu algebra decomposes as

$$A(\mathcal{A}^+) \cong \bigoplus_{v=0}^{19} \text{End}(T_v).$$

For each block v , choose elementary matrices $\tilde{E}_{ij}^{(v)}$ and write $E_{ij}^{(v)} := \Phi(\tilde{E}_{ij}^{(v)}) \in A(U)$ for their images. Some of these matrix blocks may have zero image.

Definition 8.1 (The Zhu corner). Put $e_v = E_{11}^{(v)}$ and $e_Z = \sum_{v=0}^{19} e_v \in A(U)$. The Zhu corner is

$$D := e_Z A(U) e_Z.$$

For the exceptional block $v = 10$ we order its constituents as $V(0, 20) \oplus V(1, 19)$ and select the highest line of $V(0, 20)$ for $E_{11}^{(10)}$. This is one rank-one idempotent in the full 61-dimensional source block; it does not split that block into two vertices. Appendix J.2 explains the source projectors and the cross-constituent matrix units.

The following observation reduces semisimplicity of $A(U)$ to semisimplicity of its smaller corner D .

Lemma 8.2. *The idempotent e_Z is a full idempotent:*

$$A(U) e_Z A(U) = A(U).$$

In particular, D is Morita equivalent to $A(U)$ and so if D is semisimple, then $A(U)$ is as well.

Proof. Fullness follows from the identity

$$1 = \sum_{v,i} E_{ii}^{(v)} = \sum_{v,i} E_{i,1}^{(v)} e_Z E_{1,i}^{(v)}.$$

□

We now reconstruct $A(U)$ from the numerical matrix spaces and the corners of D .

For $a \in U$ and blocks w, v of $A(\mathcal{A}^+)$, define

$$c_{w \rightarrow v}^{i,j}(a) := E_{1i}^{(v)} [a] E_{j1}^{(w)} \in e_v D e_w.$$

The full class can be recovered from these entries:

$$[a] = \sum_{v,w,i,j} E_{i1}^{(v)} c_{w \rightarrow v}^{i,j}(a) E_{1j}^{(w)}.$$

Collecting the corner entries gives

$$X_{w \rightarrow v}([a]) \in \text{Hom}_{\mathbb{C}}(T_w, T_v) \otimes_{\mathbb{C}} (e_v D e_w).$$

As the ordered pair of blocks varies, these maps preserve multiplication and the $\mathfrak{g}$ -action, as proved below.

The first tensor factor records where an entry sits in a known affine matrix; the second records its unknown value in D . For example, a contribution $E_{23}^{v,w} \otimes d$ means the actual element $E_{21}^{(v)} d E_{13}^{(w)}$. Multiplying it by $E_{31}^{w,s} \otimes c$ gives $E_{21}^{v,s} \otimes dc$. Thus the usual matrix-entry calculation and the unknown coefficient multiplication are separate operations. This lets us calculate numerical matrices of the affine generators before knowing the simple blocks of U .

Proposition 8.3 (Matrix realization of $A(U)$). *The corner entries define an isomorphism of associative algebras*

$$X: A(U) \xrightarrow{\sim} \bigoplus_{t,v} \text{Hom}_{\mathbb{C}}(T_t, T_v) \otimes_{\mathbb{C}} e_v D e_t.$$

For $F \in \text{Hom}_{\mathbb{C}}(T_t, T_v)$, $G \in \text{Hom}_{\mathbb{C}}(T_s, T_t)$, $d \in e_v D e_t$, and $c \in e_t D e_s$, multiplication is

$$(F \otimes d)(G \otimes c) = (F \circ G) \otimes dc.$$

Products with nonmatching intermediate blocks are zero. The unit is $\sum_t \text{id}_{T_t} \otimes e_t$.

This isomorphism is $\mathfrak{g}$ -equivariant, where $\mathfrak{g}$ acts on $A(U)$ by the Zhu commutator and on the right-hand side by

$$x \cdot (F \otimes d) = (\rho_v(x)F - F\rho_t(x)) \otimes d.$$

In particular, the factors $e_v D e_t$ carry the trivial action in this realization.

The last assertion specifies the action transported by X : the affine generator acts on the numerical Hom space and leaves the coefficient slot fixed. It is a statement about this realization, whose verification follows below; no independent Lie algebra action on an arbitrary abstract corner is being assumed.

Proof. The images of the affine matrix units satisfy

$$E_{ij}^{(t)} E_{kl}^{(t)} = \delta_{jk} E_{il}^{(t)}, \quad E_{ij}^{(v)} E_{kl}^{(t)} = 0 \quad (v \neq t), \quad \sum_{t,i} E_{ii}^{(t)} = 1.$$

These identities remain valid even when a source block vanishes in $A(U)$.

Write $E_{ij}^{v,t} \in \text{Hom}_{\mathbb{C}}(T_t, T_v)$ for the numerical matrix unit in the chosen bases. For $b \in A(U)$, its corner matrix is

$$X_{t \rightarrow v}(b) = \sum_{i,j} E_{ij}^{v,t} \otimes E_{1i}^{(v)} b E_{j1}^{(t)}.$$

The second factor belongs to $e_v D e_t$, since

$$e_v E_{1i}^{(v)} = E_{1i}^{(v)}, \quad E_{j1}^{(t)} e_t = E_{j1}^{(t)}, \quad e_v A(U) e_t = e_v D e_t.$$

Thus these matrices define the stated linear map X .

Define a linear map in the opposite direction by

$$\Psi(E_{ij}^{v,t} \otimes d) = E_{i1}^{(v)} d E_{1j}^{(t)} \quad (d \in e_v D e_t).$$

Equivalently, for a numerical matrix $F = (F_{ij})$,

$$\Psi(F \otimes d) = \sum_{i,j} F_{ij} E_{i1}^{(v)} d E_{1j}^{(t)}.$$

For $b \in A(U)$, the matrix-unit identities give

$$\begin{aligned} \Psi(X(b)) &= \sum_{v,t,i,j} E_{i1}^{(v)} (E_{1i}^{(v)} b E_{j1}^{(t)}) E_{1j}^{(t)} \\ &= \sum_{v,t,i,j} E_{ii}^{(v)} b E_{jj}^{(t)} = b. \end{aligned}$$

Conversely, extracting the corner entries of $\Psi(E_{ij}^{v,t} \otimes d)$ gives

$$E_{1k}^{(w)} (E_{i1}^{(v)} d E_{1j}^{(t)}) E_{\ell 1}^{(s)} = \delta_{wv} \delta_{st} \delta_{ki} \delta_{\ell j} d.$$

Hence $X \circ \Psi$ is also the identity, proving that X and Ψ are mutually inverse.

We next check multiplication. For composable tensors $F \otimes d$ and $G \otimes c$ as in the statement,

$$\begin{aligned}\Psi(F \otimes d)\Psi(G \otimes c) &= \sum_{i,j,k,\ell} F_{ij}G_{k\ell}E_{i1}^{(v)}d(E_{1j}^{(t)}E_{k1}^{(t)})cE_{1\ell}^{(s)} \\ &= \sum_{i,j,\ell} F_{ij}G_{j\ell}E_{i1}^{(v)}dcE_{1\ell}^{(s)} \\ &= \Psi((F \circ G) \otimes dc),\end{aligned}$$

where we used $E_{1j}^{(t)}E_{k1}^{(t)} = \delta_{jk}e_t$ and $de_tc = dc$. If the intermediate blocks differ, the corresponding product of matrix units is zero. This proves the multiplication formula. Moreover,

$$\Psi\left(\sum_t \text{id}_{T_t} \otimes e_t\right) = \sum_{t,i} E_{i1}^{(t)}e_tE_{1i}^{(t)} = \sum_{t,i} E_{ii}^{(t)} = 1.$$

Thus X is an isomorphism of unital associative algebras.

Finally, let $x \in \mathfrak{g}$. In the source Zhu algebra, the affine generator class acts on T_t by $\rho_t(x)$. Its image in $A(U)$ therefore satisfies

$$[x] = \sum_{t,i,j} \rho_t(x)_{ij}E_{ij}^{(t)}.$$

In particular, its corner entries are

$$E_{1i}^{(v)}[x]E_{j1}^{(t)} = \begin{cases} \rho_t(x)_{ij}e_t, & v = t, \\ 0, & v \neq t. \end{cases}$$

Consequently,

$$X([x]) = \sum_t \rho_t(x) \otimes e_t.$$

Since X preserves multiplication, it also preserves commutators. On the (v, t) tensor block, this gives

$$\begin{aligned}\left[\sum_u \rho_u(x) \otimes e_u, F \otimes d\right] &= \rho_v(x)F \otimes e_vd - F\rho_t(x) \otimes de_t \\ &= (\rho_v(x)F - F\rho_t(x)) \otimes d.\end{aligned}$$

This is precisely the usual action on $\text{Hom}_{\mathbb{C}}(T_t, T_v)$ with trivial action on the multiplicity factor e_vDe_t .

This proves the asserted equivariance. $\square$

Each T_t is irreducible as a $\mathfrak{g}$ -module except for the exceptional block, which has two constituents. We keep these constituent labels in every numerical matrix.

Notation 8.1. Recall that $T_t \cong \bigoplus_{\lambda \in \Lambda_t} V(\lambda)$ and so there is a decomposition of $\mathfrak{g}$ -modules:

$$\text{Hom}_{\mathbb{C}}(T_t, T_v) \cong \bigoplus_{\lambda \in \Lambda_t, \mu \in \Lambda_v} \text{Hom}_{\mathbb{C}}(V(\lambda), V(\mu)).$$

Here Λ_x denotes the highest weights of T_x for a block x .

For $b \in A(U)$, write $X_{t \rightarrow v}^{\lambda, \mu}(b)$ for the component of $X_{t \rightarrow v}(b)$ in $\text{Hom}_{\mathbb{C}}(V(\lambda), V(\mu)) \otimes e_vDe_t$.

8.2 Normalized highest-weight matrices

To extract coefficients in D from the matrix realization, we must fix the numerical matrix in each highest-weight line. The normalization removes the scalar ambiguity and makes both coefficient extraction and numerical multiplication explicit. Recall that

$$\mathfrak{g} = A_1 \oplus A_1, \quad A_1 = \mathfrak{sl}_2.$$

As motivation for the two-factor construction, we first describe the normalization for one copy of A_1 .

Definition 8.4 (Normalization of highest-weight matrices). In each irreducible A_1 -module $V(n)$, use the fixed unnormalized lowering basis

$$v_0, f v_0, \dots, f^n v_0.$$

The space $\text{Hom}_K(V(n), V(m))$ contains a one-dimensional highest-weight line of weight ℓ precisely when

$$\ell = |m - n|, |m - n| + 2, \dots, m + n.$$

Put

$$\delta = \frac{\ell - m + n}{2}.$$

Normalize its highest-weight matrix by requiring that the entry in row 1, column $\delta + 1$ is 1. Its potentially nonzero entries lie in positions $(i + 1, i + \delta + 1)$, for $0 \leq i \leq \min(m, n - \delta)$. Writing these entries as r_i , they are determined by

$$r_0 = 1, \quad r_{i+1} = r_i \frac{(i + \delta + 1)(n - i - \delta)}{(i + 1)(m - i)}, \quad 0 \leq i < \min(m, n - \delta). \quad (8.1)$$

All other entries are zero.

Rows refer to the target lowering basis and columns to the source lowering basis. Matrix entries are numbered from one, while the powers f^i start at zero. For example, $C_{4 \rightarrow 4}^2$ has entries $1, 3/2, 3/2, 1$ in positions $(1, 2), (2, 3), (3, 4), (4, 5)$. The entry $(1, 3)$ of its square is $1 \cdot 3/2$, so $C_{4 \rightarrow 4}^2 C_{4 \rightarrow 4}^2 = (3/2) C_{4 \rightarrow 4}^4$.

For $\mathfrak{g} = A_1 \oplus A_1$, take the tensor product of the two normalized one-factor matrices, with the second factor varying fastest in the ordered tensor-product basis.

Notation 8.2 (Normalized highest-weight matrices). Denote the normalized one-factor matrix by $C_{n \rightarrow m}^\ell$. For $\lambda \in \Lambda_t$, $\mu \in \Lambda_v$, and $\nu = (\nu_1, \nu_2)$, write

$$C_{t \rightarrow v}^{\lambda, \mu}(\nu) = C_{\lambda_1 \rightarrow \mu_1}^{\nu_1} \otimes C_{\lambda_2 \rightarrow \mu_2}^{\nu_2} \in \text{Hom}_K(V(\lambda), V(\mu)),$$

whenever both one-factor matrices exist. This is the normalized highest-weight matrix of weight ν in the indicated constituent pair. The constituent labels are retained separately, including both constituents of the exceptional block.

Lemma 8.5 (Highest-weight coefficient factorization). Let $[z] \in A(U)$ be highest of $\mathfrak{g}$ -weight ν . For each constituent pair on which type ν occurs, there is a unique coefficient $\xi_{t \rightarrow v}^{\lambda, \mu}(z) \in e_v D e_t$ such that

$$X_{t \rightarrow v}^{\lambda, \mu}([z]) = C_{t \rightarrow v}^{\lambda, \mu}(\nu) \otimes \xi_{t \rightarrow v}^{\lambda, \mu}(z).$$

If type ν does not occur, that component is zero. An admissible coefficient is allowed to be zero.

Proof. The matrix realization is $\mathfrak{g}$ -equivariant, and the coefficient factor carries the trivial $\mathfrak{g}$ -action. The numerical factor therefore lies in the corresponding highest-weight line. The fixed normalization determines its coefficient uniquely. $\square$

The normalization also fixes the numerical multiplication table. A product of two highest-weight matrices is either zero or a scalar multiple of the normalized matrix of the summed weight. When that output type occurs, the scalar is read at the entry where the output matrix is normalized to 1. For $A_1 \oplus A_1$, the structure coefficient is the product of the two one-factor coefficients. We state the resulting multiplication table in Subsection 8.3; the detailed matrix construction and normalization conventions are given in Appendix L.

8.3 The numerical highest-weight algebra

Put

$$\mathcal{M}_{\text{num}} = \bigoplus_{t,v} \text{Hom}_K(T_t, T_v),$$

with ordinary block composition. Its highest-weight subalgebra is

$$\mathcal{H}_{\text{num}} = \{F \in \mathcal{M}_{\text{num}} : E_s \cdot F = E_t \cdot F = 0\}.$$

It is closed under composition because the Lie algebra action is a derivation. A basis consists of all the normalized matrices $C_{t \rightarrow v}^{\lambda, \mu}(\nu)$ for all occurring highest weights, not only the weights of the sixteen chosen vectors.

For one Lie algebra factor, the multiplication rule for composable matrices with admissible output weight is

$$\boxed{C_{m \rightarrow p}^b C_{n \rightarrow m}^a = \frac{\binom{r+s}{r} \binom{n-s}{r}}{\binom{m}{r}} C_{n \rightarrow p}^{a+b}} \quad (8.2)$$

$$r = \frac{b - p + m}{2}, \quad s = \frac{a - m + n}{2}.$$

If the output weight is inadmissible, the product is zero. Products with mismatching intermediate constituent labels are zero. For two Lie algebra factors, multiply the two one-factor structure coefficients.

Proof. The product is highest of weight $a + b$. Its highest-weight space in $\text{Hom}_K(V(n), V(p))$ is at most one-dimensional. The normalized output entry is in row 1, column $r + s + 1$. At that position the product has the single contribution from row $r + 1$, column $r + s + 1$ of the right-hand matrix. Iterating (8.1) gives this entry as $\binom{r+s}{r} \binom{n-s}{r} / \binom{m}{r}$. If the output type does not occur, highest-weight covariance makes the whole product zero. $\square$

These multiplication rules, together with

$$1 = \sum_{\text{labelled constituents}} C_{n \rightarrow n}^0,$$

give a complete generators-and-relations presentation for the numerical highest-weight algebra (using the corresponding tensor-product identity matrices for two factors). Every word reduces to a linear combination of the displayed basis matrices, and those matrices are linearly independent. The

constituent labels in these rules are numerical labels; the coefficient quiver will still have twenty block vertices.

Numerical composition also requires matching the intermediate Lie algebra module constituent. Formal coefficient paths will only require matching the affine block. These differ at the exceptional block, where two constituents share one vertex. A zero numerical composition contributes zero to an extracted relation; it does not make its accompanying formal path zero. Only a proved coefficient relation can do that.

8.4 Lowering and the remaining tensor-product components

The preceding multiplication rule concerns products of two highest matrices. The relation families also contain highest vectors formed from lowered inputs. Their numerical evaluation is fixed by the same bases. For a rectangular matrix define

$$\mathcal{L}(C) = \rho_{\text{target}}(f)C - C\rho_{\text{source}}(f).$$

Thus the matrix of an unnormalized lowering power is $\mathcal{L}^i(C)$. The two Lie algebra factors act independently.

For composable highest matrices A, B of one-factor weights a, b and $0 \leq k \leq \min(a, b)$, the product of the highest-weight tensor-product component indexed by k is evaluated by

$$\sum_{i=0}^k u_i \mathcal{L}^i(A) \mathcal{L}^{k-i}(B), \quad u_0 = 1, \quad u_{i+1} = -u_i \frac{(k-i)(b-k+i+1)}{(i+1)(a-i)}.$$

Its highest weight is $a+b-2k$. If that type occurs between the outer constituents, the expression is a scalar multiple of the normalized output matrix. That scalar is its entry at the output normalization position. If the type does not occur, the expression is zero. For two factors use the product of the two Clebsch–Gordan coefficients and lower in both factors. These are the Clebsch–Gordan conventions of Appendix K.

For the component of highest weight $a+b-2$, take $k=1$. The recurrence gives $u_1 = -b/a$, so the expression is $A\mathcal{L}(B) - (b/a)\mathcal{L}(A)B$. Take $A = B = C_{4 \rightarrow 4}^2$, hence $a = b = 2$. In the above lowering basis,

$$\mathcal{L}(A) = \text{diag}(-1, -1/2, 0, 1/2, 1).$$

The entry $(1, 2)$ of $A\mathcal{L}(A) - \mathcal{L}(A)A$ is $-1/2 - (-1) = 1/2$. Its output weight is two, and this entry is the normalization position of $C_{4 \rightarrow 4}^2$. The entire highest-weight expression is therefore $(1/2)C_{4 \rightarrow 4}^2$. For two factors the position for lowering offsets (δ_1, δ_2) is row 1, column $1 + \delta_1(\lambda_2 + 1) + \delta_2$. This is the same unnormalized Clebsch–Gordan convention used for the lattice descriptors, not a new rescaling of the inputs.

Consequently all needed numerical coefficients are determined by the recurrence, lowering operators, and Clebsch–Gordan coefficients. The relation calculation may record the needed input labels, output labels, and exact scalars as one reusable numerical table. The ordinary highest-weight multiplication table alone does not include the other highest-weight tensor-product components.

8.5 The independent coefficients generate the corner

Proposition 8.6 (Generators of D). *Let $z_0, \dots, z_{15}$ be the sixteen highest-weight generators of $A(U)$, together with the six affine generators of $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. For each pair of blocks t, v and weights*

λ, μ , let

$$\xi_{t \rightarrow v}^{\lambda, \mu}(z_a) \in e_v D e_t$$

denote the corresponding coefficient of $X_{t \rightarrow v}(z_a)$, whenever admissible. Then

$$D = \left\langle e_t, \xi_{t \rightarrow v}^{\lambda, \mu}(z_a) \mid 0 \leq a \leq 15, t, v, \lambda, \mu \right\rangle.$$

Proof. Let $\mathcal{B} \subseteq D$ be the $\mathbb{C}$ -subalgebra generated by the displayed idempotents and coefficients. The block space

$$\bigoplus_{t,v} \text{Hom}_{\mathbb{C}}(T_t, T_v) \otimes e_v \mathcal{B} e_t$$

is a unital subalgebra of the matrix realization. It contains the matrices of the affine generators $\sum_t \rho_t(x) \otimes e_t$ and the images of the sixteen highest generators, by coefficient factorization. The twenty-two-generator result therefore makes this subalgebra the whole matrix realization of $A(U)$. Extracting its $(1, 1)$ entries in each block gives $e_v D e_t = e_v \mathcal{B} e_t$ for every t, v . Since $\sum_t e_t$ is the unit of D , it follows that $\mathcal{B} = D$. $\square$

The argument uses the 22 associative generators before taking the corner. Their numerical matrices and coefficient entries generate the whole matrix realization, so the selected entries generate D . Affine generators have known matrices with vertex-scalar coefficients; the z_a contribute the unknown coefficients. An admissible arrow is included even when later relations prove its value zero.

9 From VOA identities to a finite presentation of the Zhu corner

We have now arrived at the crucial section: constructing a presented algebra that maps onto D . As mentioned at the start, the idea is to use a particular path algebra presentation and prove that the resulting algebra is semisimple. This will imply that D , and hence $A(U)$, is semisimple.

The construction keeps several algebras distinct. In the following table, $(-)_\mathbb{C}$ means scalar extension through the fixed embedding $K \subset \mathbb{C}$. Every displayed evaluation map is surjective.

Object	Coefficients	Role and map
KQ	K	Formal coefficient paths; $(KQ)_\mathbb{C} \twoheadrightarrow D$.
$KQ/\mathcal{I}$	K	Full projected identities; its complex algebra maps onto D .
R_{red}	K	Equivalent upper presentation after vertex removal and generator substitutions.
S_R	Local DVR R	Integral rules on active arrows; $S_K = S_R \otimes_R K$.
$S_R/\mathfrak{m}S_R$	$\mathbb{F}_{10009}$	Special fiber: a computed split semisimple algebra.
$(S_K)_\mathbb{C}$	$\mathbb{C}$	Absolutely semisimple generic algebra; $(S_K)_\mathbb{C} \twoheadrightarrow D$.
$D = e_Z A(U) e_Z$	$\mathbb{C}$	Actual corner; equality with the upper algebra is proved in Section 11.

The integral algebra is finite because its cubic words can be shortened over R (Appendix K.5.2, C14). Its special fiber is semisimple by a separate confluence, trace, and splitting calculation (Appendix K.5.3, C15). Lemma 9.6 then proves generic semisimplicity. A special-fiber block can disappear when R -torsion is removed, or when the generic algebra is quotiented onto D . The later survival argument rules out both losses.

Before any of these reductions, a relation passes through five explicit steps:

$$\begin{aligned}
\text{lattice kernel vector} &\longmapsto \text{formal Zhu polynomial} \\
&\longmapsto \text{matrix with formal path coefficients} \\
&\longmapsto \text{coefficient polynomials in } KQ \\
&\longmapsto KQ/\mathcal{I}.
\end{aligned}$$

Each step is defined below. Appendix K.1.1 follows one saved relation through the complete chain.

9.1 The coefficient quiver and its evaluation

Definition 9.1 (The coefficient quiver and its path algebra). The vertices

$$Q_0 = \{0, \dots, 19\}$$

index the twenty matrix blocks of $A(\mathcal{A}^+)$. Their top spaces have the following K -models:

$$T_t = \bigoplus_{\lambda \in \Lambda_t} V(\lambda), \quad \Lambda_t = \begin{cases} \{(0, 2t)\}, & 0 \leq t \leq 9, \\ \{(0, 20), (1, 19)\}, & t = 10, \\ \{(1, 39 - 2t)\}, & 11 \leq t \leq 19. \end{cases}$$

In particular, $t = 10$ is one vertex with two highest-weight constituents.

Write $\nu_a = (\nu_{a,1}, \nu_{a,2})$ for the highest Cartan weight of z_a . These weights are

$$\begin{aligned}
(\nu_0, \dots, \nu_8) &= ((2, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 2), (2, 2), (0, 4), (0, 4)), \\
(\nu_9, \dots, \nu_{15}) &= ((1, 13), (1, 7), (1, 5), (1, 1), (0, 12), (0, 8), (0, 8)).
\end{aligned}$$

For $t, v \in Q_0$, $\lambda \in \Lambda_t$, and $\mu \in \Lambda_v$, the tuple (a, t, v, λ, μ) is *admissible* if $0 \leq a \leq 15$ and, for $k = 1, 2$,

$$|\mu_k - \lambda_k| \leq \nu_{a,k} \leq \mu_k + \lambda_k, \quad \mu_k + \lambda_k - \nu_{a,k} \in 2\mathbb{Z}.$$

These are precisely the conditions for $\text{Hom}_K(V(\lambda), V(\mu))$, with its natural Lie algebra action, to contain a highest-weight vector of weight ν_a .

The quiver Q has one formal arrow

$$x_{t \rightarrow v}^{\lambda, \mu}(z_a): t \longrightarrow v$$

for each admissible tuple. Here t, v label the source and target blocks, λ, μ label their Lie algebra module constituents, and a labels the chosen vector. The notation z_a in an arrow is a label; it does not mean evaluation in D .

A path of length $m \geq 1$, from t_0 to t_m , is a formal word

$$p = A_m \cdots A_1, \quad A_i = x_{t_{i-1} \rightarrow t_i}^{\lambda_i, \mu_i}(z_{a_i}), \quad 1 \leq i \leq m,$$

where each arrow is admissible. The traversal order is $A_1, A_2, \dots, A_m$. Only consecutive block endpoints must agree: no equality between μ_i and λ_{i+1} is required. At each vertex t , also include the length-zero path $\varepsilon_t: t \rightarrow t$.

Let $\text{Paths}(Q)$ be this set of paths. The path algebra is the vector space

$$KQ = \bigoplus_{p \in \text{Paths}(Q)} Kp$$

with the following multiplication. For two positive-length basis paths

$$\begin{aligned} p &= A_m \cdots A_1 : t_0 \longrightarrow t_m, \\ q &= B_n \cdots B_1 : v_0 \longrightarrow v_n, \end{aligned}$$

define

$$pq = \begin{cases} A_m \cdots A_1 B_n \cdots B_1, & t_0 = v_n, \\ 0, & t_0 \neq v_n. \end{cases}$$

In the first case, the resulting word traverses $B_1, \dots, B_n, A_1, \dots, A_m$, and is a path from v_0 to t_m .

For the length-zero paths, define

$$\varepsilon_v p = \delta_{v, t_m} p, \quad p \varepsilon_t = \delta_{t, t_0} p, \quad \varepsilon_v \varepsilon_t = \delta_{vt} \varepsilon_t.$$

Finally, extend multiplication K -bilinearly:

$$\left(\sum_p c_p p \right) \left(\sum_q d_q q \right) = \sum_{p, q} c_p d_q (pq), \quad c_p, d_q \in K,$$

where all sums have finite support. The unit is

$$1_{KQ} = \sum_{t \in Q_0} \varepsilon_t.$$

For example, $x_{0 \rightarrow 0}^{(0,0),(0,0)}(z_4)$ records the coefficient of the vector z_4 from block 0 to block 0, between the indicated Lie algebra module constituents. Its normalized numerical matrix is the 1×1 identity, but the arrow itself is still a free loop. There are twenty block vertices and twenty-one Lie algebra module constituents, because vertex 10 has two constituents. If $x : t \rightarrow v$ and $y : s \rightarrow t$, the word xy travels $s \rightarrow t \rightarrow v$: multiplication is right to left. The later notation $x_{k;t,s}$ likewise means an arrow $s \rightarrow t$.

The actual coefficients in Proposition 8.6 give evaluation

$$\tilde{\Xi} : KQ \otimes_K \mathbb{C} \rightarrow D, \quad \varepsilon_t \mapsto e_t, \quad x_{t \rightarrow v}^{\lambda, \mu}(z_a) \mapsto \xi_{t \rightarrow v}^{\lambda, \mu}(z_a).$$

This is surjective because these coefficients generate D . At this stage the arrows are formal and no VOA relation has yet been imposed on KQ .

9.2 Step 1: exact VOA kernel relations

We now describe the identities in U from which the path relations will be obtained. Let S_2 be the 120-dimensional space spanned by the zero-mode lowering vectors of $z_0, \dots, z_{15}$, and let T denote translation. The relation calculations compare different expressions for the same low-weight vector. Each comparison is a linear map from a formal space of expressions to a homogeneous subspace of U ; a vector in its kernel is an actual VOA identity. Every calculation is performed inside $V_{E_6 \oplus A_2}$ using explicit lattice coordinates in the required weights.

For $a, b \in S_2$, put

$$N(a, b) = \frac{a_{(-1)}b + b_{(-1)}a}{2}, \quad B(a, b) = \frac{a_{(0)}b - b_{(0)}a}{2}.$$

These have conformal weights four and three, respectively, since $\text{wt}(a_{(n)}b) = \text{wt}(a) + \text{wt}(b) - n - 1$.

The target VOA relations: three families.

1. *Symmetric weight-four relations.* We compute the kernel of

$$\Phi_{\text{sym}}: \text{Sym}^2 S_2 \longrightarrow U_4, \quad a \odot b \longmapsto N(a, b).$$

Its elements give identities $\sum_{i,j} c_{ij} N(a_i, a_j) = 0$ between symmetrized normal products.

2. *Weight-three relations.* We compare translations, affine generator normal products, and alternating zeroth products through the linear map

$$\begin{aligned} \Phi_3: S_2 \oplus (\mathfrak{g} \otimes S_2) \oplus \Lambda^2 S_2 &\longrightarrow U_3, \\ (v, h \otimes w, a \wedge b) &\longmapsto Tv + h_{(-1)}w + B(a, b). \end{aligned}$$

Thus a kernel relation expresses a linear combination of alternating zeroth products in terms of derivatives and affine generator products. Its image is the previously constructed space S_3 .

3. *Mixed weight-four relations.* Using these weight-three vectors, we compute the kernel of

$$\begin{aligned} \Phi_{\text{mix}}: S_3 \oplus (\mathfrak{g} \otimes S_3) \oplus \text{Sym}^2 S_2 &\longrightarrow U_4, \\ (u, h \otimes v, a \odot b) &\longmapsto Tu + h_{(-1)}v + N(a, b). \end{aligned}$$

These identities compare derivatives and affine generator products of weight-three vectors with normal products of weight-two vectors.

In summary, the VOA relations we wish to calculate are precisely the kernels of the maps above:

$$\ker \Phi_{\text{sym}}, \quad \ker \Phi_3, \quad \ker \Phi_{\text{mix}}.$$

A key observation is that all of the maps Φ_{sym} , Φ_3 and Φ_{mix} are $\mathfrak{g}$ -equivariant. To see this, recall that the zero modes of the affine generators satisfy

$$\begin{aligned} x_{(0)}(a_{(n)}b) &= (x_{(0)}a)_{(n)}b + a_{(n)}(x_{(0)}b), \\ x_{(0)}(Ta) &= T(x_{(0)}a) \quad (x \in \mathfrak{g}). \end{aligned}$$

Consequently, all three maps are $\mathfrak{g}$ -equivariant and hence their kernels are finite-dimensional $\mathfrak{g}$ -modules. In particular, let L_Φ be the formal domain of one of the three maps. If $\text{HW}(L_\Phi)$ denotes its highest-weight space, equivariance gives

$$\text{HW}(\ker \Phi) = \ker(\Phi|_{\text{HW}(L_\Phi)}).$$

The highest-weight vectors generate the kernel as a $\mathfrak{g}$ -module, so it suffices to compute the restricted kernel.

This reduces the lattice calculation to the highest-weight subspaces.

Reduction to highest-weight relations

1. **Determine the $\mathfrak{g}$ -module structure** of the three domains:

$$\text{Dom}(\Phi_{\text{sym}}) = \text{Sym}^2(S_2),$$

$$L_3 = \text{Dom}(\Phi_3) = S_2 \oplus (\mathfrak{g} \otimes S_2) \oplus \bigwedge^2 S_2,$$

$$L_4 = \text{Dom}(\Phi_{\text{mix}}) = S_3 \oplus (\mathfrak{g} \otimes S_3) \oplus \text{Sym}^2(S_2).$$

In particular, decompose each domain into irreducible $\mathfrak{g}$ -modules.

2. **Calculate the kernels restricted to highest-weight vectors:**

$$\ker \left(\Phi_{\text{sym}}|_{\text{HW}(\text{Sym}^2 S_2)} \right), \quad \ker \left(\Phi_3|_{\text{HW}(L_3)} \right), \quad \ker \left(\Phi_{\text{mix}}|_{\text{HW}(L_4)} \right).$$

The three kernels give 66, 408, and 446 highest-weight families, respectively.

The scalar dimensions and highest-family counts answer different questions:

Map	Domain	Image rank	Kernel	Highest families	Witness
Φ_{sym}	7260	5207	2053	66	<code>quadratic_exact.json</code>
Φ_3	7980	956	7024	408	<code>bracket_exact.json</code>
Φ_{mix}	13952	6007	7945	446	<code>mixed_exact.json</code>

Here the targets are the full lattice vector spaces in weights four, three, and four. The domain dimensions follow from $\dim S_2 = 120$, $\dim S_3 = 956$, and $\dim \mathfrak{g} = 6$. The 920 highest families generate all three kernels by zero-mode lowering, since finite-dimensional $\mathfrak{g}$ -modules are completely reducible. Projection to different source and target constituents later produces many more scalar path rows; 920 is not that row count. Appendix K.2 explains the exact expansion and kernel solve before the C11 files are used.

Location of the decompositions and kernel certificates. The two stages of the calculation are documented as follows.

1. **$\mathfrak{g}$ -module decompositions.**

Appendix A specifies the sixteen normalized highest-weight vectors and their lowering vectors. Appendix H records the decomposition of S_2 and the construction and Lie algebra actions of S_3 ; the corresponding basis certificates are listed under C03–C04 in Appendix R. The Clebsch–Gordan conventions used to construct highest-weight bases of

$$\text{Sym}^2 S_2, \quad S_2 \oplus (\mathfrak{g} \otimes S_2) \oplus \Lambda^2 S_2, \quad S_3 \oplus (\mathfrak{g} \otimes S_3) \oplus \text{Sym}^2 S_2$$

are given in Appendix K. The complete highest-weight domain descriptors, grouped by weight and retaining every multiplicity, are stored in the R12 files `shared/rank12/certificates/quadratic_discovery.json`, `shared/rank12/certificates/bracket_discovery.json`, and

`shared/rank12/certificates/mixed_discovery.json`, respectively. These files also specify the modular image-pivot data.

2. Highest-weight kernel relations.

Appendix K describes the three kernel calculations for Φ_{sym} , Φ_3 , and Φ_{mix} , which give respectively 66, 408, and 446 highest-weight relations. Their exact coefficients, domain descriptors, selected image coordinates, and mode-one corrections are stored in `shared/rank12/certificates/quadratic_exact.json`, `shared/rank12/certificates/bracket_exact.json`, and `shared/rank12/certificates/mixed_exact.json`. The relations are verified as identities of complete lattice vectors over K ; their zero-mode lowering vectors recover the full relation spaces. The certificate locators and reproduction commands are collected under C11 in Appendix R.

Output of Step 1. A highest relation R is an explicit vector in one of the three formal domains with $\Phi(R) = 0$ as a complete lattice vector. Its coefficients and Cartan weight are retained for the next step.

9.3 Step 2: formal Zhu polynomials

Next, we need to convert the VOA relations to relations in the Zhu algebra.

Three notations will be kept separate: $z_{a;i,j}$ is a lattice vector, $[z_{a;i,j}]$ its actual Zhu class, and $Z_{a;i,j}$ a free symbol before relations. The 120 symbols for all of S_2 are useful for linear conversion of the kernel descriptors. They deliberately exceed the eventual sixteen highest-weight vectors and six affine generators generating the associative algebra. The mode-one corrections below retain the lower-weight part of the Zhu identity; omitting them would only describe a leading graded relation, not the relation in $A(U)$.

The vector identities become associative polynomial identities using the Zhu formulas from the preceding section. In particular,

$$\begin{aligned} [N(a, b)] &= \frac{[a] * [b] + [b] * [a]}{2} + [c_+(a, b)], & c_{\pm}(a, b) &= \frac{a_{(1)}b \pm b_{(1)}a}{2}. \\ [B(a, b)] &= [a] * [b] - [b] * [a] - [c_-(a, b)], \end{aligned}$$

To make these identities into explicit symbolic relations, we first specify the symbols representing their correction terms.

Definition 9.2 (Formal Zhu symbols). Write the fixed lowering basis of S_2 as

$$z_{a;i,j} = (F_s)_{(0)}^i (F_t)_{(0)}^j z_a, \quad 0 \leq a \leq 15, \quad 0 \leq i \leq \nu_{a,1}, \quad 0 \leq j \leq \nu_{a,2}.$$

Thus the lowering powers use the zeroth vertex modes. Let $\mathcal{F}$ be the free unital associative K -algebra generated by $Z_{a;i,j}$ and the six affine generator symbols J_x , for $x \in \{E_s, F_s, H_s, E_t, F_t, H_t\}$. Extend $x \mapsto J_x$ linearly on $\mathfrak{g}$, and define

$$Z_2: S_2 \longrightarrow \mathcal{F}, \quad Z_2 \left(\sum_{a,i,j} \gamma_{a;i,j} z_{a;i,j} \right) = \sum_{a,i,j} \gamma_{a;i,j} Z_{a;i,j}.$$

Evaluation is the K -algebra homomorphism

$$\text{ev}: \mathcal{F} \longrightarrow A(U), \quad Z_{a;i,j} \longmapsto [z_{a;i,j}], \quad J_x \longmapsto [x].$$

Thus $\text{ev}(Z_2(u)) = [u]$. The map Z_2 is linear, not multiplicative for VOA products. The conformal class is already represented by $Z_2(\omega_U)$, since $\omega_U \in S_2$.

The closure $(S_2)_{(1)}S_2 \subseteq S_2$ gives unique expansions

$$c_{\pm}(a, b) = \sum_{d,i,j} \gamma_{d;i,j}^{\pm}(a, b) z_{d;i,j}, \quad \gamma_{d;i,j}^{\pm}(a, b) \in K.$$

To determine these coefficients, we compute the complete lattice-monomial coordinates of the correction and solve an exact linear system in the fixed S_2 basis. The resulting equality is checked in every lattice-monomial coordinate, not only in the rows used to solve the system. Consequently,

$$Z_2(c_{\pm}(a, b)) = \sum_{d,i,j} \gamma_{d;i,j}^{\pm}(a, b) Z_{d;i,j}.$$

This is an explicit linear combination of existing symbols, *not an additional formal generator*.

The displayed Zhu identities therefore give the symbolic conversions

$$\begin{aligned} Z(N(a, b)) &:= \frac{1}{2} (Z_2(a)Z_2(b) + Z_2(b)Z_2(a)) + Z_2(c_+(a, b)), \\ Z(B(a, b)) &:= Z_2(a)Z_2(b) - Z_2(b)Z_2(a) - Z_2(c_-(a, b)). \end{aligned}$$

Here Z denotes conversion of formal expressions, extended linearly on their domains. Both formulas are well-defined noncommutative polynomials in $\mathcal{F}$, whose evaluations are $[N(a, b)]$ and $[B(a, b)]$. Different expressions for the same vector can give different formal polynomials with the same evaluation.

The equivariance established above also applies to the correction maps c_+ and c_- . Thus, for a highest vector z of Cartan weight η in the symmetric or alternating domain, its total correction c_z , obtained by applying the appropriate correction map term by term, is highest of weight η , or is zero. It follows that

$$c_z = \sum_{\nu_d=\eta} \beta_d z_d, \quad Z_2(c_z) = \sum_{\nu_d=\eta} \beta_d Z_{d;0,0}.$$

Every copy of type η must be retained. This conclusion uses the highest-weight property of z in the formal domain; the separate assertion that its vector image vanishes is what supplies a relation after evaluation. After converting all summands, each proved VOA relation gives a polynomial $P_z \in \mathcal{F}$ with $\text{ev}(P_z) = 0$.

9.3.1 Converting the weight-three family

We now apply the same conversion to the three summands of L_3 . For brevity, let α, β denote indices (a, i, j) of the fixed basis, and write $z_{\alpha} = z_{a;i,j}$ and $Z_{\alpha} = Z_{a;i,j}$. An ordered basis of L_3 consists of

$$z_{\alpha}, \quad h \otimes z_{\alpha}, \quad z_{\alpha} \wedge z_{\beta} \quad (\alpha < \beta),$$

with h ranging over the fixed basis of the six weight-one generators. The first two groups represent translations and affine generator normal products, respectively.

Definition 9.3 (Weight-three Zhu conversion). Define the K -linear map $\zeta_3: L_3 \rightarrow \mathcal{F}$ on the three summands by

$$\begin{aligned} \zeta_3(v) &= -2Z_2(v), \\ \zeta_3(h \otimes v) &= Z_2(v)J_h, \\ \zeta_3(a \wedge b) &= Z_2(a)Z_2(b) - Z_2(b)Z_2(a) - Z_2(c_-(a, b)), \end{aligned}$$

where $v, a, b \in S_2$ and $h \in \mathfrak{g}$.

The factor -2 comes from translation in weight two, and the order in the second line follows from $[h_{(-1)}v] = [v] * [h]$. The last line is the alternating conversion already established. Consequently,

$$\text{ev}(\zeta_3(w)) = [\Phi_3(w)] \quad (w \in L_3).$$

Every $R \in \ker \Phi_3$ therefore gives the explicit formal relation $P_R = \zeta_3(R) \in \ker \text{ev}$.

9.3.2 Fixing polynomial representatives of S_3

The mixed family also requires a polynomial representing each vector of S_3 . Let $x_0, \dots, x_{955}$ be its ordered exact basis from Appendix H. Regard Φ_3 as a surjection onto S_3 , and fix a $\mathfrak{g}$ -equivariant linear section

$$\sigma_3: S_3 \longrightarrow L_3, \quad \Phi_3 \sigma_3 = \text{id}_{S_3}.$$

Such a section is obtained by lifting a basis of the highest-weight spaces of S_3 through the corresponding restricted maps, retaining every multiplicity, and extending by zero-mode lowering. The lifting coefficients are determined by exact linear systems in complete lattice-monomial coordinates; each identity $\Phi_3(\sigma_3(x_\ell)) = x_\ell$ is verified in every coordinate.

Concretely, Appendix H.5 selects 87 highest lifts from the recorded generation descriptors and propagates them by lowering to the 956 basis vectors. A coordinate pivot chooses independent output coordinates; the section chooses preimages in the formal input space L_3 . These are different choices. The following polynomial uses the preimages, and the section-independence argument below shows why the resulting combined relation ideal is unaffected.

Define

$$Z_3: S_3 \longrightarrow \mathcal{F}, \quad Z_3(u) := \zeta_3(\sigma_3(u)).$$

To make its meaning explicit, write the three components of a lift as

$$\sigma_3(x_\ell) = \left(v_\ell, \sum_h h \otimes w_{\ell,h}, \sum_{\alpha < \beta} r_\ell^{\alpha,\beta} z_\alpha \wedge z_\beta \right),$$

where $v_\ell, w_{\ell,h} \in S_2$ and $r_\ell^{\alpha,\beta} \in K$. Then

$$\begin{aligned} Z_3(x_\ell) = & -2Z_2(v_\ell) + \sum_h Z_2(w_{\ell,h})J_h \\ & + \sum_{\alpha < \beta} r_\ell^{\alpha,\beta} (Z_\alpha Z_\beta - Z_\beta Z_\alpha - Z_2(c_-(z_\alpha, z_\beta))). \end{aligned}$$

Every term is expressed in the original generators of $\mathcal{F}$, including the correction coordinates computed above. Thus $Z_3(u)$ introduces *no new formal generator*, and $\text{ev}(Z_3(u)) = [u]$.

9.3.3 Converting the mixed weight-four family

The corresponding basis of L_4 consists of x_ℓ , $h \otimes x_\ell$, and $z_\alpha \odot z_\beta$ with $\alpha \leq \beta$.

Definition 9.4 (Mixed weight-four Zhu conversion). Define the K -linear map $\zeta_{\text{mix}}: L_4 \rightarrow \mathcal{F}$ on its three summands by

$$\begin{aligned} \zeta_{\text{mix}}(u) &= -3Z_3(u), \\ \zeta_{\text{mix}}(h \otimes u) &= Z_3(u)J_h, \\ \zeta_{\text{mix}}(a \odot b) &= Z(N(a, b)), \end{aligned}$$

where $u \in S_3$, $h \in \mathfrak{g}$, and $a, b \in S_2$. The last line uses the symmetric conversion already defined.

Here -3 is the translation factor in weight three. Substituting the expressions for $Z_3(u)$ makes every mixed conversion an explicit polynomial in the same formal symbols. The Zhu identities give

$$\text{ev}(\zeta_{\text{mix}}(w)) = [\Phi_{\text{mix}}(w)] \quad (w \in L_4).$$

Thus each mixed kernel vector gives a polynomial in $\ker \text{ev}$.

The section fixes the expressions used in this calculation. If σ'_3 is another section, then

$$Z'_3(u) - Z_3(u) = \zeta_3((\sigma'_3 - \sigma_3)(u)), \quad (\sigma'_3 - \sigma_3)(u) \in \ker \Phi_3.$$

Hence changing the section changes the mixed polynomials only by elements of the two-sided ideal generated by the converted weight-three relations. Once those relations are imposed, the combined presentation is independent of this choice.

Covariance and the output of Step 2. Give $\mathcal{F}$ the Lie algebra action by derivations determined by

$$x \cdot Z_2(u) = Z_2(x_{(0)}u), \quad x \cdot J_y = J_{[x,y]}.$$

The correction maps and the three conversion maps are equivariant; the mixed conversion also uses the chosen equivariant section. Thus a highest kernel vector R gives a highest polynomial $P_R \in \mathcal{F}$ with

$$\boxed{\text{ev}(P_R) = 0.}$$

This polynomial is expressed entirely in the original affine generator and S_2 symbols, including every lower-mode correction.

9.4 Step 3: matrices with formal path coefficients

To substitute into P_R before knowing D , use the algebra

$$\mathcal{M}_Q = \bigoplus_{t,v} \text{Hom}_K(T_t, T_v) \otimes_K \varepsilon_v KQ \varepsilon_t,$$

with multiplication

$$(F \otimes p)(G \otimes q) = (FG) \otimes pq$$

for matching intermediate blocks, and zero otherwise. Its unit is $\sum_t \text{id}_{T_t} \otimes \varepsilon_t$. The Lie algebra action operates on the numerical factor.

There is a K -algebra homomorphism $\hat{X} : \mathcal{F} \rightarrow \mathcal{M}_Q$ specified by

$$\hat{X}(J_x) = \sum_t \rho_t(x) \otimes \varepsilon_t$$

and

$$\hat{X}(Z_{a;0,0}) = \sum_{t,v,\lambda,\mu} C_{t \rightarrow v}^{\lambda,\mu}(\nu_a) \otimes x_{t \rightarrow v}^{\lambda,\mu}(z_a),$$

where the sum runs over admissible tuples. For $Z_{a;i,j}$ lower this numerical matrix i and j times in the two Lie algebra factors, retaining the same formal arrow coefficients. This determines $\hat{X}$ on every generator of $\mathcal{F}$, and hence on every polynomial.

Substitution is multiplication in $\mathcal{M}_Q$: numerical matrices compose and formal coefficients concatenate as paths. The numerical products are evaluated using (8.2) and the products of the remaining highest-weight tensor-product components described in Subsection 8.4. The exact normalization convention agrees with Appendix L.

Notation	Type	Meaning
z	U	Lattice vector
$[z]$	$A(U)$	Actual Zhu class
Z	$\mathcal{F}$	Formal polynomial representative
$C_{t \rightarrow v}^{\lambda, \mu}$	$\text{Hom}_K(V(\lambda), V(\mu))$	Known numerical matrix
$x_{t \rightarrow v}^{\lambda, \mu}$	$\varepsilon_v K Q \varepsilon_t$	Free path coefficient
$\xi_{t \rightarrow v}^{\lambda, \mu}$	$e_v D e_t$	Evaluated coefficient

Substitution in $\mathcal{M}_Q$ is defined before any path relations are imposed. The next identity compares that formal substitution with evaluation in the actual Zhu algebra.

Let ev_{mat} replace the formal arrows and vertices by their values under $\tilde{\Xi}$. After scalar extension the defining formulas give

$$\text{ev}_{\text{mat}}(\hat{X}(P)) = X(\text{ev}(P)).$$

In particular, the output $\hat{X}(P_R)$ of Step 3 is a matrix with path coefficients whose actual evaluation is zero.

9.5 Step 4: extracting the path polynomials

Let R have highest Cartan weight η . Equivariance makes $\hat{X}(P_R)$ highest of the same weight. In each block and constituent pair, expand it in the fixed normalized highest matrix:

$$\hat{X}(P_R)_{t \rightarrow v}^{\lambda, \mu} = C_{t \rightarrow v}^{\lambda, \mu}(\eta) \otimes p_{R; t \rightarrow v}^{\lambda, \mu}, \quad p_{R; t \rightarrow v}^{\lambda, \mu} \in \varepsilon_v K Q \varepsilon_t.$$

If the output type does not occur, this component is zero. The numerical matrix is already fixed. Its coefficient polynomial is read at the entry normalized to 1: in each factor this is row 1, column $1 + (\eta_i - \mu_i + \lambda_i)/2$. For two factors use the fixed tensor-product ordering. Each exceptional constituent is treated separately.

Since evaluation of the whole matrix is zero and the normalized numerical matrices are linearly independent,

$$\tilde{\Xi}(p_{R; t \rightarrow v}^{\lambda, \mu}) = 0.$$

These coefficient polynomials are the output of Step 4. They are computed by exact arithmetic in the formal paths; there is no separate search for relations in the unknown algebra D .

A zero numerical product contributes zero to a projected relation. It does not by itself impose that the corresponding formal path is zero. Path relations come from the coefficient polynomials of the proved VOA identities and the other specified identities below.

A short complete example is the first relation in `bracket_exact.json`, group 0, relation 0. It converts to $C_s - 3Z_{1;0,0}$, where $C_s = J_{H_s}^2 + 2J_{E_s}J_{F_s} + 2J_{F_s}J_{E_s}$. On a constituent of highest weight r for the first $\mathfrak{sl}_2$ factor, its path coefficient is $r(r+2)\varepsilon_v - 3x_{v \rightarrow v}^{\lambda, \lambda}(z_1)$. This becomes a defining relation, expressing a redundant generator coefficient by the known Casimir scalar. Appendix K.1.1 gives the descriptor, vector identity, signs, and every conversion step, as well as saved symmetric and mixed examples.

9.6 Step 5: the ideal and the upper presentation

Let $\mathcal{P}_{\text{full}}$ consist of all nonzero coefficient polynomials obtained from the three relation families, together with the coefficients of the known affine identities and of the centrality equations for $\hat{X}(Z_2(\omega_U))$.

The additional source data are explicit:

- Affine blocks obey the source Casimir identities $C_s(C_s - 3) = 0$ and $\prod_{j=0}^{39} (C_t - j(j + 2)) = 0$ on the listed source tops, with the more selective source projectors of Appendix J.2.
- The affine weight-two vectors and conformal classes are substituted by the identities in Appendix K.3; for example, $[z_1] = C_s/3$.
- The total conformal class is central. After reduction its coefficient equation for $x : s \rightarrow t$ is $H_t x - x H_s = 0$.

Equivariance, the matrix-unit identities, and the prescribed normalization to coefficient one are already built into $\hat{X}$ in Subsection 9.4. They determine which coefficient is extracted, rather than contributing unproved extra relations. The listed source identities and conformal substitutions hold in the actual source action, so their extracted coefficients also evaluate to zero.

Use the affine and conformal coordinates of Appendix K. Define

$$\mathcal{I} = \langle \mathcal{P}_{\text{full}} \rangle \subseteq KQ.$$

Each polynomial evaluates to zero, so evaluation descends to

$$(KQ/\mathcal{I}) \otimes_K \mathbb{C} \twoheadrightarrow D.$$

Surjectivity is inherited from the coefficient generation result. It is enough that the relations hold and that the coefficients generate D ; equality with the entire kernel is not required. The lattice-vector relation calculations and their projected coefficients are the C11 input. **[CALC C11, p. 132]**

9.7 Vertex removal and the reduced presentation

Vertex removal occurs after the full path relations have been constructed. The C12 calculation supplies exact ideal-membership witnesses over K which kill the vertices in four successive pairs:

$$(10, 11), \quad (9, 12), \quad (8, 13), \quad (7, 14).$$

At each stage the witnesses may use the vertices killed at the earlier stages. Induction therefore gives $\varepsilon_v \in \mathcal{I}$ for every excluded vertex, and evaluation gives $e_v = 0$ in D . **[CALC C12, p. 132]** These are consequences of the path relations; the numerical matrix table does not independently remove a coefficient vertex.

For a concrete typed witness, take the vertex-10 record in `shared/rank12/certificates/global_vertex_exact.json`. If p_o denotes the projected relation with the recorded origin o and c_o its exact coefficient, the checked identity is

$$\sum_{(o, c_o) \in \text{witness_terms}} c_o p_o = \varepsilon_{10} \quad \text{in } \varepsilon_{10} KQ \varepsilon_{10}.$$

Each term is reconstructed with its declared corner, not merely as an untyped empty word. Appendix K.4 decodes an actual origin and explains the full sum. If $\varepsilon_v \in \mathcal{I}$, an incident arrow $x = \varepsilon_{t(x)} x \varepsilon_{s(x)}$ vanishes in the quotient, and so does every path through v . The cascade repeats this argument in each surviving quotient. These are exclusions of eight affine source vertices, not a count of eight ordinary U -modules.

Let Q_{surv} be obtained by deleting those vertices and their incident arrows. Because their idempotents are already zero in $KQ/\mathcal{I}$, passing to the surviving quiver preserves the upper presentation and its surjection onto D . After the affine and conformal substitutions, we use the reduced generators below. This is the point at which the twenty-vertex presentation becomes a twelve-vertex presentation. Appendix K and the C12 witnesses specify the exclusion calculation and its order.

The generators and their actual values. After the eight exclusions, the vertex labels are

$$V = \{0, 1, 2, 3, 4, 5, 6, 15, 16, 17, 18, 19\}.$$

The affine type at v is

$$(r_v, j_v) = \begin{cases} (0, 2v), & 0 \leq v \leq 6, \\ (1, 39 - 2v), & 15 \leq v \leq 19. \end{cases}$$

Continue to write $e_v = E_{11}^{(v)} \in D$. These idempotents are orthogonal and sum to the unit e_Z of D . A vertex labels an affine block, not a simple U -module. **[CALC C12, p. 132]**

The affine identities and conformal coordinates in Appendix K leave the following nine highest-weight vectors, together with the total conformal class:

k	4	8	9	10	11	12	13	14	15
$(\ell_1(k), \ell_2(k))$	(0,0)	(0,4)	(1,13)	(1,7)	(1,5)	(1,1)	(0,12)	(0,8)	(0,8)

Let Q_{red} have one loop H_v at each $v \in V$ and one arrow $x_{k;t,s} : s \rightarrow t$ for each listed vector and each pair satisfying

$$\begin{aligned} |r_t - r_s| &\leq \ell_1(k) \leq r_t + r_s, & r_t + r_s - \ell_1(k) &\in 2\mathbb{Z}, \\ |j_t - j_s| &\leq \ell_2(k) \leq j_t + j_s, & j_t + j_s - \ell_2(k) &\in 2\mathbb{Z}. \end{aligned}$$

These are precisely the conditions for the corresponding highest rectangular matrix to exist. There are 356 arrows in total.

Each arrow has a specified value in D . The value of H_v is $e_v[\omega_U]e_v$. For $x_{k;t,s}$ put

$$\delta_1 = \frac{\ell_1(k) - r_t + r_s}{2}, \quad \delta_2 = \frac{\ell_2(k) - j_t + j_s}{2}, \quad \mu = 1 + \delta_1(j_s + 1) + \delta_2.$$

In the unnormalized tensor-product lowering basis, with the second Lie algebra factor varying fastest and indices starting at one, the normalized highest rectangular matrix has entry one in row 1, column μ . Its coefficient in the Zhu class of z_k is therefore

$$\xi_{k;t,s} = E_{11}^{(t)}[z_k]E_{\mu 1}^{(s)} \in e_t D e_s.$$

The normalization recurrences and the matrix-unit reconstruction above show that these coefficients and the e_v generate D . **[CALC C11, p. 132]**

The reduced relation polynomials. Let $\mathcal{P}$ be all nonzero highest matrix coefficients of the Zhu-converted relation families above, using the affine and conformal substitutions of Appendix K, together with the relations $H_t x - x H_s$ for every generator arrow $x : s \rightarrow t$. These are 11,293 polynomials in KQ_{red} .

The total is 10,949 projected family rows and 344 centrality rows, one for each generator arrow; the remaining twelve arrows are the H_v loops. Substitution of the seven redundant highest-weight

vectors is a change of presentation, preserving the surjection onto D . Exact row operations then remove $43 + 29 + 1 = 73$ arrows, leaving $356 - 73 = 283$ active arrows. The 6,745 monic rules count rewritten relations on these active arrows, not a new count of kernel families. Appendix K.4 records the stages.

The intermediate algebra is, by definition,

$$R_{\text{red}} = KQ_{\text{red}}/\langle \mathcal{P} \rangle.$$

Here and below the brackets denote the two-sided ideal generated by the displayed relations. Evaluating the formal vertices and arrows at the idempotents and entries just specified gives

$$R_{\text{red}} \otimes_K \mathbb{C} \rightarrow D.$$

This map is well defined because the equations come from identities in the VOA, and is onto because their entries generate D .

Exact row operations over K successively express 73 generators in terms of the remaining generators and vertex idempotents. The resulting presentation has 283 active arrows and 6,745 monic quadratic rules. Its derivation and the linear eliminations are the content of C13. [CALC C13, p. 132] These eliminations are performed in characteristic zero; the integral algebra is defined from the resulting rules.

9.8 The integral presentation

Put

$$p = 10009, \quad R = \mathbb{Z}[\theta]_{(p, \theta - 2914)} \subset K, \quad \mathfrak{m} = (p, \theta - 2914)R.$$

This is the DVR of Appendix R.1, with fraction field K and residue field $\mathbb{F}_p$. [CALC C17, p. 133] The following finite data specify the generators and every relation: `shared/rank12/new_results/exact_upper_rules.json` in the rank package R12 identified in Appendix R. Let Λ be exactly its list `active_arrows`. The array `arrows` gives the endpoints and generator label of each index; its 356 records retain the original numbering. Let Q_{act} have vertices V and only the arrows indexed by Λ .

For clarity, RQ_{act} is the algebra generated by formal idempotents ε_v and formal entries x_a , subject to

$$\varepsilon_v \varepsilon_w = \delta_{vw} \varepsilon_v, \quad \sum_{v \in V} \varepsilon_v = 1, \quad x_a = \varepsilon_{t(a)} x_a \varepsilon_{s(a)}.$$

Here $s(a)$ and $t(a)$ are the initial and terminal vertices of arrow a . Thus $x_a x_b$ is zero unless $s(a) = t(b)$; a path is simply a product with matching indices.

Each record in `rules`, with `leading=[a,b]` and right-hand terms (w, c) under `rhs`, specifies the polynomial

$$\rho = x_a x_b - \sum_{(w, c) \in \text{rhs}} c(\theta) x_w.$$

A word $w = [a_1, \dots, a_n]$ means $x_w = x_{a_1} \cdots x_{a_n}$. An empty word denotes the vertex idempotent at the endpoints of that diagonal relation. A coefficient list $[c_0, \dots, c_d]$ is read in descending powers, as $c_0 \theta^d + \cdots + c_d$.

For example, the actual record `leading=[229,195]`, `rhs=[]` is the relation $x_{229} x_{195} = 0$. Arrow 195 travels $19 \rightarrow 1$ and arrow 229 travels $1 \rightarrow 5$, so this relation lies in the $(5, 19)$ corner; its empty right-hand array means zero. An empty *word* on a diagonal right-hand side instead means its vertex

unit. A saved linear elimination is $x_{80} = (-11\theta^2 + 308)x_{120}$, read from $[-11, 0, 308]$. Substituting it in a word means replacing every occurrence of x_{80} and multiplying the coefficients.

The row order used by the program has key $(-|w|, w)$: longer words lead, and equal-length words are lexicographic in their integer arrow labels. A rewrite replaces a leading word by shorter words or lexicographically larger words of the same length. This terminates at any fixed initial length. Saturation means repeatedly substituting linear eliminations in every row, collecting terms, and recomputing quadratic row echelon form until the presentation stabilizes.

Let $\mathcal{R}$ be exactly these 6,745 polynomials. All their coefficients lie in R . **[CALC C14, p. 132]**

This is a local integrality assertion, not a claim that every displayed rational denominator is prime to p . The simple root $\theta \equiv 2914 \pmod{p}$ chooses the local embedding; its Hensel lifts test cancellation in any numerator whose denominator contains p . The resulting residues agree with the modular rules. Likewise, a nonzero determinant modulo p makes the corresponding lifted determinant a unit in R . Appendix R.1 gives the coefficient algorithm, and Appendix K.5.2 applies it to the overlap equations. No flatness of S_R is needed. Define

$$S_R = RQ_{\text{act}}/\langle \mathcal{R} \rangle, \quad S_K = S_R \otimes_R K.$$

The eliminated arrows are not additional generators of S_R . When discussing these algebras, we also write e_v for the formal vertex idempotent ε_v .

For every active arrow let ξ_a be its previously specified value in D . Evaluation

$$\varepsilon_v \longmapsto e_v, \quad x_a \longmapsto \xi_a$$

annihilates $\mathcal{R}$, since these rules are exact consequences of the projected VOA identities. The successive eliminations show that the active entries still generate D . Hence evaluation gives the required surjection, using the fixed embedding $K \subset \mathbb{C}$:

$$S_K \otimes_K \mathbb{C} \twoheadrightarrow D.$$

Additional relations may hold in D ; equality with this presented algebra is not needed.

The purpose of the integral presentation. The low-weight identities give equations of degree at most two in these entries. They allow exact reduction of words, while associativity supplies further relations between different reduction orders. The next proposition proves that these relations shorten every path to a combination of 1,081 specified paths of length at most two, over R itself. This makes S_R finite as an R -module. **[CALC C14, p. 132]**

The same presentation can then be reduced modulo $\mathfrak{m}$. Computations over $\mathbb{F}_p$ prove that $S_R/\mathfrak{m}S_R$ is split semisimple. **[CALC C15, p. 132]** The prime is useful because the rule coefficients are integral there and the determinants used in the integral shortening are units there. Both conditions are verified. The integral spanning bound and the semisimple special fiber together allow Lemma 9.6 to prove that S_K is absolutely semisimple. Its complex quotient D is then semisimple, and the fullness of e_Z transfers this conclusion to $A(U)$. The descent lemma allows R -torsion; which matrix blocks survive in D is determined by the subsequent argument in Section 11.

The proof below has two outputs with different status. The integral calculation gives a spanning set of $12 + 283 + 786 = 1,081$ paths; it does not assert their independence. The special-fiber calculation gives a basis of $12 + 282 + 252 = 546$ normal paths, after completing the relations modulo $\mathfrak{m}$. Their respective methods are Appendices K.5.2 and K.5.3.

9.9 Finiteness and the semisimple special fiber

Proposition 9.5. *Put $p = 10009$ and*

$$R = \mathbb{Z}[\theta]_{(p, \theta - 2914)} \subset K.$$

This is a DVR with residue field $\mathbb{F}_p$; the field polynomial has a simple root at 2914 modulo p , as explained in Appendix R.1. [CALC C17, p. 133] There is an R -algebra S_R with the following properties.

1. $S_K = S_R \otimes_R K$ *surjects onto the actual complex corner D after extension of scalars;*
2. S_R *is finite as an R -module;*
3. *Its special fiber is split semisimple with block sizes*

$$1, 1, 1, 1, 2, 3, 4, 6, 9, 10, 10, 14.$$

In particular, its dimension is 546. The certificate also records all ranks of the affine block idempotents on these simple blocks.

Proof. The exact quadratic rules are consequences of the proved Zhu relations. [CALC C13, p. 132] Their coefficients are integral in R , [CALC C14, p. 132] so they define an algebra S_R whose generic fiber surjects onto D .

The crucial finiteness check is a reduction of every three-letter path to paths of length at most two. It comes from the associativity relations between the two quadratic reduction orders. The selected coefficient minors are units in R , so this reduction holds over R itself. Replacing a three-letter subpath shortens any longer path. Induction therefore proves that the 1,081 paths of length at most two span S_R . [CALC C14, p. 132]

For an overlap $x_a x_b x_c$, write the quadratic rules as $x_a x_b = \sum_u A_u x_u$ and $x_b x_c = \sum_v B_v x_v$. Reducing their difference gives the exact equation

$$\text{NF} \left(\sum_u A_u x_u x_c - \sum_v B_v x_a x_v \right) = 0.$$

Both the cubic coefficients and the shorter terms are retained. Within each endpoint corner, the matrix on normal cubic columns has a selected unit minor; the 53 endpoint blocks cover all 1,364 cubic columns. Solving these equations over R therefore expresses every remaining cubic in shorter paths. Appendix K.5.2 decodes the saved $(0, 0)$ overlap block and its nonzero determinant.

In the special fiber, the completed rules terminate and every critical overlap has zero remainder. [CALC C15, p. 132] The endpoint version of the Diamond Lemma [32, Theorem 1.2] gives independence as well as spanning of the irreducible paths: every ambiguity is resolvable in a compatible terminating order. The resulting 546 normal paths are therefore a basis. The regular trace pairing

$$(x, y) \longmapsto \text{Tr}(L_{xy})$$

is nondegenerate. If x belongs to the radical, then xy is nilpotent for every y , so $\text{Tr}(L_{xy}) = 0$. Nondegeneracy forces $x = 0$. Thus the special fiber is semisimple. Splitting is a further step. A separating central element has twelve distinct eigenvalues in $\mathbb{F}_p$. Interpolation produces its primitive central projectors, and the checked dimensions of their regular blocks are squares d_i^2 . The explicit block data identify these as $M_{d_i}(\mathbb{F}_p)$. For a source idempotent, its rank on the regular block is

$d_i m_v$, since the left regular module is d_i copies of the simple module. Dividing by d_i gives the recorded source multiplicity m_v . Appendix K.5.3 separates this splitting calculation from the trace test. [CALC C15, p. 132]

Appendix K gives the word counts, overlap counts, and integral coefficient checks; the witnesses and replay programs are listed in Appendix R. $\square$

9.10 Semisimplicity in characteristic zero

Specialization can lose blocks: $R/\mathfrak{m}$ has a nonzero special fiber and zero generic fiber. The next lemma shows that this is the only kind of loss we need to allow. A later Galois argument proves that every block survives here.

For instance, put $k = R/\mathfrak{m}$. The algebra $S = R \times (R/\mathfrak{m})$ has special fiber $k \times k$ but generic fiber K : its second summand is torsion. Thus a split special fiber need not have every block in the generic fiber. The lemma proves semisimplicity while allowing this precise possibility.

Lemma 9.6 (Integral semisimple descent). *Let R be a DVR with residue field $k = R/\mathfrak{m}$, and let S be an R -algebra finite as an R -module. If $S/\mathfrak{m}S$ is split semisimple, then the generic fiber is absolutely semisimple. Its absolute matrix blocks form a subcollection of the special-fiber blocks. For fixed idempotents of S , the ranks on each surviving simple block are preserved.*

Proof. Let $T = \{s \in S : \mathfrak{m}^n s = 0 \text{ for some } n\}$. Centrality of R makes T a two-sided ideal. The quotient $C = S/T$ is finite torsion-free over a DVR, hence finite free, and $C \otimes_R K = S \otimes_R K$. The map $S/\mathfrak{m}S \rightarrow C/\mathfrak{m}C$ has a two-sided ideal as kernel. In a product of simple matrix algebras such an ideal is a product of whole summands. Consequently $C/\mathfrak{m}C$ retains a subcollection of the original blocks, without shrinking a surviving block.

Pass to the $\mathfrak{m}$ -adic completion of R , retaining the notation for the completed free algebra. Completion is faithfully flat, so it suffices to establish the asserted absolute generic structure after this field extension. Newton iteration for $e^2 = e$ lifts the block idempotents, successively in complementary corners, to an orthogonal family. For distinct lifts, $e_i C e_j$ has zero special fiber, so Nakayama gives $e_i C e_j = 0$. The lifts are therefore central: different blocks cannot become linked.

The mixed-corner argument is important: it proves centrality of the lifted block units rather than presupposing it. Each mixed corner is finite over the complete local ring, and its zero residue module forces it to vanish by Nakayama.

In a component reducing to $M_d(k)$, lift a rank-one idempotent f . The module $C_i f$ is free of rank d . The action map

$$C_i \longrightarrow \text{End}_R(C_i f)$$

is an isomorphism modulo $\mathfrak{m}$ between free modules of rank d^2 . Its determinant is a unit, so $C_i \cong M_d(R)$. Thus the completed generic fiber is split semisimple. Descending along the field extension gives absolute semisimplicity and the same absolute block sizes over the original fraction field.

The completion may split a generic central simple algebra that was not already split over K . After passage to an algebraic closure the same matrix sizes occur; this is why the conclusion is *absolute* semisimplicity rather than an assertion that every block is already $M_d(K)$.

Each fixed source idempotent acts on $C_i f$ with a direct-summand image. Its rank is unchanged between the two fibers. These ranks are exactly the source profile of the surviving simple block. $\square$

Apply the lemma to Proposition 9.5. After extension of scalars to $\mathbb{C}$, S_K and its quotient D are semisimple, with dimension at most 546 and at most twelve simples.

Corollary 9.7. *The VOA U is strongly rational.*

Proof. The exact presentation, [CALC C13, p. 132] the integral spanning bound, [CALC C14, p. 132] and the split semisimple special fiber, [CALC C15, p. 132] together with Lemma 9.6, make D semisimple. Let J be the radical of $A(U)$, which is finite-dimensional by C_2 -cofiniteness. The nilpotent ideal $e_Z J e_Z$ in the semisimple corner is zero. Fullness gives $1 = \sum_i x_i e_Z y_i$, and hence, for $z \in J$,

$$z = \sum_{i,j} x_i e_Z (y_i z x_j) e_Z y_j = 0.$$

Thus $J = 0$, so $A(U)$ is semisimple as well.

All hypotheses of the rationality criterion are now available: the positive-form argument proves simplicity and self-contragredience (Section 3); the lattice grading is nonnegative and integral with $U_0 = \mathbb{C}\mathbf{1}$; Section 5 proves C_2 -cofiniteness; and the preceding full-corner argument proves semisimplicity of the finite-dimensional Zhu algebra. In particular, the criterion is being used only after Zhu semisimplicity has been established.

McRae's rationality theorem [20, Corollary 5.19] now applies: U is simple, self-contragredient and C_2 -cofinite, its grading is nonnegative and integral with $U_0 = \mathbb{C}\mathbf{1}$, and its Zhu algebra is semisimple. Thus U is strongly rational. $\square$

Rationality is now proved before module exhaustion.

Part IV

Modules and exhaustion

10 Six lattice modules

The lattice construction gives six concrete irreducible representations of U without assuming rationality. The following section uses the Zhu-algebra presentation to identify six further modules and prove exhaustion.

10.1 The modules supplied by lattice cosets

Choose minuscule representatives λ_E, λ_A of the two nonzero discriminant classes, of squared norms $4/3$ and $2/3$, and set

$$W_{e,a} = V_{L+e\lambda_E+a\lambda_A}, \quad e, a \in \{0, 1, -1\}.$$

Use simple-root coordinates in the order of Appendix B. The fixed representatives are

$$3\lambda_A = (2, 1, 0, 0, 0, 0, 0, 0), \quad 3\lambda_E = (0, 0, 4, 3, 5, 6, 4, 2).$$

Thus all nine coset labels are obtained by the stated choices of e, a , including zero. Their minimum shells are Weyl orbits of $e\lambda_E + a\lambda_A$; we sort $3p$ lexicographically to order the lowest-weight exponentials. Appendix I gives the reflection algorithm and an actual zero-mode entry.

Their nonvacuum lowest spaces have weights and dimensions $(1/3, 3)$, $(2/3, 27)$, and $(1, 81)$, according as only a , only e , or both coset labels are nonzero. Define the cyclic U -modules

$$\begin{aligned} B &= \langle W_1 \cap U_1^\perp \rangle_U, & P &= \langle (W_{0,1})_{1/3} \rangle_U, & Q &= \langle (W_{1,0})_{2/3} \rangle_U, \\ X_+ &= \langle (W_{1,1})_1 \rangle_U, & X_- &= \langle (W_{1,-1})_1 \rangle_U. \end{aligned}$$

The orthogonal complement defining B is inside the invariant U -module $U^\perp \subset W$. It is a genuine lowest space and has dimension $86 - 6 = 80$. For calculations it can be represented by the isomorphic quotient W_1/H .

Proposition 10.1. *The modules U, B, P, Q, X_+, X_- are pairwise nonisomorphic irreducible ordinary U -modules, all self-contragredient. Their lowest weights and top dimensions are*

	U	B	P	Q	X_+	X_-
h	0	1	$1/3$	$2/3$	1	1
dim top	1	80	3	27	81	81.

They give a simultaneous quotient

$$A(U) \twoheadrightarrow \mathbb{C} \oplus M_{80}(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_{27}(\mathbb{C}) \oplus M_{81}(\mathbb{C})^{\oplus 2}. \quad (10.1)$$

Proof. The zero-mode matrices in Appendix I have a one-dimensional commutant on each of the five lattice lowest spaces. The rank certificates modulo 10009 give an upper bound one on its characteristic-zero dimension; the identity gives the matching lower bound. Since the action is adjoint-closed for a positive form, [CALC C08, p. 131] a proper invariant subspace would have an invariant orthogonal complement and a nonscalar commuting projection. Hence each lowest space is irreducible. Complete reducibility of the lattice restrictions, Lemma 3.2, then makes its cyclic module irreducible at every weight.

In the commutant solve an unknown matrix entry is allowed only between equal pairs of Cartan weights. The exact weights have no aliases at the chosen prime. A selected nonzero minor gives a characteristic-zero lower bound on the equation rank, while the identity matrix lies in the exact kernel. Together these prove that the commutant consists of scalars. Positivity and adjoint closure are the extra ingredients that turn this calculation into irreducibility: a reducible action would supply an orthogonal commuting projection. Appendix I specifies both the matrix construction and the rank test.

Let $\gamma = (0, 1), (1, 0), (1, 1), (1, -1)$, write T_γ for the corresponding lattice lowest space, and put $M_\gamma = \langle T_\gamma \rangle_U$. For each pair, the exact matrices in Appendix I give a nonzero map

$$F_\gamma : T_\gamma \longrightarrow T_{-\gamma}, \quad F_\gamma o_\gamma(g) = o_{-\gamma}(g) F_\gamma$$

for all 22 generating classes g . [CALC C09, p. 131] Thus F_γ is an $A(U)$ -homomorphism. Its kernel and image are submodules of the simple tops, so it is an isomorphism. Zhu's correspondence now gives $M_\gamma \cong M_{-\gamma}$.

There is a separate duality argument. The standard invariant lattice pairing between W_γ and $W_{-\gamma}$ pairs e^p with e^{-p} by a nonzero scalar, and is nondegenerate on their whole lowest spaces. Its restriction to $M_\gamma \times M_{-\gamma}$ is therefore nonzero. The two radicals are U -submodules; simplicity makes them zero. This invariant pairing identifies $M_{-\gamma}$ with the restricted contragredient M'_γ . Consequently

$$M_\gamma \cong M_{-\gamma} \cong M'_\gamma.$$

The contragredient action and the lattice pairing are recalled in Appendix I.2.

The two constructions have different roles:

Construction	Input	Conclusion
Ordinary-top intertwiner F_γ	Two matrices for the ordinary Zhu action	$M_\gamma \cong M_{-\gamma}$.
Invariant lattice pairing B_γ	Opposite-coset lattice Heisenberg modules and contragredient action	$M_{-\gamma} \cong M'_\gamma$.

The first has no dual-action transpose in its equations. The second uses that transpose, with the signs in Appendix I.2.

For B , the involution ϕ of Proposition 3.1 preserves U, H and the positive form, hence preserves $B_1 = W_1 \cap H^\perp$. Since it preserves vertex modes, it also preserves $B = \langle B_1 \rangle_U$. For $0 \neq x \in B$, $B_W(x, \phi x) = H_\chi(x, x) > 0$. Thus the invariant bilinear form is nondegenerate on each grade of B , proving $B \cong B'$.

The cross-Hom between the two 81-dimensional tops is zero, as certified by a full-rank minor. **[CALC C08, p. 131]** Thus $X_+ \not\cong X_-$. Lowest weights or top dimensions distinguish all other pairs. Finally, the image of a finite-dimensional complex algebra on a simple space is the full matrix algebra. The six distinct maximal two-sided ideals therefore give (10.1) by Chinese remainders. $\square$

The quotient has dimension 20,261. It supplies a lower bound on $A(U)$, rather than an exhaustion theorem. The remaining work must bound the whole algebra, including any unseen simple blocks or radical.

Taking the highest-weight corner of the quotient supplied by these modules gives

$$D \rightarrow \mathbb{C} \oplus M_9(\mathbb{C}) \oplus \mathbb{C} \oplus M_4(\mathbb{C}) \oplus M_{10}(\mathbb{C})^{\oplus 2}. \quad (10.2)$$

The summands correspond respectively to U, B, P, Q, X_+, X_- .

If the lowest space decomposes under the affine source as $\bigoplus_v T_v \otimes \mathbb{C}^{m_v}$, the selected rank-one idempotent in each block gives

$$\dim e_Z \text{top}(M) = \sum_v m_v, \quad \dim \text{top}(M) = \sum_v (\dim T_v) m_v.$$

For example, Q has four source copies of sizes 1, 5, 9, 12, so its corner dimension is 4 while its top dimension is 27. Appendix J.2 gives the decompositions producing 1, 9, 1, 4, 10, 10 in the displayed corner quotient.

11 Exhaustion of the ordinary modules

11.1 The conformal weights are forced by traces

We will use three dimensions, each with a different meaning. The *corner dimension* d is the dimension of a simple D -module. The *full top dimension* is the dimension after inflation to an $A(U)$ -module. The *categorical dimension* is a pivotal trace in $\text{Rep}(U)$ and will be treated in the next subsection. Only the first two enter the ordinary matrix traces below.

Let Z denote the total conformal class in the corner, the image of $[\omega_U]$. It is central. The certificate gives an exact identity in the cocenter of S_K :

$$Z \equiv \sum_v \alpha_v e_v \pmod{[S_K, S_K]}. \quad (11.1)$$

Here $[S_K, S_K]$ is the linear span of $xy - yx$, not a two-sided ideal. Every finite-dimensional representation trace kills it, because $\text{tr}(xy) = \text{tr}(yx)$. Thus the congruence gives an equality of traces without asserting equality of the two elements in the algebra. The coefficients and source ranks are listed in Appendix K.6. Its witness expresses the difference as a sum of explicit arrow commutators, with zero residual over K . **[CALC C16, p. 133]**

If a simple corner module has dimension d , source ranks m_v , and actual lowest conformal weight h , the class Z acts as $h \text{id}$: the Zhu action of $[\omega_U]$ is L_0 on the lowest space. Taking the ordinary trace in (11.1) gives

$$hd = \sum_v \alpha_v m_v. \quad (11.2)$$

Profile preservation in Lemma 9.6 therefore fixes every possible additional lowest weight:

$$\frac{5}{13}, \quad \frac{6}{13}, \quad \frac{7}{13}, \quad \frac{8}{13}, \quad \frac{11}{13}, \quad \frac{15}{13}.$$

The full lowest-space dimension is $\sum_v s_v m_v$, where s_v is the affine source matrix size. This differs from the corner dimension d . The profile calculation is in Appendix K. **[CALC C16, p. 133]** For example, one allowed profile has $m_2 = m_{18} = 1$. Its corner dimension is 2, but its full top dimension is $5 + 8 = 13$; its conformal weight is

$$h = \frac{5/13 + 9/13}{2} = \frac{7}{13}.$$

For reference, the six possible new profiles are visible here. They are conditional on survival at this point.

h	$d = \sum m_v$	$\sum \alpha_v m_v$	Nonzero source multiplicities
5/13	1	5/13	$m_2 = 1$
6/13	1	6/13	$m_3 = 1$
7/13	2	14/13	$m_2 = m_{18} = 1$
8/13	3	24/13	$m_1 = m_3 = m_{19} = 1$
11/13	6	66/13	$m_0 = m_2 = m_3 = m_4 = m_{16} = m_{19} = 1$
15/13	14	210/13	$m_v = 1$ for $v = 0, 1, 6, 15, 16, 17, 18, 19$; $m_2 = m_3 = m_4 = 2$

The weights come from dividing the trace numerator by d ; the special-fiber numerical eigenvalue hints are not used to justify their characteristic-zero lifts.

All possible nonvacuum lowest weights, including the five known ones, are strictly positive.

11.2 Positive categorical dimensions

Lemma 11.1. *For a simple, self-contragredient, strongly rational VOA whose nonvacuum simples have strictly positive lowest weights, the existing VOA spherical dimensions are positive and equal to Frobenius–Perron dimensions. In particular this applies to U .*

Proof. The VOA trace-function S matrix agrees, up to a common sign, with the normalized categorical matrix [21, Theorems 3.9–3.10]. This statement uses all trace functions with insertions, so it does not require distinct scalar characters.

No positive Hermitian form on every ordinary U -module is being assumed. Positivity here comes first from ordinary character coefficients, then from the agreement of trace and categorical modular

matrices. Insertions make that agreement applicable even if scalar characters coincide. On the positive imaginary axis, modular transformation gives

$$\chi_Y(it) = \sum_Z S_{YZ}^{\text{tr}} \chi_Z(i/t).$$

For t small, $\chi_Z(i/t)$ has exponential leading factor $\exp(2\pi(c/24 - h_Z)/t)$ times its positive top dimension. Each nonvacuum term is smaller than the vacuum term by $\exp(-2\pi h_Z/t)$. This is the local reason that the following ratio reads the vacuum column of the S matrix.

Since every other lowest weight is positive, the vacuum alone gives the leading term as $t \rightarrow 0^+$. Thus

$$\lim_{t \rightarrow 0^+} \frac{\chi_Y(it)}{\chi_U(it)} = \frac{S_{Y0}^{\text{tr}}}{S_{00}^{\text{tr}}} = \dim_{\text{VOA}} Y$$

by [21, Proposition 3.11]. The characters are positive there, so the limit is nonnegative. A simple pivotal dimension is nonzero: its trace pairs two nonzero maps through the unique copy of the tensor unit in $Y^{**} \otimes Y^*$. Semisimplicity makes this pairing nonzero. Hence the limit is strictly positive. The vacuum asymptotic also fixes $S_{00}^{\text{tr}} > 0$.

Positive spherical dimensions give a positive dimension homomorphism of the fusion ring, so they equal the unique Frobenius–Perron homomorphism. The specified VOA pivotal structure is therefore the unique positive one [22, Proposition 8.23]. All possible nonvacuum weights of U have already been shown positive, so the argument applies before deciding which new blocks survive. $\square$

11.3 The lattice extensions and their branching identities

Before using a branching matrix, we need the extension to define a separable algebra in the modular category. The positive dimensions just proved supply this point without an assumption that every U -module is unitary.

Lemma 11.2. *Let V be $W = V_{E_6 \oplus A_2}$ or either of its two E_8 lattice extensions. The inherited vertex products make $V \downarrow_U$ a connected commutative separable algebra of trivial twist in $\text{Rep}(U)$. Its local-module category is $\text{Rep}(V)$. If b is its full branching matrix, then*

$$bS_U = S_V b, \quad bT_U = T_V b.$$

Here *connected* means $\text{Hom}_{\text{Rep}(U)}(\mathbf{1}, A_V) = \mathbb{C}$; *separable* means that multiplication has a bimodule section. A module is *local* when its action is unchanged by double braiding with the algebra. All algebra modules and local algebra modules therefore have different roles: the latter are the ordinary modules of the extension.

Proof. The restriction is a finite semisimple ordinary module by Lemma 3.2. The conformal inclusion and the actual extension products give the algebra by [19, Theorem 3.2]; [19, Theorem 3.4] identifies its local modules with $\text{Rep}(V)$. Since $V_0 = \mathbb{C}\mathbf{1}$, it is connected; since V is integrally graded, its twist is trivial.

Write $A_V = V \downarrow_U$. The normalized invariant vacuum pairing of the lattice VOA gives a non-degenerate symmetric Frobenius pairing on A_V . Normalize the counit ϵ by $\epsilon\eta = 1$ and let Δ be the corresponding coproduct. The Frobenius identities make $m\Delta$ a bimodule endomorphism of the regular module. Connectedness gives $\text{End}_{A_V}(A_V) \cong \text{Hom}_{\text{Rep}(U)}(\mathbf{1}, A_V) = \mathbb{C}$, so write $m\Delta = \lambda \text{id}_{A_V}$. The pairing ϵm and copairing $\Delta\eta$ are its symmetric duality maps. Taking their closed trace gives

$$\lambda = \epsilon m \Delta \eta = \dim_{\text{Rep}(U)} A_V.$$

This dimension is positive by Lemma 11.1. Thus

$$\frac{\Delta}{\dim_{\text{Rep}(U)} A_V} : A_V \longrightarrow A_V \otimes A_V$$

is a bimodule section of multiplication. This proves separability. The Frobenius and local-module conventions are those of [23, 19]. The categorical branching identities now apply [24, Equation (1.1)]. $\square$

11.4 One new block must survive

Lemma 11.3. *The semisimple corner has more than the six known simple blocks.*

Proof. Suppose there were exactly six simples. Since the inclusion is conformal, their lowest weights and the actual lattice lowest-weight spaces force full branching: W restricts as $U \oplus B$, and the eight nonzero lattice coset modules restrict as two copies each of P, Q, X_+, X_- . There is no other module of an eligible weight residue or higher integral lowest weight.

In the hypothetical six-module situation the complete matrix is

(e, a)	U	B	P	Q	X_+	X_-
$(0, 0)$	1	1	0	0	0	0
$(0, \pm 1)$	0	0	1	0	0	0
$(\pm 1, 0)$	0	0	0	1	0	0
$(1, 1), (-1, -1)$	0	0	0	0	1	0
$(1, -1), (-1, 1)$	0	0	0	0	0	1

Each paired row occurs twice. This is the matrix under the six-module assumption; the actual post-exhaustion branching table is in Lemma 12.1.

Let b be this branching matrix, in column order U, B, P, Q, X_+, X_- . It has equal columns for U and B , and distinguishes the other four columns. Therefore

$$\ker b = \mathbb{C}v, \quad v = (1, -1, 0, 0, 0, 0)^\top.$$

The branching identity $bS_U = S_W b$ preserves this kernel, so $S_U v = \lambda v$. Thus the difference between the vacuum and B would have to be an S -eigenvector. All six modules are self-dual, hence $S_U^2 = I$ and $\lambda = \pm 1$.

By Lemma 11.1, let $d_Y > 0$ be their categorical dimensions and $\mathcal{D}_0^2 = \sum_Y d_Y^2$. The vacuum coordinate is

$$\lambda = \frac{1 - d_B}{\mathcal{D}_0}, \quad \mathcal{D}_0^2 \geq 1 + d_B^2 > (1 - d_B)^2.$$

This contradicts $|\lambda| = 1$. $\square$

11.5 One Galois orbit forces all six new blocks

The character matrix uses $T_{MM} = e^{2\pi i(h_M - c/24)}$, with $c = 8$. It differs from the ribbon twist $e^{2\pi i h_M}$. The six candidate entries have the following labels:

h	$h - c/24$	Exponent $a \bmod 13$
5/13	2/39	5
6/13	5/39	6
7/13	8/39	7
8/13	11/39	8
11/13	20/39	11
15/13	32/39	2

The conductor is 39. By the Chinese remainder theorem each unit modulo 13 lifts to a cyclotomic unit modulo 39; its square is one modulo 3. Hence the square action fixes the common factor ζ_3^{-1} and acts on a through the six quadratic residues modulo 13. This is the modular Galois permutation of actual simples [21, Corollary 4.4], distinct from conjugating the definition field in Appendix F. Appendix R.1 and the C17 guide record the finite arithmetic.

Take one additional actual simple, whose existence is now proved. Its normalized genus-one T eigenvalue is

$$\zeta_3^{-1} \zeta_{13}^a, \quad a \in \{2, 5, 6, 7, 8, 11\}.$$

By VOA Galois symmetry, the diagonal entries of $\sigma^2(T)$ are permutations of those of T [21, Corollary 4.4]. Squares of cyclotomic units fix ζ_3^{-1} and act modulo 13 through the square subgroup $\{1, 3, 4, 9, 10, 12\}$. Its orbit on any of these six nonresidues is the entire set $\{2, 5, 6, 7, 8, 11\}$. Thus six distinct new T eigenvalues occur. **[CALC C17, p. 133]** None is an eigenvalue of a known module. The upper presentation allows at most six additional blocks, so every one survives. This simultaneously rules out loss through torsion and loss through the quotient map to the actual corner.

Theorem 11.4. *The VOA U is strongly rational and has exactly twelve simple ordinary modules, with the following lowest weights and top dimensions:*

<i>Module</i>	h	<i>Top dimension</i>	<i>Corner dimension</i>
U	0	1	1
P	1/3	3	1
$M_{5/13}$	5/13	5	1
$M_{6/13}$	6/13	7	1
$M_{7/13}$	7/13	13	2
$M_{8/13}$	8/13	14	3
Q	2/3	27	4
$M_{11/13}$	11/13	42	6
B	1	80	9
X_+, X_-	1	81 each	10 each
$M_{15/13}$	15/13	119	14

All are self-contragredient. The semisimple Zhu algebra and its full corner have dimensions 36,625 and 546, respectively; their matrix-block decompositions are in Appendix K.

Proof. The preceding arguments prove rationality and the complete simple list. Every allowed corner block survives, so the upper bound 546 is attained. The surjection $S_K \otimes_K \mathbb{C} \twoheadrightarrow D$ is between semisimple algebras with the same twelve absolute block sizes and total dimension 546, so it is an isomorphism. This equality is asserted after the indicated scalar extension. The six additional modules exist by survival and Zhu's correspondence; the concrete lattice realizations supplied above

are for the first six modules. Inflation through the source matrix units gives the full top dimensions and the matrix-block decomposition of $A(U)$ in Appendix K. [CALC C16, p. 133] The six old modules are self-contragredient by Proposition 10.1. The restricted contragredient of an ordinary simple is simple with the same lowest weight (Appendix I.2). Each new weight belongs to exactly one simple, so its contragredient is that same module. $\square$

The listed top dimensions are dimensions of Zhu representations. Categorical dimensions are computed in Section 12.

Part V

Proof of Equivalence

12 Recovering a Haagerup fusion category

Put $\mathcal{C} = \text{Rep}(U)$ with its actual VOA ribbon structure. The categorical route uses the lattice extensions themselves, including their multiplication. The modular matrix will enter only to determine a dimension, while the associator is handled separately.

12.1 The full lattice branching

Lemma 12.1. *The nine restrictions are*

$$\begin{aligned} W_{0,0} \downarrow_U &= U \oplus B, & W_{0,\pm 1} \downarrow_U &= P, & W_{\pm 1,0} \downarrow_U &= Q, \\ W_{1,1} \downarrow_U &= W_{-1,-1} \downarrow_U = X_+, & W_{1,-1} \downarrow_U &= W_{-1,1} \downarrow_U = X_-. \end{aligned}$$

Proof. Rationality makes each restriction semisimple, and the conformal inclusion preserves its actual L_0 . The twelve-module list leaves only P in weight residue $1/3$ and only Q in residue $2/3$; their tops already exhaust the corresponding lattice lowest-weight spaces. In a nonzero lattice coset module of integral lowest weight, the embedded X_+ or X_- exhausts all 81 vectors of weight one. The vacuum module cannot occur there, and no simple has a higher integral lowest weight. Finally, $W_0 = \mathbb{C}$ and $\dim W_1 = 86$ are exhausted by the embedded U and B , with dimensions 1 and $6 + 80$ in these grades. The exhaustive weight list excludes any further constituent. [CALC C08, p. 131] [CALC C16, p. 133] $\square$

The canonical coset label table, used for branching below and for the algebra vectors in Appendix O.2, is

Coset labels (e, a)	Restriction to U	Lowest lattice weight
$(0, 0)$	$U \oplus B$	0
$(0, 1), (0, -1)$	P in each coset module	$1/3$
$(1, 0), (-1, 0)$	Q in each coset module	$2/3$
$(1, 1), (-1, -1)$	X_+ in each coset module	1
$(1, -1), (-1, 1)$	X_- in each coset module	1

Each pair represents two rows of the full nine-row branching matrix. The new weights have nonzero denominator-13 residues, so none can occur in an integrally graded lattice coset module or a coset

module with residue $1/3$ or $2/3$. This explains why the same table, first considered under the six-module hypothesis, is now the proved full restriction.

The discriminant group and its quadratic form are

$$G = L^*/L = \mathbb{Z}_3^2, \quad q(e, a) = \frac{2e^2 + a^2}{3} \pmod{\mathbb{Z}}.$$

The two lines $H_+ = \langle(1, 1)\rangle$ and $H_- = \langle(1, -1)\rangle$ are isotropic. Their index-three overlattices are even unimodular lattices of rank eight, hence E_8 . The holomorphic lattice VOAs restrict as

$$V_{E_8, \pm} \downarrow_U = U \oplus B \oplus 2X_{\pm}.$$

Call the corresponding algebra objects $A_{\pm}$.

Lemma 12.2. *The actual E_8 products make $A_{\pm}$ Lagrangian algebras, and*

$$\mathcal{C} \simeq_{\text{br}} Z(\mathcal{D}_{\pm}), \quad \mathcal{D}_{\pm} = \mathcal{C}_{A_{\pm}}.$$

Proof. By Lemma 11.2, $A_{\pm}$ are connected commutative separable algebras of trivial twist, with local modules given by the actual E_8 extensions. Holomorphicity makes the local-module category Vec . The local-module dimension formula gives $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$, so A is Lagrangian. The center theorem then gives the stated braided equivalence [25, Corollaries 3.32 and 4.1]. $\square$

The algebras here include their inherited multiplication: the underlying direct sums alone would not specify an extension. In the modular category $\mathcal{C} = \text{Rep}(U)$, Lemma 11.2 supplies connected commutative separable algebras and the equivalence $\mathcal{C}_{A_{\pm}}^0 \simeq \text{Rep}(V_{E_8, \pm}) = \text{Vec}$. The dimension formula [25, Corollary 3.32] reads

$$\text{FPdim}(\mathcal{C}_A^0) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)^2}.$$

Thus its value one is exactly the Lagrangian dimension condition. The center equivalence then applies to the category of *all* right A -modules $\mathcal{C}_A$, by [25, Corollary 4.1(i)].

Fix $A = A_+$ and $\mathcal{D} = \mathcal{C}_A$, using right modules. If $r_M : M \otimes A \rightarrow M$ is the action, convert it to a left action by

$$\ell_M^- = r_M \circ c_{M, A}^{-1} : A \otimes M \longrightarrow M.$$

The braiding here is that of $\mathcal{C}$. This fixes the relative tensor product $M \otimes_A N$ and the orientation of the central free-module functor from its first use; see Section 13 for the ribbon comparison. Its induced spherical dimensions are

$$\dim_{\mathcal{D}} M = \frac{\dim_{\mathcal{C}}(\text{For } M)}{\dim_{\mathcal{C}} A} > 0$$

[23, Theorems 1.17–1.18]. We work with this positive pivotal structure. The other sign exchanges X_+ and X_- .

12.2 Why there are six simples, including three invertibles

We distinguish the four functors:

Functor	Source and target	Purpose
F	$Z(\mathcal{D}) \rightarrow \mathcal{D}$	Forget the half-braiding
I	$\mathcal{D} \rightarrow Z(\mathcal{D})$	Right adjoint of F
F_A	$\mathcal{C} \rightarrow \mathcal{D}$	Free right module $Y \mapsto Y \otimes A$
For	$\mathcal{D} \rightarrow \mathcal{C}$	Underlying U -object

Under $\mathcal{C} \simeq Z(\mathcal{D})$, F_A corresponds to F and For to I . The intermediate algebra $R_0 = \mathbf{1} \oplus B$ corresponds to W , not to the full E_8 algebra A .

Let $F : Z(\mathcal{D}) \rightarrow \mathcal{D}$ be forgetful and I its right adjoint. Under the center equivalence, $I(\mathbf{1}) = A$. Since $FI(\mathbf{1}) = \bigoplus_Y Y \otimes Y^*$, adjunction gives

$$\text{rank}(\mathcal{D}) = \dim \text{End}_{\mathcal{C}}(A) = 1^2 + 1^2 + 2^2 = 6. \quad (12.1)$$

This count uses the actual algebra's underlying object.

Indeed $\text{End}_{Z(\mathcal{D})}(I\mathbf{1}) \cong \text{Hom}_{\mathcal{D}}(FI\mathbf{1}, \mathbf{1})$. The latter has dimension one from each summand $Y \otimes Y^*$, by simplicity of Y . It therefore counts the simple objects of $\mathcal{D}$. On the $\mathcal{C}$ side the multiplicities in $A = \mathbf{1} \oplus B \oplus 2X_+$ are 1, 1, 2, giving $1 + 1 + 4 = 6$. The same three multiplicities will determine the Grothendieck algebra in Section 12.4.

The intermediate lattice VOA W gives the algebra $R_0 = \mathbf{1} \oplus B$. Its local-module category is the pointed metric-group category $\mathcal{E} = \text{Rep}(W)$ on G . The extension from W to E_8 is the algebra S supported on H_+ . Viewing an S -module as an A -module embeds $\mathcal{E}_S$ as a tensor subcategory of $\mathcal{C}_A$. Indeed, an A -linear map is R_0 -linear, hence a W -module map; its remaining condition is exactly S -linearity. This proves fullness. Successive tensor products over R_0 and S give the tensor product over A .

Lemma 12.3. *The category $\mathcal{D}$ contains distinct invertibles $\mathbf{1}, \alpha, \alpha^2$, with $\alpha^3 = \mathbf{1}$, and their pointed subcategory is the untwisted $\text{Vec}_{\mathbb{Z}_3}$. Their underlying $\mathcal{C}$ -objects are*

$$\text{For}(\mathbf{1}) = \mathbf{1} \oplus B \oplus 2X_+, \quad \text{For}(\alpha) = \text{For}(\alpha^2) = P \oplus Q \oplus X_-.$$

Proof. The three G/H_+ coset modules are distinct simple S -modules of dimension one; full faithfulness retains their distinctness in $\mathcal{D}$. Each nontrivial coset contains one lattice coset label of each type P, Q, X_- , giving their underlying objects.

The algebra $S = \bigoplus_{h \in H_+} h$ has its actual lattice-extension product inside the local R_0 -module category $\mathcal{E} = \text{Rep}(W)$. Induction transitivity identifies successive free modules over R_0 and S with free modules over A . The fully faithful embedding above retains the tensor structure, not just the three coset labels.

To fix the associator, realize the pointed metric group with trivial associator and braiding $c((e, a), (f, c)) = \zeta_3^{2ef+ac}$. Its trace is the stated quadratic form. The complementary isotropic line H_- has trivial braiding and associator, and meets each coset of H_+ exactly once.

The identification with this explicit braided model is the metric-group classification [28, Proposition 2.41 and Appendix D.1]. For example, $(1, -1)$ generates H_- , and its self-braiding is $\zeta_3^{2+1} = 1$. Its three coset labels are a coherent complement to H_+ : the quotient map $H_- \rightarrow G/H_+$ is an isomorphism. Multiplication on this line has trivial associator, so the tensor constraints of induction can be chosen coherently with its group multiplication.

Free induction restricted to H_- is therefore a tensor equivalence onto $\mathcal{E}_S$, proving the assertion.

This fixes the associator only on the pointed $\mathbb{Z}_3$ subcategory. The associator involving noninvertible objects is determined separately below.

□

The group acts freely on the six simples. Indeed, adjunction gives

$$\mathrm{Hom}_{\mathcal{D}}(FI(\mathbf{1}), \alpha) = \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{For} \alpha) = 0.$$

A fixed simple $\alpha Y \cong Y$ would contribute to $\mathrm{Hom}_{\mathcal{D}}(Y \otimes Y^*, \alpha)$, contradicting this vanishing.

Explicitly, rigidity gives $\mathrm{Hom}_{\mathcal{D}}(Y \otimes Y^*, \alpha) \cong \mathrm{Hom}_{\mathcal{D}}(Y, \alpha \otimes Y)$. A fixed point would make this space nonzero. But $Y \otimes Y^*$ is a summand of $FI(\mathbf{1}) = \bigoplus_Z Z \otimes Z^*$, whose Hom to α was just shown zero. Thus no such fixed point can occur. A free action of a group of order three has orbits of size three, so the remaining three simples form exactly one orbit.

The same holds for α^2 , and for right action by duality. Thus the simples form two three-element orbits: the invertibles already exhibited, and one further orbit.

12.3 One dimension fixes the second orbit

Let b be the full branching matrix of Lemma 12.1. Lemma 11.2 gives

$$bS_{\mathcal{C}} = S_W b, \quad bT_{\mathcal{C}} = T_W b, \quad (S_W)_{(e,a),(f,c)} = \frac{1}{3} \zeta_3^{-(ef-ac)}. \quad (12.2)$$

This is the categorical branching identity, which applies even when some scalar characters coincide. All simples are self-dual, so $S_{\mathcal{C}}^2 = I$. Their fusion matrices are real symmetric. Verlinde eigenvalues are therefore real; positivity of the vacuum row makes $S_{\mathcal{C}}$ real. It is consequently orthogonal.

The branching determines its action on the five-dimensional row space

$$E_5 = \mathrm{span}\{e_0 + e_B, e_P, e_Q, e_{X_+}, e_{X_-}\}.$$

The remaining space is $K_7 = \mathrm{span}\{v, e_{\mu_1}, \dots, e_{\mu_6}\}$, where $v = (e_0 - e_B)/\sqrt{2}$ and the μ_j are ordered by twist exponents $2j^2 \bmod 13$. Its T spectrum is

$$\zeta_3^{-1}(1, \zeta_{13}^2, \zeta_{13}^8, \zeta_{13}^5, \zeta_{13}^6, \zeta_{13}^{11}, \zeta_{13}^7).$$

For the usual orthonormal inner product on the module-label space, E_5 has normalized basis $(e_0 + e_B)/\sqrt{2}, e_P, e_Q, e_{X_+}, e_{X_-}$. It is invariant under S by the branching identity and under T by the listed weights. Since S is orthogonal and T is diagonal unitary, its orthogonal complement K_7 is invariant too. The congruence representation on this complement, rather than the full twelve-dimensional space, is what the finite-group degree calculation identifies.

Lemma 12.4. *On this seven-dimensional space, $S_{vv} = -1/\sqrt{13}$.*

Proof. The congruence theorem makes the genus-one representation factor through $\mathrm{SL}_2(\mathbb{Z}/39)$ [21, Theorem 3.10]. The 3-factor acts by a scalar character: its unipotent T^{13} is scalar, and the normal conjugates generate that factor. The character has S value one. Removing it leaves a seven-dimensional representation of $\mathrm{PSL}_2(\mathbb{F}_{13})$.

The factorization uses $\mathrm{SL}_2(\mathbb{Z}/39) \cong \mathrm{SL}_2(\mathbb{F}_3) \times \mathrm{SL}_2(\mathbb{F}_{13})$. Normal conjugates of the scalar unipotent make the 3-factor scalar. Its character has order dividing three, whereas S has order dividing four, so that character takes S to one. Finally the central $-I$ on the 13-factor acts as $S^2 = I$, giving the projective group.

The only nontrivial irreducibles of dimension at most seven are the two even Weil representations. Their degrees and their realization are checked in Appendix O. [CALC C18, p. 133] Since our T action is nontrivial, the representation has a nontrivial constituent. Maschke's theorem makes it a direct sum of irreducibles, and the finite-group degree list has no nontrivial degree below seven. Since its total dimension is seven, it is irreducible and must be one of the two degree-seven models.

The model is the space of even functions on $\mathbb{F}_{13}$. Its T action multiplies the value at x by $\zeta_{13}^{ax^2}$, with $a = 1$ or 2 . Its S action is a normalized finite Fourier transform. Our exponents select $a = 2$, the nonresidue orbit. The exponent-zero line is the delta function at zero, and the Fourier coefficient on that line is $-1/\sqrt{13}$. Appendix O gives the Fourier formula and an elementary Gauss-sum calculation fixing its sign. [CALC C18, p. 133] This line corresponds to $\mathbb{C}v$, and rescaling it does not change the diagonal entry. $\square$

Put $u = (e_0 + e_B)/\sqrt{2}$, so $e_0 = (u + v)/\sqrt{2}$. The two invariant spaces are orthogonal, hence $S_{uv} = S_{vu} = 0$ and $S_{00} = (S_{uu} + S_{vv})/2$.

Branching gives S entry $1/3$ on $(e_0 + e_B)/\sqrt{2}$. Therefore

$$S_{00} = \frac{1}{6} - \frac{1}{2\sqrt{13}}, \quad \text{Dim}(\mathcal{D}) = \sqrt{\text{Dim}(\mathcal{C})} = \frac{1}{S_{00}} = \frac{39 + 9\sqrt{13}}{2}. \quad (12.3)$$

Here $\text{Dim}(\mathcal{C}) = S_{00}^{-2}$ is the modular global dimension. The center identity $\text{Dim}(Z(\mathcal{D})) = \text{Dim}(\mathcal{D})^2$ and positivity choose the positive square root, so $\text{Dim}(\mathcal{D}) = S_{00}^{-1}$. These are categorical dimensions; the top and corner dimensions of Section 11 have no role in this projection.

The second orbit has common dimension d . It cannot be invertible, since this global dimension is not six. Hence

$$3 + 3d^2 = \text{Dim}(\mathcal{D}), \quad d = \frac{3 + \sqrt{13}}{2}, \quad d^2 = 1 + 3d. \quad (12.4)$$

The same branching and Weil calculation gives

$$\dim B = d^2, \quad \dim P = \dim Q = \dim X_+ = \dim X_- = 1 + d^2, \quad \dim \mu_j = 3d. \quad (12.5)$$

The auxiliary full matrix and its exact checks are in Appendix O. [CALC C19, p. 133]

12.4 The fusion rules follow from these constraints

Duality acts as an involution on the three noninvertibles, so choose a self-dual one ρ . Its left orbit is $\rho, \alpha\rho, \alpha^2\rho$. The free right action commutes with the left action, so it is one of the two cyclic shifts. The multiplicities $1, 1, 2$ in $I(\mathbf{1})$ are the dimensions of the irreducible representations of the Grothendieck algebra [26, Theorem 2.13]. Its block decomposition is therefore

$$K_0(\mathcal{D}) \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}). \quad (12.6)$$

It is noncommutative. If right multiplication by α gave the same shift as left multiplication, the two generators α, ρ would commute. Thus the right shift is opposite:

$$\alpha\rho = \rho\alpha^2. \quad (12.7)$$

By self-duality and the free action,

$$\dim \text{Hom}(\alpha^i, \rho^2) = \dim \text{Hom}(\alpha^i \rho, \rho) = \delta_{i,0}.$$

Thus the unit is the only invertible constituent, with multiplicity one. The relation (12.7) makes ρ^2 commute with α . Conjugation by α fixes ρ^2 and cycles the three noninvertibles. Their multiplicities must therefore be equal, say m . Positive dimensions give $d^2 = 1 + 3md = 1 + 3d$, so $m = 1$. We have proved the actual fusion rules

$$\boxed{\alpha^3 = \mathbf{1}, \quad \alpha\rho = \rho\alpha^2, \quad \rho^2 = \mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho.} \quad (12.8)$$

All three noninvertibles are consequently self-dual.

For later identification it is useful to know the free-module functor. Put $X = \rho \oplus \alpha\rho \oplus \alpha^2\rho$. The half-braiding makes every free image invariant under conjugation by α , so its noninvertible multiplicities are equal. Adjunction with the underlying objects in Lemma 12.3 fixes the invertible multiplicities, and (12.5) fixes the remainder:

U -module	Free A_+ -module
U	$\mathbf{1}$
B	$\mathbf{1} \oplus X$
X_+	$2\mathbf{1} \oplus X$
P, Q, X_-	$\alpha \oplus \alpha^2 \oplus X$
Each μ_j	X

For example, adjunction gives one unit in $F_A(B)$ from $\text{Hom}_{\mathcal{C}}(B, \text{For } \mathbf{1}) = \mathbb{C}$, and gives no α or α^2 from their underlying objects. Conjugation invariance makes its three noninvertible multiplicities equal, say m . Since a tensor functor preserves the positive dimension homomorphism, $\dim_{\mathcal{D}} F_A(B) = \dim_{\mathcal{C}} B = d^2 = 1 + 3d$. Thus $1 + 3md = 1 + 3d$ and $m = 1$, proving $F_A(B) = \mathbf{1} \oplus \rho \oplus \alpha\rho \oplus \alpha^2\rho$.

In particular ρ is an actual summand of $B \otimes A$: its endomorphism algebra is $\mathbb{C}^4$, and the three nonvacuum primitive projectors give the three noninvertibles.

This is an existence statement about projectors in a semisimple categorical endomorphism algebra. It does not assert that coordinate matrices for those VOA projectors are among the exported lattice certificates.

Adjunction also gives their underlying U -objects:

$$\text{For}(\alpha^i \rho) = B \oplus P \oplus Q \oplus X_+ \oplus X_- \oplus \bigoplus_{j=1}^6 \mu_j. \quad (12.9)$$

12.5 The associator and the classification bridge

Fusion rules alone do not identify a tensor category. We must also determine how the decompositions of triple products agree. Every object is built from α and ρ , and every word reduces by splitting ρ^2 . Once those four split maps are chosen, the associator is recorded by how left tensoring with ρ acts on them and their projections. Comparing these actions gives the classification equations below; we do not need a separate list of VOA F -symbols.

The bridge has the following local route. Coherent pointed normalization gives strict tensor words (Appendix M.1); padding embeds their actual Hom spaces faithfully in an algebra (Appendix M.2); the four summands of ρ^2 give an injective Leavitt subalgebra (Appendix M.3); typed coefficient comparisons give the classification equations (Appendix M.4); reconstruction and a recursive split prove full equivalence (Appendix M.5).

Locally, choose $s : \mathbf{1} \rightarrow \rho^2$ and $t_g : \alpha_g \rho \rightarrow \rho^2$ with inverse projections s', t'_g . For instance, $(s' \otimes \text{id}_\rho) \circ (\text{id}_\rho \otimes s) : \rho \rightarrow \rho$ is a scalar a because ρ is simple. The subsequent comparisons determine a , the normalization scalars, and the matrix entries in specified finite Hom spaces.

Proposition 12.5 (Normalization into the algebraic classification). *A complex fusion category with (12.8) and untwisted pointed subcategory $\text{Vec}_{\mathbb{Z}_3}$ is tensor equivalent to one of the Evans–Gannon categories $\mathcal{T}(\mathbb{Z}_3; \pm, \omega, M)$.*

Proof. Appendix M proves the normalization from the actual categorical splitting maps. Coherent invertible actions and the four summands of ρ^2 reduce the associator to finite coefficient comparisons. They give $a^2 + 3a = 1$, a cube root ω , and the matrix equations of Appendix N. For $\mathbb{Z}_3$, two quadratic entries already force $\omega = 1$, as shown there. **[CALC C20, p. 133]**

Evans–Gannon reconstruction [27, Theorem 2] then gives a category with the same six simple types. The appendix proves tensor compatibility of the faithful functor and expands each word morphism into matrix entries between those types. The entries are scalar or zero in both categories, so the functor is full and gives the claimed tensor equivalence. **[CALC C21, p. 133]** $\square$

Corollary 12.6. *The fusion categories $\mathcal{D}_\pm$ are tensor equivalent to $\mathcal{H}_2$ or $\mathcal{H}_3$, and $\text{Rep}(U) \simeq_{\text{br}} Z(\mathcal{H}_3)$.*

Proof. For $\mathbb{Z}_3$, Evans–Gannon’s classification has four classes: $\mathcal{H}_2, \mathcal{H}_3$ and their negative-dimension Galois associates [27, Theorem 3(ii)]. **[CALC C20, p. 133]** Our category has its positive pivotal structure. That structure is unique here even without positivity: a tensor automorphism of the identity has scalar z on ρ ; the appearances of $\mathbf{1}$ and ρ in ρ^2 force $z^2 = 1$ and $z^2 = z$, so $z = 1$. The remaining summands force its scalars on every invertible to be one. Thus a tensor equivalence cannot replace the positive dimensions by the negative branch. Only $\mathcal{H}_2, \mathcal{H}_3$ remain.

To see the pivotal uniqueness explicitly, a tensor automorphism η acts by scalar z on ρ and by z^2 on every summand of ρ^2 . The summands $\mathbf{1}$ and ρ force $z^2 = 1 = z$. The other two noninvertible summands then also have scalar one. Tensoring ρ by α shows that the scalar on α is one, and likewise for α^2 . Thus the automorphism is the identity on all six simples. The positive pivotal structure therefore selects the positive classification branch, without altering the associator on the pointed subcategory already fixed.

These two categories are Morita equivalent [11, Theorem 1.4], so their Drinfeld centers are braided equivalent. Combine this with Lemma 12.2. $\square$

13 Why the equivalence preserves the ribbon structure

The remaining point is the twist, rather than another comparison of matrices. Positivity makes the answer automatic once the braided equivalence has been established.

Section 11 used character T to establish module survival, whereas ribbon compatibility concerns the twist ϑ . Removing the common $e^{-2\pi ic/24}$ character factor gives the VOA twists. The remaining step is to compare the specified positive pivotal structures; Appendix Q gives the duality maps that make this comparison precise.

Lemma 13.1. *A complex-linear braided tensor equivalence between fusion categories with positive spherical dimensions preserves their specified pivotal structures and their corresponding ribbon twists.*

Proof. Transport the target pivotal structure through the equivalence and its canonical duality comparisons. It becomes a spherical structure on the source with positive simple dimensions. A pseudounitary fusion category has a unique positive pivotal structure [22, Proposition 8.23]; thus the transported structure is the specified source structure.

The Drinfeld isomorphism $u_X : X \rightarrow X^{**}$ is natural under braided functors with these duality comparisons. With the convention $j_X = u_X \vartheta_X$, preservation of j and u gives preservation of ϑ . Appendix Q writes the duality comparisons explicitly. $\square$

Corollary 13.2. *The braided equivalence $\text{Rep}(U) \simeq Z(\mathcal{H}_3)$ is ribbon for the existing VOA structure and the standard positive structure on the Haagerup center.*

Proof. The VOA dimensions are positive by Lemma 11.1. The standard center pivotal structure has the same dimension as its underlying object in the unitary category $\mathcal{H}_3$, hence is positive [28, Remark 2.35(i)]. Apply Lemma 13.1. $\square$

Braiding convention. For right A -modules, the tensor product uses the left action $\ell_M^- = r_M \circ c_{M,A}^{-1}$. With this convention the central free-module functor $\mathcal{C} \rightarrow Z(\mathcal{C}_A)$ is braided for the given braiding on $\mathcal{C}$ [25, Sections 3.3–3.4 and Corollary 4.1]. The tensor and Morita equivalences used above induce braided center equivalences with this orientation.

Twists versus character T . For a simple module of lowest weight h ,

$$\vartheta^{\text{VOA}} = e^{2\pi i L_0} = e^{2\pi i h} \text{id}, \quad T_{\text{character}} = e^{-2\pi i c/24} \text{diag}(\vartheta) = \zeta_3^{-1} \text{diag}(\vartheta).$$

The vacuum twist is one; its character T eigenvalue is ζ_3^{-1} . This common character factor is not part of the ribbon twist. At $c = 8$ the multiplicative anomaly $e^{2\pi i c/8}$ is one, as required for a center.

The completed dependencies can be read as follows:

Main assertion	Established input
VOA properties and positive form	Proposition 3.1
Affine Lie algebra and lattice definition	Section 2
Strong generation and C_2 -cofiniteness	Theorems 4.4, 5.2
Rationality and the twelve-module list	Corollary 9.7, Theorem 11.4
Braided Haagerup-center equivalence	Lemma 12.2, Proposition 12.5, Corollary 12.6
Preservation of the ribbon structure	Corollary 13.2

Proof of Theorem 1.1. Proposition 3.1, Theorem 4.4, and Theorem 5.2 prove the structural and finiteness claims for the explicitly defined lattice subalgebra. The full-corner argument and Theorem 11.4 prove rationality and exhaust its simples. The E_8 extensions, the actual fusion rules, and the associator normalization give the braided equivalence by Corollary 12.6. Corollary 13.2 upgrades it to the stated ribbon equivalence. $\square$

Appendix

A The sixteen certificate-normalized highest-weight vectors

This appendix specifies the actual vectors $z_0, \dots, z_{15} \in S_2$ used in the certificates, in their original order and normalization. Their Zhu classes, together with the six affine generator classes, are the 22 generators used in the reduction. The short formulas and sparse tables below define every vector without requiring the reader to reconstruct an unnamed certificate basis.

A.1 Coordinates, arithmetic, and lattice vectors

We use the same simple-root coordinates $\alpha_0, \dots, \alpha_7$ as Appendix B: the first two coordinates belong to A_2 and the last six to E_6 . A tuple $\beta = (\beta_0, \dots, \beta_7)$ denotes $\sum_i \beta_i \alpha_i$. The Gram matrix is

$$Q = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \quad (\text{A.1})$$

Thus $(\beta, \gamma) = \beta^T Q \gamma$. The cocycle, including the signs of bare exponentials, is

$$\varepsilon(\beta, \gamma) = (-1)^{\sum_i \beta_i \gamma_i + \sum_{i>j} \beta_i Q_{ij} \gamma_j}. \quad (\text{A.2})$$

Fix

$$r = \sqrt{13} > 0, \quad q = \sqrt{-3} = i\sqrt{3}, \quad \theta = r + q, \quad K = \mathbb{Q}(r, q). \quad (\text{A.3})$$

In particular, $r^2 = 13$, $q^2 = -3$, and $\theta^4 - 20\theta^2 + 256 = 0$. The sparse tables express coefficients in the basis $1, r, q, rq$ of K . The certificate instead stores a list of coefficients in descending powers of θ , with leading zeros omitted. The conversion is

$$c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 = (c_0 + 10c_2) + (c_1 + 4c_3)r + (c_1 + 36c_3)q + 2c_2rq. \quad (\text{A.4})$$

For an ordered list $\nu = ((n_1, i_1), \dots, (n_k, i_k))$, put

$$M(\beta; \nu) = \alpha_{i_1}(-n_1) \cdots \alpha_{i_k}(-n_k) e^\beta. \quad (\text{A.5})$$

An empty list gives $M(\beta; \emptyset) = e^\beta$, and $e^{\mathbf{0}} = \mathbf{1}$. All exponentials in this appendix are the bare exponentials for (A.2); no transported seven-coordinate signs are implicit in the tables.

A.2 The affine generators and the short formulas

For completeness, the roots appearing in the affine generators are

$$\begin{aligned} s &= \alpha_2, & (c_0, \dots, c_6) &= (2, 2, 5, 8, 9, 8, 5), \\ \rho_0 &= \alpha_0, & \rho_1 &= \alpha_1, & \rho_2 &= \alpha_3, \\ \rho_3 &= \alpha_5, & \rho_4 &= \alpha_6, & \rho_5 &= \alpha_7, \\ \rho_6 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6. \end{aligned}$$

Write $t = \sum_{i=0}^6 c_i \rho_i = (2, 2, 5, 10, 10, 18, 14, 8)$. The six affine generators are

$$\begin{aligned} E_s &= e^s, & F_s &= -e^{-s}, & H_s &= s(-1)\mathbf{1}, \\ E_t &= \sum_{i=0}^6 e^{\rho_i}, & F_t &= -\sum_{i=0}^6 c_i e^{-\rho_i}, & H_t &= t(-1)\mathbf{1}. \end{aligned} \quad (\text{A.6})$$

The ambient conformal vector and the level-39 Sugawara vector are

$$\omega_W = \frac{1}{2} \sum_{i,j=0}^7 (Q^{-1})_{ij} \alpha_i(-1) \alpha_j(-1) \mathbf{1}, \quad (\text{A.7})$$

$$\omega_t = \frac{1}{82} \left(\frac{1}{2} (H_t)_{(-1)} H_t + (E_t)_{(-1)} F_t + (F_t)_{(-1)} E_t \right). \quad (\text{A.8})$$

The following eight vectors have short expressions:

$$z_0 = s(-1)e^s = T E_s, \quad (\text{A.9})$$

$$z_1 = s(-1)^2 \mathbf{1} = (H_s)_{(-1)} H_s, \quad (\text{A.10})$$

$$z_2 = \frac{1}{4} \left(\frac{1}{2} (H_t)_{(-1)} H_t + (E_t)_{(-1)} F_t + (F_t)_{(-1)} E_t \right) = \frac{41}{2} \omega_t, \quad (\text{A.11})$$

$$z_3 = \frac{3}{20} z_1 + \frac{1}{5} z_2 - \frac{3}{5} \omega_W, \quad (\text{A.12})$$

$$z_5 = \sum_{i=0}^6 \rho_i(-1) e^{\rho_i} = T E_t, \quad (\text{A.13})$$

$$z_6 = (E_s)_{(-1)} E_t = \sum_{i=0}^6 \varepsilon(s, \rho_i) e^{s+\rho_i}, \quad (\text{A.14})$$

$$z_7 = (E_t)_{(-1)} E_t, \quad (\text{A.15})$$

$$z_9 = e^{\beta_1} + \frac{7-3rq}{20} e^{\beta_2} - \frac{r+3q}{2} e^{\beta_3}. \quad (\text{A.16})$$

Here $T = L_{-1}^W$ and the three lattice vectors are

$$\begin{aligned} \beta_1 &= (0, 0, 1, 2, 1, 3, 2, 1), \\ \beta_2 &= (0, 0, 2, 2, 3, 4, 3, 2), \\ \beta_3 &= (1, 1, 1, 1, 1, 2, 2, 1). \end{aligned} \quad (\text{A.17})$$

Thus z_9 is precisely the normalized seed v' of the certificates:

$$z_9 = e^{\beta_1} + \left(\frac{11}{10} - \frac{3\theta^2}{40} \right) e^{\beta_2} - \left(\frac{\theta^3}{32} + \frac{3\theta}{8} \right) e^{\beta_3}. \quad (\text{A.18})$$

The normal product in (A.15) can also be written entirely in lattice coordinates:

$$\begin{aligned} z_7 &= \sum_{\substack{0 \leq i < j \leq 6 \\ (\rho_i, \rho_j) = -1}} \varepsilon(\rho_i, \rho_j) (\rho_i - \rho_j) (-1) e^{\rho_i + \rho_j} \\ &\quad + 2 \sum_{\substack{0 \leq i < j \leq 6 \\ (\rho_i, \rho_j) = 0}} \varepsilon(\rho_i, \rho_j) e^{\rho_i + \rho_j}. \end{aligned} \quad (\text{A.19})$$

A.3 The ordering and highest weights

Write (r_a, j_a) for the highest weight of z_a under the zeroth modes of the two affine generator triples. The ordered list is

a	(r_a, j_a)	$(r_a + 1)(j_a + 1)$	a	(r_a, j_a)	$(r_a + 1)(j_a + 1)$
0	(2, 0)	3	8	(0, 4)	5
1	(0, 0)	1	9	(1, 13)	28
2	(0, 0)	1	10	(1, 7)	16
3	(0, 0)	1	11	(1, 5)	12
4	(0, 0)	1	12	(1, 1)	4
5	(0, 2)	3	13	(0, 12)	13
6	(2, 2)	9	14	(0, 8)	9
7	(0, 4)	5	15	(0, 8)	9

In particular,

$$(E_s)_{(0)} z_a = (E_t)_{(0)} z_a = 0, \quad (H_s)_{(0)} z_a = r_a z_a, \quad (H_t)_{(0)} z_a = j_a z_a. \quad (\text{A.20})$$

The zero-mode lowering vectors are the unnormalized vectors

$$(F_s)_{(0)}^p (F_t)_{(0)}^b z_a, \quad 0 \leq p \leq r_a, \quad 0 \leq b \leq j_a. \quad (\text{A.21})$$

Their 120 members form a basis of S_2 . Here “highest weight” refers to the finite-dimensional Lie algebra $\mathfrak{g}$; it does not assert that all positive affine modes act by zero.

A.4 How the certificate obtained these vectors

The source is calculation C03 in the active lattice tree `shared/lattice/`. Its exact vector file is

`shared/lattice/research/results/haagerup_c8_degree_two_exact_basis.json`.

The index a in this appendix is exactly the zero-based index in that file’s `highest_weight_vectors` array. An entry `[i, coefficients]` in its `basis_coordinates` denotes a multiple of the zero-based entry i in its `basis` array. The displayed lattice formulas are obtained by substituting those basis entries, multiplying the exact coefficients in K , and collecting equal lattice monomials.

The construction is implemented in

`shared/lattice/research/haagerup_c8_closure_exact.py`.

Its method `lift` reconstructs the ordered 120-vector basis over K from the generation words stored in

`shared/lattice/research/results/haagerup_c8_modp_basis_10009_0_1.json`.

The initial words are derivatives and normal products of affine generators, the coset conformal vector, the chosen axis, and the seed. Further words are affine generator zeroth products and first products of previously constructed weight-two vectors. Each word is reduced against the previously accepted rows of its Cartan weight, and its first remaining lattice-monomial coefficient is normalized

to one. The prime-10009 calculation supplies the words and pivots; the vectors and reductions are recomputed over K .

The method `highest_vectors` then computes the affine generator zeroth-mode matrices. On each nonnegative Cartan-weight subspace it forms the stacked equations for $(E_s)_{(0)}$ and $(E_t)_{(0)}$, and solves their joint kernel by exact row reduction over K . For each nonpivot coordinate it sets that coordinate to one and the other nonpivot coordinates to zero. This determines the displayed ordered kernel basis, including the choices among repeated highest weights. The entries called `dimensions` in the JSON are $(r_a + 1, j_a + 1)$, rather than (r_a, j_a) .

The closure results are recorded separately in

`shared/lattice/research/results/haagerup_c8_closure_exact.json`.

To check the formulas directly, apply the affine generator zeroth modes using

$$Y(e^\alpha, z)e^\beta = \varepsilon(\alpha, \beta) z^{(\alpha, \beta)} \exp\left(\sum_{n>0} \frac{\alpha(-n)}{n} z^n\right) e^{\alpha+\beta}, \quad (\text{A.22})$$

together with the Heisenberg commutator and positive-mode exponential. The tables satisfy (A.20); their unnormalized lowering vectors (A.21) have exact rank 120. The short conformal and normal-product formulas above agree with the same expanded certificate vectors.

A.5 Complete sparse expansions of the remaining vectors

For $a \in \{4, 8, 10, 11, 12, 13, 14, 15\}$, the following table for z_a is its definition as an explicit lattice vector:

$$z_a = \frac{1}{D_a} \sum_{(\beta, \nu) \text{ in the table for } z_a} p_a(\beta, \nu; r, q) M(\beta; \nu). \quad (\text{A.23})$$

The table header gives D_a , and the last column gives the integer polynomial p_a in $1, r, q, rq$. Every nonzero term is listed; unlisted coefficients are zero. The prefactor $1/D_a$ is part of the formula and preserves the certificate normalization.

For example, the first row of the z_4 table has $\beta = (-1, -1, 0, 0, 0, 0, 1, 1)$, no Heisenberg-mode factors, and coefficient 39. It therefore contributes $(39/130)e^\beta = (3/10)e^\beta$ to z_4 . A nonempty ν inserts the listed Heisenberg modes in (A.5); equal lattice vector and Heisenberg-mode labels are collected. The symbols D_a here are normalization denominators, not the coefficient spaces D_3, D_4 of strong generation.

A.5.1 The vector z_4 : $D_4 = 130$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, -1, 0, 0, 0, 0, 1, 1)$	$\emptyset$	39
$(-1, -1, 0, 0, 0, 1, 1, 0)$	$\emptyset$	39
$(-1, -1, 0, 1, 0, 1, 0, 0)$	$\emptyset$	39
$(-1, -1, 1, 1, 2, 2, 1, 1)$	$\emptyset$	-39
$(-1, 0, 0, 0, 0, 0, 0, 1)$	$\emptyset$	-39

Continued on the next page

The vector z_4 , continued; $D_4 = 130$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, 0, 0, 0, 0, 1, 0, 0)$	$\emptyset$	39
$(-1, 0, 0, 1, 0, 0, 0, 0)$	$\emptyset$	78
$(-1, 0, 1, 1, 2, 2, 1, 0)$	$\emptyset$	-78
$(0, -1, 0, 0, 0, 0, 0, 1)$	$\emptyset$	39
$(0, -1, 0, 0, 0, 1, 0, 0)$	$\emptyset$	-39
$(0, -1, 0, 1, 0, 0, 0, 0)$	$\emptyset$	-78
$(0, -1, 1, 1, 2, 2, 1, 0)$	$\emptyset$	78
$(0, 0, -1, -1, -2, -2, 0, 0)$	$\emptyset$	$15r$
$(0, 0, -1, -1, -2, -1, -1, 0)$	$\emptyset$	$15r$
$(0, 0, -1, 0, -2, -2, -1, 0)$	$\emptyset$	$-15r$
$(0, 0, -1, 0, -2, -1, -1, -1)$	$\emptyset$	$30r$
$(0, 0, 0, -1, 0, 0, 0, 1)$	$\emptyset$	$-15r$
$(0, 0, 0, -1, 0, 0, 1, 0)$	$\emptyset$	$-15r$
$(0, 0, 0, 0, 0, -1, 0, 1)$	$\emptyset$	$6r$
0	$((1, 0), (1, 2))$	65
0	$((1, 0), (1, 4))$	130
0	$((1, 0), (1, 5))$	78
0	$((1, 0), (1, 6))$	65
0	$((1, 0), (1, 7))$	52
0	$((1, 1), (1, 2))$	-65
0	$((1, 1), (1, 4))$	-130
0	$((1, 1), (1, 5))$	-78
0	$((1, 1), (1, 6))$	-65
0	$((1, 1), (1, 7))$	-52
0	$((1, 2), (1, 2))$	$-10r$
0	$((1, 2), (1, 3))$	$-21r$
0	$((1, 2), (1, 4))$	$-40r$
0	$((1, 2), (1, 5))$	$-27r$
0	$((1, 2), (1, 6))$	$-8r$
0	$((1, 2), (1, 7))$	$-4r$
0	$((1, 3), (1, 3))$	$-21r$
0	$((1, 3), (1, 4))$	$-42r$
0	$((1, 3), (1, 5))$	$-33r$
0	$((1, 3), (1, 6))$	$3r$
0	$((1, 3), (1, 7))$	$9r$
0	$((1, 4), (1, 4))$	$-40r$
0	$((1, 4), (1, 5))$	$-54r$
0	$((1, 4), (1, 6))$	$-16r$
0	$((1, 4), (1, 7))$	$-8r$
0	$((1, 5), (1, 5))$	$-18r$
0	$((1, 5), (1, 6))$	$-6r$
0	$((1, 5), (1, 7))$	$-6r$
0	$((1, 6), (1, 6))$	$-7r$
0	$((1, 6), (1, 7))$	$-7r$

Continued on the next page

The vector z_4 , continued; $D_4 = 130$

β	ν	$p_a(\beta, \nu; r, q)$
0	$((1, 7), (1, 7))$	$-4r$
$(0, 0, 0, 0, 0, 1, 0, -1)$	$\emptyset$	$6r$
$(0, 0, 0, 1, 0, 0, -1, 0)$	$\emptyset$	$-27r$
$(0, 0, 0, 1, 0, 0, 0, -1)$	$\emptyset$	$-24r$
$(0, 0, 1, 0, 2, 1, 1, 1)$	$\emptyset$	$30r$
$(0, 0, 1, 0, 2, 2, 1, 0)$	$\emptyset$	$-15r$
$(0, 0, 1, 1, 2, 1, 1, 0)$	$\emptyset$	$24r$
$(0, 0, 1, 1, 2, 2, 0, 0)$	$\emptyset$	$27r$
$(0, 1, -1, -1, -2, -2, -1, 0)$	$\emptyset$	195
$(0, 1, 0, -1, 0, 0, 0, 0)$	$\emptyset$	-195
$(0, 1, 0, 0, 0, -1, 0, 0)$	$\emptyset$	-156
$(0, 1, 0, 0, 0, 0, 0, -1)$	$\emptyset$	156
$(1, 0, -1, -1, -2, -2, -1, 0)$	$\emptyset$	-195
$(1, 0, 0, -1, 0, 0, 0, 0)$	$\emptyset$	195
$(1, 0, 0, 0, 0, -1, 0, 0)$	$\emptyset$	156
$(1, 0, 0, 0, 0, 0, 0, -1)$	$\emptyset$	-156
$(1, 1, -1, -1, -2, -2, -1, -1)$	$\emptyset$	-390
$(1, 1, 0, -1, 0, -1, 0, 0)$	$\emptyset$	390
$(1, 1, 0, 0, 0, -1, -1, 0)$	$\emptyset$	702
$(1, 1, 0, 0, 0, 0, -1, -1)$	$\emptyset$	702

A.5.2 The vector z_8 : $D_8 = 6177600$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, -1, 0, 1, 0, 1, 1, 1)$	$\emptyset$	$-335530r + 1220960q$
$(-1, -1, 1, 1, 2, 3, 2, 1)$	$\emptyset$	$335530r - 1220960q$
$(-1, 0, 0, 0, 0, 0, 1, 1, 1)$	$\emptyset$	$1342120 - 1408800r - 1677650q$ $- 375680rq$
$(-1, 0, 0, 1, 0, 1, 1, 0)$	$\emptyset$	$1342120 - 1744330r - 456690q$ $- 375680rq$
$(-1, 0, 1, 1, 2, 2, 2, 1)$	$\emptyset$	$1342120 - 1073270r - 2898610q$ $- 375680rq$
$(0, -1, 0, 0, 0, 1, 1, 1)$	$\emptyset$	$1342120 + 1408800r + 1677650q$ $- 375680rq$
$(0, -1, 0, 1, 0, 1, 1, 0)$	$\emptyset$	$1342120 + 1744330r + 456690q$ $- 375680rq$
$(0, -1, 1, 1, 2, 2, 2, 1)$	$\emptyset$	$1342120 + 1073270r + 2898610q$ $- 375680rq$
$(0, 0, -1, 0, -2, -1, 0, 0)$	$\emptyset$	$-9308195 - 1613055rq$
$(0, 0, 0, 0, 0, 0, 1, 1)$	$((1, 0))$	$671060 - 536635r - 1449305q$ $- 187840rq$
$(0, 0, 0, 0, 0, 0, 1, 1)$	$((1, 1))$	$671060 + 536635r + 1449305q$ $- 187840rq$

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The vector z_8 , continued; $D_8 = 6177600$

β	ν	$p_a(\beta, \nu; r, q)$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 2))	$5340530 - 134070rq$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 3))	$7325760 + 671060rq$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 4))	$10681060 - 268140rq$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 5))	$3676855 - 348795rq$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 6))	$8402225 - 3032325rq$
(0, 0, 0, 0, 0, 0, 1, 1)	((1, 7))	$-4725370 + 2683530rq$
(0, 0, 0, 0, 0, 1, 0, 1)	$\emptyset$	$17112030 - 4789920rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 0))	$-671060 + 872165r + 228345q$ $+ 187840rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 1))	$-671060 - 872165r - 228345q$ $+ 187840rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 2))	$5340530 - 134070rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 3))	$7325760 + 671060rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 4))	$10681060 - 268140rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 5))	$22732190 - 2280610rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 6))	$-8709805 + 1757595rq$
(0, 0, 0, 0, 0, 1, 1, 0)	((1, 7))	$-3984435 - 925935rq$
(0, 0, 0, 1, 0, 0, 0, 1)	$\emptyset$	$27102465 - 3912165rq$
(0, 0, 0, 1, 0, 0, 1, 0)	$\emptyset$	$16127800 - 5603080rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 0))	$-2013180 + 2616495r + 685035q$ $+ 563520rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 1))	$-2013180 - 2616495r - 685035q$ $+ 563520rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 2))	$-3967665 - 1747125rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 3))	$6098235 - 4564725rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 4))	$-7935330 - 3494250rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 5))	$-12012000 + 96360rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 6))	$2021565 - 974115rq$
(0, 0, 0, 1, 0, 1, 0, 0)	((1, 7))	$-3984435 - 925935rq$
(0, 0, 1, 0, 2, 2, 2, 1)	$\emptyset$	$13401661 + 467231rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 0))	$-2013180 + 1609905r + 4347915q$ $+ 563520rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 1))	$-2013180 - 1609905r - 4347915q$ $+ 563520rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 2))	$10065900 - 2817600rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 3))	$21482331 - 3155499rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 4))	$20131800 - 5635200rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 5))	$27136005 - 5554545rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 6))	$2868762 - 3252498rq$
(0, 0, 1, 1, 2, 2, 1, 1)	((1, 7))	$-17263038 + 2382702rq$
(0, 0, 1, 1, 2, 2, 2, 0)	$\emptyset$	$-21988408 + 5066232rq$
(0, 0, 1, 1, 2, 3, 1, 0)	$\emptyset$	$-7322913 + 5724027rq$
(0, 0, 1, 2, 2, 2, 1, 0)	$\emptyset$	$-11273808 + 3155712rq$
(0, 1, 0, 0, 0, 0, 0, 1)	$\emptyset$	$14092260 - 1771215r + 321555q$ $- 3944640rq$

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The vector z_8 , continued; $D_8 = 6177600$

β	ν	$p_a(\beta, \nu; r, q)$
$(0, 1, 0, 0, 0, 0, 1, 0)$	$\emptyset$	$13421200 - 3381120r - 4026360q$ $- 3756800rq$
$(0, 1, 0, 0, 0, 1, 0, 0)$	$\emptyset$	$14092260 - 764625r - 3341325q$ $- 3944640rq$
$(0, 1, 0, 1, 0, 0, 0, 0)$	$\emptyset$	$16105440 + 7084860r - 1286220q$ $- 4508160rq$
$(0, 1, 1, 1, 2, 2, 1, 0)$	$\emptyset$	$16105440 + 3058500r + 13365300q$ $- 4508160rq$
$(1, 0, 0, 0, 0, 0, 0, 1)$	$\emptyset$	$14092260 + 1771215r - 321555q$ $- 3944640rq$
$(1, 0, 0, 0, 0, 0, 1, 0)$	$\emptyset$	$13421200 + 3381120r + 4026360q$ $- 3756800rq$
$(1, 0, 0, 0, 0, 1, 0, 0)$	$\emptyset$	$14092260 + 764625r + 3341325q$ $- 3944640rq$
$(1, 0, 0, 1, 0, 0, 0, 0)$	$\emptyset$	$16105440 - 7084860r + 1286220q$ $- 4508160rq$
$(1, 0, 1, 1, 2, 2, 1, 0)$	$\emptyset$	$16105440 - 3058500r - 13365300q$ $- 4508160rq$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 0))$	$28184520 - 7889280rq$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 1))$	$-28184520 + 7889280rq$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 2))$	$1206630r + 16021590q$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 3))$	$8452800r + 10065900q$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 4))$	$2413260r + 32043180q$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 5))$	$2735880r + 22704240q$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 6))$	$-3865050r + 9982050q$
$(1, 1, 0, 0, 0, 0, 0, 0)$	$((1, 7))$	$-3703740r + 5312580q$

A.5.3 The vector z_{10} : $D_{10} = 132$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, 0, 1, 1, 1, 2, 2, 1)$	$\emptyset$	$-66 - 2q$
$(0, -1, 1, 1, 1, 2, 2, 1)$	$\emptyset$	$-66 + 2q$
$(0, 0, 0, 0, -1, 0, 1, 1)$	$\emptyset$	$-6 - 14rq$
$(0, 0, 1, 0, 1, 2, 2, 1)$	$\emptyset$	$6 - 14rq$
$(0, 0, 1, 1, 1, 1, 2, 1)$	$\emptyset$	$36 + 4rq$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 0))$	$66 + 2q$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 1))$	$66 - 2q$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 2))$	$6 + 14rq$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 3))$	12
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 4))$	$12 + 28rq$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 5))$	$-24 + 24rq$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 6))$	$-66 + 14rq$
$(0, 0, 1, 1, 1, 2, 1, 1)$	$((1, 7))$	$-36 + 4rq$

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The vector z_{10} , continued; $D_{10} = 132$

β	ν	$p_a(\beta, \nu; r, q)$
(0, 0, 1, 1, 1, 2, 2, 0)	$\emptyset$	$-36 + 4rq$
(0, 0, 1, 2, 1, 2, 1, 0)	$\emptyset$	$48 - 24rq$
(0, 0, 2, 2, 3, 3, 2, 1)	$\emptyset$	$-48 - 24rq$
(0, 1, 1, 1, 1, 1, 1, 1)	$\emptyset$	$-33 + 33r + 3q + 33rq$
(0, 1, 1, 1, 1, 2, 1, 0)	$\emptyset$	$-33 - 33r + 3q - 33rq$
(1, 0, 1, 1, 1, 1, 1, 1)	$\emptyset$	$-33 - 33r - 3q + 33rq$
(1, 0, 1, 1, 1, 2, 1, 0)	$\emptyset$	$-33 + 33r - 3q - 33rq$
(1, 1, 0, 0, -1, 0, 0, 0)	$\emptyset$	$66r + 162q$
(1, 1, 1, 0, 1, 1, 1, 1)	$\emptyset$	$-66r + 162q$
(1, 1, 1, 1, 1, 1, 1, 0)	$\emptyset$	$168q$

A.5.4 The vector z_{11} : $D_{11} = 278720$

β	ν	$p_a(\beta, \nu; r, q)$
(-1, -1, 1, 1, 1, 2, 2, 1)	$\emptyset$	$-13871r - 13351q$
(-1, 0, 1, 1, 1, 2, 1, 1)	$\emptyset$	$-13871r - 13351q$
(0, -1, 1, 1, 1, 2, 1, 1)	$\emptyset$	$13871r + 13351q$
(0, 0, 0, 0, -1, 0, 0, 1)	$\emptyset$	$26975 + 5235rq$
(0, 0, 0, 0, -1, 0, 1, 0)	$\emptyset$	$26975 + 5235rq$
(0, 0, 0, 1, -1, 0, 0, 0)	$\emptyset$	$38506 - 19166rq$
(0, 0, 1, 0, 1, 1, 2, 1)	$\emptyset$	$-42380 + 100rq$
(0, 0, 1, 0, 1, 2, 1, 1)	$\emptyset$	$-42380 + 100rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 0))$	$138710 + 14638r - 28002q + 10270rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 1))$	$138710 - 14638r + 28002q + 10270rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 2))$	$26182 - 14898rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 3))$	$-70122 - 1954rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 4))$	$52364 - 29796rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 5))$	$-57824 - 29536rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 6))$	$-57044 - 21060rq$
(0, 0, 1, 1, 1, 1, 1, 1)	$((1, 7))$	$-13884 - 12684rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 0))$	$-138710 - 767r + 41353q - 10270rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 1))$	$-138710 + 767r - 41353q - 10270rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 2))$	$793 + 20133rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 3))$	$54717 + 7289rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 4))$	$1586 + 40266rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 5))$	$70928 + 33744rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 6))$	$84019 + 26295rq$
(0, 0, 1, 1, 1, 2, 1, 0)	$((1, 7))$	$70135 + 13611rq$
(0, 0, 2, 1, 3, 3, 2, 1)	$\emptyset$	$-66248 + 17112rq$
(0, 1, 0, 0, -1, 0, 0, 0)	$\emptyset$	$-77090 - 6235r - 132275q - 31610rq$
(0, 1, 1, 0, 1, 1, 1, 1)	$\emptyset$	$200330 - 52450r + 75790q - 11070rq$
(0, 1, 1, 1, 1, 1, 1, 0)	$\emptyset$	$61620 - 8536r - 8216q - 21340rq$

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The vector z_{11} , continued; $D_{11} = 278720$

β	ν	$p_a(\beta, \nu; r, q)$
$(1, 0, 0, 0, -1, 0, 0, 0)$	$\emptyset$	$-77090 + 6235r + 132275q - 31610rq$
$(1, 0, 1, 0, 1, 1, 1, 1)$	$\emptyset$	$200330 + 52450r - 75790q - 11070rq$
$(1, 0, 1, 1, 1, 1, 1, 0)$	$\emptyset$	$61620 + 8536r + 8216q - 21340rq$
$(1, 1, 0, 0, -1, -1, 0, 0)$	$\emptyset$	$12470r + 264550q$
$(1, 1, 1, 0, 1, 1, 1, 0)$	$\emptyset$	$-104900r + 151580q$
$(1, 1, 1, 1, 1, 1, 0, 0)$	$\emptyset$	$-134442r - 129402q$

A.5.5 The vector z_{12} : $D_{12} = 384912000$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, -1, 1, 1, 1, 1, 1, 1)$	$\emptyset$	$12322194r + 63544962q$
$(-1, -1, 1, 1, 1, 2, 1, 0)$	$\emptyset$	$-12322194r - 63544962q$
$(-1, 0, 0, 0, -1, 0, 0, 0)$	$\emptyset$	$-61610970 + 31874490r + 47115370q$ $- 24440370rq$
$(-1, 0, 1, 0, 1, 1, 1, 1)$	$\emptyset$	$-61610970 + 7230102r - 79974554q$ $- 24440370rq$
$(-1, 0, 1, 1, 1, 1, 1, 0)$	$\emptyset$	$-61610970 + 19552296r - 16429592q$ $- 24440370rq$
$(0, -1, 0, 0, -1, 0, 0, 0)$	$\emptyset$	$-61610970 - 31874490r - 47115370q$ $- 24440370rq$
$(0, -1, 1, 0, 1, 1, 1, 1)$	$\emptyset$	$-61610970 - 7230102r + 79974554q$ $- 24440370rq$
$(0, -1, 1, 1, 1, 1, 1, 0)$	$\emptyset$	$-61610970 - 19552296r + 16429592q$ $- 24440370rq$
$(0, 0, 0, -1, -1, -1, 0, 1)$	$\emptyset$	$286919325 - 32757175rq$
$(0, 0, 0, -1, -1, 0, 0, 0)$	$\emptyset$	$348530295 - 8316805rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 0))$	$-61610970 + 31874490r + 47115370q$ $- 24440370rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 1))$	$-61610970 - 31874490r - 47115370q$ $- 24440370rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 3))$	$348530295 - 8316805rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 5))$	$176378700 + 11337500rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 6))$	$-172151595 + 19654305rq$
$(0, 0, 0, 0, -1, -1, 0, 0)$	$((1, 7))$	$-229535460 + 26205740rq$
$(0, 0, 0, 0, -1, 0, -1, 0)$	$\emptyset$	$-172151595 + 19654305rq$
$(0, 0, 0, 0, -1, 0, 0, -1)$	$\emptyset$	$-229535460 + 26205740rq$
$(0, 0, 1, 0, 1, 0, 1, 1)$	$\emptyset$	$278824236 - 6653444rq$
$(0, 0, 1, 0, 1, 1, 0, 1)$	$\emptyset$	$209118177 - 4990083rq$
$(0, 0, 1, 0, 1, 1, 1, 0)$	$((1, 0))$	$61610970 - 7230102r + 79974554q$ $+ 24440370rq$
$(0, 0, 1, 0, 1, 1, 1, 0)$	$((1, 1))$	$61610970 + 7230102r - 79974554q$ $+ 24440370rq$
$(0, 0, 1, 0, 1, 1, 1, 0)$	$((1, 2))$	$286919325 - 32757175rq$

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The vector z_{12} , continued; $D_{12} = 384912000$

β	ν	$p_a(\beta, \nu; r, q)$
(0, 0, 1, 0, 1, 1, 1, 0)	((1, 3))	$286919325 - 32757175rq$
(0, 0, 1, 0, 1, 1, 1, 0)	((1, 4))	$573838650 - 65514350rq$
(0, 0, 1, 0, 1, 1, 1, 0)	((1, 5))	$295014414 - 58860906rq$
(0, 0, 1, 0, 1, 1, 1, 0)	((1, 6))	$77801148 - 27767092rq$
(0, 0, 1, 0, 1, 1, 1, 0)	((1, 7))	$77801148 - 27767092rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 0))	$184832910 - 58656888r + 49288776q$ $+ 73321110rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 1))	$184832910 + 58656888r - 49288776q$ $+ 73321110rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 2))	$516454785 - 58962915rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 3))	$-110899746 - 43992666rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 4))	$1032909570 - 117925830rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 5))	$577706634 - 122609886rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 6))	$405555039 - 102955581rq$
(0, 0, 1, 1, 1, 1, 0, 0)	((1, 7))	$307336608 - 53972832rq$
(0, 1, 0, -1, -1, -1, 0, 0)	$\emptyset$	$154027425 + 146642220r - 123221940q$ $+ 61100925rq$
(0, 1, 0, 0, -1, -1, -1, 0)	$\emptyset$	$92416455 + 51018750r - 264568050q$ $+ 36660555rq$
(0, 1, 1, 0, 1, 1, 0, 0)	$\emptyset$	$92416455 + 124951914r + 116701722q$ $+ 36660555rq$
(1, 0, 0, -1, -1, -1, 0, 0)	$\emptyset$	$154027425 - 146642220r + 123221940q$ $+ 61100925rq$
(1, 0, 0, 0, -1, -1, -1, 0)	$\emptyset$	$92416455 - 51018750r + 264568050q$ $+ 36660555rq$
(1, 0, 1, 0, 1, 1, 0, 0)	$\emptyset$	$92416455 - 124951914r - 116701722q$ $+ 36660555rq$
(1, 1, 0, -1, -1, -1, -1, 0)	$\emptyset$	$-293284440r + 246443880q$
(1, 1, 0, 0, -1, -1, -1, -1)	$\emptyset$	$-395321940r + 775579980q$
(1, 1, 1, 0, 1, 0, 0, 0)	$\emptyset$	$543188268r - 13040436q$

A.5.6 The vector z_{13} : $D_{13} = 260$

β	ν	$p_a(\beta, \nu; r, q)$
(0, 0, 0, 1, 0, 2, 2, 1)	$\emptyset$	260
(0, 0, 1, 1, 2, 3, 3, 2)	$\emptyset$	$91 - 39rq$
(0, 0, 1, 2, 2, 3, 2, 2)	$\emptyset$	$-195 + 15rq$
(0, 0, 1, 2, 2, 3, 3, 1)	$\emptyset$	$-104 - 24rq$
(0, 0, 1, 2, 2, 4, 2, 1)	$\emptyset$	$156 - 24rq$
(0, 1, 1, 2, 2, 3, 2, 1)	$\emptyset$	$-312 + 108r - 156q - 72rq$
(1, 0, 1, 2, 2, 3, 2, 1)	$\emptyset$	$-312 - 108r + 156q - 72rq$
(1, 1, 0, 1, 0, 1, 1, 1)	$\emptyset$	$30r - 390q$
(1, 1, 1, 1, 2, 3, 2, 1)	$\emptyset$	$186r + 78q$

A.5.7 The vector z_{14} : $D_{14} = 7800$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, 0, 1, 2, 2, 3, 2, 1)$	$\emptyset$	$7752 + 1968r + 684q + 192rq$
$(0, -1, 1, 2, 2, 3, 2, 1)$	$\emptyset$	$-312 - 48r - 4524q - 1152rq$
$(0, 0, 0, 0, 0, 1, 2, 1)$	$\emptyset$	$8990 + 2320r - 4640q - 1160rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 0))$	$-7620 - 2100r + 1530q + 480rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 1))$	$3900 + 1140r + 390q$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 2))$	$1125 + 360r + 1080q + 315rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 3))$	$-5695 - 1400r + 4600q + 1195rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 4))$	$2250 + 720r + 2160q + 630rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 5))$	$-2415 - 600r + 1800q + 465rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 6))$	$7260 + 1920r - 2640q - 630rq$
$(0, 0, 0, 1, 0, 1, 1, 1)$	$((1, 7))$	$7945 + 2120r - 2440q - 565rq$
$(0, 0, 0, 1, 0, 2, 1, 0)$	$\emptyset$	$8645 + 2440r + 520q + 265rq$
$(0, 0, 1, 1, 2, 2, 2, 2)$	$\emptyset$	$-221 + 152r + 5096q + 1409rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 0))$	$-132 + 132r - 2214q - 672rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 1))$	$-3588 - 1092r + 4134q + 1152rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 2))$	$-6820 - 1760r + 3520q + 880rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 3))$	$-8503 - 2264r + 2728q + 637rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 4))$	$-13640 - 3520r + 7040q + 1760rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 5))$	$-8503 - 2264r + 2728q + 637rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 6))$	$251 + 88r + 424q + 121rq$
$(0, 0, 1, 1, 2, 3, 2, 1)$	$((1, 7))$	$15 + 120r + 2760q + 765rq$
$(0, 0, 1, 2, 2, 2, 2, 1)$	$\emptyset$	$472 - 64r - 4672q - 1288rq$
$(0, 0, 1, 2, 2, 3, 1, 1)$	$\emptyset$	$13206 + 3408r - 6816q - 1704rq$
$(0, 0, 1, 2, 2, 3, 2, 0)$	$\emptyset$	$7960 + 2240r + 320q + 200rq$
$(0, 1, 0, 0, 0, 1, 1, 1)$	$\emptyset$	$860 + 280r - 8810q - 2240rq$
$(0, 1, 0, 1, 0, 1, 1, 0)$	$\emptyset$	$-14560 - 4100r - 8060q - 1760rq$
$(0, 1, 1, 1, 2, 2, 2, 1)$	$\emptyset$	$7904 + 2596r - 19292q - 5216rq$
$(1, 0, 0, 0, 0, 1, 1, 1)$	$\emptyset$	$15260 + 3880r + 490q + 160rq$
$(1, 0, 0, 1, 0, 1, 1, 0)$	$\emptyset$	$34400 + 9220r - 2180q - 800rq$
$(1, 0, 1, 1, 2, 2, 2, 1)$	$\emptyset$	$11936 + 2524r + 9052q + 2656rq$
$(1, 1, 0, 0, 0, 0, 1, 1)$	$\emptyset$	$10320 + 2730r + 3090q + 720rq$
$(1, 1, 0, 0, 0, 1, 1, 0)$	$\emptyset$	$-4080 - 870r - 6210q - 1680rq$
$(1, 1, 0, 1, 0, 1, 0, 0)$	$\emptyset$	$-53040 - 14190r - 12090q - 2640rq$
$(1, 1, 1, 1, 2, 2, 1, 1)$	$\emptyset$	$-14352 - 2658r - 31434q - 8592rq$

A.5.8 The vector z_{15} : $D_{15} = 1560$

β	ν	$p_a(\beta, \nu; r, q)$
$(-1, 0, 1, 2, 2, 3, 2, 1)$	$\emptyset$	$165 + 423r + 369q - 285rq$
$(0, -1, 1, 2, 2, 3, 2, 1)$	$\emptyset$	$165 - 423r - 369q - 285rq$
$(0, 0, 0, 0, 0, 1, 2, 1)$	$\emptyset$	$740 - 610rq$

Continued on the next page

The vector z_{15} , continued; $D_{15} = 1560$

β	ν	$p_a(\beta, \nu; r, q)$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 0))	$-960 - 300r + 810q + 240rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 1))	$-960 + 300r - 810q + 240rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 2))	$1455 - 15rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 3))	$860 + 440rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 4))	$2910 - 30rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 5))	$1320 + 30rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 6))	$2055 - 525rq$
(0, 0, 0, 1, 0, 1, 1, 1)	((1, 7))	$2050 - 470rq$
(0, 0, 0, 1, 0, 2, 1, 0)	$\emptyset$	$3185 - 175rq$
(0, 0, 1, 1, 2, 2, 2, 2)	$\emptyset$	$1729 + 259rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 0))	$795 - 123r - 1179q + 45rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 1))	$795 + 123r + 1179q + 45rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 2))	$-595 + 455rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 3))	$-1948 + 542rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 4))	$-1190 + 910rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 5))	$-1948 + 542rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 6))	$-229 + 41rq$
(0, 0, 1, 1, 2, 3, 2, 1)	((1, 7))	$750 + 150rq$
(0, 0, 1, 2, 2, 2, 2, 1)	$\emptyset$	$-2348 - 308rq$
(0, 0, 1, 2, 2, 3, 1, 1)	$\emptyset$	$966 - 924rq$
(0, 0, 1, 2, 2, 3, 2, 0)	$\emptyset$	$2800 - 320rq$
(0, 1, 0, 0, 0, 1, 1, 1)	$\emptyset$	$260 - 900r - 450q - 640rq$
(0, 1, 0, 1, 0, 1, 1, 0)	$\emptyset$	$1220 - 1800r + 1980q - 880rq$
(0, 1, 1, 1, 2, 2, 2, 1)	$\emptyset$	$-535 - 1269r - 3987q - 685rq$
(1, 0, 0, 0, 0, 1, 1, 1)	$\emptyset$	$260 + 900r + 450q - 640rq$
(1, 0, 0, 1, 0, 1, 1, 0)	$\emptyset$	$1220 + 1800r - 1980q - 880rq$
(1, 0, 1, 1, 2, 2, 2, 1)	$\emptyset$	$-535 + 1269r + 3987q - 685rq$
(1, 1, 0, 0, 0, 0, 1, 1)	$\emptyset$	$1380r + 840q$
(1, 1, 0, 0, 0, 1, 1, 0)	$\emptyset$	$-420r - 60q$
(1, 1, 0, 1, 0, 1, 0, 0)	$\emptyset$	$-4020r + 3900q$
(1, 1, 1, 1, 2, 2, 1, 1)	$\emptyset$	$-3918r - 8814q$

B Lattice conventions and affine verification

B.1 The ambient lattice and its cocycle

Fix simple-root coordinates $\alpha_0, \dots, \alpha_7$, with the first two spanning A_2 and the last six spanning E_6 . The Gram matrix is

$$Q = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \quad (\text{B.1})$$

A coordinate vector $x \in \mathbb{Z}^8$ represents $\sum_i x_i \alpha_i$, and $(x, y) = x^T Q y$. For a lattice vector,

$$\text{wt}(e^\lambda) = \frac{(\lambda, \lambda)}{2}, \quad \text{wt}(\alpha_{i_1}(-n_1) \cdots \alpha_{i_m}(-n_m)e^\lambda) = \frac{(\lambda, \lambda)}{2} + \sum_j n_j.$$

Put $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. The underlying vector space is

$$W = \text{Sym}(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}_\varepsilon[L].$$

Here $\mathbb{C}_\varepsilon[L]$ is the twisted group algebra defined below, and $h(-n)$, for $n > 0$, acts by multiplication by $h \otimes t^{-n}$ in the symmetric-algebra factor. The products $\alpha_{i_1}(-n_1) \cdots \alpha_{i_m}(-n_m)e^\lambda$, with mode labels ordered lexicographically, form the *lattice-monomial basis*. Their *Heisenberg degree* is $\sum_j n_j$. We fix the normalized bimultiplicative cocycle

$$\varepsilon(a, b) = (-1)^{\sum_i a_i b_i + \sum_{i>j} a_i Q_{ij} b_j}. \quad (\text{B.2})$$

Its commutator is $(-1)^{(a,b)}$, as verified below. This fixes the signs of the root generators.

For example, $\varepsilon(s, -s) = -1$ and $F_s = -e^{-s}$ give

$$Y(E_s, z)F_s = z^{-2}(\mathbf{1} + s(-1)\mathbf{1}z + O(z^2)).$$

The coefficients of z^{-2} and z^{-1} are respectively $(E_s)_{(1)}F_s = \mathbf{1}$ and $(E_s)_{(0)}F_s = H_s$. This one coefficient extraction illustrates the primitive mode calculation used throughout the finite-dimensional computations.

The conformal vector is

$$\omega_W = \frac{1}{2} \sum_{i,j=0}^7 (Q^{-1})_{ij} \alpha_i(-1) \alpha_j(-1) \mathbf{1}. \quad (\text{B.3})$$

Thus W has central charge eight and $W \cong L_1(E_6) \otimes L_1(A_2)$. The standard simplicity and regularity results apply to this positive definite even lattice VOA [12, 13].

Why the finite enumeration is complete. Positive definiteness gives $|x_i|^2 \leq (x^T Q x)(Q^{-1})_{ii}$ by Cauchy–Schwarz. For a norm bound B , enumerate integers in the resulting finite coordinate box and retain exactly those with $x^T Q x \leq B$. Rational ellipsoid enumeration is a faster version of this same complete search. For each retained lattice vector, enumerate partitions of the remaining weight $d - (x, x)/2$, assigning a simple-root index to each negative Heisenberg mode. This constructs the whole lattice-monomial basis in weight d , not a sample of lattice vectors suggested by the defining generators.

B.2 The chosen roots

The level-one root is $s = (0, 0, 1, 0, 0, 0, 0, 0)$. Introduce the seven roots

$$\begin{aligned}\rho_0 &= (1, 0, 0, 0, 0, 0, 0, 0), & \rho_1 &= (0, 1, 0, 0, 0, 0, 0, 0), \\ \rho_2 &= (0, 0, 0, 1, 0, 0, 0, 0), & \rho_3 &= (0, 0, 0, 0, 0, 1, 0, 0), \\ \rho_4 &= (0, 0, 0, 0, 0, 0, 1, 0), & \rho_5 &= (0, 0, 0, 0, 0, 0, 0, 1), \\ \rho_6 &= (0, 0, 1, 1, 2, 2, 1, 0).\end{aligned}\tag{B.4}$$

Their Gram matrix is that of $A_2 \oplus A_5$, in the order with edges $\{0, 1\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}$, and each is orthogonal to s . Write

$$(c_0, \dots, c_6) = (2, 2, 5, 8, 9, 8, 5), \quad t := \sum_i c_i \rho_i = (2, 2, 5, 10, 10, 18, 14, 8).\tag{B.5}$$

Then $(s, s) = 2$, $(t, t) = 78$, and $(s, t) = 0$.

B.3 Lattice mode formulas

We use $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, with $T = L_{-1}^W$. The Heisenberg relations are

$$[h(m), k(n)] = m(h, k) \delta_{m+n, 0}, \quad h(0) e^\lambda = (h, \lambda) e^\lambda.$$

The twisted group algebra has multiplication $e^a e^b = \varepsilon(a, b) e^{a+b}$. On a pure exponential,

$$Y(e^a, z) e^b = \varepsilon(a, b) z^{(a, b)} \exp \left(\sum_{m > 0} \frac{a(-m)}{m} z^m \right) e^{a+b}.\tag{B.6}$$

On vectors containing negative Heisenberg modes, one also applies the positive-mode exponential

$$\exp \left(- \sum_{m > 0} \frac{a(m)}{m} z^{-m} \right).$$

The programs expand both exponential factors directly and compute their coefficients exactly.

The exponent defining ε is bilinear modulo two, so it is a normalized cocycle. Its antisymmetrization is

$$\frac{\varepsilon(a, b)}{\varepsilon(b, a)} = (-1)^{(a, b)},$$

because the diagonal contribution $2 \sum_i a_i b_i$ is even. For every root used here, $\varepsilon(\rho, -\rho) = -1$. Thus the minus signs in F_s, F_t give the positive levels. The leading principal minors of Q are 2, 3, 6, 12, 18, 15, 12, 9; hence Q is positive definite and $\det Q = 9$. **[CALC C01, p. 130]**

For either triple (E, F, H) at level k , the nonzero singular products are

$$H_{(0)} E = 2E, \quad H_{(0)} F = -2F, \quad E_{(0)} F = H, \quad H_{(1)} H = 2k\mathbf{1}, \quad E_{(1)} F = k\mathbf{1}.$$

The reverse zero-mode products are their negatives, and the reverse first-mode pairings agree. Other first-mode products vanish. All cross-products between the two triples vanish in modes zero and one; higher nonnegative modes vanish by weight. The exact checker evaluates every entry of both bracket and pairing tables in the rank-eight lattice, obtaining $k = 1, 39$. **[CALC C01, p. 130]**

For the simple affine identification, apply [29, Theorem 3.1] to each of the two simple Lie subalgebras of W_1 . Its hypotheses hold for the ambient W : it is simple and self-contragredient, with nonnegative integral grading and $W_0 = \mathbb{C}\mathbf{1}$ by the lattice construction [12]; it is regular by [13], hence C_2 -cofinite by [30, Theorem 4.5]. The theorem identifies the two generated affine VOAs with $L_1(\mathfrak{sl}_2)$ and $L_{39}(\mathfrak{sl}_2)$, using the levels computed above. Their commuting images define a nonzero map from the simple tensor product, so that map is injective. These hypotheses concern the known ambient lattice VOA, and do not assume C_2 -cofiniteness or rationality of U .

For $k = 1, 39$, write

$$\omega_k = \frac{1}{2(k+2)} \left(\frac{1}{2} H_{(-1)} H + E_{(-1)} F + F_{(-1)} E \right), \quad \omega_{\mathcal{A}} = \omega_1 + \omega_{39}. \quad (\text{B.7})$$

The checker expands this expression as lattice vectors and compares it with $\omega_W - \omega_C$, without using the Griess product table to define either Sugawara vector. **[CALC C01, p. 130]**

C The complete sparse coset basis

Put $M\gamma = \sum_{i=0}^6 \gamma_i \rho_i$, and use the ordered $A_2 \oplus A_5$ coordinates $\gamma = (\gamma_0, \dots, \gamma_6)$. The notation in the tables is

$$P_{ij} = \rho_i(-1)\rho_j(-1)\mathbf{1}, \quad X_\gamma = \eta(\gamma)e^{M\gamma}, \quad \eta(\gamma) = (-1)^{\gamma_6(\gamma_2+\gamma_3+\gamma_4+\gamma_5)}. \quad (\text{C.1})$$

In particular, an X_γ in the tables is a signed lattice exponential, not the bare exponential in the ambient eight coordinates. The matrices $M^T Q M$ and the Gram matrix of the ordered root system $A_2 \oplus A_5$ agree.

To explain the sign, let ε_ρ denote the same lower-triangular cocycle formula in these seven simple-root coordinates. The ratio between $\varepsilon(Ma, Mb)$ and $\varepsilon_\rho(a, b)$ differs on the unordered pairs $\{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\}$. The displayed quadratic sign is its coboundary:

$$\eta(a)\eta(b)\varepsilon(Ma, Mb) = \varepsilon_\rho(a, b)\eta(a+b).$$

Checking this identity on basis pairs checks its bilinear exponent identically. Thus the tables specify vectors for precisely the ambient cocycle (B.2). **[CALC C01, p. 130]**

For $k = 0, 1, 2, 3$, define

$$b_k = \sum_{i \leq j} a_{ij}^{(k)} P_{ij} + \sum_{\gamma} d_{\gamma}^{(k)} X_{\gamma}. \quad (\text{C.2})$$

The next two tables list every nonzero contribution. Unlisted coefficients are zero. Quadratic off-diagonal coefficients occur once; they are not entries of a symmetrized matrix. There are no derivative $\rho_i(-2)\mathbf{1}$ terms. **[CALC C01, p. 130]**

As a decoding example, the $(0, 1)$ quadratic row contributes $(35/3)\rho_0(-1)\rho_1(-1)\mathbf{1}$ to b_2 , with no additional factor of two. In the first exponential row, $\gamma = (-1, -1, 0, 0, 0, 1, 1)$ has $\eta(\gamma) = -1$ and $M\gamma = (-1, -1, 1, 1, 2, 2, 1)$. Its coefficient -1 in b_2 therefore contributes the positive bare exponential $e^{M\gamma}$. For the opposite row $\eta(-\gamma) = \eta(\gamma)$; its own listed coefficient must still be read separately. These tables define a chosen basis; Appendix D proves that it is the entire coset space.

C.1 Quadratic Heisenberg coefficients

(i, j)	$a_{ij}^{(0)}$	$a_{ij}^{(1)}$	$a_{ij}^{(2)}$	$a_{ij}^{(3)}$
(0, 0)	0	0	35/3	-35/6
(0, 1)	0	0	35/3	-35/6
(0, 2)	0	0	-10/3	5/6
(0, 3)	0	0	-20/3	8/3
(0, 4)	0	0	-9	9/2
(0, 5)	0	0	-28/3	16/3
(0, 6)	0	0	-20/3	25/6
(1, 1)	0	0	35/3	-35/6
(1, 2)	0	0	-20/3	25/6
(1, 3)	0	0	-28/3	16/3
(1, 4)	0	0	-9	9/2
(1, 5)	0	0	-20/3	8/3
(1, 6)	0	0	-10/3	5/6
(2, 2)	1/3	4/9	5/3	-5/6
(2, 3)	1/3	2/9	8/3	-4/3
(2, 4)	0	-4/9	2	-1
(2, 5)	-1/3	-5/9	4/3	-2/3
(2, 6)	-2/3	-1/9	2/3	-1/3
(3, 3)	1/3	2/9	8/3	-4/3
(3, 4)	0	1/9	4	-2
(3, 5)	-2/3	-2/9	8/3	-4/3
(3, 6)	-1/3	-5/9	4/3	-2/3
(4, 4)	0	1/3	3	-3/2
(4, 5)	0	1/9	4	-2
(4, 6)	0	-4/9	2	-1
(5, 5)	1/3	2/9	8/3	-4/3
(5, 6)	1/3	2/9	8/3	-4/3
(6, 6)	1/3	4/9	5/3	-5/6

C.2 Signed exponential coefficients

γ	$d_{\gamma}^{(0)}$	$d_{\gamma}^{(1)}$	$d_{\gamma}^{(2)}$	$d_{\gamma}^{(3)}$
(-1, -1, 0, 0, 0, 1, 1)	0	0	-1	1
(-1, -1, 0, 0, 0, 1, 1, 0)	0	0	-1	1
(-1, -1, 0, 0, 1, 1, 0, 0)	0	0	-1	1
(-1, -1, 1, 1, 0, 0, 0)	0	0	-1	1
(-1, 0, 0, 0, 0, 0, 1)	0	0	4	-3
(-1, 0, 0, 0, 0, 0, 1, 0)	0	0	3	-2
(-1, 0, 0, 0, 0, 1, 0, 0)	0	0	2	-1
(-1, 0, 0, 0, 1, 0, 0, 0)	0	0	1	0
(-1, 0, 1, 0, 0, 0, 0)	0	0	0	1
(0, -1, 0, 0, 0, 0, 1)	0	0	0	1
(0, -1, 0, 0, 0, 0, 1, 0)	0	0	1	0

Continued on the next page

γ	$d_\gamma^{(0)}$	$d_\gamma^{(1)}$	$d_\gamma^{(2)}$	$d_\gamma^{(3)}$
(0, -1, 0, 0, 1, 0, 0)	0	0	2	-1
(0, -1, 0, 1, 0, 0, 0)	0	0	3	-2
(0, -1, 1, 0, 0, 0, 0)	0	0	4	-3
(0, 0, -1, -1, 0, 1, 1)	-1	8/9	0	0
(0, 0, -1, 0, 0, 0, 1)	1	-1/3	0	0
(0, 0, -1, 0, 0, 1, 0)	0	5/9	0	0
(0, 0, -1, 0, 1, 0, 0)	0	5/9	0	0
(0, 0, 0, -1, 0, 0, 1)	0	8/9	0	0
(0, 0, 0, -1, 0, 1, 0)	1	0	0	0
(0, 0, 0, 0, -1, 0, 1)	0	1	0	0
(0, 0, 0, 0, 1, 0, -1)	0	5/9	0	0
(0, 0, 0, 1, 0, -1, 0)	1	0	0	0
(0, 0, 0, 1, 0, 0, -1)	0	5/9	0	0
(0, 0, 1, 0, -1, 0, 0)	0	1	0	0
(0, 0, 1, 0, 0, -1, 0)	0	8/9	0	0
(0, 0, 1, 0, 0, 0, -1)	1	-1/3	0	0
(0, 0, 1, 1, 0, -1, -1)	-1	8/9	0	0
(0, 1, -1, 0, 0, 0, 0)	0	0	10	-15/2
(0, 1, 0, -1, 0, 0, 0)	0	0	12	-8
(0, 1, 0, 0, -1, 0, 0)	0	0	9	-9/2
(0, 1, 0, 0, 0, -1, 0)	0	0	4	0
(0, 1, 0, 0, 0, 0, -1)	0	0	0	5/2
(1, 0, -1, 0, 0, 0, 0)	0	0	0	5/2
(1, 0, 0, -1, 0, 0, 0)	0	0	4	0
(1, 0, 0, 0, -1, 0, 0)	0	0	9	-9/2
(1, 0, 0, 0, 0, -1, 0)	0	0	12	-8
(1, 0, 0, 0, 0, 0, -1)	0	0	10	-15/2
(1, 1, -1, -1, 0, 0, 0)	0	0	-10	10
(1, 1, 0, -1, -1, 0, 0)	0	0	-18	18
(1, 1, 0, 0, -1, -1, 0)	0	0	-18	18
(1, 1, 0, 0, 0, -1, -1)	0	0	-10	10

D Coset dimension, products, and Virasoro checks

For the vectors e, f of (2.6), the singular products are

$$\begin{aligned}
e_{(0)}e &= Te, & e_{(1)}e &= 2e, & e_{(2)}e &= 0, & e_{(3)}e &= \frac{35}{31}\mathbf{1}, \\
f_{(0)}f &= Tf, & f_{(1)}f &= 2f, & f_{(2)}f &= 0, & f_{(3)}f &= \frac{1200}{1271}\mathbf{1}.
\end{aligned}
\tag{D.1}$$

All $e_{(n)}f$ and $f_{(n)}e$ for $n \geq 0$ vanish. For $n \geq 4$, this follows from the weight grading; the four remaining modes are computed exactly. **[CALC C01, p. 130]**

The ambient dimension check is performed in W_2 . Rational ellipsoid enumeration of $x^T Q x \leq 4$ gives 78 norm-two vectors and 702 norm-four vectors. Together with the Heisenberg vectors this gives the complete weight-two basis of dimension $8 + 36 + 8 \cdot 78 + 702 = 1370$. The simultaneous

zero-Cartan-weight subspace for s, t has dimension 86. On that whole subspace, the matrix of the conditions

$$x_{(n)}b = 0 \quad (x \in \{E_s, F_s, H_s, E_t, F_t, H_t\}, n = 0, 1, 2)$$

has rank 82 over $\mathbb{Q}$. Larger affine modes have negative output weight and vanish. Thus the kernel has dimension four. The four vectors of Appendix C lie in this kernel and are independent, so they are a basis of $(C_W)_2$. Commuting with the affine generators is equivalent to commuting with the vertex algebra they generate. **[CALC C01, p. 130]**

One can assemble the constraint matrix without knowing the answer: for each candidate column of zero Cartan weight, expand every displayed affine generator mode in the full output lattice-monomial basis and stack those coefficients. For a quadratic Heisenberg candidate, for example,

$$(E_s)_{(1)}(\alpha_i(-1)\alpha_j(-1)\mathbf{1}) = (s, \alpha_i)(s, \alpha_j)e^s.$$

This follows by commuting the two Heisenberg modes past $Y(e^s, z)$ and reading z^{-2} . The selected rank-82 minor gives a lower rank bound; the four explicit independent null vectors give the upper bound. Only after both checks do we use the first-mode table to compute $e_{(1)}e = 2e$ and the mode-three pairings to obtain the central charges.

The table below gives $((b_i)_{(1)}b_j)$ in the ordered basis (b_0, b_1, b_2, b_3) . The product is symmetric on these basis vectors; the mode-two products vanish, as checked separately. **[CALC C01, p. 130]**

(i, j)	b_0	b_1	b_2	b_3
$(0, 0)$	10	0	0	0
$(0, 1)$	$-2/3$	0	0	0
$(0, 2)$	8	0	-2	-4
$(0, 3)$	-4	0	2	4
$(1, 1)$	$16/27$	$74/9$	0	0
$(1, 2)$	0	8	$-5/3$	$-10/3$
$(1, 3)$	0	-4	$5/3$	$10/3$
$(2, 2)$	-12	-36	78	-24
$(2, 3)$	-20	0	$-43/2$	55
$(3, 3)$	36	18	6	-45

The bilinear pairings $(b_i)_{(3)}b_j = G_{ij}\mathbf{1}$ are

$$G = \begin{pmatrix} 10 & -2/3 & 8 & -4 \\ -2/3 & 104/9 & 32/3 & -16/3 \\ 8 & 32/3 & 1965 & -2245/2 \\ -4 & -16/3 & -2245/2 & 2805/4 \end{pmatrix}.$$

Here G denotes the VOA bilinear pairing, not a coordinate-wise Hermitian form on the multiplicity space of Appendix E. Using the explicit lattice vectors, the calculation verifies

$$\frac{8b_0 + 9b_1 + 2b_2 + 2b_3}{41} = \omega_W - \omega_{\mathcal{A}}.$$

It then computes modes zero through three of each ordered pair $(e, e), (f, f), (e, f), (f, e)$. These are exactly (D.1) and the vanishing mixed singular products. The mode-zero checks use the full weight-three lattice expressions; they are not inferred solely from a Griess idempotent equation. All arithmetic uses the basis $(1, r, q, rq)$ with $r^2 = 13$ and $q^2 = -3$. **[CALC C01, p. 130]**

E Highest-weight space and seed verification

E.1 Lattice vectors and arithmetic normalization

Put

$$\begin{aligned}\beta_1 &= (0, 0, 1, 2, 1, 3, 2, 1), \\ \beta_2 &= (0, 0, 2, 2, 3, 4, 3, 2), \\ \beta_3 &= (1, 1, 1, 1, 1, 2, 2, 1).\end{aligned}\tag{E.1}$$

The same line, with coefficient of u_1 normalized to one, is represented by

$$v' = u_1 + \left(\frac{11}{10} - \frac{3\theta^2}{40}\right)u_2 - \left(\frac{\theta^3}{32} + \frac{3\theta}{8}\right)u_3.\tag{E.2}$$

The calculation below verifies that $v' = ((-r + 3q)/20)^{-1}v$. **[CALC C01, p. 130]**

E.2 The eigenvalue calculation

In the ordered basis u_1, u_2, u_3 ,

$$e_{(1)} = \frac{1}{93} \begin{pmatrix} 29 & 10 & -r \\ 10 & 29 & r \\ -10r & 10r & 26 \end{pmatrix}, \quad \det(x - e_{(1)}) = (x - 13/31)^2(x - 2/31).$$

[CALC C01, p. 130]

The repeated eigenvalue has the explicitly parametrized eigenspace

$$(a, b, c) = (A - rC/20, A + rC/20, C).$$

Indeed the first two rows of $93e_{(1)} - 39I$ say $10(b - a) = rc$, and the third follows from $r^2 = 13$. The defining seed takes $A = 3q/20$, $C = 1$, and then rescaling its first coefficient gives v' . The matrix acts on columns. With the transported Hermitian form its Gram matrix is $34560 \operatorname{diag}(10, 10, 1)$, as follows from the three character values in Appendix G. For instance, the unequal entries $-r$ and $-10r$ become equal after multiplying by those Gram weights. Thus coordinate asymmetry is consistent with self-adjointness.

Let $u_i = e^{\beta_i}$, in the order of (E.1). The checker enumerates every lattice vector of norm at most four by rational ellipsoid enumeration. It then forms the complete lattice weight-two basis: eight $\alpha_i(-2)$ vectors, 36 quadratic Heisenberg vectors, all root exponentials with one Heisenberg-mode factor, and all norm-four exponentials. Its subspace with Cartan weights $(1, 13)$ is exactly $\mathbb{C}u_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_3$. For each u_i , it verifies

$$(E_s)_{(0)}u_i = (E_t)_{(0)}u_i = 0, \quad (H_s)_{(0)}u_i = u_i, \quad (H_t)_{(0)}u_i = 13u_i,$$

and $x_{(n)}u_i = 0$ for each of the six affine generators and $n = 1, 2$. All larger positive modes have negative output weight. Thus these three vectors form the whole affine highest-weight multiplicity space of type $(1, 13)$ in W_2 . **[CALC C01, p. 130]**

With columns denoting the images of u_j , the operators $(b_i)_{(1)}$ on this space are

$$\begin{aligned} B_0 &= \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & B_1 &= \begin{pmatrix} 8/9 & 8/9 & 0 \\ 8/9 & 8/9 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 7 & 0 & -1 \\ 0 & 7 & 1 \\ -10 & 10 & 20 \end{pmatrix}, & B_3 &= \begin{pmatrix} -7/2 & 0 & 1 \\ 0 & -7/2 & -1 \\ 10 & -10 & -10 \end{pmatrix}. \end{aligned}$$

These matrices are recomputed from the cocycle-corrected lattice vectors, then compared with the displayed entries. **[CALC C01, p. 130]** In particular,

$$\frac{8B_0 + 9B_1 + 2B_2 + 2B_3}{41} = \frac{23}{41}I_3.$$

Substitution of the coefficients of e and v gives (2.9). The affine Sugawara weight is independently

$$\frac{1(1+2)}{4(1+2)} + \frac{13(13+2)}{4(39+2)} = \frac{59}{41}.$$

The total weight is consistent with $59/41 + 23/41 = 2$, and $13/31 + 180/1271 = 23/41$.

The coefficient field is biquadratic. Its four embeddings send (r, q) to $(\pm r, \pm q)$, and the corresponding values of $\theta = r + q$ are distinct. Also

$$\theta^2 = 10 + 2rq, \quad \theta^3 = 4r + 36q, \quad r = \frac{9\theta}{8} - \frac{\theta^3}{32}, \quad q = \frac{\theta^3}{32} - \frac{\theta}{8}.$$

Thus $\mathbb{Q}(\theta) = \mathbb{Q}(r, q)$, with minimal polynomial $X^4 - 20X^2 + 256$. If $a = (-r + 3q)/20$, then

$$a^{-1} = -\frac{r + 3q}{2} = -\left(\frac{\theta^3}{32} + \frac{3\theta}{8}\right), \quad \frac{r + 3q}{20a} = \frac{11}{10} - \frac{3\theta^2}{40}.$$

Consequently $v' = a^{-1}v$, proving the normalization in (E.2). A nonzero rescaling of a generator gives the same generated vertex subalgebra.

F Coefficient-field conjugate choices

For $\sigma, \tau \in \{1, -1\}$, define

$$e_\sigma = \frac{2}{31}b_0 + \frac{6}{31}b_1 + \frac{2 - \sigma r}{93}b_2 + \frac{2 - 2\sigma r}{93}b_3, \quad f_\sigma = \omega_C - e_\sigma,$$

and

$$v_{\sigma, \tau} = \frac{-\sigma r + 3\tau q}{20}u_1 + \frac{\sigma r + 3\tau q}{20}u_2 + u_3.$$

The choice in Definition 2.1 is $(\sigma, \tau) = (1, 1)$. The four choices are

σ	τ	Axis	Seed coefficients in (u_1, u_2, u_3)
1	1	e_+	$((-r + 3q)/20, (r + 3q)/20, 1)$
1	-1	e_+	$((-r - 3q)/20, (r - 3q)/20, 1)$
-1	1	e_-	$((r + 3q)/20, (-r + 3q)/20, 1)$
-1	-1	e_-	$((r - 3q)/20, (-r - 3q)/20, 1)$

The affine generators and b_i have rational coefficients. Applying these four automorphisms of K to the exact defining identities gives the same central charges and seed eigenvalues for every choice. The checker also evaluates each conjugate seed eigenvalue directly. [CALC C01, p. 130] Complex conjugation fixes r and sends q to $-q$, exchanging the two seed lines for a fixed axis.

These are conjugates over the coefficient field. No complex-linear isomorphism between the generated VOAs is claimed here. This field conjugation is distinct from the modular-Galois symmetry of a rational VOA, which is used only after rationality has been proved in Section 11.

In particular no permutation of ordinary module labels is inferred from this table. The later Galois argument acts on character- T eigenvalues at conductor 39, after Corollary 9.7, and is explained independently in Section 11.

G Finite checks for the positive form

All vectors, cocycle signs, and β_i are those of the actual nine-generator definition. The replay reconstructs them directly from `shared/definition/data/definition.json`. It uses exact rational arithmetic in $\mathbb{Q}(r, q)$, with $r^2 = 13$, $q^2 = -3$, and complex conjugation $(r, q) \mapsto (r, -q)$.

The lattice character and generator images. For the positive character in (3.1), the relevant values are

$$\begin{aligned}\chi(s) &= 1, \\ (\chi(\rho_0), \dots, \chi(\rho_6)) &= (2, 2, 5, 8, 9, 8, 5), \\ (\chi(\beta_1), \chi(\beta_2), \chi(\beta_3)) &= (345600, 345600, 34560).\end{aligned}$$

The displayed conjugation formulas for the affine generators now follow term by term. For example, $\phi(E_t) = \sum_i \chi(\rho_i) e^{-\rho_i} = -F_t$. Every b_i has real rational coefficients, quadratic Heisenberg terms, and norm-four exponential terms. The quadratic terms are fixed. The cocycle-transport sign satisfies $\eta(-\gamma) = \eta(\gamma)$, so the exponential coefficient test can be performed directly in the seven ρ -coordinates:

$$d_{-\gamma}^{(i)} = d_{\gamma}^{(i)} \prod_{j=0}^6 c_j^{\gamma_j}.$$

For each pair of opposite exponential lattice vectors, the two coefficients satisfy exactly the relation imposed by ϕ . Replaying the complete sparse tables gives

$$\phi(b_0) = b_0, \quad \phi(b_1) = b_1, \quad \phi(b_2) = b_2, \quad \phi(b_3) = b_3.$$

The real coefficients of e and f then prove they are fixed. [CALC C02, p. 130]

The only nontrivial generator image. Put $u_i = e^{\beta_i}$ and $D = (F_s)_{(0)}(F_t)_{(0)}^{13}$. The exact lowering calculation is

$$\frac{Du_1}{13!} = 345600e^{-\beta_2}, \quad \frac{Du_2}{13!} = 345600e^{-\beta_1}, \quad \frac{Du_3}{13!} = -34560e^{-\beta_3}. \quad (\text{G.1})$$

[CALC C02, p. 130] Write

$$a = \frac{-r + 3q}{20}, \quad b = \frac{r + 3q}{20}, \quad v = au_1 + bu_2 + u_3.$$

Since $\bar{a} = -b$ and $\bar{b} = -a$, equation (G.1) gives

$$\frac{Dv}{13!} = 345600b e^{-\beta_1} + 345600a e^{-\beta_2} - 34560e^{-\beta_3} = -\phi(v).$$

There is no need to assume that U contains the conjugate highest-weight line: its image under ϕ is this explicit lowest-weight lowering vector already in U . For an independent normalization check,

$$H_\chi(v, v) = 345600|a|^2 + 345600|b|^2 + 34560 = 103680 > 0,$$

because $|a|^2 = |b|^2 = 1/10$. The replay also checks $v_{(3)}\phi(v) = 103680\mathbf{1}$. **[CALC C02, p. 130]**

The finite computation has three stages: fix the character and cocycle, iterate the root zero-mode formula on each of the three seed exponentials, and compare the final coefficient at the opposite lattice vector. For example, the u_3 trace ends at $-34560 \cdot 13! e^{-\beta_3}$ before division by $13!$. The negative sign is essential: it combines with the conjugate coefficients of v to give $Dv/13! = -\phi(v)$.

How to verify the lowering calculation. The certificate records all thirteen successive $(F_t)_{(0)}$ images of each u_i , followed by its $(F_s)_{(0)}$ image. At each step it uses the inherited lattice mode formula. On an exponential this is

$$Y(e^\rho, z)e^\lambda = \varepsilon(\rho, \lambda) z^{(\rho, \lambda)} \exp\left(\sum_{m>0} \frac{\rho(-m)}{m} z^m\right) e^{\rho+\lambda}.$$

For vectors containing negative Heisenberg modes, a factor $h(-n)$ is either retained or replaced by the scalar $-(\rho, h)z^{-n}$. Expanding these choices and taking the coefficient of z^{-1} gives the zero mode exactly. Thus all intermediate calculations in (G.1) are independently replayable; no saved product table or modular rank is an input.

Replay and scope. Run, from the publication root,

```
python3 verify_certificate.py C02
```

Use Python 3.12 with `sympy` and assertions enabled. The supplied witness is `shared/positive/certificates/HGR_UASSUMP_v1_report.json`. The entry point stages fresh computation and writes `replay/C02_latest.json`; it preserves the supplied witness. The witness includes all residuals, full lowering traces, input SHA-256 hashes, and execution metadata. It also checks L_1 on every member of the complete 86-dimensional ambient weight-one lattice basis. This argument does not use the later established value $\dim U_1 = 6$. The finite certificate together with the structural proof establishes the proposition in Section 3. The remaining inputs to rationality are proved separately: C_2 -cofiniteness in Section 5 and Zhu semisimplicity in Section 9.

Independent structural audit. The following file provides a separate implementation:

```
shared/positive/scripts/audit_HGR_UASSUMP_v1_independent.py
```

It imports neither the original lattice-mode implementation nor its coefficient-field class. It derives root zero modes directly from the following weight-two cases. Let ρ be a root and let $d = (\rho, \rho)$. For $(p, p) = 4$,

$$(e^\rho)_{(0)}e^p = \begin{cases} \varepsilon(\rho, p)e^{p+\rho}, & d = -1, \\ \varepsilon(\rho, p)\rho(-1)e^{p+\rho}, & d = -2, \\ 0, & d \geq 0. \end{cases}$$

For $(p, p) = 2$ and one Heisenberg-mode factor $h(-1)$, the cases used in the lowering subspace are

$$(e^\rho)_{(0)} h(-1) e^p = \begin{cases} \varepsilon(\rho, p) (h(-1) - (\rho, h) \rho(-1)) e^{p+\rho}, & d = -1, \\ -\varepsilon(\rho, p) (\rho, h) e^{p+\rho}, & d = 0, \\ 0, & d \geq 1. \end{cases}$$

The $d = -2$ case has $p = -\rho$, hence $(s, p) = 0$, and cannot arise in this $(s, p) = 1$ lowering subspace for the t -affine generators. The implementation also handles it explicitly. These closed formulas independently reproduce all 45 recorded vectors in the three lowering traces, including their initial vectors and their final $(F_s)_{(0)}$ images.

The audit independently checks every coefficient pair for opposite lattice vectors of the four b_i , reconstructs the Sugawara subtraction, and recomputes the four first-mode action matrices on the seed multiplicity space. Their positive Hermitian Gram matrix is

$$G_{\text{top}} = \text{diag}(345600, 345600, 34560), \quad B_i^\top G_{\text{top}} = G_{\text{top}} B_i.$$

Here B_i denotes the recomputed first-mode matrix of b_i . The audit also checks $(F_t)_{(0)}^* = (E_t)_{(0)}$ on an invariant 340-dimensional weight-two subspace and rejects nine deliberately corrupted conventions. Those finite adjoint checks are supplementary; **[CALC C02, p. 130]** the all-weight unitary proof is the transport argument in the main text. Run the independent audit after the main replay:

```
python3 shared/positive/scripts/audit_HGR_UASSUMP_v1_independent.py
```

Its report is

```
shared/positive/certificates/HGR_UASSUMP_v1_independent_audit_r2.json
```

H Strong-generation coordinates and mode closure

This appendix supplies the coordinate choices and mode calculations used in Section 4, together with the ordered weight-three basis and highest-weight lifts used in Section 9. The normal-space calculation makes the weight-three quotient explicit; the later relation ranks give upper bounds on new generators.

The explicit finite feedback conditions are

$$\begin{aligned} H_{(n)} S_i &\subset S_{i-n} \quad (i = 2, 3; n \geq 0), \\ (S_2)_{(1)} S_2 &\subset S_2, \quad (S_2)_{(2)} S_2 \subset H, \\ (S_2)_{(1)} S_3 &\subset S_3, \quad (S_2)_{(2)} S_3 \subset S_2, \quad (S_2)_{(3)} S_3 \subset H, \\ L_1 S_2 &\subset H, \quad L_1 S_3 \subset S_2. \end{aligned}$$

Use the lattice formula (B.6), the Heisenberg commutator, and the complete exact lattice-monomial bases. Each required product reduces with zero residual against the indicated basis. $\mathfrak{g}$ -equivariance reduces the left inputs to sixteen highest vectors; lowering spans each of their irreducible $\mathfrak{g}$ -modules. The weight-three check includes all $16 \cdot 956$ products in each of modes 1, 2, 3. The basis data, zero-residual records, and reproduction scripts are specified in Appendix R. **[CALC C03, p. 130] [CALC C04, p. 130]**

The four raising and lowering Lie algebra actions give all six zeroth actions by their commutators. Positive affine modes require no additional containment table: $TH \subset S_2$ and $(Th)_{(n+1)} = -(n +$

1) $h_{(n)}$ turn the mode-closure checks into the required affine generator containments. The remaining scalar outputs and larger modes follow from weight grading.

The L_1 containment in weight three also follows on its defining summands TS_2 , $H_{(-1)}S_2$, and $(S_2)_{(0)}S_2$. Use $[L_1, T] = 2L_0$, $[L_1, h_{(-1)}] = h_{(0)}$, and $[L_1, a_{(0)}] = 2a_{(1)} + (L_1a)_{(0)}$, together with the weight-two feedback. Skew symmetry gives $(S_2)_{(0)}H \subset S_2$, so every resulting term lies in S_2 . This includes weight-two vectors without any assumption that positive Virasoro modes act by zero on them.

H.1 The finite-dimensional $\mathfrak{g}$ -modules

The constructed space S_2 has dimension 120. Its decomposition is

$$S_2 \cong V_{2,0} \oplus 4V_{0,0} \oplus V_{0,2} \oplus V_{2,2} \oplus 2V_{0,4} \oplus V_{1,13} \oplus V_{1,7} \\ \oplus V_{1,5} \oplus V_{1,1} \oplus V_{0,12} \oplus 2V_{0,8}.$$

[CALC C03, p. 130]

H.2 The ordered weight-three basis and its certificate

There are three useful descriptions of the same vector: its raw generation word, its normalized lattice-monomial residual, and its coefficients in a zero-mode lowering basis. For example, the origin ["translation", 0] means Ty_0 ; subtracting accepted rows and dividing by its pivot gives its lattice-monomial coordinate vector. Expanding that vector in the $u_{b,i,j}$ below gives its coordinates in the Lie algebra module basis. These conversions are exact linear algebra, not changes to the underlying lattice vector. Lists of field coefficients here use descending powers of θ .

The ordered exact basis $x_0, \dots, x_{955}$ of S_3 is stored in the C04 certificate, not as a list of formulas in this appendix. Its precise locator in the LAT package is

`shared/lattice/research/results/haagerup_c8_grade_three_exact_basis.json`.

The top-level array **basis** has 956 entries, and x_ℓ is, by definition, entry **basis**[ℓ]. All array indices in this subsection start at zero. Each entry is a sparse list of pairs consisting of a lattice-monomial label and its coefficient. A label records the integral lattice coordinates in the fixed A_2 -then- E_6 simple-root order, followed by the sorted tuple of Heisenberg-mode labels (n, i) ; the latter denotes the $(-n)$ mode of the i th simple-root Heisenberg vector. The coefficient list $[c_d, \dots, c_0]$ denotes $\sum_{k=0}^d c_k \theta^k$, with rational c_k and

$$\theta = \sqrt{13} + \sqrt{-3}, \quad \theta^4 - 20\theta^2 + 256 = 0.$$

Thus the file specifies every x_ℓ as a complete lattice vector over K , including its normalization.
[CALC C04, p. 130]

Generation order and normalization. Let $y_0, \dots, y_{119}$ be the entries, in their saved order, of the array **basis** in the C03 file

`shared/lattice/research/results/haagerup_c8_degree_two_exact_basis.json`.

These form the normalized basis of S_2 in lattice-monomial coordinates; their indices are not the indices (a, i, j) of its zero-mode lowering basis. The highest vectors z_a are expressed in the y_i basis by the array `highest_weight_vectors`, using the entry `basis_coordinates` for each vector. The six affine generators are ordered as

$$(h_0, h_1, h_2, h_3, h_4, h_5) = (E_s, F_s, H_s, E_t, F_t, H_t).$$

The companion file is

`shared/lattice/research/results/haagerup_c8_grade_three_basis_modp.json`.

It contains an array `origins` in the same 956-row order. Its entries prescribe the raw vectors w_ℓ as follows:

Row range	Saved origin	Raw vector w_ℓ
$0 \leq \ell < 120$	<code>["translation", i]</code>	Ty_i
$120 \leq \ell < 634$	<code>["current_minus_one", k, i]</code>	$(h_k)_{(-1)}y_i$
$634 \leq \ell < 956$	<code>["weight_two_zero_mode", i, j]</code>	$(y_i)_{(0)}y_j$

The indices i, j, k are read from the corresponding origin entry; the order is the saved array order, not a new sorting of these tuples. Although this file was produced in characteristic 10009, it is used here only to prescribe the generation words and their order.

The basis row x_ℓ is the normalized residual of w_ℓ , rather than the raw vector itself. More precisely, reconstruct w_ℓ over K and reduce it against the previously accepted rows. At each step, take the lexicographically least lattice-monomial label in the residual. If that label is the leading pivot of an earlier row, subtract the corresponding multiple of that row and continue; otherwise stop. The comparison is the lexicographic comparison of the tuple of lattice coordinates followed by the Heisenberg-mode tuple. If r_ℓ is the resulting residual, p_ℓ its least label, and γ_ℓ the coefficient of p_ℓ , set

$$x_\ell = \gamma_\ell^{-1} r_\ell.$$

The C04 calculation reconstructs nonzero residuals with distinct normalized pivots and compares their complete coefficients with the saved rows. The reconstruction is implemented by `ExactSpan.replay` and `ExactSpan.reduce` in

`shared/lattice/research/haagerup_c8_grade_three_exact.py`.

The same class supplies the exact translation and mode operations. The finite-field discovery of the ordered origins is in `shared/lattice/research/haagerup_c8_grade_three.py`. [CALC C04, p. 130] In particular,

$$\begin{aligned} \langle x_0, \dots, x_{119} \rangle_K &= TS_2, \\ \langle x_0, \dots, x_{633} \rangle_K &= TS_2 + H_{(-1)}S_2 = N_3. \end{aligned}$$

The normal-prefix rank and the 322-dimensional quotient are checked separately in C05, as explained below. [CALC C05, p. 130]

H.3 The highest-weight lifts used in the mixed family

The mixed-relation code uses a zero-mode lowering basis of the same space S_3 , rather than the coordinate-pivot basis just defined. Its ordered list of 87 highest vectors and their formal preimages is encoded in the R12 file

shared/rank12/certificates/mixed_exact.json.

Specifically, entry `basis3[b]`, for $0 \leq b < 87$, records a highest weight (r_b, j_b) and an index κ_b in its `descriptor` field. The formal highest vector $\delta_{\kappa_b} \in L_3$ is described by entry κ_b of the top-level array `basis3_descriptors`. That array contains all 495 highest-weight domain descriptors, not merely the 87 selected ones. Its three kinds, `derivative`, `normal_current`, and `alternating`, specify the three summands of L_3 ; the `a` or `terms` entries give the explicit highest vector. A `derivative` descriptor with index a denotes z_a in the first summand, whose vector image is Tz_a . For the other two kinds, decode each stored term as its rational coefficient times, respectively,

$$h_{\epsilon;i,j} \otimes z_{a;k,l} \quad \text{or} \quad z_{a;i,j} \wedge z_{d;k,l},$$

where the affine generator tag `["current", epsilon, i, j]` means

$$h_{0;i,0} = (F_s)_{(0)}^i E_s \quad (0 \leq i \leq 2), \quad h_{1;0,j} = (F_t)_{(0)}^j E_t \quad (0 \leq j \leq 2).$$

These are the unnormalized adjoint lowering bases, as used in `Triple.current_vectors`. The selection order is obtained by traversing the `groups` of `shared/rank12/certificates/bracket_discovery.json` in their saved order and concatenating their `selected_columns` arrays. This is precisely the order retained by `basis3`. [CALC C11, p. 132]

Put

$$u_{b;i,j} := (F_s)_{(0)}^i (F_t)_{(0)}^j \Phi_3(\delta_{\kappa_b}), \quad 0 \leq i \leq r_b, \quad 0 \leq j \leq j_b.$$

The exact bracket calculation identifies these selected images and their zero-mode lowering vectors as a basis of S_3 , with

$$\sum_{b=0}^{86} (r_b + 1)(j_b + 1) = 956.$$

They determine the equivariant section used in Section 9 by the explicit rule

$$\sigma_3(u_{b;i,j}) = (F_s)_{(0)}^i (F_t)_{(0)}^j \delta_{\kappa_b},$$

where the zero modes on the right act on the formal domain L_3 . Equivariance of Φ_3 gives $\Phi_3 \sigma_3(u_{b;i,j}) = u_{b;i,j}$, and hence $\Phi_3 \sigma_3 = \text{id}_{S_3}$.

For an x_ℓ in the coordinate-pivot basis, the corresponding lift is obtained by its unique expansion in the $u_{b;i,j}$ basis:

$$x_\ell = \sum_{b,i,j} \eta_{\ell;b,i,j} u_{b;i,j} \quad \implies \quad \sigma_3(x_\ell) = \sum_{b,i,j} \eta_{\ell;b,i,j} (F_s)_{(0)}^i (F_t)_{(0)}^j \delta_{\kappa_b}.$$

The coefficients $\eta_{\ell;b,i,j}$ are determined by exact linear algebra in the complete lattice coordinates. The certificate encodes σ_3 through the selected descriptors and zero-mode lowering; it does not contain a separate table of the 956 expressions $\sigma_3(x_\ell)$. The implementation is `Mixed.__init__` and `Mixed.u3` in `shared/rank12/certificates/mixed_weight4.py`, using `Triple.image` in `shared/rank12/certificates/bracket_relations.py`. Thus the stored descriptors supply the actual formal lifts used for Zhu conversion in the mixed calculation. [CALC C11, p. 132]

Paths in the organized certificate bundle. Every displayed path is already relative to the extracted publication root, including its `shared/` prefix. No additional LAT or R12 directory is to be inserted. The corresponding entry points are C04 for the exact basis and feedback, C05 for the normal prefix, and C11 for the highest-weight lifts and relations. From the organized bundle root, run

```
python3 verify_certificate.py C04
python3 verify_certificate.py C05
python3 verify_certificate.py C11
```

The prefix checker is `shared/lattice/check_normal_prefix.py`. It checks the rank and support interface of the basis and bracket certificates; the complete lattice-vector identities are reconstructed by the separate C04 and C11 replays.

H.4 The projected Jacobi relations

Detailed proof of Proposition 4.2. First remove brackets with a normal expression as input. The derivation and normal-product identities, with Proposition 4.1, give

$$(S_2)_{(0)}N_3 \subset N_4, \quad (S_2)_{(0)}N_4 \subset N_5, \quad (N_3)_{(0)}S_3 \subset N_5. \quad (\text{H.1})$$

The affine weight-two vectors and ω_W also contribute only normal expressions. The mode expansions below cover both input orders.

For weight four, the map

$$\mu : E \otimes D_3 \longrightarrow S_4/N_4, \quad a \otimes x \longmapsto a_{(0)}x$$

is onto. Jacobi kills the rows

$$J(a, b, c) = a \otimes \beta(b, c) + b \otimes \beta(c, a) + c \otimes \beta(a, b). \quad (\text{H.2})$$

Indeed, replace β by the actual zeroth product modulo N_3 . The commutator identity and skew symmetry give the cyclic relation modulo $TS_3 \subset N_4$.

The selected Jacobi relations leave three coefficient representatives, denoted D_4 . Exact elimination defines a projection $\Gamma : E \otimes D_3 \rightarrow D_4$ and a lift ρ with

$$a_{(0)}x \equiv \rho\Gamma(a \otimes x) \pmod{N_4}. \quad (\text{H.3})$$

[CALC C06, p. 131] This proves the assertion about weight four. There may be further relations; the three representatives are an upper bound on new generators.

For weight five, any $r = \sum_i a_i \wedge b_i \in \ker \beta$ and $x \in D_3$ give a coefficient row

$$j(r, x) = \sum_i (a_i \otimes \Gamma(b_i \otimes x) - b_i \otimes \Gamma(a_i \otimes x)) \in E \otimes D_4. \quad (\text{H.4})$$

Under $a \otimes d \mapsto a_{(0)}\rho(d) \pmod{N_5}$, its image is

$$\sum_i ((a_i)_{(0)}(b_i)_{(0)}x - (b_i)_{(0)}(a_i)_{(0)}x) = \left(\sum_i (a_i)_{(0)}b_i \right)_{(0)} x.$$

The vector in parentheses lies in N_3 , and (H.1) therefore makes the image zero. The second relation matrix has full column rank: these rows span $E \otimes D_4$. [CALC C06, p. 131] Thus every weight-five obstruction vanishes. Together with (H.1) and the affine and conformal terms, this gives (4.2).

The relations above are defined over K and follow from actual vertex identities. Nonzero minors modulo 10009 certify their independence over K and supply the exact inverses used in Γ and $\ker \beta$. This gives an upper bound on new generators because the independent rows are relations between them. The next subsection records the ranks, selected columns, and lifts. $\square$

H.5 The representatives and their integral lifts

The three rank assertions used in the main proof are [CALC C06, p. 131]

Map or relation matrix	Columns	Selected rank	Consequence
β	4,656	322	Onto D_3
Quartic Jacobi relations	31,234	31,231	Three representatives
Quintic Jacobi relations	291	291	No new representative

Here $4,656 = \dim \Lambda^2 E$, $31,234 = \dim(E \otimes D_3)$, and $291 = \dim(E \otimes D_4)$.

In the original highest-vector list of S_2 , choose indices

$$4, 8, 9, 10, 11, 12, 13, 14, 15.$$

For each highest-weight vector z of type (r, j) , take the unnormalized lowering vectors $(F_s)_{(0)}^a (F_t)_{(0)}^b z$, in order $0 \leq a \leq r$, $0 \leq b \leq j$. Their 97 vectors span E , with decomposition

$$V_{0,0} \oplus V_{0,4} \oplus V_{1,13} \oplus V_{1,7} \oplus V_{1,5} \oplus V_{1,1} \oplus V_{0,12} \oplus 2V_{0,8}.$$

A nonzero 97×97 complement minor gives $S_2 = B_2 + K\omega_W + E$. [CALC C06, p. 131] Use the ordered exact S_3 basis $x_0, \dots, x_{955}$ fixed in Subsection H.2. Its first 120 rows come from TS_2 and the next 514 from $H_{(-1)}S_2$, so the first 634 rows lie in N_3 . They span it: the derivative and affine-generator normal-product inputs have total dimension 840, and their 18 exact highest-weight relation families have total dimension 206. Each relation uses only normal columns and has its own nonpivot column, so their lowering vectors give 206 independent input relations. This bounds $\dim N_3$ by 634; the first 634 basis rows give the reverse bound. The final 322 rows consequently give a basis of S_3/N_3 , denoted D_3 . This is a characteristic-zero quotient calculation. [CALC C05, p. 130]

The rank statements have two logically distinct ingredients. The complete lattice/VOA identities show that each row is a valid relation. The selected minors show that sufficiently many such rows are independent. Their indices are distinct and weight-homogeneous; their covered columns are exactly the quartic complement of the three listed free columns, or all quintic columns. Nonzero determinants without these coverage checks would not give the stated quotient bound. The three quartic representatives require no converse independence assertion in the VOA.

At $p = 10009$ use $r \mapsto 825$, $q \mapsto 2089$, and $\theta \mapsto 2914$. Work in the DVR $R = \mathbb{Z}[\theta]_{(p, \theta - 2914)}$, whose DVR property is proved in Appendix R.1. The ordered S_3 basis has coefficients in R and distinct normalized coordinate pivots. Hence reduction of any integral normal input uses only R -coefficients; its last 322 coordinates are zero by the calculation above. The same normal prefix therefore gives a direct summand over R , with free quotient of rank 322. The selected bracket minor is a unit and defines an exact inverse. For each nonpivot pair $a \wedge b$, subtract its exact expansion in that pivot-pair basis to obtain an element of $\ker \beta$. This defines every kernel row before reduction modulo p .

The 4,334-dimensional kernel is not constructed by interpreting saved finite-field entries as rational coefficients.

The selected quartic Jacobi rows give a unit 31,231-column minor. The free columns, in $(E \text{ index}, D_3 \text{ index})$ notation, are

$$(96, 287), \quad (96, 147), \quad (96, 145).$$

Projection along their exact row span defines Γ . If a is the 96th E vector, the lifts are

$$q_- = a_{(0)}x_{921}, \quad q_0 = a_{(0)}x_{781}, \quad q_+ = a_{(0)}x_{779}.$$

Their Cartan weights are $(0, -2), (0, 0), (0, 2)$. They span S_4 modulo N_4 ; independence is unnecessary.

The quintic rows (H.4) use the exact inverse bracket matrix and exact Γ . Their 291×291 determinant has nonzero reduction. The stored block computation and its inverse therefore specialize correctly even if entries that vanish modulo p do not vanish over K . **[CALC C06, p. 131]** The independent verifier reconstructs 149 quartic and 47 quintic square minors with a separate Bareiss determinant algorithm. All are nonzero. It also checks all 147,440 modular quartic Jacobi rows against the saved remainder map. **[CALC C06, p. 131]**

H.6 Affine generator and normal-product closure

For a typical derivative input, $a_{(0)}Tb = T(a_{(0)}b)$ lies in $TS_3 \subset N_4$ when $a, b \in S_2$. For a typical affine generator normal input, the derivation formula gives

$$a_{(0)}(h_{(-1)}b) = (a_{(0)}h)_{(-1)}b + h_{(-1)}(a_{(0)}b).$$

Skew symmetry and finite mode closure put the first factor $a_{(0)}h$ in S_2 , up to affine generator derivatives, while the second bracket lies in S_3 . Thus both terms have the weight-four normal forms used in N_4 . The following expansions include the reversed orders and the weight-five cases as well.

We first justify the normal-input containments (H.1). A zeroth mode commutes with T and acts as a derivation on normal products. Applying this to the definitions of N_3 and N_4 gives $(S_2)_{(0)}N_3 \subset N_4$ and $(S_2)_{(0)}N_4 \subset N_5$. Reversed normal products differ by derivatives of the singular products already controlled by finite mode closure.

For the remaining containment, $(Tb)_{(0)}v = 0$, while

$$(h_{(-1)}b)_{(0)}v = \sum_{j \geq 0} h_{(-1-j)}b_{(j)}v + \sum_{j \geq 0} b_{(-1-j)}h_{(j)}v.$$

The first $j = 0$ term uses $b_{(0)}v \in S_4$; the second uses $h_{(0)}v \in S_3$. All other terms use finite mode closure and derivatives of lower-weight normal products. Each lies in N_5 . Hence $(N_3)_{(0)}S_3 \subset N_5$.

The finite mode closure and the definitions give

$$H_{(m)}S_4 \subset S_{4-m} \quad (m \geq 0).$$

For a new zeroth product, expand

$$h_{(m)}(a_{(0)}x) = a_{(0)}h_{(m)}x + \sum_{j=0}^m \binom{m}{j} (h_{(j)}a)_{(m-j)}x.$$

Here $h_{(0)}a \in S_2$, $h_{(1)}a \in H$, $h_{(2)}a$ is scalar, and higher terms vanish. For $m = 1$, the new term lies in $(S_2)_{(0)}S_2 \subset S_3$; for $m = 2, 3$ it uses the stated feedback. Translation and normal-product terms follow from the same commutator identity and the normal-product formula. The latter also gives $(B_2)_{(0)}S_3 \subset N_4$ and $(B_2)_{(0)}S_4 \subset N_5$; derivatives of affine generators have zeroth mode zero, and $(\omega_W)_{(0)} = T$.

All parenthesized normal expressions of weights at most four lie in the constructed spaces. At weight five their span is N_5 : the root splits are $1 + 4$ and $2 + 3$, and reversals differ by derivatives of controlled singular products. This justifies the normal spans used in (H.1).

H.7 Recovering positive S_2 -modes on S_4

The purpose of this subsection is to turn zeroth-mode closure into all the positive-mode closure needed by Lemma 4.3. Each commutator isolates the next mode, except at the zero coefficient where an L_2 commutator is used. It is a formal VOA deduction from the finite input, not an additional unrecorded high-weight computation.

The definitions and low feedback give

$$L_1S_4 \subset S_3, \quad L_2S_4 \subset S_2, \quad L_1N_5 \subset S_4.$$

For every weight-two vector a ,

$$\begin{aligned} [L_1, a_{(n)}] &= (2 - n)a_{(n+1)} + (L_1a)_{(n)}, \\ [L_2, a_{(n)}] &= (3 - n)a_{(n+2)} + 3(L_1a)_{(n+1)} + (L_2a)_{(n)}. \end{aligned}$$

Use $L_1a \in H$, $L_2a \in K\mathbf{1}$, $[L_1, T] = 2L_0$, and $[L_2, T] = 3L_1$. On $a_{(0)}x$, for instance, $L_1(a_{(0)}x) = a_{(0)}L_1x + 2a_{(1)}x + (L_1a)_{(0)}x$. On $(S_2)_{(-1)}S_3$, the only additional L_1 term is $3(S_2)_{(0)}S_3 \subset S_4$.

For $a \in S_2$, $x \in S_4$, the established zeroth-product containment and these inclusions yield successively

$$\begin{aligned} 2a_{(1)}x &= L_1(a_{(0)}x) - a_{(0)}L_1x - (L_1a)_{(0)}x \in S_4, \\ a_{(2)}x &= L_1(a_{(1)}x) - a_{(1)}L_1x - (L_1a)_{(1)}x \in S_3, \\ 2a_{(3)}x &= L_2(a_{(1)}x) - a_{(1)}L_2x - 3(L_1a)_{(2)}x - (L_2a)_{(1)}x \in S_2, \\ -a_{(4)}x &= L_1(a_{(3)}x) - a_{(3)}L_1x - (L_1a)_{(3)}x \in H. \end{aligned}$$

The fifth product is scalar; higher ones vanish by weight. The L_2 step is needed because the coefficient of $a_{(3)}$ in $[L_1, a_{(2)}]$ is zero. These computations verify every singular $H + S_2$ action on the chosen finite set, as required by Lemma 4.3.

H.8 The all-weight induction and normal form

Detailed proof of Lemma 4.3. A normal word is a product of derivatives of generators; its parenthesization is initially arbitrary. We first prove singular closure using a well-founded weight order, and only then change parenthesizations to a right-nested form. This separates the all-weight closure claim from the normal-form claim.

The zeroth modes of A preserve $\mathcal{N}(G)$: they commute with T , are derivations of every normal product, and send each generator into $\mathcal{N}(G)$ by hypothesis 1. Delete vacuum factors from normal expressions. Prove closure under singular products $x_{(n)}y$, $n \geq 0$, by induction on the ordered pair $(\text{wt}(x) + \text{wt}(y), \text{wt}(x))$. The vacuum cases are zero. Derivatives reduce the first coordinate, by

$$(Tp)_{(n)}y = -np_{(n-1)}y, \quad x_{(n)}Tq = T(x_{(n)}q) + nx_{(n-1)}q.$$

For a normal product in the first argument, use

$$(p_{(-1)}q)_{(n)}y = \sum_{j \geq 0} p_{(-1-j)}q_{(n+j)}y + \sum_{j \geq 0} q_{(n-1-j)}p_{(j)}y.$$

Every inner singular product, and every remaining outer singular product, has smaller total input weight. Outer negative products are normal expressions.

For any first argument x , expand a normal product in the second:

$$x_{(n)}(p_{(-1)}q) = p_{(-1)}(x_{(n)}q) + \sum_{j=0}^n \binom{n}{j} (x_{(j)}p)_{(n-1-j)}q.$$

The nested products have smaller total input weight because p, q have positive weight. In each outer product in the correction sum the total weight has dropped by $j + 1$. Thus both inputs reduce to generators. When the first is in A , hypothesis 1 applies.

For a remaining generator, hypothesis 2 reduces the problem to a bracket $a_{(0)}b$. Its action is

$$(a_{(0)}b)_{(n)}y = a_{(0)}(b_{(n)}y) - b_{(n)}(a_{(0)}y).$$

Here $\text{wt}(b) = \text{wt}(a_{(0)}b) - \text{wt}(a) + 1$ is strictly smaller than the weight of the bracket. In the first term, $b_{(n)}y$ therefore has smaller total input weight, and the outer $a_{(0)}$ preserves $\mathcal{N}(G)$. In the second term, the total input weight is unchanged but the first input is b . This is the purpose of the second induction coordinate. Both terms are controlled, proving closure.

For a concrete comparison, if the second input is $p_{(-1)}q$, the outer inputs in the correction term have total weight $\text{wt}(x) + \text{wt}(p) + \text{wt}(q) - j - 1$. Thus their first induction coordinate decreases. For a bracket first input, only the second coordinate may decrease; this is why an induction solely on total weight would not explain that case.

To obtain the usual right-nested normal form, assign each generator its conformal weight as degree and let differentiation preserve degree. A homogeneous normal expression has degree at most its conformal weight. A singular product of two generators therefore lowers degree by at least one. The two mode identities just used extend this bound from generator pairs to all normal expressions. Quasi-associativity and skew symmetry therefore make the associated graded algebra commutative and associative. Its monomials lift to right-nested normal products. Subtracting their leading symbols and inducting on the filtration degree proves ordinary strong generation. $\square$

More explicitly, if F_p bounds the sum of undifferentiated generator weights in a normal expression, the mode identities give

$$(F_p)_{(n)}F_q \subset F_{p+q-1} \ (n \geq 0), \quad (F_p)_{(n)}F_q \subset F_{p+q} \ (n < 0).$$

Induct on total leaf weight, then derivative count: a singular product lowers the former, and sesquilinearity lowers the latter.

H.9 How the coset-square witnesses are computed

The input is the exact 120-vector and 956-vector bases and the explicit axes e, f . The output sought is a full lattice identity placing each square in the span of terms already known to vanish in the reduced C_2 algebra. Only zero-Cartan-weight columns are needed, since both axes are invariant vectors.

Construct the candidate columns. There are 58 derivatives Tx_i of weight-three zero-weight basis vectors and 238 affine generator products $(h_k)_{(-1)}x_i$ whose Cartan weights add to zero. For each pair of weight-two highest-weight vectors of irreducible $\mathfrak{g}$ -modules of the same nontrivial type (j, k) , take the invariant normal-product sum

$$\sum_{a=0}^j \sum_{b=0}^k (-1)^{a+b} ((F_s)_{(0)}^a (F_t)_{(0)}^b z)_{(-1)} ((F_s)_{(0)}^{j-a} (F_t)_{(0)}^{k-b} z').$$

The raising relations cancel adjacent terms, so this is an invariant vector. The saved ordering gives 14 such columns. The total is $58 + 238 + 14 = 310$; independence of all 310 is neither asserted nor required.

Reduce in complete lattice-monomial coordinates. Expand the candidates with the lattice mode formulas. Traverse them in the saved order, subtract previously accepted normalized rows, and divide each nonzero residual by its first lattice-monomial coefficient. The span of derivatives and normal products with affine generators has rank 217; the invariant products increase the accepted row count to 222. Call the resulting rows $b_0^{\text{sq}}, \dots, b_{221}^{\text{sq}}$. Expand $e_{(-1)}e$ and $f_{(-1)}f$, solve for their coefficients in these rows, and reassemble every ambient coefficient independently. Both residuals are exactly zero over K .

Read a witness. The coefficient file is `shared/lattice/research/results/haagerup_c8_c2_axis_exact.json`. Its `echelon_basis_origins` and `exact_normalization_pivots` reconstruct the rows by the preceding reduction, and `quadratic_witnesses` gives their coefficients. For instance, row 6 comes from `["derivative_S3", 22]` with normalization pivot 2: it is the normalized residual of Tx_{22} after earlier rows. The e^2 witness includes the term

$$\frac{86485}{674622} b_6^{\text{sq}}.$$

This is one term of the recorded sum, not a stand-alone square identity. The file supplies all other terms in descending θ -coefficient order; the independent audit replays all 222 rows and reconstructs the full sum. **[CALC C07, p. 131]**

Why this span vanishes in the reduced quotient. Derivatives have zero C_2 class. The class of a normal product with a weight-one generator has that generator's nilpotent class as a factor. Each invariant normal-product sum consists of products of vectors in nontrivial irreducible $\mathfrak{g}$ -modules, whose classes are in the Poisson nilradical by Section 5. Thus every accepted row vanishes in the reduced quotient. The two square identities prove that the axis classes vanish there too. Strong generation, proved independently, is the additional input that turns these finite identities into C_2 -cofiniteness.

I The lattice lowest-space matrices

Lowest-weight lattice vectors and matrix construction

Let Q be the simple-root Gram matrix of Appendix B. Choose $\lambda_A = Q^{-1}e_0$ and $\lambda_E = Q^{-1}e_2$, where e_i is the coordinate unit vector (indices start at zero). Their triples are the integer vectors displayed

in Section 10. For coset label (e, a) start with $v_0 = 3(e\lambda_E + a\lambda_A)$ and repeatedly apply

$$v \mapsto v - (Qv)_i e_i, \quad 0 \leq i \leq 7.$$

This enumerates the finite Weyl orbit. The chosen fundamental weights are minuscule, so these orbits are exactly the minimum shells, of sizes 3, 27, and 81 for the nonzero coset labels. Sort the vectors $v = 3p$ lexicographically. Then $p = \lambda + \beta$, with $\beta = (v - v_0)/3 \in L$, fixes the module cochain through $\varepsilon(\alpha, \beta)$; it is not obtained by feeding a fractional lattice vector into the lattice cocycle.

For an actual entry take coset label $(1, 0)$ and $p = \lambda_E - s$, with $s = \alpha_2$. Since $(s, \lambda_E) = 1$, both p and $p + s$ lie in the minimum shell. Here $\beta = -s$ and $\varepsilon(s, -s) = -1$, so $o(E_s)e^{\lambda_E - s} = -e^{\lambda_E}$. This is a lowest-space entry of the matrix of an affine generator. Expanding each of the sixteen highest-weight vectors by the same recipe gives the other matrices.

The unknown commutant entries join equal exact Cartan-weight pairs. The constraints $Mo(g) - o(g)M = 0$ are linear in those unknowns. Selected minors certify their rank after reduction at the good prime, and all field coefficients are regular there. The tables below give the rank lower bound; the identity supplies the exact one-dimensional kernel.

The following formulas determine every entry of the 22 zero-mode matrices used in Proposition 10.1. Write a minimum-norm lattice vector in a lattice coset as $p = \lambda + \beta$, with $\beta \in L$. With the lattice-module cocycle convention, the lowest-space contributions are

$$\begin{aligned} o(e^\alpha)e^p &= \varepsilon(\alpha, \beta)e^{p+\alpha}, \\ o(h_i(-1)e^\alpha)e^p &= (h_i, p)\varepsilon(\alpha, \beta)e^{p+\alpha}, \\ o(h_i(-2)\mathbf{1})e^p &= -(h_i, p)e^p, \\ o(h_i(-1)h_j(-1)\mathbf{1})e^p &= (h_i, p)(h_j, p)e^p. \end{aligned}$$

The first line is used when the destination is another minimum-norm lattice vector in the same coset; the second is for a root α . Other contributions vanish. For $B_1 \cong W_1/H$, the full lattice mode engine is used at output weight one, and the invariance of H is checked before quotienting. The two affine generator triples and their Cartan weights are also checked. **[CALC C08, p. 131]**

At $p = 10009$, $\theta \mapsto 2914$, the commutant systems have **[CALC C08, p. 131]**

Top	Cartan-compatible variables	Equation rank	Nullity
B_1	282	281	1
Each 3-dimensional top	3	2	1
Each 27-dimensional top	41	40	1
Each 81-dimensional top	331	330	1

The exact Cartan weights have no modular aliasing. All selected coordinate denominators are good at the prime. Thus the minors prove characteristic-zero rank lower bounds, not just modular nullities. The independent verifier reconstructs all 21 selected rank minors. Example determinants for B, P, Q, X_+, X_- are 5307, 10008, 7033, 2058, 9805 modulo 10009. The cross-Hom of X_+ and X_- has a 331×331 nonzero minor, with determinant 3629. **[CALC C08, p. 131]**

I.1 The exact opposite-coset maps

For opposite-coset identifications, positive modular Hom dimensions would be insufficient. The four maps F_γ are solved over K , with rows in the target top and columns in the source top:

Coset labels $\gamma \rightarrow -\gamma$	Top dimension	Lie algebra Hom dimension	Nonzero entries
$(0, 1) \rightarrow (0, -1)$	3	1	3
$(1, 0) \rightarrow (-1, 0)$	27	4	41
$(1, 1) \rightarrow (-1, -1)$	81	14	331
$(1, -1) \rightarrow (-1, 1)$	81	14	331

The Lie algebra Hom spaces reduce to dimension one after imposing the sixteen weight-two equations. **[CALC C09, p. 131]** The first map is I_3 .

For instance, the saved entry `[0,0,["1"]]` in the first pair of `shared/modules/exact_pair_audit.json` means row 0 of coset label $(0, -1)$, column 0 of coset label $(0, 1)$, with coefficient 1. Rows and columns are zero based in these arrays. For an affine generator g , its residual is $(I_3 o_{(0,1)}(g) - o_{(0,-1)}(g) I_3)_{rc} = 0$; the saved map and the independently reconstructed matrices check the same identity for every entry and all 22 classes. For every map, all 22 residuals

$$F_\gamma o_\gamma(g) - o_{-\gamma}(g) F_\gamma$$

vanish over K . Their determinants modulo 10009 are, in the same order, 1, 1513, 803, 5805, all nonzero. **[CALC C09, p. 131]** These are maps between two ordinary top actions; the pairing used to identify the contragredient is a separate object.

The complete entry lists are in the actual-module package's `shared/modules/exact_pair_audit.json`. Its bases indexed by lattice vectors are the lexicographically ordered Weyl orbits returned by `shared/modules/top_audit.py`, with lattice vectors stored as $3p$. An entry `[r, c, [a_0, \dots, a_m]]` means $(F_\gamma)_{rc} = a_0 \theta^m + \dots + a_m$. The script `shared/modules/exact_pair_audit.py` reconstructs the zero-mode matrices from the formulas above and solves the four systems over K . The bundle's `shared/modules/check_top_identifications.py` reads the saved maps, checks every exact residual against freshly reconstructed matrices, and tests invertibility at the good prime. These finite checks establish the top identifications; Zhu's simple-module correspondence is the theorem that identifies their generated simple modules.

I.2 Contragredients and the invariant lattice pairing

For an ordinary module $M = \bigoplus_{n \geq 0} M_{h+n}$, the restricted contragredient is the graded dual $M' = \bigoplus_{n \geq 0} M_{h+n}^*$ with action

$$\langle Y_{M'}(a, z) \ell, m \rangle = \langle \ell, Y_M(e^{zL_1}(-z^{-2})^{L_0} a, z^{-1}) m \rangle. \quad (\text{I.1})$$

This is the standard contragredient definition [20, Section 2.2, equation (2.1)]. In particular, L_0 acts by the transpose of L_0 on M , so the lowest weight stays h . If M is simple, so is M' : a proper nonzero submodule of M' would have a proper nonzero annihilator in M , invariant by (I.1). The finite dimensions of the grades ensure both assertions about the annihilator.

The standard lattice construction supplies a bilinear pairing

$$B_\gamma : W_\gamma \times W_{-\gamma} \longrightarrow \mathbb{C}$$

[12]. In the module cocycle convention used above, normalize $B_\gamma(e^p, e^{-p}) = 1$, set it to zero on other pairs of lowest-weight exponentials, and extend to Heisenberg-mode factors by

$$B_\gamma(h(-n)x, y) = -B_\gamma(x, h(n)y), \quad n > 0.$$

For example, in the lowest-weight bases the pairing entry joining e^{λ_E} to $e^{-\lambda_E}$ is 1, and it is zero for any other target lattice vector. This is a bilinear duality pairing, not the positive Hermitian form. For a weight-one generator g with $L_1 g = 0$, invariance gives $o_{M'}(g) = -o_M(g)^\top$ in dual bases; for the conformal element it gives the positive transpose of L_0 . These signs come from the factor $(-z^{-2})^{L_0}$ in (I.1).

The Heisenberg relations make this nondegenerate on each paired Heisenberg module component. For the exponential vertex operators, the adjoint sign is supplied by

$$\varepsilon(\alpha, -\beta - \alpha) = (-1)^{(\alpha, \alpha)/2} \varepsilon(\alpha, \beta).$$

Together with the adjoints of the Heisenberg modes, this gives invariance directly from the two exponential factors in (B.6). Invariance takes the form

$$B_\gamma(x, Y(a, z)y) = B_\gamma(Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})x, y).$$

Restrict this identity to the cyclic U -modules $M_\gamma, M_{-\gamma}$. The pairing is still nonzero on the tops, and its radicals are submodules, so simplicity makes both radicals zero. It pairs equal conformal grades, each finite-dimensional; hence $y \mapsto B_\gamma(-, y)$ is an isomorphism $M_{-\gamma} \rightarrow M'_\gamma$. This proves the second isomorphism in $M_\gamma \cong M_{-\gamma} \cong M'_\gamma$ independently of the top-intertwiner computation. **[CALC C09, p. 131]**

J Affine source identities and the full idempotent

J.1 The cubic seed witness

The Heisenberg degree in $(e^a)_{(n)}e^b$ is $-n - 1 - (a, b)$. Since $(\beta_i, \beta_i) = 4$ and $(\beta_i, \beta_j) = 2$ for $i \neq j$, the two seed products have only the following nonzero contributions:

Step	Surviving lattice vectors	Heisenberg degree
$v'_{(-3)}v'$	$\beta_j + \beta_k, j \neq k$	$2 - 2 = 0$
$v'_{(-5)}(v'_{(-3)}v')$	$\beta_1 + \beta_2 + \beta_3 = w + s$	$4 - 4 = 0$
$(F_s)_{(2)}e^{w+s}$	w	$-3 - (-3) = 0$

Repeated lattice vectors give negative Heisenberg degree and hence zero. For every permutation (i, j, k) of $(1, 2, 3)$, the fixed cocycle gives

$$-\varepsilon(-s, w + s)\varepsilon(\beta_i, \beta_j + \beta_k)\varepsilon(\beta_j, \beta_k) = 1.$$

The minus sign is the one in $F_s = -e^{-s}$. Thus all six terms add, and their common coefficient is $\gamma_2\gamma_3$. Reducing this product by the field polynomial gives the complete vector identity

$$(F_s)_{(2)}(v'_{(-5)}(v'_{(-3)}v')) = 6\gamma_2\gamma_3e^w = \left(\frac{39\theta^3}{160} - \frac{243\theta}{40}\right)e^w. \quad (\text{J.1})$$

Both seed coefficients are nonzero, so there is no cancellation to check beyond these six cocycle signs. **[CALC C10, p. 131]**

J.2 Affine orbits and source idempotents

The twenty local affine orbits are represented by $(0, j)$, $j = 0, 2, \dots, 38$. Their paired affine weight difference is $10 - j/2$. Only $j = 20$ ties, giving the exceptional top $(0, 20) \oplus (1, 19)$ of dimension 61. Consequently the source Lie algebra module types on the whole Zhu algebra are

$$(0, 0), (0, 2), \dots, (0, 20), \quad (1, 1), (1, 3), \dots, (1, 19).$$

The exceptional two types have equal multiplicities in any source representation. In this paragraph, E_i, F_i, H_i denote their Zhu classes, for $i \in \{s, t\}$. With $C_i = H_i^2 + 2H_i + 4F_iE_i$ and $p_0 = 1 - C_s/3$, $p_1 = C_s/3$, the source identities in $A(U)$ are

$$\begin{aligned} C_s(C_s - 3) &= 0, & p_0 \prod_{a=0}^{10} (C_t - (2a)(2a+2)) &= 0, \\ p_1 \prod_{a=0}^9 (C_t - (2a+1)(2a+3)) &= 0, & E_t^{21} = F_t^{21} &= 0. \end{aligned} \tag{J.2}$$

The affine Sugawara class $\Omega = [\omega_{\mathcal{A}}]$ satisfies

$$\prod_{n \in \mathcal{T}} (41\Omega - n) = 0, \quad \mathcal{T} = \{0, 2, 6, 11, 12, 14, 19, 20, 26, 30, 35, 42, 46, 56, 59, 72, 74, 90, 91, 110\}.$$

This class need not be central in $A(U)$; the total conformal class is. **[CALC C10, p. 131]**

For $\mathcal{J}_0 = \{0, 2, \dots, 20\}$ and $\mathcal{J}_1 = \{1, 3, \dots, 19\}$, set

$$q_{r,j}(x) = \prod_{k \in \mathcal{J}_r \setminus \{j\}} \frac{x - k(k+2)}{j(j+2) - k(k+2)}, \quad \ell_j(x) = \prod_{m=-j, -j+2, \dots, j-2} \frac{x - m}{j - m}.$$

The Casimir C_s has eigenvalues 0, 3, so p_r selects the type for the first $\mathfrak{sl}_2$ factor. The Lagrange polynomial $q_{r,j}$ is one at $j(j+2)$ and zero at every other allowed Casimir of the second Lie algebra factor. Within the selected type, ℓ_j is one at the highest Cartan weight j and zero at its other weights. For example, $p_0 q_{0,2}(C_t) \ell_2(H_t)$ selects exactly the highest line in type $(0, 2)$. These projectors act blockwise; cross-constituent matrix units in the exceptional full block come from its full $A(\mathcal{A}^+)$ action, not from the zero-mode raising and lowering operators alone.

Then $e_{r,j} = p_r q_{r,j}(C_t) \ell_r(H_s) \ell_j(H_t)$ selects the highest line of that type. The sum over all permitted types except $(1, 19)$ is e_Z : in the exceptional source block it uses $(0, 20)$. This is one rank-one idempotent per source matrix block and hence is full. These formulas define it independently of whether a source block survives in $A(U)$. **[CALC C10, p. 131]**

The known tops have finite $\mathfrak{g}$ -module decompositions

$$\begin{aligned} U_0 &: (0, 0), \\ B_1 &: (1, 9) + (1, 5) + (1, 3) + (0, 10) + (0, 8) + (0, 6) + 2(0, 4) + (0, 2), \\ P_{1/3} &: (0, 2), \\ Q_{2/3} &: (1, 5) + (0, 8) + (0, 4) + (0, 0), \\ (X_{\pm})_1 &: (1, 7) + (1, 5) + (1, 3) + (0, 10) + (0, 8) + 2(0, 6) + (0, 4) + 2(0, 2). \end{aligned}$$

These decompositions contain respectively 1, 9, 1, 4, 10, 10 highest lines, giving the corner dimensions in (10.2). **[CALC C08, p. 131] [CALC C10, p. 131]**

For example, the lowest space $Q_{2/3}$ has four selected highest lines, one in each of types $(1, 5), (0, 8), (0, 4), (0, 0)$. Its corner dimension is therefore 4, whereas its full lowest-space dimension is $12 + 9 + 5 + 1 = 27$. In general, if type v has size s_v and multiplicity m_v , the two counts are $\sum_v m_v$ and $\sum_v s_v m_v$. This is the conversion used in Sections 10 and 11.

K Zhu coordinates, reduction sizes, and source profiles

K.1 Zhu conversion

Zhu conversion retains the lower-mode correction. For weight-two a, b , the star-product and commutator identities give directly

$$\left[\frac{a_{(-1)}b + b_{(-1)}a}{2} \right] = \frac{[a] * [b] + [b] * [a]}{2} + \left[\frac{a_{(1)}b + b_{(1)}a}{2} \right], \quad (\text{K.1})$$

$$\left[\frac{a_{(0)}b - b_{(0)}a}{2} \right] = [a] * [b] - [b] * [a] - \left[\frac{a_{(1)}b - b_{(1)}a}{2} \right]. \quad (\text{K.2})$$

For example, adding the commutator formula in both orders gives $[a_{(0)}b] + [b_{(0)}a] = -[a_{(1)}b] - [b_{(1)}a]$; substitution into the two star products proves (K.1). This avoids discarding higher-translation terms in a skew-symmetry expansion. For an affine generator, $[h_{(-1)}b] = [b] * [h]$.

K.1.1 A saved relation through all five steps

All array indices in this example are zero based. In `shared/rank12/certificates/bracket_exact.json`, group 0, relation 0 has `relation=[[1,["3/2"]],[18,["1"]]]` and an empty alternating mode-one correction. Descriptor 1 is Tz_1 . Descriptor 18 is

$$(E_s)_{(-1)}(F_s)_{(0)}^2 z_0 - ((F_s)_{(0)}E_s)_{(-1)}(F_s)_{(0)}z_0 + ((F_s)_{(0)}^2 E_s)_{(-1)}z_0.$$

Since $z_0 = TE_s$, $(F_s)_{(0)}E_s = -H_s$ and $(F_s)_{(0)}^2 E_s = -2F_s$, the exact vector relation is

$$\frac{3}{2}Tz_1 - 2(E_s)_{(-1)}TF_s - (H_s)_{(-1)}TH_s - 2(F_s)_{(-1)}TE_s = 0. \quad (\text{K.3})$$

Its complete weight-three lattice-monomial residual is zero; the input does not rely on a selected subset of coordinates.

Translation gives $[Tz_1] = -2[z_1]$ and $[Th] = -[h]$ for an affine generator. Affine generator normal products reverse the star order, $[h_{(-1)}b] = [b] * [h]$. Thus (K.3) becomes

$$P = -3Z_{1;0,0} + J_{H_s}^2 + 2J_{F_s}J_{E_s} + 2J_{E_s}J_{F_s} = C_s - 3Z_{1;0,0}.$$

This is a formal polynomial whose actual Zhu evaluation is zero. On a Lie algebra module constituent $\lambda = (r, j)$ inside vertex v , $\hat{X}(P)$ has component

$$\text{id}_{V(r,j)} \otimes (r(r+2)\varepsilon_v - 3x_{v \rightarrow v}^{\lambda,\lambda}(z_1)).$$

The identity matrix is the normalized highest matrix of type $(0, 0)$, with entry $(1, 1)$ equal to one. Extraction therefore gives precisely the displayed coefficient polynomial. It is one of the generators of $\mathcal{I}$, because it was obtained from a proved VOA kernel identity. The substitution $[z_1] = C_s/3$

removes this redundant coefficient later. This example illustrates all five steps without claiming a new source-vertex exclusion.

The other families can be read in the same convention. Let N_d be the symmetric normal-product expression specified by descriptor d of `quadratic_exact.json`. Its first relation is

$$-16380N_1 + 16380N_{23} + 6720N_{24} - 256N_{38} + 36N_{78} - 280N_{112} + N_{152} = 0.$$

For example, $N_1 = N(z_0, (F_s)_{(0)}^2 z_0) - N((F_s)_{(0)} z_0, (F_s)_{(0)} z_0) + N((F_s)_{(0)}^2 z_0, z_0)$. On replacing each N by its symmetrized star polynomial, the additional *total* correction is $-65520Z_{1;0,0} + 384Z_{2;0,0}$. Both the weight-four vector residual and the weight-two correction residual are checked exactly. The latter term must not be dropped.

Finally, group 0, relation 0 of `mixed_exact.json` is $3D_0 + 2D_{193} + D_{196} = 0$, with no mode-one correction. Here $D_0 = Tu_0$ and $u_0 = Tz_1$; descriptors 193, 196 are the recorded invariant normal products of affine generators with the highest-weight vectors u_{75}, u_{76} in weight three. Their affine generator terms use coefficients 1, -1 , 1 in the unnormalized lowering basis. Conversion sends $3D_0$ to $-9Z_3(u_0)$ and each affine generator product $h_{(-1)}u$ to $Z_3(u)J_h$. The chosen section expresses every $Z_3(u)$ in the original symbols, so no new associative generator appears.

K.2 The lattice-vector relation families

The computation uses three $\mathfrak{g}$ -equivariant families:

Map or relation space	Kernel dimension	Highest families
Symmetrized normal products of S_2	2,053	66
Translations, affine generator products, and alternating zeroth products	7,024	408
Mixed weight-four relations	7,945	446

All 920 highest families are identities of complete lattice vectors over K . Their zero-mode lowering vectors span the full $\mathfrak{g}$ -equivariant families. The first map has domain dimension 7,260 and rank 5,207. The second has domain dimension 7,980 and rank 956.

The mixed map has domain dimension 13,952 and rank 6,007. To obtain these kernels, enumerate the highest input descriptors in each Lie algebra module type, including every multiplicity; expand each descriptor as a complete lattice vector; choose independent image columns; and solve for the dependent columns over K . Modular discovery only proposes the image columns. The exact solve verifies both the coefficient relations and their residuals in lattice coordinates. Lowering the highest kernel vectors recovers their entire irreducible $\mathfrak{g}$ -modules. Mode-one corrections are separately expanded and solved in the exact 120-vector S_2 basis, with full lattice-monomial residual zero. This last check certifies the lower-weight part used in Zhu conversion.

Each asserted relation is checked separately with exact zero residual. [CALC C11, p. 132]

K.3 $\mathfrak{g}$ -equivariance and conformal coordinates

The sixteen highest-weight vectors have types

$$(2, 0), 4(0, 0), (0, 2), (2, 2), 2(0, 4), (1, 13), (1, 7), (1, 5), (1, 1), (0, 12), 2(0, 8).$$

Lowering vectors are obtained using unnormalized powers of the lowering operators. For a highest tensor of weights m, n with component index k , use $c_0 = 1$ and

$$\frac{c_{i+1}}{c_i} = -\frac{(k-i)(n-k+i+1)}{(i+1)(m-i)}.$$

Take the product of these coefficients in the two Lie algebra factors. In the symmetric square, the sum of the component indices for the two Lie algebra factors is even; in the exterior square, it is odd. For a highest rectangular matrix of weight ℓ , between source weights n and m , put $\delta = (\ell - m + n)/2$ and normalize the first entry to one. Its one-based nonzero positions are $(i+1, i+\delta+1)$ with

$$\frac{r_{i+1}}{r_i} = \frac{(i+\delta+1)(n-i-\delta)}{(i+1)(m-i)}.$$

It exists for $|m-n| \leq \ell \leq m+n$ with even $m+n-\ell$. The matrix lowering action is $F_m R - R F_n$; take Kronecker products for two factors. Their coefficients define the corner arrows.

The complete lattice-vector conformal coordinates are

$$\omega_s = z_1/4, \quad \omega_t = 2z_2/41, \quad \omega_U = z_1/4 + z_2/3 - 5z_3/3.$$

Their source Zhu coordinates are

$$\begin{aligned} [z_0] &= -[E_s], & [z_1] &= C_s/3, & [z_2] &= C_t/8, & [z_5] &= -[E_t], \\ [z_6] &= [E_s] * [E_t], & [z_7] &= [E_t] * [E_t], \\ [z_3] &= C_s/20 + C_t/40 - 3[\omega_U]/5. \end{aligned}$$

The free generator labels are 4, 8, 9, 10, 11, 12, 13, 14, 15.

The first display concerns equalities of lattice vectors; the second concerns their Zhu classes, with C_s, C_t the Lie algebra Casimir classes. On each source constituent these Casimirs act by $r(r+2)$ and $j(j+2)$. Thus seven of the original sixteen highest-weight vectors are substituted by known polynomials in the affine generators and the total conformal class. The latter is retained through its vertex loops H_v . Normalized multiplication and its remaining highest-weight tensor-product components are given in Appendix L.

Every projected relation uses the exact mode-one corrections in (K.1)–(K.2); translations contribute -2 or -3 according to the original vector weight. **[CALC C11, p. 132]**

K.4 Exclusions and the quadratic presentation

Number the twenty source vertices by $j = 2v$, $0 \leq v \leq 19$. Explicit exact combinations of the projected relations kill the vertices in four successive pairs:

$$(10, 11), \quad (9, 12), \quad (8, 13), \quad (7, 14).$$

After these exclusions, the surviving source vertices are $0, \dots, 6, 15, \dots, 19$, with 356 arrows. They correspond to Lie algebra module types $(0, 0), (0, 2), \dots, (0, 12)$ and $(1, 9), (1, 7), \dots, (1, 1)$. A source vertex is not a U -simple: a single simple top can occupy several vertices.

Here is a decoding of the first unit witness. The vertex-10 record in `shared/rank12/certificates/global_vertex_exact.json` has 83 terms $[o, c]$ and verifies $\sum_o c p_o = \varepsilon_{10}$ in the $(10, 10)$ corner. One genuine term has origin `["mixed", 0, 15, 20, 20]` and coefficient

6982268347739/2287521789574258944000. The origin means mixed family, highest-weight group 0, relation 15, target constituent 20, source constituent 20. In the initial constituent order, index 20 is (1, 19), belonging to vertex 10. The verifier reconstructs that kernel relation and its correction, applies the normalized matrices, and checks all path endpoints before adding the coefficient. All 83 terms are included in the identity, not merely the one printed here. The empty constant is stored with its (10, 10) type; it cannot be substituted at another vertex. The subsequent three files record the next pairs with the preceding exclusions as verified premises.

As a quadratic-substitution example, `exact_upper_rules.json` records $x_{80} = (-11\theta^2 + 308)x_{120}$ in the (6, 17) corner. Consequently a composable term $x_b x_{80}$ becomes $(-11\theta^2 + 308)x_b x_{120}$ with the same endpoints. The replacement is made in every row, and the resulting row echelon system is solved again. This is the saturation procedure whose stage ranks follow.

The successive exact identity witnesses at the excluded vertex pairs use (83, 111), (80, 98), (80, 98), and (80, 95) relation terms. Each is checked after the preceding exclusions. [CALC C12, p. 132] Quadratic saturation then processes 11,293 rows: 10,949 nonzero family projections and 344 centrality equations. Its stages are

Stage	Exact row rank	New eliminated arrows
0	9,859	43
1	7,950	29
2	6,787	1
3	6,745	0

No source idempotent is killed at these stages. [CALC C13, p. 132] The exact portable rules and all 73 linear eliminations are in `shared/rank12/new_results/exact_upper_rules.json` of the rank package. The coefficient polynomial at 2914 has derivative $7285 \neq 0$ modulo 10009, giving the unramified local specialization. [CALC C17, p. 133] Local integrality tests use a Hensel lift where a canonical rational denominator contains 10009; this tests valuation rather than mistakenly excluding an integral field element. [CALC C14, p. 132]

K.5 The integral word bound and the special fiber

K.5.1 Coefficient regularity

The monic quadratic presentation has 283 arrows and 6,745 rules. All 151,115 right-side coefficients are integral at the chosen prime. [CALC C14, p. 132]

The input is the portable exact quadratic rule array, not its modular discovery table. A descending coefficient list represents a polynomial in θ . Appendix R.1 describes the Hensel residue test when numerator and denominator have positive p -valuation. Each residue is compared with the modular rule before any modular minor is used.

K.5.2 Cubic shortening over the DVR

After quadratic reduction, the numbers of normal paths are

Length	0	1	2
Number of paths	12	283	786

There are 1,364 normal cubic paths. For an overlap with both adjacent quadratic pairs reducible, associativity equates the two reduction orders. The resulting relations contain 53 selected square minors with unit determinants in R ; together they cover all 1,364 normal cubic columns. Solving expresses every cubic path as a combination of paths of length at most two. Thus $12 + 283 + 786 = 1,081$ paths span S_R as an R -module. **[CALC C14, p. 132]**

More explicitly, for a selected overlap abc use the exact rules $ab = \sum_u A_u u$, $bc = \sum_v B_v v$ and form

$$\Delta_{abc} = \text{NF} \left(\sum_u A_u u c - \sum_v B_v a v \right).$$

Normal form here reduces quadratic subwords only, retaining all coefficients of both cubic and shorter words. Associativity makes $\Delta_{abc} = 0$ in the presented algebra. For each endpoint pair, collect the cubic coefficients into M and the shorter terms into b . A unit minor of M solves $M(\text{cubics}) = -b$ over R . Replacing a cubic subword in a longer path reduces its length; induction yields the finite spanning assertion.

The first block in `shared/rank12/new_results/finite_bound_certificate.json` has endpoints $[0,0]$, twelve cubic columns, and twelve selected overlaps, starting with $[0,0,0]$. For this overlap the two reductions are $\text{NF}((\sum_u A_u u)x_0)$ and $\text{NF}(x_0(\sum_u A_u u))$, where $\sum_u A_u u$ is the complete saved rule for $[0,0]$. Thus the row coefficient on a normal cubic w is $\sum_u A_u (\text{NF}(ux_0)_w - \text{NF}(x_0u)_w)$; the shorter coefficients are kept by the same formula. This describes the row with its coefficients, not only its word labels. The selected matrix determinant is $8842 \bmod 10009$. The recorded matrix hash pins its entries; replay reconstructs them from the rules. The 53 blocks together cover all 1,364 cubic columns.

K.5.3 Confluence and the special-fiber basis

In the special fiber, completion leaves $12+282+252 = 546$ normal paths. The order is degree followed by the fixed lexicographic path order, so reduction terminates. All 184,016 quadratic overlaps and 1,653 higher overlaps resolve with zero remainder. The Diamond Lemma [32, Theorem 1.2], with the endpoint units fixed, says that a compatible terminating system with all overlap and inclusion ambiguities resolved has its irreducible paths as a basis. This gives dimension 546, not merely a spanning upper bound.

K.5.4 Trace test, central projectors, and splitting

The regular trace pairing has rank 546, hence the special fiber is semisimple. Its center has dimension twelve; a separating central element gives twelve orthogonal projectors summing to one. Their checked regular and source ranks identify the split matrix blocks and the profiles used below. **[CALC C15, p. 132]**

If Z_{sep} is the separating central element with distinct eigenvalues $\lambda_i \in \mathbb{F}_p$, its projectors can be obtained by

$$P_i = \prod_{j \neq i} \frac{Z_{\text{sep}} - \lambda_j}{\lambda_i - \lambda_j}.$$

This central element distinguishes all twelve blocks; the conformal element alone has only ten eigenvalues. The projectors are checked as orthogonal central idempotents summing to one. Their regular-block ranks are d_i^2 , and the checked split block actions give $M_{d_i}(\mathbb{F}_p)$. A source idempotent has rank $d_i m_v$ on that regular block, giving m_v on a simple module.

For example, the first block in `shared/rank12/new_results/modular_blocks.json` has $d_i = 1$ and separating eigenvalue 3063. Its projector is $\varepsilon_0 + 5204x_0 + 4255x_{12} + 432x_{304}x_{319}$ over $\mathbb{F}_p$. The record `[0, [], 1]` is the typed unit at vertex 0; the other records supply their vertex, word, and residue coefficient. This is modular data, not a displayed exact characteristic-zero projector. The descent lemma transports the possible sizes and profiles, allowing torsion loss.

K.6 Conformal trace and surviving block profiles

The coefficients are

v	α_v	v	α_v
0	0	6	86/13
1	1/3	15	4/13
2	5/13	16	35/39
3	6/13	17	0
4	89/39	18	9/13
5	54/13	19	41/39

To construct the witness, expand the total conformal loops and the source units in the exact quadratic normal coordinates. Expand composable arrow commutators in the same coordinates, solve the resulting linear system over K , and retain a sparse solution. Discovery chooses candidates, but the conclusion is checked by reducing the *raw* difference with the exact rules. For example, the record with `arrows=[98,135]` and `coefficient_descending=["131/165110400"]` in `shared/rank12/new_results/trace_conformal_witness.json` contributes

$$\frac{131}{165110400}(x_{98}x_{135} - x_{135}x_{98}).$$

The two products occupy their own diagonal corners; the global identity is a sum in the full algebra. All 134 commutators are included before the exact zero residual is checked. Taking a simple-module trace kills each commutator and gives $hd = \sum_v \alpha_v m_v$.

The total conformal class satisfies (11.1). Its exact witness is a sum of 134 arrow commutators, with zero residual over K . Taking traces with the following profiles gives the possible weights. **[CALC C16, p. 133]**

The surviving source sizes are

v	Type	s_v	v	Type	s_v
0	(0,0)	1	6	(0,12)	13
1	(0,2)	3	15	(1,9)	20
2	(0,4)	5	16	(1,7)	16
3	(0,6)	7	17	(1,5)	12
4	(0,8)	9	18	(1,3)	8
5	(0,10)	11	19	(1,1)	4

The potential new profiles, conditional on survival, have

h	d	Nonzero ranks m_v
5/13	1	$m_2 = 1$
6/13	1	$m_3 = 1$
7/13	2	$m_2 = m_{18} = 1$
8/13	3	$m_1 = m_3 = m_{19} = 1$
11/13	6	$m_0 = m_2 = m_3 = m_4 = m_{16} = m_{19} = 1$
15/13	14	$m_0 = m_1 = m_6 = m_{15} = m_{16} = m_{17} = m_{18} = m_{19} = 1$, and $m_2 = m_3 = m_4 = 2$

Substitute these into (11.2) to recover the weights, and into $\sum_v s_v m_v$ to recover top dimensions. **[CALC C16, p. 133]**

For reference, the six additional profiles give

h	Full top dimension	Corner dimension	Twist exponent modulo 13
5/13	5	1	5
6/13	7	1	6
7/13	13	2	7
8/13	14	3	8
11/13	42	6	11
15/13	119	14	2

The matrix blocks after exhaustion are

$$\begin{aligned}
A(U) &\cong \mathbb{C} \oplus M_3 \oplus M_5 \oplus M_7 \oplus M_{13} \oplus M_{14} \oplus M_{27} \oplus M_{42} \\
&\quad \oplus M_{80} \oplus M_{81}^{\oplus 2} \oplus M_{119}, \\
D &\cong \mathbb{C}^4 \oplus M_2 \oplus M_3 \oplus M_4 \oplus M_6 \oplus M_9 \oplus M_{10}^{\oplus 2} \oplus M_{14}.
\end{aligned}$$

Their dimensions are 36,625 and 546. **[CALC C16, p. 133]**

The total conformal class consequently has minimal polynomial

$$t(t-1)(3t-1)(3t-2)(13t-5)(13t-6)(13t-7)(13t-8)(13t-11)(13t-15),$$

up to a nonzero scalar. Its degree is ten because three simples have lowest weight one. The top dimensions above are dimensions of Zhu representations, not categorical dimensions.

For completeness, the all-even actual quotient after exhaustion is

$$A(U)/A(U)p_1A(U) \cong \mathbb{C} \oplus M_3 \oplus M_5 \oplus M_7.$$

Its dimension is 84, and its full corner has dimension four. In particular the 5/13 and 6/13 blocks are nonzero in the actual quotient, not merely in the modular presentation.

This all-even quotient is an additional structural check after exhaustion. The proof of survival uses branching and Galois symmetry, not this supplementary quotient dimension.

L Normalized highest-weight matrices and their products

This appendix specifies the numerical matrix calculus used in Subsection 8.2. Throughout, $\mathfrak{g} = A_1 \oplus A_1$, where $A_1 = \mathfrak{sl}_2$. We first give the formulas for one copy of A_1 , then take tensor products.

L.1 The one-factor normalization

In $V(n)$ use the basis $v_i = f^i v_0$, for $0 \leq i \leq n$. The A_1 -action is

$$fv_i = v_{i+1}, \quad ev_i = i(n-i+1)v_{i-1}, \quad hv_i = (n-2i)v_i,$$

with vectors outside the stated range interpreted as zero. On $\text{Hom}_K(V(n), V(m))$, the action is

$$x \cdot C = \rho_m(x)C - C\rho_n(x).$$

For $\ell = |m-n|, |m-n|+2, \dots, m+n$, put

$$\delta = \frac{\ell - m + n}{2}.$$

The normalized highest-weight matrix $C_{n \rightarrow m}^\ell$ has entries

$$C_{n \rightarrow m}^\ell(i+1, i+\delta+1) = r_i, \quad 0 \leq i \leq \min(m, n-\delta),$$

and all other entries zero, where

$$r_i = \frac{\binom{\delta+i}{i} \binom{n-\delta}{i}}{\binom{m}{i}}.$$

In particular, $r_0 = 1$.

The weight equation $h \cdot C = \ell C$ forces the indicated positions. The highest-weight equation $e \cdot C = 0$ gives

$$(i+1)(m-i)r_{i+1} = (i+\delta+1)(n-i-\delta)r_i.$$

Starting with $r_0 = 1$ yields the displayed entry formula. Thus the normalization determines the matrix uniquely.

L.2 The multiplication table

Consider two composable normalized matrices

$$C_{m \rightarrow p}^b, \quad C_{n \rightarrow m}^a,$$

and put

$$r = \frac{b-p+m}{2}, \quad s = \frac{a-m+n}{2}.$$

Their product is

$$C_{m \rightarrow p}^b C_{n \rightarrow m}^a = \frac{\binom{r+s}{r} \binom{n-s}{r}}{\binom{m}{r}} C_{n \rightarrow p}^{a+b}$$

when the output type occurs. Otherwise the product is zero.

Indeed, the A_1 -action on matrices is a derivation, so the product is highest of weight $a+b$. If that type occurs, its highest-weight line is one-dimensional. The normalized output position is row 1, column $r+s+1$. At this position the product has the single contribution

$$C_{m \rightarrow p}^b(1, r+1) C_{n \rightarrow m}^a(r+1, r+s+1).$$

The first factor is 1, and the entry formula for the second gives the displayed structure coefficient. If the output type does not occur, the product must vanish.

For example, the fixed normalization gives

$$C_{4 \rightarrow 4}^2 C_{4 \rightarrow 4}^2 = \frac{3}{2} C_{4 \rightarrow 4}^4.$$

L.3 Two factors and the algebra presentation

For $\lambda \in \Lambda_t$, $\mu \in \Lambda_v$, and $\nu = (\nu_1, \nu_2)$, use

$$C_{t \rightarrow v}^{\lambda, \mu}(\nu) = C_{\lambda_1 \rightarrow \mu_1}^{\nu_1} \otimes C_{\lambda_2 \rightarrow \mu_2}^{\nu_2},$$

with the second factor varying fastest.

In a source of type (n_1, n_2) , a lowering pair (i, j) has one-based column $1 + i(n_2 + 1) + j$. For example, in type $(2, 4)$ the pair $(1, 2)$ is column 8. An output matrix whose normalized offsets are $(1, 2)$ has entry one in row 1, column 8, so its coefficient is read there. This is the convention for path-coefficient extraction. At the exceptional block, numerical products must match the intermediate constituent, although formal paths match only the intermediate block. A numerical zero is therefore not itself a formal path relation.

Tensor-product multiplication shows that the two-factor structure coefficient is the product of the two one-factor coefficients.

All admissible matrices of this form constitute a basis of the numerical highest-weight algebra. Products with nonmatching intermediate block or constituent labels are zero. For matching labels, use the multiplication rule above in each A_1 factor. The identity is

$$1 = \sum_t \sum_{\lambda \in \Lambda_t} C_{t \rightarrow t}^{\lambda, \lambda}((0, 0)).$$

These multiplication rules and the identity relation give a complete generators-and-relations presentation: every word reduces to a linear combination of the basis matrices, and those matrices are linearly independent.

L.4 Lowering and the remaining tensor-product components

The ordinary multiplication table concerns products of highest matrices. To evaluate highest-weight expressions involving lowered inputs, define

$$\mathcal{L}(C) = \rho_m(f)C - C\rho_n(f) \quad (C \in \text{Hom}_K(V(n), V(m))).$$

An unnormalized lowering power is represented by $\mathcal{L}^i(C)$.

Let A, B be composable highest matrices of weights a, b . For $0 \leq k \leq \min(a, b)$, the product associated with the highest-weight component of weight $a + b - 2k$ is

$$\sum_{i=0}^k u_i \mathcal{L}^i(A) \mathcal{L}^{k-i}(B),$$

where

$$u_0 = 1, \quad u_{i+1} = -u_i \frac{(k-i)(b-k+i+1)}{(i+1)(a-i)} \quad (0 \leq i < k).$$

The recurrence is obtained by applying e and canceling consecutive terms. The resulting matrix is highest of weight $a + b - 2k$.

For a worked lowered example set $k = 1$ and $A = B = C_{4 \rightarrow 4}^2$. Then $u_0 = 1$, $u_1 = -1$, and $\mathcal{L}(A) = \text{diag}(-1, -1/2, 0, 1/2, 1)$. The resulting entries in positions $(1, 2), (2, 3), (3, 4), (4, 5)$ are $1/2, 3/4, 3/4, 1/2$, respectively. Hence

$$A\mathcal{L}(A) - \mathcal{L}(A)A = \frac{1}{2}C_{4 \rightarrow 4}^2.$$

The scalar is read at $(1, 2)$, whereas the ordinary product $A^2 = (3/2)C_{4 \rightarrow 4}^4$ was read at $(1, 3)$. This distinguishes the component with $k = 1$ from the product of highest-weight matrices.

If this output type occurs, the expression is a scalar multiple of its normalized highest matrix. Read the scalar at the output entry normalized to 1. If the output type does not occur, the expression is zero. For $A_1 \oplus A_1$, perform this construction independently in the two factors and multiply their scalar coefficients.

The entry formulas, ordinary products, and products of the remaining highest-weight components therefore determine every numerical coefficient needed in the relation calculation by exact arithmetic. These are numerical matrix identities; relations among formal paths are obtained separately by extracting coefficients from the converted VOA identities.

M The associator normalization in detail

We give the detailed proof of Proposition 12.5 in five stages. Let $\mathcal{T}$ be a complex fusion category with the fusion rules (12.8) and untwisted pointed subcategory $\text{Vec}_{\mathbb{Z}_3}$. Write $\alpha_g = \alpha^g$ for $g \in \mathbb{Z}_3$.

M.1 Coherent tensor words

Fix coherent multiplication isomorphisms for the pointed subcategory. Choose isomorphisms $w_g : \alpha_g \rho \rightarrow \rho \alpha_{-g}$. The two ways to move $\alpha_g \alpha_h$ across ρ differ by a scalar 2-cocycle. Since $H^2(\mathbb{Z}_3, \mathbb{C}^\times) = 0$, rescale the w_g so that these two ways agree. Concretely, choose the map for a generator, rescale it by a cube root so that moving three copies past ρ gives the identity, and define the other maps by composition. This is the vanishing of Izumi's $c_{1,2}$ obstruction [31, Section 2.1]; the pointed associator has already made $c_{0,3}$ zero.

The tensor-word category can now be transported to a strict category with objects

$$M = \mathbb{Z}_3 \rtimes_{-1} \mathbb{N}, \quad (g, n)(h, m) = (g + (-1)^n h, n + m),$$

representing $\alpha_g \rho^n$. Its tensor structure maps are obtained by moving every invertible to the left and combining group labels. The only critical overlaps are three group labels, and two group labels crossing one ρ ; these are precisely pointed associativity and the normalized 2-cocycle identity. Consequently the rewriting is coherent and gives strict associativity. In particular $\alpha_g \alpha_h = \alpha_{g+h}$ and $\alpha_g \rho = \rho \alpha_{-g}$ are literal object equalities. Morphism spaces retain the actual spaces in $\mathcal{T}$.

In detail, three group labels reduce consistently because their multiplication maps obey $m_{g+h,k} \circ (m_{g,h} \otimes 1) = m_{g,h+k} \circ (1 \otimes m_{h,k})$. For two labels crossing ρ , write the discrepancy as $c(g, h)$. Its normalized cocycle equation is $c(g, h)c(g + h, k) = c(h, k)c(g, h + k)$, with $c(0, g) = c(g, 0) = 1$. Rescaling w_g by a one-cochain replaces c by its coboundary; the chosen cube-root normalization makes $c(g, h) = 1$. These resolve the two possible overlaps in moving invertibles left. Transport then makes the word product strict, while leaving each actual Hom space and its composition intact. This transports the tensor structure; it does not assert a trivial associator for the full fusion category.

M.2 A faithful algebra of padded morphisms

Here is an explicit algebra that makes the subsequent coefficient calculation legitimate. Let $\Gamma = \mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}$, and denote $u_n = (0, n) = \rho^n$. For $\gamma \in \Gamma$ put

$$\mathcal{B}_\gamma = \varinjlim_n \text{Hom}_{\mathcal{T}}(u_n, \gamma u_n), \quad \mathcal{B} = \bigoplus_{\gamma \in \Gamma} \mathcal{B}_\gamma, \quad (\text{M.1})$$

where n is sufficiently large that both objects lie in M , and the transition map is right tensoring by ρ . These maps are injective for a simple reason: $\mathbf{1}$ is a split summand of ρ^2 . If $f \otimes \text{id}_\rho = 0$, tensor once more by ρ and restrict to that unit summand to recover $f = 0$.

For $x = (g, n)$, right tensor by $\alpha_{-(-1)^n g}$ to make the source of $f : x \rightarrow y$ equal to u_n . This embeds its morphism space into degree yx^{-1} of (M.1); call the embedding ι . It is invariant under further right padding: an invertible pad cancels in this normalization, and a ρ pad gives the transition map. To multiply $\iota(f)$, for $f : x \rightarrow y$, and $\iota(h)$, for $h : u \rightarrow v$, choose a common right multiple $vp = xq$ in M and set

$$\iota(f)\iota(h) = \iota((f \otimes \text{id}_q) \circ (h \otimes \text{id}_p)). \quad (\text{M.2})$$

For example, $v = \rho^2$ and $x = \alpha\rho$ have the common right multiple $v\mathbf{1} = x(\alpha\rho) = \rho^2$. An instructive typed product takes $f = s' : \rho^2 \rightarrow \mathbf{1}$ and $h = \text{id}_\rho \otimes s : \rho \rightarrow \rho^3$. Writing $\lambda_\rho(\iota(s))$ for the image of $\text{id}_\rho \otimes s$, the choice $p = \mathbf{1}$, $q = \rho$ gives

$$\iota(s')\lambda_\rho(\iota(s)) = \iota((s' \otimes \text{id}_\rho) \circ (\text{id}_\rho \otimes s)),$$

an endomorphism of ρ . Thus multiplication can require padding even when the written algebra product has only two factors.

Such multiples always exist: match the powers of ρ , then correct the group label. Any two choices have a further common right multiple; cancellation in $M \subset \Gamma$ makes their two pairs of pads agree there. Thus (M.2) is independent of the choice, and the same argument proves associativity. Identity morphisms all give the nonzero unit 1. Left tensoring respects padding and induces algebra endomorphisms

$$\lambda_x(\iota f) = \iota(\text{id}_x \otimes f), \quad \lambda_x \lambda_y = \lambda_{xy}. \quad (\text{M.3})$$

The tensor-product identity is

$$\iota(f \otimes h) = \iota(f)\lambda_x(\iota(h)), \quad f : x \rightarrow y, \quad h : u \rightarrow v.$$

Indeed, the right side is represented by $(f \otimes \text{id}_v) \circ (\text{id}_x \otimes h)$. Interchange therefore makes $\iota(f)$ an intertwiner from λ_x to λ_y , and makes the resulting functor strict tensor. Injectivity on each original Hom space gives faithfulness.

The roles are separate: the split unit in ρ^2 makes each padding map injective; common right multiples make the product independent of its representatives; further common multiples make triple products associative; and interchange gives tensor compatibility. Together they yield a faithful functor from the actual word category, rather than a claim about arbitrary endomorphism intertwiners of $\mathcal{B}$.

M.3 Split maps and the Leavitt algebra

Choose splitting inclusions $s : \mathbf{1} \rightarrow \rho^2$, $t_g : \alpha_g \rho \rightarrow \rho^2$ and inverse projections s', t'_g . Their images in $\mathcal{B}$ obey

$$s's = 1, \quad s't_g = t'_g s = 0, \quad t'_g t_h = \delta_{g,h}, \quad ss' + \sum_g t_g t'_g = 1. \quad (\text{M.4})$$

They therefore define a unital map from the Leavitt algebra $L(1,4)$ to $\mathcal{B}$. This map is injective, since $L(1,4)$ is simple [33, Theorem 3.11 and Corollary 3.13(iii)]. Thus reduced Leavitt monomials can be compared without introducing additional relations.

The primes denote inverse projections in an algebraic direct-sum splitting; they need not be Hermitian adjoints. The first relations recover a summand after projecting back to it, and the last

reconstructs the identity on ρ^2 . Since the resulting map is unital and the unit of $\mathcal{B}$ is nonzero, its kernel cannot be all of the simple Leavitt algebra. Its kernel is therefore zero. This is the comparison tool needed below; simplicity of all ambient intertwiner spaces is not asserted.

The intertwiner identities and (M.4) also give the endomorphism fusion decomposition

$$\lambda_{\rho^2}(z) = szs' + \sum_g t_g \lambda_{\alpha_g \rho}(z) t'_g, \quad z \in \mathcal{B}.$$

This is equation (4.2) of [27] for our explicitly constructed algebra.

M.4 Typed coefficient definitions and equations

We now perform the coefficient comparison from [27, Section 4, proof of Theorem 1] in the actual categorical morphism spaces. We do not invoke that theorem on $\mathcal{B}$: it requires simplicity of all endomorphism intertwiners in the ambient algebra, which has not been asserted for $\mathcal{B}$. Instead, each expression below is a composite of categorical maps with a specified source and target. Its comparison space is therefore the following finite-dimensional categorical Hom space, embedded injectively in $\mathcal{B}$.

For example, the four actions to be determined have the types

Expression	Categorical Hom space	Dimension
$\lambda_\rho(s)$	$\text{Hom}(\rho, \rho^3)$	4
$\lambda_\rho(s')$	$\text{Hom}(\rho^3, \rho)$	4
$\lambda_\rho(t_g)$	$\text{Hom}(\alpha_{-g}\rho^2, \rho^3)$	11
$\lambda_\rho(t'_g)$	$\text{Hom}(\rho^3, \alpha_{-g}\rho^2)$	11

Indeed, the fusion rules give

$$\rho^3 \cong \mathbf{1} \oplus \alpha \oplus \alpha^2 \oplus 4\rho \oplus 3\alpha\rho \oplus 3\alpha^2\rho.$$

The first pair of Hom spaces therefore has dimension 4. For the second pair, $\alpha_{-g}\rho^2$ contains one invertible and one copy of each noninvertible, giving dimension $1 + 4 + 3 + 3 = 11$. Composing with the split projections reduces these spaces to scalar spaces and to

$$\text{Hom}(\mathbf{1}, \rho^2) = \mathbb{C}s, \quad \text{Hom}(\alpha_g \rho, \rho^2) = \mathbb{C}t_g, \quad \text{Hom}(\rho^2, \alpha_h \rho^2) = \mathbb{C}ss' \delta_{h,0} \oplus \bigoplus_k \mathbb{C}t_{h+k}t'_k. \quad (\text{M.5})$$

Products in (M.5) have the right padding prescribed by (M.2). For example, $s'\lambda_\rho(s)$ is the endomorphism

$$(s' \otimes \text{id}_\rho) \circ (\text{id}_\rho \otimes s) : \rho \rightarrow \rho,$$

and $t'_g \lambda_\rho(s)$ has source ρ and target $\rho^2 \alpha_g$; its comparison vector is $t_g \otimes \text{id}_{\alpha_g}$. Likewise $t'_h \lambda_\rho(t_0)$ lies in the third space of (M.5).

To remove any ambiguity in the raw coefficients, define them by the following cut-downs, using the padded product (M.2) throughout. Before the pointed character ψ is shown trivial, the unprimed

expansions are

$$\lambda_\rho(s) = as + b \sum_g \psi(g) t_g t_g, \quad (\text{M.6})$$

$$\lambda_\rho(s') = a' s' + b' \sum_g \psi(-g) t'_g t'_g, \quad (\text{M.7})$$

$$\lambda_\rho(t_g) = \psi(-g) c s t'_{-g} + d_0 t_{-g} s s' + \sum_{h,k} M_{h+g,k+g} t_h t_{h+k+g} t'_k, \quad (\text{M.8})$$

$$\lambda_\rho(t'_g) = d'_0 s s' t'_{-g} + \psi(g) c' t_{-g} s' + \sum_{h,k} M'_{k+g,h+g} t_k t'_{g+h+k} t'_h. \quad (\text{M.9})$$

The group indices are in $\mathbb{Z}_3$. Each action has the source and target in the preceding four-row Hom table. The coefficients are uniquely specified by projection onto the splitting basis, equivalently by

$$\begin{aligned} s' \lambda_\rho(s) &= a, & \lambda_\rho(s') s &= a', \\ t'_0 \lambda_\rho(s) &= b t_0, & \lambda_\rho(s') t_0 &= b' t'_0, \\ s' \lambda_\rho(t_0) &= c t'_0, & s' \lambda_\rho(t'_0) &= d'_0 s' t'_0, \\ t'_h \lambda_\rho(t_g) &= d_0 \delta_{h,-g} s s' + \sum_k M_{h+g,k+g} t_{h+g+k} t'_k, \\ t'_h \lambda_\rho(t'_g) &= \psi(g) c' \delta_{g,-h} s' + \sum_k M'_{k+g,h+g} t'_{g+h+k} t'_k. \end{aligned}$$

For the penultimate line, right transport of the source to ρ^2 places the comparison in $\text{Hom}(\rho^2, \alpha_{h+g} \rho^2)$, with the basis (M.5). In particular, at $g = h = 0$ its ss' coefficient is d_0 and its $t_k t'_k$ coefficient is $M_{0,k}$. The last line likewise defines c' and the primed matrix entries. These conventions are the cut-down identities [27, Equations (4.16)–(4.18)], now interpreted in our actual categorical Hom spaces.

Here is one full scalar comparison. The intertwiner identity gives $s' \lambda_{\rho^2}(s) = ss'$ in the transported space $\text{Hom}(\rho^2, \rho^2)$. Substitute (M.9) and the cut-downs:

$$s' \lambda_{\rho^2}(s) = a^2 + bc \left(3d_0 ss' + \sum_g M_{0,g} \sum_k t_k t'_k \right).$$

Replace $\sum_k t_k t'_k$ by $1 - ss'$. The constant coefficient and the ss' coefficient give

$$bc \sum_g M_{0,g} = -a^2, \quad a^2 + 3bcd_0 = 1.$$

The st'_0 coefficient of $t'_0 \lambda_{\rho^2}(s) = \lambda_\rho(s) t'_0$ gives $a = bcd_0$, hence $a^2 + 3a = 1$. The primed comparison gives $(a')^2 + 3a' = 1$. In particular both a and a' are nonzero, as are b, c, d_0 and their primed partners. These are the comparisons [27, Equations (4.19)–(4.20) and preceding coefficient comparison].

Preservation of $s's = 1$ gives another transparent equation:

$$\lambda_\rho(s') \lambda_\rho(s) = aa' + 3bb' = 1,$$

since each $t'_g t_h$ reduces to $\delta_{g,h}$. Preservation of the complete splitting identity gives $aa' + 3cc' = 1$ and $d_0 d'_0 = 1$ from its constant and diagonal coefficients.

For the normalization, replace $s \mapsto \lambda s$ and $s' \mapsto \lambda^{-1} s'$, leaving t_g, t'_g fixed. All splitting identities remain valid, and the coefficients change by

$$b \mapsto \lambda b, \quad c \mapsto \lambda^{-1} c, \quad b' \mapsto \lambda^{-1} b', \quad c' \mapsto \lambda c'.$$

Choose $\lambda^2 = c/b$, possible because $b, c \neq 0$. Then $c = b$, and $bb' = cc'$ forces $c' = b'$. The equations $a = bcd_0$, $a' = b'c'd'_0$ now give $aa' = (bb')^2$. Put $x = bb'$. Then $x^2 + 3x = 1$. If a, a' were the two different roots, $aa' = -1$, so $x^2 = -1$ and $-1 + 3x = 1$, giving $x = 2/3$, a contradiction. Thus $a = a'$, and $x = a$ follows by comparing $x^2 + 3x = 1$ with $a^2 + 3a = 1$.

The nonzero scalar comparison $s'\lambda_\rho(s) = a$ then forces the pointed character to be trivial, as in [27, Equation (4.6) and the discussion following (4.19)]. The remaining linear and quadratic coefficient comparisons give $b' = bd_0$, $M'_{g,h} = M_{h,g}$, and $d_0^3 = 1$. Their exact reduction is [27, Equations (4.20)–(4.27)]. Set $\delta = a^{-1}$ and $\omega = d_0$. Then $b^2 = a/\omega = 1/(\omega\delta)$, and the actions are exactly the normalized ones in Appendix N.

The matrix constraints come from the same typed intertwiner identities: the row sum from the $t'_0\lambda_{\rho^2}(s)$ comparison; the quadratic identity from the complete splitting relation; cyclic symmetry from the $st'_{h-g}t'_h$ coefficients of $t'_0\lambda_{\rho^2}(t_g) = \lambda_\rho(t_g)t'_0$; and the quartic identity from the $t_h t_{i-g+h} t'_{k+i} t'_k$ coefficient of the same identity with t_{-g} . The full equations and their index conventions are in Appendix N; the imported final reductions are [27, Equations (4.7)–(4.10) and the final paragraph of Section 4]. They are mathematical coefficient deductions. C20 checks the finite solution system and C21 checks the normalized actions; neither replaces this bridge. [CALC C20, p. 133] [CALC C21, p. 133] The displayed actions show that every λ_ρ and λ_{α_g} preserves the embedded $L(1, 4)$.

M.5 Full equivalence on tensor words

The source is the tensor-word subcategory of the given fusion category $\mathcal{T}$; the target is the word subcategory in the reconstruction [27, Theorem 2]. Both have the six simple types $\alpha_g, \alpha_g\rho$. For word powers zero and one their Hom spaces are scalar on equal types and zero on unequal types, so this is the base case of the comparison.

Here is a direct proof that every word morphism lies in this algebra and that the functor is full. For $x = \alpha_g\rho^n$ with $n \geq 2$, split the last ρ^2 . The resulting summands have powers $n - 2$ and $n - 1$, so repetition terminates at the six simple types.

For instance, $\alpha_g\rho^3$ splits as $\alpha_g\rho$ and the three words $\alpha_{g-h}\rho^2$, by moving α_h across the remaining ρ . Each power has dropped from three to one or two. This gives an explicit induction on the word power, with all splitting maps in the preserved Leavitt algebra.

The inclusions and projections are composites of left tensors of s, s', t_h, t'_h . Their images belong to the preserved Leavitt algebra.

Write the resulting split maps as $q_i^x : X_i \rightarrow x$ and $p_i^x : x \rightarrow X_i$, where X_i is simple. They satisfy $p_i^x q_j^x = \delta_{ij} \text{id}_{X_i}$ and $\sum_i q_i^x p_i^x = \text{id}_x$. For any $f : x \rightarrow y$,

$$f = \sum_{i,j} q_i^y (p_i^y f q_j^x) p_j^x.$$

Each middle map is zero between different simple types and scalar between equal types. Hence every $\iota(f)$ lies in the Leavitt algebra, defining the faithful tensor functor on every word Hom space. Evans–Gannon reconstruction gives the same six simple types with exactly these scalar and zero Hom spaces. The same formula therefore expresses every target morphism as the image of a source

morphism. This proves fullness. Direct sums and idempotent completion give the required tensor equivalence.

On direct sums the functor acts on the resulting block matrices. Full faithfulness transports idempotents and their images, so passing to additive and idempotent completions reaches both fusion categories. Faithfulness came from the padded embedding; fullness came from the displayed split-map expansion. The two conclusions are therefore independently accounted for.

□

N The normalized categorical presentation

N.1 Actions, conventions, and their status

The parameters $a = \delta^{-1}$ and $\omega = d_0$ are defined by the scalar comparisons of Appendix M.4. Inverse rescaling chooses $c = b$, after which $b' = \omega b$ and $d'_0 = \omega^{-1}$. These are normalization and constraint statements, not definitions of a new adjoint operation: primes still mean splitting projections. The positive branch $\delta > 0$ is selected only when this presentation is applied to the positive category in Section 12.5.

These endomorphism formulas are consequences of the actual categorical splitting maps, not a fit to modular data. Let $G = \mathbb{Z}_3$, $a = \delta^{-1}$, $a^2 + 3a = 1$, $\omega^3 = 1$, and $b^2 = 1/(\omega\delta)$. On the embedded Leavitt generators the normalized actions are

$$\lambda_\rho(s) = as + b \sum_g t_g t_g, \quad \lambda_\rho(s') = as' + \omega b \sum_g t'_g t'_g, \quad (\text{N.1})$$

$$\lambda_\rho(t_g) = bst'_{-g} + \omega t_{-g} s s' + \sum_{h,k} M_{h+g,k+g} t_h t_{h+k+g} t'_k, \quad (\text{N.2})$$

$$\lambda_\rho(t'_g) = \omega b t_{-g} s' + \omega^{-1} s s' t'_{-g} + \sum_{h,k} M_{k+g,h+g} t_k t'_{g+h+k} t'_h. \quad (\text{N.3})$$

Here $d_0 d'_0 = 1$ forces the inverse coefficient. **[CALC C21, p. 133]** The pointed action fixes s, s' and translates t_h, t'_h by $2g$, with indices modulo three.

N.2 Matrix constraints and coefficient origins

The matrix equations are

$$M_{g,h} = \omega M_{-h,g-h} = \omega^{-1} M_{h-g,-g}, \quad (\text{N.4})$$

$$\sum_h M_{h,0} = -\omega^{-1} a, \quad (\text{N.5})$$

$$\sum_g M_{h+g,k} M_{k,g} = \delta_{h,0} - a \delta_{k,0}, \quad (\text{N.6})$$

$$\sum_{l,m} M_{l,m} M_{l+g,h} M_{h+m,l+i} M_{i,k+m} = M_{h-g,i-g} \delta_{k,g} - \omega^{-1} a \delta_{h,0} M_{i,k} - \omega a M_{g,h} \delta_{i,0}. \quad (\text{N.7})$$

These are equations (4.7)–(4.10) of [27]. The reconstruction theorem is applied to these verified generator actions on the embedded Leavitt algebra. The coefficient comparisons that obtain the actions are made in the original categorical Hom spaces, as typed in Appendix M; they do not require a simplicity assertion for all intertwiners in the larger padded algebra.

Each matrix equation comes from a splitting identity. The following table records the identities whose coefficients are compared; the cyclic symmetry also uses the quadratic identity.

Matrix condition	Splitting identity
Row sum	$t'_0 \lambda_{\rho^2}(s) = \lambda_{\rho}(s) t'_0$
Quadratic identity	$\lambda_{\rho}(ss' + \sum_g t_g t'_g) = 1$
Cyclic symmetry and $\omega^3 = 1$	$t'_0 \lambda_{\rho^2}(t_g) = \lambda_{\rho}(t_g) t'_0$
Quartic identity	$t'_0 \lambda_{\rho^2}(t_{-g}) = \lambda_{\rho}(t_{-g}) t'_0$

The actions on s, s' are defined by the dimension-four comparison spaces; those on t_g, t'_g by the dimension-eleven spaces in Appendix M.4. The row-sum, symmetry, quadratic, and quartic lines are constraints obtained after normalization, not additional arbitrary choices of M . For the quartic line, g, h, i, k are fixed labels and l, m run independently over the three group elements. Expansion of two applications of λ_{ρ} gives four matrix factors; summing their intermediate labels l, m gives the left-hand coefficient. The right-hand coefficient comes from the shorter splitting terms. The compared reduced word is $t_h t_{i-g+h} t'_{k+i} t'_k$ in the $t'_0 \lambda_{\rho^2}(t_{-g})$ identity, as identified in [27, final paragraph of Section 4].

For a representative comparison, $s' \lambda_{\rho^2}(s) = ss'$ gives

$$bc \sum_g M_{0,g} = -a^2 = 3bcd_0 - 1,$$

and the st'_0 coefficient of $t'_0 \lambda_{\rho^2}(s) = \lambda_{\rho}(s) t'_0$ gives $a = bcd_0$. Hence $a^2 + 3a = 1$. Since $a \neq 0$, applying the pointed action to $s' \lambda_{\rho}(s) = a$ forces the character on s to be trivial. Preservation of the Leavitt relations gives $aa' + 3bb' = aa' + 3cc' = 1$ and $d_0 d'_0 = 1$. Inverse rescaling of dual splitting maps makes $c = b$, $c' = b'$, so $aa' = (bb')^2$. Opposite roots would give $aa' = -1$ and $bb' = 2/3$, a contradiction. With the same root the equations give $a = a' = bb'$. Further comparisons yield $b' = bd_0$, $M'_{g,h} = M_{h,g}$ and $d_0^3 = 1$.

For $\mathbb{Z}_3$, two quadratic entries already force $\omega = 1$.

An entry fixed by the order-three cyclic index permutation satisfies $M_{g,h} = \omega M_{g,h}$. For a non-trivial cube root this forces that entry to vanish. These fixed positions are exactly $(0,0), (1,2), (2,1)$ in the following calculation.

If $\omega \neq 1$, cyclic symmetry makes the entries at $(0,0), (1,2), (2,1)$ zero and determines the matrix from $x = M_{0,1}$ and $y = M_{0,2}$:

$$M = \begin{pmatrix} 0 & x & y \\ \omega x & \omega^2 y & 0 \\ \omega y & 0 & \omega^2 x \end{pmatrix}.$$

The quadratic identity with $(h,k) = (0,0)$ and $(0,1)$ now gives

$$\omega(x^2 + y^2) = 1 - a, \quad \omega(x^2 + y^2) = 1.$$

Thus $a = 0$, contradicting $a^2 + 3a = 1$. No elimination computation is needed to exclude the nontrivial cube roots of unity. **[CALC C20, p. 133]**

The split-map expansion at the end of Appendix M then gives full faithfulness on every word Hom space, and hence the tensor equivalence.

O Auxiliary modular data and fusion checks

O.1 The seven-dimensional Fourier model

Enumerate determinant-one 2×2 matrices over $\mathbb{F}_{13}$, identifying g with $-g$ by taking the lexicographically smaller representative. Conjugation partitions the 1092 elements into nine classes. For class sums K_i , count products in two classes and divide the count in class k by its size to obtain $K_i K_j = \sum_k n_{ij}^k K_k$. The exported matrices are multiplication by these sums in the nine-dimensional rational center of the group algebra.

For an actual entry, class 1 has size 91, and the identity coefficient of K_1^2 is 91. It is matrix 1, row 0, column 1 in `shared/category/verification/psl2_13_certificate.json`. A square-free separating central operator gives rational central projectors. For a projector with factor degree f and identity coefficient β , the complex irreducible degree satisfies $d^2 = 1092\beta/f$. The factor degree counts the Galois-conjugate characters in that rational block. The resulting list passes the completeness check

$$1^2 + 2 \cdot 7^2 + 3 \cdot 12^2 + 13^2 + 2 \cdot 14^2 = 1092.$$

The exact class-algebra calculation for $\mathrm{PSL}_2(\mathbb{F}_{13})$ gives complex irreducible degrees

$$1, 7, 7, 12, 12, 12, 13, 14, 14.$$

[CALC C18, p. 133] The two degree-seven representations act on even functions on $\mathbb{F}_{13}$. For $a = 1, 2$, put $G(a) = \sum_x \zeta_{13}^{ax^2}$ and write their generators as

$$(T_a f)(x) = \zeta_{13}^{ax^2} f(x), \quad (S_a f)(x) = \frac{G(a)}{13} \sum_y \zeta_{13}^{-2axy} f(y).$$

The finite-group certificate verifies these actions against both generators on all 1092 group elements. The seven T eigenvalues are distinct. The S entries from the zero eigenline to every other eigenline are nonzero, so each representation is irreducible. Their T spectra distinguish them, and the degree list shows that they exhaust the possibilities in dimension seven. **[CALC C18, p. 133]**

Use the orthonormal even-function basis $\delta_0, (\delta_j + \delta_{-j})/\sqrt{2}, 1 \leq j \leq 6$. The implementation uses unnormalized sums $\delta_j + \delta_{-j}$ for exact cyclotomic arithmetic; diagonal change of basis gives this orthonormal model. The zero-line diagonal entry is unchanged. The code evaluates a matrix on every canonical group element and checks right multiplication by both generators S, T . The simple T spectrum and nonzero S_{0j} entries prove irreducibility: an invariant subspace is a sum of T eigenlines, and the unitary S cannot separate any of them from the zero line.

Here the Fourier sign can be obtained without a choice of square root. Since -1 is a square modulo 13, $G(a)$ is real. The change of variables $u = x - y, v = x + y$ gives

$$|G(a)|^2 = \sum_{u,v} \zeta_{13}^{auv} = 13.$$

The square and nonsquare orbits partition the nonzero residues, so $G(1) + G(2) = 0$. Moreover

$$G(1) = 1 + 4 \left(\cos \frac{2\pi}{13} + \cos \frac{6\pi}{13} + \cos \frac{8\pi}{13} \right) > 0 :$$

the first cosine is greater than $1/2$, the second is positive, and the third is greater than $-1/2$. Thus $G(1) = \sqrt{13}$ and $G(2) = -\sqrt{13}$. The coefficient of the delta function at zero in its own S_a image is $G(a)/13$. This proves the diagonal entry used in Lemma 12.4. **[CALC C18, p. 133]**

O.2 The complete modular matrix

Use the order

$$(U, B, P, Q, X_+, X_-, \mu_1, \dots, \mu_6),$$

with $(\mu_1, \dots, \mu_6) = (M_{15/13}, M_{8/13}, M_{5/13}, M_{6/13}, M_{11/13}, M_{7/13})$. Set $s_{\mp} = 1/6 \mp 1/(2\sqrt{13})$. The upper 6×6 block is

$$\begin{pmatrix} s_- & s_+ & 1/3 & 1/3 & 1/3 & 1/3 \\ s_+ & s_- & 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -1/3 & 2/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & 2/3 & -1/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 & -1/3 & 2/3 & -1/3 \\ 1/3 & 1/3 & -1/3 & -1/3 & -1/3 & 2/3 \end{pmatrix}.$$

The other entries are

$$S_{0\mu_j} = \frac{1}{\sqrt{13}}, \quad S_{B\mu_j} = -\frac{1}{\sqrt{13}}, \quad S_{P\mu_j} = S_{Q\mu_j} = S_{X_+\mu_j} = S_{X_-\mu_j} = 0,$$

$$S_{\mu_i\mu_j} = -\frac{\zeta_{13}^{4ij} + \zeta_{13}^{-4ij}}{\sqrt{13}}, \quad \vartheta = \text{diag}(1, 1, \zeta_3, \zeta_3^2, 1, 1, \zeta_{13}^{2j^2} : 1 \leq j \leq 6).$$

The j -th nonzero Fourier line has twist exponent $2j^2 \bmod 13$, namely 2, 8, 5, 6, 11, 7. These are the weights of the ordered μ_j modulo integers. The matrix ϑ here is the ribbon twist; character T is $\zeta_3^{-1}\vartheta$. The zero line corresponds to $(e_U - e_B)/\sqrt{2}$, and the other eigenvectors are rescaled to make the vacuum row positive.

These formulas follow from the orthogonal branching decomposition and the even Weil representation in Lemma 12.4. The rescalings by complex numbers of modulus one on the six nontrivial Weil eigenlines are fixed by symmetry and the strictly positive vacuum row. In exact arithmetic the script verifies $S^2 = 1$, $(S\vartheta)^3 = 1$, and all 12^3 Verlinde coefficients as nonnegative integers. **[CALC C19, p. 133]**

Writing $\Sigma_\mu = \bigoplus_j \mu_j$, the particular products relevant to A_+ are

$$\begin{aligned} B^2 &= U \oplus B \oplus P \oplus Q \oplus X_+ \oplus X_- \oplus \Sigma_\mu, \\ BX_+ &= B \oplus P \oplus Q \oplus 2X_+ \oplus X_- \oplus \Sigma_\mu, \\ X_+^2 &= U \oplus 2B \oplus P \oplus Q \oplus 2X_+ \oplus X_- \oplus \Sigma_\mu. \end{aligned}$$

Thus $\dim \text{End}_A(F_A(B)) = 1 + N_{BB}^B + 2N_{BX_+}^B = 4$, confirming the split established categorically in Section 12.4. Both vectors $e_0 + e_B + 2e_{X_\pm}$ are fixed by S . **[CALC C19, p. 133]**

For a direct Verlinde example, orthogonality and self-duality give $N_{BB}^U = \sum_j S_{Bj}^2 = 1$. The displayed products give $N_{BB}^B = N_{BX_+}^B = 1$, so the free- B endomorphism dimension is $1 + 1 + 2 = 4$. The full 12^3 calculation checks consequences of the modular formulas. Actual lattice multiplication and the main categorical arguments establish the algebra objects, fusion category, and associator equivalence.

P Independent positive-solution check

To avoid confusing the algebra A with the classification matrix, denote the latter by M . In the $\omega = 1$ presentation it has the shape

$$M = \begin{pmatrix} c & p & q \\ p & q & f \\ q & g & p \end{pmatrix}, \quad a = \frac{1}{\dim \rho}, \quad a^2 + 3a = 1.$$

The script forms all linear, quadratic and quartic matrix equations, not only a selected necessary subset. The resulting quotient has dimension eight. A separating multiplication operator has a square-free degree-eight characteristic polynomial, so the quotient is reduced and has exactly eight labeled points. Multiplication by a has characteristic polynomial $(x^2 + 3x - 1)^4$. Inversion sends (c, p, q, f, g) to (c, q, p, g, f) , and imposing its fixed-point equations gives the unit ideal. **[CALC C20, p. 133]**

The polynomial ring is over $\mathbb{Q}$ with lexicographic variable order (g, f, e, d, c, a) ; program variables d, e are the manuscript's p, q . For example, the row-sum equation is $a + c + d + e = 0$, or $a + c + p + q = 0$ in the manuscript variables. The complete matrix equations give 52 distinct nonzero polynomials and a nine-element Gröbner basis. Its leading monomials leave the eight standard monomials

$$1, a, c, ac, d, ad, f, af.$$

Their exponent vectors are recorded in the declared order in `shared/category/verification/haagerup_classification_check.json`. Gröbner reduction of the products of $32a + c + 2d + 4e + 8f + 16g$ with these monomials gives an eight-dimensional multiplication operator. Its square-free degree-eight characteristic polynomial has eight distinct eigenvalues over $\mathbb{C}$. The quotient is therefore reduced and is a product of eight copies of $\mathbb{C}$, not merely a scheme of length eight.

The inversion-fixed ideal is the unit ideal, so inversion pairs the eight labeled points into four orbits. Multiplication by a has characteristic polynomial $(x^2 + 3x - 1)^4$; the positive reciprocal-dimension root selects four points and two orbits. Their interpretation as tensor-equivalence classes uses [27, Theorems 2(b) and 3(ii)].

For the positive root, write $r = \sqrt{13}$ and $a = (r - 3)/2$. The four labeled points are the two sign choices in each of the following rows:

$$\begin{aligned} c &= (2 - r)/3, & p &= (5 - r \pm \sqrt{6 + 6r})/12, & q &= (5 - r \mp \sqrt{6 + 6r})/12, & f &= g = (1 + r)/6; \\ c &= (7 - r)/6, & p &= q = (1 - r)/6, & f &= (r - 1)/12 \pm i\sqrt{2 + 2r}/4, & g &= \bar{f}. \end{aligned}$$

Every displayed matrix is Hermitian. The two signs in each row are interchanged by inversion, giving exactly two positive tensor-equivalence classes by the reconstruction equivalence criterion. The names $\mathcal{H}_3$ and $\mathcal{H}_2$, respectively, are the identifications in [27, Theorem 3(ii)]. This table lists the classification alternatives; it does not assert which matrix is produced by either lattice gluing. **[CALC C20, p. 133]**

Q Explicit ribbon comparisons

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be the braided tensor equivalence in the lemma. This F is the equivalence being compared, rather than the center forgetful functor of Section 12. The maps below first compare chosen duals, then double duals, then pivotal structures, and finally twists. All braidings have the orientation fixed by $\ell_M^- = r_M c_{M,A}^{-1}$ in the module category.

Explicit duality comparisons for Lemma 13.1. Choose duals with evaluation $\text{ev}_X : X^* \otimes X \rightarrow \mathbf{1}$ and coevaluation $\text{coev}_X : \mathbf{1} \rightarrow X \otimes X^*$. Write $F_2(X, Y) : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ and $F_0 : \mathbf{1} \rightarrow F(\mathbf{1})$ for the tensor constraints. The canonical duality comparison $\delta_X : F(X^*) \rightarrow F(X)^*$ is characterized by

$$\text{ev}_{F(X)} \circ (\delta_X \otimes \text{id}_{F(X)}) = F_0^{-1} \circ F(\text{ev}_X) \circ F_2(X^*, X),$$

and the corresponding coevaluation identity. The double-dual comparison is

$$\kappa_X = (\delta_X^*)^{-1} \circ \delta_{X^*} : F(X^{**}) \longrightarrow F(X)^{**}.$$

The evaluation equation characterizes $\delta_X : F(X^*) \rightarrow F(X)^*$ by the uniqueness of a dual object. Its dual has type $\delta_X^* : F(X)^{**} \rightarrow F(X^*)^*$, so the displayed composition defining κ_X has exactly the stated source $F(X^{**})$ and target $F(X)^{**}$. This comparison is required before pivotal maps with those targets can be equated.

Transport the target pivotal structure $j^{\mathcal{B}}$ to the source: full faithfulness determines j^{tr} uniquely from

$$\kappa_X \circ F(j_X^{\text{tr}}) = j_{F(X)}^{\mathcal{B}}.$$

Transport of duality and trace makes j^{tr} spherical, with $\dim_{j^{\text{tr}}}(X) = \dim_{j^{\mathcal{B}}}(F(X)) > 0$ for every simple X . The specified source structure is also positive. Both are the unique positive pivotal structure of the pseudounitary source category [22, Proposition 8.23], so $j^{\text{tr}} = j^{\mathcal{A}}$.

Having identified the pivotal structures through κ_X , it remains to compare the braiding-defined Drinfeld maps $u_X : X \rightarrow X^{**}$. The following formula does that comparison locally and then recovers the twist.

Use the Drinfeld isomorphism and twist convention of [28, Section 2.8.2, equations (12)–(16)]:

$$\begin{aligned} u_X &= (\text{ev}_X \otimes \text{id}_{X^{**}}) \circ (c_{X, X^*} \otimes \text{id}_{X^{**}}) \circ (\text{id}_X \otimes \text{coev}_{X^*}), \\ j_X &= u_X \circ \vartheta_X, \quad \vartheta_X = u_X^{-1} \circ j_X. \end{aligned}$$

Associativity and unit constraints are suppressed here. Braidedness and the duality comparisons give $\kappa_X \circ F(u_X^{\mathcal{A}}) = u_{F(X)}^{\mathcal{B}}$. Therefore

$$u_{F(X)}^{\mathcal{B}} \circ F(\vartheta_X^{\mathcal{A}}) = \kappa_X \circ F(j_X^{\mathcal{A}}) = j_{F(X)}^{\mathcal{B}} = u_{F(X)}^{\mathcal{B}} \circ \vartheta_{F(X)}^{\mathcal{B}}.$$

Cancel the Drinfeld isomorphism to obtain $F(\vartheta_X^{\mathcal{A}}) = \vartheta_{F(X)}^{\mathcal{B}}$ for every object X . $\square$

R Exact certificates, provenance and reproduction

The finite part of the proof uses exact arithmetic over K . This appendix lists the assertion needed at each step and the program that verifies it. The companion computational package is available from https://github.com/Doctorfm98/Haagerup_V0A/releases; see the reading guide for download instructions and the index of all 21 certificates. This separate package contains the defining data, selected witnesses and verification programs. Running the programs creates the replay records. The manifest records byte counts and SHA-256 hashes. Each supplied finite equality is checked with exact arithmetic.

R.1 Arithmetic conventions and good-prime logic

The coefficient field is $K = \mathbb{Q}[\theta]/(\theta^4 - 20\theta^2 + 256)$ with the complex embedding $\theta = \sqrt{13} + i\sqrt{3}$. The prime is $p = 10009$, with $r \mapsto 825$, $q \mapsto 2089$, $\theta \mapsto 2914$. The field polynomial vanishes at 2914 and its derivative is $7285 \neq 0$ modulo p . **[CALC C17, p. 133]**

The local ring is $R = \mathbb{Z}[\theta]_{(p, \theta - 2914)}$. To see directly why it is a DVR, write the field polynomial as $F(X)$. The identity

$$F(\theta) - F(2914) = (\theta - 2914)u(\theta)$$

has $u(\theta) \equiv F'(2914) \neq 0$ in the residue field, so $u(\theta)$ is a unit in R . Since $F(\theta) = 0$ and $p \mid F(2914)$, it follows that $\theta - 2914 \in pR$. Thus the maximal ideal is pR . The ring is a one-dimensional Noetherian local domain, finite over $\mathbb{Z}$ before localization; its principal maximal ideal makes it a DVR. In particular its residue field is $\mathbb{F}_p$, and a nonzero minor modulo p has unit determinant in this ring.

A nonzero determinant modulo this prime proves a characteristic-zero lower bound for the rank of an exact matrix defined and regular there. It proves an upper bound on a quotient only when the rows are already valid characteristic-zero relations. The strong-generation argument uses exactly this second situation. In the Zhu argument, integral quadratic rules plus unit cubic minors prove a finite model over the local DVR; the proof then removes torsion and lifts split semisimple blocks. Neither step treats an arbitrary modular ambient-span dimension as a characteristic-zero upper bound.

The source files explicitly specify coefficient order. Number-field lists must be interpreted using that metadata: the original SymPy vector representation and the portable four-coordinate arithmetic need not use the same list order. The recurrence formulas, source-vector order and selected minors are part of the certificate definitions.

For a rank comparison, the quartic relation matrix has 31,234 columns and a selected nonzero 31,231-row minor. This is a rank lower bound. Since the rows are valid characteristic-zero relations, their quotient has dimension at most $31,234 - 31,231 = 3$. A modular span of arbitrary ambient vectors would not justify that quotient conclusion.

Coefficient encodings can be decoded explicitly. In the exact S_2, S_3 basis files, kernel files, and `exact_upper_rules.json`, $[-11, 0, 308]$ means $-11\theta^2 + 308$, with descending powers and leading zeros omitted. Internally K4 in `shared/rank12/certificates/fast_exact.py` stores ascending numerators (n_0, n_1, n_2, n_3) and denominator d , meaning $(n_0 + n_1\theta + n_2\theta^2 + n_3\theta^3)/d$. Thus θ has internal tuple $(0, 1, 0, 0)$ but serialized descending list $[1, 0]$. The C07 nilradical audit also uses four ascending slots internally. Modular minor and projector coefficients are residues, not exact lifts. Use each file's metadata and decoder rather than assuming a universal list order.

R.2 Calculation locators and verification entries

The identifiers below are the targets of the sans-serif **CALC** markers throughout the paper. All file paths refer to the root of the companion computational package downloaded from https://github.com/Doctorf98/Haagerup_V0A/releases, as explained in the reading guide. The input data and substantive programs are under `shared/`; `CERTIFICATES.md` indexes all 21 individual reader guides. Each entry below also gives the exact path to its guide. Use Python 3.12 with assertions enabled and install `requirements.txt`. The root command `python3 verify_all.py --jobs 3` runs all finite checks in dependency order. Each command below runs from that same root; the runner selects the required arithmetic backend. One full rank-twelve run verifies C13–C16 together using the source families checked in C11.

C01. Definition, coset axes and seed *Reader guide.* `certificates/C01_definition_coset_axes_seed/README.md`. *Finite input.* Gram positivity, the fixed cocycle, levels 1, 39, the four-dimensional coset space (rank 82 on all 86 zero-Cartan-weight vectors), the axis singular products, and the chosen seed eigenvalues. *Specification.* Appendices B, D, and E; Definition 2.1. *Witnesses.* `shared/definition/data/definition.json`; `shared/definition/certificates/definition/coset_rank_witness.json`. *Verify.* `python3 verify_certificate.py C01`. *Generated replay record.* `replay/C01_latest.json`. *Human check.* Expand the displayed lattice vectors by the exponential and Heisenberg formulas; check each residual and the selected coset minor.

C02. Positive form and the lowering identity *Reader guide.* `certificates/C02_positive_form_lowering/README.md`. *Finite input.* All nine conjugation images, the three thirteen-step lowering traces, and $L_1 W_1 = 0$. The second implementation checks 150 assertions, a 340-dimensional subspace, and nine corrupted conventions. *Specification.* Appendix G, Equation (G.1); Proposition 3.1. *Witnesses.* `shared/positive/certificates/HGR_UASSUMP_v1_report.json`; `shared/positive/certificates/HGR_UASSUMP_v1_independent_audit_r2.json`. *Verify.* `python3 verify_certificate.py C02`. *Generated replay record.* `replay/C02_latest.json`. *Human check.* Follow the three lowering traces with the closed root-action formulas printed in the appendix. The all-weight positive form, simplicity and duality follow from transport and invariance in the main proof.

C03. Weight-two basis and closure *Reader guide.* `certificates/C03_weight_two_basis_closure/README.md`. *Finite input.* The ordered 120-vector space, its $\mathfrak{g}$ -module decomposition, all required mode containments for the affine and weight-two generators. *Specification.* Appendices A.4 and H.1; Proposition 4.1. *Witnesses.* `shared/lattice/research/results/haagerup_c8_degree_two_exact_basis.json`; `shared/lattice/research/results/haagerup_c8_closure_exact.json`. *Verify.* `python3 verify_certificate.py C03`. *Generated replay record.* `replay/C03_latest.json`. *Human check.* Read the generation words, reconstruct the vectors, and reduce each required product against their normalized coordinate pivots. The driver replaces the primitive product engine; the generation words and linear algebra are reused. The selected complement minor is checked in C06.

C04. Weight-three basis and feedback *Reader guide.* `certificates/C04_weight_three_basis_feedback/README.md`. *Finite input.* The ordered 956-vector basis, Lie algebra actions, and 15,296 products in each of modes 1, 2, 3, with exact zero residuals. *Specification.* Appendices H.2 and H.3; Proposition 4.1. *Witnesses.* `shared/lattice/research/results/haagerup_c8_grade_three_exact_basis.json`; `shared/lattice/research/results/haagerup_c8_grade_three_exact.json`. *Verify.* `python3 verify_certificate.py C04`. *Generated replay record.* `replay/C04_latest.json`. *Human check.* Review the complete product loops and output-weight bounds, then compare the rebuilt basis and all feedback residuals.

C05. The normal prefix and quotient *Reader guide.* `certificates/C05_normal_prefix_quotient/README.md`. *Finite input.* 840 normal inputs, 206 independent relations in 18 highest families, and the first 634 basis rows spanning N_3 . The remaining 322 rows represent the free quotient. *Specification.* Appendices H.2 and H.5; Section 4. *Witnesses.* `shared/rank12/certificates/bracket_discovery.json`; `shared/rank12/certificates/bracket_exact.json`; `shared/lattice/research/results/haagerup_c8_grade_three_basis_modp.json`. *Verify.* `python3 verify_certificate.py C05`. *Generated replay record.* `replay/C05_latest.json`. *Human check.* Check the support of each normal relation and its distinguished nonpivot column; compare the prefix origins with the exact basis. Full lattice residuals are independently regenerated in C04 and C11; this checker verifies their rank/support interface.

C06. Bracket and Jacobi obstructions *Reader guide.* `certificates/C06_bracket_jacobi_obstructions/README.md`. *Finite input.* Bracket rank 322; quartic valid-relation rank 31,231 in 31,234 columns; quintic rank 291 in 291 columns. The independent audit checks 149 quartic and 47 quintic square minors and all 147,440 quartic remainder rows. *Specification.* Appendices H.4, H.5, and H.7; Lemma 4.3. *Witnesses.* `shared/strong_generation/low_bracket.npz`; `shared/strong_generation/jacobi_degree3_minor_witnesses.json`; `shared/strong_generation/jacobi_weight5_witnesses.json`. *Verify.* `python3 verify_certificate.py` C06. *Generated replay record.* `replay/C06_latest.json`. *Human check.* Derive the relation rows by Jacobi before checking their selected determinants. Before determinants, the checker verifies weight homogeneity, distinct row and column indices, the exact quartic complement of three free columns, and all quintic columns. Full mode rebuilds the tensors; `-audit-only` checks the supplied tensors and minors. The all-weight conclusion is Lemma 4.3.

C07. The two coset-square identities *Reader guide.* `certificates/C07_coset_square_identities/README.md`. *Finite input.* 222 exact rows from 310 specified nilradical/derivative columns; complete lattice-vector reductions of $e_{(-1)}e$ and $f_{(-1)}f$. The supplementary mixed product is retained. *Specification.* Appendix H.9; Lemma 5.1 and Theorem 5.2. *Witnesses.* `shared/lattice/research/results/haagerup_c8_c2_axis_exact.json` (coefficient reductions); `shared/lattice/research/results/haagerup_c8_c2_nilradical_audit.json` (audit). *Verify.* `python3 verify_certificate.py` C07. *Generated replay record.* `replay/C07_latest.json`. *Human check.* Inspect the witness columns and verify that each is a derivative, an affine generator product, or an invariant product of nontrivial irreducible $\mathfrak{g}$ -modules. The nilradical argument proves why these identities imply C_2 -cofiniteness after strong generation.

C08. Simple lattice tops and their distinction *Reader guide.* `certificates/C08_lattice_tops/README.md`. *Finite input.* The 22 zero-mode matrices, scalar commutants, all 21 selected rank minors, and the nonzero cross-Hom minor distinguishing X_+ and X_- . *Specification.* Appendix I; Proposition 10.1. *Witnesses.* `shared/modules/top_audit.json`; `shared/modules/independent_top_audit.json`. *Verify.* `python3 verify_certificate.py` C08. *Generated replay record.* `replay/C08_latest.json`. *Human check.* Reconstruct the matrices on lowest-weight spaces from the printed formulas and inspect the Cartan-compatible commutant equations. The positive adjoint-closed action turns a scalar commutant into top irreducibility; the main proof then establishes cyclic-module simplicity.

C09. Exact opposite-coset maps *Reader guide.* `certificates/C09_opposite_charge_maps/README.md`. *Finite input.* Four maps over K , all 88 generator-intertwining residuals, and determinants 1, 1513, 803, 5805 modulo 10009. The saved-map check also verifies opposite minimum-norm coset vectors and rejects four entry mutations. *Specification.* Appendices I.1 and I.2; Proposition 10.1. *Witnesses.* `shared/modules/exact_pair_audit.json`. *Verify.* `python3 verify_certificate.py` C09. *Generated replay record.* `replay/C09_latest.json`. *Human check.* Use the stated entry order to multiply each saved map by the freshly reconstructed matrices. These maps identify ordinary tops. The separate invariant lattice pairing and Zhu correspondence prove self-contragredience in the text.

C10. The affine extension and source blocks *Reader guide.* `certificates/C10_affine_extension_source_blocks/README.md`. *Finite input.* The cubic seed witness, its nonzero coefficient and highest type (1, 39), the twenty local orbit tops, source identities, and known corner dimensions. *Specification.* Appendices J.1 and J.2; Subsection 8.1. *Witnesses.* `shared/affine/affine_extension_audit.json`. *Verify.* `python3 verify_certificate.py` C10. *Generated replay record.* `replay/C10_latest.json`. *Human check.* Check the six cocycle signs, affine weights and orbit minima; substitute each permitted Lie algebra module type into the source identities. The matrix-unit identity in the main text proves fullness, including when a source block vanishes.

C11. Full lattice relations and Zhu coordinates *Reader guide.* `certificates/C11_full_state_zhu_relations/README.md`. *Finite input.* 66 symmetric, 408 bracket and 446 mixed highest families, each an identity of complete lattice vectors, with the mode-one Zhu corrections and conformal coordinates. *Specification.* Appendices K.1.1, K.2, and K.3; Subsections 9.2–9.6. *Witnesses.* `shared/rank12/certificates/quadratic_exact.json`; `shared/rank12/certificates/bracket_exact.json`; `shared/rank12/certificates/mixed_exact.json`; `shared/rank12/certificates/central_coordinates.json`. *Verify.* `python3 verify_certificate.py C11`. *Generated replay record.* `replay/C11_latest.json`. *Human check.* Take a relation descriptor, expand every term as a lattice vector, and retain the displayed mode-one correction when converting it to a Zhu relation.

C12. Eight exact vertex exclusions *Reader guide.* `certificates/C12_vertex_exclusions/README.md`. *Finite input.* Explicit unit-identity witnesses kill the pairs (10,11), (9,12), (8,13) and (7,14) in that order. *Specification.* Appendix K.4; Subsection 9.7. *Witnesses.* `shared/rank12/certificates/global_vertex_exact.json`; `shared/rank12/certificates/cascade_vertex_exact_1.json`; `shared/rank12/certificates/cascade_vertex_exact_2.json`; `shared/rank12/certificates/cascade_vertex_exact_3.json`. *Verify.* `python3 verify_certificate.py C12`. *Generated replay record.* `replay/C12_latest.json`. Every relation origin and composable projected path is checked in its declared corner (v, v) . The unit at each vertex remains distinct, and all preceding exclusions are reverified in the same exact run.

Human check. Sum each listed characteristic-zero combination of projected relations and obtain exactly its vertex identity, after the preceding exclusions.

C13. Exact quadratic saturation *Reader guide.* `certificates/C13_quadratic_saturation/README.md`. *Finite input.* 11,293 projected rows; exact stage ranks 9859, 7950, 6787, 6745; all 73 linear eliminations and 6745 portable quadratic rules. *Specification.* Appendix K.4; Subsection 9.8. *Witnesses.* `shared/rank12/new_results/quadratic_saturation_exact.json`; `shared/rank12/new_results/exact_upper_rules.json`. *Verify.* `python3 verify_certificate.py C13`. *Generated replay record.* `replay/RANK12_latest.json`. *Human check.* Trace each reduction rule to the projected VOA relations and compare its exact coefficients with the portable rule list. The full driver rederives the upper rules from the archived families verified in C11.

C14. Integral finiteness over the DVR *Reader guide.* `certificates/C14_integral_finiteness/README.md`. *Finite input.* 151,115 coefficients regular at the prime, and 53 unit-minor blocks covering all 1364 normal cubic columns. Thus 1081 paths span the integral algebra. *Specification.* Appendices K.5.1, K.5.2, and R.1; Proposition 9.5. *Witnesses.* `shared/rank12/new_results/finite_bound_certificate.json`. *Verify.* `python3 verify_certificate.py C14`. *Generated replay record.* `replay/RANK12_latest.json`. *Human check.* Form both reductions of each selected overlap, check the unit determinants, and solve for every cubic path over the local ring. This is an integral spanning bound before using the special fiber; longer paths shorten by the induction in the proof.

C15. Semisimple special fiber *Reader guide.* `certificates/C15_semisimple_special_fiber/README.md`. *Finite input.* All 184,016 quadratic and 1653 higher overlaps, the 546 normal paths and trace-pairing rank, and the twelve split blocks with their source ranks. *Specification.* Appendices K.5.3 and K.5.4; Proposition 9.5 and Lemma 9.6. *Witnesses.* `shared/rank12/new_results/finite_diamond_mod.json`; `shared/rank12/new_results/finite_structure_mod.json`; `shared/rank12/new_results/modular_blocks.json`. *Verify.* `python3 verify_certificate.py C15`. *Generated replay record.* `replay/RANK12_latest.json`. *Human check.* Review the terminating order and complete overlap loops, then check the trace matrix and central projectors. Characteristic-zero semisimplicity follows from the proved integral descent lemma, after removal of torsion.

C16. Conformal trace and source profiles *Reader guide.* `certificates/C16_conformal_trace_source_profiles/README.md`. *Finite input.* 134 exact arrow commutators give the conformal cocenter identity with zero residual. Source ranks determine all possible new weights and full top dimensions. *Specification.* Appendix K.6; Equations (11.1)–(11.2). *Witnesses.* `shared/rank12/new_results/trace_conformal_witness.json`; `shared/rank12/new_results/trace_conformal_exact.json`; `shared/rank12/new_results/modular_blocks.json`. *Verify.* `python3 verify_certificate.py C16`. *Generated replay record.* `replay/RANK12_latest.json`. *Human check.* Reduce the sum of the raw commutators exactly; substitute each source profile into the printed trace formula. The profile list becomes the actual module list only after branching and the Galois argument.

C17. Good prime and the Galois orbit *Reader guide.* `certificates/C17_good_prime_galois_arithmetic/README.md`. *Finite input.* The field polynomial has a simple root at 2914 modulo 10009, with derivative 7285. The square subgroup modulo 13 has orbit $\{2, 5, 6, 7, 8, 11\}$ on any one of these exponents, and fixes the common order-three character factor. *Specification.* Appendix R.1 and Subsection 11.5; Theorem 11.4. *Witnesses.* `shared/category/check_conceptual_arithmetic.py`. *Verify.* `python3 verify_certificate.py C17`. *Generated replay record.* `replay/C17_latest.json`. *Human check.* Multiply each of the six exponents by the six quadratic residues modulo 13. The permutation of actual module eigenvalues uses the cited VOA Galois theorem, after strong rationality.

C18. Finite group and the even Weil models *Reader guide.* `certificates/C18_finite_group_weil_models/README.md`. *Finite input.* All 1092 elements of $\mathrm{PSL}_2(13)$, its rational class algebra and irreducible degrees, and the two exact degree-seven models. *Specification.* Appendix O.1; Lemma 12.4. *Witnesses.* `shared/category/verification/psl2_13_certificate.json`. *Verify.* `python3 verify_certificate.py C18`. *Generated replay record.* `replay/C18_latest.json`. *Human check.* Inspect the group enumeration and class algebra; check the Fourier matrices on the generators. The Gauss-sum sign also has the printed elementary proof.

C19. Auxiliary modular and condensation arithmetic *Reader guide.* `certificates/C19_modular_condensation_arithmetic/README.md`. *Finite input.* The full modular formulas, both algebra vectors, all 12^3 Verlinde coefficients, and the displayed induction products. *Specification.* Appendix O.2; Section 12.4. *Witnesses.* `shared/category/verification/e8_condensation_checks.json`. *Verify.* `python3 verify_certificate.py C19`. *Generated replay record.* `replay/C19_latest.json`. *Human check.* Substitute the exact Fourier entries into the branching and Verlinde identities. These checks confirm the formulas. Actual lattice multiplication and the categorical arguments establish the algebras and fusion category.

C20. Classification equations and labeled solutions *Reader guide.* `certificates/C20_classification_normalization/README.md`. *Finite input.* All linear, quadratic and quartic classification equations; the reduced eight-point quotient and inversion orbits. The short two-entry argument excludes $\omega \neq 1$. *Specification.* Appendices M.4, N.2, and P; Proposition 12.5. *Witnesses.* `shared/category/verification/haagerup_classification_check.json`; `shared/category/check_normalization_phases.py`. *Verify.* `python3 verify_certificate.py C20`. *Generated replay record.* `replay/C20_latest.json`. *Human check.* Compare coefficients of the typed splitting identities; use the two printed quadratic entries to exclude nontrivial cube roots of unity. The eight-point arithmetic checks the classification alternatives; the cited reconstruction and classification theorems identify their categories.

C21. Normalized Leavitt actions *Reader guide.* `certificates/C21_leavitt_reconstruction/README.md`. *Finite input.* 138 exact checks of the normalized actions, Leavitt relations, fusion decomposition, group action and crossing; a sign mutation is rejected. *Specification.* Appendices M.2, M.3, M.4, M.5, and N.1; Proposition 12.5. *Witnesses.* `shared/category/check_leavitt_reconstruction.py`. *Verify.* `python3 verify_certificate.py C21`. *Generated replay record.* `replay/C21_latest.json`. *Human check.* Apply the normalized actions to all eight Leavitt generators and reduce both sides in the Leavitt normal form. The faithful padded embedding and the full split-map functor are proved in the appendix; these deductions are not replaced by finite residuals.

R.3 A second lattice-product implementation

The additional audit evaluates primitive products using the following normal-product identity. For $h \in \mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, write

$$h^-(z) = \sum_{n < 0} h(n)z^{-n-1}, \quad h^+(z) = \sum_{n \geq 0} h(n)z^{-n-1}.$$

For $d \geq 1$ and a lattice-VOA vector u ,

$$Y(h(-d)u, z) = \frac{\partial_z^{d-1} h^-(z)}{(d-1)!} Y(u, z) + Y(u, z) \frac{\partial_z^{d-1} h^+(z)}{(d-1)!}.$$

Iterate this identity for $u = h_{i_2}(-d_2) \cdots h_{i_k}(-d_k)e^\alpha$. For each Heisenberg-mode factor in the first input, choose its negative-mode or nonnegative-mode summand; apply the latter to the second input and expand the negative-mode exponential. For input lattice vector β and requested output weight w , the total Heisenberg degree is exactly $w - (\alpha + \beta, \alpha + \beta)/2$. This bounds the expansion; it does not impose an additional truncation.

The routine `shared/lattice/independent_wick.py` imports neither supplied lattice-mode engine. It regenerates the two low-weight bases, their feedback checks, and the 920 source relation families using the supplied generation words and linear algebra. Modular pivots only choose an ordering; every resulting relation is evaluated on full characteristic-zero vectors. The reproduced mathematical data agree with the archived data. The driver and exact comparisons are included in the bundle, with their scope recorded separately from the all-weight and categorical proofs.

R.4 Pinning the two large exact bases

The SHA-256 values of the lattice basis inputs are

```
shared/lattice/research/results/haagerup_c8_degree_two_exact_basis.json:
390f7471538845638d852fd84e36a56d972d419577cbde29c4d758a0719a449c

shared/lattice/research/results/haagerup_c8_grade_three_exact_basis.json:
6c7f5e5ef19e23626247c1b2b6042ae13fe8595c6485a296e19837fea5c47ad2
```

The manifest in the companion bundle pins the exact rule data, source relation tables, selected determinant files and verification programs. Timestamps and elapsed times are execution metadata; semantic comparison is used when a replay rewrites those fields.

R.5 Reproduction order and scope

Verify the publication manifest before replaying, and use a writable copy of the extracted tree. Install `requirements.txt` with Python 3.12, with assertions enabled. From the root of the extracted GitHub computational package, run

```
python3 verify_manifest.py
python3 verify_all.py --jobs 3
```

The runner regenerates the definition and structural checks, the low-weight bases and feedback, the normal quotient and strong-generation tensors, the module and affine data, all 920 relation families,

the typed vertex exclusions, the full upper presentation, and the categorical arithmetic. It enforces their dependency order and selects SymPy’s pure-Python backend for the rank-twelve computation. C13–C16 share one full replay; no additional archive extraction is required.

There are 21 logical scopes and 18 scheduled computations, because the four rank-twelve scopes share one derivation. Shipped basis and rule files are inspectable witnesses; full replay rebuilds the specified components. Fresh summaries and logs are generated reports, and a saved status flag is not a replay. C05 checks a rank/support interface using lattice-vector premises verified by C04 and C11. C06 full mode rebuilds tensors before testing minors; audit-only mode checks the supplied tensors and witnesses. These modes have different replay scopes.

C06 checks distinct minor rows and columns, the exact quartic complement of three free columns, complete quintic coverage and weight homogeneity before checking determinants. C12 keeps the vertex units typed, checks every origin and path endpoint, and re-verifies the complete preceding exclusion prefix in the same exact run.

The root `README.md` gives installation and reproduction commands. The individual guides linked from `CERTIFICATES.md` identify the active inputs, substantive programs, exact output gates and human checks. Reproduction records and command logs are written under `replay/`. Each certificate proves its listed finite assertion. Rationality and category identification also require the mathematical arguments in Sections 4, 9, and 11–13, including the all-weight, descent, survival and categorical reconstruction steps.

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