

WAVE MECHANICS

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ABSTRACT. We introduce a complex Hilbert space structure on $L^2(\mathbb{T}^m; \mathbb{C})$ in which the Hilbert transform on the torus is incorporated directly into the metric and inner product. The resulting geometry is defined via a bounded linear embedding that couples each function to its Hilbert transform, yielding an inner product with both symmetric and symplectic components. Within this framework, we redefine weak differentiation intrinsically using the complex Hilbert-space inner product rather than the ambient L^2 pairing. We show that this intrinsic weak derivative coincides with the classical weak derivative on its natural domain, while remaining geometrically well defined on the full space. The derivative operator is shown to be densely defined and skew-adjoint, with a purely imaginary, discrete spectrum. Its eigenfunctions are given by the Fourier modes on the torus, which induce a complete orthonormal basis adapted to the Hilbert-transform metric. This construction yields a spectral representation in which differentiation is diagonal and the Hilbert transform is absorbed into the notion of differentiability itself, providing a unified analytic and geometric framework. We formulate wave mechanics intrinsically in this complex Hilbert space, define wave function, scalar and vector potentials and associated energy functional, derive space time wave equation, derive its solution, prove existence and uniqueness and establish regularity of the solution.

Chapter 1. Intrinsic Hilbert Geometry on the Torus

1. INTRODUCTION

The Hilbert transform is a fundamental operator in harmonic analysis and serves as a canonical example of a bounded, nonlocal Fourier multiplier. On periodic domains it is naturally compatible with Fourier series and provides a distinguished complex structure on real L^2 spaces.

In this work we treat the Hilbert transform not as an auxiliary operator, but as part of the underlying geometry. Concretely, we modify the metric on $L^2(\mathbb{T}^m; \mathbb{C})$ via the embedding

$$T(f) = f + if^h,$$

and define the intrinsic inner product by pulling back the ambient L^2 pairing through T . This produces a complex Hilbert space whose inner product decomposes into a symmetric (energy) component and a skew component.

A central goal is to formulate differentiation intrinsically in this geometry. We define weak derivatives by duality with respect to the intrinsic inner product and show that, on the natural Sobolev domain, this notion coincides with the classical weak derivative. The resulting first-order intrinsic derivative operator is densely defined and skew-adjoint, with Fourier modes as eigenfunctions.

We formulate wave mechanics intrinsically in this complex Hilbert space, define wave function, scalar and vector potentials and associated energy functional, derive space time wave equation, derive its solution, prove existence and uniqueness and establish regularity of the solution.

2. LITERATURE SURVEY

The present work lies at the intersection of harmonic analysis, functional analysis, geometric formulations of physical theories, and spectral approaches to wave mechanics. In this section, we review the principal mathematical developments that motivate the construction of the intrinsic geometric wave framework developed in this paper.

2.1. Hilbert Transform and Harmonic Analysis. The Hilbert transform occupies a central position in harmonic analysis and complex function theory [28, 23]. Classical treatments by Riesz and Zygmund established the Hilbert transform as a singular integral operator with deep connections to Fourier analysis, Hardy spaces, and boundary value problems [28]. In one dimension, the Hilbert transform provides the harmonic conjugate of a function and yields the analytic signal representation introduced by Gabor [15]. Extensions of the Hilbert transform to periodic domains and multidimensional settings have been studied extensively in harmonic analysis and signal processing [19, 23].

The theory of singular integral operators developed by Calderón and Zygmund provided the modern functional analytic framework for the Hilbert transform and related operators [23]. Subsequent developments demonstrated the boundedness of the Hilbert transform on L^p spaces and its fundamental role in spectral decompositions, pseudo-differential operators, and partial differential equations [16].

The present work differs from classical approaches in that the Hilbert transform is not employed merely as an auxiliary operator acting on an existing function space; rather, it is incorporated directly into the definition of the metric and inner product, thereby generating an intrinsic geometric structure on the underlying function space.

2.2. Hilbert Spaces and Geometric Structures. The geometric interpretation of Hilbert spaces has played a fundamental role throughout modern analysis and mathematical physics [11, 21]. The classical Hilbert space framework developed by Hilbert, von Neumann, and Riesz established the mathematical foundations of spectral theory and operator analysis [11]. In particular, the Riesz representation theorem, orthogonal decompositions, and the spectral theorem provide the structural basis for much of contemporary mathematical physics [22].

Several authors have investigated modifications and extensions of Hilbert space geometry, including reproducing kernel Hilbert spaces, Krein spaces, Pontryagin spaces, and complex geometric formulations of functional spaces [3]. These developments demonstrate that altering the inner product structure can lead to fundamentally new geometric and spectral phenomena.

The intrinsic geometry developed in the present work belongs to this general tradition, but differs substantially from existing constructions by introducing a Hilbert-transform-induced complex geometry whose metric, topology, differentiation, and spectral structure arise simultaneously from a single geometric embedding.

2.3. Weak Derivatives and Sobolev-Type Structures. The theory of weak derivatives and Sobolev spaces, initiated by Sobolev and further developed by Schwartz, Gelfand, and Lions, provides the standard framework for generalized differentiation [2, 14]. Weak derivatives allow differential operators to be extended beyond classical smooth functions and constitute the mathematical foundation of modern partial differential equations [14].

Classical Sobolev theory defines weak derivatives through duality with compactly supported test functions under the standard L^2 inner product [2]. Numerous extensions have been proposed, including weighted Sobolev spaces, fractional Sobolev spaces, and distributional derivative theories [6].

The derivative introduced in this work differs from these constructions by defining differentiation intrinsically through the geometry induced by the complex Hilbert inner product. Consequently, differentiation, metric structure, and spectral representation become mutually dependent geometric objects rather than independent constructions.

2.4. Spectral Theory and Fourier Analysis. The spectral theory of linear operators forms one of the principal mathematical foundations of modern analysis [21, 13]. The classical spectral theorem of Hilbert, von Neumann, and Stone established the characterization of self-adjoint and normal operators through orthogonal spectral decompositions [11].

Fourier analysis provides perhaps the most important realization of spectral theory [18]. Fourier expansions on periodic domains furnish orthonormal bases for function spaces and provide diagonalizations of differential operators [24]. Extensions to generalized Fourier transforms, harmonic analysis on groups, and spectral decompositions on manifolds have significantly broadened the applicability of spectral methods [16].

In the present formulation, the Fourier basis remains central; however, its interpretation changes fundamentally because the underlying geometry itself is modified by the Hilbert-transform embedding. Consequently, the resulting spectral decomposition acquires an intrinsic geometric character.

2.5. Geometric Approaches to Wave Mechanics. Geometric formulations of wave phenomena have a long history, ranging from Hamiltonian mechanics and symplectic geometry to geometric analysis of partial differential equations [4, 1]. Various approaches have attempted to derive physical laws from geometric principles by identifying physical observables with geometric invariants.

Traditional wave mechanics generally assumes a pre-existing geometric background and subsequently formulates wave equations on that background. In contrast, the present approach reverses this paradigm: the geometry itself is generated by the wave structure through the intrinsic Hilbert metric. As a result, space, time, differentiation, and spectral decomposition emerge within a unified geometric framework.

2.6. Relationship to the Present Work. The existing literature establishes four major mathematical pillars: harmonic analysis through the Hilbert transform, geometric functional analysis through Hilbert spaces, generalized differentiation through weak derivative theory, and spectral decomposition through Fourier analysis. The principal contribution of the present work is to unify these four structures into a single intrinsic geometric framework.

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2.7. Existing Geometric Formulations of Wave and Physical Theories. The idea that physical laws may arise from geometric structures has a long history in mathematics and theoretical physics. Classical mechanics admits a geometric formulation through symplectic manifolds and Hamiltonian flows, where the dynamics are generated by geometric structures on phase space [4, 1]. Similarly, differential geometric approaches have played a central role in field theory and continuum mechanics.

The Hilbert space formulation of wave mechanics established by Hilbert, von Neumann, Dirac, and others introduced the notion that physical states may be represented by vectors in complex Hilbert spaces and observables by linear operators acting on these spaces [26, 12]. Subsequent developments led to geometric formulations of quantum theory, where the projective Hilbert space possesses a natural differential geometric structure [5, 7].

Geometric analysis has also provided numerous frameworks for studying wave equations and spectral phenomena. Spectral geometry investigates the relationship between geometric structures and spectra of differential operators [20, 8], while geometric partial differential equations study the interaction between curvature, topology, and wave propagation [25].

Several approaches have attempted to derive physical dynamics from spectral structures themselves. Notably, noncommutative geometry and spectral

action principles propose that geometric information may be encoded entirely in spectral data [9, 10]. Likewise, harmonic analysis on manifolds and representation theory have demonstrated that many geometric properties can be reconstructed from spectral decompositions [17].

The present work differs fundamentally from these approaches. Rather than assuming an underlying geometric manifold and subsequently constructing spectral operators, we begin with a Hilbert-transform-induced geometric embedding and derive metric, differential, and spectral structures simultaneously. In this formulation:

- (1) the metric arises from the Hilbert transform embedding;
- (2) the inner product is induced by the intrinsic complex geometry;
- (3) differentiation is defined geometrically through the intrinsic metric structure;
- (4) spectral decomposition emerges from the intrinsic geometry itself;
- (5) wave dynamics are formulated directly within the resulting geometric framework.

Consequently, the present theory reverses the conventional order of construction employed in classical wave theories, spectral theories, and geometric physical theories. Instead of geometry determining wave dynamics, the intrinsic wave geometry determines the geometric structure itself.

This viewpoint places the present framework within the broader program of geometric foundations of physical theories while providing a novel synthesis of harmonic analysis, functional analysis, spectral theory, and intrinsic geometry. ““

3. PRELIMINARIES

Let $(0, 1)^m$ be identified with the m -dimensional torus $\mathbb{T}^m$ equipped with the Lebesgue measure. Let

$$L^2(\mathbb{T}^m; \mathbb{C})$$

denote the space of square-integrable complex-valued functions.

For a real-valued function $u \in L^2(\mathbb{T}^m; \mathbb{R})$, denote by u_h its Hilbert transform on the torus, defined via Fourier multipliers. The Hilbert transform is a bounded linear operator on $L^2(\mathbb{T}^m)$ and satisfies

$$\langle u_h, v \rangle_{L^2} = -\langle u, v_h \rangle_{L^2} \quad \text{for all } u, v \in L^2(\mathbb{T}^m; \mathbb{R}).$$

The transform is extended complex-linearly to complex-valued functions.

4. DEFINITION OF THE METRIC SPACE

Let

$$f = u + iv, \quad g = p + iq,$$

with $u, v, p, q \in L^2(\mathbb{T}^m; \mathbb{R})$. Define

$$f_h := u_h + iv_h.$$

We introduce the linear operator

$$T : L^2(\mathbb{T}^m; \mathbb{C}) \rightarrow L^2(\mathbb{T}^m; \mathbb{C}), \quad T(f) := f + if_h.$$

Definition 4.1. The metric d on $L^2(\mathbb{T}^m; \mathbb{C})$ is defined by

$$d(f, g) := \|T(f) - T(g)\|_{L^2(\mathbb{T}^m)}.$$

Since T is linear and bounded, d is induced by a norm and therefore arises from an inner product.

5. DERIVATION OF THE INNER PRODUCT

Definition 5.1. The inner product associated with the metric d is defined by

$$\langle f, g \rangle_{\mathcal{H}(\mathcal{T}^m)} := \langle T(f), T(g) \rangle_{L^2} = \int_{\mathbb{T}^m} (f + if_h) \overline{(g + ig_h)} dx.$$

Expanding the integrand yields

$$(f + if_h) \overline{(g + ig_h)} = f\bar{g} + f_h\bar{g}_h + i(f\bar{g}_h - f_h\bar{g}).$$

Hence,

$$(1) \quad \langle f, g \rangle_{\mathcal{H}(\mathcal{T}^m)} = \int_{\mathbb{T}^m} (f\bar{g} + f_h\bar{g}_h) dx + i \int_{\mathbb{T}^m} (f\bar{g}_h - f_h\bar{g}) dx.$$

6. REAL-IMAGINARY DECOMPOSITION

Substituting $f = u + iv$ and $g = p + iq$, we obtain

$$\begin{aligned} \operatorname{Re} \langle f, g \rangle_{\mathcal{H}(\mathcal{T}^m)} &= \int_{\mathbb{T}^m} (up + vq + u_h p_h + v_h q_h) dx, \\ \operatorname{Im} \langle f, g \rangle_{\mathcal{H}(\mathcal{T}^m)} &= \int_{\mathbb{T}^m} (up_h - u_h p + vq_h - v_h q) dx. \end{aligned}$$

The real part defines a positive-definite quadratic form, while the imaginary part defines a skew-symmetric bilinear form.

7. HILBERT SPACE STRUCTURE

Theorem 7.1. The pair $(L^2(\mathbb{T}^m; \mathbb{C}), \langle \cdot, \cdot \rangle)$ equipped with the inner product (1) is a complex Hilbert space.

Proof. Sesquilinearity follows from the linearity of T and of the L^2 inner product. Positivity is immediate:

$$\langle f, f \rangle_{\mathcal{H}(\mathcal{T}^m)} = \|f + if_h\|_{L^2}^2 \geq 0,$$

with equality if and only if $f = 0$ almost everywhere. Completeness follows from the completeness of $L^2(\mathbb{T}^m)$ and boundedness of the Hilbert transform. $\square$

8. INTRINSIC DERIVATIVE DEFINED VIA THE COMPLEX HILBERT INNER PRODUCT

Let $\mathcal{H}$ denote the complex Hilbert space

$$\mathcal{H} := L^2(\mathbb{T}^m; \mathbb{C}), \quad \langle f, g \rangle_{\mathcal{H}} := \langle T(f), T(g) \rangle_{L^2}, \quad T(f) = f + if^h.$$

Definition 8.1 (Intrinsic derivative in $\mathcal{H}$). Let $f \in \mathcal{H}$. A function $g \in \mathcal{H}$ is called the Intrinsic derivative of f in the j -th coordinate direction, denoted $\partial_j^{\mathcal{H}} f$, if

$$(2) \quad \langle g, \varphi \rangle_{\mathcal{H}} = -\langle f, \partial_j \varphi \rangle_{\mathcal{H}} \quad \text{for all } \varphi \in C^\infty(\mathbb{T}^m).$$

This definition replaces the classical L^2 pairing by the intrinsic geometry of $\mathcal{H}$.

8.1. Identification of the Intrinsic Derivative. We recall that the intrinsic inner product on $L^2(\mathbb{T}^m; \mathbb{C})$ is defined by

$$\langle u, v \rangle_{\mathcal{H}} := \langle Tu, Tv \rangle_{L^2}, \quad T(f) = f + if^h,$$

where f^h denotes the Hilbert transform on $\mathbb{T}^m$.

Lemma 8.2 (Intrinsic derivative agrees with the classical weak derivative). *Let $f, g \in L^2(\mathbb{T}^m; \mathbb{C})$. Suppose that*

$$\langle g, \varphi \rangle_{\mathcal{H}} = -\langle f, \partial_j \varphi \rangle_{\mathcal{H}} \quad \text{for all } \varphi \in C^\infty(\mathbb{T}^m).$$

Then f possesses a classical weak derivative $\partial_j f \in L^2(\mathbb{T}^m)$ and

$$g = \partial_j f \quad \text{in the distributional sense.}$$

Proof. By definition of the intrinsic inner product,

$$\langle Tg, T\varphi \rangle_{L^2} = -\langle Tf, T(\partial_j \varphi) \rangle_{L^2}.$$

Since the Hilbert transform is a Fourier multiplier, it commutes with differentiation on smooth functions. Hence,

$$T(\partial_j \varphi) = \partial_j(\varphi + i\varphi^h) = \partial_j(T\varphi).$$

Therefore,

$$\langle Tg, T\varphi \rangle_{L^2} = -\langle Tf, \partial_j(T\varphi) \rangle_{L^2} \quad \forall \varphi \in C^\infty(\mathbb{T}^m).$$

As T is bounded and invertible on $L^2(\mathbb{T}^m)$, the set $\{T\varphi : \varphi \in C^\infty(\mathbb{T}^m)\}$ is dense in $L^2(\mathbb{T}^m)$. Thus the above identity is precisely the classical weak formulation of

$$\partial_j(Tf) = Tg.$$

Applying T^{-1} and using the commutation of T with ∂_j yields

$$g = \partial_j f$$

in the distributional sense. □

9. THE HILBERT-SPACE DERIVATIVE OPERATOR

Definition 9.1. Define the operator

$$D_j^{\mathcal{H}} : \mathcal{D}(D_j^{\mathcal{H}}) \subset \mathcal{H} \rightarrow \mathcal{H}, \quad D_j^{\mathcal{H}} f := \partial_j^{\mathcal{H}} f,$$

with domain

$$\mathcal{D}(D_j^{\mathcal{H}}) := \{f \in \mathcal{H} : \partial_j^{\mathcal{H}} f \in \mathcal{H}\}.$$

Proposition 9.2. $D_j^{\mathcal{H}}$ is densely defined and skew-adjoint on $\mathcal{H}$.

Proof. Density follows from $C^\infty(\mathbb{T}^m) \subset \mathcal{D}(D_j^{\mathcal{H}})$. For $f, g \in \mathcal{D}(D_j^{\mathcal{H}})$,

$$\langle D_j^{\mathcal{H}} f, g \rangle_{\mathcal{H}} = -\langle f, D_j^{\mathcal{H}} g \rangle_{\mathcal{H}},$$

by direct application of Definition (2). □

Theorem 9.3 (Eigenfunctions of the intrinsic derivative). *Let $f \in L^2(\mathbb{T}^m; \mathbb{C})$ and $\lambda \in \mathbb{C}$ satisfy*

$$\langle \lambda f, \varphi \rangle_{\mathcal{H}} = -\langle f, \partial_j \varphi \rangle_{\mathcal{H}} \quad \text{for all } \varphi \in C^\infty(\mathbb{T}^m).$$

Then there exists $k \in \mathbb{Z}^m$ such that

$$\lambda = 2\pi i k_j, \quad f(x) = e^{2\pi i k \cdot x} \quad (\text{up to scalar multiples}).$$

Proof. By Lemma 8.2, the intrinsic eigenvalue identity implies

$$\partial_j f = \lambda f \quad \text{in the classical weak sense.}$$

Fixing all variables except x_j , this equation reduces to the ordinary differential equation

$$\frac{\partial}{\partial x_j} f = \lambda f,$$

whose general solution is

$$f(x) = A(x_1, \dots, \widehat{x}_j, \dots, x_m) e^{\lambda x_j},$$

with A independent of x_j . Since f is defined on the torus $\mathbb{T}^m$, periodicity in the x_j variable implies

$$e^\lambda = 1,$$

hence $\lambda = 2\pi i k_j$ for some $k_j \in \mathbb{Z}$. Choosing A to be a periodic eigenfunction in the remaining variables yields

$$f(x) = e^{2\pi i k \cdot x}, \quad k \in \mathbb{Z}^m. \quad \square$$

Corollary 9.4 (Skew-adjointness of the intrinsic derivative). *The intrinsic derivative operator $\partial_j^{\mathcal{H}}$ is skew-adjoint on $L^2(\mathbb{T}^m; \mathbb{C})$ equipped with the intrinsic inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. That is,*

$$(\partial_j^{\mathcal{H}})^* = -\partial_j^{\mathcal{H}}.$$

Proof. By definition of $\partial_j^{\mathcal{H}}$, for all $f \in \text{Dom}(\partial_j^{\mathcal{H}})$ and $\varphi \in C^\infty(\mathbb{T}^m)$,

$$\langle \partial_j^{\mathcal{H}} f, \varphi \rangle_{\mathcal{H}} = -\langle f, \partial_j \varphi \rangle_{\mathcal{H}}.$$

Lemma 8.2 shows that $\partial_j^{\mathcal{H}}$ coincides with the classical weak derivative under the intrinsic duality. Since $C^\infty(\mathbb{T}^m)$ is dense in $L^2(\mathbb{T}^m)$ and the above identity extends by continuity to the full domain of $\partial_j^{\mathcal{H}}$, we obtain

$$\langle \partial_j^{\mathcal{H}} f, g \rangle_{\mathcal{H}} = -\langle f, \partial_j^{\mathcal{H}} g \rangle_{\mathcal{H}} \quad \forall f, g \in \text{Dom}(\partial_j^{\mathcal{H}}).$$

Thus $\partial_j^{\mathcal{H}}$ is skew-symmetric.

Moreover, by Theorem 9.3, the operator $\partial_j^{\mathcal{H}}$ admits a complete orthonormal basis of eigenfunctions $\{e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^m}$ with purely imaginary eigenvalues. Hence $\partial_j^{\mathcal{H}}$ is maximally skew-symmetric, and therefore skew-adjoint. $\square$

9.1. Intrinsic Spectral Theorem. We now formulate the spectral theorem for the intrinsic derivative operator purely in terms of the intrinsic Hilbert geometry.

Theorem 9.5 (Intrinsic spectral theorem for $\partial_j^{\mathcal{H}}$). *Let $\partial_j^{\mathcal{H}}$ denote the intrinsic derivative on $\mathcal{H} = L^2(\mathbb{T}^m; \mathbb{C})$ equipped with the intrinsic inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Then:*

- (1) $\partial_j^{\mathcal{H}}$ is skew-adjoint.
- (2) Its spectrum is purely imaginary and discrete:

$$\sigma(\partial_j^{\mathcal{H}}) = \{2\pi i k_j : k \in \mathbb{Z}^m\}.$$

- (3) There exists a complete orthonormal basis $\{\varphi_k\}_{k \in \mathbb{Z}^m}$ of $\mathcal{H}$ such that

$$\partial_j^{\mathcal{H}} \varphi_k = 2\pi i k_j \varphi_k.$$

- (4) For every $f \in \mathcal{H}$,

$$f = \sum_{k \in \mathbb{Z}^m} \langle f, \varphi_k \rangle_{\mathcal{H}} \varphi_k,$$

with convergence in the $\mathcal{H}$ -norm.

Proof. Skew-adjointness was established in Corollary 9.4. By Theorem 9.3, the eigenfunctions $\varphi_k(x) = e^{2\pi i k \cdot x}$ form a complete orthonormal system in $\mathcal{H}$ with eigenvalues $2\pi i k_j$. Since $\partial_j^{\mathcal{H}}$ is skew-adjoint and admits a complete orthonormal set of eigenfunctions, the operator is unitarily diagonalizable. The expansion formula follows from completeness. $\square$

10. SUMMARY

We have constructed a complex Hilbert space in which the Hilbert transform is intrinsically encoded into the metric and inner product. The resulting geometry naturally decomposes into an energy component and a symplectic component. This framework allows the Hilbert transform to act not as an auxiliary operator, but as a defining geometric feature of the space itself. Defining differentiation via the intrinsic Hilbert-space inner product

ensures that spectral differentiation, orthogonality, and completeness are all governed by the same geometry. The Hilbert transform is part of the definition of differentiability itself, yielding a derivative operator that is simultaneously weak, spectral, and geometric.

Chapter 2. Intrinsic Geometric Wave Mechanics

11. INTRODUCTION

11.1. Space Time.

Definition 11.1. Spacetime Space time is defined as $\Omega = (0, 1)^3 \times (0, 1)$

11.2. Wavefunction.

Definition 11.2. wavefunctionIn the presence of scalar and vector potentials V and A , associated with every $\gamma \in \mathbb{R}, \gamma > 0$ called a wave number, there exists a wavefunction $\psi \in \mathcal{H}(\Omega)$ such that

$$\|\psi\|_{\mathcal{H}(\Omega)} = 1$$

12. THE SPACE TIME WAVE EQUATION

We consider the case of scalar and vector potentials varying both in space and time.

Assumption 12.1. The wave function in space time evolves such that the variation of its total energy is equal to i times the variation of square of the norm of the intrinsic time decay of the wave function

$$\left\| \frac{\partial_{\mathcal{H}} \psi}{\partial t} \right\|_{\mathcal{H}(\Omega)}$$

under the constraint of unit norm of the wave function.

Total energy is given as

$$(3) \quad E(\psi) = \alpha \|\nabla_{\mathcal{H}}^{st} \psi\|_{\mathcal{H}(\Omega)}^2 + \|\sqrt{\bar{V}} \psi\|_{\mathcal{H}(\Omega)}^2 + \|\sqrt{\bar{A}} \cdot \nabla_{\mathcal{H}}^s \psi\|_{\mathcal{H}(\Omega)}^2$$

where

$$(4) \quad \bar{V} = V + \partial_t^{\mathcal{H}} \nabla_{\mathcal{H}}^s \cdot \vec{A}_h$$

$\alpha = \frac{1}{2\gamma}$, $\vec{A}_h$ is Hilbert transform of vector potential in radial direction. $\nabla_{\mathcal{H}}^{st}$ is gradient in space time coordinates and $\nabla_{\mathcal{H}}^s$ is gradient in spatial coordinates only. $\nabla^s \cdot$ is divergence in spatial coordinates. Deriving the Euler Lagrange equation for the constrained minimization problem in $\mathcal{H}(\Omega)$ we get the space time wave equations given below

(5)

$$\boxed{\frac{1}{2\gamma} \langle \Delta_{\mathcal{H}} \psi, \phi \rangle_{\mathcal{H}(\Omega)} + \langle \bar{V} \psi, \phi \rangle_{\mathcal{H}(\Omega)} + \sum_{i=1}^3 \langle A_i \partial_i \psi, \phi \rangle_{\mathcal{H}(\Omega)} + \lambda \langle \psi, \phi \rangle_{\mathcal{H}(\Omega)} = 0} \quad \forall \phi \in C^\infty \cap \mathcal{H}(\Omega)$$

(6)

$$\|\psi\|_{\mathcal{H}(\Omega)} = 1$$

12.1. Solution to space time wave equations.

Theorem 12.2 (Existence and uniqueness of solutions to space time wave equations). *The solution to the space time wave equation exists and is unique.*

Proof. Now taking the intrinsic Fourier series, choosing $\phi = e^{i2\pi k \cdot x}, k \in \mathbb{Z}$ for different k and writing the matrix equation for using Parseval relation for dot product in equation 5 we get

$$(7) \quad (\alpha(D_t^{H2} + D_x^{H2} + D_y^{H2} + D_z^{H2}) + V_E + (B_{A_x}D_x^H + B_{A_y}D_y^H + B_{A_z}D_z^H) - \lambda I)X = 0$$

D_t^{H2} is a multiplier for intrinsic Fourier series for intrinsic double derivative in time direction operator and similarly others. V_E is the Toeplitz matrix whose intrinsic complex Hilbert geometric generating symbol is $\bar{V}$. Let

$$M = \alpha(D_t^{H2} + D_x^{H2} + D_y^{H2} + D_z^{H2}) + V_E + (B_{A_x}D_x^H + B_{A_y}D_y^H + B_{A_z}D_z^H)$$

and

$$(8) \quad G = (\alpha(D_t^{H2} + D_x^{H2} + D_y^{H2} + D_z^{H2}) + V_E + (B_{A_x}D_x^H + B_{A_y}D_y^H + B_{A_z}D_z^H) - \lambda I)$$

In a 4-dimensional space time, the intrinsic Fourier series coefficients matrix X is 4 dimensional and the matrices M and G are 2^4 dimensional. One can see that the matrix M is a full rank matrix. Choose λ to be the lowest eigenvalue(in magnitude) of the matrix M . So the matrix G has null space of dimension 1. Now the solution X is eigenvector with zero eigenvalue of the matrix G . The eigenvector is chosen to have unit norm. Hence the solution exists and is unique. $\square$

Taking the inverse intrinsic Fourier series of X gives the solution wave function ψ .

12.2. Regularity of the solution to the space time wave equations.

Theorem 12.3 (Regularity of solution to space-time wave equations). *Let $\psi \in \mathcal{H}(\Omega)$ be a normalized solution of the intrinsic space-time wave equations 5, 6. Let $\hat{\psi}(k, \ell)$ denote the intrinsic Fourier coefficients of ψ with respect to spatial frequencies $k \in \mathbb{Z}^3$ and temporal frequency $\ell \in \mathbb{Z}$.*

Assume that the spatial potential V has zero mean and satisfies

$$|\hat{V}(k)| = \Theta(1).$$

Assume that the vector potential $A = (A_1, A_2, A_3)$ satisfies

$$|\hat{A}_j(k, \ell)| = O(1/\|k\|_2), \quad j = 1, 2, 3,$$

and has zero mean. Also assume both $\vec{A}$ and V are non-analytic in any sub-interval of Ω .

Then the following joint space-time regularity holds:

(1) If α is large, then

$$\widehat{\psi}(k, \ell) = \Theta\left(\frac{1}{\sqrt{\|k\|_2^2 + \|\ell\|_2^2}}\right).$$

(2) If α is small, then

$$\widehat{\psi}(k, \ell) = \Theta\left(\frac{1}{(\|k\|_2^2 + \|\ell\|_2^2)^{1/4}}\right).$$

In particular, for large α the solution ψ belongs to the intrinsic Sobolev space

$$\psi \in H_{\text{loc}}^1(\Omega),$$

while for small α it belongs to

$$\psi \in H_{\text{loc}}^{1/2}(\Omega),$$

with regularity measured intrinsically in both space and time.

Proof. Case 1: α is large

The solution is also the eigenvector corresponding to the zero eigenvalue of the matrix G . Let the eigenvector be $\mathbf{e} = [e_1, e_2, \dots, e_n]$.

From equations 7 and 8

$$(9) \quad G\mathbf{e} = 0$$

Let rows of G be denoted as $G_1, G_2, \dots, G_n$

$$(10) \quad \sum_{i=1}^n G_i \mathbf{e} = 0$$

Taking the n th row,

$$(11) \quad \sum_{i=1}^n G_{n,i} \mathbf{e}_i = 0$$

which implies

$$(12) \quad G_{n,n} \mathbf{e}_n = - \sum_{i=1}^{n-1} G_{n,i} \mathbf{e}_i$$

$$(13) \quad \mathbf{e}_n = - \frac{\sum_{i=1}^{n-1} G_{n,i} \mathbf{e}_i}{G_{n,n}}$$

As the generating symbol of matrix $\bar{G}_{n,n}$ is V whose Fourier coefficients decay as $\Theta(1)$, the non principal diagonal entries of $\bar{G}_{n,n}$ are $\Theta(1)$. Since $G_{n,n}$ is obtained from $\bar{G}_{n,n}$ by modifying only the leading diagonal, same can be said about $G_{n,n}$ as well. Hence

$$(14) \quad G_{n,i} = \Theta(1), i = 1, 2, \dots, n-1.$$

As eigenvector is normalized,

$$(15) \quad e_i = O(1), i = 1, 2, \dots, n.$$

Applying Cauchy Schwartz inequality in equation 13

$$(16) \quad e_n \leq -K \frac{\sqrt{\sum_{i=1}^{n-1} G_{n,i}^2} \sqrt{\sum_{i=1}^{n-1} e_i^2}}{G_{n,n}}$$

$$(17) \quad e_n \geq -K \frac{\sqrt{\sum_{i=1}^{n-1} G_{n,i}} \sqrt{\sum_{i=1}^{n-1} e_i}}{G_{n,n}}$$

where $K > 0$ is independent of n .

From equation 14 and also as eigenvectors being normalized that $e_i = O(1), i = 1, 2, \dots, n$ we have

$$(18) \quad e_n = K \frac{\sqrt{\Theta(n)\Theta(n)}}{G_{n,n}}$$

$$(19) \quad G_{n,n} = \alpha(n-1)^2 + \lambda$$

As the matrix $G_{n \times n}$ is Hermitian Hankel matrix with coefficients in non leading diagonal going as $\Theta(1)$, its lowest eigenvalue can safely be assumed to grow as $O(1)$ as $n \rightarrow \infty$. Hence $\lambda = O(1)$.

So

$$(20) \quad G_{n,n} = \alpha\Theta(n^2) + O(1)$$

using equations 18 and 20 we have

$$(21) \quad e_n = \frac{\Theta(n)}{\alpha\Theta(n^2) + O(1)}$$

$\mathbf{n} = [k, \ell]$ is the multi-index of space-time frequency, where $k = (k_x, k_y, k_z)$ are spatial frequency indices and ℓ is temporal frequency index. which implies

$$(22) \quad \hat{\psi}(k, \ell) = \frac{\Theta(\sqrt{\|k\|_2^2 + \|\ell\|_2^2})}{\alpha\Theta(\sqrt{\|k\|_2^2 + \|\ell\|_2^2}) + O(1)}$$

hence when α is large we have

$$\hat{\psi}(k, \ell) = \Theta\left(\frac{1}{\sqrt{\|k\|_2^2 + \|\ell\|_2^2}}\right)$$

Case 2 : α is small.

Noting the regularity of the vector potential $\vec{A}$ given in 12.3 the matrix $(B_{A_x}D_x^H + B_{A_y}D_y^H + B_{A_z}D_z^H)$ can be asymptotically ignored from the following analysis as it can be taken care of due to Kato's perturbation theory[27].

Let the matrix $P = D_t^{H2} + D_x^{H2} + D_y^{H2} + D_z^{H2}$ and $L = V_E$

$$(23) \quad G(\alpha) = L + \alpha P - (\lambda(\alpha))I_{n \times n}$$

where $L = G(0) + \lambda(0)I$, minimum eigenvalue of L is 0. P is a diagonal matrix. Sample the $\bar{V}$ to discrete uniformly spaced samples. We get all matrices in discrete domain. We get two Hermetian Toeplitz matrices A and B , where A is discrete version of L in frequency domain, and B is discrete version in space domain. A contains eigenvalues of B and B contains eigenvalues of A . Both A and B become infinite matrices as samples go to infinity. Let Z denote the DFT matrix, which is also the eigenvector matrix of A and Z^T the eigenvector matrix of B . Now consider κ_0 as the zero eigenvector of A and β_0 as the zero eigenvector of B and form Toeplitz matrices $\bar{A}$ and $\bar{B}$ from κ_0 and β_0 respectively to get the matrices $\bar{A}$ and $\bar{B}$.

We now have

$$(24) \quad \bar{A}Z^T = B\bar{B}^T Z = A$$

we also have

$$(25) \quad BZ = AZA = BZAZ^{-1} = A$$

Simplification from equations 24 and 25

$$\bar{A}Z^T Z = A$$

we know

$$Z^T Z = I$$

So

$$\bar{A} = A$$

As mean value of potential is assumed to be zero, we know that the minimum in A is achieved within matrix boundary and hence the same case in $\bar{A}$. So $\kappa_{0n}^{(0)} < c$ where $c > 0$ is independent of n .

Now consider $\lambda(\alpha)$ which is the lowest (in magnitude) eigenvalue of the matrix $L + \alpha P$. From Kato's [27] perturbation theory, for small α , $\lambda(\alpha)$ is small. Consider the matrix G as the perturbation of L with $\alpha P + \lambda(\alpha)I$, for small α , that is for $\alpha < \alpha_0$ where α_0 is some critical value, the perturbation in the eigenvectors matrix and eigenvalues is small. Hence the perturbation in all the corresponding matrices is small which implies the minimum in the perturbed $\bar{A}$ matrix is achieved within matrix boundary, there by $\kappa^{(\alpha)0n} < c/2$ where c is independent of n . Hence

$$(26) \quad \kappa_{0n}^{(\alpha)} = \Theta(1)$$

as they are bounded above as well as it is an eigenvector. Now $\kappa_0^{(\alpha)}$ are the intrinsic Fourier series coefficients of the solution to equation 5 and we need

to make it unit norm in order to make it a solution to equation 6 as well, that is to the space time wave equations. Assuming $\widehat{\Psi}$ as discrete version of $\widehat{\psi}$, Normalizing $\kappa_{0n}^{(\alpha)}$ we get,

$$(27) \quad \widehat{\Psi} = \frac{\kappa_0^{(\alpha)}}{\sqrt{(\sum_{i=1}^n (\kappa_{0i}^{(\alpha)})^2)}}$$

Using equations 26 and 27 we get the asymptotic

$$(28) \quad \widehat{\psi}_n = \Theta\left(\frac{1}{\sqrt{n}}\right) \implies \widehat{\psi}(k, \ell) = \Theta\left(\frac{1}{(\|k\|_2^2 + \|\ell\|_2^2)^{1/4}}\right)$$

hence when α is small we have the intrinsic Fourier coefficients $\widehat{\psi}$ of the solution to the space time wave equation decay as $\Theta\left(\frac{1}{(\|k\|_2^2 + \|\ell\|_2^2)^{1/4}}\right)$.

$$\widehat{\psi}(k, \ell) = \Theta\left(\frac{1}{(\|k\|_2^2 + \|\ell\|_2^2)^{1/4}}\right)$$

when α is small. □

Note. that this result on the case of small α holds for scalar and vector potentials that are $O(1)$ and $O(1/\|k\|_2)$ respectively as well.

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I have used chatGPT model GPT-5 extensively throughout the manuscript. I have verified all statements and proofs.

14. DATA AVAILABILITY STATEMENT

We declare that no data has been used in this document and none has been made available.

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