

MEASURABLE INDEPENDENCE DENSITY EQUALS THE FINITE INDEPENDENCE-RATIO INFIMUM IN THE EUCLIDEAN PLANE

ÁKOS DÚCZ

ABSTRACT. Let $m_1(\mathbb{R}^2)$ be the supremum of upper densities of Lebesgue-measurable subsets of the plane containing no pair of points at distance one. We show, using the spectral rigidity theorem in OpenAI's recent proof that the plane is not five-colorable, that

$$m_1(\mathbb{R}^2) = \inf_G \frac{\alpha(G)}{|V(G)|},$$

where G ranges over nonempty finite unit-distance graphs in the plane. The argument constructs an isometry-invariant law on independent subsets of the countable algebraic plane, projects the occupancy indicator onto the continuous spectral factor without changing its expectation, and extracts a measurable independent set with arbitrarily small density loss. As a consequence, if $f(n)$ is the least independence number among unit-distance graphs on n vertices, then $f(n)/n \rightarrow m_1(\mathbb{R}^2)$. The proof is nonquantitative and uses the cited spectral rigidity result as a black box.

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1. INTRODUCTION

For a Lebesgue-measurable set $A \subseteq \mathbb{R}^2$, write

$$\bar{\delta}(A) = \limsup_{R \rightarrow \infty} \frac{|A \cap [-R, R]^2|}{(2R)^2},$$

where $|\cdot|$ denotes planar Lebesgue measure. A set is *unit-avoiding* if it contains no two points at Euclidean distance one. Define

$$m_1(\mathbb{R}^2) = \sup\{\bar{\delta}(A) : A \subseteq \mathbb{R}^2 \text{ is measurable and unit-avoiding}\}.$$

A finite unit-distance graph is the graph induced by a finite point set in $\mathbb{R}^2$, with an edge between each pair at distance one. Let $\alpha(G)$ be its independence number, and put

$$\alpha_{\text{fin}} = \inf \left\{ \frac{\alpha(G)}{|V(G)|} : G \text{ is a nonempty finite unit-distance graph in } \mathbb{R}^2 \right\}.$$

A standard averaging inequality gives $m_1(\mathbb{R}^2) \leq \alpha_{\text{fin}}$; see, for example, [1, 2]. Our result supplies the reverse inequality.

Theorem 1.1. *Using the spectral rigidity theorem of [4, Theorem 2.3],*

$$m_1(\mathbb{R}^2) = \alpha_{\text{fin}}.$$

Date: October 2026.

2020 Mathematics Subject Classification. Primary 52C10; Secondary 05C15, 05C72.

Key words and phrases. Unit-distance graphs, independence ratio, measurable sets, spectral rigidity, invariant random independent sets.

Moreover, for every $\varepsilon > 0$ there is a measurable, periodic, strictly unit-avoiding set $A \subseteq \mathbb{R}^2$ of density at least $\alpha_{\text{fin}} - \varepsilon$.

The input from [4] was developed to pass from arbitrary proper colorings to weak measurable colorings. Here the constraints concern only pairs of occupied sites, rather than all color classes. The additional point needed for the density problem is that projection onto the continuous factor is a *conditional expectation*; it preserves the one-site occupancy probability. The spectral rigidity theorem then preserves the forbidden-pair correlations, and positivity allows us to pass from a real-valued field to its support.

The equality identifies the asymptotic finite independence ratio with the measurable independence density, a relation conjectured in [2, Conjecture 1]. Previous work using geometric fractional colorings and averaging, including [1, 3, 2], provides context but does not by itself imply this equality. Our proof depends essentially on the nontrivial spectral statement of [4]; we neither reprove nor independently verify that theorem here.

2. FINITE CONFIGURATIONS AND AN INVARIANT LAW

We first record the easy inequality and then construct a probability law on independent sets that retains the finite-graph bound.

Lemma 2.1. $m_1(\mathbb{R}^2) \leq \alpha_{\text{fin}}$.

Proof. Fix a finite unit-distance graph with vertex set $\{v_1, \dots, v_n\} \subset \mathbb{R}^2$, and a measurable unit-avoiding set A . For every $x \in \mathbb{R}^2$,

$$\sum_{j=1}^n \mathbf{1}_A(x + v_j) \leq \alpha(G).$$

Integrate over $Q_R = [-R, R]^2$. For each fixed v_j , the difference between $\int_{Q_R} \mathbf{1}_A(x + v_j) dx$ and $|A \cap Q_R|$ is $O_G(R)$, since the symmetric difference of Q_R and $Q_R + v_j$ has area $O_G(R)$. Thus

$$n|A \cap Q_R| \leq \alpha(G)|Q_R| + O_G(R).$$

Divide by $|Q_R|$, take the limit superior, and then the infimum over G . $\square$

Set

$$F = \overline{\mathbb{Q}} \cap \mathbb{R}, \quad E = F + iF, \quad K = \{u \in E : |u| = 1\}.$$

Here E is a countable dense additive subgroup of $\mathbb{R}^2 \cong \mathbb{C}$, and K is its countable group of algebraic unit rotations. As in [4], the semidirect product $E \rtimes K$ acts on configurations indexed by E . Denote the compact metrizable configuration space of independent sets by

$$\Omega = \left\{ \eta \in \{0, 1\}^E : \eta(x)\eta(y) = 0 \text{ whenever } |x - y| = 1 \right\}.$$

The action of translations and rotations preserves Ω .

Lemma 2.2. *There exists an $E \rtimes K$ -invariant Borel probability measure μ on Ω such that*

$$q := \int_{\Omega} \eta(0) d\mu(\eta) \geq \alpha_{\text{fin}}.$$

Proof. Choose a Følner sequence $(F_n)_{n \geq 1}$ of finite nonempty subsets of the discrete abelian group $(E, +)$. For each n , the induced unit-distance graph on F_n has an independent set $I_n \subseteq F_n$ with

$$|I_n| \geq \alpha_{\text{fin}}|F_n|.$$

Let $\eta_n \in \Omega$ indicate I_n , extended by zero to all of E . Write $\tau_t \eta(x) = \eta(x + t)$ and set

$$\nu_n = \frac{1}{|F_n|} \sum_{t \in F_n} \delta_{\tau_t \eta_n}.$$

Then

$$\int \eta(0) d\nu_n = \frac{|I_n|}{|F_n|} \geq \alpha_{\text{fin}}.$$

For every fixed $z \in E$, the total-variation distance between $(\tau_z)_* \nu_n$ and ν_n is at most $|F_n \triangle (F_n + z)|/|F_n|$, which tends to zero. Compactness of Ω gives a weak subsequential limit ν that is translation invariant and has origin marginal at least α_{fin} ; evaluation $\eta \mapsto \eta(0)$ is continuous.

Finally, average ν over a Følner sequence in the abelian group K . Each rotated measure remains translation invariant, since rotations conjugate translations to translations. A weak limit μ is therefore invariant under both E and K . All rotations fix the origin, so every measure in this last averaging process has exactly the same origin marginal as ν . $\square$

Remark 2.3. We do not need to know whether every finite unit-distance graph has an algebraic-coordinate realization. The inequality $\alpha(G[F_n])/|F_n| \geq \alpha_{\text{fin}}$ holds simply because F_n is itself a finite point configuration in the plane.

3. A DENSITY-PRESERVING SPECTRAL TRANSFER

We isolate the only deep input. Let T_z be the Koopman translation operators associated with $(E, +)$ on $L^2(\mu)$. The rotation operators fix $f(\eta) = \eta(0)$ and conjugate translations to rotated translations. The space of L^2 -continuous vectors is

$$\mathcal{H}_c = \left\{ v \in L^2(\mu) : \lim_{\substack{z \rightarrow 0 \\ z \in E}} \|T_z v - v\|_2 = 0 \right\}.$$

By [4, Lemma 4.1], $\mathcal{H}_c = L^2(\mathcal{F}_c)$ for an invariant probability factor $\mathcal{F}_c$, and the restricted translation representation extends strongly continuously from E to all $x \in \mathbb{R}^2$. Its orthogonal projection is conditional expectation onto $\mathcal{F}_c$. This lemma applies to any probability-preserving E -action, not only to a coloring system.

Proposition 3.1 (One-coordinate transfer). *Let μ be any $E \rtimes K$ -invariant probability measure on Ω , and let $q = \int \eta(0) d\mu$. Subject to [4, Theorem 2.3], there is a jointly measurable function*

$$H : \mathbb{R}^2 \times \Omega \longrightarrow [0, 1]$$

such that for almost every $x \in \mathbb{R}^2$,

$$(3.1) \quad \int_{\Omega} H(x, \eta) d\mu(\eta) = q,$$

and for every positive integer N ,

$$(3.2) \quad \int_{\Omega} \int_{Q_N} \int_{S^1} H(x, \eta) H(x + u, \eta) d\sigma(u) dx d\mu(\eta) = 0,$$

where σ is normalized arc-length measure on the unit circle.

Proof. Let $f(\eta) = \eta(0)$, and let

$$p = \mathbb{E}(f \mid \mathcal{F}_c), \quad b = f - p.$$

Conditional expectation yields $0 \leq p \leq 1$ and $\int p d\mu = q$. Both p and b are rotation-fixed, because f is and $\mathcal{F}_c$ is rotation invariant.

The translation spectral measure of b is supported outside the Borel subgroup of ordinary continuous characters of the discrete additive group E ; this is the spectral characterization of $\mathcal{H}_c$ in [4, Lemma 4.1]. Rotation-fixity makes this spectral measure K -invariant. The wild-character rigidity theorem [4, Theorem 2.3] therefore implies that it is a nonnegative scalar multiple of Haar measure on the compact dual group of E . It follows that

$$\langle b, T_z b \rangle = 0 \quad (z \in E \setminus \{0\}).$$

The continuous subspace and its orthogonal complement are translation invariant, so their translation cross terms vanish. Independence of every configuration in Ω gives, for all $u \in K$,

$$0 = \langle f, T_u f \rangle = \langle p, T_u p \rangle + \langle b, T_u b \rangle = \langle p, T_u p \rangle.$$

The group K is dense in S^1 ; using the strong continuity of the extended translations on $\mathcal{H}_c$ gives

$$(3.3) \quad \langle p, T_u p \rangle = 0 \quad (u \in S^1).$$

Moreover, $0 \leq T_x p \leq 1$ as an element of $L^2(\mu)$ for every $x \in \mathbb{R}^2$: the extended operators are strong limits of positive unital translation operators, as proved in [4, Lemma 4.1].

The strongly continuous map $x \mapsto T_x p$ from $\mathbb{R}^2$ to the separable Hilbert space $L^2(\mu)$ admits jointly measurable representatives. For completeness, on each bounded square approximate this map in the product L^2 norm by finite-valued measurable maps to $L^2(\mu)$; after choosing measurable representatives of their finitely many values, a subsequence converges almost everywhere on the product space. Exhausting $\mathbb{R}^2$ by squares and clamping the resulting representative to $[0, 1]$ produces a jointly measurable field $H(x, \eta)$ representing $T_x p$ for almost every x . This is the measurable-field construction of [4, Proposition 4.2].

Since the extended operators preserve the constant function, $\int T_x p d\mu = \int p d\mu = q$, proving (3.1). For every $x \in \mathbb{R}^2$ and $u \in S^1$, the L^2 inner product of $T_x p$ and $T_{x+u} p$ is zero by (3.3). For each fixed u , outside a null set of x , the field represents both these vectors; hence

$$\int H(x, \eta) H(x + u, \eta) d\mu(\eta) = 0$$

for almost every x . Fubini in u and then Tonelli give (3.2); the integrands are nonnegative. $\square$

4. ELIMINATING EXCEPTIONAL PAIRS AND OBTAINING PERIODIC DENSITY

A measurable set $B \subseteq \mathbb{R}^2$ will be called *weakly unit-avoiding* if

$$\int_{Q_N} \int_{S^1} \mathbf{1}_B(x) \mathbf{1}_B(x + u) d\sigma(u) dx = 0 \quad \text{for every integer } N \geq 1.$$

Lemma 4.1 (Density-point properness). *If $B \subseteq \mathbb{R}^2$ is measurable and weakly unit-avoiding, then the set B^* of Lebesgue density-one points of B is measurable, differs from B by a null set, and contains no pair of points at distance one.*

Proof. The first assertions are the Lebesgue density theorem. Suppose that $x, y \in B^*$ and $y = x + u_0$ with $|u_0| = 1$. For small $h > 0$, let Z be uniform in the disk $B(x, h)$, and independently let U be uniform in an arc of the unit circle of positive length contained in $\{u : |u - u_0| < h\}$. Since x is a density-one point,

$$\Pr(Z \notin B) = o(1) \quad (h \downarrow 0).$$

For each fixed U , the variable $Z + U$ is uniform in a disk of radius h . Consequently its mixture distribution has density at most $1/(\pi h^2)$ and is supported on $B(y, 2h)$. Since y is a density-one

point,

$$\Pr(Z + U \notin B) \leq \frac{|B(y, 2h) \setminus B|}{\pi h^2} = o(1).$$

For sufficiently small h there is positive probability that both Z and $Z + U$ belong to B . But the joint distribution of (Z, U) is absolutely continuous with respect to $dx d\sigma(u)$ on a bounded region, contradicting weak unit avoidance. $\square$

Proposition 4.2. *If μ is as in Proposition 3.1 with origin marginal q , then for every $\varepsilon > 0$ there exists a measurable, periodic, strictly unit-avoiding $A \subseteq \mathbb{R}^2$ satisfying $\delta(A) \geq q - \varepsilon$.*

Proof. Apply Proposition 3.1 and, for each sample η , set

$$B_\eta = \{x \in \mathbb{R}^2 : H(x, \eta) > 0\}.$$

By nonnegativity and (3.2), almost every B_η is weakly unit-avoiding: indeed $H(x, \eta)H(x + u, \eta) = 0$ for almost every (x, u) , simultaneously on all Q_N . Thus the density-point set B_η^* is strictly unit-avoiding and has the same measure as B_η in every bounded measurable region.

For each fixed $R > 0$, since $\mathbf{1}_{B_\eta}(x) \geq H(x, \eta)$ and (3.1) holds,

$$\int_\Omega |B_\eta \cap Q_R| d\mu(\eta) \geq \int_{Q_R} \int_\Omega H(x, \eta) d\mu(\eta) dx = q|Q_R|.$$

Given $\gamma > 0$, choose a sample from the conull set of weakly unit-avoiding samples with

$$|B_\eta^* \cap Q_R| \geq (q - \gamma)|Q_R|.$$

Set $C = B_\eta^* \cap Q_R$, $L = 2R + 2$, and repeat this truncated set periodically:

$$A = \bigcup_{z \in L\mathbb{Z}^2} (C + z).$$

Within each cell it is unit-avoiding; distinct copies lie at distance at least 2 from each other, so A is strictly unit-avoiding. Its periodic density satisfies

$$\delta(A) = \frac{|C|}{L^2} \geq (q - \gamma) \left(\frac{2R}{2R + 2} \right)^2.$$

First choose R sufficiently large and then $\gamma > 0$ sufficiently small that the right side is at least $q - \varepsilon$. (When $q \leq \varepsilon$ the empty set suffices.) $\square$

Proof of Theorem 1.1. By Lemma 2.2, there is an isometry-invariant law on independent subsets of E with one-site marginal $q \geq \alpha_{\text{fin}}$. Proposition 4.2 gives periodic measurable unit-avoiding subsets of $\mathbb{R}^2$ with densities arbitrarily close to q . Hence $m_1(\mathbb{R}^2) \geq q \geq \alpha_{\text{fin}}$. Lemma 2.1 gives the reverse inequality. $\square$

5. CONSEQUENCES AND LIMITATIONS

For $n \geq 1$ let

$$f(n) = \min\{\alpha(G) : G \text{ is a unit-distance graph on } n \text{ distinct planar points}\}.$$

Corollary 5.1. *The first assertion of [2, Conjecture 1] follows from Theorem 1.1:*

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = m_1(\mathbb{R}^2).$$

Proof. By definition, $f(n)/n \geq \alpha_{\text{fin}}$. Fix $\gamma > 0$, and choose a finite unit-distance graph G on s vertices with $\alpha(G)/s < \alpha_{\text{fin}} + \gamma$. For $n = ks + r$, $0 \leq r < s$, take k sufficiently separated translated copies of G , together with r isolated points separated from all copies. Then

$$f(n) \leq k\alpha(G) + r,$$

and so $\limsup_{n \rightarrow \infty} f(n)/n \leq \alpha(G)/s < \alpha_{\text{fin}} + \gamma$. Let $\gamma \downarrow 0$ and apply Theorem 1.1. $\square$

Corollary 5.2. *For every $\varepsilon > 0$ there exists a finite planar unit-distance graph G with*

$$\frac{\alpha(G)}{|V(G)|} < 0.2470 + \varepsilon.$$

Proof. Ambrus, Csiszárík, Matolcsi, Varga, and Zsámboki proved $m_1(\mathbb{R}^2) \leq 0.2470$ [1]. Apply Theorem 1.1 and the definition of the infimum. $\square$

Remark 5.3. The argument does not identify the common constant, prove that its supremum or infimum is attained, or bound the size of a finite graph witnessing a given approximation. In particular, it does not establish the separate conjecture that $m_1(\mathbb{R}^2)$ equals the density of Croft's construction. The reduction is conditional on the spectral rigidity input [4, Theorem 2.3], rather than an independent proof of that difficult result.

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