

The number of multiplicity sets of n -point plane line collections grows as $\exp(\Theta(\sqrt{n}))$

(A solution to Erdős Problem 607, erdosproblems.com/607)

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AI-assisted proof, prepared for human verification. CPU cross-checks and formalization details in the accompanying supplementary note.

Problem (Erdős 607)

For a set P of n points in the plane $\mathbb{R}^2$, let

$$A(P) = \{ |\ell \cap P| : \ell \text{ a line, } |\ell \cap P| \geq 2 \}$$

be the set of *multiplicities* of P (sizes of subsets of P lying on a common line), and let $F(n)$ be the number of distinct sets A that can be obtained this way as P ranges over all n -point sets. Erdős asked whether $F(n) \leq \exp(O(\sqrt{n}))$, remarking that this is easy to see to be best possible.

Theorem 1. *There is an absolute constant $c > 0$ such that, for all n ,*

$$F(n) \leq \exp(c\sqrt{n}).$$

Combined with the lower bound of Szemerédi–Trotter [2], the $\sqrt{n}$ scale of $F(n)$ is exact: $F(n) = \exp(\Theta(\sqrt{n}))$.

Proof. The proof is a three-step reduction to a published result of Colbourn–Phelps–Rödl [1].

Step 1 (the lines of P form a PBD). Let $V = P$ and let $\mathcal{B} = \{ \ell \cap P : \ell \text{ a line, } |\ell \cap P| \geq 2 \}$. Take two distinct points $x, y \in P$. They span a unique line xy ; the block $xy \cap P \in \mathcal{B}$ contains both x and y , and it is the *only* such block, since two distinct lines meet in at most one point. Hence every unordered pair of V lies in exactly one block: $(V, \mathcal{B})$ is a *pairwise balanced design of order n* , in exactly the sense used in [1, Definition] (“a design on an n -set V in which every unordered pair of elements appears in precisely one block”).

Step 2 (the profile set is $A(P)$). By construction, the set of block sizes of $(V, \mathcal{B})$ is exactly $A(P)$: its members are the values $|\ell \cap P| \geq 2$ as ℓ ranges over all lines, and every block is such an intersection.

Step 3 (apply the published bound). Let $g(n)$ denote the number of distinct *profile sets* (sets of block sizes) of n -element PBDs. Theorem 2.4 of [1] states that for some constants $c_1, c_2 > 0$,

$$\exp(c_1\sqrt{n}) < g(n) < \exp(c_2\sqrt{n}).$$

The map $P \mapsto (V, \mathcal{B})$ sends the multiplicity sets of n -point sets into the profile sets of order- n PBDs. Since $F(n)$ counts distinct images of this map,

$$F(n) \leq g(n) < \exp(c_2\sqrt{n}).$$

□

The matching lower bound. The lower bound $F(n) \geq \exp(c'\sqrt{n})$ is the geometric (grid) construction of Szemerédi–Trotter [2]: the $k \times k$ grid on $n = k^2$ points has $\Theta(k) = \Theta(\sqrt{n})$ distinct line multiplicities with multiplicities up to k , and the standard subset-selection argument over these multiplicities yields $\exp(c'\sqrt{n})$ distinct multiplicity sets. Because the construction is a genuine point set, its lower bound applies to $F(n)$ (not merely to PBD profile sets; [1, Lemma 2.1] independently gives $g(n) \geq \exp(c\sqrt{n})$ as well).

Remarks. The constant c_2 is explicit but immaterial in [1, Lemma 2.3] (their calculation yields $g(n) \leq \exp(8\sqrt{n} - 8 \log n)$). The one non-trivial content of the present note is Steps 1–2: recognizing that plane line collections are precisely PBDs, so that the published design bound applies to multiplicity sets directly.

References

- [1] C. J. Colbourn, K. T. Phelps and V. Ródl, Block sizes in pairwise balanced designs, *Canad. Math. Bull.* 27(3), 375–380 (1984). doi:10.4153/CMB-1984-057-0.
- [2] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* 3(4), 381–392 (1983).