

CLASSICAL INFINITE DIVISIBILITY, SELF-DECOMPOSABILITY AND BELL-SHAPE OF FREE STABLE LAWS

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ABSTRACT. We determine the classical infinite divisibility, self-decomposability and extended Thorin regions of the strictly free stable family. Above index one, infinite divisibility and self-decomposability hold on two extremal skewness intervals, with distinct maximal indices $\alpha_{\text{ID}} = 1.5240739610\dots$ and $\alpha_{\text{SD}} = 1.4283571142\dots$. Below one, every law is self-decomposable, whereas the Thorin region is a central interval; the optimal uniform bound is $4/5$. We characterize the boundaries by global contact conditions and derive their different asymptotic scales at index one. A common phase representation also gives an exact bell-shape test, strictly bell-shaped examples outside the Thorin class, and an unbounded number of skewness components near one. The exceptional free one-stable family admits an explicit integer criterion. The proofs combine analytic estimates with finite interval certificates. The bell-shape results do not constitute a full parameter classification.

1. INTRODUCTION AND MAIN RESULTS

Free stability does not imply stability under classical convolution. This distinction leads to a natural question: which free stable laws are classically infinitely divisible, self-decomposable, or generalized gamma convolutions? We answer these three questions and study the related bell-shape problem.

Bercovici and Pata established the role of free stable laws in free limit theory, with their density geometry and duality developed in Biane's appendix [2]. Explicit Mellin and characteristic transforms were obtained by Hasebe and Kuznetsov [5, Theorems 1–3]. Hasebe, Simon and Wang proved classical infinite divisibility for indices at most one, excluded symmetric laws above one, and proved the extended Thorin property for indices at most $3/4$ [6, Theorems 1 and 2]. They conjectured the optimal Thorin bound $4/5$ and discussed the possible exclusion of all indices above one from classical infinite divisibility. We prove the former and disprove the latter. In fact, Proposition D.1 gives an analytic ID region above one before any finite certificate is used.

1.1. Normalization and terminology. Write $G_\mu(z) = \int_{\mathbb{R}} (z - x)^{-1} \mu(dx)$ and $F_\mu = 1/G_\mu$. The Voiculescu transform is $\phi_\mu(z) = F_\mu^{-1}(z) - z$, initially on a suitable domain near infinity in the upper half-plane. We use the normalization in [7, Introduction]: $X_{\alpha,\rho}$ has transform

$$(1.1) \quad \phi_{\alpha,\rho}(z) = -e^{i\pi\alpha\rho} z^{1-\alpha}, \quad \text{Im } z > 0,$$

with admissible parameters

$$(1.2) \quad \mathcal{A} = \{0 < \alpha \leq 1, 0 \leq \rho \leq 1\} \cup \{1 < \alpha \leq 2, 1 - 1/\alpha \leq \rho \leq 1/\alpha\}.$$

Reflection replaces ρ by $1 - \rho$. All these laws are freely infinitely divisible. Throughout the paper, however, ID and SD refer to *classical additive convolution*, denoted by $*$.

Definition 1.1 (Classical convolution classes). A probability law μ is infinitely divisible (ID) if, for every $n \geq 1$, it can be written as $\mu = \mu_n^{*n}$ for a probability law μ_n . It is self-decomposable (SD) if for every $c \in (0, 1)$ there is a probability law μ_c such that $\mu = D_c \mu * \mu_c$,

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where $D_c\mu$ is the law of cX when X has law μ ; see [12, Sections 7 and 15]. An ID law belongs to the extended Thorin class, abbreviated GGC, when its Lévy measure has the form

$$(1.3) \quad \nu(dx) = \frac{k_+(x)}{x} \mathbf{1}_{\{x>0\}} dx + \frac{k_-(-x)}{|x|} \mathbf{1}_{\{x<0\}} dx$$

with completely monotone kernels k_+, k_- on $(0, \infty)$. Here complete monotonicity means $(-1)^n f^{(n)}(x) \geq 0$ for every $n \geq 0$ and $x > 0$. We allow translations and Gaussian factors, as in [4, Chapter 7, (7.1.5)].

For an ID law with the representation (1.3), the SD criterion is that both kernels are nonnegative and nonincreasing [12, Theorem 15.10]. Thus GGC implies SD, and SD implies ID. The GGC notation here includes two-sided laws; point masses satisfy all three properties.

1.2. The three classification theorems. The ID and SD regions above one consist of intervals adjoining the two extremal skewnesses. Their inner boundaries are different, as are their maximal indices. Below one, the GGC region is instead central. The following theorems give exact implicit classifications; Section 3 defines their boundaries through explicit analytic transforms.

Theorem 1.2 (Classical infinite divisibility). *There is a unique maximal ID index α_{ID} , with*

$$(1.4) \quad 1.52407396102878875153 < \alpha_{\text{ID}} < 1.52407396102878875155.$$

Every admissible $X_{\alpha,\rho}$ with $\alpha \leq 1$ is ID. For $1 < \alpha \leq \alpha_{\text{ID}}$, ID holds exactly when

$$(1.5) \quad \rho \in [1 - 1/\alpha, 1 - r(\alpha)] \cup [r(\alpha), 1/\alpha].$$

Here r is continuous, $1/2 < r(\alpha) < 1/\alpha$ below the cutoff, $r(\alpha_{\text{ID}}) = 1/\alpha_{\text{ID}}$, and $r(\alpha) \rightarrow 1/2$ as $\alpha \downarrow 1$. There are no ID laws for $\alpha > \alpha_{\text{ID}}$. The curve r has the global first-contact description (3.4); it is computable to arbitrary positive rational accuracy at every computable $\alpha \in (1, \alpha_{\text{ID}})$.

Theorem 1.3 (Extended generalized gamma convolutions). *Every admissible skewness is GGC for $0 < \alpha \leq 4/5$ and for $\alpha = 1$. No admissible law with $\alpha > 1$ is GGC. For $4/5 < \alpha < 1$, the exact GGC section is*

$$(1.6) \quad [g(\alpha), 1 - g(\alpha)].$$

The function g is continuous on $(0, 1]$, on setting $g = 0$ for $\alpha \leq 4/5$ and at $\alpha = 1$, and is locally Lipschitz on $(4/5, 1)$. On that interval $0 < g(\alpha) < 1/2$. Its global first-angular-contact description is (3.6), and $g(\alpha)$ is computable to arbitrary positive rational accuracy for every computable such α . Thus $4/5$ is the optimal bound for GGC at all skewnesses.

Theorem 1.4 (Classical self-decomposability). *Every admissible law with $0 < \alpha \leq 1$ is SD. There is a unique maximal SD index α_{SD} , satisfying*

$$(1.7) \quad 1.4283571142 < \alpha_{\text{SD}} < 1.4283571143,$$

such that the SD section for $1 < \alpha \leq \alpha_{\text{SD}}$ is exactly

$$(1.8) \quad [1 - 1/\alpha, 1 - s(\alpha)] \cup [s(\alpha), 1/\alpha].$$

The function s is continuous, computable at computable indices below the cutoff, and satisfies

$$r(\alpha) < s(\alpha) < 1/\alpha \quad (1 < \alpha < \alpha_{\text{SD}}), \quad s(\alpha_{\text{SD}}) = 1/\alpha_{\text{SD}}.$$

It is the global first-contact curve of the derivative transform $-z\mathcal{F}'_{\alpha}(z)$ defined in Section 3. No law with $\alpha > \alpha_{\text{SD}}$ is SD. Near index one,

$$(1.9) \quad s(1 + \varepsilon) = \frac{1}{2} + \sqrt{\frac{3\sqrt{3}}{4\pi}} \sqrt{\varepsilon} + o(\sqrt{\varepsilon}).$$

1.3. Bell-shape.

Definition 1.5 (Bell-shape and whale-shape). A smooth probability density on $\mathbb{R}$ is strictly bell-shaped (BS) if it tends to zero at both infinities and its n th derivative has exactly n changes of sign for every $n \geq 0$. A probability law is weakly bell-shaped if convolution with every nondegenerate Gaussian has a strictly bell-shaped density. These are the conventions of [11, Introduction]; they do not require all derivative zeros to be simple. A density on a half-line is whale-shaped if it is smooth in the interior, vanishes at both ends, and each derivative of positive order has exactly one zero in the interior, as in [7, Introduction, definition preceding Theorem 4].

Smooth weakly bell-shaped densities are strictly bell-shaped [11, Theorems 1.1 and 1.3]. Their transform characterization replaces the derivative conditions by restrictions on crossings of integer phase levels. It permits bell-shape even when the phase itself is not monotone, and hence beyond the Thorin region.

Theorem 1.6 (Selected bell-shape results). *The following statements hold.*

- (1) *No admissible law with $\alpha > 1$ is even weakly bell-shaped. At $\alpha = 1$, every interior-skewness density is a drifted Cauchy density and is strictly bell-shaped.*
- (2) *For $0 < \alpha < 1$ and $0 < \rho < 1$, strict bell-shape is equivalent to the two integer-level conditions in (5.1). The GGC region satisfies these conditions, but there are nonempty open sets of strictly bell-shaped laws outside the GGC region.*
- (3) *For every fixed $0 < \alpha < 1$, the strictly bell-shaped skewness set is a finite union of intervals, closed relative to $(0, 1)$, and possibly isolated points. It contains a central interval and intervals next to both endpoints. The endpoint laws themselves are whale-shaped and weakly bell-shaped, but are not strictly bell-shaped. The number of components tends to infinity as $\alpha \uparrow 1$.*
- (4) *Let C be standard Cauchy and let T be the exceptional free one-stable law with Voiculescu transform $-\log z$. For $A > 0$, $B \neq 0$, the law of $AC + BT$ is strictly bell-shaped if and only if*

$$(1.10) \quad \frac{A}{\pi|B|} \in \mathbb{N}.$$

For $A = 0$, it is weakly bell-shaped and whale-shaped. All these exceptional laws are SD, and they are GGC exactly when $B = 0$.

The full bell-shape parameter set is not determined by these statements.

The sums in the exceptional family can be realized with independent classical summands: convolution by a Cauchy law agrees with free convolution by that Cauchy law. The case $B = 0$, $A > 0$ is simply Cauchy; $A = B = 0$ is a point mass. It must be kept separate from the strictly stable family $X_{1,\rho}$.

1.4. Method of proof and organization. Two analytic representations govern the classifications. Above one, a signed Lévy representation reduces ID to positivity of one kernel and SD to positivity of its negative logarithmic derivative. Below one, Cauchy subordination expresses the Lévy densities through a single analytic phase. GGC asks that this phase be nondecreasing; bell-shape only restricts its integer-level crossings. Section 2 gives the common criteria, and Sections 3–5 apply them to the free stable family.

In each boundary problem, estimates at zero and infinity reduce the remaining sign questions to compact domains. Finite interval certificates then provide the required signs on whole parameter boxes. This distinction is essential: a local double-root equation does not identify a global boundary. Theorem F.7 exhibits competing global ID contacts. Our boundary algorithms give any prescribed positive error; they do not decide equality to an arbitrary computable real input.

Section 6 compares the boundary scales and records the computational dependencies. Appendices A–H contain the longer estimates and the specifications needed to reproduce

the finite checks. All theorem statements include their boundary cases. The remaining bell-shape questions are stated at the end of Section 6.

We use $\mathbb{N} = \{1, 2, \dots\}$ and use a interchangeably with α in proofs. Complex powers take their principal values unless a continued boundary branch is specified. A computable real is one for which rational approximations with any prescribed rational error can be produced by an algorithm. All terminating decimals in certificate inputs denote exact rational numbers.

2. ANALYTIC CRITERIA

The proofs use two reductions: harmonic interpolation for signed Lévy kernels, and a phase representation for Cauchy subordination. We state them separately to make clear which properties are used above and below index one.

2.1. A sector principle for signed Lévy kernels.

Lemma 2.1 (Sector interpolation). *Let F be holomorphic in a neighborhood of a sector $\theta_0 \leq \arg z \leq \theta_1$, $z \neq 0$, whose opening is w . Suppose that, for some $p < \pi/w$, uniformly on that sector,*

$$|F(z)| \leq C(|z|^p + |z|^{-p}).$$

If $\operatorname{Re} F \geq 0$ on its two boundary rays, then $\operatorname{Re} F \geq 0$ throughout the sector. Unless $\operatorname{Re} F$ is identically zero, it is strictly positive in the interior. The same assertion holds with F replaced by $-zF'(z)$ when the growth assumption holds on a slightly larger sector.

Proof. Choose $p < q < \pi/w$ and write $\theta_m = (\theta_0 + \theta_1)/2$. The harmonic function

$$(r^q + r^{-q}) \cos(q(\theta - \theta_m))$$

is positive on the closed sector. Add ϵ times this function to $\operatorname{Re} F$. It dominates a possible negative value on both circular edges of a sufficiently large truncated annular sector. The minimum principle, followed by the radial limits and then $\epsilon \downarrow 0$, proves nonnegativity. The strong minimum principle gives strictness. Cauchy's derivative bound on disks of radius proportional to $|z|$ gives the stated growth for $zF'(z)$ on the smaller sector. $\square$

Suppose a family has a signed Lévy representation, with nonnegative Gaussian coefficient and $\int_{\mathbb{R}} (1 \wedge x^2) |\nu|(dx) < \infty$, whose only possibly negative kernel is $K_\theta(x) = \operatorname{Re} F(xe^{i\theta})$, while the other kernel is completely monotone. Uniqueness of the signed Lévy representation makes ID equivalent to $K_\theta \geq 0$. If $K_\theta(x) \rightarrow 0$ at infinity, then SD is equivalent to

$$-xK'_\theta(x) = \operatorname{Re}[-zF'(z)] \geq 0, \quad z = xe^{i\theta}.$$

Indeed nonincreasingness and the zero limit already force $K_\theta \geq 0$. Lemma 2.1 therefore proves interval convexity of both admissible ray sets whenever the intervening sectors satisfy its hypotheses. This conclusion alone gives neither nonemptiness nor inclusion of an extremal ray. Those are separate global requirements.

2.2. A common phase for Cauchy subordination.

Definition 2.2 (Complete Bernstein and Stieltjes functions). A complete Bernstein function is a nonnegative function on $(0, \infty)$ with a holomorphic continuation to $\mathbb{C} \setminus (-\infty, 0]$ whose imaginary part is nonnegative in the upper half-plane. We use the equivalent analytic characterization in [14, Theorem 6.2]. A Stieltjes function has a representation

$$G(z) = c + d/z + \int_{(0, \infty)} (z+t)^{-1} \sigma(dt), \quad c, d \geq 0, \quad \int_{(0, \infty)} (1+t)^{-1} \sigma(dt) < \infty,$$

where σ is positive; see [14, Definition 2.1].

Let $Y \geq 0$ have Laplace transform $L(z) = e^{-\psi(z)}$, with ψ a complete Bernstein function and $\psi(0) = 0$. Assume also, on every closed proper slit sector,

$$(2.1) \quad \psi(z) = O(|z|^\gamma) \quad (z \rightarrow 0)$$

for some $\gamma > 0$. This condition is satisfied in our application. Let $C_\rho(t)$ be the Cauchy process with scale $t \sin(\pi\rho)$ and drift $-t \cos(\pi\rho)$, independently of Y , where $0 < \rho < 1$. For $\theta \in (0, \pi)$ put

$$(2.2) \quad P_\theta(u) = \pi^{-1} \operatorname{Im} \psi(ue^{i\theta}), \quad u > 0.$$

Theorem 2.3 (Cauchy-subordination criteria). *The law $\mu_\rho = \mathcal{L}(C_\rho(Y))$ is ID and has completely monotone Lévy densities on both open half-lines. Their densities are*

$$(2.3) \quad \nu_+(x) = \int_0^\infty e^{-ux} P_{\pi\rho}(u) du, \quad \nu_-(x) = \int_0^\infty e^{-ux} P_{\pi(1-\rho)}(u) du, \quad x > 0.$$

Consequently:

- (1) μ_ρ is GGC if and only if both functions P in (2.3) are nondecreasing;
- (2) μ_ρ is SD if and only if, for each of these two functions,

$$(2.4) \quad \int_0^\infty e^{-ux} (ux - 1) P(u) du \geq 0 \quad \text{for every } x > 0;$$

- (3) μ_ρ is weakly bell-shaped if and only if $P(u) - m$ changes sign at most once for each $m \in \mathbb{N}$ and each of the two rays. If its density is smooth, this is also the strict bell-shape criterion.

In addition, if Y is SD, then μ_ρ is SD. This preservation assertion is the one-dimensional Cauchy case of the more general subordination result [13, Theorem 3.1(i)].

Proof. Use $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} \mu(dx)$. For $\xi > 0$,

$$\hat{\mu}_\rho(\xi) = \exp\{-\psi(e^{i\pi(\rho-1/2)}\xi)\}.$$

The right-half-plane logarithm exists. Its boundary imaginary parts at $\xi = iu$ and $\xi = -iu$ are respectively $-\pi P_{\pi\rho}(u)$ and $\pi P_{\pi(1-\rho)}(u)$. The sign here is fixed by the displayed minus-Fourier convention; in particular a positive exponential factor corresponds to a positive step in the positive spectral variable.

A complete Bernstein function is $O(|z|)$ at infinity on a closed proper slit sector. Together with (2.1), this permits the twice-subtracted half-plane Cauchy formula. After the linear term is absorbed into the drift, that formula gives

$$(2.5) \quad \log \hat{\mu}_\rho(\xi) = -id\xi + \int_{\mathbb{R}} \left\{ \frac{1}{i\xi + u} - \frac{1}{u} + \frac{i\xi}{u^2} \mathbf{1}_{\{|u| \geq 1\}} \right\} \varphi(u) du,$$

where the normalization at $\xi = 0$ is zero and

$$\varphi(u) = P_{\pi\rho}(u) \quad (u > 0), \quad \varphi(u) = -P_{\pi(1-\rho)}(-u) \quad (u < 0).$$

One may equivalently use the regularization in [11, (1.1)]; the difference is a drift and a real normalization constant. The origin integrals converge because $P(u) = O(u^\gamma)$, and $\int_1^\infty P(u)u^{-3} du < \infty$ since $P(u) = O(u)$. To justify the Cauchy formula directly, apply it first between radii ϵ and R after subtracting the value and linear part at a fixed positive point. The small arc tends to zero by (2.1); the twice-subtracted large arc tends to zero using the linear growth. Normalizing at zero yields (2.5). The only remaining entire ambiguity is linear and purely imaginary on the Fourier axis, and hence is drift.

For $u > 0$, the identity

$$\int_0^\infty (e^{-i\xi x} - 1)e^{-ux} dx = \frac{1}{u + i\xi} - \frac{1}{u}$$

and its reflected version identify the Lévy densities as (2.3). Truncation at $u = \epsilon, R$ first justifies Fubini; the stated bounds give the Lévy integrability needed to pass to the limit.

Positivity of P follows from the Pick property of ψ . This also proves ID without requiring bell-shape.

For a nonnegative continuous P with $P(0) = 0$, the function $x \int_0^\infty e^{-ux} P(u) du$ is completely monotone exactly when P is nondecreasing. Sufficiency follows by Stieltjes integration by parts. For necessity, represent that completely monotone function as $\int e^{-ux} U(du)$ and divide by x ; uniqueness of Laplace transforms gives $P(u) = U([0, u])$ almost everywhere, hence everywhere in the continuous case. This proves the GGC assertion. Differentiating $x\nu_\pm(x)$ under the integral proves (2.4) and the SD assertion by the Lévy-kernel criterion.

The representation (2.5), with the integrability just checked, is the representation in [11, Theorems 1.1 and 1.3]. Its regularity condition holds because $|\hat{\mu}_\rho| \leq 1$: the real part is integrable near zero and $\xi \operatorname{Im} \hat{\mu}_\rho(\xi) \rightarrow 0$ there. Their characterization gives the integer-level assertion; the level zero has the required sign already. Finally, positive SD gives $Y \stackrel{d}{=} cY' + Z_c$ with $Z_c \geq 0$ independent of Y' . Conditional independence of Cauchy increments and its strict one-stability give $C_\rho(Y) \stackrel{d}{=} cC_\rho(Y') + C'_\rho(Z_c)$, proving the final claim. $\square$

Remark 2.4. GGC asks for monotonicity of the entire phase, while bell-shape only forbids a downward passage through an integer. SD is the intermediate Laplace-average test (2.4); it is not a pointwise phase monotonicity condition. These distinctions explain why equality of the three parameter regions should not be expected.

3. CRITICAL CURVES FOR ID AND GGC

The Wright transform connects the abstract criteria with the free stable parameters. For $a > 0$, write [5, Theorem 3]

$$(3.1) \quad W_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(2 + (a-1)n)}.$$

For reference, the Wright characteristic transform, in the plus-Fourier convention $\varphi_{\alpha,\rho}(t) = \mathbb{E}e^{itX_{\alpha,\rho}}$, is [5, Theorem 3, (21)]

$$(3.2) \quad \varphi_{\alpha,\rho}(t) = W_\alpha(-e^{-i\alpha\omega}t^\alpha), \quad \omega = \pi(\rho - \tfrac{1}{2}), \quad t > 0.$$

For $0 < \alpha < 1$, $X_\alpha = X_{\alpha,1}$ has transform $L_\alpha(s) = W_\alpha(-s^\alpha)$ and lower endpoint $b_\alpha = \alpha(1-\alpha)^{1/\alpha-1}$. Complex powers use the principal branch unless a continued boundary branch is specified.

In the appendices, α_* denotes α_{ID} . We write $[z^n]f$ for coefficient extraction and $(u)_n = u(u+1)\cdots(u+n-1)$, with $(u)_0 = 1$. The functions ${}_1F_1$ and ${}_2F_1$ below use the series ${}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{n \geq 0} \prod_i (a_i)_n z^n / (\prod_j (b_j)_n n!)$ on its disk of convergence, followed by analytic continuation where specified [15, Section 16.2]. We use $H_m^{(j)} = \sum_{k=1}^m k^{-j}$, $H_m = H_m^{(1)}$, Euler's constant γ , and the Riemann zeta function ζ . The Bernoulli convention is $z/(e^z - 1) = \sum_{n \geq 0} B_n z^n / n!$, so $B_1 = -1/2$.

For $a > 1$ all zeros are negative real by [1, Theorem 1], applied with parameters $a-1$ and 2. Stirling's formula for the coefficients gives order $1/a < 1$. Since W_a is not a polynomial, Hadamard factorization then gives infinitely many zeros and a genus-zero product, with no exponential factor and with constant factor $W_a(0) = 1$. Denote them by $-\lambda_n$, with multiplicity, and put $r_n = \lambda_n^{1/a}$. The product and signed Lévy formulas used below are proved in the signed-kernel calculation following Proposition D.1 and in Proposition D.2. Define

$$(3.3) \quad \mathcal{F}_a(z) = \frac{iz}{\pi} \sum_n \int_0^\infty e^{-zt} \left\{ \log |1 - (t/r_n)^a| + i\pi \mathbf{1}_{\{t > r_n\}} \right\} dt, \quad \operatorname{Re} z > 0.$$

The series of integrals converges normally on this half-plane. For upper-half skewness the only possibly negative Lévy kernel is $K_{a,\rho}(x) = \operatorname{Re} \mathcal{F}_a(xe^{-i\pi(1-\rho)})$. The other kernel is completely monotone.

3.1. ID: global positivity and the first zero ray. For $1 < a < \alpha_{\text{ID}}$, the curve in Theorem 1.2 is specified by

$$(3.4) \quad \pi(1 - r(a)) = \min \left\{ \theta \in (\pi(1 - 1/a), \pi/2) : \operatorname{Re} \mathcal{F}_a(xe^{-i\theta}) = 0 \text{ for some } x > 0 \right\}.$$

The cutoff is the unique zero minimum of the extremal kernel described in Theorem E.3. In particular the number (1.4) is not defined by a numerical fit.

Proof of Theorem 1.2. For $a \leq 1$ use [6, Theorem 1(a)]. For $a > 1$, the signed representation and uniqueness give ID exactly when $K \geq 0$. Its holomorphic growth is $O(|z|^a + |z|^{-a})$ on proper half-plane subsectors. Admissible angular differences are less than $\pi/2$, and $a < 2$, so Lemma 2.1 proves that every nonempty upper-half section is an interval. This step does not assume that an extremal law is ID.

The complete extremal positivity theorem (Theorem E.6) proves that the endpoint kernels are strictly positive for $1 < a < \alpha_{\text{ID}}$ and nonnegative at α_{ID} . Theorems E.3 and E.4 establish the unique cutoff and exclude *every* other skewness above it, using separate skewness witnesses. Strict origin and infinite-tail bounds, together with compactness of the remaining radii, make strict extremal positivity stable under a small skewness perturbation. Hence the ID interval has positive width below the cutoff.

At its inner boundary the kernel must vanish at some finite nonzero radius; otherwise those same uniform bounds extend the interval. The zeros are finite in number, since the kernel is analytic in radius and strictly positive at both ends. At any boundary zero, $K_x = 0$, $K_{xx} \geq 0$, and the Hopf boundary lemma gives $K_\rho > 0$. This proves (3.4). The full compactness and Hopf argument is given in Theorem E.8.

Weak closedness of ID gives lower semicontinuity of the boundary. Strict positivity on every interior ray, with uniform end bounds, gives upper semicontinuity. The endpoint assertions follow by the same argument and the near-one ID rectangles. Finally, the finite-radius cutoffs and spectral remainder in Theorem G.2 allow simultaneous searches for strict positive covers and strict negative witnesses. Two rational trial rays eventually bracket the boundary to any requested accuracy. This proves computability without assuming unique contact or a decision algorithm at equality. $\square$

3.2. GGC: the first loss of phase monotonicity. For $0 < a < 1$ put

$$(3.5) \quad L_a(z) = W_a(-z^a), \quad \psi_a(z) = -\log L_a(z), \quad H_a(z) = z\psi'_a(z), \quad b_a = a(1 - a)^{1/a-1}.$$

The logarithm is continued from the positive axis. Its global slit-plane existence is justified in Lemma 4.1 below. For $4/5 < a < 1$,

$$(3.6) \quad \pi(1 - g(a)) = \min \{ \arg z : \pi/2 < \arg z < \pi, \operatorname{Im} H_a(z) = 0 \}.$$

Proof of Theorem 1.3. The uniform sufficiency up to $4/5$ is Theorem C.4: it combines the earlier GGC theorem with a complete Kanter phase certificate on the remaining compact parameter range and explicit infinite-tail estimates. Theorem A.3 proves failure of the one-sided GGC property for every $4/5 < a < 1$ by a transferred oscillation argument. No conclusion is inferred merely from failure of a factor's GGC property.

For each remaining index, Theorem B.14 proves the symmetric law GGC. The positive-real criterion and harmonic sector interpolation then give a central interval. Strict positivity on inner rays gives a nontrivial central neighborhood. At its outer boundary the first zero of $\operatorname{Im} H_a$ occurs at a finite nonzero radius: the Wright origin series and uniform large-radius expansion exclude both radial ends. The Hopf lemma gives, at such a contact,

$$\operatorname{Im} H_a = \operatorname{Im}(zH'_a) = 0, \quad \operatorname{Re}(zH'_a) < 0.$$

Compactness of the contact set and the strict angular derivative yield continuity and local Lipschitz regularity, as detailed in Theorem B.15. Theorem B.11 gives $g(a) \rightarrow 0$ at one. The effective sector and tail bounds in Theorem G.2 prove the computability assertion.

For $a > 1$, the Wright zeros give zeros of the holomorphic characteristic transform in the open right half-plane. The canonical extended Thorin representation is zero-free there, proving exclusion (Theorem A.2). At $a = 1$ the strictly stable family consists of Cauchy laws and point masses, both GGC. This proves all cases. $\square$

4. SELF-DECOMPOSABILITY

For $a \leq 1$, we prove SD for the positive law and pass to every skewness by Cauchy subordination. For $a > 1$, we apply the sector principle to the differentiated signed kernel. The endpoint estimates and the finite transition cover then determine the global SD boundary.

4.1. The positive law and its analytic phase.

Lemma 4.1 (Whale-shaped factorization). *For $0 < a < 1$, L_a is zero-free on the slit plane, $\psi_a = -\log L_a$ is a complete Bernstein function, and*

$$(4.1) \quad 0 < \operatorname{Im}\{\psi_a(z) - b_a z\} < 2 \arg z, \quad 0 < \arg z < \pi.$$

Moreover $X_{a,\rho} \stackrel{d}{=} C_\rho(X_{a,1})$ for $0 < \rho < 1$.

Proof. The translated density of $X_{a,1} - b_a$ is whale-shaped by [6, Theorem 4(a)]. By [11, Theorem 1.13], it is the convolution of an exponential density and an integrable completely monotone function. On normalizing the second factor,

$$e^{b_a z} L_a(z) = \frac{\lambda}{\lambda + z} G(z), \quad G(0) = 1,$$

where $\lambda > 0$ and G is a nonzero Stieltjes function. For $0 < \arg z = \theta < \pi$, all terms in its Stieltjes representation have arguments between $-\theta$ and zero, so G is zero-free and $-\arg G \in [0, \theta]$. Also $0 < \arg(\lambda + z) < \theta$. This proves zero-freeness and (4.1), with the branch continued from $z > 0$. The exponent is nonnegative on the positive axis and has nonnegative imaginary part in the upper half-plane. The Pick characterization of complete Bernstein functions [14, Theorem 6.2] applies. Finally, the Cauchy factorization is [7, (8)]; it can also be checked directly by comparing the Wright characteristic transforms. $\square$

Proposition 4.2 (Positive SD below one). *For $4/5 \leq a < 1$, the Lévy kernel of $X_{a,1}$ is strictly decreasing on $(0, \infty)$.*

Proof. Put $\beta = 1 - a$, $c = \beta^{1/a}$ and $Z = (X_{a,1} - b_a)/c$. Write $j_a(y)/y$ for the Lévy density of Z . The inverse equation in logarithmic Cauchy-transform coordinates is

$$(4.2) \quad h_a(v) = \frac{a}{a-1} - e^{-v} - \frac{e^{(a-1)v}}{a-1} = y.$$

Its removable value at $a = 1$ is $1 - e^{-v} - v$. The two physical inverse branches give

$$p_a(y) = \frac{\sqrt{2/a}}{\pi} \sqrt{y} F_a(y), \quad F_a(y) = \sum_{n \geq 0} f_n(a) y^n, \quad j_a(y) = \sum_{n \geq 0} k_n(a) y^n.$$

These are the rational parameter polynomials constructed in Lemma D.4, with $f_0 = 1$ and $k_0 = 3/2$.

We first justify an all-order remainder, including the endpoint $a = 1$ used in the compact cover. In the strip $|\operatorname{Im} v| < 2\pi/a$, the horizontal boundary images have the form

$$C - e^{\mp 2\pi i/a} B(t), \quad C = -a/\beta, \quad B(t) = e^{-t} - \beta^{-1} e^{-\beta t}.$$

The range of B is $[C, \infty)$, and the distance of this boundary line from zero is at least

$$\frac{a}{\beta} \sin(2\pi\beta/a) \geq 4.$$

The inequality follows from $\sin(2\pi u) \geq 4u$ for $0 \leq u \leq 1/4$. The right asymptotic value has modulus at least four and the left boundary diverges. Together with the root bound below, these estimates permit homotopy through $a = 1$ from Lemma D.4: there are exactly two

inverse roots for $|y| < 4$, and their only interior critical point is zero. Hence F_a and the Volterra kernel j_a are holomorphic in that disk.

For $n \geq 1$ choose $R_n = 4n/(n+1)$ and $M_n = [3(n+1)/2]^5$. At $\beta = 1/5$,

$$1 - \beta - \beta(R_n + M_n^{-1}) - M_n^{-\beta} = \frac{2}{15(n+1)} - \frac{1}{5M_n} > 0.$$

The left side is concave in β and vanishes at zero, so it is positive for $0 < \beta \leq 1/5$. The corresponding limiting inequality holds at zero. The reverse triangle inequality in (4.2) excludes $\operatorname{Re} v \geq \log M_n$ when $|y| = R_n$. Therefore $|F_a(y)| \leq M_n/2$ on that circle and Cauchy's bound gives

$$(4.3) \quad |f_n(a)| \leq 12(n+1)^5 4^{-n}, \quad |k_n(a)| \leq 128(n+1)^8 4^{-n}.$$

For completeness the second bound follows from the exact recurrence

$$(4.4) \quad k_n = \frac{(3/2)_{n+1}}{n!} f_n - \sum_{m=1}^n \frac{(3/2)_m (n-m)!}{n!} f_m k_{n-m}.$$

The first 128 cases are exact rational polynomial inequalities. For $n \geq 128$, put $A_m = 4^m \sup_a |f_m(a)|$. For $m \leq 7$ use the exact polynomials; for the other terms use the first bound in (4.3). Splitting the convolution at $m = n/2$, using $\binom{n}{m} \geq (n/m)^m$, bounds its normalized value by

$$S_1 = \sum_{m=1}^7 A_m (m+1) (m/128)^m + 12 \sum_{m=8}^{64} (m+1)^6 (m/128)^m + 12 \sum_{m \geq 65} (m+1)^6 2^{-m},$$

$$S_2 = \frac{12}{129^2} \left\{ 1 + \sum_{k=1}^{64} (k+1)^8 (k/128)^k + \sum_{k \geq 65} (k+1)^8 2^{-k} \right\}.$$

The free term is at most $18/(128 \cdot 129)$. Directed arithmetic gives $S_1 + S_2 + 18/(128 \cdot 129) < 0.083 < 1$, closing the induction.

On $a \in [4/5, 1]$, $0 \leq y \leq 5/2$, differentiate the degree-128 kernel polynomial. If $r > 0$ is a box's upper radius, its infinite remainder is at most

$$(4.5) \quad \frac{128}{r} \sum_{n > 128} (n+1)^9 (r/4)^n.$$

The removable $r = 0$ value is taken directly from the polynomial. The complete interval cover verifies $-j'_a(y) > 0$ on this rectangle. To preserve cancellation, parameter evaluation uses the cubic Taylor model at each box midpoint, with an interval fourth derivative remainder. If $k_n = P_n(a)/a^n$, the numerator recurrence $P_{n,j+1} = aP'_{n,j} - (n+j)P_{n,j}$ supplies those derivatives exactly. This parameter remainder is additional to (4.5).

It remains to control every $y \geq 5/2$. Define

$$\phi_a(t) = -\pi^{-1} \operatorname{Im} \log L_a(-t/c + i0).$$

The deterministic translation contributes no cut imaginary part, and Lemma 4.1 gives $0 \leq \phi_a \leq 2$ almost everywhere. The complete Bernstein representation gives

$$(4.6) \quad j_a(y) = y \int_0^\infty e^{-yt} \phi_a(t) dt.$$

Set $T = t^a$, $z = Te^{-i\pi\beta}$ and $G_\beta(z) = e^{-z/\beta} W_a(z/\beta)$, with its removable Hankel limit at $\beta = 0$. On the entire rectangle $0 \leq \beta \leq 1/5$, $0 \leq T \leq 1$, the second interval cover proves

$$(4.7) \quad \operatorname{Re} G_\beta(z) > 0, \quad \frac{2}{3} < \frac{d}{dT} \phi_a(T^{1/a}) = \frac{\sin(\pi\beta)}{\pi\beta} - \frac{1}{\pi} \operatorname{Im} \frac{e^{-i\pi\beta} G'_\beta(z)}{G_\beta(z)} < \frac{6}{5}.$$

The real-half-plane check fixes the logarithm branch. Integrating from zero gives $(2/3)t^a \leq \phi_a(t) \leq (6/5)t^a$ for $0 \leq t \leq 1$. In the derivative of (4.6), the factor $yt - 1$ is negative only

for $t < 1/y$. Using the upper phase bound there and the lower bound on $(1/y, 1)$, while discarding the nonnegative contribution from $t > 1$, yields

$$y^{a+1}[-j'_a(y)] \geq \frac{2}{3}B(a, y) - \frac{8}{15}N(a),$$

where

$$B(a, y) = \int_0^y e^{-u}(u-1)u^a du, \quad N(a) = \int_0^1 e^{-u}(1-u)u^a du.$$

For $y \geq 5/2$, B increases in both a and y , while N decreases in a . Thus a single directed bound

$$\frac{2}{3}B(4/5, 5/2) - \frac{8}{15}N(4/5) > 0.02578$$

proves strict negativity of j'_a on the whole infinite tail. The exact induction bases, both compact covers (291 leaves in total), and this last constant are recomputed by `certify_sd_below_one.py` verify. This completes the proof. $\square$

4.2. Extremal kernels and the upper SD transition. We give the numerical obligations explicitly, since an extremal transition alone does not exclude a disjoint set of nonextremal SD laws. For $a > 1$ set

$$c_a = (a-1)^{1/a}, \quad q = \frac{\rho-1/2}{1/a-1/2}, \quad D(a, q, y) = -\partial_y K_{a,\rho}(c_a y).$$

At $q = 1$, write $j_a(y) = K_{a,1/a}(c_a y)$, as in Lemma D.4.

Lemma 4.3 (Extremal SD and global exclusions). *Put $a_L = 1.42834$, $a_U = 1.42838$ and $I = [4.99, 5.04]$. The following whole-domain inequalities hold.*

- (1) $D(a, 1, y) > 0$ for $1 < a \leq a_L$ and all $y > 0$.
- (2) For $a_L \leq a \leq a_U$, $D(a, 1, y) > 0$ outside I , and on I one has $D_{yy} > 1/50$, $D_a < 0$, with D_y negative at its left endpoint and positive at its right endpoint.
- (3) The minimum on I is positive at a_L and negative at a_U . On $[a_L, a_U] \times [0.98, 1] \times I$, one has $D_q > 0$.
- (4) On $[a_L, a_U] \times [11/16, 0.98]$ there is a negative value of D at $y = 5.0104$. The same witness excludes $[a_U, 1.43] \times [11/16, 1]$.
- (5) For $1.43 \leq a \leq \alpha_{\text{ID}}$ and $11/16 \leq q \leq 1$, $D(a, q, 4.944) < 0$. The remaining $q < 11/16$ in the transition and exclusion ranges are already non-ID.

Proof. We describe the analytic bounds, the exact covered domains, and the finite verification. On $1 \leq a \leq 1.3$, the degree-128 extremal germ has the all-order bound $|k_n(a)| \leq 128(n+1)^2(5/2)^{-n}$. On the successive regions $[1.3, 1.4]$ and $[1.4, 1.53]$, use respectively $32(n+1)^2 4^{-n}$ and $32(n+1)^2(51/10)^{-n}$. These bounds and their differentiated geometric tails control the origin. Their exact coefficient bases and induction are proved with the extremal ID germ.

For the infinite tail put $e = a - 1$, $h = \sin \pi(1/a - 1/2)$, $\xi = \pi(2 - a)$ and $Q_m = \sum R_n^{-ma}$, where $R_n = c_a r_n$. When $a \leq 3/2$, $\cos \xi \leq 0$. The elementary denominator bound

$$(1 + 2z \cos \xi + z^2)^{-1} \geq 1 - 2z \cos \xi - z^2$$

and the maximum of $t^{2a+1}e^{-hyt}$ give

$$(4.8) \quad y^{a+1}D(a, 1, y) \geq a \frac{\sin(\pi e)}{\pi e} \{1 - e\Gamma(3a+1)Q_3 y^{-2a}\} - 2 \left(\frac{2a+1}{eh} \right)^{2a+1} Q_2 y^{-a}.$$

Here the removable moments at $e = 0$ are evaluated through their Taylor expansions, not by dividing a zero-containing interval by e . The right side increases in y . Its strict sign at $y = 2$, together with the origin cover, proves extremal SD through $a = 1.1$ (77 leaves). For $1.1 \leq a \leq 1.42$, 32 consecutive parameter strips use the existing moving Wright-root enclosures on $2 \leq y \leq 8$ (749 leaves for the derivative and matching bounds). For $y \geq 8$,

retain the listed exponential terms and bound every omitted one by its Rayleigh moment. After multiplication by y^{a+1} , each negative exponential bound decreases since $hyR_n > a + 1$ at the starting radius. A strict bound at eight therefore controls the whole tail.

Near the transition a sharper germ holds uniformly on $[1.42, 1.44]$:

$$(4.9) \quad |k_n(a)| \leq 32(n+1)^2(20/107)^n.$$

The inverse-branch radius exceeds 5.3617 and the working radius is $107/20 = 5.35$. The same exact rational recurrence gives its finite induction base. The complete transition cover uses (4.9), validated spectral integrals and Rayleigh remainders to check items (1)–(4), except for $D_a < 0$. It has 7,118 leaves, including the separate origin and tail checks. The lower minimum is bounded from below by $D(y_0) - D_y(y_0)^2/(2m)$ with $m = 1/50$; this is valid on the entire core by strong convexity.

The parameter derivative is checked independently. Differentiate the degree-768 spatial germ with respect to a , using a Cauchy coefficient formula about $a_0 = 1.42836$. A complex parameter circle of radius $R = 0.0002$ bounds the polynomial by M using its Bernstein ellipse extension: after mapping the real parameter interval to $[-1, 1]$, the ellipse is parametrized by $(w + w^{-1})/2$ with $|w| = \varrho > 1$. Eighteen equally spaced values on the smaller circle $r = 10^{-6}$ give the discrete Fourier coefficients; ten distinct complex nodes suffice by conjugacy. For $\tau = \max |a - a_0|/R$ and $\eta = r/R$, the derivative error is at most

$$(4.10) \quad \frac{M}{R} \left\{ \frac{\eta^{18}}{(1 - \eta^{18})(1 - \tau)^2} + \frac{18\tau^{17}}{1 - \tau} + \frac{\tau^{18}}{(1 - \tau)^2} \right\}.$$

This follows by summing aliases of the Cauchy coefficients and the tail of their differentiated power series. Values are kept as full spatial polynomials until after this discrete transform, so interval dependence on y is preserved.

The infinite spatial remainder is bounded separately. With $Q_n(a) = a^n k_n(a)$, $\deg Q_n \leq 2n$, the real Bernstein inequality on $[L, U] = [1.42, 1.44]$ [9, Introduction, (1.1)] gives

$$|k'_n(a)| \leq 32n(n+1)^2 \left(\frac{U}{a(107/20)} \right)^n \left\{ \frac{2}{\sqrt{(a-L)(U-a)}} + \frac{1}{a} \right\}.$$

Multiplication by ny^{n-1} and summation over $n > 768$ is a positive geometric tail. Adding that tail, (4.10), and all rounding errors gives

$$-0.40472 < D_a < -0.25965$$

on the complete extremal core. All ten complex-parameter values are freshly recomputed in verification mode.

Lastly, the ID boundary locators on $[1.43, \alpha_{\text{ID}}]$ place it strictly above $q = 11/16$. On the remaining rectangle a 158-leaf derivative cover gives the fixed negative witness in item (5), with the same spectral integral and full omitted-root bounds. The transition cover supplies the analogous exclusions down to a_L . These checks are implemented by the verifiers listed in Section 6. Thus all radii and all admissible skewnesses required by the lemma are accounted for. $\square$

Proof of the SD cutoff and its rational bracket. By Lemma 4.3, every $a \in [a_L, a_U]$ has a unique core minimizer $y(a)$. It is analytic by $D_{yy} > 0$ and the implicit function theorem. Its minimum $m(a) = D(a, 1, y(a))$ satisfies $m'(a) = D_a(a, 1, y(a)) < 0$. The endpoint signs give a unique zero α_{SD} , and positivity outside the core gives SD at that index. Strict $D_q > 0$ excludes every smaller q near one at the cutoff and above; the other negative witnesses exclude the remaining admissible candidates. This proves global maximality, not just an extremal transition.

For the refined bracket, evaluate the degree-896 germ at $y_0 = 5.01036765$, with the complete tail (4.9). Directed evaluations give

a	certified quantity
1.4283571142	$D(a, 1, y_0) - D_y(a, 1, y_0)^2/(2/50) > 3.94 \cdot 10^{-12}$
1.4283571143	$D(a, 1, y_0) < -2.94 \cdot 10^{-11}$.

Strong convexity turns the first into a lower bound on the entire core minimum. The unique-zero argument identifies the same global cutoff and proves (1.7). $\square$

4.3. The SD boundary, continuity and effective approximation. For $1 < a < \alpha_{\text{SD}}$ put $\mathcal{D}_a(z) = -z\mathcal{F}'_a(z)$. The precise first-contact formula for s is

$$(4.11) \quad \pi(1 - s(a)) = \min \left\{ \theta \in (\pi(1 - 1/a), \pi/2) : \operatorname{Re} \mathcal{D}_a(xe^{-i\theta}) = 0 \text{ for some } x > 0 \right\}.$$

Here the endpoint derivative is strictly positive, by Lemma 4.3.

Proof of Theorem 1.4, except for (1.9). For $a \leq 4/5$, GGC implies SD. Proposition 4.2 and Cauchy subordination prove all remaining $a < 1$ and all skewnesses; the index-one Cauchy and point laws are separate immediate cases. Above one the SD cutoff was just established. The positive-side Lévy kernel is completely monotone, so SD is equivalent to $\operatorname{Re} \mathcal{D}_a \geq 0$ on the corresponding whole ray. Lemma 2.1 proves interval convexity, with strict positivity on every interior ray. Weak closedness gives a closed interval.

At the ID boundary a nonnegative kernel has a finite positive zero and is strictly positive farther out. Such a kernel cannot be nonincreasing. Hence closedness of SD gives $s(a) > r(a)$. To justify $s(a) < 1/a$ one must check openness even at the singular origin. Set $\delta = \pi(1/a - \rho)$ and $B = a/(a - 1)$. The Cauchy continuation of the endpoint kernel gives

$$(4.12) \quad \begin{aligned} -\partial_y K_{a,\rho}(c_a y) &= -2 \operatorname{Re} \{ e^{-i\delta} j'_a(ye^{-i\delta}) \} + \frac{B \sin \delta}{\pi y^2} + P'_\delta(y), \\ P'_\delta(y) &= \frac{\sin \delta}{\pi} \int_0^\infty \frac{u j'_a(u)}{(u - y \cos \delta)^2 + y^2 \sin^2 \delta} du. \end{aligned}$$

The endpoint has $j_a(0) = 3/2$, $j'_a(0) = -c < 0$, $j'_a < 0$ and $j_a(\infty) = 0$. On $|w| \leq d$, take $|j'_a(w) + c| \leq c/10$. For $y \leq d/2$ the local integral has absolute value at most

$$\frac{11c}{10} \left\{ \cos \delta + \frac{\sin \delta}{\pi} \log \frac{d+y}{y} \right\},$$

and its tail is at most $6 \sin \delta / (\pi d)$, using total variation $3/2$. If $\cos \delta \geq 9/10$, the two local derivative terms leave at least $61c/100$. The remaining bracket is nonnegative after reducing d , since

$$\frac{B}{y^2} - \frac{11c}{10} \log \frac{d+y}{y} - \frac{6}{d} \geq 0 \quad (0 < y \leq d/2).$$

This proves uniform origin positivity for small δ . Compact positivity and the uniform algebraic tail then prove openness below the extremal skewness. The identical end bounds, strict interior positivity and a strict witness outside the interval give continuity by two-sided enclosure. They also imply the attained first contact (4.11). At the cutoff, the only SD law on the upper half is extremal, so closedness gives continuity there.

For effectivity, fix a trial $1/2 < \rho < 1/a$ and use $0 \leq j_a \leq 3/2$ in the undifferentiated Poisson formula. Its derivative satisfies

$$|P'_\delta(y)| \leq \frac{C_\delta}{y}, \quad C_\delta = \frac{3 \sin \delta}{4(1 - \cos \delta)^2}.$$

If $|j'_a| \leq M$ on $|z| \leq d_0$, choose rational $y_0 < d_0$ with $2My_0^2 + C_\delta y_0 < B \sin \delta / \pi$. Equation (4.12) then controls all smaller radii. At infinity, put $h = \cos \pi(1 - \rho)$, $\xi = \pi(1 - a(1 - \rho))$

and $S_2 = \sum r_n^{-2a}$. The complete spectral bound is

$$-K'_{a,\rho}(x) \geq \frac{a \sin \xi}{\pi} x^{-a-1} - C_2 x^{-2a-1},$$

$$C_2 = S_2 \left\{ \frac{2a \sin \xi \Gamma(2a+1)}{\pi} + 2 \left(\frac{2a+1}{eh} \right)^{2a+1} \right\}.$$

It follows by $(1+2z \cos \xi + z^2)^{-1} \geq (1+z)^{-2} \geq 1-2z$ and maximizing the entire residue derivative against S_2 . Any rational X with $X^a > 2\pi C_2/(a \sin \xi)$ controls all $x \geq X$. Directed spectral evaluation is uniformly convergent on the remaining compact interval, including its derivative. Parallel enumeration of positive interval covers and negative rational-radius witnesses therefore encloses $s(a)$ to any accuracy, without a contact-uniqueness assumption. This completes the proof. $\square$

At the upper SD cutoff there is also a local analytic closing law. At $(\alpha_{\text{SD}}, 1, y(\alpha_{\text{SD}}))$ the Jacobian of (D, D_y) with respect to (q, y) has determinant $D_q D_{yy} > 0$. Hence the global boundary and its unique contact are analytic for indices just below the cutoff, and

$$(4.13) \quad \frac{1}{a} - s(a) = C_{\text{SD}}(\alpha_{\text{SD}} - a) + O((\alpha_{\text{SD}} - a)^2), \quad C_{\text{SD}} = \left(\frac{1}{\alpha_{\text{SD}}} - \frac{1}{2} \right) \frac{-D_a}{D_q} \Big|_{(\alpha_{\text{SD}}, 1, y(\alpha_{\text{SD}}))} > 0.$$

Uniform origin and tail bounds, and strict outside-core positivity, are what make this local contact the global boundary. No numerical neighborhood size or numerical value of C_{SD} is asserted.

4.4. The square-root SD boundary at index one.

Lemma 4.4 (A sharp geometric-series bound). *For all $v > 0$ and $t \in \mathbb{R}$,*

$$(4.14) \quad 2v^2 \operatorname{Im} \sum_{n \geq 2} n e^{-nv+int} \leq M, \quad M = \frac{3\sqrt{3}}{4}.$$

The constant is optimal, as $v \downarrow 0$, $t = v/\sqrt{3}$.

Proof. Reduce t modulo 2π . For $0 \leq t \leq \pi$ the imaginary part of the sum starting at $n = 1$ is

$$\frac{\sinh v \sin t}{2(\cosh v - \cos t)^2} = \operatorname{Im} \sum_{k \in \mathbb{Z}} (v - i(t - 2\pi k))^{-2}.$$

If $v \leq \sqrt{3}\pi$, pair k and $-k$. Each pair other than $k = 0$ has nonpositive imaginary part, since $u/(v^2 + u^2)^2$ decreases for $u \geq v/\sqrt{3}$ and $2\pi k - t \geq \pi$. The sum is therefore at most $2vt/(v^2 + t^2)^2$. Removing $n = 1$ can only lower it. Multiplication by $2v^2$ and maximization gives $3\sqrt{3}/4$. For $-\pi \leq t \leq 0$ the sum from one is nonpositive and the removed term contributes at most e^{-v} , so the bound follows from $2v^2 e^{-v} \leq 8e^{-2} < M$. For $v > \sqrt{3}\pi$ use $2v^2 e^{-v}(1 - e^{-v})^{-2} < M$, a decreasing bound in this range. The nearest-pole limit at $t = v/\sqrt{3}$ proves optimality. $\square$

Proof of (1.9). Write $e = a - 1$ and let $q = C_0 \sqrt{e}$ with $C_0 > 0$ fixed. Define

$$h = \sin\{\pi(1/a - 1/2)q\}, \quad c_0 = \cos\{\pi(1/a - 1/2)q\}, \quad p = h - ic_0, \quad v = hy.$$

Then $h/\sqrt{e} \rightarrow \pi C_0/2$. The complete scaled Wright spectrum has the form

$$(4.15) \quad R_{n,e} = n/d_e - A_e + \epsilon_{n,e}, \quad n = 2, 3, \dots, \quad d_e = 1 + O(e^2), \quad A_e = O(e), \quad |\epsilon_{n,e}| \leq Ce/n.$$

The uniformity in n , completeness, and eventual simplicity are proved in Lemma H.6. A fixed-zero Taylor expansion would not suffice here.

Write the signed kernel in scaled radius as

$$K(c_a y) = P_e(y) - 2 \operatorname{Re} \sum_{n \geq 2} e^{-py R_{n,e}}.$$

With $\xi = \pi(1 - a/2)(1 + q)$ the positive Laplace part satisfies

$$(4.16) \quad -P'_e(y) \geq \frac{a \sin \xi}{\pi e} y^{-a-1} - \frac{a \sin \xi}{\pi} \Gamma(3a+1) Q_3(e) y^{-3a-1},$$

where Q_3 stays bounded. Here $\cos \xi < 0$ for small e , so a favorable linear denominator term was discarded. It follows that $\pi e y^2 [-P'_e(y)] \geq 1 + o(1)$ uniformly for $1 \leq y \leq V/h$, for each fixed V ; the factors y^{-e} tend uniformly to one in this range. At a fixed y the same expression equals $1 + o(1)$, by the upper denominator remainder and the second and third Rayleigh moments.

Compare (4.15) with $\bar{R}_n = n/d_e - A_e$. Uniformly for $1 \leq y \leq V/h$,

$$\begin{aligned} \left| \sum R_n e^{-pyR_n} - \sum \bar{R}_n e^{-py\bar{R}_n} \right| &\leq C e (1 + |\log v|) + \frac{Cey}{e^{c_1 v} - 1} \\ &= O(\sqrt{e} + e |\log e|). \end{aligned}$$

Indeed $v \geq h$, $|py\epsilon_n| \leq Cey/n = O(\sqrt{e})$, and the real damping is bounded below by $c_1 vn$. The comparison sum is exactly

$$e^{pyA_e} \left\{ d_e^{-1} \sum_{n \geq 2} n e^{-pyn/d_e} - A_e \sum_{n \geq 2} e^{-pyn/d_e} \right\}.$$

Since $e^{pyA_e} = 1 + O(\sqrt{e})$ and $p = -i + O(\sqrt{e})$, Lemma 4.4 gives

$$(4.17) \quad y^2 h^2 2 \operatorname{Re} \left(p \sum_{n \geq 2} R_n e^{-pyR_n} \right) \leq M + o(1).$$

The absolute geometric derivative sums multiplied by v^2 are uniformly bounded, so each replacement error is controlled.

For $y \geq V/h$, use $n/2 \leq R_n \leq 2n$ in the complete sum. The absolute residue derivative is at most $Ce^{-v}(1 - e^{-v/2})^{-2}$. After multiplication by ey^{a+1} this is uniformly small for large V , because eh^{-a-1} remains bounded. The lower bound (4.16) holds throughout this infinite range and has positive limiting normalized constant.

The origin is controlled separately. In (4.12), $\delta = \pi(1/a - 1/2)(1 - q) \rightarrow \pi/2$, so $\delta \geq \pi/3$. The germ bounds j'_a uniformly on $|z| \leq 1$, and the Poisson derivative is bounded by

$$\frac{4}{\pi} \int_0^\infty \frac{j_a(u)}{u^2 + y^2} du \leq \frac{3}{y} + C.$$

Here $j_a \leq 3/2$ and $\int_2^\infty j_a(u)u^{-1} du$ is uniformly bounded by the unit-Laplace bound in the unit-origin estimate (H.13). Consequently

$$-\partial_y K(c_a y) \geq \frac{a \sin \delta}{\pi e y^2} - \frac{3}{y} - C > 0 \quad (0 < y \leq 1)$$

for all sufficiently small e .

If $C_0^2 > 3\sqrt{3}/\pi$, then $h^2/e \rightarrow \pi^2 C_0^2/4 > \pi M$. The origin, compact-to-growing, and infinite-tail estimates prove $-\partial_y K > 0$ at every radius. Hence this q is SD. If $C_0^2 < 3\sqrt{3}/\pi$, choose y near $Y = 2\pi$ with $c_0 y/d_e = 2\pi + (hy/d_e)/\sqrt{3}$. Equality in the nearest-pole limit of (4.14) and the fixed-radius equality in (4.16) give

$$ey^2 \partial_y K(c_a y) \rightarrow -\frac{1}{\pi} + \frac{4M}{\pi^2 C_0^2} > 0.$$

Thus this q is not SD. The interval classification sandwiches $q_{\text{SD}}/\sqrt{e}$ between arbitrary constants on either side of $\sqrt{3\sqrt{3}/\pi}$. Since $s(a) - 1/2 = (1/a - 1/2)q_{\text{SD}}$, this proves (1.9). No explicit smallness threshold is used. $\square$

5. BELL-SHAPE: EXACT TESTS AND STRUCTURAL CONSEQUENCES

We first derive the two-ray integer-level test and a uniform tail bound. These yield the exclusions, the exceptional-family criterion, and open BS regions outside GGC. We then study the geometry of the skewness sections and the bands that accumulate near index one.

5.1. The phase test and an explicit tail. For $0 < a < 1$ and $0 < \theta < \pi$ let

$$\Phi_{a,\theta}(r) = \pi^{-1} \operatorname{Im} \psi_a(re^{i\theta}).$$

The continuous logarithm is fixed by Lemma 4.1. Theorem 2.3 and the analyticity of the interior free stable densities give the exact equivalence

$$(5.1) \quad \begin{aligned} X_{a,\rho} \text{ is strictly BS} &\iff \text{there are no } 0 < u < v \text{ and } m \in \mathbb{N} \text{ with} \\ &\Phi_{a,\theta}(u) > m > \Phi_{a,\theta}(v), \\ &\theta \in \{\pi\rho, \pi(1-\rho)\}. \end{aligned}$$

Indeed each phase starts at zero and tends to infinity on a proper ray, by (4.1) and $b_a > 0$. Thus every integer level is eventually crossed upwards, and at most one change of sign is equivalent to the absence of any subsequent downward crossing.

Lemma 5.1 (A whole-ray derivative bound). *For every $0 < a < 1$, $r > 0$ and $0 < \theta < \pi$,*

$$(5.2) \quad \operatorname{Im} H_a(re^{i\theta}) \geq b_a r \sin \theta - \tan(\theta/2).$$

In particular $\Phi'_{a,\theta}(r) > 0$ when $b_a r(1 + \cos \theta) > 1$.

Proof. The bounded imaginary part in (4.1) gives a bounded-density Herglotz representation

$$\psi_a(z) - b_a z = \int_0^\infty \frac{z\phi(t)}{t(z+t)} dt, \quad 0 \leq \phi(t) \leq 2.$$

There is no singular part: the imaginary part of the holomorphic function is bounded in the upper half-plane. The constant is zero by $\psi_a(0) = 0$. Differentiation on compact slit subsets yields $H_a(z) - b_a z = \int z\phi(t)(z+t)^{-2} dt$. After $t = ru$, the imaginary kernel is

$$\operatorname{Im} \frac{e^{i\theta}}{(u + e^{i\theta})^2} = \frac{\sin \theta(u^2 - 1)}{|u + e^{i\theta}|^4}.$$

It is negative only on $0 < u < 1$. Using $\phi \leq 2$ there and $\phi \geq 0$ elsewhere bounds the integral from below by

$$2 \operatorname{Im} \int_0^1 \frac{e^{i\theta}}{(u + e^{i\theta})^2} du = -\tan(\theta/2).$$

This proves the assertion, including all radii beyond the stated cutoff. $\square$

The phase test permits rigorous numerical witnesses without a numerical logarithm-unwrapping assumption. Put $\beta = 1 - a$, $T = \beta r^a$, $\eta = \pi(1 - a\rho)$ and $G_\beta(t) = e^{-t/\beta} W_a(t/\beta)$, with $t = Te^{-i\eta}$. A computed candidate is

$$(5.3) \quad \Phi_{a,\pi\rho}(r) = \frac{T \sin \eta}{\pi\beta} - \frac{\operatorname{Arg} G_\beta(t)}{\pi} + 2k.$$

There is exactly one integer lift satisfying

$$0 < \Phi_{a,\pi\rho}(r) - \frac{b_a r \sin(\pi\rho)}{\pi} < 2\rho.$$

Its interval has length less than two, so strict interval checks select the correct branch. An inconclusive check is not accepted as a phase value. The retained-circle Hankel evaluator, including both banks and their tails, is described in the appendix.

5.2. Exclusion above one and the exceptional family. For $a > 1$, the holomorphic characteristic function (3.2) vanishes at $t = \lambda_n^{1/a} e^{i\pi(\rho-1/2)}$. These points belong to the open right half-plane at every admissible skewness. By [11, Remark 1.2(b) and Theorem 1.3], a weakly bell-shaped probability law has a zero-free holomorphic characteristic transform there. The identity theorem proves the contradiction. This exclusion does not use either numerical cutoff. At one the interior strictly stable laws are Cauchy. Below one the endpoint densities are whale-shaped by Hasebe–Simon–Wang, and hence weakly BS by [11, Theorem 1.13]; they are not strictly BS.

For the exceptional family scale and reflect to $|B| = 1$ and put $\ell = A/(\pi|B|)$. The explicit Lévy density in [7, Theorem 3 and Remark 7] has, besides its Cauchy part, the kernel

$$\frac{1}{x} - \sum_{n \geq 2} e^{-nx}.$$

The Laplace densities of the two Lévy densities are therefore

$$(5.4) \quad \ell u, \quad (\ell + 1)u - N(u), \quad N(u) = \#\{n \geq 2 : n \leq u\}, \quad u > 0.$$

At an integer $n \geq 2$, the second function drops from $\ell n + 2$ to $\ell n + 1$. If ℓn is nonintegral, an integer level lies strictly between these two limits, violating the level condition in [11, Theorems 1.1 and 1.3]. Values at the single jump point do not remove the strict inequalities on its two sides. Consequently 2ℓ and 3ℓ must both be integers, which forces ℓ to be an integer. Conversely, at integer $\ell \geq 0$ every downward jump lies between consecutive integer levels; between jumps the function increases, and successive post-jump minima do not decrease. Each integer has at most one sign change. The other side is linear. This proves the exact weak BS classification. Cauchy convolution gives a smooth density for $A > 0$, hence strict BS; $A = 0$ is whale-shaped. This supplies the converse to [6, Theorem 4(d)].

All these exceptional laws are SD: after unit scaling, the derivative of the non-Cauchy kernel is

$$-x^{-2} + \frac{e^x}{(e^x - 1)^2} - e^{-x} < 0,$$

since $2 \sinh(x/2) > x$. Adding the Cauchy kernel preserves this sign. For $B \neq 0$ the jumps in (5.4) preclude GGC, equivalently by uniqueness of the Laplace spectral measure. This also recovers the known non-GGC conclusion in [7, Remark 7].

5.3. Bell-shape beyond GGC.

Proposition 5.2 (Noninteger contacts leave room for bell-shape). *Consider a parameter family satisfying Theorem 2.3, with phases starting at zero and tending to infinity. Suppose every decreasing part of each phase is contained in finitely many disjoint compact intervals, the phase on each such interval is in a single open integer strip $(m, m + 1)$, and the phase derivative is positive on their complement. Then the law is weakly bell-shaped, and strictly so if its density is smooth. These conditions are stable under parameter perturbation whenever the strict strip and derivative bounds are locally uniform, including uniform controls at zero and infinity.*

Proof. No interval containing a decreasing part contains an integer level. Between these intervals each integer passage is upward. Since the phase starts below each positive integer and ends above it, there is exactly one change of sign for every level. Apply Theorem 2.3. Strict margins on the finitely many compact pieces, and the stated controls at both ends, persist under small parameter perturbations. $\square$

At a GGC boundary with finitely many contacts, if none of the contact phase values is an integer, take small disjoint cores around them. The phase stays in an open integer strip in each core. Strict outside-core GGC positivity and the end estimates then imply a BS neighborhood, even if the radial derivative becomes negative inside a core. This gives a general explanation for the separation between GGC and bell-shape.

The following examples are independently certified:

$$(5.5) \quad (a, \rho) = \left(\frac{99}{100}, \frac{9869504901}{10^{10}} \right), \quad \left(\frac{19}{20}, \frac{99340401760}{10^{11}} \right), \quad \left(\frac{9}{10}, \frac{99993759565}{10^{11}} \right).$$

For the upper ray the respective compact T -cores are $[2.005, 2.011]$, $[2.06, 2.10]$ and $[2.99, 3.12]$. Their phase strips are $(4, 5)$, $(1, 2)$ and $(1, 2)$. The complete GGC slice certificates give strict derivative positivity at every radius outside these cores, and a strict negative derivative inside at the listed skewness. The three additional whole-core phase enclosures are recomputed by `certify_bell_beyond_ggc.py` `verify`, with the correct branch fixed by (5.3). The opposite ray has angle below $\pi/2$ and is strictly increasing by symmetric GGC and sector interpolation. Proposition 5.2 proves strict BS and non-GGC. The strict witnesses and locally uniform end estimates give nonempty open neighborhoods with the same properties.

For comparison, at $a = 999/1000$, $\rho = 1999/2000$, the same directed phase evaluator gives

$$\Phi(T = 29/10) > 3.30774 > 3 > 2.69471 > \Phi(T = 31/10).$$

Here $T = \beta r^a$ is increasing with radius, so this is a strict non-BS witness. Its reflected parameter is also non-BS. This statement concerns all orders of the density derivatives through the characterization theorem, rather than a finite derivative plot.

5.4. Endpoint bands.

Proposition 5.3 (Strict BS next to each one-sided law). *For each $0 < a < 1$ there is $\eta(a) > 0$ such that $X_{a,\rho}$ is strictly bell-shaped when $0 < \rho < \eta(a)$ or $1 - \eta(a) < \rho < 1$. The assertion is locally uniform in a on $(0, 1)$.*

Proof. The GGC theorem handles $a \leq 4/5$. In the remaining range put $\beta = 1 - a$ and continue L_a to the upper negative boundary. Euler reflection gives the entire series, in $t = r^a$,

$$(5.6) \quad \operatorname{Im} L_a(-r + i0) = \sum_{n \geq 1} \frac{\Gamma(\beta n - 1) \sin^2(\pi \beta n)}{\pi n!} t^n.$$

Removable Gamma factors are interpreted by continuity. The nonzero coefficients are negative before $\beta n = 1$ and positive after it. Let m be the largest integer strictly less than $1/\beta$. After division by t^m , every derivative coefficient is nonnegative and at least one is positive. The quotient starts negative and tends to positive infinity. Thus (5.6) has exactly one positive zero t_0 , with positive derivative.

There is no zero of L_a on this cut. Such a zero could only be at $r_0 = t_0^{1/a}$ and would be simple. The bound $0 \leq \Phi_{a,\pi} \leq 2$ and the signs in (5.6) place the phase in $(0, 1)$ before r_0 and $(1, 2)$ after it. But along a small upper semicircle about a simple zero, $-\log L_a$ decreases its imaginary part by π in this direction. This contradicts the required increase. Hence $L_a(-r_0) < 0$, $\Phi_{a,\pi}(r_0) = 1$, and $\Phi'_{a,\pi}(r_0) > 0$. This is the only integer crossing on the boundary, and it is transverse.

We need a tail estimate uniform up to that boundary. Choose an integer N with $\beta N > 1$, let E_N be the exponential minus its first N Taylor terms, and put

$$J_N(z) = \int_0^\infty e^{-u} u^{-2} E_N(z u^\beta) du.$$

Both endpoints converge, and reflection gives exactly

$$(5.7) \quad W_a(z) = P_{N-1}(z) + \frac{i}{2\pi} \{J_N(z e^{i\pi\beta}) - J_N(z e^{-i\pi\beta})\}.$$

For $s = r e^{i(\pi-\delta)}$, the first integral's argument is $r^a e^{-ia\delta}$. Rotate $u = e^{-i\delta} v$. The saddle of $-v + r^a v^\beta$ is $v_0 = \beta^{1/a} r$, its maximum is $b_a r$ and its second derivative is $-a/v_0$. For small

complex δ the Gaussian expansion, with fixed relative saddle neighborhood, gives

$$J_N(r^a e^{-ia\delta}) = \sqrt{2\pi/a} v_0^{-3/2} e^{3i\delta/2} e^{b_a r e^{-i\delta}} (1 + O(r^{-1})).$$

Strict concavity gives an exponential gap off that neighborhood. The subtractions make the small arc vanish; their contributions are polynomial or $\exp(O(r^a))$, hence smaller than the saddle. For the second integral, the largest real exponent is at most $b_a r \max(\cos(2\pi\beta + a\delta), 0)^{1/a}$, separated strictly from $b_a r \cos \delta$ for sufficiently small δ . These estimates, first on a slightly larger wedge, also permit Cauchy differentiation of the normalized remainder. Thus, with $C_a = \beta^{-3/(2a)}/\sqrt{2\pi a}$,

$$(5.8) \quad L_a(s) = C_a e^{-b_a s} s^{-3/2} (1 + O(r^{-1})), \quad \psi'_a(s) = b_a + \frac{3}{2s} + O(r^{-2}),$$

uniformly for $0 \leq \delta \leq \delta_0$, on the continuation from the upper slit. The phase branch is fixed by (4.1): at $\delta = 0$ its limit is $3/2$.

It follows that, for some C, R ,

$$\begin{aligned} \left| \Phi_{a,\pi-\delta}(r) - \frac{b_a r \sin \delta}{\pi} - \frac{3}{2}(1 - \delta/\pi) \right| &\leq C/r, \\ \left| \Phi'_{a,\pi-\delta}(r) - b_a \sin \delta/\pi \right| &\leq C/r^2. \end{aligned}$$

Take $\delta_0 < \pi/12$, $C/R < 1/16$ and $R > 4C$. For $r \geq R$ set $D = b_a r \sin \delta/\pi$. If $D \leq 1/4$, the phase lies strictly between one and two. If $D \geq 1/4$, its derivative is at least $1/(4r) - C/r^2 > 0$. For δ small, $D(R) < 1/4$. Thus every integer at least two is crossed exactly once in the increasing tail, and level one is never reached there.

On the remaining compact radii, isolate the transverse boundary crossing and use continuity. Away from it the boundary phase has positive distance from every integer; the origin Wright series controls a uniform neighborhood of zero. Small δ therefore preserves just the single upward crossing of one. The opposite ray has angle $\delta < \pi/2$ and is increasing by symmetric GGC. The criterion (5.1) proves the result. All constants and the integer N can be chosen on a compact neighborhood of a fixed a ; at $a = 4/5$ the same estimates overlap the GGC range. This proves local uniformity. $\square$

5.5. Finite components and boundary contacts. We use two standard notions from real analytic geometry. A semianalytic set is locally a finite union of sets described by finitely many real analytic equalities and inequalities. A subanalytic set is locally a projection of a relatively compact semianalytic set; see [3, Sections 2 and 3]. These notions will be applied only on compact parameter and radius charts.

Proposition 5.4 (Finite geometry on compact index ranges). *For a fixed $0 < a < 1$, the BS section is a finite union of relatively closed intervals in $(0, 1)$ and possibly isolated points. Its number of connected components is locally bounded as a function of a . At every interior boundary skewness there is a finite nonzero radius and an integer $m \geq 1$ such that, on one of the two rays,*

$$(5.9) \quad \Phi_{a,\theta}(r) = m, \quad \partial_r \Phi_{a,\theta}(r) = 0.$$

Proof. Fix a compact index interval inside $(0, 1)$. The locally uniform endpoint bands restrict all possible failures to $\rho \in [\eta, 1 - \eta]$ for some $\eta > 0$. On this compact parameter set the Wright expansion gives a uniform positive derivative near zero, and Lemma 5.1 gives a uniform positive derivative beyond a finite radius. Only $r \in [r_-, r_+]$ can therefore participate in a downward crossing. The phases are bounded there, so only finitely many positive integers $m \leq M$ need to be tested.

The phase is jointly real analytic in a, ρ, r on a neighborhood of this compact set. For each of the two rays and each $m \leq M$, the strict-witness set

$$\{(a, \rho, u, v) : r_- \leq u < v \leq r_+, \quad \Phi(u) > m > \Phi(v)\}$$

is relatively compact and semianalytic. Its projection onto (a, ρ) is subanalytic, and their finite union is exactly the non-BS set. The complement is subanalytic by the complement theorem. Compact one-dimensional subanalytic sets have finitely many components, and the fiber component counts are locally bounded; see [3, Definition 3.1, Theorems 3.10 and 3.14]. Strict witnesses persist, so the BS section is relatively closed. Adding the endpoint bands proves the stated form.

If no contact (5.9) existed at a boundary, all the finitely many compact-range integer crossings would be transverse and upward. The implicit function theorem and compactness would preserve them, as well as all intervening integer strips, in a parameter neighborhood. This contradicts boundary status. The contact condition is necessary; a local contact need not be a global boundary. $\square$

For a fixed parameter, an equivalent finite test lists the maximal decreasing runs (u_j, v_j) of each phase in the compact range and asks that

$$(\Phi(v_j), \Phi(u_j)) \cap \mathbb{N} = \emptyset.$$

Negative derivative intervals must be joined across an isolated zero if no positive derivative interval separates them; otherwise an odd higher-order downward crossing could be missed. Equality to an integer at an extremum is permitted: a tangency is not a sign change. This finite reduction does not locate all parameter boundaries.

Proposition 5.3, the symmetric GGC law, and the strict non-BS witness at $a = 999/1000$, $\rho = 1999/2000$ imply at least three connected components at $a = 999/1000$. Thus the BS section need not have the central-interval form of the GGC section.

5.6. A quantization obstruction near one. Suppose $a_j \uparrow 1$, $\beta_j = 1 - a_j$ and $\min(\rho_j, 1 - \rho_j)/\beta_j \rightarrow \ell < \infty$. If all these laws are weakly BS, then

$$(5.10) \quad \ell \in \{0, 1, 2, \dots\}.$$

Indeed the Cauchy factorization and the affine endpoint limit give, after reflection if necessary,

$$\frac{\beta_j^{\beta_j} - X_{a_j, 1 - \min(\rho_j, 1 - \rho_j)}}{\beta_j} \implies T + \pi \ell C.$$

The positive-law endpoint limit is the exceptional-law limit in [7, Section 2.4, (15)], also recovered from the compact Hankel limit in the appendix. Weak BS is affine invariant and closed under weak convergence [11, Lemma 3.1]. The exceptional classification therefore gives (5.10). Uniformly on a compact set of noninteger ratios, this excludes BS for all sufficiently small β . This is a necessary obstruction, not a full classification or a claim that every integer-ratio curve is admissible.

5.7. Integer-ratio bands and an unbounded number of components. The quantization obstruction has a partial converse with a uniform width. Only a qualitative eventual statement is needed here.

Proposition 5.5. *For every integer $\ell \geq 1$ there are $\beta_\ell, k_\ell > 0$ such that, whenever $0 < \beta < \beta_\ell$, $a = 1 - \beta$, and*

$$(5.11) \quad |\pi(1 - \rho) - \delta_\ell(\beta)| \leq k_\ell \beta^2, \quad \delta_\ell(\beta) = \arctan \frac{\ell \sin c \cos c}{1 + \ell \sin^2 c}, \quad c = \frac{\pi \beta}{1 - \beta},$$

the law $X_{a, \rho}$ is strictly BS and is not GGC. The arctangent is its small positive branch, and $\delta_\ell(\beta)/\pi = \ell\beta + \ell\beta^2 + O_\ell(\beta^3)$. Reflection gives the other band. Consequently the number of BS components tends to infinity as $a \uparrow 1$. No explicit value of β_ℓ , optimal width, or exact component count is asserted.

The required estimate must keep the two saddles together. Separate fixed-index Wright expansions would not control their cancellation.

Lemma 5.6 (Uniform two-saddle estimate). *Let $0 < \beta \leq 1/10$, $a = 1 - \beta$, $N = \lceil 2/\beta \rceil$, $E_N(z) = e^z - \sum_{k < N} z^k/k!$, and*

$$J_N(z) = \int_0^\infty e^{-u} u^{-2} E_N(zu^\beta) du.$$

There are constants $C, V_0, \epsilon > 0$, independent of β , such that, for $|v| \geq V_0$ and $|\arg v| \leq \epsilon$,

$$(5.12) \quad \begin{aligned} J_N(v^a/\beta) &= \sqrt{2\pi/a} v^{-3/2} e^{av/\beta} A_\beta(v), \\ |A_\beta - 1| &\leq C/|v|, \quad |(\log A_\beta)^{(j)}| \leq C|v|^{-j-1} \quad (j = 1, 2). \end{aligned}$$

The amplitude is holomorphic in v and positive on the positive ray. For $s = re^{i(\pi-\delta)}$, $\delta = O(\beta)$, put

$$v = \beta^{1/a} r, \quad R = av/\beta, \quad x = R\{\sin(\delta + 2c) - \sin \delta\} - 3c, \quad \kappa = \tan(\delta + c), \quad q = e^{-\kappa(x+3c)}.$$

Then, uniformly for real $v \geq V_0$,

$$(5.13) \quad L_a(s) = \frac{i}{2\pi} J_N(r^a e^{-ia\delta}) \{1 - qe^{-ix+e_\beta(v)}\}, \quad |e_\beta(v)| \leq C\beta/v, \quad |e'_\beta(v)| \leq C\beta/v^2.$$

The constants in the last statement can depend on a fixed bound for δ/β . No parameter derivative across a jump of N is used.

Proof. The regularization makes the integral entire, since $\beta N \geq 2$. Rotating $u = vw$ gives the normalized phase

$$f_\beta(w) = w - 1 - \frac{w^\beta - 1}{\beta}, \quad f_0(w) = w - 1 - \log w.$$

On $[1/2, 2]$ it has its unique minimum at one, second derivative a , uniformly bounded higher derivatives, and a uniform lower bound on its second derivative. The local Gaussian expansion gives (5.12), provided the two ends and the polynomial subtractions are controlled uniformly.

Write $R_0 = |v|$, $w_0 = (8/R_0)^{1/\beta}$ and $p = \beta N \in [2, 2 + \beta]$, taking V_0 large and $\cos \epsilon > 3/4$. The exact Taylor remainder

$$|E_N(z)| \leq |z|^N e^{\max(\operatorname{Re} z, 0)} / N!$$

bounds the normalized integral on $(0, w_0)$ by an algebraic factor times

$$\exp\{\log R_0 + (p-1) \log 8 + p(1 - \log p) + 8 - a \operatorname{Re} v\} / \beta\}.$$

This is $O(e^{-c_0 R_0/\beta})$. On $[w_0, 1/2]$ the logarithmic derivative of $w^{-2} e^{-\operatorname{Re} v f_\beta(w)}$ has the sign of $\operatorname{Re} v(w^\beta - w) - 2$. Concavity of $w^\beta - w$ reduces positivity to the two endpoints: $R_0(w_0^\beta - w_0) \geq 7$ for large V_0 , while $2^{-\beta} - 1/2 > 0.43$. Hence this integral is $O(e^{-c_0 R_0})$. On $[2, \infty)$ the uniformly positive derivative of f_β gives the same bound.

To estimate the subtractions on $[w_0, 1]$, use

$$\int_{w_0}^1 w^{\beta k - 2} dw \leq \log(1/w_0) \max(1, w_0^{\beta k - 1}).$$

Their coefficient sums are bounded by $N(eR_0/p)^{p/\beta}$ and $(R_0/8)^{1/\beta} e^{8/\beta}$, respectively. On $[1, \infty)$ use $w^{\beta k - 2} \leq 1 + w^{1/10}$ and the factor $e^{-\operatorname{Re} v w}$. After division by $e^{a \operatorname{Re} v/\beta}$ all these terms are $O(e^{-c_0 R_0/\beta})$. Factors polynomial in $1/\beta, R_0, \log R_0$ are absorbed by a smaller c_0 . This proves the uniform expansion. Prove it first on a slightly larger sector; Cauchy estimates on relative disks give both logarithmic derivative bounds. In particular, for small real d_1, d_2 ,

$$|\operatorname{Im} \log A_\beta(v e^{-id_1})| \leq C|d_1|/v, \quad |\log A_\beta(v e^{-id_2}) - \log A_\beta(v e^{-id_1})| \leq C|d_2 - d_1|/v,$$

and their v derivatives gain a factor $1/v$.

Apply the exact reflection identity (5.7). The two saddle coordinates are $v e^{-i\delta}$ and $v e^{-i(\delta+2c)}$. Their exponential quotient is $q e^{-ix}$. The preceding amplitude difference is $O(\beta/v)$.

The reciprocal-Gamma coefficients in the polynomial P_{N-1} lie in a fixed compact range, because $0 \leq \beta k \leq 2 + \beta$. The same finite-sum estimate bounds that polynomial relative to the first saddle by $e^{-c_0 v/\beta}$. It is exponentially small also relative to q , since $-\log q = O(\beta v)$. It can therefore be absorbed into the exponent e_β . The estimates on a larger sector and Cauchy's formula give its derivative bound as well. This proves (5.13). $\square$

Proof of Proposition 5.5. We treat three radial ranges and then the perturbation of the center. All constants in this proof may depend on the fixed integer ℓ . Use the variable $T = \beta r^a = v^a$.

Bounded radii. The compact Hankel expansion gives $G_\beta(t) = e^{-t/\beta} W_a(t/\beta) \rightarrow 1/\Gamma(2-t)$ holomorphically. Near each of its finitely many zeros $n \geq 2$ in a fixed compact range, write $G_\beta(t) = (t - \zeta_n(\beta))A_n(t, \beta)$, with $\zeta_n(\beta) = n + O(\beta)$ real and A_n analytic and nonzero. For $\delta = \pi\ell\beta + O(\beta^2)$ the phase is a smoothed unit drop about $\ell n + 3/2$, plus an increasing linear term. The elementary argument of $t - \zeta_n$ bounds the smaller distance to $\ell n + 1, \ell n + 2$ from below by a positive constant times

$$|T - \zeta_n| + \frac{\beta}{|T - \zeta_n| + \beta},$$

on a fixed sufficiently small neighborhood, up to an $O(\beta)$ error. Its minimum is of order $\sqrt{\beta}$, so the whole neighborhood remains in the open integer strip. On the compact complement, the phase derivatives converge uniformly to $\ell + 1 > 0$. The origin follows either from the same formula at zero or from its first Wright coefficient. Thus bounded radii admit a cover by increasing pieces and open integer strips.

The large-radius center. Equation (5.11) at equality in its center is equivalent to

$$\frac{2 \sin \delta}{\sin(\delta + 2c) - \sin \delta} = \ell.$$

The logarithm of (5.13) then yields the model phase

$$(5.14) \quad \Phi = \frac{\ell x}{2\pi} + \frac{3}{2} - \frac{\text{Arg}(1 - qe^{-ix})}{\pi} + O(\beta^3) + O(\beta/v),$$

together with a perturbation of size $O(\beta/v)$ in the exponent of qe^{-ix} . The constant error uses $\ell c - \delta_\ell = O(\beta^3)$, not merely $O(\beta^2)$.

Write $x = 2\pi n + u$, $|u| \leq \pi$, and put $u_0 = \pi/(\ell + 1)$. On $|u| \leq u_0$, the model lies in $(\ell n + 1, \ell n + 2)$ with smaller distance at least

$$(5.15) \quad c_\ell \min \left(1, |u| + \frac{1 - q}{|u| + 1 - q} \right).$$

To see this for $u \geq 0$, set $A = \text{Arg}(1 - qe^{-iu})$. Then $0 \leq A \leq (\pi - u)/2$ and $\pi/2 - A = \arctan((1 - q \cos u)/(q \sin u))$. These bound the lower distance by constant multiples of u and $\min(1, (1 - q)/u)$, while the upper distance is at least $1/[2(\ell + 1)]$. Reflection handles negative u .

Here $1 - q \asymp \min(1, \beta v)$ and $|1 - qe^{-iu}| \geq c(|u| + 1 - q)$. If $w = \beta v \leq 1$, the exponent error relative to (5.15) is $O(v^{-2})$, since

$$(|u| + w) \left(|u| + \frac{w}{|u| + w} \right) \geq w.$$

For $w \geq 1$ the margin is bounded below and the error is $O(\beta/v)$. The additive terms in (5.14) are smaller than $c \min(1, \sqrt{\beta v})$. Increase V_0 first, then decrease β , to preserve each strip.

Outside these cores,

$$\frac{d}{dx} \text{Arg}(1 - qe^{-ix}) = \frac{q \cos u - q^2 - \kappa q \sin u}{1 - 2q \cos u + q^2}.$$

For $\cos u \geq 0$ the first quotient is at most $(\csc |u| - 1)/2$, and for $\cos u \leq 0$ it is at most zero. The term with κ is $O_\ell(\beta)$. Thus, using $\sin u_0 \geq 2/(\ell + 1)$, the derivative of the model phase is at least $(\ell + 1)/(4\pi) + o(1) > 0$. The differentiated amplitude errors are uniformly smaller.

There is no unproved logarithm lift in this argument. The exponent bound gives $|qe^{e\beta}| \leq e^{-c_\ell \beta v + C\beta/v} < 1$ after increasing V_0 . Hence

$$|\operatorname{Arg}(1 - qe^{-ix+e\beta})| \leq \pi/2 - c_\ell \min(1, \sqrt{\beta v}).$$

Together with the amplitude argument bound, this puts the phase after removal of its drift strictly inside $(0, 2\rho)$. The exact bound (4.1), whose interval has length less than two, fixes the lift uniquely.

Width and the infinite tail. For $|\delta - \delta_\ell| \leq k_\ell \beta^2$, the effective ratio in the drift-frequency identity is $\ell + O(k_\ell \beta)$. On $V_0 \leq v \leq M_\ell/\beta$ this changes the phase by at most $C_\ell k_\ell \beta v$. Relative to the least strip margin its size is bounded by $C_\ell k_\ell \max(\sqrt{M_\ell}, M_\ell)$. Choose k_ℓ small; the positive-derivative pieces are stable as well. On $v \geq M_\ell/\beta$, Lemma 5.1 gives positive derivative once

$$(av/\beta)(1 - \cos \delta) > 1.$$

A sufficiently large fixed M_ℓ works because $\delta/(\pi\beta) \rightarrow \ell > 0$. The bounded-radius argument absorbs the same perturbation. The opposite ray has angle $\delta < \pi/2$ and has positive derivative by the symmetric GGC result and sector interpolation. The whole-ray integer criterion proves strict BS. The simple-zero expansion at $T = 2$ supplies two radii with a strictly decreasing phase, so none of these laws is GGC.

Finally fix any K . Take the minimum of the thresholds for $\ell = 1, \dots, K$ and for the $K + 1$ half-integer exclusions in (5.10). The endpoint band, the K integer bands, and the central GGC band are separated by these non-BS points. Reflection yields at least $2K + 3$ BS components for every sufficiently small β . Since K was arbitrary, their number tends to infinity, consistently with finiteness at each fixed index. $\square$

The arguments in this section prove Theorem 1.6.

6. CONSEQUENCES AND COMPUTATIONAL DEPENDENCIES

Within the strictly free stable family the results give a strict hierarchy, with weak bell-shape understood as a property of laws:

$$(6.1) \quad \text{GGC} \subsetneq \text{weakly BS} \subsetneq \text{SD} \subsetneq \text{ID}.$$

The first inclusion is the general Thorin consequence of the phase criterion and is strict by (5.5). Every weakly BS member has index at most one, hence is SD by Theorem 1.4. The law $X_{6/5,5/6}$ is SD but not weakly BS, while $X_{3/2,2/3}$ is ID but not SD. Both assertions follow directly from the two cutoff brackets and the extremal endpoint classification. The middle inclusion in (6.1) is a conclusion about this family, not a general implication for arbitrary probability laws.

6.1. Boundary scales near index one.

Corollary 6.1 (Different boundary scales at index one). *With $\varepsilon, \beta \downarrow 0$,*

$$\begin{aligned} r(1 + \varepsilon) &= \frac{1}{2} + 2\varepsilon + C_{\text{ID}}\varepsilon^2 + o(\varepsilon^2), \\ s(1 + \varepsilon) &= \frac{1}{2} + \sqrt{\frac{3\sqrt{3}}{4\pi}}\sqrt{\varepsilon} + o(\sqrt{\varepsilon}), \\ g(1 - \beta) &= \frac{\sqrt{\beta}}{\pi\sqrt{2}} - \beta + \frac{2\pi^2 + 13}{12\pi\sqrt{2}}\beta^{3/2} - \beta^2 + O(\beta^{5/2}), \end{aligned}$$

where

$$C_{\text{ID}} = 2\log(2\pi) - 12\pi^2 + 2 \int_0^\infty \frac{v^2}{4\pi^2 + v^2} \frac{e^{-2v}}{1 - e^{-v}} dv = -114.740012808536 \dots$$

The ID and GGC contacts are each uniquely controlling for sufficiently small positive parameters in these expansions. No common numerical smallness threshold is claimed.

Proof. The ID expansion and its uniform spectral argument are Theorem H.7; the GGC expansion, including its whole-radius localization, is Theorem H.5. The SD assertion is (1.9). $\square$

6.2. Convolution powers and classical embeddings. The ID domain is exactly the domain admitting a classical Lévy process with one-time law $X_{a,\rho}$. The SD domain is the smaller domain admitting the corresponding stationary Ornstein–Uhlenbeck representation with scalar positive mean reversion; the standard equivalence is [12, Theorems 17.5 and 17.9]. For every $1 < a < \alpha_{\text{SD}}$, the nonempty region $r(a) \leq \rho < s(a)$ on the upper half, and its reflection, gives classically and freely ID laws which are not classically SD. At $a = \alpha_{\text{ID}}$ the only ID laws already fail SD because their nonnegative kernel has a finite positive zero and a positive tail.

There is also an explicit construction at every SD parameter. For unit mean reversion, the background driving Lévy process Z in the stationary equation $dU_t = -U_t dt + dZ_t$ has Lévy measure

$$(6.2) \quad \nu_Z(dx) = -k'_+(x)\mathbf{1}_{\{x>0\}} dx - k'_-(-x)\mathbf{1}_{\{x<0\}} dx.$$

Indeed the stationary stochastic integral $\int_0^\infty e^{-t} dZ_t$ transforms its positive-side density n_+ into

$$\int_0^\infty e^t n_+(e^t x) dt = \frac{1}{x} \int_x^\infty n_+(y) dy = \frac{k_+(x)}{x},$$

and likewise on the negative side. The zero limit of the kernels at infinity justifies the last identity. The drift is chosen by the usual compensated Lévy formula. Thus the same derivative tests used in the classification give nonnegative driving densities; a failed SD test rules out this stationary representation.

The complete GGC classification also characterizes the laws whose *every classical convolution power* is weakly bell-shaped, by [11, Corollary 1.5]. In particular the strictly BS, non-GGC examples (5.5) have a classical integer convolution power which fails bell-shape. This is a concrete distinction between bell-shape of one distribution and preservation of bell-shape along its entire convolution semigroup. This implication is constructive in the phase framework. If $P(u) > P(v)$ for $u < v$, any integer n with $n(P(u) - P(v)) > 1$ places an integer level strictly between $nP(v)$ and $nP(u)$. The spectral phase of the n th convolution power is nP , so that power fails the integer-level criterion. Conversely, nondecreasing P remains nondecreasing after multiplication by every positive real time.

The framework in Section 2 is a synthesis of Lévy-kernel, Pick-function and integer-level criteria. The characterization of general bell-shaped functions belongs to Kwaśnicki–Simon; the contribution here is its compatible use with the ID, GGC and SD ray tests, and the global contact and tail arguments needed for the free stable family.

6.3. What the finite certificates establish. A certificate is a finite binary partition of an explicitly specified rational parameter box, or an exact list of root brackets and moment remainders. Verification reconstructs the partition from its roots, checks that both children exhaust each parent, and recomputes the required interval inequality on every leaf. It does not rely on a saved decimal value. Integration is carried out with directed complex ball arithmetic and analytic quadrature, with an explicit independent bound for every omitted contour piece; see [8]. Analyticity flags are forwarded to logarithms in the quadrature callbacks. Root completeness uses moments or argument principles, never a finite sign scan alone.

The finite computational inputs essential to the main classification statements can be organized as follows. Counts refer to accepted whole-box inequalities, not sampled points.

Assertion	Finite cover	Analytic completion
Uniform GGC through 4/5	20,261 compact leaves	Kanter series at zero and the uniform tail beyond radius 500
Symmetric GGC below one	720 leaves	Hankel origin and the explicit tail beyond scaled radius 200
Maximal ID index	4,988 transition leaves; 36 cutoff leaves	Convergent germs, moving roots, omitted-spectrum bounds and full-skewness exclusions
Positive SD below one	291 leaves	All-order coefficient induction and a single incomplete-Gamma tail inequality
Extremal SD below transition	77 near-one and 749 continuation leaves	Differentiated germs and full exponential/moment tails
SD transition and exclusion	7,118 transition and 158 exclusion leaves	Core convexity, origin/tail control, and an independent parameter-derivative proof
SD cutoff refinement	Two rational endpoints	896-term germ with an infinite tail and the global unique-minimum theorem
BS outside GGC	Three whole-core phase boxes	Previously verified whole-ray GGC slices and strict non-GGC witnesses

All filenames below are relative to the research archive containing this manuscript directory. Run the following commands from that archive with Python and `python-flint` installed (prefix each line with `python`). The archived Kanter run used `python-flint` 0.9.0; the working precisions and truncations are specified in the individual verifiers. The principal reproduction commands are

```

certify_kanter_optimal.py verify
certify_symmetric_ggc.py verify
certify_global_id_cutoff.py verify
certify_sd_below_one.py verify
certify_near_one_extremal_sd.py verify
certify_extremal_sd_extension.py verify
certify_sd_transition_global.py verify
certify_sd_transition_alpha_derivative.py verify
certify_large_alpha_non_sd.py verify
certify_sd_cutoff_refinement.py verify
certify_bell_beyond_ggc.py verify
check_bell_phase_witnesses.py

```

The mathematical specifications in the appendix and Section 4 explain their formulas and remainder bounds. The full archive retains the exact rational partitions, coefficient polynomials, root certificates, programs and versioned verification records. Files under `manuscript_audit_runs/` record a fresh run of all twelve commands on 4 October 2026; all returned success, and the original dependencies were preserved. Eight additional upstream recomputations, recorded under `manuscript_dependency_runs/`, also passed: these check the exact germ generation, the remaining endpoint ID covers, the SD transition germ, the competing ID contacts, and the three whole-ray GGC slices used by the non-GGC BS examples. Some commands also rerun their prerequisite covers. The archived provenance audit reports 1,069 recorded dependency hashes across 134 verification records. These are the execution records of 3–4 October 2026. References in the appendices to successful or fresh runs refer to those archived computations. Their saved outputs and dependency hashes are included in the accompanying archive. Hash consistency, numerical recomputation, and the analytic arguments are distinct parts of the evidence.

6.4. Scope of the assertions. The main classification results do not assume a unique global radial contact at every index. The explicit ID crossover in the appendix shows why such an assumption would be false. The convergence and strict-margin searches prove boundary computability, while remaining silent about finite-time equality decisions for arbitrary real inputs.

The bell-shape statements proved here leave a finite contact problem at each fixed index, but they do not solve all of those problems. In particular, no globally earliest bell-shape cusp, exact global component count, or complete bell-shape classification is asserted. Unfinished exploratory covers are not inputs to any theorem in this manuscript. These limitations concern the scope of the claims, rather than an extrapolation from numerical evidence.

APPENDIX A. OBSTRUCTIONS TO THE THORIN PROPERTY

The first obstruction uses zeros of the characteristic transform above one. The second transfers a cut-phase oscillation to the positive free stable law below one. These arguments prove necessity independently of the positive certificates in the next two appendices.

A.1. A zero-free obstruction for every $\alpha > 1$.

Lemma A.1. *An extended GGC characteristic function, restricted to $t > 0$, has a holomorphic zero-free continuation to $\operatorname{Re} t > 0$. For $\Psi = -\log \varphi$, that continuation satisfies*

$$(A.1) \quad \operatorname{Re} \Psi'(t) \geq 0, \quad \operatorname{Re} t > 0.$$

Proof. The canonical representation is

$$\varphi(t) = \exp \left\{ iat - \frac{\sigma^2 t^2}{2} - \int_{\mathbb{R} \setminus \{0\}} \left[\log(1 - it/u) + \frac{itu}{1 + u^2} \right] U(du) \right\},$$

where U is positive, logarithmically integrable at zero and u^{-2} -integrable at infinity; see [10, Section 3, p. 4]. The logarithm has a holomorphic branch on the right half-plane, since its argument never vanishes there. The compensated integrand is $O(u^{-2})$ at infinity and $O(1 + |\log |u||)$ at zero, locally uniformly in t . Hence the exponent is holomorphic and its exponential is zero-free. For $t = x + iy$, differentiation gives

$$\operatorname{Re} \Psi'(t) = \sigma^2 x + \int \frac{x}{(u + y)^2 + x^2} U(du) \geq 0.$$

□

Theorem A.2. *Every admissible $X_{\alpha, \rho}$ with $1 < \alpha \leq 2$ is not an extended GGC.*

Proof. For $\alpha > 1$, W_α has negative real zeros $-\lambda_n$ by [1, Theorem 1]; see also [7, Remark 3]. Set $r_n = \lambda_n^{1/\alpha}$. The right-half-plane continuation in (3.2) vanishes at

$$t_n = r_n e^{i\omega}, \quad -e^{-i\alpha\omega} t_n^\alpha = -\lambda_n.$$

Admissibility gives $|\omega| < \pi/2$. Thus these zeros lie in $\operatorname{Re} t > 0$, contradicting Lemma A.1 and the identity theorem. □

No determination of the ID region above one is used in this proof.

A.2. The one-sided obstruction at 4/5.

Theorem A.3. *For every $4/5 < \alpha < 1$, $X_{\alpha, 1}$ is not GGC. Its reflection $X_{\alpha, 0}$ is not extended GGC.*

This improves the non-explicit exclusion near one in Corollary 1 of [7]. The proof below includes the step needed to transfer the oscillation; non-GGC of the Kanter factor alone would not imply the result.

A.2.1. *An exact differential relation.* Fix $\alpha \in (4/5, 1)$, and put

$$\beta = 1 - \alpha, \quad p = \alpha/\beta, \quad v_0 = \beta^{1/\alpha}, \quad b = \alpha\beta^{\beta/\alpha}, \quad \vartheta = 2\pi\beta/\alpha \in (0, \pi/2).$$

Let U be uniform on $(0, 1)$ and define the Kanter random variable

$$K_\alpha = \frac{\sin(\pi\alpha U) \sin^{(1-\alpha)/\alpha}(\pi(1-\alpha)U)}{\sin^{1/\alpha}(\pi U)}, \quad 0 < \alpha < 1.$$

Its normalization and transform are those of [7, (6) and Section 2.3]:

$$F_\beta(s) = \mathbb{E}e^{-sK_\alpha} = \sum_{n \geq 0} \frac{(-s^\alpha)^n}{n! \Gamma(1 - \beta n)}.$$

The coefficient identity for reciprocal Gamma functions gives

$$(A.2) \quad F_\beta(s) = L_\alpha(s) - \frac{\beta}{\alpha} s L'_\alpha(s).$$

Indeed $1/\Gamma(1 - \beta n) = (1 - \beta n)/\Gamma(2 - \beta n)$. The identity remains valid at reciprocal-Gamma zeros and on either side of the slit.

A.2.2. *The complex saddle.* The lower-cut representation from [7] reads

$$(A.3) \quad F_\beta(-r - i0) = 1 + \frac{1}{2\pi i} \int_0^\infty e^{-ru} \left(e^{ru^\beta e^{2\pi i \beta}} - e^{ru^\beta} \right) \frac{du}{u}.$$

Let $q(v) = v^\beta - v$, and define the convergent integral

$$J(\lambda) = \int_0^\infty \frac{e^{\lambda q(v)} - e^{-\lambda v}}{v} dv, \quad \operatorname{Re} \lambda > 0.$$

Near zero its integrand is $O(v^{\beta-1})$. In the first exponential of (A.3), subtract e^{-ru} before rotating $u = e^{i\vartheta}v$. The equality $\beta\vartheta + 2\pi\beta = \vartheta$ then gives

$$(A.4) \quad B(r) := \operatorname{Re} F_\beta(-r - i0) = 1 + \frac{1}{2\pi} \operatorname{Im} J(re^{i\vartheta}).$$

The subtracted integrand makes the small arc vanish. The large arc vanishes because $\vartheta < \pi/2$ and $\beta < 1$. This subtraction also fixes the contour constant unambiguously.

The function q has its unique maximum at v_0 , with

$$q(v_0) = b, \quad q''(v_0) = -\alpha/v_0.$$

Laplace's method in the fixed complex direction $\arg \lambda = \vartheta$ yields

$$(A.5) \quad J(re^{i\vartheta}) = \sqrt{\frac{2\pi}{\alpha v_0}} r^{-1/2} e^{-i\vartheta/2} e^{bre^{i\vartheta}} + O(r^{-3/2} e^{br \cos \vartheta}).$$

Here constants may depend on the fixed α . To justify the error, localize near v_0 and expand q and $1/v$. The complex Gaussian has strictly positive real damping. Outside that neighborhood the real exponent has a fixed gap below its maximum. Near zero bound the subtracted integrand by a constant times $rv^{\beta-1}$ with that gap; at infinity $q(v) < 0$. These pieces are absorbed by the displayed error. Thus no convergence of an infinite asymptotic series is presumed.

Set

$$C = (2\pi\alpha v_0)^{-1/2}, \quad \kappa = b \cos \vartheta > 0, \quad \nu = b \sin \vartheta > 0.$$

It follows that

$$(A.6) \quad B(r) = Cr^{-1/2} e^{\kappa r} \sin(\nu r - \vartheta/2) + O(r^{-3/2} e^{\kappa r}).$$

A.2.3. *Transferring the oscillation.* Let $A(r) = \operatorname{Re} L_\alpha(-r - i0)$. Equation (A.2) becomes

$$B = A - \frac{r}{p} A', \quad A(r) = r^p \left\{ c - p \int_{r_0}^r u^{-p-1} B(u) du \right\}.$$

For $\lambda = be^{i\vartheta}$, integration by parts gives

$$\int_{r_0}^r e^{\lambda u} u^{-p-3/2} du = \lambda^{-1} e^{\lambda r} r^{-p-3/2} (1 + O(r^{-1})) + O(1).$$

The integrated error from (A.6) is $O(e^{\kappa r} r^{-p-5/2}) + O(1)$. Since $\kappa > 0$, the homogeneous polynomial term is eventually negligible. Consequently

$$(A.7) \quad \operatorname{Re} L_\alpha(-r - i0) = -\frac{pC}{b} r^{-3/2} e^{\kappa r} \left\{ \sin(\nu r - 3\vartheta/2) + O(r^{-1}) \right\}.$$

Taking sequences where the sine is $+1$ or -1 proves the presence of both signs arbitrarily far along the cut.

A.2.4. *The Thorin contradiction.* If X_α were GGC, its drift would be b , the lower endpoint of its support. The density at that endpoint is proportional to $(x - b)^{1/2}$. Equivalently, the endpoint expansion in Proposition 6 of [7], and Watson's lemma also for the first derivative, give

$$e^{bs} L_\alpha(s) \sim c_\alpha s^{-3/2}, \quad \lim_{s \rightarrow \infty} s \left(-\frac{L'_\alpha(s)}{L_\alpha(s)} - b \right) = \frac{3}{2}.$$

For a GGC this limit is the total Thorin mass, so $U((0, \infty)) = 3/2$. The product representation on the lower cut gives, for almost every r ,

$$L_\alpha(-r - i0) = \exp \left\{ br - \int_0^\infty \log |1 - r/u| U(du) \right\} e^{i\pi U((0, r))}.$$

For all sufficiently large r , $1/2 < U((0, r)) \leq 3/2$, and hence the real part is nonpositive. This contradicts the open intervals of positive real part supplied by (A.7). Almost-everywhere boundary values are sufficient. This proves Theorem A.3.

Remark A.4. At $\alpha = 4/5$, $\vartheta = \pi/2$ and $\kappa = 0$. The homogeneous polynomial can no longer be discarded. The proof does not exclude this endpoint.

Corollary A.5. *If $c < 1$ and $X_{\alpha, \rho}$ is GGC for every $0 < \alpha \leq c$ and every $\rho \in [0, 1]$, then $c \leq 4/5$. The sufficiency of this endpoint is established in Theorem C.4 below.*

Corollary A.6. *For each fixed $4/5 < \alpha < 1$, some $\varepsilon_\alpha > 0$ satisfies*

$$\rho \in [0, \varepsilon_\alpha] \cup [1 - \varepsilon_\alpha, 1] \implies X_{\alpha, \rho} \text{ is not extended GGC.}$$

Proof. Extended GGC laws are closed under weak convergence. A sequence of GGC laws approaching either one-sided endpoint would contradict Theorem A.3. $\square$

No explicit ε_α , or exclusion of all interior skewnesses, is implied by this argument.

A.3. **A certified two-sided example.** For $0 < \alpha < 1$, define

$$H_\alpha(s) = -s \frac{L'_\alpha(s)}{L_\alpha(s)} = -\alpha z \frac{W'_\alpha(z)}{W_\alpha(z)}, \quad z = -s^\alpha,$$

at regular points. For $q = e^{i\pi(1/2-\rho)}$, $\varphi_{\alpha, \rho}(t) = L_\alpha(qt)$ on $t > 0$. If the law is extended GGC, (A.1), continued to $t = -ir$, therefore requires

$$(A.8) \quad \operatorname{Im} H_\alpha(re^{i\pi\rho}) \geq 0 \quad (r > 0)$$

where the boundary expression is regular. Reflection gives the corresponding ray with angle $\pi(1 - \rho)$. These necessary tests alone are not a parameter classification; the global criteria and critical curves are established below.

Proposition A.7. *$X_{9/10, 999999/1000000}$, and its reflection, is ID but not extended GGC.*

ID follows from the original theorem. The script `certify_two_sided_non_ggc.py` gives the outward enclosure

$$-0.000858987369327726 < \operatorname{Im} H_{9/10}(60e^{i\pi 999999/1000000}) < -0.000858987369327724,$$

which violates (A.8). This is an Arb interval computation, with explicit truncation error as follows.

For $n \geq N > 1/\beta$, $\beta = 1 - \alpha$, reflection gives

$$\left| \frac{z^n}{n! \Gamma(2 - \beta n)} \right| \leq A_n = \frac{r^{\alpha n} \Gamma(\beta n - 1)}{\pi n!}.$$

Concavity under a Gamma law gives $\Gamma(x + \beta)/\Gamma(x) \leq x^\beta$; see [15, (5.6.4)]. Therefore

$$A_{n+1}/A_n \leq q_N = r^\alpha \beta^\beta N^{-\alpha} < 1.$$

Truncation before $n = N$ has errors at most

$$\frac{A_N}{1 - q_N}, \quad A_N \left\{ \frac{N}{1 - q_N} + \frac{q_N}{(1 - q_N)^2} \right\}$$

for W and zW' , respectively. At $N = 400$, $r = 60$, $\alpha = 9/10$, these are below $4.14 \cdot 10^{-185}$ and $1.657 \cdot 10^{-182}$. Both errors are included in the interval division.

APPENDIX B. THE CENTRAL GGC INTERVAL

We prove the ray criterion, establish a nonempty central interval, and characterize its first angular contact. The final moment construction is an independent obstruction and is not needed for the classification.

B.1. The GGC skewness set is a central interval. This section first gives the interval structure. Nonemptiness at every index is proved subsequently in Theorem B.14.

Lemma B.1 (Uniform logarithmic derivative bounds). *For every fixed $0 < \alpha < 1$, uniformly on any closed sector $|\arg s| \leq \pi - \varepsilon$,*

$$H_\alpha(s) = \frac{s^\alpha}{\Gamma(\alpha)} + O(|s|^{2\alpha}) \quad (s \rightarrow 0), \quad H_\alpha(s) = b_\alpha s + \frac{3}{2} + o(1) \quad (|s| \rightarrow \infty).$$

In particular L_α has no zeros sufficiently close to zero or sufficiently far from zero in each such sector.

Proof. The first assertion is the Wright series. For the second, the proof of Theorem 1 in [7] shows that $y^{-1}f_\alpha(b_\alpha + y)$ is completely monotone. Hence, for a positive locally finite measure ν ,

$$f_\alpha(b_\alpha + y) = y \int_0^\infty e^{-yu} \nu(du), \quad L_\alpha(s) = e^{-b_\alpha s} g(s), \quad g(s) = \int_0^\infty (s + u)^{-2} \nu(du).$$

The identity for $g(s)$ extends to the slit plane by normal convergence of the generalized Stieltjes integral. The known endpoint behavior is $f_\alpha(b_\alpha + y) \sim c\sqrt{y}$, $c > 0$. For completeness, the positive-measure Laplace continuity argument applied to $\nu_R(A) = R^{-1/2}\nu(RA)$ gives

$$\int e^{-tu} \nu_R(du) = R^{-1/2} \frac{f_\alpha(b_\alpha + t/R)}{t/R} \longrightarrow ct^{-1/2}, \quad \nu_R \longrightarrow \frac{c}{\sqrt{\pi}} u^{-1/2} du$$

vaguely on $[0, \infty)$. One can apply the ordinary continuity theorem to the exponentially tilted finite measures. Moreover, $\nu([0, T]) \leq e \int e^{-u/T} \nu(du) = O(\sqrt{T})$ as $T \rightarrow \infty$. This bound controls both the small and large tails when integrating $(z + u)^{-2}$ or $(z + u)^{-3}$, uniformly for $|z| = 1$, $|\arg z| \leq \pi - \varepsilon$. Consequently

$$g(s) \sim \frac{c\sqrt{\pi}}{2} s^{-3/2}, \quad g'(s) \sim -\frac{3c\sqrt{\pi}}{4} s^{-5/2}.$$

Thus $-sL'_\alpha(s)/L_\alpha(s) = b_\alpha s - sg'(s)/g(s)$ has the asserted asymptotic. The nonzero leading terms prove the zero exclusions. This proof does not assume GGC or zero-freeness in the remaining compact annulus. $\square$

Lemma B.2 (Positive-real converse). *Suppose a characteristic function restricted to $t > 0$ is $\exp(-\Psi(t))$, with $\Psi(0+) = 0$. If $D = \Psi'$ extends holomorphically to $\operatorname{Re} t > 0$, $\operatorname{Re} D \geq 0$, and $D(t) = O(t^{\delta-1})$ as $t \downarrow 0$ for some $\delta > 0$, then its law is extended GGC.*

Proof. The half-plane Herglotz representation has the form

$$D(t) = \sigma^2 t + ic + \int_{\mathbb{R}} \left(\frac{1}{t + iu} + \frac{iu}{1 + u^2} \right) U(du), \quad U \geq 0, \quad \int \frac{U(du)}{1 + u^2} < \infty.$$

The bound at zero and positivity give

$$\int_0^1 \operatorname{Re} D(x) dx = \frac{\sigma^2}{2} + \frac{1}{2} \int_{\mathbb{R}} \log(1 + u^{-2}) U(du) < \infty.$$

In particular $U(\{0\}) = 0$ and its logarithmic moment at zero is finite. Integrating the representation and using $\Psi(0+) = 0$ gives precisely the canonical extended Thorin representation in Lemma A.1. This also verifies the needed integrability, rather than treating positive-realness alone as sufficient without a normalization at zero. $\square$

Proposition B.3 (An exact ray criterion). *Fix $0 < \alpha < 1$, $0 < \rho < 1$, and $m = \max(\rho, 1 - \rho)$. The law $X_{\alpha, \rho}$ is extended GGC if and only if both of the following hold:*

- (1) $L_\alpha(s) \neq 0$ for $s \neq 0$, $|\arg s| \leq \pi m$.
- (2) For all $r > 0$,

$$\operatorname{Im} H_\alpha(re^{i\pi\rho}) \geq 0, \quad \operatorname{Im} H_\alpha(re^{i\pi(1-\rho)}) \geq 0.$$

The zero condition is part of this criterion; checking ray signs on a grid is not sufficient.

Proof. Put $q = e^{i\pi(1/2-\rho)}$. The logarithmic derivative is $D_\rho(t) = H_\alpha(qt)/t$. If the law is GGC, Lemma A.1 gives zero-freeness in $-\pi\rho < \arg s < \pi(1 - \rho)$, and reflection supplies the other half of the stated symmetric sector. There cannot be a zero on one of its bounding rays either: a zero of multiplicity k at $t_0 \in i\mathbb{R} \setminus \{0\}$ would give $D_\rho(t_0 + x) = -k/x + O(1)$, contradicting $\operatorname{Re} D_\rho \geq 0$. All these boundary points are ordinary analytic points of the Wright formula, since $0 < \rho < 1$. The boundary values at $t = ir$ and $t = -ir$ give the two inequalities.

Conversely the zero condition makes D_ρ holomorphic on the right half-plane and continuous on its boundary except at zero. The two inequalities say that its real part is nonnegative there. Lemma B.1 gives $D_\rho(t) = O(1)$ at infinity and $D_\rho(t) = O(|t|^{\alpha-1})$ at zero, uniformly in that half-plane. Choose $1 - \alpha < p < 1$. Adding $\epsilon \operatorname{Re}(t^p + t^{-p})$, applying the minimum principle on a truncated half-annulus, and then sending the radii to zero and infinity gives $\operatorname{Re} D_\rho \geq 0$ after $\epsilon \downarrow 0$. Indeed $\cos(p \arg t) \geq \cos(p\pi/2) > 0$, so the barrier controls both circular boundaries. Lemma B.2 finishes the proof. $\square$

Theorem B.4 (Connectedness of the GGC skewness section). *For each fixed $0 < \alpha < 1$, the set*

$$\mathcal{G}_\alpha = \{\rho \in [0, 1] : X_{\alpha, \rho} \text{ is extended GGC}\}$$

is either empty or a closed interval $[g_\alpha, 1 - g_\alpha]$, where $0 \leq g_\alpha \leq 1/2$. If $4/5 < \alpha < 1$ and the set is nonempty, then $g_\alpha > 0$.

Proof. Suppose first that $\rho_0 \in \mathcal{G}_\alpha \cap (1/2, 1)$. Proposition B.3 makes $u(s) = \operatorname{Im} H_\alpha(s)$ harmonic in $0 < \arg s < \pi\rho_0$, continuous up to its two rays, and nonnegative on both: it is zero on the positive real axis. The bounds from Lemma B.1 are $O(|s|^\alpha)$ at zero and $O(|s|)$ at infinity, uniformly in the closed sector. Choose $1 < p < 1/\rho_0$ and use the positive harmonic barrier

$$\epsilon(r^p + r^{-p}) \cos(p(\theta - \pi\rho_0/2)), \quad s = re^{i\theta}.$$

Its cosine has a strictly positive lower bound on the closed sector. The minimum principle on truncated sectors, followed by removal of the barrier, gives $u(s) \geq 0$ throughout the sector. Therefore the two ray inequalities hold for every $\rho \in [1 - \rho_0, \rho_0]$, and the zero condition is inherited from the larger sector. Proposition B.3 proves GGC on that whole interval.

If an endpoint 0 or 1 belongs to $\mathcal{G}_\alpha$, the positive GGC representation gives $H_\alpha(s) = b_\alpha s + \int s/(s+t) U(dt)$. This has nonnegative imaginary part in the upper half-plane and L_α is zero-free on the slit plane, so the same ray criterion proves $\mathcal{G}_\alpha = [0, 1]$. Reflection handles points below one half. Finally the class of extended GGC laws is closed under weak convergence, so $\mathcal{G}_\alpha$ is closed. These facts include the possibilities of an empty set or the singleton $\{1/2\}$. For $\alpha > 4/5$, Theorem A.3 excludes both endpoints; closedness then gives $g_\alpha > 0$ whenever the set is nonempty. $\square$

The interval theorem reduces the classification to nonemptiness and the location of g_α . The following stability result controls interior skewnesses near a known one-sided GGC index. Theorem B.14 supplies nonemptiness throughout the remaining range, and Theorem B.15 identifies the boundary.

Proposition B.5 (Stability away from the one-sided endpoints). *Suppose $0 < \alpha_0 < 1$ and $X_{\alpha_0,1}$ is GGC. For every $\eta \in (0, 1/2)$, there exists $\varepsilon > 0$ such that*

$$|\alpha - \alpha_0| < \varepsilon, \quad \eta \leq \rho \leq 1 - \eta \implies X_{\alpha,\rho} \text{ is extended GGC.}$$

In particular, for each such η , the GGC theorem of [6, Theorem 2] extends to some rectangle $(3/4, 3/4 + \varepsilon_\eta) \times [\eta, 1 - \eta]$. This is an existence statement; it supplies no explicit numerical value of ε_η .

Proof. The small- and large- s estimates in Lemma B.1 hold uniformly when α ranges over a compact neighborhood of α_0 in $(0, 1)$. For the large- s statement, the square-root endpoint expansion is locally uniform in α : this follows either from the explicit expansion in Proposition 6 of [7], or from the analytic inverse at its nondegenerate quadratic critical point. Its leading coefficient is positive and continuous. The positive-measure scaling proof of Lemma B.1 then applies uniformly, using the uniform bound $\nu_\alpha([0, T]) \leq ef_\alpha(b_\alpha + 1/T)T$. On bounded annuli, the Wright series and its derivative are jointly continuous in (α, s) .

For $q = e^{i\pi(1/2-\rho)}$ at $\alpha = \alpha_0$, the positive Thorin representation gives, throughout $\operatorname{Re} t \geq 0$, $t \neq 0$,

$$D_{\alpha_0,\rho}(t) = b_{\alpha_0}q + \int \frac{1}{t+u/q} U(du), \quad \operatorname{Re} D_{\alpha_0,\rho}(t) \geq b_{\alpha_0} \sin(\pi\eta) > 0.$$

All the corresponding qt lie in the fixed closed slit sector $|\arg s| \leq \pi(1-\eta)$. There L_{α_0} is zero-free. Uniform estimates exclude zeros for all nearby α near zero and infinity; compactness and joint continuity exclude them in the remaining annulus. At infinity $D_{\alpha,\rho}(t) = b_\alpha q + o(1)$, with a uniform strict positive real-part margin. At zero,

$$D_{\alpha,\rho}(t) = \frac{q^\alpha t^{\alpha-1}}{\Gamma(\alpha)} (1 + O(|t|^\alpha)), \quad |\arg(q^\alpha t^{\alpha-1})| \leq \frac{\pi}{2} - \alpha\pi\eta,$$

again giving uniform strict positivity. On the intervening compact half-annulus, joint continuity preserves the displayed positive margin at α_0 , after shrinking ε . Lemma B.2 proves the claim. Taking $\alpha_0 = 3/4$ uses the original theorem. $\square$

Corollary B.6 (Certified skewness intervals). *For each row in the following table, all $\rho \in [0, \epsilon] \cup [1 - \epsilon, 1]$ give ID laws which are not extended GGC:*

α	ϵ	r	strict enclosure of $\operatorname{Im} H_\alpha(re^{i\pi(1-\epsilon)})$
9/10	1/20000	45	$(-0.001128, -0.001127)$
19/20	1/200	50	$(-0.282110, -0.282109)$
99/100	1/100	212	$(-3.893375, -3.893374)$

Proof. The script `certify_ggc_skewness_exclusions.py` verifies these outward enclosures with 150-digit Arb arithmetic and the explicit Wright tail bounds given earlier. The reusable evaluator is `wright_interval.py`; the certificate records each rational input and strict enclosing decimal interval. Each negative value excludes $\rho_* = 1 - \epsilon$. If any $\rho \in [\rho_*, 1]$ were GGC, Theorem B.4 would force ρ_* to be GGC as well. Reflection gives the other interval. Thus an entire parameter interval is excluded by a single rigorous witness and the structural theorem, not by interpolation of a numerical grid. $\square$

Theorem B.7 (Endpoint exclusion on every linear scale). *For each finite $M > 0$, there exists $\alpha_M < 1$ such that*

$$\alpha_M < \alpha < 1, \quad \min(\rho, 1 - \rho) \leq M(1 - \alpha) \implies X_{\alpha, \rho} \text{ is not extended GGC.}$$

Consequently, along any sequence of nonempty sections $\mathcal{G}_\alpha = [g_\alpha, 1 - g_\alpha]$ with $\alpha \uparrow 1$,

$$\frac{g_\alpha}{1 - \alpha} \longrightarrow +\infty.$$

The threshold α_M is not made explicit by this proof.

Proof. Let $\beta = 1 - \alpha$, $\delta = 1 - \rho$, and let C be standard symmetric Cauchy independent of X_α . The Cauchy factorization in [7] gives

$$X_{\alpha, 1-\delta} \stackrel{d}{=} X_\alpha \{\cos(\pi\delta) + \sin(\pi\delta)C\}.$$

The known exceptional-index limit is $(\beta^\beta - X_\alpha)/\beta \Rightarrow T$. If $\delta/\beta \rightarrow \ell < \infty$, then $X_\alpha \rightarrow 1$ in probability, and Slutsky's theorem, preserving independence of C , yields

$$\frac{\beta^\beta - X_{\alpha, 1-\delta}}{\beta} \Rightarrow T - \pi\ell C \stackrel{d}{=} T + \pi\ell C.$$

Indeed the additional location correction is $X_\alpha(1 - \cos(\pi\delta))/\beta = o_P(1)$. By Theorem 3 of [7], the negative-side Lévy kernel of this limit is

$$k_-(x) = \frac{1 + \ell}{x} - \frac{e^{-2x}}{1 - e^{-x}} = \int_0^\infty e^{-ux} \left((1 + \ell) du - \sum_{n \geq 2} \delta_n(du) \right).$$

Uniqueness of the Laplace transform shows that it cannot be CM: the negative atoms cannot be canceled by a multiple of Lebesgue measure. Thus the limit is not extended GGC for any finite ℓ . If the first assertion failed for some M , a sequence of GGC laws with $\alpha \uparrow 1$, $\delta/\beta \in [0, M]$ would have a subsequence with $\delta/\beta \rightarrow \ell \in [0, M]$. Extended GGC is closed under affine transformations and weak limits, a contradiction. Reflection covers the endpoint zero. The last assertion follows by taking $\rho = g_\alpha$, since a nonempty section includes its endpoints by closedness. $\square$

This result does not decide GGC for a fixed interior ρ as $\alpha \uparrow 1$; that is a different limit regime.

B.1.1. A stronger square-root endpoint obstruction. The first zero of the reciprocal Gamma function gives a sharper scale than the preceding weak-limit argument.

Theorem B.8. *For every $0 < c < 1/(\pi\sqrt{2})$, there is $\alpha_c < 1$ such that*

$$\alpha_c < \alpha < 1, \quad \min(\rho, 1 - \rho) \leq c\sqrt{1 - \alpha} \implies X_{\alpha, \rho} \text{ is not extended GGC.}$$

In particular, along any sequence of nonempty GGC sections approaching one,

$$\liminf_{\alpha \uparrow 1} \frac{g_\alpha}{\sqrt{1 - \alpha}} \geq \frac{1}{\pi\sqrt{2}}.$$

A matching upper bound and a sharper expansion are proved in Theorem H.5.

Proof. Write $\beta = 1 - \alpha$, $f(x) = 1/\Gamma(2 - x)$, and

$$G_\beta(t) = e^{-t/\beta} W_{1-\beta}(t/\beta).$$

We first show, uniformly for $t = 2e^{-i\eta}$ and $|\eta| \leq C\sqrt{\beta}$ with fixed C , that

$$(B.1) \quad G_\beta(t) = -(t - 2) + O(\beta), \quad G'_\beta(t) = -1 + O(\sqrt{\beta}).$$

Let $\lambda = 2/\beta$, $z = t/\beta$, and let N be Poisson with mean λ . Absolute values of the complex Poisson sums are bounded by $e^{\lambda - \operatorname{Re} z}$ times expectation under this real Poisson law. This prefactor is bounded because $\lambda - \operatorname{Re} z = 2(1 - \cos \eta)/\beta = O(1)$. On $1 \leq x \leq 3$, Taylor's formula and $f(2) = 0, f'(2) = -1$ give

$$f(x) = -(x - 2) + O((x - 2)^2).$$

The full complex Poisson sum of the linear term is exactly $-(t - 2)$, and $\mathbb{E}(\beta N - 2)^2 = 2\beta$. This gives the first error bound once the tails are controlled.

For $\beta N < 1$, f and its first derivative are bounded and the Poisson probability is exponentially small in $1/\beta$. For $n \geq n_0 = \lceil 3/\beta \rceil$, reflection bounds $|f(\beta n)| \leq \Gamma(\beta n - 1)/\pi$. The successive ratio of $\lambda^n \Gamma(\beta n - 1)/n!$ is at most $\lambda \beta^\beta n^{\beta-1}$, which is bounded strictly below one at n_0 for small β . Stirling's formula at n_0 then gives a bound $O(e^{-k/\beta})$ after multiplication by $e^{-\lambda}$, for any fixed $0 < k < 2 - 3(1 + \log(2/3))$. Polynomial factors are absorbed by reducing k . This also controls the tails of the subtracted linear term.

Termwise differentiation of the entire series gives

$$G'_\beta(t) = e^{-z} \sum_{n \geq 0} \frac{z^n}{n!} \frac{f(\beta(n+1)) - f(\beta n)}{\beta}.$$

On the middle interval, its final factor equals $-1 + O(|\beta n - 2| + \beta)$, and $\mathbb{E}|\beta N - 2| \leq \sqrt{2\beta}$. The same tail estimate applies to the shifted Gamma bound, even with the extra factor $1/\beta$. This proves (B.1).

Now choose $\delta = c\sqrt{\beta}$, $\rho = 1 - \delta$, and

$$s_\beta = (2/\beta)^{1/(1-\beta)} e^{i\pi\rho}, \quad \eta = \pi(\beta + (1-\beta)\delta), \quad z = -s_\beta^{1-\beta} = (2/\beta)e^{-i\eta}, \quad t = \beta z.$$

Since $t - 2 = -2i\pi c\sqrt{\beta} + O(\beta)$, (B.1) implies $G_\beta(t) \neq 0$ for small β and

$$\frac{tG'_\beta(t)}{G_\beta(t)} = \frac{i}{\pi c\sqrt{\beta}} + O(1).$$

The exact logarithmic derivative identity is

$$H_{1-\beta}(s_\beta) = -(1-\beta) \left(z + \frac{tG'_\beta(t)}{G_\beta(t)} \right).$$

Consequently

$$(B.2) \quad \lim_{\beta \downarrow 0} \sqrt{\beta} \operatorname{Im} H_{1-\beta}(s_\beta) = 2\pi c - \frac{1}{\pi c} < 0.$$

The necessary ray condition excludes $\rho = 1 - c\sqrt{\beta}$. The central-interval theorem and reflection extend this exclusion to both full endpoint bands. Letting $c \uparrow 1/(\pi\sqrt{2})$ gives the liminf assertion. $\square$

The file `check_sqrt_endpoint_scale.py` supplies strict point enclosures consistent with (B.2); these checks are not needed for the analytic proof and do not establish a converse. For example, at $\beta = 10^{-4}$, $c = 1/5$ the expression on the left of (B.2) is strictly negative. This gives the whole non-GGC range $\alpha = 9999/10000$, $\rho \in [0, 1/500] \cup [499/500, 1]$.

B.1.2. GGC returns on every fixed interior interval near index one. The endpoint obstruction does not persist at a fixed distance from $\rho = 0, 1$. The following uniform rescaling argument proves this, without confusing weak convergence to Cauchy with openness of the GGC class.

Lemma B.9 (Compact Hankel limit). *Let $G_\beta(t) = e^{-t/\beta} W_{1-\beta}(t/\beta)$. As $\beta \downarrow 0$,*

$$G_\beta(t) = \frac{1}{\Gamma(2-t)} + O(\beta), \quad G'_\beta(t) = \left(\frac{1}{\Gamma(2-t)} \right)' + O(\beta),$$

locally uniformly on the whole complex plane.

Proof. Take a Hankel contour $\mathcal{H}$ around the negative real axis, with circular part of radius one. Hankel's reciprocal-Gamma formula, termwise integration, and $0 < \beta < 1$ give the convergent integral

$$G_\beta(t) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^u u^{-2} \exp\left(t \frac{u^\beta - 1}{\beta}\right) du.$$

The powers use the corresponding boundary logarithms along the two straight parts of $\mathcal{H}$. For $0 < \beta \leq 1/4$,

$$\left| \frac{u^\beta - 1}{\beta} - \log u \right| \leq \frac{\beta}{2} |\log u|^2 e^{\beta |\log u|}.$$

On either straight part $u = -r \pm i0$, $r \geq 1$, the right side is bounded by a constant times $\beta(\log r + \pi)^2 r^{1/4}$. If $|t| \leq M$, this error is sublinear in r , while $|e^u| = e^{-r}$. The integrands, their first t derivatives, and their differences divided by β therefore have an integrable majorant independent of β , for example a constant times $e^{-r/2}$ for large r . The circular part is uniformly bounded directly. Dominated convergence with the same difference bound gives the stated $O(\beta)$ estimates. At $\beta = 0$ the limiting integral is Hankel's formula for $1/\Gamma(2-t)$. $\square$

Lemma B.10 (Uniform endpoint scaling). *For $0 < \beta \leq 1/5$, put $a = 1 - \beta$, $c_\beta = \beta^{1/a}$,*

$$Y_\beta = \frac{X_a - b_a}{c_\beta}, \quad P_\beta(v) = \mathbb{E} e^{-v Y_\beta} = e^{av/\beta} W_a(-v^a/\beta).$$

For every fixed $\eta > 0$, uniformly in $0 < \beta \leq 1/5$ and $|\arg v| \leq \pi - \eta$,

$$P_\beta(v) \sim \frac{v^{-3/2}}{\sqrt{2\pi a}}, \quad P'_\beta(v) \sim -\frac{3v^{-5/2}}{2\sqrt{2\pi a}} \quad (|v| \rightarrow \infty).$$

In particular, sufficiently far out in any such sector, $P_\beta \neq 0$ and $-vP'_\beta/P_\beta = 3/2 + o(1)$, uniformly even as $\beta \downarrow 0$.

Proof. Let p_β be the density of Y_β . The free Cauchy transform equation obtained from (2) of [7], with $u = -G_{Y_\beta}$ near its left support endpoint, is

$$y = h_\beta(u) := 1 - \frac{1}{u} + \frac{u^{-\beta} - 1}{\beta}.$$

This is analytic jointly near $u = 1$, including $\beta = 0$, where $h_0(u) = 1 - 1/u - \log u$. We have $h_\beta(1) = h'_\beta(1) = 0$ and $h''_\beta(1) = -a$. The local square-root inverse therefore gives

$$p_\beta(y) = \frac{1}{\pi} \sqrt{\frac{2}{a}} \sqrt{y} + O(y^{3/2}) \quad (y \downarrow 0),$$

with a uniform remainder for $0 < \beta \leq 1/5$. Indeed the analytic inverse is constructed in a common neighborhood because $a \geq 4/5$; its coefficients and its convergence radius can be chosen uniformly on the compact parameter interval including zero. For $\beta > 0$, the branch with $\Im u > 0$ is the physical Cauchy-transform branch at the known support edge. The even powers of $\sqrt{y}$ are real and do not contribute to the density.

Use the shifted gamma mixture from [7, Proof of Theorem 1], with shape two. The unit-rate gamma component has Laplace transform $(1 + s)^{-2}$. Affine scaling gives

$$\frac{p_\beta(y)}{y} = \int_0^\infty e^{-yu} \nu_\beta(du), \quad P_\beta(v) = \int_0^\infty (v + u)^{-2} \nu_\beta(du).$$

The uniform density estimate implies $\nu_\beta([0, T]) = O(\sqrt{T})$ uniformly for large T , using $\nu_\beta([0, T]) \leq eT p_\beta(1/T)$. For bounded T , the bound $\nu_\beta([0, T]) \leq (1 + T)^2 P_\beta(1) \leq (1 + T)^2$ is also uniform. For any sequence $R \rightarrow \infty$, the Laplace transforms of the measures $R^{-1/2} \nu_\beta(R du)$ satisfy the same uniform convergence argument as in Lemma B.1. Taking a subsequence of β in the compact interval $[0, 1/5]$ identifies every possible limiting measure as

$$\frac{1}{\pi\sqrt{\pi}} \sqrt{\frac{2}{1 - \beta_*}} u^{-1/2} du.$$

The uniform cumulative-mass bounds control both ends of the Stieltjes integrals and their derivatives on every closed slit sector. This proves the two uniform asymptotics and their ratio. No unproved uniform steepest-descent expansion is used here. $\square$

Theorem B.11 (Fixed interior skewness near one). *For every $0 < \eta < 1/2$, there exists $\beta_\eta > 0$ such that*

$$1 - \beta_\eta < \alpha < 1, \quad \eta \leq \rho \leq 1 - \eta \implies X_{\alpha, \rho} \text{ is extended GGC.}$$

Thus the GGC sections are nonempty for all indices sufficiently close to one, and their critical endpoints satisfy

$$g_\alpha \longrightarrow 0 \quad (\alpha \uparrow 1), \quad \liminf_{\alpha \uparrow 1} \frac{g_\alpha}{\sqrt{1 - \alpha}} \geq \frac{1}{\pi\sqrt{2}}.$$

This argument gives no explicit β_η . The sharper boundary expansion is Theorem H.5.

Proof. Write $\beta = 1 - \alpha$, $a = \alpha$, $v = c_\beta s$, and $t = -v^a = -\beta s^a$. Lemma B.10 gives, uniformly on $|\arg s| \leq \pi(1 - \eta)$,

$$H_a(s) = \frac{a}{\beta} v - \frac{v P'_\beta(v)}{P_\beta(v)} = \frac{a}{\beta} v + \frac{3}{2} + o(1) \quad (|v| \rightarrow \infty).$$

Choose $R \geq 1$ so large that throughout $|v| \geq R$, $0 < \beta \leq 1/5$, this error has absolute value less than one, $P_\beta \neq 0$, and $4R \sin(\pi\eta) > 1$. The imaginary part of H_a is then strictly positive on both required rays, whose angles lie between $\pi\eta$ and $\pi(1 - \eta)$.

For $|v| \leq R$, the points $t = -v^a$ lie in the fixed compact set

$$\mathcal{K} = \{0\} \cup \{|t| \leq R, |\arg t| \geq \pi\eta\}.$$

The function $f(t) = 1/\Gamma(2 - t)$ has no zeros on this set, because its zeros are the positive integers $2, 3, \dots$. Lemma B.9 shows that for small enough β , G_β is zero-free there and G'_β/G_β is uniformly bounded. The exact identity

$$H_a(s) = as^a \left(1 + \beta \frac{G'_\beta(t)}{G_\beta(t)} \right)$$

therefore preserves strictly positive imaginary part on both rays: the arguments of s^a stay between $4\pi\eta/5$ and $\pi(1 - \eta)$. This includes arbitrarily small s , so there is no missing intermediate or zero-radius regime. The same compact argument excludes zeros of L_a on the whole stated sector. Combined with the large- v part, all conditions of Proposition B.3 hold. The limit of g_α follows by taking arbitrarily small fixed η , and Theorem B.8 provides the lower bound. $\square$

B.1.3. *A certificate covering every symmetric law below one.* The two compact integrals below remove the singular dependence on β before interval evaluation. Together with an explicit unbounded-tail argument, they prove nonemptiness at every remaining index, not just near $4/5$ and one.

Lemma B.12 (An explicit symmetric tail). *For $0 < \beta \leq 1/5$, $a = 1 - \beta$, and $r \geq 200$,*

$$\operatorname{Im} H_a(ir/c_\beta) > 4r - 450 > 0.$$

Moreover, L_a has no zeros in $\operatorname{Re} s \geq 0$.

Proof. Use $p_\beta, P_\beta, h_\beta$ from Lemma B.10, and set $j_\beta(y) = p_\beta(y)/y$. We first give explicit bounds

$$(B.3) \quad p_\beta(y) < \sqrt{y}, \quad -j'_\beta(y) > \frac{1}{10}y^{-3/2}, \quad 0 < y \leq 1/200.$$

Write $u = 1 + w$. The coefficients of the inverse equation are

$$h_\beta(1 + w) = \sum_{n \geq 2} h_n w^n, \quad h_n = (-1)^n \left(\frac{(1 + \beta)_{n-1}}{n!} - 1 \right).$$

They satisfy $h_2 = -a/2$, $|h_3|/a \leq 7/10$, $|h_4|/a \leq 81/100$, and $|h_n|/a \leq 5/4$ for $n \geq 5$. At $R = 3/20$, the leading quadratic exceeds the sum of the remaining terms by at least

$$\frac{4}{5} \left(\frac{R^2}{2} - \frac{7}{10}R^3 - \frac{81}{100}R^4 - \frac{5}{4} \frac{R^5}{1 - R} \right) > 0.00669 > 1/200.$$

Rouché's theorem puts the two inverse roots in $|w| < R$ for the stated y . Choose the upper physical branch and write $y = t^2$, $w = tV(t)$. Define the remainders by

$$h_\beta(1 + w) = -\frac{a}{2}w^2(1 + \mathcal{R}(w)), \quad h'_\beta(1 + w) = -aw(1 + \mathcal{S}(w)).$$

The same coefficient bounds give

$$|\mathcal{R}| < 0.26, \quad |\mathcal{S}| < 0.414, \quad |\mathcal{S} - \mathcal{R}| < 0.158.$$

For clarity, the respective bounds before rounding are

$$\begin{aligned} & 2 \left(\frac{7}{10}R + \frac{81}{100}R^2 + \frac{5}{4}R^3/(1 - R) \right), \\ & \frac{21}{10}R + \frac{81}{25}R^2 + \frac{5}{4}R^3\{5/(1 - R) + R/(1 - R)^2\}, \\ & \frac{7}{10}R + \frac{81}{50}R^2 + \frac{5}{4}R^3\{3/(1 - R) + R/(1 - R)^2\}. \end{aligned}$$

Thus

$$V = i\sqrt{2/a(1 + \mathcal{R})}^{-1/2}, \quad |V| < 1.84, \quad \operatorname{Im} V > 1.24, \quad |tV'| = \left| V \frac{\mathcal{R} - \mathcal{S}}{1 + \mathcal{S}} \right| < 0.5.$$

Since $p_\beta(t^2) = t \operatorname{Im} V/\pi$, these prove (B.3); indeed $-j'_\beta(y)y^{3/2} = \operatorname{Im}(V - tV')/(2\pi) > 0.1$.

Write $j_\beta(y) = \int e^{-yu} \nu_\beta(du)$. For $T \geq 200$, (B.3) gives $\nu_\beta([0, T]) \leq e\sqrt{T}$. Splitting the Stieltjes integral at $r, 2r, 4r, \dots$ gives

$$|P'_\beta(ir)| \leq 2e \left(1 + \frac{\sqrt{2}}{1 - 2^{-5/2}} \right) r^{-5/2}.$$

The elementary bound $(1 + t^2)^2 \leq 6e^t$ yields

$$|P_\beta(ir)| \geq -\operatorname{Im} P_\beta(ir) = \frac{2}{r^2} \int \frac{u/r}{(1 + (u/r)^2)^2} \nu_\beta(du) \geq -\frac{j'_\beta(1/r)}{3r^3} > \frac{1}{30}r^{-3/2}.$$

Consequently

$$\left| r \frac{P'_\beta(ir)}{P_\beta(ir)} \right| < 60e \left(1 + \frac{\sqrt{2}}{1 - 2^{-5/2}} \right) < 450.$$

The identity $H_a(ir/c_\beta) = air/\beta - irP'_\beta(ir)/P_\beta(ir)$ proves the stated tail, since $a/\beta \geq 4$. Finally $P_\beta(s) = \int (s+u)^{-2} \nu_\beta(du)$ is zero-free in $\operatorname{Re} s > 0$: its integrands lie in a common open half-plane. On the imaginary boundary its imaginary part is strictly negative in the upper half-plane and strictly positive in the lower. The measure is not concentrated at zero, by (B.3). Multiplication by the exponential preserves zero-freeness, and $L_a(0) = 1$. $\square$

Lemma B.13 (Validated parabolic contour). *For $0 \leq \beta \leq 1/5$, define the removable function $E_\beta(z) = (e^{\beta z} - 1)/\beta$, with $E_0(z) = z$. For $r > 0$, set*

$$v = ir, \quad q = 1 + x/\sqrt{r}, \quad w = 1 + iq, \quad u = rw^2/2, \\ \ell = 2 \log \left(1 + \frac{(1+i)x}{2\sqrt{r}} \right), \quad \Phi = u - v - vE_\beta(\ell).$$

The logarithm displayed here has its continuous principal value on the real x contour. Put

$$F = \int_{-\infty}^{\infty} e^\Phi w^{-3} dx, \quad J = \int_{-\infty}^{\infty} E_\beta(\ell) e^\Phi w^{-3} dx.$$

For $\beta > 0$,

$$P_\beta(ir) = \frac{2}{\pi r^{3/2}} F, \quad P'_\beta(ir) = -\frac{2a}{\pi r^{3/2}} J, \quad \frac{\beta \operatorname{Im} H_a(ir/c_\beta)}{ar} = 1 + \beta \operatorname{Re}(J/F).$$

The errors on truncating the two integrals to $[-T, T]$ are at most

$$B_T = \frac{e^{-3T^2/10}}{3T/10}, \quad 17B_T,$$

respectively, uniformly in both parameters. These formulas and bounds also have a regular $\beta = 0$ limit.

Proof. Deform the Hankel contour to $u = (\sqrt{ir} + ix/\sqrt{2})^2$. Its ends approach the two negative rays, where e^u dominates the sublinear power in the exponent. It stays off the cut and winds around zero once. Combining the exponential shift before taking $\beta \rightarrow 0$ gives exactly Φ . Differentiation in v with the contour held fixed multiplies its integrand by $-aE_\beta(\ell)$, which proves the identities.

For $q \geq 0$, the function $\operatorname{Re} \Phi/r$ has value and derivative zero at $q = 1$, and second derivative

$$-1 + (2\beta - 1)2^{1-\beta} \operatorname{Re}\{e^{ia\pi/2}(1+iq)^{-2a}\} \leq -1.$$

For $q \leq 0$, the cosine in $\operatorname{Re}(e^{ia\pi/2}(1+iq)^{2\beta})$ is at least $\sin(\beta\pi/2)$, so

$$\frac{\operatorname{Re} \Phi}{r} \leq \frac{1 - q^2}{2} - 2^{-\beta} \frac{\sin(\beta\pi/2)}{\beta} < -\frac{4}{5} - \frac{q^2}{2}.$$

The ratio at zero is interpreted by continuity. Its lower bound $13/10$ follows from $\sin z/z \geq 1 - z^2/6$ and is checked by directed ball arithmetic. Combining both halves gives $\operatorname{Re} \Phi \leq -3x^2/10$ everywhere.

If $z = \sqrt{1+q^2} \geq 1$, then

$$|E_\beta(\ell)w^{-3}| \leq e^{d/5}(2 \log z + d)z^{-13/5} < 17, \quad d = \log 2 + 3\pi/2.$$

The right side is decreasing in $z \geq 1$, and its value at one is less than 17. Also $|w^{-3}| \leq 1$. The elementary Gaussian tail estimate now gives the two error bounds. $\square$

Theorem B.14 (Every symmetric law below one is GGC). *For every $0 < \alpha \leq 1$, $X_{\alpha,1/2}$ is extended GGC. For $4/5 \leq \alpha < 1$, the proof uses a validated computation on 720 exact rational parameter boxes, together with Lemma B.12. In particular, $\mathcal{G}_\alpha$ is nonempty for every $0 < \alpha < 1$.*

Proof. Theorem C.4 covers $\alpha \leq 4/5$; the Cauchy law covers $\alpha = 1$. In the remaining range put $\beta = 1 - \alpha \in (0, 1/5]$. The zero condition in Proposition B.3 follows from Lemma B.12.

For small scaled radius, use G_β from Lemma B.9, $t = -Te^{ia\pi/2}$, and the normalized test

$$Q_0(\beta, T) = \operatorname{Im} \left[e^{ia\pi/2} \{1 + \beta G'_\beta(t)/G_\beta(t)\} \right] > 0, \quad 0 \leq \beta \leq 1/5, \quad 0 \leq T \leq 1.$$

This covers $0 < r \leq 1$, since $T = r^a$. Both G and G' are enclosed by their radius-one Hankel contours; the straight parts are truncated at 80. For $u = -z \pm i0$, $z \geq 80$,

$$|E_\beta(\log u)| \leq (\log z + \pi)z^{1/5} \leq z/2.$$

The resulting two-sided tail errors for G, G' are respectively $2e^{-40}/(\pi 80^2)$ and $e^{-40}/(\pi 80)$.

For $1 \leq r \leq 200$, enclose the integrals in Lemma B.13 at $T = 12$ and check

$$Q_1(\beta, r) = 1 + \beta \operatorname{Re}(J/F) > 0.$$

For numerical stability only, the phase is evaluated in the exactly equivalent form

$$\Phi = \frac{v\ell^2}{2} \{ {}_1F_1(1; 3; \ell) - \beta {}_1F_1(1; 3; \beta\ell) \}, \quad E_\beta(\ell) = \ell {}_1F_1(1; 2; \beta\ell).$$

Thus neither division by a parameter interval containing zero nor subtraction of the large saddle terms is needed.

The file `symmetric_ggc_certificate.json` records a recursive binary partition of the two complete rectangles into 720 boxes. The functions in `symmetric_hankel.py` use directed complex ball arithmetic and validated analytic quadrature at 45 decimal digits, adding every displayed truncation error. All logarithms in quadrature forward its analyticity flag. The separate command `certify_symmetric_ggc.py verify` reconstructs the exact partition from its roots and reevaluates every box, accepting only strictly positive whole-box lower bounds. That verification passed; its hashes and constant bounds are saved in `symmetric_ggc_verification.json`. Finally $r \geq 200$ is covered by Lemma B.12. Both ray inequalities are the same when $\rho = 1/2$, so Proposition B.3 proves the claim. This is a computer-assisted proof, not merely a numerical scan or a proof-assistant formalization. $\square$

B.1.4. The critical curve and its first angular contact. The structural statements can now be made without an empty-section qualification. They also specify which contact must be selected in any numerical investigation of the critical curve.

Theorem B.15 (Critical GGC curve). *There is a continuous function $g : (0, 1] \rightarrow [0, 1/2]$ such that*

$$X_{\alpha, \rho} \text{ is extended GGC} \iff g(\alpha) \leq \rho \leq 1 - g(\alpha), \quad 0 < \alpha \leq 1.$$

It satisfies

$$g(\alpha) = 0 \quad (0 < \alpha \leq 4/5 \text{ or } \alpha = 1), \quad 0 < g(\alpha) < 1/2 \quad (4/5 < \alpha < 1),$$

is locally Lipschitz on $(4/5, 1)$, and obeys

$$\liminf_{\alpha \uparrow 1} \frac{g(\alpha)}{\sqrt{1 - \alpha}} \geq \frac{1}{\pi\sqrt{2}}.$$

For $4/5 < \alpha < 1$, the following angular minimum exists and is attained at a finite, nonzero radius:

$$(B.4) \quad \pi(1 - g(\alpha)) = \min\{\arg s : \pi/2 < \arg s < \pi, L_\alpha(s) \neq 0, \operatorname{Im} H_\alpha(s) = 0\}.$$

Every minimizer s_ satisfies*

$$(B.5) \quad \operatorname{Im} H_\alpha(s_*) = 0, \quad \operatorname{Im}\{s_* H'_\alpha(s_*)\} = 0, \quad \operatorname{Re}\{s_* H'_\alpha(s_*)\} < 0.$$

Thus the relevant contact is the smallest angular one, not an arbitrarily chosen solution of the two equality equations. The theorem is an exact variational characterization. Theorem G.2 below proves arbitrary-accuracy computability, and Theorem H.5 gives the sharp endpoint expansion.

Proof. Existence of a closed nonempty central section follows from Theorems B.4 and B.14. The positive and negative one-sided results give its stated zero and positive ranges. At every $4/5 < \alpha < 1$, the certificate and the two analytic radius regimes prove strict positivity of $\text{Im } H_\alpha(ir)$ at every $r > 0$. The small- and large-radius bounds in Lemma B.1 stay strictly positive on rays in a neighborhood of $\pi/2$. On the remaining compact annulus, both strict positivity and zero-freeness persist after a small increase in sector angle. The exact ray criterion then gives $g(\alpha) < 1/2$.

Write $\theta_* = \pi(1 - g(\alpha))$. By the sector minimum principle used in Theorem B.4, $U_\alpha(s) = \text{Im } H_\alpha(s)$ is nonnegative on $0 \leq \arg s \leq \theta_*$, and is strictly positive in the interior by the strong minimum principle and its nonconstant small-radius asymptotic. On the outer ray it is nonnegative. If it had no zero there, its uniform small- and large-radius signs, and compactness of the intervening annulus, would allow a further increase in angle. The zero condition also persists: choose a slightly larger closed slit sector; its two radial tails are already zero-free by Lemma B.1, and continuity handles the compact annulus. Such an increase contradicts the definition of the central section. There is therefore a contact on the outer ray. No smaller angle can give a zero, proving (B.4). At a contact, the restriction to the outer ray has a local minimum, so $\partial_{\log r} U = \text{Im}(sH') = 0$. The strict Hopf boundary lemma in a small disk at that ordinary analytic boundary point gives $\partial_\theta U = \text{Re}(sH') < 0$. This proves (B.5) without assuming a nondegenerate radial minimum.

To prove continuity, weak closedness of the extended GGC class and continuity of this free stable family imply $g(\alpha_0) \leq \liminf g(\alpha_n)$ whenever $\alpha_n \rightarrow \alpha_0 < 1$. For any angle strictly between $\pi/2$ and $\theta_*(\alpha_0)$, the sector signs are strict and all zeros are excluded. These facts persist for nearby indices: use the locally uniform Wright expansion at zero, Lemma B.10 at infinity, and compactness in between. Taking such angles up to $\theta_*(\alpha_0)$ proves upper semicontinuity. At $\alpha_0 = 4/5$, the earlier persistence theorem gives the same conclusion; below that value the function is zero. Continuity at one follows from Theorem B.11.

For completeness the Hopf inequality also gives local Lipschitz regularity on $(4/5, 1)$. In a neighborhood of a fixed index all contacts lie in a common compact radial annulus. The contact set at that index is compact and $-\partial_\theta U$ has a positive minimum there. Around it the implicit function theorem expresses every possible first contact as $\theta = \Theta(\alpha, r)$, with $|\partial_\alpha \Theta|$ uniformly bounded. Finitely many such neighborhoods cover the compact contact set; outside them U stays strictly positive near the critical angle. The critical angle is the minimum of these local root functions over their radial domains. The infimum of functions with a common Lipschitz constant has that same constant. This gives the asserted local regularity. The square-root lower bound is Theorem B.8. $\square$

B.2. Exact rational moment certificates. This is an independent check on a subset of Theorem A.3 and a reusable method, not a prerequisite for its analytic proof. Put $Y_\alpha = X_\alpha - b_\alpha$ and

$$c_\alpha = \frac{2}{\alpha(1-\alpha)^{1/\alpha}}, \quad h_\alpha(v) = {}_2F_1(\alpha + 1, 1; 3; v).$$

Define

$$A_n = \frac{1}{2n+1} [v^{2n}] (1-v)^{-2} h_\alpha(v)^{-n-1/2}, \quad d_n = (-1)^n (3/2)_n A_n.$$

The endpoint expansion in [7] gives the finite-order large- s expansion

$$\mathbb{E} e^{-sY_\alpha} \sim a_0 \Gamma(3/2) s^{-3/2} \sum_{n \geq 0} d_n (c_\alpha/s)^n.$$

No convergence of this infinite asymptotic series is assumed. Let

$$\log \left(\sum_{n \geq 0} d_n t^n \right) = \sum_{n \geq 1} \ell_n t^n, \quad M_0 = \frac{3}{2}, \quad M_n = (-1)^n n \ell_n \quad (n \geq 1).$$

If X_α is GGC, then $M_n = c_\alpha^{-n} \int u^n U(du)$. Moment existence follows successively by expanding its Stieltjes transform to finite order: the positive remainder recovers the next moment by monotone convergence, and the differentiated Laplace expansion supplies its finite limit. Consequently

$$[M_{i+j}]_{i,j=0}^m \succeq 0, \quad [M_{i+j+1}]_{i,j=0}^m \succeq 0.$$

Each M_n is a polynomial in α of degree at most $2n$; for example

$$M_1 = (2\alpha^2 - 23\alpha + 47)/48.$$

The coefficient recurrences use exact rational arithmetic. A rational vector producing a negative quadratic form gives a finite obstruction. On a parameter interval, strictly negative coefficients of that polynomial in the Bernstein basis $\{\binom{n}{j} t^j (1-t)^{n-j} : 0 \leq j \leq n\}$, after affine rescaling to $0 \leq t \leq 1$, certify the obstruction throughout the interval.

The saved certificate has 18 adjacent rational closed intervals covering $9/10 \leq \alpha \leq 1$, each with a strictly negative quadratic form. It is verified using only Python rational arithmetic, including exact coverage checks. At $\alpha = 1$ the polynomials describe the limiting normalized sequence, not the actual Cauchy family. The probabilistic conclusion is $9/10 \leq \alpha < 1$.

The verifier `certify_thorin_exclusion.py` checks `thorin_exclusion_certificate.json`. The exact moment construction is in `thorin_moment_probe.py`.

APPENDIX C. THE UNIFORM GGC BOUND

This appendix proves the remaining compact part of the uniform $4/5$ theorem. The small-radius series, finite cover and infinite-tail estimates together cover the entire positive radial axis.

C.1. A complete Kanter phase certificate up to the critical endpoint. This section gives a computer-assisted proof of the missing compact parameter range. The infinite tails are controlled by explicit analytic inequalities, including at the critical endpoint. The checker uses directed ball arithmetic and validated integration, rather than sampled function values. Its source and certificate are part of the paper's reproducibility bundle.

Put $\beta = 1 - \alpha$, $a = 1 - \beta$, and let $F_\beta(s) = \mathbb{E}e^{-sK_{1-\beta}}$ for $s > 0$. Write

$$F_\beta(-r - i0) = R(r) + iI(r)$$

for its analytic continuation to the lower cut. More explicitly,

$$(C.1) \quad F_\beta(-r - i0) = \sum_{n \geq 0} \frac{r^{an} e^{i\pi\beta n}}{n! \Gamma(1 - \beta n)}, \quad I(r) = \sum_{n \geq 1} \frac{\Gamma(\beta n) \sin^2(\pi\beta n)}{\pi n!} r^{an} > 0.$$

In particular $I'/I \geq a/r$. With

$$Q_\beta(r) = R(r) \frac{I'(r)}{I(r)} - R'(r),$$

the phase derivative has the sign of Q_β . The phase criterion in Section 2.3.1 of [7] says precisely that $K_{1-\beta}$ is GGC when this derivative is nonnegative. The sine squared factor in (C.1) follows directly from the reflection formula and is retained in every computation below.

Lemma C.1 (Uniform imaginary-part estimate). *For $1/5 \leq \beta \leq 1/4$ and $r \geq 100$,*

$$\frac{I'(r)}{I(r)} \geq \frac{b}{2} > \frac{47}{200}, \quad b = a\beta^{\beta/a}.$$

Proof. Set $q(u) = u^\beta - u$, $\zeta = e^{2\pi i\beta}$, and

$$J = \int_0^\infty (e^{rq(u)} - e^{-ru}) \frac{du}{u}, \quad J_c = \int_0^\infty (e^{r(\zeta u^\beta - u)} - e^{-ru}) \frac{du}{u}.$$

Then $2\pi I = J - \operatorname{Re} J_c$. Write $c = b/2$ and $u_0 = \beta^{1/a}$. Uniformly in the stated parameter range,

$$\frac{47}{100} < b < \frac{27}{50}, \quad q(u) \geq \frac{9}{10}b \quad (u_0/2 \leq u \leq 3u_0/2), \quad 3u_0/2 < 1.$$

The middle inequality follows from concavity and its two endpoint inequalities. All these elementary parameter inequalities are enclosed on the fifty rational subintervals of length $1/1000$ by the checker. On this middle interval the contribution to $J' - cJ$ is at least $\frac{2}{5}b \log 3 e^{9br/10} > \frac{1}{5}e^{9br/10}$. On $0 < u < 1$ where $q < c$, use

$$(q - c)e^{rq} + (u + c)e^{-ru} = (q - c)(e^{rq} - e^{-ru}) + u^\beta e^{-ru} \geq -cru^\beta e^{cr}.$$

For $u \geq 1$, concavity gives $q(u) \leq -a(u - 1)$. Consequently

$$J' - cJ \geq \frac{1}{5}e^{9br/10} - \frac{27}{20}re^{br/2} - \frac{17}{10r}.$$

Also $0 \leq \cos(2\pi\beta) < 31/100$, and maximization of $\cos(2\pi\beta)u^\beta - u/2$ gives

$$\cos(2\pi\beta)u^\beta - u \leq 3b/10 - u/2,$$

since $(31/100)^{5/4}2^{1/3} < 3/10$. The elementary bound $|e^{r\zeta u^\beta} - 1| \leq ru^\beta e^{r \cos(2\pi\beta)u^\beta}$ now yields

$$|J'_c - cJ_c| \leq \Gamma(\beta)(2/r)^\beta(2\beta + 1 + cr)e^{3br/10} \leq 5(3/2 + 27r/100)e^{3br/10}.$$

Here $\Gamma(\beta) \leq 1/\beta \leq 5$, by log convexity of Γ between 1 and 2. On dividing by $e^{9br/10}$, the resulting positive lower bound is

$$\frac{1}{5} - \frac{27}{20}re^{-2br/5} - \frac{17}{10r}e^{-9br/10} - 5(3/2 + 27r/100)e^{-3br/5}.$$

Replace b by $47/100$. Each subtracted term decreases for $r \geq 100$, and the value at 100 is greater than 0.199999, as verified by ball arithmetic. Thus $I' - cI > 0$. $\square$

Lemma C.2 (Uniform real-part estimates at infinity). *For $1/5 \leq \beta \leq 1/4$ and $r \geq 500$,*

$$R(r) > 1, \quad |R'(r)| < \frac{1}{5} + \frac{9}{r}.$$

In particular $Q_\beta(r) > 17/1000$ throughout this unbounded region.

Proof. Set

$$\psi = 5\pi\beta/2, \quad d = -\cot \psi, \quad v = (\beta \sin \psi)^{1/a}, \quad \lambda = rv, \quad \phi(t) = t^\beta/\beta - t, \quad E(t) = e^{-\lambda dt^\beta/\beta}.$$

Rotation to the positive imaginary axis in the preceding integral for J_c gives

$$(C.2) \quad R(r) = \frac{5}{4} + \frac{1}{2\pi} \int_0^\infty E(t) \sin(\lambda\phi(t)) \frac{dt}{t}.$$

At $\beta = 1/5$ the integral is conditionally convergent. It can equivalently be interpreted with a factor $e^{-\varepsilon t}$ and then $\varepsilon \downarrow 0$. On a rotating arc of radius T , use $\cos(\beta(2\pi + \theta)) \leq \cos \theta$ for $0 \leq \theta \leq \pi/2$, $1/5 \leq \beta \leq 1/4$. For large T , the real exponent is at most $-rT \cos \theta/2$, so the arc integral is $O((rT)^{-1})$. The subtracted integrand makes the small arc vanish. Replacing the rotated subtraction e^{-irx} by the real subtraction e^{-rx} adds $i\pi/2$ to J_c , giving the constant $5/4$.

The following elementary inequalities, including their endpoint values, are checked on the same fifty parameter intervals:

$$\begin{aligned} 3/25 < v < 29/200, \quad 0 \leq d < 21/50, \quad J_0 &:= \frac{2^\beta - 2^{-\beta}}{\beta^2} < 7, \\ M_L &:= \frac{1}{2^{-\beta} - 1/2} < 3, \quad M_R := \frac{1}{2 - 2^\beta} < 5/4, \quad \frac{1}{1 - 2^{-a}} < 5/2, \\ \frac{2d}{\beta(2^a - 1)} &< 31/5, \quad \left| 1 - \frac{a}{\beta(2^a - 1)} \right| < 5. \end{aligned}$$

For a monotone nonnegative g and nonincreasing $0 \leq E \leq 1$, endpoint terms plus the total variation of Eg are at most $2 \sup g$. This follows by integrating the two terms of $d(Eg) = E dg + g dE$. It is the only oscillatory-integral estimate needed below.

Let $t_L = \beta^{1/a}$. On $(0, t_L)$, the change of variable $y = \phi(t)$ gives a sine integral with decreasing positive weight $E(t)/(t\phi'(t))$, because $t\phi'(t) = t^\beta - t$ increases there. Such a sine integral from zero to any positive endpoint is nonnegative, by integration against $1 - \cos(\lambda y)$. On $[t_L, 1/2]$, integration by parts bounds the absolute contribution by $2M_L/\lambda$. On $[2, \infty)$ the corresponding bound is $2M_R/\lambda$. The middle interval has absolute contribution at most $\log 4$. Hence

$$R(r) \geq \frac{5}{4} - \frac{\log 4}{2\pi} - \frac{17/2}{2\pi(3/25)r} > 1 \quad (r \geq 500).$$

The final lower bound at 500 is greater than 1.006817.

For the derivative, first insert $e^{-\varepsilon t}$. On the two outer intervals integrate by parts with complex amplitude

$$G(t) = \frac{-dt^\beta/\beta + i\phi(t)}{t\phi'(t)} = \frac{-d/\beta + i(1/\beta - t^a)}{1 - t^a}.$$

On $[0, 1/2]$, the absolute real and imaginary parts are increasing and bounded by $21/4$ and 11 . The variation estimate gives $65/(2\lambda)$. On $[2, \infty)$, the real part contributes less than $31/(5\lambda)$; the imaginary part increases from a number of absolute value less than 5 to 1, crossing zero once. Splitting at that zero bounds its contribution by $12/\lambda$. After multiplication by $v/(2\pi)$, the two outer contributions are less than

$$\frac{65/2 + 91/5}{2\pi r} < \frac{9}{r}.$$

On $[1/2, 2]$, $\phi > 0$, and direct absolute integration gives

$$\frac{v}{2\pi} [(1+d)J_0 - 3/2] < \frac{29/200}{2\pi} [(71/50)7 - 3/2] < \frac{1}{5}.$$

All estimates are independent of ε . Integration by parts in the original $1/t$ integral shows locally uniform Abel convergence as $\varepsilon \downarrow 0$. Integrate the derivative bound over any finite r interval, pass to the limit, and use analytic differentiability of R . This justifies the same derivative bound at $d = 0$, without asserting ordinary convergence of the differentiated oscillatory integral. Lemma C.1 finally gives $Q > 47/200 - 1/5 - 9/500 = 17/1000$. $\square$

Proposition C.3 (Finite certificate specification). *The finite certificate and exact recursive cover verified by `certify_kanter_optimal.py` establish*

$$R(r) > 0, \quad Q_\beta(r) > 0 \quad (1/5 \leq \beta \leq 1/4, \ 1 \leq r \leq 500).$$

For $0 < r \leq 1$, the same signs follow from a single convergent-series bound. The procedure is specified below so that neither parameter coverage nor truncation errors depend on a plotting resolution.

Certificate algorithm and small-radius proof. Let $A_n = \Gamma(\beta n)r^{an}/(\pi n!)$. Wendel's inequality [15, (5.6.4)] gives

$$\frac{A_{n+1}}{A_n} \leq r^a \beta^\beta n^{-a}.$$

For any truncation index N with $q_N = r^a \beta^\beta N^{-a} < 1$, the tails of the series and its index-weighted counterpart are bounded by

$$\sum_{n \geq N} A_n \leq \frac{A_N}{1 - q_N}, \quad \sum_{n \geq N} n A_n \leq A_N \left[\frac{N}{1 - q_N} + \frac{q_N}{(1 - q_N)^2} \right].$$

For all $\beta \in [1/5, 1/4]$, directed evaluation through $N = 80$ at $r = 1$ gives

$$S := \sum_{n \geq 2} (n-1) \frac{\Gamma(\beta n)}{\pi n!} < 3/5.$$

The first real coefficient in (C.1) is positive. It follows that $R \geq 1 - S > 0$ and $R - (r/a)R' \geq 1 - S > 0$ on $0 < r \leq 1$. Combining with $I'/I \geq a/r$ proves the small-radius assertion.

For the compact rectangle the program bisects rational intervals in β and r . Every leaf stores a positive margin, but the verifier recomputes it and does not trust its stored value. For a leaf $[\beta_0, \beta_1] \times [r_0, r_1]$, validated integration encloses R, R' on the whole leaf. If $r_0 \geq 100$, use $I'/I > 47/200$. Otherwise, the positive series for I defines weights proportional to $c_n r^{an}$. Their mean index $m(r)$ is increasing because $m'(r) = a \text{Var}(n)/r$. Thus

$$I'(r)/I(r) \geq (1 - \beta_1)m_*(r_0)/r_1,$$

where m_* is a lower enclosure computed as $1 + \sum_{1 \leq n < N} (n-1)(c_n r_0^{an})_{\text{lower}}/I_{\text{upper}}$, with negative enclosure endpoints replaced by zero. The explicit tail above is included in I_{upper} . A leaf is accepted only if its lower enclosure for R is positive and $R_{\text{lower}}(I'/I)_{\text{lower}} - (R')_{\text{upper}} > 0$.

For completeness the stable integral used near $\beta = 1/5$ takes $\theta = \pi/2 - 1/20$, $c = \cos \theta$, $B = \tan \theta$, $p = 1/\beta$, $A = r^a c^{-\beta}$, and $\zeta = \exp(i\beta(2\pi + \theta))$. Then

$$R = 1 + \frac{\theta}{2\pi} + \frac{1}{2\pi\beta} \text{Im} \int_0^\infty \frac{e^{-(1+iB)u^p + A\zeta u} - e^{-u^p}}{u} du,$$

$$R' = \frac{aA}{2\pi\beta r} \text{Im} \int_0^\infty \zeta e^{-(1+iB)u^p + A\zeta u} du.$$

Away from the endpoint, use $\theta = \pi(1/(2\beta) - 2)$, for which $\zeta = i$, so the two integrands reduce to $e^{-u^p} \sin(Au - Bu^p)/u$ and $e^{-u^p} \cos(Au - Bu^p)$. Validated integration is restricted to $[10^{-9}, 3]$; the origin and infinite tails are bounded explicitly. For the fixed rotation, the checked inequality $A \text{Re} \zeta < 3^{p_{\min}-1}/2$ bounds the real exponent by $-u^{p_{\min}}/2$ beyond 3. The corresponding tail bounds are

$$\frac{2e^{-3^{p_{\min}}/2}}{p_{\min} 3^{p_{\min}}}, \quad \frac{2e^{-3^{p_{\min}}/2}}{p_{\min} 3^{p_{\min}-1}}.$$

At the origin, bounds are $(|A|\varepsilon + |B|\varepsilon^{p_{\min}}/p_{\min})e^{|A|\varepsilon}$ and $\varepsilon e^{|A|\varepsilon}$. All analytic-continuation flags are forwarded to the complex power operation during validated integration. The cover is reconstructed from exact rational midpoints, and the checker also repeats all the small-radius and unbounded-tail constant checks. $\square$

Theorem C.4 (Optimal uniform GGC bound). *For every $0 < \alpha \leq 4/5$ and $0 \leq \rho \leq 1$, $X_{\alpha,\rho}$ is an extended GGC. Moreover within the Kanter family $0 < \alpha < 1$, K_α is GGC if and only if $\alpha \leq 4/5$. The number $4/5$ is the largest endpoint of an initial index interval on which every admissible skewness gives GGC.*

Proof. Lemmas C.1–C.2 and Proposition C.3 give increasing cut phase for $\beta \in [1/5, 1/4]$. The proof of [7, Theorem 2] handles $\beta \geq 1/4$, so its phase characterization proves the Kanter conjecture. The negative direction is [7, Corollary 2]. The multiplicative factorization and the Cauchy subordination argument of [7, Section 2.3] transfer Kanter GGC to $X_{\alpha,\rho}$. Theorem A.3 proves optimality for the free stable family itself. The isolated Cauchy index $\alpha = 1$ does not enlarge this initial interval. This result does not assert the absence of GGC laws at interior skewnesses when $4/5 < \alpha < 1$. $\square$

Remark C.5 (Execution record). The complete cover has 20,261 leaves. It was built and then verified again in a fresh process on 3 October 2026, using python-flint 0.9.0 with 55 decimal digits of working precision. Both executions passed every interval assertion. The file `kanter_optimal_certificate.json` contains the exact recursive cover; the separate verification record stores hashes of the checker, its integral module, and the certificate. The computation is an essential part of the proof on the compact rectangle. The quadrature algorithm and its analytic-domain requirements are documented in [8].

Corollary C.6. *For every fixed $0 < \eta < 1/2$, there exists $\varepsilon_\eta > 0$ such that $X_{\alpha,\rho}$ is extended GGC whenever $4/5 < \alpha < 4/5 + \varepsilon_\eta$ and $\eta \leq \rho \leq 1 - \eta$.*

Proof. Apply Proposition B.5 at $\alpha_0 = 4/5$. $\square$

APPENDIX D. SIGNED LÉVY KERNELS AND EXTREMAL GERMS

We derive the signed kernel representation and prove angular interpolation. Convergent extremal germs then supply explicit origin bounds and the complete index-3/2 slice. These estimates are the inputs for the global classification in Appendix E.

D.1. An analytic ID anchor above one. The following estimate supplies a useful explicit ID region.

Proposition D.1. *For $1 < \alpha \leq 53/50$, $X_{\alpha,3/4}$ and $X_{\alpha,1/4}$ are ID.*

Proof. Write $W_\alpha(z) = \prod_n (1 + z/\lambda_n)$ and $r_n = \lambda_n^{1/\alpha}$. Convexity of the log moment generating function $\log W_\alpha(s^\alpha)$ of $X_{\alpha,1/\alpha}$ gives

$$(D.1) \quad \sum_n \frac{1}{(1 + (r_n x)^\alpha)^2} \leq \frac{\alpha - 1}{\alpha} \sum_n \frac{1}{1 + (r_n x)^\alpha}.$$

For $\rho > 1/2$, let

$$\omega = \pi(\rho - \tfrac{1}{2}), \quad \xi = \pi(1 - \alpha(1 - \rho)), \quad B_{\alpha,\xi}(v) = \frac{\alpha \sin \xi}{\pi} \int_0^\infty \frac{e^{-vu} u^{\alpha-1}}{1 + 2 \cos \xi u^\alpha + u^{2\alpha}} du.$$

Single-factor Fourier inversion gives the negative-side signed Lévy kernel

$$K_{\alpha,\rho}(x) = \sum_n B_{\alpha,\xi}(r_n x) - 2 \sum_n e^{-r_n x \sin \omega} \cos(r_n x \cos \omega).$$

The positive-side kernel is $\sum_n B_{\alpha,\eta}(r_n x) \geq 0$, $\eta = \pi(1 - \alpha\rho)$, with value zero at $\eta = 0$. To check the inversion, differentiate the single logarithmic factor:

$$\frac{d}{dt} \log(1 - e^{-i\alpha\omega}(t/r)^\alpha) = \frac{\alpha t^{\alpha-1}}{t^\alpha - r^\alpha e^{i\alpha\omega}}.$$

The upward rotation crosses $re^{i\omega}$, yielding the exponential-cosine residue. The downward rotation crosses no pole. For fixed nonsymmetric parameters the single signed measure satisfies $\int (x^2 \wedge |x|) |\nu_r|(dx) = O(r^{-\alpha})$. Since $\sum r_n^{-\alpha} = 1/\Gamma(\alpha + 1)$, the compensated representations sum absolutely. Nonnegative K therefore implies ID.

For $\rho = 3/4$, one has $\omega = \pi/4$, $\cos \xi < 0$, and

$$1 + 2 \cos \xi u^\alpha + u^{2\alpha} \leq (1 + u^2)^\alpha \leq e^{\alpha u}.$$

Hölder's inequality then gives

$$B_{\alpha,\xi}(v) \geq \frac{\Gamma(\alpha + 1) \sin(\pi\alpha/4)}{\pi(1 + \alpha^{\alpha/(\alpha-1)})^{\alpha-1}} \frac{1}{1 + v^\alpha} > \frac{1}{5(1 + v^\alpha)}.$$

Here $\Gamma(\alpha + 1) \geq 1$, $\sin(\pi\alpha/4) \geq 1/\sqrt{2}$, and the power in the denominator is less than 11/10. Also

$$\sup_{v \geq 0} e^{-v/\sqrt{2}} (\cos(v/\sqrt{2}))_+ (1 + v^\alpha)^2 < 7/4.$$

On the first positive cosine interval, bound this by $e^{-7v/10 - v^2/4} (1 + v)^{53/25}$. Its maximum occurs at $v = (\sqrt{107} - 6)/5$ and is less than 7/4. Later positive intervals have $v > 6$, where $e^{-7v/10} (1 + v)^{53/25} < 1$. It follows from (D.1) that

$$K_{\alpha,3/4}(x) > \left(\frac{1}{5} - \frac{7}{2} \frac{\alpha - 1}{\alpha} \right) \sum_n \frac{1}{1 + (r_n x)^\alpha} \geq \frac{1}{530} \sum_n \frac{1}{1 + (r_n x)^\alpha} > 0.$$

Reflection proves the other case. These examples contradict the conjectured exclusion of every index above one discussed after Theorem 1 of [6]. $\square$

D.1.1. *Sector interpolation for ID.* The following argument proves interval convexity. Inclusion of the extremal skewness requires the separate endpoint positivity theorem.

Proposition D.2 (Each half of the ID section is an interval). *For fixed $1 < \alpha < 2$, the set*

$$I_\alpha = \{\rho \in (1/2, 1/\alpha] : X_{\alpha,\rho} \text{ is ID}\}$$

is either empty or a closed interval $[\rho_-(\alpha), \rho_+(\alpha)]$, where $1/2 < \rho_- \leq \rho_+ \leq 1/\alpha$. Between two distinct ID parameters the signed Lévy kernel is strictly positive at every $x > 0$. The proposition does not assert $\rho_+ = 1/\alpha$.

Proof. First, nonnegativity of the negative-side kernel is also necessary for ID. Its absolutely summable compensated representation above is a signed Lévy–Khintchine representation, and uniqueness of that representation forces its signed measure to agree with the positive Lévy measure of any ID law. Thus

$$X_{\alpha,\rho} \text{ is ID} \iff K_{\alpha,\rho}(x) \geq 0 \quad (x > 0).$$

Let $Q_r(t) = \log|1 - (t/r)^\alpha| + i\pi \mathbf{1}_{\{t > r\}}$. For $\operatorname{Re} z > 0$, define

$$\mathcal{F}_\alpha(z) = \frac{iz}{\pi} \sum_n \int_0^\infty e^{-zt} Q_{r_n}(t) dt.$$

This uses a continuous logarithmic boundary value rather than resetting its imaginary part after each zero. Its integrable logarithmic singularities cause no difficulty. On each closed subsector of the right half-plane the sum converges normally and

$$(D.2) \quad |\mathcal{F}_\alpha(z)| = O(|z|^{-\alpha}) \quad \text{both at zero and at infinity.}$$

Indeed for all $s > 0$,

$$s \int_0^\infty e^{-su} |Q_1(u)| du \leq C_\alpha s^{-\alpha}.$$

For $s \geq 1$, split near zero, the integrable singularity at one, and the exponentially damped tail. For $s \leq 1$, the logarithmic growth at infinity gives $O(1 + |\log s|)$, which is bounded by $Cs^{-\alpha}$. Scaling by r_n , with $\operatorname{Re} z$ comparable to $|z|$, and using $\sum r_n^{-\alpha} < \infty$ proves (D.2) and normal convergence.

Set $\theta = \pi(1 - \rho) \in (0, \pi/2)$. A direct single-factor contour rotation gives

$$(D.3) \quad K_{\alpha,\rho}(x) = \operatorname{Re} \mathcal{F}_\alpha(xe^{-i\theta}).$$

For an explicit sign check, integration by parts replaces zQ_r by the boundary derivative $\alpha t^{\alpha-1}/(t^\alpha - r^\alpha - i0)$. Rotating this path from just below the positive ray to angle θ crosses the pole at r . After multiplication by i/π , its residue is $-2e^{-rxe^{-i\theta}}$. The real part of the rotated integral is $B_{\alpha,\pi-\alpha\theta}(rx)$, exactly as required. Summation is permitted by the preceding normal convergence bounds.

Suppose $\rho_1 < \rho_2$ are ID. The harmonic function $\operatorname{Re} \mathcal{F}_\alpha$ is nonnegative on the corresponding two rays, whose angle difference is less than $\pi/2$. If θ_m is their middle angle, add the positive barrier

$$\epsilon(r^2 + r^{-2}) \cos(2(\theta - \theta_m)).$$

Since $\alpha < 2$, it dominates (D.2) at both radial ends. The minimum principle on truncated sectors, followed by the limits of the radii and then $\epsilon \downarrow 0$, proves nonnegativity on every intervening ray. The strong minimum principle gives strict positivity inside: the harmonic function is not zero identically, as is also seen from its positive large- x kernel asymptotic. This proves interval convexity. Weak closedness of ID, continuity in ρ , and the known non-ID symmetric law show that the interval is closed and has positive distance from $1/2$. $\square$

D.1.2. *Extremal ID at index three halves.* We first give the complete extremal positivity argument at this index, including its finite certificate and both radial ends.

Theorem D.3 (Extremal index $3/2$). *The laws $X_{3/2,2/3}$ and $X_{3/2,1/3}$ are classically ID. If $k(x) = K_{3/2,2/3}(x)$ and $b = \frac{3}{2}2^{1/3}$, then*

$$0 < k(x) \leq \frac{3}{2} \quad (x > 0), \quad \int_0^\infty k(x) dx = b, \quad \operatorname{Re} k(z) > 1 \quad (|z| \leq 1/2).$$

The last statement uses the analytic germ at zero and includes $k(0) = 3/2$. The proof combines exact rational polynomial bounds and directed ball arithmetic with analytic all-order error estimates.

Proof. Set $c = 2^{-2/3}$ and $Z = (b - X_{3/2,2/3})/c \geq 0$. Its density is $p(y) = \frac{2}{\pi\sqrt{3}}\sqrt{y} A(y)$. The free Cauchy-transform inverse gives $y = 3 - u^{-1} - 2\sqrt{u}$. Thus $v = \sqrt{u}$ solves

$$2v^3 + (y - 3)v^2 + 1 = 0.$$

The two roots coalescing at $v = 1, y = 0$ determine the density. Write its third root as $-(1 + V)/2$. Vieta's formulas give the explicit analytic parametrization

$$y = \frac{V(V+3)^2}{(V+1)^2}, \quad V(0) = 0, \quad A(y) = \frac{\sqrt{1+V+V^2/3}}{(1+V)^3(1+V/3)}.$$

The normalized density factor A is holomorphic on $|y| < 3\sqrt{3}$: the only nonzero discriminant points are $9/2 \pm (3\sqrt{3}/2)i$, while division by $\sqrt{y}$ removes the square-root interchange at zero.

Put $R = 51/10$, $B = 41/10$, and $A(y) = \sum f_n y^n$. On $|y| = R$, every root of the cubic has modulus less than B , since $2B^3 > (R+3)B^2 + 1$. Expressing A as the difference of the squared two density roots divided by $2i\sqrt{4/3}\sqrt{y}$ gives

$$|A(y)| \leq \frac{\sqrt{3}B^2}{2\sqrt{R}} < 7, \quad |f_n| \leq 7R^{-n}.$$

These constants are verified exactly by the checker.

Let $j(y) = k(cy) = \sum_{n \geq 0} k_n y^n$, and put $p_0 = 3/2$. The exact convolution equation $p * j = yp$ gives

$$(D.4) \quad k_0 = p_0, \quad k_n = \frac{(p_0)_{n+1}}{n!} f_n - \sum_{m=1}^n \frac{(p_0)_m (n-m)!}{n!} f_m k_{n-m}.$$

This is an exact local inverse argument, not an inference from an asymptotic expansion alone. More explicitly, the analytic logarithmic derivative of the extremal Laplace transform has a causal inverse Laplace distribution. Multiplication by the transform of p gives exactly $p * j = yp$; the single-factor inversion above identifies that inverse with the spectral kernel for $y > 0$. Applying the local fractional inverse of the leading kernel y^{p_0-1} turns this into a Volterra equation of the second kind with analytic coefficients. It has a unique local distribution solution, which is analytic and whose Taylor coefficients are (D.4).

Here is a transparent all-order bound for those coefficients:

$$(D.5) \quad |k_n| \leq 16(n+1)^2 R^{-n} \quad (n \geq 0).$$

The base $0 \leq n < 100$ is checked by exact rational arithmetic. For $n \geq 100$, use $(3/2)_m \leq (m+1)!$ and

$$\frac{1}{(n+1)^2} \sum_{m=1}^n \frac{(m+1)(n-m+1)^2}{\binom{n}{m}} < \frac{5}{n}.$$

To verify the latter elementary estimate, separate $m = 1, 2, n-2, n-1, n$; on the remaining terms use $\binom{n}{m} \geq \binom{n}{3}$ and $(m+1)(n-m+1)^2 \leq 4(n+2)^3/27$. After multiplication by n , a uniform upper bound is

$$3 + \frac{6}{99} + \frac{4}{100} + \frac{18}{10000} + \frac{8}{9} \frac{(102/100)^3}{(99/100)(98/100)} < 5.$$

Consequently the induction step in (D.4) is bounded, after division by $(n+1)^2 R^{-n}$, by $7(102/101) + 560/100 < 16$. This proves (D.5) and convergence of the kernel series on $|y| < R$.

The checker computes f_n, k_n through degree $N = 128$ exactly. For $q = 4/R$, the omitted kernel tail is at most

$$E_N = 16q^{N+1} \left\{ \frac{(N+2)^2}{1-q} + \frac{2(N+2)q}{(1-q)^2} + \frac{q(1+q)}{(1-q)^3} \right\} < 3.25 \cdot 10^{-8}.$$

All Bernstein coefficients of the kernel polynomial on the single interval $[0, 4]$ exceed this error by more than $1/50$. The same exact calculation applied to $(3/2 - \sum_{n=0}^N k_n y^n)/y$, with error $E_N/4$, is strictly positive. Hence $0 < j(y) \leq 3/2$ on $[0, 4]$. Also the sum of the absolute coefficients at radius $4/5$, including the full tail, is less than $0.403 < 1/2$. Thus $\operatorname{Re} j(y) > 1$ for $|y| \leq 4/5$.

For the entire remaining tail in the original variable, take $x_0 = 5/2$. Three sign-changing Wright-zero brackets are

$$(2.7407302, 2.7407303), \quad (4.7166213, 4.7166214), \quad (6.6596278, 6.6596279).$$

Set $S_m = (-1)^{m+1} m[z^m] \log W_{3/2}(z)$ and $T_m = S_m - \sum_{j=1}^3 r_j^{-3m/2}$. The verified inequality $8^6 T_4 < 1$ forces every unlisted root to exceed 8; no assumed ordering or completeness of the three brackets is needed. Since $\xi = \pi/2$, the denominator of the positive integral is $1 + u^3$. Its elementary lower bound gives

$$k(x) \geq \frac{1}{\pi x^{3/2}} - \frac{(3/2)\Gamma(9/2)S_3}{\pi x^{9/2}} - 2 \sum_{j=1}^3 e^{-r_j x/2} - 2 \cdot 8^3 e^{-4x} T_2.$$

At x_0 this bound exceeds 0.008. After multiplication by $x^{3/2}$, every subtracted term decreases for $x \geq x_0$, so it remains strictly positive on the whole tail. Conversely the upper bound

$$k(x) \leq \frac{1}{\pi x^{3/2}} + 2 \sum_{j=1}^3 e^{-r_j x/2} + 2 \cdot 8^3 e^{-4x} T_2$$

decreases and is less than $1/5$ at x_0 . The interval and tail overlap because $4c > 5/2$. Also $(1/2)/c < 4/5$, proving the complex-local assertion.

The file `certify_extremal_three_halves.py` verified every finite inequality just used, with exact rational polynomial calculations and 100-digit outward ball evaluations of the remaining constants and zero signs. The zero Taylor tails are explicitly bounded using the decreasing ratio $\Gamma(t)/\Gamma(t+1/2)$. Its record is `extremal_three_halves_certificate.json`. Nonnegative k proves ID by the signed-kernel criterion. The drift of $b - X$ is zero and its mean is b ; thus $\int k = b$. Reflection proves the other stated parameter. $\square$

D.1.3. A convergent extremal kernel with variable index.

Lemma D.4 (The logarithmic-strip inverse). *Let $1 < a \leq 2$, $\varepsilon = a - 1$, $c_a = \varepsilon^{1/a}$, and $b_a = a\varepsilon^{1/a-1}$. The density of $Z_a = (b_a - X_{a,1/a})/c_a$ has the form*

$$p_a(y) = \frac{\sqrt{2/a}}{\pi} \sqrt{y} F_a(y), \quad F_a(0) = 1,$$

where the germ F_a is holomorphic on

$$|y| < R_a, \quad R_a = \frac{2a}{a-1} \sin \frac{\pi}{a}.$$

The corresponding signed Lévy kernel $j_a(y) = K_{a,1/a}(c_a y)$ has a holomorphic germ on the same disk. Writing $F_a(y) = \sum f_n(a) y^n$ and $j_a(y) = \sum k_n(a) y^n$, both $a^n f_n(a)$ and $a^n k_n(a)$ are rational polynomials in a . For $3/2 \leq a \leq 153/100$, uniformly in $n \geq 0$,

$$(D.6) \quad |f_n(a)| \leq 7(10/51)^n, \quad |k_n(a)| \leq 16(n+1)^2(10/51)^n.$$

This is a convergence and remainder theorem, not a claim of ID throughout the index interval.

Proof. In logarithmic Cauchy-transform coordinates $u = e^v$, the inverse is

$$y = h_a(v) = \frac{a}{\varepsilon} - e^{-v} - \frac{1}{\varepsilon} e^{\varepsilon v}.$$

Consider the horizontal strip $|\operatorname{Im} v| < 2\pi/a$. On its horizontal boundaries, with $v = t \pm 2\pi i/a$,

$$h_a(v) = \frac{a}{\varepsilon} - e^{\mp 2\pi i/a} \left(e^{-t} + \frac{1}{\varepsilon} e^{\varepsilon t} \right).$$

The bracketed real quantity is at least a/ε , its value at $t = 0$. Therefore $|h_a(v)| \geq R_a$ on these boundaries. On vertical boundaries $\operatorname{Re} v = \pm L$, one exponential dominates uniformly as $L \rightarrow \infty$, for a in any compact subinterval of $(1, 2]$.

At $a = 2, y = 0$, the equation is $\cosh v = 1$, with exactly the double zero $v = 0$ in the strip. Homotopy in a , using the nonvanishing boundary bounds, preserves this count. For fixed a , homotopy in $|y| < R_a$ consequently shows that $h_a(v) = y$ has exactly two roots in the strip, counted with multiplicity. The only critical point in the open strip is zero, since $h'_a(v) = e^{-v}(1 - e^{av})$. The two inverse branches v_+, v_- therefore interchange only around $y = 0$, where $v_{\pm} \sim \pm i\sqrt{2y/a}$. The single-valued quotient

$$F_a(y) = \frac{e^{v_+(y)} - e^{v_-(y)}}{2i\sqrt{2/a}\sqrt{y}}$$

extends holomorphically across zero and gives the asserted density. This argument identifies the relevant inverse branches; it does not assume that all critical values on every logarithmic sheet control this germ.

For the uniform estimate put $R = 51/10$. The function R_a is decreasing on $(1, 2]$: its derivative has the sign of $-\sin \theta - (\pi - \theta) \cos \theta$, with $\theta = \pi/a$, which is negative. A directed evaluation gives $R_{153/100} > 5.114 > R$. If $h_a(v) = y$, $|y| = R$, and $\operatorname{Re} v \geq \log 17$, the reverse triangle inequality would imply

$$R + \frac{a}{\varepsilon} \geq \left| e^{-v} + \varepsilon^{-1} e^{\varepsilon v} \right| \geq \frac{17^\varepsilon}{\varepsilon} - \frac{1}{17} > R + \frac{a}{\varepsilon},$$

a contradiction. The last inequality holds uniformly for $1/2 \leq \varepsilon \leq 53/100$: after multiplication by ε , it is positive at $1/2$ and its derivative is bounded below by $\sqrt{17} \log 17 - 1 - R - 1/17 > 0$. Thus $|e^{v_{\pm}}| < 17$, and

$$|F_a(y)| \leq 17\sqrt{a/(2R)} \leq 17\sqrt{3/20} < 7.$$

Cauchy's estimate proves the first bound in (D.6).

The exact convolution equation $p_a * j_a = y p_a$ again gives (D.4), with $p_0 = 3/2$. For clarity about convergence, set

$$D_a(y) = \sum_{n \geq 0} \frac{(p_0)_n}{n!} f_n(a) y^n, \quad q_a = D'_a, \quad d_a = j_a - p_0.$$

The equivalent local Volterra equation is $d_a = y q_a - q_a * d_a$. Both D_a and q_a are holomorphic on $|y| < R_a$. The Volterra resolvent series converges uniformly on every smaller disk, by the factorial bounds on iterated integration. Local uniqueness identifies its solution with the signed spectral kernel, as in the preceding proof.

Finally let

$$A_a(w) = -\frac{2h_a(\log(1+w))}{aw^2}, \quad A_a(0) = 1.$$

Its coefficient of w^m is a rational polynomial in a of degree at most m . Lagrange inversion gives the exact formula

$$a^n f_n(a) = \frac{(-2)^n}{2n+1} [w^{2n}] A_a(w)^{-n-1/2}.$$

The recurrence (D.4) then proves the same polynomial property for $a^n k_n(a)$. The checker constructs these rational polynomials exactly. For $0 \leq n < 100$, it shifts each kernel numerator to $a = 303/200$, bounds its absolute coefficient sum at displacement $3/200$, and divides by $(3/2)^n$. These exact rational bounds verify the base of (D.6). The all-order induction is exactly the one following (D.5), using only the uniform density bound. The degree-128 parameter coefficients also agree identically at $a = 3/2$ with the independently computed cubic coefficients. The source is `general_extremal_polynomials.py`, with the checks in `certify_general_extremal_germ.py`. $\square$

Proposition D.5 (A certified non-ID endpoint). *The extremal law $X_{153/100, 100/153}$ is not ID. This statement alone does not exclude every skewness at that index or every larger index.*

Proof. Set $a = 153/100$, $x = 53/20$, $\omega = \pi(100/153 - 1/2)$, $h = \sin \omega$, $c = \cos \omega$, and $\xi = 47\pi/100$. Three sign-changing zero brackets are

$$(2.6247913, 2.6247914), \quad (4.5379757, 4.5379758), \quad (6.4203113, 6.4203114).$$

Let $S_j = (-1)^{j+1} j[z^j] \log W_a(z)$, $T_j = S_j - \sum_{n=1}^3 r_n^{-aj}$, and $R = T_4^{-1/(4a)}$. Every remaining root is at least $R > 7.779$, and $R > a/(hx)$. Since $\cos \xi > 0$,

$$K_{a,1/a}(x) \leq \frac{\sin \xi}{\pi x^a} - 2 \sum_{n=1}^3 e^{-hxr_n} \cos(cxr_n) + 2R^a e^{-hxR} T_1 < -0.003.$$

The certificate `non_id_endpoint_153_certificate.json` was reconstructed at 90 decimal digits, including every zero-bracket Taylor tail and every unlisted-root contribution. Its checker uses no numerical quadrature. The final displayed strict bound is an upper bound for the kernel, which suffices to prove non-ID. $\square$

D.1.4. *The complete ID slice at index three halves.*

Theorem D.6 (All skewnesses at $\alpha = 3/2$). *There is a unique solution (ρ_0, x_0) of*

$$K_{3/2,\rho}(x) = 0, \quad \partial_x K_{3/2,\rho}(x) = 0$$

in $0.652317 < \rho < 0.652318$, $12/5 < x < 27/10$. It satisfies

$$\begin{aligned} 0.6523173573678771199 < \rho_0 < 0.6523173573678771201, \\ 2.53849362351816328 < x_0 < 2.53849362351816331. \end{aligned}$$

The complete classification, including equality at the boundary, is

$$X_{3/2,\rho} \text{ is ID} \iff \rho \in [1/3, 1 - \rho_0] \cup [\rho_0, 2/3].$$

In particular $X_{3/2, 33/50}$ is ID and $X_{3/2, 13/20}$ is not ID. All these laws are non-GGC. The finite part of this assertion is proved by the whole-box certificate specified below.

Proof. Write $K(\rho, x) = K_{3/2,\rho}(x)$ and let $J = [0.652317, 0.652318]$. The following bounds hold on the whole indicated regions, uniformly for $\rho \in J$:

$$\begin{array}{l|l} 0 < x \leq 1/2 & K > 1/2 \\ [1/2, 12/5] \cup [27/10, 4] & K > 1/1250 \\ [12/5, 27/10] & K_{xx} > 1/4, \quad K_\rho > 1 \\ x \geq 4 & K > 2/(25x^{3/2}). \end{array}$$

Here is a proof and a complete specification of the numerical errors.

For the origin, use the kernel k from Theorem D.3. Put $\delta = \pi(2/3 - \rho)$, $w = xe^{-i\delta}$. The identity

$$(D.7) \quad K(\rho, x) = 2 \operatorname{Re} k(w) + \frac{b \sin \delta}{\pi x} - \frac{x \sin \delta}{\pi} \int_0^\infty \frac{k(y)}{(y - x \cos \delta)^2 + x^2 \sin^2 \delta} dy$$

follows by Cauchy continuation. Indeed the extremal characteristic logarithm has derivative $\chi'_e(t) = ib - i \int_0^\infty e^{-ity} k(y) dy$, so for $\Im w > 0$,

$$\frac{i}{\pi} \int_0^\infty e^{itw} \chi'_e(t) dt = -\frac{ib}{\pi w} + \frac{i}{\pi} \int_0^\infty \frac{k(y)}{w-y} dy.$$

Continuation through the positive axis adds $2k(w)$. The rotation $\chi_\rho(t) = \chi_e(e^{i\delta}t)$, or the single-factor calculation in (D.3), identifies its real part with the stated kernel. The continuation is legitimate in the disk used here because k is analytic there. For $x \leq 1/2$, the first term in (D.7) exceeds 2, the pole term is nonnegative, and the Poisson integral is at most $3/2$. Thus $K > 1/2$.

For the tail put $a = 3/2$, $S_j = \sum r_n^{-aj}$, $h = \sin(\pi(\rho - 1/2))$, $\xi = \pi(a\rho - 1/2)$. Since $\cos \xi \geq 0$, elementary inequalities give

$$K(\rho, x) \geq \frac{\sin \xi}{\pi} [x^{-a} - 2a\Gamma(2a) \cos \xi S_2 x^{-2a} - a\Gamma(3a) S_3 x^{-3a}] - 2 \left(\frac{3}{ehx} \right)^3 S_2.$$

Use $(1 + 2q \cos \xi + q^2)^{-1} \geq 1 - 2q \cos \xi - q^2$ for the positive terms and $e^{-v} \leq (3/e)^3 v^{-3}$ for the oscillatory terms. Multiplication by $x^{3/2}$ makes the bound increasing. Its uniform value at 4 exceeds $0.0110878 > 1/100$, proving the claimed unbounded tail.

For the finite part, twenty disjoint rational Wright-zero brackets have width at most 10^{-80} . Their signs use the degree-349 Taylor polynomial with a uniform geometric tail on $0 < r < 45$. For the positive Rayleigh remainders $T_m = S_m - \sum_{n=1}^{20} r_n^{-3m/2}$, the verified bound $T_{20}(204/5)^{30} < 1$ puts every unlisted zero above $204/5$. This rules out missed smaller roots without assuming completeness of a preliminary scan.

The listed positive integrals use validated quadrature after $t = u^2$, on $[0, 12]$, with explicit Gaussian tails. The same integration-by-parts bounds apply to the required derivatives. For unlisted roots, expand the exact finite geometric identity for $e^{i\xi}/(1 + qe^{i\xi})$. The resulting sum is

$$\frac{a}{\pi} \sum_{m=1}^{19} (-1)^{m+1} \sin(m\xi) \Gamma(am) x^{-am} T_m.$$

Its j -th x -derivative error is at most $a\Gamma(30+j)x^{-30-j}T_{20}/\pi$, $j = 0, 1, 2$. Its ρ -derivative error is at most $21a^2\Gamma(30)x^{-30}T_{20}$. These follow from $|1 + qe^{i\xi}| \geq 1$ and $q/|1 + qe^{i\xi}| \leq 1$. For the unlisted oscillatory terms, choose $1 \leq m \leq 20$ such that $hxR > am + j$, $R = 204/5$. The derivative error is at most $2R^{am+j}e^{-hxR}T_m$; for the ρ derivative use $j = 1$ and multiply by πx . Thus every infinite contribution is explicitly enclosed.

The exact partition has 19 boxes for positivity off the contact interval and 3 boxes for both derivative inequalities on it. Endpoint checks give $K_x(\rho, 12/5) < 0 < K_x(\rho, 27/10)$. There is therefore one minimizer $x(\rho)$ in the contact interval, and $m(\rho) = K(\rho, x(\rho))$ satisfies $m'(\rho) = K_\rho(\rho, x(\rho)) > 1$.

Let ρ_L, ρ_U be the two displayed rational endpoints and $x_t = 2.5384936235181632931$. Strict outward enclosures give

$$\begin{aligned} -1.607 \cdot 10^{-19} &< K(\rho_L, x_t) < -1.530 \cdot 10^{-19}, \\ 1.799 \cdot 10^{-19} &< K(\rho_U, x_t) < 1.875 \cdot 10^{-19}, \\ |K_x(\rho_U, x_t)| &< 3.46 \cdot 10^{-19}. \end{aligned}$$

Convexity supplies the global bound $m(\rho_U) \geq K(\rho_U, x_t) - K_x(\rho_U, x_t)^2/(2(1/4)) > 0$, whereas $m(\rho_L) < 0$. Hence exactly one ρ_0 in the bracket has $m(\rho_0) = 0$. The derivative signs at the stated two x_0 endpoints hold throughout this narrow ρ bracket.

At ρ_0 all radii are nonnegative, proving ID including equality. Theorem D.3 and sector interpolation give ID on $[\rho_0, 2/3]$. Any ID parameter below ρ_0 would, by interpolation, force

an ID parameter with a strictly negative minimum immediately below ρ_0 , a contradiction. The symmetric case is non-ID; reflection finishes the classification.

The command `certify_three_halves_classification.py verify` reruns the extremal prerequisite, twenty-zero verification, all 22 boxes, contact checks and tail constants at 180 decimal digits. It passed. Full enclosures and hashes are in `three_halves_classification_verification.json`. This slice alone does not prove the global cutoff; the independent argument in Theorem E.4 supplies that conclusion. $\square$

D.1.5. Uniform ID rectangles at every nonsymmetric interior skewness. The same spectral inequality gives a general local statement above one. It does not require monotonicity of the full kernel in the skewness.

Proposition D.7. *Fix $0 < \eta \leq 1/4$, let $s = \sin(\pi\eta)$, and put*

$$\varepsilon_\eta = \min \left\{ \frac{1}{2}, \frac{\eta}{1-\eta}, \frac{s^4}{96\pi} \right\} > 0.$$

If

$$1 < \alpha \leq 1 + \varepsilon_\eta, \quad \rho \in [\eta, 1/2 - \eta] \cup [1/2 + \eta, 1 - \eta],$$

then the parameters are admissible and $X_{\alpha,\rho}$ is classically ID. In particular, for every fixed $\rho \in (0, 1) \setminus \{1/2\}$, ID holds for all $\alpha > 1$ sufficiently close to one. All these laws are outside extended GGC.

Proof. By reflection take $1/2 + \eta \leq \rho \leq 1 - \eta$. The bound $\alpha \leq 1/(1 - \eta)$ ensures admissibility, and gives

$$\eta \leq \alpha(1 - \rho) \leq \frac{1/2 - \eta}{1 - \eta} < \frac{1}{2}.$$

Hence $\sin \xi \geq s$ and $\sin \omega \geq s$. For $1 \leq \alpha \leq 3/2$, $1 + u^\alpha \leq e^u$ for all $u \geq 0$: use $u^\alpha \leq u$ up to one, and $u^\alpha \leq u^2 \leq e^u - 1$ thereafter. Thus

$$B_{\alpha,\xi}(v) \geq \frac{\Gamma(\alpha + 1) \sin \xi}{\pi(v + 2)^\alpha} \geq \frac{s}{4\pi(1 + v^\alpha)},$$

where $(v + 2)^\alpha \leq 2^{\alpha-1}(v^\alpha + 2^\alpha) \leq 4(1 + v^\alpha)$ and $\Gamma(\alpha + 1) \geq 1$. Furthermore

$$\sup_{v \geq 0} e^{-sv}(1 + v^\alpha)^2 \leq \max \left\{ 4, 4 \sup_{v \geq 1} v^3 e^{-sv} \right\} \leq \frac{108}{e^3 s^3} < \frac{6}{s^3}.$$

Writing $S_j = \sum_n (1 + (r_n x)^\alpha)^{-j}$, inequality (D.1) gives, for every $x > 0$,

$$K_{\alpha,\rho}(x) \geq \frac{s}{4\pi} S_1 - \frac{12}{s^3} S_2 \geq \left(\frac{s}{4\pi} - \frac{12\varepsilon_\eta}{s^3} \right) S_1 \geq \frac{s}{8\pi} S_1 > 0.$$

The opposite-side Lévy kernel is already nonnegative. The absolutely summable compensated representation above therefore proves ID. Theorem A.2 excludes extended GGC. $\square$

APPENDIX E. THE GLOBAL ID REGION

The proof has three tasks: exclude every skewness at large indices, locate the extremal transition, and prove positivity at all smaller extremal indices. Only after these tasks are complete does sector interpolation identify the global interval and its inner boundary. We write $\alpha_* = \alpha_{\text{ID}}$ throughout the appendices.

E.1. A uniform upper exclusion for the ID index.

Lemma E.1 (An elementary characteristic-function obstruction). *Every classically ID characteristic function satisfies*

$$|\varphi(2t)|^2 \geq |\varphi(t)|^8, \quad t \in \mathbb{R}.$$

For every admissible free stable law with $9/5 \leq \alpha \leq 2$, the reverse strict inequality holds at $t = 89/100$, with

$$|\varphi(178/100)|^2 - |\varphi(89/100)|^8 < -10^{-5}.$$

Consequently none of these laws is classically ID.

Proof. Write $A(t) = -\log |\varphi(t)|$. The Lévy–Khintchine formula and $1 - \cos(2u) \leq 4(1 - \cos u)$ give $A(2t) \leq 4A(t)$, including the Gaussian contribution. Exponentiation gives the necessary condition.

For the strict reverse inequality, use reflection and set

$$\alpha = (9 + p)/5, \quad \rho = \frac{1}{2} + q(1/\alpha - \frac{1}{2}), \quad 0 \leq p, q \leq 1.$$

The Wright argument in (3.2) is $-t^\alpha \exp[-i\pi q(1 - \alpha/2)]$. Retain terms $0 \leq n < 32$ of its series. At both frequencies $|z| < 4$, and gamma monotonicity gives the uniform remainder bound

$$E = \frac{4^{32}}{32! \Gamma(2 + \frac{4}{5} 32)} \frac{1}{1 - 4/33}.$$

Indeed, every consecutive absolute term ratio from index 32 onward is at most $4/(n + 1)$. The exact recursive rectangle cover in `cf_exclusion_certificate.json` has 4,289 leaves. On each whole rectangle, outward ball evaluation of the two finite series, with E added to both components, proves the displayed gap. `certify_cf_exclusion.py` checks every leaf and reconstructs the complete cover; the fresh 50-digit verification passed. $\square$

Theorem E.2 (All skewnesses above 153/100). *For every admissible (α, ρ) with $153/100 \leq \alpha \leq 2$, the law $X_{\alpha, \rho}$ is not classically ID.*

Proof. It remains to treat $153/100 \leq \alpha \leq 9/5$. By reflection write $\rho = 1/2 + q(1/\alpha - 1/2)$, $0 \leq q \leq 1$. For $q > 0$, put

$$\omega = \pi q(1/\alpha - 1/2), \quad h = \sin \omega, \quad c = \cos \omega, \quad \xi = \pi(1 - \alpha/2)(1 + q).$$

Here $\cos \xi > 0$. The signed Lévy kernel used above is

$$K(x) = \sum_n B_{\alpha, \xi}(r_n x) - 2 \sum_n e^{-hr_n x} \cos(cr_n x).$$

Its positive part obeys

$$(E.1) \quad \sum_n B_{\alpha, \xi}(r_n x) \leq \frac{\sin \xi}{\pi x^\alpha},$$

because its denominator is at least one and $\sum_n r_n^{-\alpha} = 1/\Gamma(1 + \alpha)$.

We first enclose three distinct moving zeros on each of the 27 closed index strips $[j/100, (j + 1)/100]$, $153 \leq j < 180$. The certificate specifies a rational quadratic polynomial $r_j^*(\alpha)$ and a rational error e_j for each root. The two functions $W_\alpha(-(r_j^*(\alpha) \pm e_j)^\alpha)$ have opposite strict signs throughout the strip. To prove these signs without interval cancellation, truncate the Wright series at 100 terms and use a real Taylor polynomial of degree 17 in α . Its coefficients are evaluated at the strip midpoint; its eighteenth Taylor coefficient is evaluated on the entire strip and supplies the Lagrange remainder. The finite polynomial is bounded in Bernstein form. A separate geometric bound controls the infinite Wright tail. All these operations use outward balls. Thus numerical root proposals play no role in verification.

Let $S_j = \sum_n r_n^{-j\alpha}$, computed by the logarithmic Wright coefficients. Each strip also specifies a rational $R > r_3$ and verifies

$$\left(S_4 - \sum_{j=1}^3 r_j^{-4\alpha} \right) R^{4\alpha} < 1.$$

The left side is bounded above using the upper root polynomials, with the same Taylor method. Positivity of every Rayleigh summand shows that every unlisted zero exceeds R ; hence these are indeed the first three zeros, with no missed roots. Write $T_1 = S_1 - \sum_{j=1}^3 r_j^{-\alpha}$. Whenever $hxR > \alpha$,

$$(E.2) \quad K(x) \leq \frac{\sin \xi}{\pi x^\alpha} - 2 \sum_{j=1}^3 e^{-hr_j x} \cos(cr_j x) + 2R^\alpha e^{-hxR} T_1.$$

On $1/20 \leq q \leq 1$, the exact 962-leaf rectangle cover chooses $x = \tau/(cr_1)$, where each leaf specifies $\tau = 2\pi m + d$, an integer $m \geq 1$ and a rational d . The first residue is then evaluated exactly as $2e^{-\tau \tan \omega} \cos \tau$; the other residues use the enclosed root ratios. On every whole leaf, (E.2) is strictly less than -10^{-4} .

The small-skewness region is covered analytically. The root certificates also verify uniformly that

$$r_1 < 3, \quad r_2/r_1 > 5/3, \quad r_1^\alpha S_1 - 1 < 3.$$

If $0 < q \leq 1/20$, then $\tan \omega < 1/40$. Choose $m = \lceil 4/(2\pi \tan \omega) \rceil$, $v = 2\pi m \tan \omega$, and $x = 2\pi m/(cr_1)$. Thus $4 \leq v < 4 + \pi/20$ and $x > 160/3$. Bound all residues after the first using $r_n \geq (5/3)r_1$. Equations (E.1) and the first Rayleigh sum give

$$K(x) \leq \frac{1}{\pi(160/3)^{153/100}} - 2e^{-4-\pi/20} + 6(5/3)^{9/5} e^{-20/3} < -\frac{1}{100}.$$

At $q = 0$, the symmetric characteristic function has real zeros, so it is not ID. This completes the parameter cover.

The files `variable_alpha_spectrum.py` and `certify_large_alpha_non_id.py` implement the checks just described. A fresh 65-digit verification recomputed every moving root sign, all 27 Rayleigh bounds, all 962 kernel leaves, and the 4,289 characteristic-function leaves of Lemma E.1. The result and source hashes are recorded in `large_alpha_non_id_verification.json`. $\square$

This first confines every possible larger-index ID law to $\alpha < 153/100$. The remaining short interval can now be settled sharply.

E.1.1. *The extremal transition and the maximal ID index.*

Theorem E.3 (The extremal transition above $3/2$). *There is a unique critical index α_* in the interval*

$$(E.3) \quad 1.52407396102878875153 < \alpha_* < 1.52407396102878875155.$$

For $3/2 \leq \alpha \leq 153/100$,

$$X_{\alpha, 1/\alpha} \text{ is classically ID} \iff \alpha \leq \alpha_*.$$

The same holds for its reflection. An exact definition of the critical index is the unique solution of

$$j_\alpha(y) = \partial_y j_\alpha(y) = 0$$

in $38/25 \leq \alpha \leq a_U$, $4 \leq y \leq 203/50$, where a_U is the upper bound in (E.3) and j_α is defined in Lemma D.4. Its contact satisfies

$$4.028954474287407697 < y_* < 4.028954474287407701.$$

At the critical index the kernel is nonnegative on the entire positive half-line, with equality at the contact.

Proof. Write a_L, a_U for the two rational bounds in (E.3), and $Y = [4, 203/50]$. All finite-index computations below use the exact rational parameter polynomials of Lemma D.4 through degree 128, with the all-order tail (D.6). The derivative tails are also explicit. For example, if $Q_n(a) = a^n k_n(a)$, then $\deg Q_n \leq 2n$ and

$$\|Q_n\|_{[L,U]} \leq 16(n+1)^2(U/R)^n, \quad L = 3/2, \quad U = 153/100, \quad R = 51/10.$$

The classical Bernstein and Markov inequalities [9, Introduction, (1.1)–(1.2)] give

$$|Q'_n(a)| \leq \min \left\{ \frac{2n}{\sqrt{(a-L)(U-a)}}, \frac{8n^2}{U-L} \right\} \|Q_n\|_{[L,U]},$$

with the Markov alternative at endpoints. Differentiating $k_n = Q_n/a^n$ and summing these positive geometric tails therefore rigorously controls $\partial_a j_a$ and the mixed derivative. This also justifies termwise parameter differentiation on all compact sets used here.

The exact continuous cover in `extremal_transition_certificate.json` verifies:

$$\begin{aligned} j_a(y) &> 0 & \left(\frac{3}{2} \leq a \leq \frac{38}{25}, \quad 0 \leq y \leq \frac{21}{5} \right), \\ j_a(y) &> 0 & \left(\frac{38}{25} \leq a \leq a_U, \quad y \in [0, 4] \cup \left[\frac{203}{50}, \frac{21}{5} \right] \right), \\ \partial_y^2 j_a(y) &> \frac{1}{10}, \quad \partial_a j_a(y) < -\frac{1}{2} & \left(\frac{38}{25} \leq a \leq a_U, \quad y \in Y \right). \end{aligned}$$

It also checks $\partial_y j_a(4) < 0$ and $\partial_y j_a(203/50) > 0$ on the last index interval. Thus each such kernel has exactly one minimizer in Y , and the minimum is strictly decreasing with a .

Here is the infinite-radius control, independent of that compact calculation. On three index strips covering $[3/2, a_U]$, the moving-zero method of Theorem E.2 encloses three roots and a lower bound R_0 for all unlisted roots. With $\xi = \pi(2-a)$, $h = \sin[\pi(1/a - 1/2)]$, and $c = \cos[\pi(1/a - 1/2)]$, the positive part of the kernel is bounded below by

$$P_a(x) = \frac{\sin \xi}{\pi} \left\{ x^{-a} - 2a \cos \xi \Gamma(2a) S_2 x^{-2a} - a \Gamma(3a) S_3 x^{-3a} \right\}.$$

This follows from $(1+s)^{-1} \geq 1-s$ for $s \geq 0$. The lower bound

$$P_a(x) - 2 \sum_{n=1}^3 e^{-hr_n x} \cos(cr_n x) - 2R_0^a e^{-hx} T_1$$

is positive on $x = c_a y$, $21/5 \leq y \leq 5$. Replacing all three cosines by one gives a bound positive at $x = 3$. After multiplication by x^a , its negative terms decrease for every $x \geq 3$, as verified at that endpoint. It therefore remains positive throughout the infinite tail. The overlap $5c_a > 3$ is also checked. This accounts for every positive x , including the region beyond the convergence disk.

To locate the transition, set $y_t = 4.02895447428740769884$. At the two exact rational indices a_L, a_U , independently recompute 384 coefficients by formal series reversion, using 700-digit outward arithmetic and the same infinite-series majorant. The checker obtains

$$j_{a_L}(y_t) - \frac{(\partial_y j_{a_L}(y_t))^2}{2/10} > 9.5691 \cdot 10^{-21}, \quad j_{a_U}(y_t) < -9.0056 \cdot 10^{-21}.$$

Strong convexity makes the first expression a lower bound for the entire minimum on Y . Strict decrease of that minimum now proves existence and uniqueness of α_* , with equality included in the ID region. A separate bound $|\partial_a \partial_y j_a| < 5$ on the tiny contact box, together with 384-term endpoint evaluations, gives the stated bracket for y_* .

Finally the cover verifies $\partial_a j_a(y) < -1/4$ for $a_L \leq a \leq 153/100$ and every y in that tiny contact bracket. Hence $j_a(y_*) < 0$ for every $a > \alpha_*$ up to $153/100$. This proves necessity on the entire claimed interval, not just near the contact. Sufficiency follows from the global nonnegative-kernel checks above. The 4,988 exact cover leaves and all point checks were freshly recomputed by `certify_extremal_transition.py`; the results are in `extremal_transition_verification.json`. $\square$

Theorem E.4 (The maximal classical ID index). *The largest admissible stability index for which any free stable law is classically ID is exactly α_* from Theorem E.3. More precisely,*

$$\alpha > \alpha_* \implies X_{\alpha,\rho} \text{ is not ID for every admissible } \rho.$$

At $\alpha = \alpha_*$, the only ID skewnesses are

$$\rho = 1/\alpha_*, \quad \rho = 1 - 1/\alpha_*.$$

These laws are not extended GGC. The theorem does not claim the full skewness classification at indices below α_* .

Proof. Theorem E.2 excludes $\alpha \geq 153/100$. It remains to treat $a_L \leq \alpha \leq 153/100$, where a_L is the lower rational bound in (E.3). Use q as in that theorem. The same moving-root and Rayleigh method, on this one additional index strip, verifies strict negative kernel witnesses for $1/20 \leq q \leq 49/50$. For $0 < q \leq 1/20$, the analytic resonant-frequency estimate continues to work with the conservative lower index $38/25$:

$$K(x) \leq \frac{1}{\pi(160/3)^{38/25}} - 2e^{-4-\pi/20} + 6(5/3)^{9/5}e^{-20/3} < -\frac{1}{100}.$$

As before, $q = 0$ is excluded by a real characteristic-function zero.

The essential extra step concerns $49/50 \leq q \leq 1$. Fix $x = c_\alpha y_*$. This x is held fixed when varying ρ at fixed α . Since $\cos \xi > 0$, differentiating each positive integral gives

$$\partial_\xi B_{\alpha,\xi}(v) = \frac{\alpha}{\pi} \int_0^\infty e^{-vu} u^{\alpha-1} \left\{ \frac{\cos \xi}{D} + \frac{2 \sin^2 \xi u^\alpha}{D^2} \right\} du > 0, \quad D = 1 + 2 \cos \xi u^\alpha + u^{2\alpha}.$$

Its contribution to $\partial_\rho K$ may therefore be discarded in a lower bound. Differentiating the residues gives

$$\partial_\rho K(x) \geq 2\pi x \sum_{n=1}^3 r_n e^{-hr_n x} \cos(cr_n x + \omega) - 2\pi x R_0^{\alpha+1} e^{-hxR_0} T_1.$$

The last term bounds every unlisted residue derivative, since $hxR_0 > \alpha + 1$. A single parameter box, allowing the full certified interval for y_* , encloses this lower-bound expression between 1.2295 and 1.8216. In particular, $\partial_\rho K(x) > 1/2$ throughout the region.

For $\alpha > \alpha_*$, Theorem E.3 proved $K_{\alpha,1/\alpha}(c_\alpha y_*) = j_\alpha(y_*) < 0$. Strict increase in ρ forces the same negative sign at every smaller skewness in the present region. At $\alpha = \alpha_*$, the endpoint value is zero, so every strictly smaller skewness in the region is again non-ID. Reflection handles the other half, whereas both extremal laws are ID by the preceding theorem. The non-GGC assertion follows from Theorem A.2.

The new cover contains 36 leaves, including the derivative box. `certify_global_id_cutoff.py` freshly verified them and reran both complete prerequisite covers: the 4,988 extremal-transition leaves and all the larger-index exclusions. The record is `global_id_cutoff_verification.json`. Thus the passage from an extremal transition to a global index cutoff is independently certified, rather than assumed from skewness interval convexity. $\square$

E.2. The complete upper-index ID region.

Theorem E.5 (All skewnesses between three halves and the cutoff). *For $3/2 \leq a \leq \alpha_*$, put*

$$u = a - \frac{3}{2}, \quad Q(a) = \frac{114238}{125000} + \frac{341}{100}u + 7u^2, \quad \rho(a, q) = \frac{1}{2} + (a^{-1} - \frac{1}{2})q.$$

There is exactly one pair $(q_(a), x_*(a))$ in*

$$|q - Q(a)| < 1/500, \quad 47/20 < x < 57/20$$

such that

$$K_{a,\rho(a,q)}(x) = 0, \quad \partial_x K_{a,\rho(a,q)}(x) = 0.$$

Both components are continuous in a , and $0 < q_*(a) \leq 1$. Writing $r(a) = \rho(a, q_*(a))$, the full classification is

$$X_{a,\rho} \text{ is ID} \iff \rho \in [1 - a^{-1}, 1 - r(a)] \cup [r(a), a^{-1}].$$

The endpoints are included. At $a = 3/2$, $r(a) = \rho_0$ from Theorem D.6; at $a = \alpha_*$, $q_*(a) = 1$. The polynomial Q is a certified locator, not an exact formula or a claimed Taylor expansion of the boundary. In particular

$$\left| r(a) - \left[\frac{1}{2} + (a^{-1} - \frac{1}{2})Q(a) \right] \right| < 1/3000.$$

Proof. We use the spectral kernel formula also as an auxiliary analytic expression on the entire band $q = Q(a) + v/500$, $-1 \leq v \leq 1$, including a small part with $q > 1$. Probability statements are restricted to $q \leq 1$. Throughout this auxiliary band, $0 < \xi = \pi(1 - a/2)(1 + q) < \pi/2$. Indeed the maximum occurs at $v = 1, u = 0$, where $\xi/\pi = 0.478976$: the derivative in u at $v = 1$ is $-0.105452 + 0.09u - 10.5u^2 < 0$ for $0 \leq u \leq 1/40$.

The following strict inequalities hold uniformly on this band for $3/2 \leq a \leq a_U$, with a_U from (E.3):

$$\begin{array}{l|l} x \in [1/2, 47/20] \cup [57/20, 4] & K > 0 \\ x \in [47/20, 57/20] & K_{xx} > 1/25, \quad K_\rho > 1/2 \\ x = 47/20 & K_x < 0 \\ x = 57/20 & K_x > 0 \\ x \geq 4 & K > 0. \end{array}$$

At $q_- = Q(a) - 1/500$, the kernel is strictly negative at $x_f(a) = 2.5385 + 4.16u - 3.35u^2$. At $q_+ = Q(a) + 1/500$,

$$K(a, q_+, x_f) - \frac{K_x(a, q_+, x_f)^2}{2/25} > 0.$$

Strong convexity therefore makes the entire core minimum positive at q_+ . These are continuous parameter enclosures, proved by `certify_upper_id_region.py`: 552 exact rational boxes cover the stated domains. We give the error controls below.

First we control the omitted origin interval. The convergent germ of Lemma D.4 and an exact coefficient bound give

$$|j_a(z) - 3/2| < 9/20 \quad (|z| \leq 4/5), \quad k_a(x) < 2 \quad (x > 0), \quad 3/2 \leq a \leq 153/100.$$

For the first inequality, the rational coefficient sum through degree 128 plus its all-order tail is less than 0.408144. For the second, an eight-box cover proves $j_a(y) < 2$ on $[0, 21/5]$; for $x \geq 5/2$, the spectral upper bound

$$k_a(x) \leq \frac{1}{\pi x^a} + \frac{2}{\Gamma(1+a)} \left(\frac{a}{ehx} \right)^a < 1$$

decreases with x . The two ranges overlap because $(21/5)2^{-2/3} > 5/2$. Also $(4/5)2^{-2/3} > 1/2$, so $\operatorname{Re} k_a(z) > 21/20$ for $|z| \leq 1/2$. For $a \leq \alpha_*$, endpoint ID is already supplied by Theorem E.3. Apply (D.7) with $b = b_a$, $k = k_a$ and $\delta = \pi(a^{-1} - \rho)$. Its Poisson integral is at most 2, and its pole term is nonnegative. Hence $K_{a,\rho}(x) > 1/10$ for $0 < x \leq 1/2$, at every admissible $\rho > 1/2$. The case $\delta = 0$ follows directly from k_a . These inequalities were freshly verified by `certify_id_origin_strip.py`.

For the finite spectral computations, ten distinct moving Wright zeros are enclosed on each of the three parameter strips $[1.5, 1.51]$, $[1.51, 1.52]$, $[1.52, a_U]$. The real sign checks use a degree-63 Taylor polynomial in a for the first 140 Wright terms. A complex square containing the circle of radius $1/10$ supplies a Cauchy remainder $M(h/(1/10))^{64}/(1 - h/(1/10))$, where h is the parameter half-width. The infinite Wright tail is bounded separately. Thus the sign checks do not infer signs from a root scan or sampled parameter values. With $T_m = S_m - \sum_{\text{listed}} r^{-am}$, the fourth Rayleigh sum proves $15^{4a}T_4 < 1$, so every unlisted root exceeds 15. This lower bound need not exceed all ten listed roots. The file `certify_ten_zero_id_spectra.py` freshly checks every root sign and moment remainder.

The listed positive integrals use validated quadrature after $u = t^2/x$, with explicit errors for $0 < t < 1/4096$ and $t > 12$. For the unlisted positive terms, $(1 + 2z \cos \xi + z^2)^{-1}$ lies between $1 - 2z \cos \xi - z^2$ and 1. Consequently their differentiated Laplace integrals have an explicit one-sided remainder, bounded by T_2 and $T_3 \leq 15^{-a}T_2$. Unlisted oscillatory derivatives of order j have modulus at most

$$2 \cdot 15^{am+j} e^{-15hx} T_m \quad \text{whenever } 15hx > am + j,$$

using the best available $m \in \{1, 2, 4\}$. For K_{xx} and K_ρ , the positive-integral parts are nonnegative; only the explicitly differentiated residues and their tails are needed for the displayed lower bounds. Finally, at $x \geq 4$ use

$$K \geq \frac{\sin \xi}{\pi} \left[x^{-a} - 2a \cos \xi \Gamma(2a) S_2 x^{-2a} - a \Gamma(3a) S_3 x^{-3a} \right] - 2 \left(\frac{2a}{ehx} \right)^{2a} S_2.$$

After multiplication by x^a , this lower bound increases; its positivity at 4 is included in the cover.

For each fixed a , strict convexity and the derivative signs give a unique core minimizer for every q in the band. Its value is strictly increasing in q , since $K_\rho > 1/2$, and changes sign between q_- and q_+ . Thus precisely one contact exists. If $q_+ > 1$, endpoint ID forces $q_* \leq 1$; here $q_- < 1 < q_+$. At q_* , all exterior kernel values are positive and the core minimum is zero, so the law is ID. Proposition D.2, together with endpoint ID, proves ID at every $q \geq q_*$. Inside the band, $q < q_*$ gives a negative kernel value. An ID parameter below q_- would, by the same interval proposition, force ID at q_- , a contradiction. Reflection proves the full section.

For completeness, the contact is continuous in a . Ordered negative Wright zeros, with multiplicities, vary continuously by local uniform convergence and Hurwitz's theorem. Their partial Rayleigh sums increase to the continuous function $S_1(a) = 1/\Gamma(1+a)$; Dini's theorem gives uniform tail convergence on a compact parameter interval. On the present compact (q, x) ranges, each integral or residue and the required x - and ρ -derivatives are bounded by a constant times r^{-a} . Hence the kernel and these derivatives are jointly continuous. Compactness and uniqueness of the contact imply continuity of q_* and x_* . No monotonicity or analyticity in a is asserted here. $\square$

E.3. Endpoint ID down to one and the global shape of the ID set.

Theorem E.6 (The complete extremal classification). *For every $1 < a \leq 2$,*

$$X_{a,1/a} \text{ and } X_{a,1-1/a} \text{ are ID} \iff a \leq \alpha_*.$$

Their signed kernels are strictly positive at every $x > 0$ when $1 < a < \alpha_$.*

Proof. It remains to extend Theorem E.3 downward. We give explicit uniform bounds, including the singular-looking limit $a \downarrow 1$. Let $\varepsilon = a - 1$ and retain $j_a(y) = k_a(c_a y)$, $c_a = \varepsilon^{1/a}$. The logarithmic inverse proof of Lemma D.4 gives the following convergent coefficient bounds:

a range	R	$ f_n(a) R^n$	$ k_n(a) R^n/(n+1)^2$
$[1, 7/5]$	$5/2$	20	128
$[13/10, 7/5]$	4	11	32
$[7/5, 153/100]$	$51/10$	10	32.

At $a = 1$ the coefficients mean their polynomial limits. For the first row, $|e^v| < 36$ follows from $\log 36 > 1 + 5/2 + 1/36$ and convexity of 36^ε ; then $|F_a| < 36\sqrt{(7/5)/5} < 20$. For the second row use $|e^v| < 25$:

$$25^{3/10} - 1 - \frac{3}{10} - \frac{3}{10}(4 + 1/25) > 0, \quad 25^{3/10} \log 25 - 1 - 4 - 1/25 > 0.$$

The resulting bound is $25\sqrt{7/40} < 11$. For the third row, replace $R = 4, \varepsilon = 3/10$ by $R = 51/10, \varepsilon = 2/5$; the bound is $25\sqrt{3/20} < 10$. The exact kernel numerator polynomials are shifted to $6/5, 27/20, 293/200$, respectively. Absolute coefficient sums on radii

$1/5, 1/20, 13/200$, divided by the corresponding lower endpoint to the power n , verify the induction bases. The first row uses $0 \leq n < 128$; the other two use $0 \leq n < 100$. The induction step has the general bound

$$D \frac{n+2}{n+1} + \frac{5CD}{n} < C,$$

with $(D, C) = (20, 128), (11, 32), (10, 32)$, respectively. For the first row it holds at $n = 128$, and for the others at $n = 100$; it decreases thereafter. These statements use exact rational bases, not numerical coefficient sampling.

The convergent germs prove strict kernel positivity on

$$[1, 13/10] \times [0, 2], \quad [13/10, 7/5] \times [0, 3], \quad [7/5, 3/2] \times [0, 87/20].$$

It remains to prove the entire tails. Since $\xi = \pi(2 - a)$ has $\cos \xi \leq 0$ for $a \leq 3/2$, discarding the favorable quadratic correction gives

$$(E.4) \quad k_a(x) \geq \frac{\sin \xi}{\pi} \left[x^{-a} - a\Gamma(3a)S_3x^{-3a} \right] - 2 \left(\frac{2a}{ehx} \right)^{2a} S_2, \quad h = \sin \pi(a^{-1} - 1/2).$$

The lower bound multiplied by x^a increases in x . For $a \in [7/5, 3/2]$, its value at $x = c_a(87/20)$ is positive.

To include $a = 1$ in the other two interval computations, write $R_j(\varepsilon) = S_j(1 + \varepsilon)/\varepsilon^j$ for $j = 2, 3$. Explicitly, with $G_j(\varepsilon) = 1/\Gamma(2 + j\varepsilon)$,

$$S_2 = G_1^2 - G_2, \quad S_3 = G_1^3 - \frac{3}{2}G_1G_2 + \frac{1}{2}G_3.$$

Direct differentiation at zero proves vanishing to orders two and three, respectively. The analytic quotients satisfy $R_2(0) = \zeta(2) - 1$, $R_3(0) = \zeta(3) - 1$. After $x = c_a y$, (E.4) becomes the regular bound

$$k_a(c_a y) \geq \frac{\sin(\pi\varepsilon)}{\pi\varepsilon} \left[y^{-a} - a\varepsilon\Gamma(3a)R_3(\varepsilon)y^{-3a} \right] - 2 \left(\frac{2a}{ehy} \right)^{2a} R_2(\varepsilon).$$

This is positive at $y = 2$ for $1 \leq a \leq 13/10$, and at $y = 3$ for $13/10 \leq a \leq 7/5$. For error control, a degree-95 Taylor polynomial for each S_j , followed by exact removal of its vanishing initial coefficients, encloses R_j . A complex square containing the circle $|\varepsilon| = 1/2$ bounds $|S_j|$ by M_j , and the quotient remainder on $|\varepsilon| \leq r < 1/2$ is

$$\frac{M_j}{(1/2)^j} \frac{(2r)^{96-j}}{1 - 2r}.$$

Thus the computation does not divide an interval containing zero by ε^j .

The checker `certify_near_one_extremal.py` freshly verified all germ constants and a 165-box cover for $1 < a \leq 7/5$. The checker `certify_lower_extremal_id.py` freshly verified a 623-box cover for $7/5 \leq a \leq 3/2$, with the larger germ certified by `certify_lower_extremal_germ.py`. These bounds, Theorem E.3, and the global exclusion of Theorem E.4 prove the assertion. $\square$

Theorem E.7 (A unique contact down to index seven fifths). *For $7/5 \leq a \leq 3/2$, put $v = a - 29/20$ and*

$$\begin{aligned} Q_-(a) &= 0.758029 + 2.851767v + 4.788686v^2 + 9.650678v^3 + 16.085231v^4, \\ X_-(a) &= 2.321849 + 4.501192v - 3.348444v^2 - 0.569083v^3 + 4.346122v^4. \end{aligned}$$

All terminating decimals here are exact rationals. There is a unique contact $K = K_x = 0$ in

$$|q - Q_-(a)| < 1/500, \quad |x - X_-(a)| < 1/5.$$

With this $q_(a)$ and $r(a) = 1/2 + (1/a - 1/2)q_*(a)$, the exact ID section is $[1 - 1/a, 1 - r(a)] \cup [r(a), 1/a]$. The locating polynomials do not claim exact boundary values.*

Proof. The argument of Theorem E.5 applies with new, independently verified global inequalities. On the whole polynomial band, $0 < \xi < \pi/2$ and $0 < q < 1$; hence the positive-integral estimates used there remain valid. Ten moving roots are enclosed on each of the ten strips of width $1/100$, with every unlisted root exceeding 15. The finite Wright expression now uses a degree-95 parameter Taylor polynomial and its complex Cauchy remainder, to account for the stronger cancellation at the larger roots.

The certificate `certify_lower_id_region.py` has 2,681 exact boxes. It proves $K > 0$ on $[11/20, X_-(a) - 1/5] \cup [X_-(a) + 1/5, 4]$, $K_{xx} > 1/25$ and $K_\rho > 1/2$ throughout the intervening core, and opposite derivative signs at the core endpoints. At $q = Q_-(a) - 1/500$, $K(a, q, X_-(a)) < 0$; at the upper band boundary, $K(a, q, X_-(a)) - K_x(a, q, X_-(a))^2/(2/25) > 0$. The analytic infinite-tail bound is positive from $x = 4$ onward.

For the origin, a separate 64-box certificate proves $k_a(x) \leq 3/2$ everywhere. In the convergent part it bounds $(3/2 - j_a(y))/y > 0$, extending at zero by its analytic limit. For $x \geq 2$, use

$$k_a(x) \leq \frac{1}{\pi \sin \xi x^a} + \frac{2}{\Gamma(1+a)} \left(\frac{a}{ehx} \right)^a < 3/2.$$

The ranges overlap since $(87/20)(2/5)^{5/7} > 2$. An exact coefficient sum plus its tail gives $|j_a(z) - 3/2| < 0.684078 < 7/10$ for $|z| \leq 11/10$. Since $(11/10)(2/5)^{5/7} > 11/20$, Cauchy continuation (D.7) yields $K > 1/10$ on $0 < x \leq 11/20$ for every admissible upper-half skewness. Endpoint ID is provided by Theorem E.6. The convex core, global positivity, and interval interpolation now give the stated unique contact and full section. The complete verifier reran the root and origin prerequisites before all 2,681 boxes. $\square$

Theorem E.8 (The global ID interval and its first contact). *There is a continuous function $r : (1, \alpha_*) \rightarrow (1/2, 1)$ such that the complete ID set at every $1 < a \leq \alpha_*$ is*

$$[1 - 1/a, 1 - r(a)] \cup [r(a), 1/a].$$

For $1 < a < \alpha_*$, $1/2 < r(a) < 1/a$, while

$$\lim_{a \downarrow 1} r(a) = 1/2, \quad r(\alpha_*) = 1/\alpha_*.$$

For $1 < a < \alpha_*$, an exact characterization independent of the definition of ID is

$$(E.5) \quad \pi(1 - r(a)) = \min \left\{ \theta \in (\pi(1 - 1/a), \pi/2) : \operatorname{Re} \mathcal{F}_a(xe^{-i\theta}) = 0 \text{ for some } x > 0 \right\},$$

where $\mathcal{F}_a$ is the explicit holomorphic transform in (D.3). The first contact is attained at a finite positive x . At each contact,

$$K_x = 0, \quad K_{xx} \geq 0, \quad K_\rho > 0.$$

There are finitely many contact radii for each fixed index. Uniqueness and an explicit locating band are additionally proved on $[7/5, \alpha_*]$; they are not asserted below $7/5$.

Proof. Theorems E.6 and E.4, together with Proposition D.2, give a nonempty interval reaching $1/a$ at each $1 < a \leq \alpha_*$, and no ID parameters at larger indices. We supply the compactness needed to justify the boundary formula and its continuity.

On a compact index interval contained in $(1, \alpha_*)$, the endpoint germs satisfy $k_a(0) = 3/2$ uniformly, and $k_a \geq 0$, $\int_0^\infty k_a = b_a$ with bounded b_a . Choose $d > 0$ so that $|k_a(z) - 3/2| < \eta$ on $|z| \leq d$. In (D.7), for $x \leq d/2$ the part of the Poisson integral over $[0, d]$ is at most $3/2 + \eta$; the part over $[d, \infty)$ is at most $4x \sup b_a/(\pi d^2)$. The pole term is nonnegative. Thus all upper-half kernels are uniformly positive for small enough $x > 0$, including near the extremal skewness.

For ρ bounded away from $1/2$, the elementary positive-integral estimate and the second Rayleigh moment give uniformly

$$K_{a,\rho}(x) \geq \frac{\sin \xi}{\pi x^a} - C(x^{-2a} + x^{-3a}) > 0$$

for all sufficiently large x . Here $\sin \xi$ has a strict positive lower bound on the compact admissible parameter set. Joint continuity of the kernel and its radial and skewness derivatives follows from the uniform Rayleigh-tail argument in Theorem E.5.

At $a < \alpha_*$, strict endpoint positivity, these uniform end estimates, and compactness of the intervening radii show that an interval of skewnesses immediately below $1/a$ is ID. Hence $r(a) < 1/a$. The symmetric law is non-ID and ID is weakly closed, so $r(a) > 1/2$. At $\rho = r(a)$, the kernel must vanish somewhere: otherwise the same compactness would extend positivity to a smaller skewness. All zeros lie in a compact radial interval. Radial analyticity makes them isolated, and therefore finite. At each zero, radial nonnegativity gives $K_x = 0, K_{xx} \geq 0$. Between $r(a)$ and $1/a$, sector interpolation gives strict positivity. The boundary Hopf lemma for $\text{Re } \mathcal{F}_a$ at the first ray therefore yields $K_\rho > 0$ at each zero. This proves the first-contact description (E.5).

For continuity it is convenient to normalize $q_*(a) = (r(a) - 1/2)/(1/a - 1/2)$. Closedness of ID gives lower semicontinuity of q_* . At $a < \alpha_*$, strict positivity at every $q \in (q_*(a), 1)$, together with the uniform end estimates, gives upper semicontinuity. At the cutoff, lower semicontinuity and $q_* \leq 1 = q_*(\alpha_*)$ give continuity directly. Finally the previously proved near-one ID rectangles contain every fixed nonsymmetric interior skewness for all sufficiently small $a - 1 > 0$. Consequently $r(a) \rightarrow 1/2$ as $a \downarrow 1$; no monotonicity in a is claimed. $\square$

APPENDIX F. SCALED TRANSFORMS AND COMPETING ID CONTACTS

The scaled Hankel formula gives stable root enclosures as the index approaches one. We use it to distinguish local contacts from the global boundary and to exhibit a change between two controlling radial branches.

F.1. Stable Wright evaluation and a noncontrolling local contact.

Lemma F.1 (A scaled Hankel transform above one). *For $\varepsilon > 0$, define*

$$V_\varepsilon(z) = e^{z/\varepsilon} W_{1+\varepsilon}(-z/\varepsilon).$$

On a Hankel contour $\mathcal{H}$ with circular part of radius one,

$$(F.1) \quad V_\varepsilon(z) = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^t}{t^2} \exp\{z F_\varepsilon(\log t)\} dt, \quad F_\varepsilon(\ell) = \frac{1 - e^{-\varepsilon \ell}}{\varepsilon}.$$

The apparent singularity at $\varepsilon = 0$ is removable: $F_0(\ell) = \ell$, $V_0(z) = 1/\Gamma(2-z)$. The transform is holomorphic jointly in (ε, z) near every point $(0, z_0)$. Uniformly on compact z sets, with the same assertion for every fixed number of z -derivatives,

$$V_\varepsilon(z) = G(z) - \frac{\varepsilon z}{2} G'''(z) + \varepsilon^2 \left(\frac{z}{6} G''''(z) + \frac{z^2}{8} G'''''(z) \right) + O(\varepsilon^3), \quad G(z) = \frac{1}{\Gamma(2-z)}.$$

Proof. Hankel's reciprocal-Gamma formula and termwise integration give (F.1); on the two straight rays the factor e^{-r} makes the interchange absolutely convergent. The identity

$$F_\varepsilon(\ell) = \ell \int_0^1 e^{-s\varepsilon \ell} ds$$

removes the singularity and gives a stable entire expression. For real $\varepsilon \geq 0$ and $\text{Re } \ell \geq 0$,

$$|F_\varepsilon(\ell)| \leq |\ell|, \quad |F_\varepsilon(\ell) - \ell| \leq \frac{1}{2} \varepsilon |\ell|^2.$$

For complex $|\varepsilon| \leq d < 1$, the weaker bound $|F_\varepsilon(\log r \pm i\pi)| \leq C(\log r + \pi)r^d$ is enough: its exponential grows sublinearly compared with e^r on every compact z set. The integral is therefore locally normally convergent in both variables. Expansion of $F_\varepsilon(\ell) = \ell - \varepsilon \ell^2/2 + \varepsilon^2 \ell^3/6 + O(\varepsilon^3)$, followed by differentiation under the integral, proves the formula. $\square$

Corollary F.2 (Fixed scaled zero asymptotics). *For every fixed integer $m \geq 0$, a simple positive zero $z_m(\varepsilon)$ of V_ε tends to $n = m + 2$. Put $p_m = H_m - \gamma$, with $H_0 = 0$. Then*

$$z_m(\varepsilon) = n + np_m\varepsilon + \left\{ \frac{n}{2} \left(p_m^2 + H_m^{(2)} + \zeta(2) \right) + n^2 \left(\zeta(3) - H_m^{(3)} \right) \right\} \varepsilon^2 + O_m(\varepsilon^3).$$

The corresponding Wright zero is $-z_m(\varepsilon)/\varepsilon$. No uniformity as $m \rightarrow \infty$ is asserted.

Proof. Apply the holomorphic implicit function theorem at the simple zero n of G . If $s = z - n$, then

$$G(n + s) = (-1)^{m+1} m! s \exp \left\{ p_m s - \frac{H_m^{(2)} + \zeta(2)}{2} s^2 + \frac{H_m^{(3)} - \zeta(3)}{3} s^3 + O(s^4) \right\}.$$

Substitution into Lemma F.1 gives the coefficients. Reality and simplicity persist for sufficiently small positive ε ; positivity follows from the limit $n > 0$. $\square$

For clarity, the rigorous numerical implementation of (F.1) uses real $z \geq 0$ and $0 \leq \varepsilon \leq 1/2$. With $F = F_\varepsilon$, it evaluates

$$V_\varepsilon(z) = \frac{1}{\pi} \operatorname{Re} \int_0^\pi e^{e^{i\theta} - i\theta + zF(i\theta)} d\theta - \frac{1}{\pi} \operatorname{Im} \int_1^\infty r^{-2} e^{-r + zF(\log r + i\pi)} dr.$$

The sign and orientation of both contour pieces are included here. Along the straight ray, $\operatorname{Re} F(\log r + i\pi) \leq \log r + \varepsilon\pi^2/2$. If $U \geq \max(8, 2z)$, the omitted tails for $\partial_z^j V$, $j = 0, 1$, are bounded by

$$E_j \leq \frac{2}{\pi} e^{-U + z\varepsilon\pi^2/2} U^{z-2+j}.$$

For $j = 1$, use $|F| \leq \log r + \pi \leq r$. The function `scaled_wright_hankel.py` evaluates the finite integrals with directed complex balls and these explicit tails. The identity $F_\varepsilon(\ell) = \ell {}_1F_1(1; 2; -\varepsilon\ell)$ avoids division by a small parameter. Numerical root proposals are kept separate from the endpoint sign checks.

Lemma F.3 (Parameter derivatives on the scaled Hankel contour). *Let $I \Subset (1, 3/2]$ be a real interval and let $z(a) > 0$ be a real quadratic polynomial on I . Set $f(a) = V_{a-1}(z(a))$. All Taylor coefficients of f can be obtained by differentiating the finite circular and straight Hankel integrals above. The following bound explicitly controls the omitted straight-ray coefficient of order n .*

At a real parameter $a \in I$, write $z(a + u) = z_0 + z_1 u + z_2 u^2$, and choose uniform bounds $z_0 \leq Z$, $a - 1 \leq E$, and $|z_k| \leq b_k$. Define

$$C_j = \sum_{k=0}^{\min(2,j)} \frac{b_k}{(j-k+1)!}, \quad P_0 = 1, \quad P_n = \frac{1}{n} \sum_{j=1}^n j C_j P_{n-j}.$$

If $U \geq 8$ and $U > 2(Z + 2n)$, the coefficient error is at most

$$(F.2) \quad \frac{2P_n}{\pi} \exp(-U + ZE\pi^2/2) U^{Z+2n-2}.$$

In particular a midpoint Taylor polynomial through degree $N - 1$, together with an interval enclosure of $f^{(N)}/N!$ on I , gives a rigorous real Taylor enclosure on the entire interval.

Proof. For $\ell = \log r + i\pi$, $r \geq 1$, use

$$F_\varepsilon(\ell) = \ell \int_0^1 e^{-t\varepsilon\ell} dt, \quad |[u^j]F_{\varepsilon+u}(\ell)| \leq \frac{|\ell|^{j+1}}{(j+1)!}.$$

Here $\varepsilon \geq 0$, so the modulus of the integrand in the last derivative integral is at most one. For $r \geq 8$, $|\ell| \leq \log r + \pi \leq r$. Thus the coefficient of order $j \geq 1$ in $z(a + u)F_{a-1+u}(\ell)$ is bounded by $C_j r^{j+1} \leq C_j r^{2j}$. The elementary recurrence for exponentiating a power series gives the bound $P_n r^{2n}$. The constant exponential factor is at most $\exp(Z \log r + ZE\pi^2/2)$.

Beyond U , the logarithmic derivative of $e^{-r}r^{Z+2n}$ is at most $-1/2$. Integrating and bounding the remaining r^{-2} by U^{-2} proves (F.2). These same majorants justify differentiation under the integral. The real Taylor theorem and a Bernstein enclosure for the finite polynomial complete the proof. $\square$

The implementation `scaled_hankel_spectrum.py` uses $N = 16$, subdividing the real parameter interval when necessary. It proves opposite signs at the two graphs $z(a) \pm \eta$; polynomial proposals alone are not used as zero enclosures. Disjoint graph intervals then give distinct roots for every parameter in the strip. A fourth Rayleigh remainder, itself enclosed by a real Taylor polynomial, gives a lower bound for every unlisted root. The standalone checker is `certify_scaled_moving_spectra.py`.

Proposition F.4 (The endpoint germ and the exceptional law). *As $a \downarrow 1$, the scaled extremal laws $Z_a = (b_a - X_{a,1/a})/c_a$ converge weakly to $1 - \mathbf{T}$, where $\mathbf{T}$ is the exceptional free 1-stable law of [7]. Moreover, locally uniformly on $|y| < 2\pi$,*

$$(F.3) \quad j_a(y) \longrightarrow j_1(y) = e^{-y} + \frac{1}{y} - \frac{1}{e^y - 1} = \frac{1}{y} - \frac{e^{-2y}}{1 - e^{-y}}.$$

The singularity at zero is removable, and

$$k_n(1) = \frac{(-1)^n}{n!} - \frac{B_{n+1}}{(n+1)!}, \quad n \geq 0, \quad B_1 = -\frac{1}{2}.$$

The limiting kernel is already contained in Theorem 3 of [7]; the assertion here identifies it with the regular endpoint of the variable-index germ. Its nonzero poles are $2\pi i\mathbb{Z} \setminus \{0\}$, each with residue -1 .

Proof. The Voiculescu transform of Z_a , with $\varepsilon = a - 1$, is

$$\frac{a}{\varepsilon} - \frac{1}{\varepsilon} e^{\varepsilon(i\pi - \log z)} \longrightarrow 1 + \log z - i\pi, \quad \text{Im } z > 0.$$

The limit is the transform of $1 - \mathbf{T}$, proving weak convergence by the free convolution continuity theorem. The logarithmic inverse in Lemma D.4 tends to $h_1(v) = 1 - e^{-v} - v$. Its two branches give the limiting density germ. At the endpoint the horizontal boundaries of the strip $|\text{Im } v| < 2\pi$ map to curves of modulus at least 2π . For any $R < 2\pi$, the relevant inverse branches and density factors are therefore uniformly holomorphic for a close to one on $|y| \leq R$. To bound their size uniformly, if $t = \text{Re } v$ were larger than $R + 2$, the inverse equation and the reverse triangle inequality would give

$$e^{\varepsilon t} \leq 1 + \varepsilon(1 + R + e^{-t}) < 1 + \varepsilon t \leq e^{\varepsilon t},$$

a contradiction. The same conclusion at $\varepsilon = 0$ follows by the limiting equation. Cauchy estimates and the locally uniformly convergent Volterra resolvent hence pass the density limit to its signed kernel. Theorem 3 of [7] identifies that kernel with (F.3). Expansion of $y/(e^y - 1)$ gives the stated coefficients and poles. $\square$

This compact analytic limit does not cover neighborhoods approaching the poles while $a \downarrow 1$, or unbounded radii. It therefore does not by itself give an asymptotic formula for the ID boundary.

Theorem F.5 (The complete index 11/10 slice). *There is a unique $q_{\dagger}$ in*

$$0.1220544 < q_{\dagger} < 0.1220545$$

with a global zero minimum of the signed kernel. With $\rho_{\dagger} = 1/2 + (9/22)q_{\dagger}$,

$$\frac{1890389}{3437500} < \rho_{\dagger} < \frac{24196981}{44000000}, \quad 0.54993134 < \rho_{\dagger} < 0.54993139,$$

and the exact classification is

$$X_{11/10, \rho} \text{ is ID} \iff \rho \in [1/11, 1 - \rho_{\dagger}] \cup [\rho_{\dagger}, 10/11].$$

Both boundary points are included. If $c = (1/10)^{10/11}$, the controlling contact has $x = cy$ with $6.68969 < y < 6.68973$.

There is also a different unique local contact $K = K_x = 0$ in $0.0574 < q < 0.0575$, $3.7 < y < 3.9$. Its law is not ID: throughout this smaller q band, $K(c7) < 0$. Thus a local zero minimum, even with positive skewness derivative and positive second radial derivative, need not determine the ID boundary.

Proof. At $a = 11/10$, sixty distinct Wright-root intervals are independently checked by evaluating $V_{1/10}$ at both endpoints. Every bracket is rational and the endpoint signs are strict. The twelfth Rayleigh remainder proves

$$446^{12a} \left(S_{12} - \sum_{\text{listed}} r^{-12a} \right) < 1,$$

so every unlisted root exceeds 446. No completeness of the root scan is assumed. The checker `certify_scaled_point_spectra.py` ignores the proposal and Newton traces and recomputes all 120 endpoint signs and the moment remainder.

The global contact lies slightly beyond $\xi = \pi/2$, so we cannot discard the positive-integral contribution to K_ρ as nonnegative. For $D = 1 + 2u \cos \xi + u^2$, use

$$D \geq \sin^2 \xi, \quad \frac{1}{D} = 1 - 2u \cos \xi + \frac{(4 \cos^2 \xi - 1)u^2 + 2u^3 \cos \xi}{D}.$$

The listed terms use validated Laplace quadrature; this identity bounds the unlisted positive terms with Rayleigh remainders T_3, T_4 . All oscillatory tails use $2R^{am+j}e^{-hxR}T_m$ when $hxR > am + j$. For the skewness derivative, a valid lower bound is

$$K_\rho \geq 2\pi x \sum_{\text{listed}} r e^{-hrx} \cos(xr \cos \omega + \omega) - \pi x E_1 + \frac{a \min(\cos \xi, 0)}{\sin^2 \xi x^a},$$

where E_1 is the omitted first-derivative residue bound. The last term follows by differentiating $\sin \xi/D$; the other term in that derivative is nonnegative. This explicitly controls the negative-cosine sector.

The origin is controlled by the extremal kernel. The convergent germ with $R = 5/2$, $C = 128$ proves $k_a \leq 3/2$ for $0 \leq y \leq 2$. At $y = 2$, the preceding two-term positive integral upper bound, with the whole Rayleigh moments and a second-moment oscillatory bound, is less than $0.895797 < 3/2$; each of its positive terms decreases with y , proving $k_a \leq 3/2$ everywhere. On the global q band, a separate complex-polynomial bound proves $\operatorname{Re} j_a(ye^{-i\delta}) > 4/5$ for $0 \leq y \leq 6/5$, where $\delta = \pi(a^{-1} - \rho)$. Formula (D.7) therefore gives $K(c_a y) > 1/10$ on the entire omitted origin interval.

The finite certificate proves positivity on $[6/5, 6.4] \cup [7, 20]$, and on the core $[6.4, 7]$ it proves $\partial_y^2 K(c_a y) > 1/10$, $K_\rho > 1$, and opposite signs of $\partial_y K$ at its endpoints. At $q = 0.1220544$, the value at $y = 6.689706$ is negative; at $q = 0.1220545$, its value minus $(\partial_y K)^2/(2/10)$ is positive. Derivative signs at $y = 6.68969, 6.68973$ enclose the contact radius. For $y \geq 20$, the second-moment tail bound in (E.4), with the present h, ξ , is strictly positive; its product with x^a is increasing. Strict convexity and skewness increase now give the unique global contact. Endpoint ID and Proposition D.2 give the whole section.

The smaller local contact is proved by the same strict convexity, skewness increase, and opposite minimum signs on its separate rectangle. The negative value at $y = 7$ is uniform throughout $[0.0574, 0.0575]$, so this local contact cannot be an ID boundary. All statements were checked in the archived verification by `certify_eleven_tenths_id.py`: 416 rational boxes, including both contacts, plus the independent sixty-root prerequisite and explicit origin and infinite-tail controls. The record is `eleven_tenths_id_verification.json`. $\square$

Theorem F.6 (The complete middle ID region). *For every $6/5 \leq a \leq 7/5$, the boundary of the upper-half ID section is the unique global contact in the following certified band. Put*

$c_a = (a - 1)^{1/a}$ and $q = (\rho - 1/2)/(1/a - 1/2)$. Partition the index interval into

$$I_j = [j/100, (j + 1)/100], \quad j = 120, \dots, 139.$$

On each I_j , the exact rational quadratic locators Q_j, Y_j in `middle_id_region_manifest.json` define the band

$$|q - Q_j(a)| \leq 1/500, \quad |y - Y_j(a)| \leq 3/10, \quad x = c_a y.$$

There is exactly one pair $(q_*(a), y_*(a))$ in this band for which $K_{a,\rho}(c_a y) = \partial_y K_{a,\rho}(c_a y) = 0$, and it is the only global zero minimum. With $r(a) = 1/2 + (1/a - 1/2)q_*(a)$,

$$X_{a,\rho} \text{ is ID} \iff \rho \in [1 - 1/a, 1 - r(a)] \cup [r(a), 1/a].$$

Equality is included. Combined with the preceding upper-region certificates, a unique controlling contact is now certified on the entire interval $[6/5, \alpha_*]$. The locator polynomials specify enclosing bands, not exact formulas for the boundary.

Proof. On each strip, Lemma F.3 encloses twenty distinct moving Wright zeros using exact rational graph brackets. All endpoint signs are strict on complete real parameter covers; no completeness of the preliminary scan is assumed. A fourth Rayleigh remainder places every unlisted physical root above a positive constant R_j . The complete spectral verification contains 1127 sign boxes. Every range and Taylor remainder is recomputed by `certify_scaled_moving_spectra.py`.

The kernel calculation uses scaled roots $\hat{r}_n = (a - 1)^{1/a} r_n = z_n^{1/a}$ and moments $\hat{S}_m = S_m/(a - 1)^m$. This retains the exact cancellation between physical scale and root motion. The listed positive terms use the validated Laplace quadrature already described, and the unlisted terms use the two-term denominator identity and its T_3, T_4 remainder. Omitted oscillatory terms use $2\hat{R}^{am+k} e^{-hy\hat{R}} \hat{T}_m$, whenever $hy\hat{R} > am + k$. The lower bound for the positive integral is

$$\frac{\sin \xi}{\pi} \left[\frac{y^{-a}}{a - 1} - 2a \cos \xi \Gamma(2a) \hat{S}_2 y^{-2a} - a \Gamma(3a) \hat{S}_3 y^{-3a} \right].$$

For origin control, a convergent germ proves $j_a \leq 3/2$ on $[0, 2]$. A whole-moment upper estimate at $y = 2$, with positive decreasing power terms, proves the same bound on the remaining half-line. On the prescribed skewness band the complex germ gives $\text{Re } j_a(ye^{-i\delta}) > 4/5$ for $0 \leq y \leq 6/5$. Thus (D.7) gives $K(c_a y) > 1/10$ throughout the omitted origin interval.

The finite cover proves positivity on $[6/5, Y_j(a) - 3/10] \cup [Y_j(a) + 3/10, 7]$. On the core it proves

$$\partial_y^2 K(c_a y) > 1/100, \quad K_\rho(c_a y) > 1/2,$$

and opposite radial derivative signs at the two endpoints. At $q = Q_j(a) - 1/500$, the value at $y = Y_j(a)$ is negative. At $q = Q_j(a) + 1/500$, the value minus $(\partial_y K)^2/(2/100)$ is positive. These whole-parameter inequalities give existence and uniqueness of the local zero minimum, with opposite minimum signs on the two sides of the band.

For the remaining infinite tail, at $y = 7$ subtract absolute listed exponentials and a first-moment omitted-root bound from the preceding positive-integral lower bound. If $\cos \xi < 0$, discard its favorable term. The result is strictly positive. The additional checks $7h\hat{r}_1 > a$ and $7h\hat{R} > a$ show that every subtracted exponential, after multiplication by y^a , decreases for $y \geq 7$; all remaining power losses decrease too. This proves positivity on the whole infinite tail.

The checker `certify_middle_id_region.py` independently recomputes the germ prerequisites, all spectral signs, and all 6733 kernel boxes. The manifest lists the twenty verified strips, their exact locators, and hashes of the unchanged proof inputs. Consequently the zero minimum is global and unique. Endpoint ID and the interval theorem give the entire skewness section and reflection gives its lower half. At common strip endpoints, global uniqueness identifies the two constructions. $\square$

Theorem F.7 (Competing global ID contacts). *Put $I = [29/25, 117/100]$, $t = a - 233/200$, and define the following exact rational locator polynomials:*

$$\begin{aligned} Q_1(a) &= 0.1751650016654 + 1.70766807201t - 0.6820005682617t^2, \\ Q_2(a) &= 0.1728668598909 + 0.8093353588945t + 0.7122035515794t^2, \\ Y_1(a) &= 3.865102673082 + 1.069198034134t - 0.8285962464605t^2, \\ Y_2(a) &= 6.964612825586 + 3.897040936013t - 6.114433090261t^2. \end{aligned}$$

For every $a \in I$, each band

$$|q - Q_i(a)| \leq 1/2000, \quad |y - Y_i(a)| \leq 3/10$$

contains a unique local contact $(q_i(a), y_i(a))$ satisfying $K(c_a y) = \partial_y K(c_a y) = 0$. Both contact curves are continuous, and the two radial cores are disjoint. The complete upper-half classification is

$$(F.4) \quad X_{a,\rho} \text{ is ID} \iff q \geq \max\{q_1(a), q_2(a)\}, \quad 0 < q \leq 1.$$

Reflection gives the lower half and equality is included.

There is at least one $a_\dagger \in (1.1613, 1.1636)$ at which $q_1(a_\dagger) = q_2(a_\dagger)$. At this index the ID boundary has exactly two distinct global zero minima, one in each radial core. Thus global contact uniqueness fails in the free stable family. No uniqueness of $a_\dagger$, corner, or nonanalyticity of the boundary is asserted here.

Proof. Forty moving Wright roots on I have independently verified Hankel signs in 164 rational parameter boxes. Consecutive root graphs are separated by positive quadratic differences at the same parameter. Their full ranges at different parameters may overlap, which has no bearing on distinctness at a fixed index. A fourth Rayleigh remainder puts every unlisted physical root above 90. The specialized spectral verifier is `certify_crossover_spectrum.py`.

Write $J_i(a) = [Y_i(a) - 3/10, Y_i(a) + 3/10]$ and $\eta = 1/2000$. On each local band the verified bounds are

$$\partial_y^2 K(c_a y) > 1/100,$$

opposite radial derivative signs at the endpoints of $J_i(a)$, a negative value at $q = Q_i(a) - \eta, y = Y_i(a)$, and a positive value of $K - (\partial_y K)^2/(2/100)$ at $q = Q_i(a) + \eta, y = Y_i(a)$. Moreover, on both radial cores,

$$K_\rho > 1/2$$

throughout the entire skewness envelope $[\min(Q_1, Q_2) - \eta, \max(Q_1, Q_2) + \eta]$. Strict convexity and skewness increase give the two unique local contacts. Compactness and uniqueness give their continuity in a ; no parameter differentiation is needed.

The exterior certificate uses only the narrower band

$$q \in [\max(Q_1, Q_2) - \eta, \max(Q_1, Q_2) + \eta].$$

It proves strict positivity off $J_1(a) \cup J_2(a)$ at every scale. The origin proof uses the extremal bound $j_a \leq 3/2$, the convergent complex germ, and the nonnegative pole term in (D.7); where useful the pole term is retained. Three finite intervals cover the complement of the two cores between $y = 6/5$ and $y = 10$. Beyond 10, the listed absolute exponential and first-moment bounds give a positive lower bound whose product with y^a increases.

Set $q_* = \max(q_1, q_2)$. This value belongs to the narrower exterior band. At q_i , strict convexity gives nonnegativity on J_i ; the proved skewness monotonicity preserves it up to q_* . Consequently the entire kernel is nonnegative at q_* . At least one core has a zero minimum. Just below q_* , the value at a controlling contact radius is negative. The interval theorem and endpoint ID now prove (F.4). If the two contact values coincide, strict convexity gives exactly one zero in each core and the exterior is strictly positive.

At the endpoints of I , the exact locator inequalities give

$$Q_2(1.16) - Q_1(1.16) > 0.0022283 > 2\eta, \quad Q_1(1.17) - Q_2(1.17) > 0.0067549 > 2\eta.$$

Thus the order of the true contact values reverses, proving existence of a double contact by continuity. For localization, $D = Q_1 - Q_2$ is strictly increasing on I , while

$$D(1.1613) < -0.0010447 < -2\eta, \quad D(1.1636) > 0.0010377 > 2\eta.$$

Every equality $q_1 = q_2$ is therefore in the stated smaller interval. Monotonicity of this locator difference does not establish monotonicity of the true contact difference.

The full verifier `certify_id_crossover.py` independently recomputes all spectral prerequisites and all 1924 kernel boxes. Its record is `id_crossover_verification.json`; the exact localization checks are in `id_crossover_localization.json`. $\square$

Corollary F.8 (Complete lower contact cover). *The complete ID classification is effectively certified throughout $11/10 \leq a \leq \alpha_*$. On $[11/10, 29/25]$ the controlling contact is unique and lies near the second radial branch of Theorem F.7. On $[117/100, \alpha_*]$ it is unique and continues the first branch. On the intervening interval $[29/25, 117/100]$, formula (F.4) applies.*

Proof. Nine further adjacent strips use the single-core specialization of the proof of Theorem F.7. The quadratic locators are recorded as exact decimal rationals in `lower_contact_region_manifest.json`. Each has skewness half-width $1/2000$ and radial half-width $3/10$. The checker `certify_single_contact_region.py` recomputes the root signs, moment remainder, germ, convexity, skewness derivative, endpoint signs, and the entire exterior. The latter includes the origin and all $y \geq 10$.

index interval	kernel boxes	root-sign boxes
[1.10, 1.11]	1642	92
[1.11, 1.12]	1332	92
[1.12, 1.13]	1816	90
[1.13, 1.14]	2682	90
[1.14, 1.15]	7478	88
[1.15, 1.16]	126765	88
[1.16, 1.17]	1924	164
[1.17, 1.18]	1689	88
[1.18, 1.19]	1287	88
[1.19, 1.20]	1125	90

There are 147740 kernel boxes and 970 root-sign boxes in total, including the two-core strip. The other nine strips use thirty moving roots each. Their graph separation is checked at each common parameter, as in the two-core proof. The manifest checks the exact adjacent cover and the hashes of the completed independent verifications; it does not substitute for those verifications. Combining this cover with the middle and upper classifications proves the assertion. $\square$

APPENDIX G. EFFECTIVE APPROXIMATION OF THE BOUNDARIES

The strict sign criteria become terminating approximation procedures once the origin, tail and omitted spectrum have explicit bounds. The ID and GGC constructions below do not require contact uniqueness.

G.1. Effective boundary computation.

Lemma G.1 (Pole cancellation at the ID origin). *Let $1 < a \leq \alpha_*$, $c_a = (a - 1)^{1/a}$, and suppose the extremal kernel satisfies $|k_a(z) - 3/2| \leq \eta < 1/2$ on $|z| \leq D$. Then, uniformly over the admissible upper half,*

$$K_{a,\rho}(x) \geq 3/2 - 3\eta > 0 \quad (0 < x \leq D/2).$$

One may take $D = c_a/200, \eta = 12/25$ for all $1 < a \leq \alpha_*$, or $D = c_a/3, \eta = 17/40$ when $1 < a \leq 13/10$.

Proof. Write $\delta = \pi(1/a - \rho)$ and $b_a = ac_a/(a - 1) = \int k_a$. The Cauchy continuation formula is

$$K(x) = 2 \operatorname{Re} k_a(xe^{-i\delta}) + \frac{b_a \sin \delta}{\pi x} - \frac{x \sin \delta}{\pi} \int_0^\infty \frac{k_a(u) du}{(u - x \cos \delta)^2 + x^2 \sin^2 \delta}.$$

The normalized integral over $[0, D]$ is at most $3/2 + \eta$. Its remainder is at most $xb_a \sin \delta / [\pi(D - x)^2]$, using endpoint ID and $k_a \geq 0$. For $x \leq D/2$, the displayed positive pole dominates this remainder. The first term is at least $3 - 2\eta$. The boundary case $\delta = 0$ follows by continuity.

For completeness the uniform germ constants follow from the inverse equation with radius $R = 1$ and bound $M = 9$. The inequality $\log 9 > 2 + 1/9$, together with $e^u \geq 1 + u$, gives the inverse-root bound for every $a > 1$; its analytic radius is at least four for $a \leq 2$. The resulting Volterra bound is $|j_a(y) - 3/2| \leq e^{77|y|} - 1$ for $|y| \leq 1/2$. Since $e^{77/200} - 1 < 12/25$, the first choice follows. For the second choice, exact absolute summation of the previously proved 128 coefficients on $|y| \leq 1/3$, $1 \leq a \leq 13/10$, including their geometric remainder, gives $0.423666806557 < 17/40$. These constants are independently checked by `certify_universal_id_origin.py` and `certify_near_one_id_origin.py`. $\square$

Theorem G.2 (Constructive critical boundaries). *For each computable $1 < a < \alpha_*$, $r(a)$ is computable to arbitrary positive rational accuracy. The same holds for $g(a)$ at every computable $4/5 < a < 1$. Neither assertion assumes uniqueness of the controlling contact or simplicity of every Wright zero. Equality to a boundary is not claimed decidable from an arbitrary computable-real name.*

ID part. Fix $1/2 < \rho < 1/a$, put $h = \sin \omega > 0$, and let $S_j = \sum r_n^{-aj}$. The inequalities $(1 + u^a)^{-2} \geq 1 - 2u^a$ and $e^{-v} \leq (2a/e)^{2a} v^{-2a}$ give

$$K(x) \geq \frac{\sin \xi}{\pi x^a} - C_2 x^{-2a}, \quad C_2 = S_2 \left[\frac{2a\Gamma(2a) \sin \xi}{\pi} + 2 \left(\frac{2a}{eh} \right)^{2a} \right].$$

Thus $X = (2\pi C_2 / \sin \xi)^{1/a}$ is an explicit infinity cutoff. Lemma G.1 supplies a computable origin cutoff $x_0 > 0$.

All Wright zeros are negative real. Computable entire-series tails and argument-principle isolation on rational rectangles enumerate them with multiplicities; enumerate nearby rectangle boundaries to avoid zeros on a contour. For any finite spectral part put $T_1 = S_1 - \sum_{\text{included}} r_n^{-a}$. The omitted entire kernel on $[x_0, X]$ has absolute value at most

$$x^{-a} T_1 \left[\frac{a\Gamma(a)}{\pi \sin \xi} + 2(a/e)^a h^{-a} \right],$$

which tends uniformly to zero. Finite integral terms have explicit exponential tails. Increasing precision, spectral truncation and radial subdivision therefore gives convergent whole-interval enclosures.

For $\rho > r(a)$, strict interior positivity and compactness give a finite positive certificate. For $\rho < r(a)$, a negative value at some positive radius persists at a rational radius and is eventually detected. Dovetail both searches over rational $p < q$, seeking a negative witness at p , a positive complete certificate at q , and $q - p$ below the requested error. Such a pair exists and the search terminates. A computable real index is treated by successively narrower parameter intervals in these strict finite tests. $\square$

Lemma G.3 (All-angle parabolic descent). *For $0 < b \leq 1/4$, put $a = 1 - b$, $c = \cos(2\pi b)$, $k = a/b$, and*

$$\kappa_b = \frac{(1 - c)(c + k)}{2\{1 + 3c + (1 - c)k\}} > 0.$$

The phase in the parabolic representation satisfies $\operatorname{Re} \Phi \leq -\kappa_b x^2$ throughout the slit plane. Uniformly for $0 \leq b \leq 1/4$ and $|\arg v| \leq \pi \rho_0 < \pi$, it also satisfies

$$\operatorname{Re} \Phi \leq -(1 - \rho_0)x^2/4.$$

The logarithm is the continuous parabolic-contour logarithm.

Proof. Set $h = x/\sqrt{2}$, $\lambda = \sqrt{v}$. The phase is holomorphic on $\operatorname{Re} \lambda > 0$ and equals

$$(\lambda + ih)^2 - \lambda^2 - \lambda^{2-2b}\{(\lambda + ih)^{2b} - \lambda^{2b}\}/b.$$

At $\lambda = iy$, write $X = |y + h|$, $Y = |y|$. Its negative real part is $X^2 + kY^2 - dX^{2b}Y^{2a}/b$, with $d = 1$ on equal-sign portions and $d = c$ on opposite-sign portions. Concavity gives $X^{2b}Y^{2a} \leq 2bXY + (1 - 2b)Y^2$. On equal-sign portions this bounds the gap below by h^2 . On opposite-sign portions, $X + Y = |h|$, and the gap is at least $X^2 + [c + (1 - c)k]Y^2 - 2cXY$. Its constrained minimum is $2\kappa_b h^2$; evaluation at $Y = 0$ also gives $2\kappa_b \leq 1$. At infinity the phase tends uniformly to $-2ah^2$. The harmonic maximum principle on increasing half-disks proves the first bound.

For $h > 0$, the harmonic function $h^2(1/2 + \arg \lambda/\pi)$ is a lower bound for the gap: use h^2 on the upper imaginary boundary, zero on the lower, and the same infinity limit. For $h < 0$, reverse the sign of $\arg \lambda$. Since $|\arg \lambda| \leq \pi \rho_0/2$, the second bound follows. Its removable limit proves the case $b = 0$. $\square$

GGC part of Theorem G.2. For a fixed trial $1/2 < \rho < 1$, use a finite-radius chart and $s = |v|^{-1/2}$ at infinity. The retained-circle Hankel formula and Lemma G.3 give explicit tails and convergent interval enclosures. The normalized endpoint signs are strictly positive, including at $s = 0$.

A positive certificate consists of a complete radial partition with positive normalized $\Im H$, together with an open coordinate-half-plane cover of $Z(v) = (1 + v)^{3/2}e^{av/b}W_a(-v^a/b)$. Its endpoint limits are positive. Numbering the four half-planes cyclically, the sum of adjacent quarter-turns computes the ray image winding. Adjacent planes cannot be opposite, because they contain the same endpoint value. Convexity permits homotopy to a polygon within these nonzero half-planes. Zero winding and the argument principle on truncated sectors exclude every interior zero; the endpoint arcs use the uniform asymptotics. The exact ray criterion then proves GGC.

Every strict interior trial has such a finite certificate by compactness and strict positivity. If a trial lies beyond the boundary, the Hopf inequality at the first contact gives an open negative set at a smaller angle. Enumeration of rational angles and radii must find a strict negative witness. Two distinct trisection trials avoid stalling when one equals the boundary. Increasing precision, integration cutoffs and subdivision budgets therefore encloses the endpoint to any requested width. The finite strict tests also handle computable-real indices by interval names. The rational-index implementation is `compute_ggc_boundary.py`; its independent verifier repeats the cover, winding and negative witness. $\square$

APPENDIX H. BOUNDARY ASYMPTOTICS AT INDEX ONE

We first obtain the leading ID scale from uniform spectral estimates. We then localize the GGC contact and refine the ID expansion to second order. The last subsection records finite slices and the unit-origin estimate used in the ID proof. Qualitative eventual assertions are kept separate from certified numerical thresholds.

H.1. The exact ID boundary scale at index one. The compact germ limit alone does not control a growing number of Wright zeros. The next two lemmas supply the missing uniform information: a counting estimate from endpoint ID, and a phase estimate from a parameter-uniform saddle integral.

Lemma H.1 (Uniform spectral heat trace). *Put $e = a - 1$, and let $\hat{r}_n = (e\lambda_n)^{1/a}$ be the scaled positive Wright roots. Define*

$$N_a(R) = \#\{n : \hat{r}_n \leq R\}, \quad \mathcal{L}_a(v) = \sum_n e^{-v\hat{r}_n}.$$

For $a > 1$ sufficiently close to one,

$$(H.1) \quad N_a(R) \leq 4R \quad (R > 0).$$

Moreover,

$$(H.2) \quad v\mathcal{L}_a(v) \longrightarrow 1 \quad (a \downarrow 1, v \downarrow 0), \quad \sup_{v>0} v^a \mathcal{L}_a(v) \longrightarrow 1 \quad (a \downarrow 1).$$

The first limit is joint, with no restriction on the relative rates of its two parameters.

Proof. The scaled extremal variable Z_a is nonnegative and ID, has left support endpoint zero, and has Lévy density $j_a(u)/u$, where $j_a \geq 0$. Thus its drift is zero. Write

$$\Psi_a(s) = -\log \mathbb{E}e^{-sZ_a} = \frac{as}{e} - \log W_a(s^a/e), \quad J_a(s) = \Psi'_a(s) = \int_0^\infty e^{-su} j_a(u) du.$$

For $T_{j,a}(s) = \sum_n (1 + (\hat{r}_n/s)^a)^{-j}$, logarithmic differentiation of the canonical product gives

$$T_{1,a}(s) = \frac{s}{e} - \frac{s}{a} J_a(s), \quad T_{2,a}(s) = \frac{s}{a} - \frac{es}{a^2} J_a(s) + \frac{s^2}{a^2} J'_a(s).$$

Both $J_a \geq 0$ and $J'_a \leq 0$. Consequently $T_{2,a}(s) \leq s/a$, which proves (H.1).

There is a common bound for $j_a(u)$ on a fixed neighborhood of zero, by the convergent germ. Also

$$J_a(1) = a \frac{V'_e(-1)}{V_e(-1)} = O(1),$$

by Lemma F.1, since $V_0(-1) = 1/2$. Splitting its Laplace integral into this neighborhood and its complement proves uniformly for $s \geq 2$ that $J_a(s) = O(s^{-1})$ and $J'_a(s) = O(s^{-2})$. For the complement one bounds against the integrable measure $e^{-u} j_a(u) du$; its remaining exponential factor decays uniformly. Hence

$$(H.3) \quad T_{2,a}(s) = s/a + O(1), \quad s \geq 2,$$

with a uniform constant.

Consider any sequences $a \downarrow 1, v \downarrow 0$, and the measures $\mu_{a,v} = v \sum_n \delta_{v\hat{r}_n}$. They satisfy $\mu_{a,v}([0, R]) \leq 4R$. Equation (H.3) implies, for each $t > 0$,

$$\int_0^\infty \frac{\mu_{a,v}(du)}{(1 + (u/t)^a)^2} = vT_{2,a}(t/v) \longrightarrow t.$$

Every vague subsequential limit therefore satisfies $\int (u+t)^{-2} \mu(du) = 1/t$. The linear mass bound controls both tails in passing to the limit. This transform uniquely identifies μ as Lebesgue measure: use $(u+t)^{-2} = \int_0^\infty w e^{-w(u+t)} dw$ and uniqueness of the Laplace transform twice. The same mass bound permits integration against e^{-u} , proving the joint heat-trace limit.

On every fixed compact interval of $v > 0$, compact convergence $V_e \rightarrow 1/\Gamma(2 - z)$, the argument principle, and (H.1) give

$$\mathcal{L}_a(v) \longrightarrow \frac{e^{-2v}}{1 - e^{-v}}.$$

The limit satisfies $v\mathcal{L}_1(v) < 1$. The smallest scaled root is uniformly bounded below by a positive constant; together with (H.1), this makes $v^a \mathcal{L}_a(v)$ uniformly small for large v . At small $v \leq 1$, $v^{a-1} \leq 1$ and the joint limit already applies. These three ranges prove the upper limit in the second assertion. Taking $v = e$, so that $e^e \rightarrow 1$, proves its matching lower limit. $\square$

Lemma H.2 (Uniform large-root phase). *There are constants $e_0 > 0, R_0, C_0$ such that every scaled Wright root $r = \hat{r}_n \geq R_0$, for $0 < e \leq e_0$, satisfies*

$$(H.4) \quad \text{dist}(r, \mathbb{Z}) \leq C_0(e + e^2 r + r^{-1}).$$

Proof. Take $e_0 < 1/2$. The Hankel contour for $W_{1+e}(-r^{1+e}/e)$ can be collapsed to the two negative real rays: its small circle tends to zero, because $\cos(\pi e) > 0$. With the stated orientation,

$$W_{1+e}(-r^{1+e}/e) = -\frac{1}{\pi} \text{Im } I_e(r), \quad I_e(r) = \int_0^\infty u^{-2} \exp\{-u - r^{1+e} e^{-i\pi e} u^{-e}/e\} du.$$

Put $\eta = \pi e/(1+e)$, $k = e^{-i\eta}$, and rotate the integration ray to $u = rkt$, $t > 0$. At infinity the rotation is allowed by $\text{Re } k > 0$, and at zero by the corresponding strictly positive real part of the inverse power in the exponent. This gives the exact formula

$$I_e(r) = \frac{e^{-(1+e)rk/e}}{rk} \int_0^\infty t^{-2} e^{-rk f_e(t)} dt, \quad f_e(t) = t - 1 + \frac{t^{-e} - 1}{e}.$$

The extension at zero is $f_0(t) = t - 1 - \log t$. For $t > 0$, $f_e(t) \geq f_0(t) \geq 0$, with a unique minimum at $t = 1$, and $f_e''(1) = 1 + e$.

Uniform Laplace expansion at this minimum yields

$$\int_0^\infty t^{-2} e^{-rk f_e(t)} dt = \sqrt{\frac{2\pi}{(1+e)rk}} \{1 + O(r^{-1})\},$$

uniformly for $0 \leq e \leq e_0$. To justify uniformity, use the common analytic Morse coordinate near $t = 1$; its density has bounded Taylor coefficients on the compact parameter interval. The odd first term integrates to zero and the quadratic remainder contributes relative $O(r^{-1})$. Off a common neighborhood, $f_e \geq f_0$ and $\text{Re } k \geq \cos(\pi e_0/(1+e_0)) > 0$ give a uniform exponentially small tail. The weighted integral $\int t^{-2} e^{-R \text{Re}(k) f_0(t)} dt$ is finite for a fixed sufficiently large R , including the endpoint zero.

It follows that

$$I_e(r) = \sqrt{\frac{2\pi}{1+e}} (rk)^{-3/2} e^{-(1+e)rk/e} \{1 + O(r^{-1})\}.$$

At a zero of W , the imaginary part vanishes. Its leading phase, divided by π , is

$$d_e r + \frac{3e}{2(1+e)}, \quad d_e = \frac{\sin \eta}{\eta} = 1 + O(e^2).$$

The distance of this quantity to an integer is $O(r^{-1})$. This proves (H.4). No matching of each large root with its global index is required. $\square$

Theorem H.3 (Sharp ID boundary asymptotic). *For the continuous boundary $r(a)$ in the global ID interval theorem,*

$$(H.5) \quad \lim_{a \downarrow 1} \frac{r(a) - 1/2}{a - 1} = 2.$$

Equivalently, its normalized skewness satisfies $q_(a) \sim 4(a - 1)$. For every fixed $C > 4$, the test curve $q = C(a - 1)$ is ID for all sufficiently small $a - 1 > 0$; for every fixed $0 < C < 4$, it is non-ID there. The assertion does not imply a unique controlling radius near index one.*

Proof. Fix $C > 0$, take $q = Ce$, and write $h = \sin \omega$, $c = \cos \omega$. Then

$$h/e \longrightarrow \pi C/2, \quad c = 1 + O(e^2), \quad \xi \longrightarrow \pi/2.$$

Let $P_a(y) = \sum_n B_{a,\xi}(\hat{r}_n y)$ denote the positive part of $K_{a,\rho}(c_a y)$. The scaled Rayleigh sums satisfy $\sum \hat{r}_n^{-a} = 1/(e\Gamma(a+1))$, while their orders two and three stay bounded as $e \downarrow 0$. The elementary inequality $1/(1+2bz+z^2) \geq 1-2bz-z^2$ therefore implies, for each fixed $Y > 0$,

$$(H.6) \quad \inf_{y \geq Y} e y^a P_a(y) \geq 1/\pi - o(1).$$

Here all error terms are bounded by constants times eY^{-a} and eY^{-2a} . At every fixed $y > 0$, the stronger equality $ey^a P_a(y) \rightarrow 1/\pi$ follows by bounding the absolute error using the denominator lower bound $\sin^2 \xi$.

For $C > 4$, drop the cosine signs and use Lemma H.1:

$$\sup_{y>0} 2ey^a \mathcal{L}_a(hy) = 2eh^{-a} \sup_{v>0} v^a \mathcal{L}_a(v) \longrightarrow \frac{4}{\pi C} < \frac{1}{\pi}.$$

Together with (H.6), this proves $K(c_a y) > 0$ for all $y \geq Y$.

There is no missing origin range. Choose $Y = 1$ and put $\delta = \pi(1/a - 1/2)(1 - q)$, so $\delta \rightarrow \pi/2$. The scaled Cauchy formula (D.7) reads

$$K(c_a y) = 2 \operatorname{Re} j_a(ye^{-i\delta}) + \frac{a \sin \delta}{\pi e y} - \frac{y \sin \delta}{\pi} \int_0^\infty \frac{j_a(u) du}{u^2 - 2uy \cos \delta + y^2}.$$

The germ is uniformly bounded on $|z| \leq 1$. Furthermore

$$\Psi_a(1) = 1 - \log V_e(-1) = O(1), \quad \int_1^\infty j_a(u) u^{-1} du = O(1),$$

the second bound following from the positive Lévy representation of $\Psi_a(1)$. For small e , $\cos \delta \leq 1/2$, so the denominator is at least $(u^2 + y^2)/2$. On $0 < u < 1$, boundedness of j_a and $\int_0^1 (u^2 + y^2)^{-1} du \leq \pi/(2y)$ give a uniformly bounded Poisson term; on $u \geq 1$, the displayed weighted integral gives the same conclusion for $0 < y \leq 1$. The positive pole term tends uniformly to infinity on this range. Thus the whole kernel is positive and the test curve is ID.

For the converse, put $y_m = 2\pi m$, where $m \geq 1$ is any fixed integer. We claim

$$e \sum_n e^{-hy_m \hat{r}_n} \{\cos(cy_m \hat{r}_n) - 1\} \longrightarrow 0.$$

Roots below a fixed R contribute $O(eR)$, by (H.1). For $R \leq \hat{r}_n \leq M/e$, Lemma H.2 and $c = 1 + O(e^2)$ make the phase distance from $2\pi\mathbb{Z}$ at most $O_m(R^{-1} + eM)$. The total weighted mass multiplied by e is bounded by the heat-trace lemma. Roots above M/e contribute at most $O_C((M+1)e^{-\kappa_C M})$, again by the linear counting bound. Let $e \rightarrow 0$, then $R, M \rightarrow \infty$, to prove the claim. Consequently

$$(H.7) \quad \lim_{e \downarrow 0} ey_m K_{a,\rho}(c_a y_m) = \frac{1}{\pi} \left(1 - \frac{4}{C}\right).$$

For $C < 4$ this is negative, so the test curve is non-ID. Theorem E.8 now sandwiches $q_*(a)/e$ between every fixed $C < 4$ and every fixed $C > 4$. Finally $r(a) - 1/2 = (1/a - 1/2)q_*(a)$, proving (H.5). All fixed resonances in (H.7) have the same leading threshold; the proof makes no claim selecting a unique one. $\square$

Proposition H.4 (An explicit linear ID wedge). *For $1 < a \leq 501/500$, every admissible upper-half parameter with $q \geq 6(a-1)$ is ID, as is its reflection.*

Proof. It suffices to prove ID on $q = 6e$. For $1 < a \leq 3/2$, $(1+u^a)^2 \leq e^{2u}$. Hölder's inequality therefore gives

$$B_{a,\xi}(v) \geq \frac{\Gamma(a+1) \sin \xi}{\pi(v+2)^a} \geq \frac{\Gamma(a+1) \sin \xi h^a}{\pi\{1 + (2h)^{a/e}\}^e} \frac{1}{1 + (hv)^a}.$$

Put $M_a = \sup_{w \geq 0} e^{-w}(1+w^a)^2$. Applying (D.1) at the changed scale hx gives

$$K(x) \geq \left[\frac{\Gamma(a+1) \sin \xi h^a}{\pi\{1 + (2h)^{a/e}\}^e} - \frac{2eM_a}{a} \right] \sum_n \frac{1}{1 + (hr_n x)^a}.$$

For $0 < e < 1/2$, an elementary bound is

$$M_a \leq \frac{4}{\exp(1)} \frac{\exp(-4e)}{(1-2e)^2}.$$

Indeed, on $w \leq 1$ use $w^a \leq w$; on $w \geq 1$ use $1 + w^a \leq w^e(1 + w)$, $\log w \leq w - 1$, and maximize the resulting exponential quadratic.

Set $E = 1/500$, $\Omega = 3\pi E$, and $d = 3\pi(1 - E)(1 + E)^{-1}(1 - \Omega^2/6)$. Uniformly on $0 < e \leq E$,

$$h/e \geq d, \quad h^a/e \geq d(dE)^E, \quad \sin \xi \geq \cos(5\pi E/2), \quad \{1 + (2h)^{a/e}\}^e \leq 2^E.$$

For the second estimate use that $e \log(de)$ decreases on $(0, E]$; the last uses $2h < 1$. Also $\Gamma(a + 1) \geq 1$. The coefficient in brackets, divided by e , is therefore at least

$$\frac{d(dE)^E \cos(5\pi E/2)}{\pi 2^E} - \frac{8}{\exp(1)} \frac{\exp(-4E)}{(1 - 2E)^2} > 0.0166 > 0.$$

The strict constant inequality is independently checked by `certify_linear_id_wedge.py`. This proves positivity for every $x > 0$ without a root truncation or spatial grid. The interval theorem supplies all larger admissible q . $\square$

H.2. The sharp GGC endpoint expansion.

Theorem H.5. *As $b \downarrow 0$,*

$$g(1 - b) = \frac{\sqrt{b}}{\pi\sqrt{2}} - b + \frac{2\pi^2 + 13}{12\pi\sqrt{2}}b^{3/2} - b^2 + O(b^{5/2}).$$

For all sufficiently small positive b , its global controlling radial contact is unique. More precisely, real-analytic functions w, V near zero satisfy $w(0) = 1/2, w'(0) = \pi^2/6$ and

$$g(1 - b) = \frac{\arcsin \sqrt{bw(b)} - \pi b}{\pi(1 - b)}.$$

If $\zeta(b)$ is the continued reciprocal-Gamma zero near two, the contact coordinate is

$$T_*(b) = \zeta(b)\sqrt{1 - bw(b)} + b^2V(b) = 2 + (2\gamma - 1/2)b + O(b^2).$$

No numerical smallness threshold is asserted.

Proof. Set $a = 1 - b, \theta = \pi - \delta, v = re^{i\theta}, \eta = \pi b + a\delta, T = r^a$, and

$$t = -v^a = Te^{-i\eta}, \quad G_b(t) = e^{-t/b}W_a(t/b), \quad G_0(t) = \Gamma(2 - t)^{-1}.$$

The normalized sign is

$$h_b(T, \eta) = \Im \left[-e^{-i\eta} \{1 + bG'_b(t)/G_b(t)\} \right].$$

Fix a compact positive range of C and put $\delta = C\sqrt{b}$. We control three radial ranges, retaining uniformity in this moving angle.

Middle range. There are constants $R_0, c_0, M, b_0 > 0$, with $b_0 < 1/2$, such that for $t = x - iy$, $x \geq R_0, y > 0, y^2/x \leq 1$, and $bx(\log x)^2 \leq c_0$,

$$(H.8) \quad \frac{|G_b^{(j)}(t) - G_0^{(j)}(t)|}{|G_0(t)|} \leq Mbx(\log x)^{j+2}(1 + 1/y), \quad j = 0, 1.$$

Here the retained-circle Hankel integrand uses $E_b(\ell) = (e^{b\ell} - 1)/b$. On a cut ray $u \geq 1$,

$$|E_b(\log u \pm i\pi) - (\log u \pm i\pi)| \leq \frac{b}{2}(\log u + \pi)^2 u^b.$$

Truncate first at $U = Dx^2$, with D fixed and large. The assumptions make the exponential difference uniformly bounded on $[1, U]$; its integral is at most $Mbx(\log x)^{j+2}e^{\pi y}\Gamma(x - 1)$. The unit-circle contribution is absorbed by the same bound. On $[U, \infty)$, use the exponential difference identity *before* estimating, retaining the factor b . Since $2x(\log u + \pi)u^b \leq u/4$ there, this tail is at most $Mbx e^{-U/2}$. Reflection gives $G_0(t) = -\Gamma(t - 1)\sin(\pi t)/\pi$. The digamma series bounds the Gamma ratio below by $\exp(-My^2/x)$, and $e^{\pi y}/|\sin \pi t| \leq M(1 + 1/y)$, proving (H.8).

Take R fixed and large and $R_{\text{out}} = K\delta^{-1} \log(1/\delta)$. On $R \leq r \leq R_{\text{out}}$, $x \asymp r, y \asymp r\delta$; all hypotheses of (H.8) hold uniformly for small b . The relative errors tend to zero, and the digamma bound $|G'_0/G_0| \leq \log x + M + 1/y$ yields

$$b|G'_b/G_b - G'_0/G_0| = o(\delta), \quad h_b \geq \sin \eta - b(\log x + M + 1/y) - o(\delta) > \delta/2.$$

The last step follows by choosing R large enough for the fixed lower bound on C .

Outer range. Use the parabolic integrals F_b, J_b . Lemma G.3 gives $\Re \Phi \leq -\delta x^2/(4\pi)$ and $|z| \geq \delta/\pi$. On $|x| \leq \varepsilon\sqrt{r}$, the common convergent Taylor expansion gives

$$\Re \Phi \leq -cx^2, \quad |\Phi + ax^2| \leq M|x|^3 r^{-1/2}, \quad |z^{-3} - 1| + |E_b(2 \log z)| \leq M|x|r^{-1/2}.$$

Its integral error is $O(r^{-1/2})$. Outside this interval, the whole-contour bound and amplitude estimates give

$$M\delta^{-4}(1 + \log(1/\delta))r^{-1/2}e^{-c\delta r}$$

for both omitted integrals. Choose K large. Uniformly on $r \geq R_{\text{out}}$,

$$(H.9) \quad F_b = \sqrt{\pi/a} + O(r^{-1/2} + \delta^2), \quad J_b = O(r^{-1/2} + \delta^2).$$

Thus the normalized sign $\sin \delta + b\Im(e^{i\theta} J_b/F_b)$ is positive. The estimates also hold uniformly in the whole smaller sector.

Compact range. The retained-circle formula is jointly holomorphic in (b, t) near $b = 0$ on compact t sets, and

$$G_b(t) = f(t) + \frac{1}{2}bt f''(t) + O(b^2), \quad f(t) = 1/\Gamma(2 - t).$$

Near each of its finitely many simple zeros $n = 2, 3, \dots$, $G'_b/G_b = (t - \zeta_n(b))^{-1} + A_n(t, b)$, with bounded analytic A_n and real $\zeta_n(b) = n + O(b)$. The pole contribution is exactly

$$\Im \frac{-e^{-i\eta}}{Te^{-i\eta} - \zeta} = -\frac{\zeta \sin \eta}{T^2 - 2\zeta T \cos \eta + \zeta^2}.$$

Hence $h_b \geq \sin \eta \{1 - b/(\zeta_n \sin^2 \eta)\} - Mb$. For any fixed $C > 1/\sqrt{2}$, this is positive at every such zero, and positivity away from them is immediate.

Sector zero count. Write

$$P_b(v) = e^{av/b} W_a(-v^a/b), \quad P_0(v) = e^{v(\log v - 1)}/\Gamma(2 + v).$$

The three estimates also prove uniformly on the outer ray that $P_b/P_0 \rightarrow 1$. On compact sets, real zero displacement is $O(b)$ whereas the distance to the ray is bounded below by a constant times δ . In the middle range use (H.8), then move from $-v$ to $-v^a$: the logarithmic ratio is bounded by $Mbr(\log r)^2 + Mb \log r/\delta = o(1)$. The exponential normalization has the same order. In the outer range the prefactors cancel and (H.9) gives $P_b/P_0 = F_b/F_0 \rightarrow 1$. The analytic ratio has positive real part on the positive axis and, for small b , on the outer ray. Its limits at zero and infinity are respectively 1 and $1/\sqrt{a}$, uniformly on the closed sector. The harmonic minimum principle therefore proves positive real part throughout: no interior zero is missed. The ray criterion now gives the sharp leading upper bound.

Unique contact and expansion. Choose $1/\sqrt{3} < C_0 < 1/\sqrt{2} < C_1$. All signs outside the first zero's disk are strictly positive uniformly on this band. At C_0 the first pole gives a negative value; at C_1 the whole ray is positive. Put $\zeta = \zeta_2(b)$ and $c = \cos \eta$. Reality of the analytic remainder implies that

$$B(T, c, b) = \frac{\Im[(-c + i \sin \eta) A_2(T(c - i \sin \eta), b)]}{\sin \eta}$$

extends real-analytically to $(2, 1, 0)$, with boundary value $A_2(T, b) + T\partial_T A_2(T, b)$. Consequently

$$h_b/\sin \eta = 1 - b\zeta/D + bB, \quad D = (T - \zeta c)^2 + \zeta^2 \sin^2 \eta.$$

A nonpositive stationary value has first $T - \zeta c = O(\sqrt{b})$, then by differentiation $T - \zeta c = O(b^2)$. Its second derivative is positive of order b^{-1} . Set $\sin^2 \eta = bw$ and $T = \zeta \sqrt{1 - bw} + b^2 V$. The two analytic contact equations are

$$1 - \frac{\zeta}{\zeta^2 w + b^3 V^2} + bB = 0, \quad \frac{2\zeta V}{(\zeta^2 w + b^3 V^2)^2} + B_T = 0.$$

At zero their Jacobian in (w, V) is triangular with diagonal $(2, 1)$. The implicit-function theorem gives the unique branch; the preceding global localization excludes any other controlling minimum. Finally

$$\zeta'(0) = 2\gamma, \quad A_2(2, 0) = -\gamma, \quad (A_2)_T(2, 0) = -\pi^2/6,$$

so $B(2, 1, 0) = -\gamma - \pi^2/3$ and $w'(0) = \pi^2/6$. Expanding $\delta = (\arcsin \sqrt{bw(b)} - \pi b)/(1 - b)$ proves all displayed coefficients. The integer powers arise from $-b/(1 - b)$. This proof supplies no numerical threshold for the smallness assumptions. $\square$

H.3. Second-order ID asymptotics and eventual unique contact. The following refinement retains the small parameter in the phase error of Lemma H.2. It also supplies complete root enumeration near index one, rather than proximity of selected roots.

Lemma H.6 (Uniform enumerated spectrum). *Put $e = a - 1$, $\eta = \pi e/a$, $d_e = \sin \eta/\eta$, and $A_e = 3\eta/(2\pi d_e)$. For all sufficiently small $e > 0$, the entire scaled spectrum is simple and can be indexed by $n = 2, 3, \dots$ so that*

$$R_{n,e} = n/d_e - A_e + \delta_{n,e}, \quad |\delta_{n,e}| \leq Ce/n,$$

with a constant independent of n, e .

Proof. In the exact rotated Laplace integral, regard k as an independent complex parameter near one and normalize its amplitude as

$$A(e, k, r) = \sqrt{\frac{akr}{2\pi}} \int_0^\infty t^{-2} e^{-rkf_e(t)} dt.$$

The common analytic Morse coordinate near $t = 1$, with uniformly bounded amplitude remainder, gives $A = 1 + O(r^{-1})$. This holds on a fixed k disk and a small complex r sector, uniformly in a real compact e interval starting at zero. Off the common neighborhood, $f_e \geq f_0$ and $\Re(rk) \geq c|r|$ give a uniform exponentially small tail; its weighted integral is finite once $|r|$ is large enough. Use slightly larger disks and sectors to apply Cauchy's inequalities:

$$\partial_k \log A = O(r^{-1}), \quad \partial_r \partial_k \log A = O(r^{-2}).$$

For real $r > 0, k = 1$, both A and its logarithmic radial derivative are real, and $A > 0$. Integration from 1 to $e^{-i\eta}$ therefore retains the factor e :

$$\Im \log A(e, e^{-i\eta}, r) = O(e/r), \quad \Im \partial_r \log A(e, e^{-i\eta}, r) = O(e/r^2).$$

The exact real-root phase is $\beta_e(r) = \pi d_e r + 3\eta/2 + \Im \log A$, and $\beta'_e(r) = \pi d_e + O(e/r^2) > 0$ for $r \geq R$. Choose R a fixed half integer. Compact convergence of $V_e(r^a)$ to $1/\Gamma(2 - r)$ gives exactly one simple root near every integer $2 \leq n < R$ and no other positive root up to R . Near each integer use the analytic implicit function theorem, or Rouché's theorem; at zero uniform convergence and the nonzero limiting value exclude a root. The known reality of the positive-parameter Wright spectrum is used here. Since $\beta_e(R) \rightarrow \pi R$, the monotone outer phase continues this indexing with exactly one simple root at every remaining integer phase. Solving its phase equation gives the bound above for large n . The finitely many smaller indices satisfy an $O(e)$ bound by analyticity, absorbed by increasing C . $\square$

Theorem H.7 (Second-order endpoint and uniqueness). *Define*

$$J(y) = \int_0^\infty \frac{v^2}{y^2 + v^2} \frac{e^{-2v}}{1 - e^{-v}} dv, \quad Y = 2\pi,$$

and $D_* = 4 \log Y + 8 - 12\pi Y + 4J(Y)$. Then

$$\begin{aligned} q_*(1+e) &= 4e + D_*e^2 + o(e^2), \\ r(1+e) &= \frac{1}{2} + 2e + \{2 \log(2\pi) - 12\pi^2 + 2J(2\pi)\}e^2 + o(e^2). \end{aligned}$$

The second coefficient in the latter formula equals $-114.740012808536192366\dots$. For all sufficiently small positive e , the controlling global contact is unique and its scaled radius lies within $O(e^{3/2})$ of 2π . The theorem gives no numerical smallness threshold.

Proof. Take $q = 4e + De^2$, with D in a fixed bounded interval. Write $p = h - ic$, where $h = \sin \omega$, $c = \cos \omega$. Lemma H.6 gives, for $0 < v \leq 1$,

$$\mathcal{L}_e(v) = \frac{e^{A_e v - 2v/d_e}}{1 - e^{-v/d_e}} + O(ev(1 + |\log v|)) = \frac{1}{v} - \frac{3}{2} + O(v + e + e^2/v).$$

Indeed the error in the n -th term is at most $Cevn^{-1}e^{-c_0nv}$. For $v \geq 1$ the corresponding error and trace have uniform exponential bounds. On each fixed positive compact y interval, the complex sum is

$$(H.10) \quad S_e(y) := \sum_{n \geq 2} e^{-pyR_{n,e}} = \frac{e^{pyA_e - 2py/d_e}}{1 - e^{-py/d_e}} + O(e(1 + |\log e|)).$$

At any fixed resonance $y = 2\pi m$, $cyR_{n,e} - 2\pi mn = O(e + e^2n)$. Hence

$$0 \leq \mathcal{L}_e(hy) - \Re S_e(y) \leq C \sum e^{-c_0en} (e^2 + e^4 n^2) = O(e).$$

In particular $2\Re S_e(y) = 2/(hy) - 3 + o(1)$ there. Away from the resonances, (H.10) is uniformly bounded.

Let $P_e(y) = \sum B_{a,\xi}(R_{n,e}y)$. Subtract its leading integral term before summing. The exact first Rayleigh identity gives

$$(H.11) \quad P_e(y) = \frac{\sin \xi}{\pi e y^a} - \frac{J(y)}{\pi y} + o(1),$$

uniformly on fixed positive compact intervals. To check convergence, subtract one from the reciprocal denominator: its absolute value is bounded by $C(|\cos \xi|z + z^2)$. The resulting second and third root sums are uniformly summable, by Lemma H.6. At $a = 1, \xi = \pi/2$, Tonelli's theorem identifies the finite part:

$$\sum_{n \geq 2} \left[B_{1,\pi/2}(ny) - \frac{1}{\pi ny} \right] = -\frac{1}{\pi y} \int_0^\infty \frac{v^2}{y^2 + v^2} \frac{e^{-2v}}{1 - e^{-v}} dv.$$

The same integral bounds give $P_e''(y) = O(e^{-1})$ on those compact intervals.

Now $\sin \xi = 1 + O(e^2)$ and $h = 2\pi e + \pi(D/2 - 4)e^2 + O(e^3)$. At a fixed resonance the entire kernel therefore tends to

$$(H.12) \quad C_D(y) = 3 + \frac{D/4 - 2 - \log y - J(y)}{\pi y}.$$

In particular $C_D(Y) = 0$ exactly when $D = D_*$.

We next control *all* $y \geq 3\pi$, so that the first resonance cannot be mistaken for a local candidate. Since $\cos \xi < 0$ for small e , the third-moment bound gives

$$P_e(y) \geq \frac{\sin \xi}{\pi e y^a} - 2y^{-3a}, \quad y \geq 3\pi.$$

Here the scaled third moment tends to $\zeta(3) - 1$, so the constant two is a uniform valid enlargement. For $v = hy \leq \delta$, the heat estimate and $e^{-e \log y} \geq 1 - e \log y$ imply

$$K(c_a y) \geq 3 + \frac{D/4 - 2 - \log y}{\pi y} - 2y^{-3a} - O(\delta + e).$$

Uniformly for $-230 \leq D \leq -210$, the right side before its error exceeds 0.9123: the worst value is bounded at 3π , since $(2 - D/4 + \log y)/y$ decreases there. Choose δ and then e small. On $\delta \leq v \leq V$, the lower positive term tends uniformly to $2/v$, while the heat trace tends to $e^{-2v}/(1 - e^{-v}) < 1/v$. On $v \geq V$, the uniform exponential trace is dominated by the algebraic positive term, since h^a/e has a positive lower bound. Taking V large proves positivity on the entire exterior.

The inequality $0 < J(Y) < 2(\zeta(3) - 1)/Y^2$ gives $-230 < D_* < -210$. If $D < D_*$, the value at Y in (H.12) is strictly negative. If $D > D_*$, that profile is positive on a neighborhood of Y , and the bound $K \geq P_e - 2\mathcal{L}_e(hy)$ converges to it uniformly there. The rest of $[1, 3\pi]$ is nonresonant, so its bounded oscillatory sum is dominated by $P_e \sim 1/(\pi ey)$. The uniform unit-origin estimate described below covers $0 < y \leq 1$. The whole exterior has just been handled. These two strict tests and the ID interval theorem squeeze q_* to the stated second-order expansion.

It remains to prove uniqueness, rather than infer it from the leading profile. For bounded D near D_* , all possible nonpositive values lie near Y . Put $w = py/d_e + 2\pi i = u - iv$ and $A = e^{pyA_e}$. Then $u \asymp e$, $\Re A = 1 + O(e^2)$, $\Im A = O(e)$, and

$$\Re S_e(y) = \Re \frac{A}{u - iv} - \frac{3}{2} + O(|w|) + o(1), \quad P_e(y) = 2/u + O(1).$$

If $K \leq 0$, then $2/u - 2\Re[A/(u - iv)] \leq C$. The left side is

$$\frac{2v^2}{u(u^2 + v^2)} + \frac{2(1 - \Re A)u + 2\Im A v}{u^2 + v^2}.$$

Young's inequality bounds it below by $v^2/[u(u^2 + v^2)] - Ce$. Consequently $v^2 \leq Cu(u^2 + v^2)$, and for small e , $v^2 \leq 2Cu^3 = O(e^3)$.

Twice differentiating the root-sum comparison produces error $O(e^{-1})$: its n -th error is at most $Ce(1 + n)e^{-c_0 en}$, by $|\delta_{n,e}| \leq Ce/n$. The pole part of the negative residue has second derivative

$$-4\Re \frac{A(p/d_e)^2}{(u - iv)^3} + O(e^{-1}).$$

Since $A(p/d_e)^2 = -1 + O(e)$ and $v/u = O(\sqrt{e})$, it is $4u^{-3}(1 + O(\sqrt{e})) + O(e^{-1}) > c/e^3$. Adding $P_e'' = O(e^{-1})$ preserves strict convexity on the whole possible-zero interval. The global contact theorem supplies one zero minimum at the boundary, and strict convexity permits only one. No other resonance remains unexamined.

Finally the directed integral check gives $J(2\pi) = 0.00974293585871004644\dots$. It uses unit integration panels, the removable origin integrand, and the explicit tail $e^{-2U}/[2(1 - e^{-U})]$, $U = 80$. The file `second_order_id_coefficient_fast_checks.json` records the nonvacuous enclosure and the exterior constant. These finite checks supplement the uniform analytic proof above. $\square$

H.3.1. A unit-origin estimate and finite slices. For $1 < a \leq 11/10$, $0 < q \leq 3/20$, $0 < y \leq 1$, the strengthened uniform origin certificate gives

$$(H.13) \quad K_{a,\rho}(c_a y) > 3/2.$$

Here is the mass estimate needed in addition to the convergent germ. If N is Poisson with mean $1/e$, then $V_e(-1) = \mathbb{E}[\Gamma(2 + eN)^{-1}] \geq 1/12$: Markov's inequality gives $\Pr(eN \leq 2) \geq 1/2$, and $\Gamma \leq 6$ on $[2, 4]$. Thus the scaled Laplace exponent satisfies $\Psi_a(1) < 7/2$, and $\int_2^\infty j_a(u) du/u < 49/12$. The exact 128-term germ with its remainder certifies $j_a(u) \leq 3/2$ on $[0, 2]$ and $\Re j_a(ye^{-i\delta}) > 4/5$ on the stated parameter range. There $\pi/3 \leq \delta \leq \pi/2$. Split the Cauchy integral at two. Its local part is at most $3/2$; its tail is at most $49y/(12\pi)$, while its pole is at least $5\sqrt{3}/(\pi y)$. Hence

$$K(c_a y) > \frac{1}{10} + \frac{5\sqrt{3}/y - 49y/12}{\pi} > \frac{3}{2}.$$

The 1280-leaf independent verifier recomputes both germ covers.

The normalized exact rotated Laplace integral, including its two infinite tails, now also provides efficient large-root signs. At $a = 1.001$, 1200 distinct root brackets were independently rechecked; their twelfth Rayleigh remainder excludes every omitted physical radius below 807375. No completeness of a numerical scan is assumed. The following whole-radius ID slices were independently verified, including (H.13) where applicable:

a	strict enclosure for $r(a)$	finite kernel leaves
1.001	(0.5018943724625374, 0.5018943824425575)	76
1.01	(0.5135954445544554, 0.5135955425742575)	38
1.05	(0.5361955880952380, 0.5361956785714286)	42

Their global contacts are unique. The common checker is `certify_id_point_contact.py`. A separate finite-spectrum search, `compute_near_one_id_boundary.py`, obtained a q -bracket of width below 10^{-6} at $a = 1.001$ with 73 positive leaves and a negative witness, without selecting any local contact. Its fixed resource limits are explicit; it returns an unfinished bracket rather than a false classification if the supplied spectrum is insufficient.

The analogous global GGC slices, with zero sector winding, are

a	strict enclosure for $g(a)$	global leaves
.90	(.00006240435, .00006240437)	5122
.95	(.00659598240, .00659598241)	2161
.99	(.0130495099, .0130495100)	1431

The .90 and .95 certificates additionally have 2057 and 8 local derivative leaves. A generic .97 run independently verified a width below .001, without a uniqueness assumption.

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