

Mobility Edge for Sparse Complex Gaussian Matrices

Patrick Lopatto

AUTHOR'S NOTE

I have been interested in the problem considered in this manuscript since the completion of [7]. It seemed clear that the transfer operator strategy in [7] was the right way to approach the problem, but also that there were significant model-specific challenges in implementing it in the context of sparse random matrices. For various reasons, my attention became focused on other things, but I always intended to return to this question.

Shortly after the release of GPT-5.6 Sol, I became curious about whether the model could complete the details of the argument given an appropriate high-level sketch of the proof, and indeed it could (after exchanging many messages). I had hoped to produce a polished, human-written version of this manuscript, but given the current pace of mathematical progress and recent improvements in LLM writing abilities, it does not feel appropriate to delay the dissemination of these results for the rather long time that would take. The current document, while not perfect, should be accessible to those with the relevant background. Further, given the implications these results have for certain predictions in the physics literature, as described in the introductory material below, I thought it was best to release them now.

With the exception of this note and small tweaks to the introduction to add some recent references, the entire document is machine-written. I have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited. I welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

P.L. was partially supported by NSF grant DMS-2450004.

ABSTRACT

We consider the sparse Hermitian random matrix supported on the Erdős–Rényi graph $G(n, p/n)$, whose nonzero entries are independent complex Gaussian variables of variance $1/p$, together with its local limit, a weighted Poisson–Galton–Watson tree. We prove that for every fixed and sufficiently large p , the limiting operator on the tree conditioned on survival has exactly one positive mobility edge $E_c(p)$: almost surely, its spectrum is purely absolutely continuous on $(-E_c(p), 0) \cup (0, E_c(p))$ and pure point outside $[-E_c(p), E_c(p)]$. Moreover, $E_c(p)/\sqrt{p}$ converges to $\sqrt{\pi}/2$ as p tends to infinity. Thus the mobility edge lies far in the tail of the spectrum, rather than near the edge 2 of the limiting semicircle law, as had been predicted in the physics literature; the spectrum is absolutely continuous in a region whose spectral mass tends to zero as $p \rightarrow \infty$, through a resonance mechanism discovered by Aizenman and Warzel. The mobility edge is the unique zero of a free energy associated with a distinguished real solution of the cavity equation, and this free energy crosses zero transversally. We also show that the giant component of the finite graph inherits the dichotomy, in the sense that the local spectral masses in small energy windows are uniformly integrable over vertices below $E_c(p)$ and concentrate on a vanishing fraction of vertices above it. The proofs combine a transfer-operator description of the free energy, the Simon–Wolff criterion applied to the squared moduli of the Gaussian weights, a resonance argument, and an analysis of the cavity equation at large degree.

Contents

1	Introduction	1
1.1	Background	1
1.2	Model and main results.	4
1.3	Discussion	8
1.4	Notation and conventions.	10
1.5	Organization	11
2	Outline of the proof	12
2.1	Cavity variables and the two alternatives	12
2.2	The free energy.	13
2.3	Negative free energy implies localization	14
2.4	Positive free energy implies delocalization	15
2.5	The free energy at large degree	16
2.6	Assembly and the finite graph	18
PART I	The limiting tree and the cavity equation	19
3	The weighted tree and its operator	20
3.1	The finite model and the marked tree	21
3.2	Finite removals	22
3.3	Essential self-adjointness	25
3.4	Resolvents and the cavity recursion	32
3.5	Gauge and parity symmetries.	37
3.6	The bipartite factorization.	39
3.7	Compactly supported eigenvectors	45
4	Spine decompositions and fresh squared moduli	49
4.1	The Ulam–Harris construction	49
4.2	The Mecke formula and survival	52
4.3	Spine decompositions.	55
4.4	Conditioned endpoint laws	63
4.5	Fresh squared moduli	64
4.6	Selections at a vertex and the backbone	71

5	Real solutions of the cavity equation	77
5.1	Real solutions and their normalizations	77
5.2	A priori properties of real solutions	80
5.3	Regular energies and boundary values	84
5.4	Recursive tree processes	92
5.5	Fractional moments at positive imaginary part	95
5.6	Fixed-depth convergence	97
6	The transfer operator and the free energy	108
6.1	Transfer matrices and projective coordinates.	108
6.2	The Perron–Frobenius eigenvalue of the transfer operator	112
6.3	The free energy.	125
6.4	Products of three independent factors.	130
6.5	Path moments	136
PART II	Spectral type from the sign of the free energy	142
7	Rank-one perturbations of the squared row norms	143
7.1	Borel transforms and rank-one perturbations	144
7.2	Measurability of resolvent entries	150
7.3	Row deletion and the square blocks.	150
7.4	The pure point criterion.	154
7.5	The absolutely continuous criterion	156
8	Negative free energy implies localization	160
8.1	The formal inverse column	161
8.2	The many-to-one identity	163
8.3	Summability of the forward columns	165
8.4	Rerooted columns	166
8.5	Proofs of the main results.	169
9	Resonances along paths	172
9.1	Transfer matrices along a path	172
9.2	Endpoint estimates	177
9.3	Blocks and regular growth	181
9.4	Averaging over fresh factors	185
9.5	Passage to the real axis	187
10	Positive free energy implies delocalization	189
10.1	Outline of the proof	189
10.2	Amplification at the root	192
10.3	Resonant vertices	198
10.4	The first moment	201

10.5	The second moment	204
10.6	Proof of the delocalization criterion	211
PART III	The free energy at large degree	213
11	The principal solution	214
11.1	The law map	215
11.2	Estimates for compound Poisson sums	218
11.3	The class $\mathfrak{S}_{p,\theta}$ and its invariance	221
11.4	The mixed metric	224
11.5	Column estimates for the linearized map	228
11.6	The principal fixed point	232
11.7	Small parameter and the half-line	237
11.8	Real solutions and the real boundary law	239
12	Analyticity of the principal solution and bounds on its derivatives	245
12.1	The scaling $g = 1/E$	245
12.2	An analytic Banach space for the law map	252
12.3	Invertibility of the linearization	257
12.4	Analyticity of the principal law	261
12.5	Analyticity of the free energy at $s = 1$	264
12.6	Quantitative derivative bounds	267
13	Asymptotics of the free energy	279
13.1	The normalized transfer operator	280
13.2	Inputs from the principal solution	283
13.3	Elementary estimates	285
13.4	The lower bound	289
13.5	Six regions and the block matrix	290
13.6	Block estimates	292
13.7	Balancing a nonnegative matrix	297
13.8	The upper bound	299
13.9	Proof of the leading asymptotics	301
13.10	The free energy at large energies	301
14	Uniqueness of the critical energy	304
14.1	Main results and strategy	304
14.2	Elementary lemmas	306
14.3	A positive rank-two renewal criterion	309
14.4	Parameters and inputs	312
14.5	The L^2 version of the transfer operator	314
14.6	A positive decomposition of the square	320
14.7	Cutoff bounds and a polynomial inverse	327

14.8	The renewal matrix	330
14.9	Proofs of the main results.	334
15	The free energy at small energies	336
15.1	The critical transfer operator with two scalings	336
15.2	Proof of the lower bound	337
PART IV	Proofs of the main results	340
16	Proofs of the main theorems	341
16.1	Choice of constants	341
16.2	Proof of Theorem 1.1	342
16.3	Proof of Theorem 1.2	343
17	The giant component	344
17.1	Setting and notation	344
17.2	Rooted marked graphs and resolvents.	345
17.3	Local convergence on the giant component	353
17.4	The density of states	359
17.5	Eigenvalue counts and the transfer of phases	365
	References	371

Introduction

1.1 Background

Disordered quantum systems often exhibit one of two contrasting behaviors. In a *localized* phase, eigenfunctions are concentrated in bounded regions of space and the system is an insulator. In a *delocalized* phase, eigenfunctions are spread out and transport is possible. Anderson proposed in [26] that sufficiently strong disorder localizes all eigenfunctions of a tight-binding Hamiltonian. At weaker disorder, localized and delocalized states may coexist at different energies, separated by *mobility edges*, a notion introduced by Mott [103]; see also [65] and the review [71]. Localization is by now well understood mathematically in many regimes, for instance through the fractional moment method of Aizenman and Molchanov [8, 9]. We refer to [13] for an account. Delocalization is much harder to prove, and tree graphs are the main setting in which both phases, and the transition between them, are accessible.

Trees. The Anderson model on the Bethe lattice was studied by Abou-Chacra, Anderson, and Thouless in [2], who wrote self-consistent equations for the local Green function that are exact on the tree. They predicted that the localized phase corresponds to a real solution of these equations, and they located its stability limit by linearizing the equations around this solution. The case of energies near the band edges was treated in [1]. Kunz and Souillard obtained exact results on the transition between extended and localized states on the Bethe lattice [93], and Miller and Derrida studied localization on trees through a weak-disorder expansion [99]. This linear stability criterion has since been reformulated in several ways: through the free energy of a directed polymer on the tree [40, 62, 89, 102], through supersymmetric field theory [66, 100, 136], and through the largest eigenvalue of an integral operator acting on real Green functions [43, 109, 127]. In these formulations the criterion is evaluated at the symmetric point of a symmetry of the linearized kernel, which goes back to [2]; see [109] for a discussion. A pedagogical account of the cavity approach, including its analysis at large connectivity, is given in [111]. Biroli, Semerjian, and Tarzia computed the mobility edge and the density of states of the Anderson model on Bethe lattices with the cavity method in [43]. More recently, the delocalized phase on trees and random regular graphs has been studied in detail, in particular the question of its ergodicity [39, 40, 44, 45, 59, 89, 130, 131]. The critical properties of the transition and the localized phase near it were studied in [114, 128].

On the mathematical side, Klein proved that the Anderson model on the Bethe lattice has absolutely continuous spectrum at weak disorder [87, 88]; other proofs and extensions were given in [14, 63, 72, 80, 84]. Aizenman and Warzel developed a resonance mechanism for delocalization in [11, 12]. They proved that the spectrum is absolutely continuous wherever a certain free-energy function, denoted $\varphi(1; E)$ in [12] and describing the decay of fractional moments of the Green function along paths, exceeds the logarithm of the inverse branching number. As a consequence, at weak disorder the absolutely continuous spectrum extends well beyond the band of the hopping operator, into a Lifshitz-tail regime [95] of very small density of states (see [30] for Lifshitz tails on the Bethe lattice). For bounded potentials there is no mobility edge of the usual kind at weak disorder [10]; see also the survey [135]. Bapst analyzed this free energy at large connectivity in [29] and confirmed the prediction of [2] for the critical disorder. Building on these works, Aggarwal and Lopatto proved in [4] that mobility edges exist for the Anderson model on the Bethe lattice with unbounded potential at large degree, near the locations predicted in [2]. Liu and Lopatto transferred this result to random regular graphs in [96].

For random trees, much less is known. The spectral measure of the unweighted adjacency operator of the Poisson–Galton–Watson tree, the local limit of the Erdős–Rényi graph, has a dense set of atoms [36, 119], and its continuous part was studied in [53, 57, 120]. Absolutely continuous spectrum for multi-type Galton–Watson trees close to trees of finite cone type was established by Keller [83]. Arras and Bordenave proved in [28] that, conditionally on survival, the adjacency operator of a Poisson–Galton–Watson tree with large mean degree has a nontrivial absolutely continuous spectral component, which carries most of the spectral mass.

Sparse random matrices. The spectra of sparse random matrices, with a bounded mean number of nonzero entries per row, were first studied in physics in [115, 116]. Fyodorov and Mirlin showed that such matrices undergo a localization transition, and derived equations for a mobility edge and for the inverse participation ratio in [73, 74, 101]; numerical studies include [69, 70]. Biroli and Monasson proposed in [38] that localized states in the tails of the spectra of random graphs are created by rare local defects, such as vertices of abnormally large degree; see also [121], and [132] for the analogous picture for non-Hermitian sparse matrices. In the limit of high connectivity, Silva and Metz solved the resolvent equations of sparse random graphs analytically and found that all eigenvectors are extended in this limit [122]. The cavity method was developed for sparse matrices in [91, 117, 126], and mobility edges in the tails of the spectra of sparse matrices were computed numerically in [91, 92, 98, 125]. Tarzia studied the tails of the spectrum of Erdős–Rényi graphs in the critical regime in [129]. Most relevant for us, Cugliandolo, Schehr, Tarzia, and Venturelli studied in [58] the giant component of Erdős–Rényi graphs with real Gaussian edge weights, which is the real symmetric analogue of the model considered here. They found a delocalized but weakly multifractal phase at small p ,

separated by a mobility edge from localized states in the tails. The multifractality of extended states in weighted random networks was also studied by Baron and Rogers in [31].

The mathematical literature on sparse Erdős–Rényi graphs is extensive when the mean degree grows with the number of vertices; see for example [33, 34, 67, 76]. In the critical regime where the mean degree is of order $\log n$, Alt, Ducatez, and Knowles proved in a series of works [18–22] that eigenvectors are delocalized in the middle of the spectrum and localized near high-degree vertices at the spectral edges, with a mobility edge between the two regimes. See also the recent work [78]. For bounded mean degree, the global eigenvalue distribution of weighted random graphs was studied in [50, 85], and the extreme eigenvalues and their eigenvectors in [55, 79]. For the model (1.1) itself, Afanasiev showed in [3], using supersymmetry, that for $p > 2$ the second correlation function of characteristic polynomials has a GUE-type limit inside an energy interval and factorizes outside it.

Heavy-tailed matrices. A mobility edge was predicted by Cizeau and Bouchaud for Lévy matrices, whose entries are in the domain of attraction of an α -stable law with $\alpha \in (0, 1)$ [56]. Biroli, Bouchaud, and Potters showed that the largest eigenvalues of heavy-tailed matrices are governed by their largest entries [42]. Tarquini, Biroli, and Tarzia computed its location in [127], through the stability of the real solution of the cavity equations, and Biroli and Tarzia used cavity equations to determine the phase diagram of a Lévy version of the Rosenzweig–Porter ensemble [41]. The eigenvalues and eigenvectors of Lévy matrices were studied rigorously in [5, 6, 32, 48, 49]. Aggarwal, Bordenave, and Lopatto proved in [7] that Lévy matrices have a mobility edge, in the sense of the imaginary part of the resolvent of their local limit and of eigenvector participation ratios, at the location predicted in [127], and that it is unique for α close to 0 or 1.

Recent advances. Several recent preprints produced by an internal OpenAI large language model announce results on longstanding problems in localization and spectral universality. For the Anderson model with independent uniform site potentials, these include purely absolutely continuous spectrum on a nonempty energy interval at sufficiently weak disorder on $\mathbb{Z}^d$ for every fixed $d \geq 3$ [104], and pure-point spectrum throughout the spectrum at every positive disorder strength on $\mathbb{Z}^2$ [107]. For random regular graphs of every fixed degree $d \geq 3$, further preprints announce fixed-energy GOE bulk universality for the adjacency matrix [106] and for the Anderson model with sufficiently weak independent uniform diagonal disorder [105]. The fixed-energy GOE statistics result for the adjacency matrix was obtained independently and simultaneously in [77].

This paper. In this article we establish that the sparse Hermitian matrix with complex Gaussian weights on $G(n, p/n)$ has exactly one positive mobility edge in its local limit,

for every fixed sufficiently large p , and we determine its location to leading order. Its location, of order $\sqrt{p}$, is far in the tail of the spectrum, which contradicts the large- p behavior predicted in the physics literature; we discuss this in Section 1.3. We also derive consequences for the giant component of the finite graph.

1.2 Model and main results

The model

Fix $p > 1$. For each integer $n > p$, let G_n be an Erdős–Rényi random graph on the vertex set $\{1, \dots, n\}$ in which each edge is present independently with probability p/n . Conditional on G_n , let $(w_{ij})_{1 \leq i < j \leq n}$ be independent standard complex Gaussian random variables, so that each w_{ij} has density $\pi^{-1}e^{-|z|^2}$ with respect to Lebesgue measure on $\mathbb{C}$. We consider the $n \times n$ Hermitian matrix H_n defined by

$$H_n(i, j) = \frac{\mathbf{1}_{\{ij \in E(G_n)\}}}{\sqrt{p}} w_{ij}, \quad H_n(j, i) = \overline{H_n(i, j)}, \quad H_n(i, i) = 0, \quad (1.1)$$

for $1 \leq i < j \leq n$. Throughout the paper, p is fixed while n tends to infinity, and statements about large p concern the resulting limiting objects.

Since $p > 1$, the graph G_n has with high probability a unique giant component, which contains a positive fraction of the vertices, together with linearly many finite components. We write gc for the vertex set of the giant component, that is, of the largest connected component of G_n (if there are several, the one containing the smallest vertex; see Definition 3.1), $N_n = |\text{gc}|$ for its size, and H_n^{gc} for the restriction of H_n to gc . Since the finite components produce a positive density of localized eigenvectors for trivial reasons, the natural object of study is the giant component.

The local weak limit of H_n is described by a random tree. Let $\mathcal{T}$ be a Poisson–Galton–Watson tree with mean offspring p and root o , and attach to each edge e an independent weight $h_e = w_e/\sqrt{p}$, where the w_e are standard complex Gaussian variables; we set $h_{vu} = \overline{h_{uv}}$. Let $\mathcal{H}$ be the weighted adjacency operator of $\mathcal{T}$,

$$(\mathcal{H}f)(v) = \sum_{u \sim v} h_{vu} f(u).$$

Although the degrees in $\mathcal{T}$ are unbounded, $\mathcal{H}$ is almost surely essentially self-adjoint on finitely supported vectors (Theorem 3.10), and we use the same notation for its closure. A uniformly chosen vertex of G_n converges locally to the root of $(\mathcal{T}, (h_e))$, and a uniformly chosen vertex of the giant component converges locally to the root of the same tree conditioned on the event $\mathcal{S}$ that it is infinite. We write $\mathbb{P}$ for the law of the weighted tree, $\varrho_p = \mathbb{P}(\mathcal{S})$ for its survival probability, and $\mathbb{P}^\infty = \mathbb{P}(\cdot \mid \mathcal{S})$ for the survival-conditioned law.

For an interval I , we say that $\mathcal{H}$ has *purely absolutely continuous spectrum* (respectively *pure point spectrum*) in I if, for every vector f , the restriction to I of the spectral measure of f is absolutely continuous with respect to Lebesgue measure (respectively a countable sum of point masses). Intervals of the first kind are regions of delocalization, and intervals of the second kind are regions of localization. An energy separating a region of one kind from a region of the other kind is a *mobility edge*.

Main results

Set

$$\mu_* = \mathbb{E}|w| = \frac{\sqrt{\pi}}{2}, \quad w \sim \mathcal{N}_{\mathbb{C}}(0, 1). \quad (1.2)$$

Theorem 1.1. *There exists $p_0 < \infty$ such that for every $p \geq p_0$ there exists $E_c(p) > 0$ with the following properties. Almost surely with respect to $\mathbb{P}^\infty$, the operator $\mathcal{H}$ has purely absolutely continuous spectrum in*

$$(-E_c(p), 0) \cup (0, E_c(p))$$

and pure point spectrum in

$$(-\infty, -E_c(p)) \cup (E_c(p), \infty).$$

Moreover,

$$\lim_{p \rightarrow \infty} \frac{E_c(p)}{\sqrt{p}} = \mu_* = \frac{\sqrt{\pi}}{2}. \quad (1.3)$$

Finally, $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has no eigenvector with finite support and nonzero eigenvalue, and it has infinitely many linearly independent eigenvectors with finite support and eigenvalue zero.

Thus, for every fixed large p , the positive half-line contains exactly one mobility edge. The theorem makes no statement about the spectral type at the two energies $\pm E_c(p)$. The energy zero is exceptional: it is an eigenvalue of infinite multiplicity, created by pairs of leaves attached to a common vertex (localized states of this kind in diluted models go back to Kirkpatrick and Eggarter [86]), and Theorem 1.1 does not describe the spectral measure at zero.

The mobility edge lies far in the tail of the spectrum. As $p \rightarrow \infty$, the limiting empirical spectral measure of H_n converges to the semicircle law on $[-2, 2]$, as a computation with the moment method shows, while $E_c(p)$ is of order $\sqrt{p}$. Eigenvalues near $E_c(p)$ therefore come from rare local configurations with unusually large edge weights. The constant μ_* arises from a first-moment computation, which we explain in Section 2.5: at the energy $E = \xi\sqrt{p}$, a typical path of length L in the tree contributes a product of about L factors $|w_e|/(\xi p)$ to the resolvent, and there are about p^L such paths. Their total contribution therefore grows or decays like $(\mu_*/\xi)^L$.

Our second result identifies $E_c(p)$ as the unique zero of a function defined through the cavity equation, which is a recursive distributional equation for the diagonal resolvent

entry at the root. Let $\Gamma = \langle \delta_o, (\mathcal{H} - z)^{-1} \delta_o \rangle$ for z in the upper half-plane. Its law satisfies the cavity equation

$$\Gamma \stackrel{d}{=} - \left(z + \frac{1}{p} \sum_{j=1}^N Y_j \Gamma_j \right)^{-1}, \quad N \sim \text{Pois}(p), \quad Y_j \sim \text{Exp}(1),$$

where the variables on the right are independent and the Γ_j are distributed as Γ . For $E > 2$, let $g(E) = (E - \sqrt{E^2 - 4})/2$ and $\theta = g(E)^2$. For a real energy E , the variable $B = -g(E)/\Gamma$ formally satisfies

$$B \stackrel{d}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j}. \quad (1.4)$$

For every real solution of this equation we define in Section 6 a function $\varphi(s)$, $s \in (0, 2)$, as the logarithm of the Perron–Frobenius eigenvalue $\lambda(s)$ of a positive transfer operator on the real projective line. We call it the *free energy*, as in [4, 12]. For $s \in (0, 1)$, $\varphi(s)$ is the exponential growth rate of the expected generation sums of the s -th powers of the formal path products. Its limiting value $\varphi(1) = \lim_{s \uparrow 1} \varphi(s)$ decides the spectral type.

Theorem 1.2. *Fix $\xi_0 \in (0, \mu_*)$. There exists $p_1(\xi_0) \geq p_0$ such that for every $p \geq p_1(\xi_0)$ the following holds.*

- (i) *For every $E \geq \xi_0 \sqrt{p}$, the iterates of the map defined by the right-hand side of (1.4), started from the point mass at one, converge weakly to a solution $\nu_{p,E}$ of (1.4).*
- (ii) *Let $\varphi_E(1)$ be the free energy of $\nu_{p,E}$ at $s = 1$. The function $E \mapsto \varphi_E(1)$ is continuous on $[\xi_0 \sqrt{p}, \infty)$. It is positive on $[\xi_0 \sqrt{p}, E_c(p))$, vanishes at $E_c(p)$, and is negative on $(E_c(p), \infty)$, where $E_c(p)$ is the energy of Theorem 1.1. It is real-analytic in a neighborhood of $E_c(p)$, and its derivative at $E_c(p)$ is negative.*
- (iii) *For every compact set $\Xi \subset [\xi_0, \infty)$,*

$$\lim_{p \rightarrow \infty} \sup_{\xi \in \Xi} \left| \exp \varphi_{\xi \sqrt{p}}(1) - \frac{\mu_*}{\xi} \right| = 0.$$

Theorem 1.2 is an instance of the general principle, going back to Abou-Chacra, Anderson, and Thouless [2], that the mobility edge on a tree is located by the linear stability of the real solutions of the cavity equation. Its content is that, at large p , the free energy at $s = 1$ of the relevant solution changes sign exactly once, and transversally. The asymptotic statement (iii) gives (1.3).

The giant component

We now state the consequences for the finite graph. Since the spectral type of a finite matrix is trivial, we describe localization through the distribution of spectral weight over vertices in small energy windows. For an interval $I \subset \mathbb{R}$ and a vertex $v \in \text{gc}$, let

$$\mu_{n,v}(I) = \langle \delta_v, \mathbf{1}_I(H_n^{\text{gc}}) \delta_v \rangle, \quad \bar{\mu}_n(I) = \frac{1}{N_n} \text{Tr } \mathbf{1}_I(H_n^{\text{gc}}),$$

so that $\bar{\mu}_n(I)$ is the average of $\mu_{n,v}(I)$ over $v \in \text{gc}$. When $\bar{\mu}_n(I) > 0$, let $P_{I,n}(v) = \mu_{n,v}(I) / \bar{\mu}_n(I)$ be the normalized local mass, and for $R > 0$ define the deficit

$$\mathcal{D}_{I,n}(R) = 1 - \frac{1}{N_n} \sum_{v \in \text{gc}} \min\{P_{I,n}(v), R\}.$$

Set $\mathcal{D}_{I,n}(R) = 1$ when $\bar{\mu}_n(I) = 0$. When $\bar{\mu}_n(I) > 0$, the deficit is the excess normalized mass above the truncation level R , averaged over the vertices.

Definition 1.3. Let $I_h(E) = (E - h, E + h]$. The giant component is *uniformly integrable delocalized* at E if, for every $\delta > 0$,

$$\lim_{R \rightarrow \infty} \limsup_{h \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > \delta) = 0.$$

It is *spectrally condensed* at E if, for every $R < \infty$ and every $\delta > 0$,

$$\lim_{h \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > 1 - \delta) = 1.$$

In words, the giant component is uniformly integrable delocalized at E if the normalized local masses in small windows around E are uniformly integrable over the vertices, and it is spectrally condensed at E if almost all of the normalized mass is carried by a vanishing fraction of the vertices. In both definitions the limit $n \rightarrow \infty$ is taken before the window width h tends to zero. These notions concern averaged spectral projections, and they do not describe individual eigenvectors.

Theorem 1.4. *The following statements hold.*

- (i) *For every $p > 1$ and every compact set $J \subset \mathbb{R} \setminus \{0\}$, there exist constants $0 < c_{p,J} \leq C_{p,J} < \infty$ such that, for every fixed interval $I \subset J$ of positive length,*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(c_{p,J} N_n |I| \leq \text{Tr } \mathbf{1}_I(H_n^{\text{gc}}) \leq C_{p,J} N_n |I|\right) = 1.$$

- (ii) *Let $p \geq p_0$, where p_0 is as in Theorem 1.1. The giant component is uniformly integrable delocalized at Lebesgue-almost every E with $0 < |E| < E_c(p)$, and spectrally condensed at Lebesgue-almost every E with $|E| > E_c(p)$.*

1.3 Discussion

Comparison with the physics literature. Theorem 1.1 shows that, for large p and apart from the eigenvalue zero, the localized states of the limiting operator are confined to energies $|E| > E_c(p) \approx \mu_*\sqrt{p}$, far beyond the semicircle bulk $[-2, 2]$. This disagrees with the large- p behavior described in the physics literature on this model. For the real symmetric version of (1.1) restricted to the giant component, Cugliandolo, Schehr, Tarzia, and Venturelli state that the mobility edge tends to the edge 2 of the semicircle law as $p \rightarrow \infty$ [58, Section V]. This statement is an extrapolation: their numerical study covers $p \in [1.6, 5]$, and the mobility edge in their phase diagram is drawn only for $p \lesssim 2.5$ [58, Section V and Fig. 1]. Similarly, Baron and Rogers describe the eigenvalues of weighted Erdős–Rényi graphs outside the continuous part of the spectrum, which ends at $2 + O(p^{-1})$, as produced by well-connected vertices or heavily weighted edges, with eigenvectors exponentially localized around these defects [31, Section VIII]. This is the picture of Biroli and Monasson [38].

Theorem 1.1 shows that this picture is not correct at large p on the limiting tree. At energies $2 < |E| < E_c(p)$, rare local configurations with large edge weights do produce spectral weight, but they are not isolated from each other: their number grows exponentially with the distance, and resonances between distant configurations produce absolutely continuous spectrum. This is the mechanism discovered by Aizenman and Warzel for the Anderson model on regular trees, where extended states appear through disorder-enabled resonances well beyond the band of the hopping operator [11, 12]. Since the spectral mass of these energies tends to zero as $p \rightarrow \infty$, this delocalization is difficult to observe in numerical studies of moderate system size. The available numerical values are nonetheless consistent with a mobility edge that grows with p : for real Gaussian weights of variance $1/p$ on the whole graph, Kühn found the localization threshold at approximately 2.295 for $p = 2$ and 2.581 for $p = 4$ [91]. The first-moment heuristic described after Theorem 1.1 also applies to real Gaussian weights, and predicts a mobility edge near $\mathbb{E}|w|\sqrt{p} = \sqrt{2p/\pi}$ in that case; we do not prove this. At energies of order $\sqrt{p}$, the relevant real solution of the cavity equation is concentrated near $-1/E$, which explains why the leading-order location of the mobility edge is given by Anderson’s upper limit approximation, as reformulated in [2]; see also the forward approximation of [110]. This is in contrast with the critical disorder of the Anderson model on the Bethe lattice at large connectivity, where the upper limit approximation misses the correct constant because of correlations along paths [109].

Relation to previous rigorous results. Our proof follows the general strategy of [4, 7, 12]: the spectral type at an energy E is decided by the sign of an exponential growth rate of fractional moments of the Green function along paths, which is computed through the Perron–Frobenius eigenvalue of a transfer operator. Aizenman and Warzel normalize this

rate per vertex, so that the threshold for their free energy $\varphi(1; E)$ is minus the logarithm of the branching number; as in [4, 7], we sum over generations, so that the threshold for our free energy $\varphi(1)$ is zero. There are four main differences.

First, the model has no random potential. In [4, 12], the random potential at a vertex is the rank-one parameter in the Simon–Wolff criterion [124] and in spectral averaging. Here the randomness is entirely in the hopping amplitudes. We use instead that the tree is bipartite, so that $\mathcal{H}^2$ is a direct sum of two operators of the form A^*A , and that A^*A is a sum of rank-one projections onto the normalized rows of A , weighted by the squared row norms. For complex Gaussian weights these squared norms have Gamma densities and are independent of the row directions, so they can play the role of the random potential.

Second, our localization argument does not use a continuity argument in the energy. In [4, 7], the transfer operator describes the fractional moments only if the boundary values of the resolvent are already known to be real, and localization is propagated from very large energies by a continuity argument in the energy. We show instead that any real solution of the cavity equation with negative free energy at $s = 1$ produces, on an enlarged probability space, a square-summable vector F with $(\mathcal{H} - E)F = \delta_o$. This gives the input of the Simon–Wolff criterion directly, at every energy where the free energy at $s = 1$ is negative, without identifying the actual boundary values of the resolvent.

Third, the delocalization criterion applies to arbitrary real boundary laws, not only to the solution of the cavity equation that we construct. This allows us to treat all small positive energies, where we do not identify the real solutions of the cavity equation, by proving that every real solution there has positive free energy at $s = 1$.

Fourth, we prove that the mobility edge is unique on the whole positive half-line for every fixed large p . In [4], the mobility edges are shown to be isolated through a strict monotonicity estimate for the Perron–Frobenius eigenvalue near each predicted location. In [7], uniqueness is derived from an explicit formula for the critical eigenvalue, and is proved for α near 0 and 1. For the present model no explicit formula is available, and the relevant cavity law has rare cancellation events whose contribution to the critical eigenvalue must be controlled together with their energy derivatives. We obtain uniqueness from an exact positive renewal decomposition of the square of the critical operator, which reduces the monotonicity of the Perron–Frobenius eigenvalue to that of a 2×2 matrix, and from estimates on the energy derivatives of the cavity law.

Concerning finite graphs, Theorem 1.4 concerns averaged local spectral masses in small energy windows, in the spirit of the localization and delocalization criteria for heavy-tailed matrices in [7, 48] and their analogues for random regular graphs in [96]. Related transfers of delocalization from the absolutely continuous spectrum of a limiting tree to finite graphs are the quantum ergodicity theorems for regular graphs [23, 24] and for general graphs converging to trees [25], and the quantitative delocalization results of

[46, 54]. We do not prove statements about individual eigenvectors of H_n .

The complex Gaussian assumption. The assumption that the weights are complex Gaussian is used in two ways. First, the squared moduli $Y_e = |w_e|^2$ of the weights are exponential random variables, independent of the phases, and the cavity variables depend on the weights only through these moduli. Exponential laws make many estimates explicit, in particular negative moment bounds for affine combinations of independent squared moduli, which we use throughout. Second, as explained above, the squared row norms of the bipartite square have Gamma densities independent of the row directions, which provides the rank-one randomness needed for the Simon–Wolff criterion and for spectral averaging. For real Gaussian weights the squared row norms are still independent of the directions, and we expect our methods to extend, with additional work and with μ_* replaced by $\sqrt{2/\pi}$, but we have not checked the details.

Open questions. We mention several questions that our results leave open. The first concerns small p : does the limiting operator have a unique mobility edge for every $p > 1$, or only for p large? For small p , the analysis of [58] suggests that the delocalized phase has unusual properties. The second concerns the spectral type at the energies $\pm E_c(p)$ and at zero. The third concerns finite graphs: it would be interesting to prove localization and delocalization of individual eigenvectors of H_n^{gc} in the two phases, and to determine their local eigenvalue statistics, in analogy with [6, 21]. Finally, it would be natural to study the multifractal properties of the delocalized eigenvectors predicted in [31, 58].

1.4 Notation and conventions

We write $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ for the upper half-plane, and $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta > 0$. For a self-adjoint operator $\mathcal{K}$ and a vector f , the spectral measure of f is the measure μ with $\langle f, (\mathcal{K} - z)^{-1} f \rangle = \int (\lambda - z)^{-1} \mu(d\lambda)$. The letters c and C denote positive constants whose dependence on other parameters is indicated in each statement; within a proof they may change from line to line. We write $\text{Pois}(\lambda)$ and $\text{Exp}(1)$ for the Poisson law with mean λ and the exponential law with mean one, and $\stackrel{\text{d}}{=}$ for equality in distribution. For a tree vertex v other than the root, v^- is its parent and $|v|$ its generation, and $\mathcal{T}_v$ is the subtree of descendants of v , rooted at v . For an edge e , we call $Y_e = |w_e|^2 = p|h_e|^2$ the squared modulus of the edge weight. The squared moduli have the $\text{Exp}(1)$ law, and are independent of the phases $w_e/|w_e|$, which are uniform on the unit circle. Notation that is used only in one section is introduced in that section.

1.5 Organization

Section 2 outlines the proofs of our main results. Part I develops the basic theory for a fixed degree p : the operator on the tree (Section 3), spine decompositions and estimates for fresh squared moduli (Section 4), real solutions of the cavity equation (Section 5), and the transfer operator and the free energy (Section 6). Part II shows that the sign of the free energy at $s = 1$ determines the spectral type at a fixed degree: Section 7 collects the rank-one perturbation arguments, Section 8 proves that negative free energy implies localization, and Sections 9 and 10 prove that positive free energy is incompatible with real boundary values. Part III studies the free energy at large degree: we construct the principal solution of the cavity equation in Section 11, prove its analyticity and derivative bounds in Section 12, determine the leading asymptotics of the free energy at $s = 1$ in Section 13, prove the uniqueness of its zero in Section 14, and treat small energies in Section 15. Finally, Part IV contains the proofs of Theorems 1.1 and 1.2 in Section 16 and of Theorem 1.4 in Section 17.

Outline of the proof

In this section we explain the main steps of the proofs of Theorems 1.1 and 1.2, and indicate where each step is carried out. The discussion is informal, but the objects and identities that appear are the ones used in the proofs. We only consider positive energies, since $\mathcal{H}$ is unitarily equivalent to $-\mathcal{H}$ (Proposition 3.21). Throughout, $z = E + i\eta$ with $E > 0$ and $\eta > 0$.

2.1 Cavity variables and the two alternatives

Let $\mathcal{T}$ be the rooted Poisson–Galton–Watson tree with mean degree p , with edge weights $h_e = w_e / \sqrt{p}$, and let $\mathcal{H}$ be its weighted adjacency operator. Although the degrees of $\mathcal{T}$ are unbounded, $\mathcal{H}$ is essentially self-adjoint on finitely supported vectors (Theorem 3.10), and its resolvent $G(z) = (\mathcal{H} - z)^{-1}$ is canonically defined. For a vertex $v \neq o$, let $\Gamma_v(z)$ denote the diagonal resolvent entry at v of the operator restricted to the subtree of descendants of v . The Schur complement formula gives the cavity recursion

$$\Gamma_v(z) = -\left(z + \sum_{u: u^- = v} |h_{vu}|^2 \Gamma_u(z)\right)^{-1}, \quad (2.1)$$

and, since the subtrees of distinct children are independent copies of $\mathcal{T}$, the root variable $\Gamma_o = G_{oo}$ satisfies

$$\Gamma_o \stackrel{d}{=} -\left(z + \frac{1}{p} \sum_{j=1}^N Y_j \Gamma_j\right)^{-1}, \quad N \sim \text{Pois}(p), \quad Y_j \sim \text{Exp}(1), \quad (2.2)$$

with all variables on the right independent and $\Gamma_j \stackrel{d}{=} \Gamma_o$. Off-diagonal resolvent entries factor along paths: if $o = v_0, v_1, \dots, v_L = v$ is the path from the root to v , then

$$G_{ov}(z) = (-1)^L \Gamma_o(z) \prod_{j=1}^L h_{v_{j-1}v_j} \Gamma_{v_j}(z). \quad (2.3)$$

These facts are proved in Section 3.

At Lebesgue-almost every energy E , the boundary values $\Gamma_v(E + i0)$ exist, and a zero–one law holds: under the survival-conditioned law $\mathbb{P}^\infty$, either $\text{Im} \Gamma_o(E + i0) > 0$

almost surely, or $\text{Im } \Gamma_o(E + i0) = 0$ almost surely (Proposition 5.11). In the first case, the full diagonal boundary values satisfy $\text{Im } G_{vv}(E + i0) > 0$ at every vertex of the infinite tree. If this alternative holds for Lebesgue-almost every energy in an interval I , a spectral averaging argument shows that the spectral measures are absolutely continuous in I (Proposition 7.11). In the second case, which we call the *no-ac hypothesis* following [12], all cavity variables at $E + i0$ are finite and real, and the inverse cavity variables $D_v = -1/\Gamma_v(E + i0)$, which we call denominators, solve the real cavity equation

$$D \stackrel{\text{d}}{=} E - \frac{1}{p} \sum_{j=1}^N \frac{Y_j}{D_j}. \quad (2.4)$$

A law on $\mathbb{R} \setminus \{0\}$ satisfying (2.4) is called a *real solution* at energy E . The proof decides between the two alternatives by attaching to every real solution a number, the value $\varphi(1)$ at $s = 1$ of its free energy, and showing that its sign determines the spectral type.

2.2 The free energy

By (2.3), the size of G_{ov} is governed by products of the variables $|h_{v_{j-1}v_j}\Gamma_{v_j}|$ along paths. Under the no-ac hypothesis these products can be generated by a transfer operator. Fix a scaling $g > 0$ and set $B = gD$ and $\beta = g^2Y/p$, so that (2.4) becomes a recursion for B in which a child with state $x = 1/B$ is mapped to the parent state $(B' - \beta x)^{-1}$, where B' collects the contributions of the other children. Each step of this recursion is the projective action of the matrix

$$M(B, \beta) = \beta^{-1/2} \begin{pmatrix} 0 & 1 \\ -\beta & B \end{pmatrix}, \quad \det M(B, \beta) = 1. \quad (2.5)$$

The Mecke formula for Poisson processes turns sums over vertices of generation L into expectations over a single path, at the cost of a factor p per generation (Proposition 4.8). For $s \in (0, 2)$ this leads to the positive transfer operator

$$(\mathcal{Q}_s F)(x) = p \mathbb{E} [\|M(B, \beta)v_x\|^{-s} F(f_{B, \beta}(x))], \quad v_x = \frac{(x, 1)}{\sqrt{1 + x^2}},$$

acting on continuous functions on the real projective line $\mathbb{RP}^1$, where $f_{B, \beta}$ is the projective action of $M(B, \beta)$ and B is distributed according to the real solution. We show in Section 6 that $\mathcal{Q}_s^3$ is compact and strongly positive, so that $\mathcal{Q}_s$ has a simple dominant Perron–Frobenius eigenvalue $\lambda(s)$. The *free energy* is $\varphi(s) = \log \lambda(s)$, and its value $\varphi(1)$ at $s = 1$ decides the spectral type. We have

$$\varphi(s) = \log p + \lim_{L \rightarrow \infty} \frac{1}{L} \log \mathbb{E} \int_{\mathbb{RP}^1} \|M_L \cdots M_1 v\|^{-s} m(\mathrm{d}v),$$

where $M_1, M_2, \dots$ are independent copies of $M(B, \beta)$ and m is the uniform measure on $\mathbb{RP}^1$. For $s < 1$, the quantity $\varphi(s)$ is also the exponential growth rate of the expected generation sums of s -th powers of the path products in (2.3).

The free energy is convex, real-analytic, and independent of the scaling. Its key property is the symmetry

$$\varphi(s) = \varphi(2 - s), \quad (2.6)$$

which follows from the change of variables $v \mapsto Mv/\|Mv\|$ on $\mathbb{RP}^1$, whose Jacobian is $\|Mv\|^{-2}$, and from the fact that a 2×2 matrix of determinant one has the same singular values as its inverse. Then $\varphi'(1) = 0$, the point $s = 1$ is the minimum of φ , and φ decreases from $\varphi(0+) = \log p$ to $\varphi(1)$ on $(0, 1]$. The value $\varphi(1)$ is the analogue, for the present model, of the criterion of Abou-Chacra, Anderson, and Thouless [2], of the free-energy criterion of Aizenman and Warzel [12], and of the quantity $\lambda(E, \alpha)$ in the work of Aggarwal, Bordenave, and Lopatto on Lévy matrices [7]. A symmetry of this kind for the linearized cavity kernel goes back to [2]; see [109]. As in [4, 7], and unlike [12], we sum over generations rather than normalizing per vertex, and the threshold is zero.

2.3 Negative free energy implies localization

Suppose that some real solution π at energy E has $\varphi_\pi(1) < 0$. We do not need to know that π is the law of the actual boundary values. By Kolmogorov's extension theorem, we may enlarge the probability space so that every vertex v carries a real variable D_v with law π , the variables satisfy the recursion (2.4) pathwise on the tree, and the tree itself keeps its law (Lemma 5.13). Set $\Gamma_v = -1/D_v$ and define the formal inverse column

$$F(o) = \Gamma_o, \quad F(v) = \Gamma_o \prod_{j=1}^L (-h_{v_j v_{j-1}} \Gamma_{v_j}).$$

The recursion gives $(\mathcal{H} - E)F = \delta_o$ pointwise. By the many-to-one formula, $\mathbb{E} \sum_{|v|=L} |F(v)|^s$ grows at the exponential rate $\varphi_\pi(s)$, and since $\varphi_\pi(1) < 0$ we may choose $s < 1$ with $\varphi_\pi(s) < 0$. Then $F \in \ell^s \subset \ell^2$ almost surely. In particular, F lies in the domain of $\mathcal{H}$ and $\delta_o = (\mathcal{H} - E)F$ lies in the range of $\mathcal{H} - E$. By the spectral theorem, $\int \mu_o(d\lambda)/(\lambda - E)^2 < \infty$, where μ_o is the spectral measure at the root. The same holds at every vertex and on every component of the tree with one vertex removed (Proposition 8.3).

These reciprocal-square integrals are the input of the Simon–Wolff criterion [124]. Since the model has no random potential, we need another source of rank-one randomness. It is supplied by the complex Gaussian weights. The tree is bipartite, so $\mathcal{H}^2 = A^*A \oplus AA^*$, and

$$A^*A = \sum_y U_y |v_y\rangle\langle v_y|,$$

where y runs over the odd generations, v_y is the normalized vector of edge weights adjacent to y , and U_y is its squared norm. Conditional on the tree, the squared row norm U_y has a Gamma density and is independent of the direction v_y and of all other rows (Proposition 3.23). Thus each U_y is the coupling constant of a rank-one perturbation with an absolutely continuous law, and the Simon–Wolff argument applies in the square blocks. This yields pure point spectrum on every open set of energies at which a real solution with negative free energy at $s = 1$ exists (Theorem 8.4).

2.4 Positive free energy implies delocalization

Suppose now that the no-ac hypothesis holds at a regular energy E , and let π be the law of the boundary denominators. We show that $\varphi_\pi(1) \leq 0$ (Theorem 10.1). Together with the spectral averaging argument mentioned above, this proves pure absolute continuity on any interval where, for almost every regular energy, all possible real boundary laws have positive free energy at $s = 1$ (Corollary 10.2). The argument follows the resonance strategy of Aizenman and Warzel [12]: we look for vertices at large depth n that are strongly coupled to the root, and show that their contributions force $\text{Im } \Gamma_o$ to be large. Variants of this strategy were used in [4, 7].

Suppose, for a contradiction, that $0 < \varphi_\pi(1)$. We always have $\varphi_\pi(1) < \log p$ (Proposition 6.15). By convexity and symmetry, there exists $s \in (1/2, 1)$ such that

$$\gamma = -\varphi'_\pi(s) > 0, \quad F = \varphi_\pi(s) - (1-s)\gamma > 0.$$

Consider a path of length n from the root, and the event that the product of the transfer matrices along it is of size about $e^{\gamma n}$. After an exponential tilt of the path law, this is a moderate deviation event for a sum of bounded increments, with probability about $e^{-(\log p - \varphi_\pi(s) - s\gamma)n}$. Since there are about p^n paths, the expected number of such paths is about $e^{(\varphi_\pi(s) + s\gamma)n}$. Along such a path, an exact continuant identity shows that $|G_{ox}/G_{xx}| \gtrsim e^{-\gamma n}$, where x is the endpoint. At x we attach a pendant leaf whose squared edge weight t has an exponential density. Since the diagonal entry G_{xx} is a Möbius function of t with a real pole, the event $|G_{xx}| \geq T$ has probability of order T^{-1} . With $T = e^{(\gamma + \delta)n}$, we obtain an expected number

$$\mathbb{E} N_n \geq e^{(F - \delta)n - o(n)}$$

of resonant endpoints x , each satisfying $|G_{ox}| \geq e^{\delta n}$. A surviving child of x , whose imaginary part is at least a fixed quantile ζ_α of the distribution of $\text{Im } \Gamma$, then gives $\text{Im } \Gamma_o \geq e^{2\delta n - o(n)} \zeta_\alpha$ by the Ward identity.

The main difficulty is the second moment. For two resonant endpoints x and y whose paths split at depth j , the two pendant tunings are nearly independent unless the Green function between the two branches is large. This interaction is controlled

by the off-diagonal entry of the effective 2×2 Feshbach matrix at x and y , which is the bilinear form of a product of transfer matrices along the path from x to y . To average this product we reserve, near the split and near both endpoints, three consecutive factors whose squared moduli are not constrained by any of the events retained in the second moment. Three fresh factors suffice to smooth an arbitrary fixed bilinear form (Proposition 6.18). The interaction contribution is then bounded by the norm moments of the two arms, which decay at the rate $\varphi_\pi(s) - \log p$ per step. The resulting exponent is an affine function of j , and the choice of s makes it negative at $j = 0$ and at $j = n$. Thus $\mathbb{E}[N_n(N_n - 1)] \leq e^{o(n)}(\mathbb{E}N_n)^2$, and the Paley–Zygmund inequality gives $\mathbb{P}(N_n > 0) \geq e^{-o(n)}$.

To conclude, we need to raise this subexponentially small probability to a fixed level. We use the Poisson structure of the tree: the root has many children, and each of them independently carries a copy of the rare event. An amplification argument, which uses only a fixed number of amplifying children per generation and a reserve of other children that keeps the cavity variables in a fixed compact region, raises the probability to a fixed number $q_* > \alpha$ after $o(n)$ generations, at the cost of a factor $e^{-o(n)}$ in the lower bound for $\text{Im } \Gamma_o$. We obtain $\mathbb{P}^\infty(\text{Im } \Gamma_o \geq e^{2\delta n - o(n)} \zeta_\alpha) \geq q_* > \alpha$, which contradicts the definition of the quantile ζ_α once n is large. All estimates are first made for the real recursive tree process at a fixed depth and then transferred to a small positive imaginary part η , chosen after n .

2.5 The free energy at large degree

By the previous two steps, Theorem 1.1 reduces to determining the sign of the free energy at $s = 1$ of the relevant real solutions. This is where the degree p is taken large.

The principal solution. For $E > 2$ it is convenient to use the scaling $g(E) = (E - \sqrt{E^2 - 4})/2$, in which the real cavity equation becomes

$$B \stackrel{\text{d}}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j}, \quad \theta = g(E)^2.$$

At energies $E = \xi\sqrt{p}$ with ξ of order one, we have $\theta \asymp 1/p$, so the sum is an average of about p terms of size θ/p , and one expects $B \approx 1$. This is correct except on rare events where some B_j is close to zero. In Section 11 we construct, for every compact window of scaled energies ξ and all large p , a solution $\nu_{p,E}$ whose law is concentrated within $p^{-1/4}$ of one, has density at most $p^A e^{-ap}$ near zero, and has a tail decaying like $p^A e^{-ap}/L$. We denote by $\mathfrak{S}$ the class of laws with these three properties. The solution $\nu_{p,E}$, which we call the *principal solution*, is the unique fixed point of the cavity map in $\mathfrak{S}$, on which the map is contracted by a metric adapted to these three regions. The same construction covers all

energies $E \geq M_0\sqrt{p}$, where the cavity map is a contraction in the same class. We then show that at large p and energies $E = \xi\sqrt{p}$ in a fixed window, every real solution of the cavity equation is the principal solution (Proposition 11.27). The proof is a one-big-jump bootstrap: it starts from the bound $\sqrt{p+1}$ on the density near zero, which every real solution satisfies, and improves it until the solution lies in the class $\mathfrak{S}$. In particular, under the no-ac hypothesis the boundary law can only be the principal solution.

The location of the mobility edge. The leading behavior of the free energy at $s = 1$ of the principal solution is easy to guess. Since $B \approx 1$, we have $|\Gamma_v| \approx g(E) \approx E^{-1}$ along a typical path. Each step of (2.3) contributes a factor $|h_e|/E = |w_e|/(\xi p)$, and there are about p choices of the next vertex. The expected sum of the products over the p^L paths of length L is therefore about

$$\left(p \cdot \frac{\mathbb{E}|w|}{\xi p}\right)^L = \left(\frac{\mu_*}{\xi}\right)^L, \quad \mu_* = \mathbb{E}|w| = \frac{\sqrt{\pi}}{2}.$$

This suggests $\exp \varphi_{\xi\sqrt{p}}(1) \approx \mu_*/\xi$, and in Section 13 we prove that this is correct uniformly on compact sets of ξ (Theorem 13.1). The lower bound follows from the behavior of the transfer operator on states near one. The upper bound is harder, because the rare cancellations $B \approx 0$ produce large states, which may feed back into the core region of states near one. We decompose the state space into six regions and show that the only order-one transitions between the five other regions form an acyclic graph, so that every return to the core region passes through a transition of small weight. A finite-dimensional diagonal balancing then bounds the growth rate.

Uniqueness of the crossing. The leading asymptotics show that the free energy at $s = 1$ changes sign near $\mu_*\sqrt{p}$, but by themselves they do not exclude several sign changes in a window of width $o(\sqrt{p})$. In Section 12 we show that the principal law is a real-analytic function of the energy, with polynomially bounded derivatives in suitable norms, and that its first two derivatives are small when tested against smooth functions supported near one. In Section 14 we work with the L^2 version of the critical transfer operator, on a fixed space on which the energy enters only through the law of B and through exponential densities. We then decompose its square exactly as

$$A_\xi^2 = G_\xi + h_c \otimes \ell_c + h_t \otimes \ell_t,$$

where A_ξ is the critical operator on this space, the first rank-one term is a positive operator describing the region where B is close to 1, the second is its image under an involution J that exchanges this region with the region where B is close to 0, and G_ξ is a positive operator with small spectral radius. The Perron–Frobenius eigenvalue of A_ξ^2 is then determined by the Perron–Frobenius eigenvalue of a 2×2 positive renewal matrix. Its diagonal entries are equal, by the symmetry under J , and decrease strictly in ξ . Its

off-diagonal entries are positive, and one of them is exponentially small. A positivity argument, based on Glaeser's inequality, controls the derivative of this Perron–Frobenius eigenvalue without dividing by the possibly tiny gap between the two components. This shows that, for scaled energies in a fixed neighborhood of μ_* , the free energy at $s = 1$ has exactly one zero, at which it crosses zero transversally and downward. Outside this neighborhood, its sign is given by the leading asymptotics and, at large energies, by the large-energy estimate. Together, these facts give exactly one zero on the half-line, and the crossing is transverse (Theorem 14.1 and Corollary 14.2).

Small energies. For energies below a small multiple of $\sqrt{p}$, the real cavity equation need not have a principal solution, and we do not try to identify the real boundary law. Instead we show in Section 15 that *every* real solution at an energy $0 < E \leq \xi_{\text{in}}\sqrt{p}$ has free energy at $s = 1$ at least a fixed $\delta > 0$ (Proposition 15.2). The proof bounds the critical eigenvalue from below by $p\mathbb{E}\|M\|^{-1}$ for two scalings and shows, using only the recursion, that one of the two lower bounds exceeds one. Since the delocalization criterion applies to arbitrary real boundary laws, this gives absolute continuity at all small positive energies for the same fixed p .

2.6 Assembly and the finite graph

The proof of Theorem 1.1 in Section 16 combines these results. On $(0, \xi_{\text{in}}\sqrt{p}]$, every possible real boundary law has positive free energy at $s = 1$, and the delocalization criterion shows that the spectrum is absolutely continuous. At larger energies of order $\sqrt{p}$, a real boundary law must be the principal solution, whose free energy at $s = 1$ is positive below $E_c(p)$ and negative above it. Above $E_c(p)$ the principal solution is a real solution with negative free energy at $s = 1$, which gives pure point spectrum, including at all large energies. The two absolutely continuous regions overlap, which leaves $E_c(p)$ as the only transition.

Finally, in Section 17 we transfer the conclusions to the giant component of the finite graph. A uniformly chosen vertex of the giant component converges locally to the survival-conditioned tree, jointly for finitely many vertices, and the local spectral measures converge accordingly. The density of states of the limiting operator is bounded above and below on compact sets away from zero. On the absolutely continuous side, the normalized local spectral masses are uniformly integrable over vertices, by Lebesgue differentiation and Scheffé's lemma; on the pure point side, they concentrate on a vanishing fraction of vertices. This proves Theorem 1.4.

PART I

The limiting tree and the cavity equation

The weighted tree and its operator

In this section, we define the finite model and the marked Poisson–Galton–Watson tree, construct the self-adjoint operator of the tree and of the forests obtained from it by finitely many deletions, and collect the algebraic identities for its resolvent that are used throughout the paper. Section 3.1 defines the models. Section 3.2 defines finite removals and shows that marks which are not inspected by a selection remain independent standard Gaussians. Section 3.3 proves essential self-adjointness of all these operators on finitely supported vectors, identifies their domains, and proves convergence of Dirichlet truncations. Section 3.4 proves the Schur complement formula (the cavity recursion), the path product formula, and the Ward identity. Section 3.5 records a gauge transformation and the parity symmetry. Section 3.6 develops the bipartite factorization of the square of the operator and its decomposition into rank-one terms indexed by rows, which are the basis of the rank-one arguments in Section 7. Finally, Section 3.7 shows that on the survival event the operator has no finitely supported eigenvector with nonzero eigenvalue, while finitely supported eigenvectors with eigenvalue zero occur infinitely often.

Conventions. For a countable set S , we write $\ell^2(S)$ for the Hilbert space of square-summable functions $f: S \rightarrow \mathbb{C}$, with the inner product

$$\langle f, g \rangle = \sum_{s \in S} \overline{f(s)} g(s),$$

which is antilinear in the first argument. We write δ_s for the coordinate vector at $s \in S$ and $c_c(S)$ for the space of finitely supported functions on S . For $S' \subset S$, we identify $\ell^2(S')$ with the subspace of $\ell^2(S)$ of functions vanishing off S' , and we write $P_{S'}$ for the orthogonal projection onto it. For a self-adjoint operator T and a vector f , we write μ_f^T for the spectral measure of f , defined by $\langle f, \phi(T)f \rangle = \int \phi d\mu_f^T$ for bounded Borel functions ϕ . For a Borel set $I \subset \mathbb{R}$, we say that T has *pure point spectrum in I* if $\text{Ran } \mathbf{1}_I(T)$ is contained in the closed span of the eigenvectors of T , and that T is *purely absolutely continuous in I* if the restriction of μ_f^T to I is absolutely continuous with respect to Lebesgue measure for every vector f . Since the closed span of the eigenvectors consists of the vectors whose spectral measure is purely atomic [113, Section VII.2], the operator T has pure point spectrum in I if and only if the restriction of μ_f^T to I is purely atomic for every f . Spectral parameters are written $z = E + i\eta$ with $E, \eta \in \mathbb{R}$, and $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$.

3.1 The finite model and the marked tree

A standard complex Gaussian variable is a random variable w with density $\pi^{-1}e^{-|\zeta|^2}$ with respect to Lebesgue measure on $\mathbb{C}$. Writing $\zeta = re^{i\vartheta}$, Lebesgue measure on $\mathbb{C}$ is $r dr d\vartheta$, and in the variables $t = r^2$ and $\vartheta \in [0, 2\pi)$ the law of w is $e^{-t} dt \otimes d\vartheta / (2\pi)$. This implies that $Y = |w|^2$ has the law $\text{Exp}(1)$, the phase $w/|w|$ is uniform on the unit circle, and the two are independent. Also $\bar{w}$ and $-w$ are standard complex Gaussian variables.

Definition 3.1. Let $p > 0$ and let $n \geq 1$ be an integer with $n > p$. Let G_n be the Erdős–Rényi graph on $[n] = \{1, \dots, n\}$, in which each pair $\{i, j\}$ is an edge independently with probability p/n . Let $(w_{ij})_{1 \leq i < j \leq n}$ be independent standard complex Gaussian variables, independent of G_n . Define the Hermitian $n \times n$ matrix H_n by

$$H_n(i, j) = \frac{\mathbf{1}_{\{ij \in E(G_n)\}}}{\sqrt{p}} w_{ij}, \quad H_n(j, i) = \overline{H_n(i, j)}, \quad H_n(i, i) = 0, \quad 1 \leq i < j \leq n.$$

The *giant component* $\text{gc} \subset [n]$ is the vertex set of a connected component of G_n of maximal size; if there are several, we take the one containing the smallest vertex label. Thus gc is a deterministic function of G_n . We set $N_n = |\text{gc}|$ and write $H_n^{\text{gc}} = (H_n(i, j))_{i, j \in \text{gc}}$ for the restriction of H_n to $\ell^2(\text{gc})$.

Throughout the paper, the limit $n \rightarrow \infty$ is taken with p fixed. The local limit of G_n seen from a uniformly chosen vertex is the Poisson–Galton–Watson tree, and seen from a uniformly chosen vertex of gc it is that tree conditioned on survival; this is made precise in Proposition 17.13. We now define the tree and its marks on an explicit probability space.

Let $\mathbb{U} = \bigcup_{k \geq 0} \mathbb{N}^k$ be the set of finite words over $\mathbb{N} = \{1, 2, \dots\}$, with the empty word o , and let $\mathbb{U}_* = \mathbb{U} \setminus \{o\}$. For $u \in \mathbb{U}$ and $j \in \mathbb{N}$, we write uj for the concatenation, $|u|$ for the length of u , and u^- for the word obtained from $u \neq o$ by deleting its last letter.

Definition 3.2. Let $p > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying three independent families: random variables $(N_u)_{u \in \mathbb{U}}$, independent with the law $\text{Pois}(p)$; random variables $(w_u)_{u \in \mathbb{U}_*}$, independent standard complex Gaussians; and an auxiliary random element $\mathbf{a}$ of a standard Borel space.

(i) The *genealogy* is the rooted tree $\mathcal{T}$ with root o , vertex set

$$V = \{j_1 j_2 \cdots j_k \in \mathbb{U} : k \geq 0, j_i \leq N_{j_1 \cdots j_{i-1}} \text{ for } 1 \leq i \leq k\},$$

and edges $\{u^-, u\}$ for $u \in V \setminus \{o\}$. It is a PGW(p) tree: the children of $v \in V$ are $v1, \dots, vN_v$. The *generation* of v is $|v|$, the vertex v^- is the *parent* of $v \neq o$, and we write $u \sim v$ when u and v are adjacent. The degree of v is $\deg(v) = N_v + \mathbf{1}_{\{v \neq o\}}$. We write $N = N_o$ for the number of children of the root.

- (ii) The edge $\{u^-, u\}$ is identified with the word u , and w_u is its *mark*; we also write w_e for the mark of an edge e . The *edge weights* are

$$h_{u^-u} = \frac{w_u}{\sqrt{p}}, \quad h_{uu^-} = \overline{h_{u^-u}}, \quad u \in V \setminus \{o\},$$

and $h_{uv} = 0$ when u and v are not adjacent. The *squared modulus* of the mark of the edge u is $Y_u = |w_u|^2 = p|h_{u^-u}|^2$, which has the law $\text{Exp}(1)$ and is independent of the phase $w_u/|w_u|$. The pair $(\mathcal{T}, (h_{uv})_{u,v \in V})$ is the *marked tree*.

- (iii) For $v \in V$, the *forward subtree* $\mathcal{T}_v$ is the subtree of $\mathcal{T}$ induced on $V(\mathcal{T}_v) = \{vu : u \in \mathbb{U}\} \cap V$, rooted at v . In particular $\mathcal{T}_o = \mathcal{T}$.
- (iv) The *survival event* is $\mathcal{S} = \{|V| = \infty\}$, and $\varrho_p = \mathbb{P}(\mathcal{S})$. When $p > 1$, we have $\varrho_p \in (0, 1)$, and ϱ_p is the unique solution in $(0, 1]$ of $1 - \varrho = e^{-p\varrho}$. For $p > 1$ we set $\mathbb{P}^\infty = \mathbb{P}(\cdot \mid \mathcal{S})$ and write $\mathbb{E}^\infty$ for the corresponding expectation.

We write $\mathcal{G}_0 = \sigma((N_u)_{u \in \mathbb{U}}, \mathfrak{a})$ for the σ -algebra generated by the genealogy variables and the auxiliary randomness.

The marks w_u with $u \notin V$ do not enter the marked tree; including them makes the edge marks of $\mathcal{T}$ a subfamily of an independent family indexed by the fixed set $\mathbb{U}_*$. Conditionally on $\mathcal{G}_0$, the marks of the edges of $\mathcal{T}$ are independent standard complex Gaussians, and the same holds under $\mathbb{P}^\infty$, since $\mathcal{S} \in \mathcal{G}_0$. The value of ϱ_p is the classical extinction computation for Galton–Watson trees [133, Chapter 3]. The auxiliary element $\mathfrak{a}$ is used in later sections for selections by independent ranks; in Section 4, it is realized as a family $(\zeta_v)_{v \in \mathbb{U}}$ of independent auxiliary marks indexed by the words.

The families indexed by disjoint subsets of $\mathbb{U}$ are independent. We record the consequence that is used to derive distributional identities. For $j \in \mathbb{N}$, let θ_j be the *shift* which maps the families above to $((N_{ju})_{u \in \mathbb{U}}, (w_{ju})_{u \in \mathbb{U}_*})$. The genealogy and marks built from θ_j by Definition 3.2 form a marked tree $\theta_j\mathcal{T}$, and if $j \leq N_o$, then $\theta_j\mathcal{T}$ is the forward subtree $\mathcal{T}_j$ relabelled by $ju \mapsto u$. The shifted families use the variables indexed by words beginning with j , other than w_j , and we conclude that

$$\theta_1\mathcal{T}, \theta_2\mathcal{T}, \dots \text{ are independent copies of } \mathcal{T}, \text{ independent of } (N_o, (w_j)_{j \geq 1}). \quad (3.1)$$

3.2 Finite removals

Many arguments in this paper delete a vertex, an edge, or a few selected leaves from the tree and study the resulting operator. We now define these operations and the class of selections under which the retained marks keep their law.

Definition 3.3.

- (i) A *finite removal* of $\mathcal{T}$ is given by a pair (F, D) , called its *removed set*, where $F \subset V$ is a finite set of vertices and D is a finite set of edges of $\mathcal{T}$; we also write (F, D) for the finite removal. The *deleted edges* are the edges in D and the edges with an endpoint in F ; all other edges of $\mathcal{T}$ are *retained*. The *reduced forest* $\mathcal{T}^{F,D}$ is the graph with vertex set $V \setminus F$ whose edges are the retained edges. A *component* of (F, D) is the vertex set of a connected component of $\mathcal{T}^{F,D}$.
- (ii) An *inspected set* is a random finite set $Q \subset \mathbb{U}_*$ such that $\{Q = q\} \in \mathcal{G}_0^q$ for every finite $q \subset \mathbb{U}_*$, where $\mathcal{G}_0^q = \mathcal{G}_0 \vee \sigma(w_u : u \in q)$. Its σ -algebra is

$$\mathcal{G}_Q = \{A \in \mathcal{F} : A \cap \{Q = q\} \in \mathcal{G}_0^q \text{ for every finite } q \subset \mathbb{U}_*\}.$$

- (iii) A random finite removal (F, D) is *selected without inspecting the retained marks* if there exists an inspected set Q such that $\{(F, D) = (f, d)\} \in \mathcal{G}_Q$ for all finite sets f, d , and such that almost surely no edge with label in Q is retained. If this holds with $Q = \emptyset$, that is, if (F, D) is a function of the genealogy and the auxiliary randomness, we say that (F, D) is *selected from the genealogy*.

An inspected set is the set of labels of the marks that a selection procedure looks at; the procedure may decide which marks to look at using what it has seen so far, and $\mathcal{G}_Q$ is the information available at the end. The finite removals used later are the following.

- (a) *Deleting a vertex y* : $(F, D) = (\{y\}, \emptyset)$. Its components are the forward subtrees $V(\mathcal{T}_{y_j})$, $1 \leq j \leq N_y$, and, if $y \neq o$, the set $V \setminus V(\mathcal{T}_y)$.
- (b) *Deleting an edge $\{v^-, v\}$* : $(F, D) = (\emptyset, \{v\})$. Its components are $V(\mathcal{T}_v)$ and $V \setminus V(\mathcal{T}_v)$. In particular, every forward subtree is a component of a finite removal.
- (c) *Deleting selected pendant leaves*: F is a finite set of leaf children (children ℓ with $N_\ell = 0$) chosen as a function of the genealogy and of auxiliary ranks carried by $\mathbf{a}$, and $D = \emptyset$. This finite removal is selected from the genealogy.
- (d) *Dirichlet balls*: deleting the finitely many edges between generations R and $R + 1$ leaves the ball $\{v \in V : |v| \leq R\}$ as the component of o .

Lemma 3.4. *Let Q be an inspected set. For every $A \in \mathcal{G}_Q$, every finite $q \subset \mathbb{U}_*$, and every bounded measurable function Ψ of the family $(w_u)_{u \in \mathbb{U}_* \setminus q}$, we have*

$$\mathbb{E} \left[\mathbf{1}_{A \cap \{Q=q\}} \Psi((w_u)_{u \notin q}) \right] = \mathbb{P}(A \cap \{Q = q\}) \mathbb{E} \left[\Psi((w_u)_{u \notin q}) \right]. \quad (3.2)$$

The same identity holds with $\mathbb{P}, \mathbb{E}$ replaced by $\mathbb{P}^\infty, \mathbb{E}^\infty$ when $p > 1$. If (F, D) is selected without inspecting the retained marks, then conditionally on $\mathcal{G}_Q$ the marks of the retained edges are independent standard complex Gaussian variables, under $\mathbb{P}$ and under $\mathbb{P}^\infty$.

Proof. Fix A and q . By the definition of $\mathcal{G}_Q$, the event $A \cap \{Q = q\}$ belongs to $\mathcal{G}_0^q = \sigma((N_u)_{u \in \mathbb{U}}, \mathbf{a}, (w_u)_{u \in q})$. Since the three families in Definition 3.2 are independent and

the marks are independent of each other, $\mathcal{G}_0^q$ is independent of $(w_u)_{u \notin q}$, and (3.2) follows. Since $S \in \mathcal{G}_0$, the event $A \cap S$ belongs to $\mathcal{G}_Q$; applying (3.2) to it and dividing by $\mathbb{P}$ gives the identity under $\mathbb{P}^\infty$, since $\mathbb{E}[\Psi] = \mathbb{E}^\infty[\Psi]$ by the independence of $S \in \mathcal{G}_0$ and the marks.

For the last assertion, let $(w'_u)_{u \in \mathbb{U}_*}$ be an independent copy of the marks, defined on a product extension of the probability space, and set $\tilde{w}_u = w_u$ for $u \notin Q$ and $\tilde{w}_u = w'_u$ for $u \in Q$. For fixed q , the family $((w_u)_{u \notin q}, (w'_u)_{u \in q})$ has the law of $(w_u)_{u \neq 0}$ and is independent of $\mathcal{G}_0^q$, by the argument above and since w' is independent of everything else. For $A \in \mathcal{G}_Q$ and every bounded measurable function Ψ of the whole family, we then have

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A \Psi(\tilde{w})] &= \sum_q \mathbb{E}[\mathbf{1}_{A \cap \{Q=q\}} \Psi((w_u)_{u \notin q}, (w'_u)_{u \in q})] \\ &= \sum_q \mathbb{P}(A \cap \{Q=q\}) \mathbb{E}[\Psi(w)] = \mathbb{P}(A) \mathbb{E}[\Psi(w)], \end{aligned}$$

where the sums run over the countably many finite sets q . This shows that $\tilde{w}$ is independent of $\mathcal{G}_Q$ and is a family of independent standard complex Gaussians. The set of retained edges is a function of $\mathcal{T}$ and (F, D) , and in particular it is $\mathcal{G}_Q$ -measurable. Since it is almost surely disjoint from Q , the retained marks coincide with the corresponding entries of $\tilde{w}$. The conditional law, given $\mathcal{G}_Q$, of the entries of an independent i.i.d. family indexed by a $\mathcal{G}_Q$ -measurable random set is the product law over that set [81, Theorem 6.4]. This proves the claim under $\mathbb{P}$, and the same argument applies under $\mathbb{P}^\infty$. $\square$

For a finite removal selected from the genealogy, conditionally on $\mathcal{G}_0$ all marks, deleted or retained, are independent standard complex Gaussians. For the pendant leaves in (c), this means that the squared moduli of the marks of the deleted leaf edges are independent $\text{Exp}(1)$ variables, independent of the genealogy and of all retained marks.

We next record some elementary facts about the shape of the components. For a component C of a finite removal and adjacent vertices $u, w \in C$, the *branch* $C_{u \rightarrow w}$ is the component containing u of the graph obtained from $\mathcal{T}^{F,D}$ by also deleting the edge $\{u, w\}$.

Lemma 3.5. *Let (F, D) be a finite removal of $\mathcal{T}$.*

- (i) *The reduced forest $\mathcal{T}^{F,D}$ has finitely many components. Every component C is the vertex set of an induced subtree of $\mathcal{T}$: every edge of $\mathcal{T}$ with both endpoints in C is retained.*
- (ii) *Let C be a component and let $u, w \in C$ be adjacent. Then $C_{u \rightarrow w}$ is a component of the finite removal $(F, D \cup \{\{u, w\}\})$, and it consists of the vertices $x \in C$ such that the path in $\mathcal{T}$ from x to w contains u . The neighbors of u in C lie in distinct components of $(F \cup \{u\}, D)$, and the component containing a neighbor x is $C_{x \rightarrow u}$.*
- (iii) *Let C be as in (ii), let $u, w \in C$ be adjacent, and let C' be a component of a finite removal (F', D') with $F \subset F'$ and $D \subset D'$. If $C_{u \rightarrow w} \subset C'$ and $w \in C'$, then $C'_{u \rightarrow w} = C_{u \rightarrow w}$.*

Proof. Every edge of a tree disconnects it, and every edge of a forest disconnects the component that contains it. By induction on the number of deleted edges, deleting them from $\mathcal{T}$ produces a forest with exactly one more component than deleted edges. The vertices of F are isolated in this forest, and its other components are the components of (F, D) ; this proves the first claim of (i). If an edge $\{u, v\}$ of $\mathcal{T}$ with $u, v \in C$ were deleted, then the path from u to v in the connected graph $\mathcal{T}^{F, D}$ restricted to C would avoid this edge, and together with the edge it would form a cycle in $\mathcal{T}$. This proves (i).

For (ii), the first assertion holds because adding $\{u, w\}$ to D changes only the component C , which splits into two. Since C is a subtree, paths in C are paths in $\mathcal{T}$. A vertex $x \in C$ is connected to u in C without using the edge $\{u, w\}$ exactly when the path from x to u does not contain w , and this holds exactly when the path from x to w is the path from x to u followed by the edge $\{u, w\}$. This proves the description of $C_{u \rightarrow w}$. If x and x' are distinct neighbors of u in C , then they lie in distinct components after deleting u , since the unique path between them is x, u, x' . The component of x after deleting u consists of the vertices of C connected to x by a path avoiding u , and this is $C_{x \rightarrow u}$ by the description just proved.

For (iii), we have $C' \subset C$, because every edge retained by (F', D') is retained by (F, D) . Since a path in C' which avoids the edge $\{u, w\}$ is such a path in C , we have $C'_{u \rightarrow w} \subset C_{u \rightarrow w}$. Conversely, $C_{u \rightarrow w}$ induces a connected subtree of $\mathcal{T}$ by (i) and (ii), it is contained in C' , and C' is an induced subtree by (i). Then every vertex of $C_{u \rightarrow w}$ is joined to u by a path in C' avoiding $\{u, w\}$, which gives $C_{u \rightarrow w} \subset C'_{u \rightarrow w}$. $\square$

3.3 Essential self-adjointness

The weighted adjacency operator of $\mathcal{T}$ need not be bounded. In this section, we show that it is essentially self-adjoint on finitely supported vectors, simultaneously for all components of all finite removals. The argument follows [51, Appendix A]: by [51, Lemma A.3], an exhaustion of a weighted tree by finite connected sets whose boundary weights are uniformly bounded implies essential self-adjointness, and such an exhaustion of the Poisson weighted infinite tree is constructed in [51, Proposition A.2] by a subcritical exploration; see also [52] for the unweighted case. We include the proof for our model for completeness.

Hermitian networks

A *locally finite Hermitian network* is a countable set $\mathcal{V}$ with weights $k: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ such that $k_{vu} = \overline{k_{uv}}$ and $k_{vv} = 0$ for all u, v , and such that $\{u : k_{vu} \neq 0\}$ is finite for every v . Its *formal operator* acts on all functions $f: \mathcal{V} \rightarrow \mathbb{C}$ by

$$(T_{\text{form}}f)(v) = \sum_{u \in \mathcal{V}} k_{vu}f(u),$$

which is a finite sum. The *minimal operator* T_0 is the restriction of T_{form} to $c_c(\mathcal{V})$, viewed as an operator in $\ell^2(\mathcal{V})$, and the *maximal domain* is $\{f \in \ell^2(\mathcal{V}) : T_{\text{form}}f \in \ell^2(\mathcal{V})\}$. For a finite set $S \subset \mathcal{V}$, the *boundary operator* B_S is defined by

$$(B_S f)(v) = \mathbf{1}_S(v) \sum_{u \notin S} k_{vu} f(u), \quad f \in \ell^2(\mathcal{V}).$$

Since only the finitely many $u \notin S$ with a neighbor in S enter this sum, B_S is a bounded operator of finite rank.

On the marked tree, $(V, (h_{uv})_{u,v \in V})$ is a locally finite Hermitian network. If C is a component of a finite removal, then $(C, (h_{uv})_{u,v \in C})$ is a locally finite Hermitian network as well, and by Lemma 3.5(i) its weights are those of the retained edges inside C . We write $\mathcal{H}_{C,\text{form}}$ and $\mathcal{H}_{C,0}$ for its formal and minimal operators, and $\mathcal{H}_{\text{form}}$ and $\mathcal{H}_0$ when $C = V$.

Lemma 3.6. *Let $(\mathcal{V}, k)$ be a locally finite Hermitian network. Then T_0 is symmetric, and T_0^* is the restriction of T_{form} to the maximal domain. If T_0 is essentially self-adjoint, then its closure T satisfies*

$$\text{Dom}(T) = \{f \in \ell^2(\mathcal{V}) : T_{\text{form}}f \in \ell^2(\mathcal{V})\}, \quad Tf = T_{\text{form}}f \text{ for } f \in \text{Dom}(T).$$

Proof. Let $g \in c_c(\mathcal{V})$ and $f \in \ell^2(\mathcal{V})$. Only the finitely many pairs (u, v) with $u \in \text{supp } g$ and $k_{uv} \neq 0$ contribute to the following sums. Exchanging these finite sums and using $\overline{k_{vu}} = k_{uv}$, we have

$$\langle T_0 g, f \rangle = \sum_v \overline{\sum_u k_{vu} g(u)} f(v) = \sum_u \overline{g(u)} \sum_v k_{uv} f(v) = \langle g, T_{\text{form}} f \rangle. \quad (3.3)$$

Taking $f \in c_c(\mathcal{V})$, we see that T_0 is symmetric. If f is in the maximal domain, then (3.3) implies that $f \in \text{Dom}(T_0^*)$ and $T_0^* f = T_{\text{form}} f$. Conversely, if $f \in \text{Dom}(T_0^*)$, then applying (3.3) with $g = \delta_v$, we have $(T_0^* f)(v) = (T_{\text{form}} f)(v)$ for every v , that is, $T_{\text{form}} f = T_0^* f \in \ell^2$. If T_0 is essentially self-adjoint, then its closure T is self-adjoint and $T = T^* = T_0^*$, since an operator and its closure have the same adjoint. $\square$

Lemma 3.7. *Let $(\mathcal{V}, k)$ be a locally finite Hermitian network. Suppose that there exist finite sets $S_1 \subset S_2 \subset \dots$ with $\bigcup_n S_n = \mathcal{V}$ and $M = \sup_n \|B_{S_n}\| < \infty$. Then T_0 is essentially self-adjoint on $c_c(\mathcal{V})$.*

Proof. By [113, Theorem VIII.3 and its corollary], the symmetric operator T_0 is essentially self-adjoint if $\ker(T_0^* - z) = \{0\}$ for $z = i$ and $z = -i$. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and $\psi \in \ker(T_0^* - z)$. By Lemma 3.6, we have $T_{\text{form}} \psi = z\psi$ pointwise. Let $S \subset \mathcal{V}$ be finite. Multiplying the equation at v by $\overline{\psi(v)}$ and summing over $v \in S$, we obtain

$$z \|P_S \psi\|^2 = \sum_{v \in S} \sum_{u \in S} \overline{\psi(v)} k_{vu} \psi(u) + \sum_{v \in S} \sum_{u \notin S} \overline{\psi(v)} k_{vu} \psi(u) = \sum_{u, v \in S} \overline{\psi(v)} k_{vu} \psi(u) + \langle P_S \psi, B_S \psi \rangle.$$

The first sum on the right is real, because the matrix $(k_{vu})_{u,v \in S}$ is Hermitian. Taking imaginary parts, and using that $B_S \psi = B_S P_{V \setminus S} \psi$, we find

$$|\operatorname{Im} z| \|P_S \psi\|^2 \leq |\langle P_S \psi, B_S P_{V \setminus S} \psi \rangle| \leq \|B_S\| \|P_S \psi\| \|P_{V \setminus S} \psi\|. \quad (3.4)$$

We apply (3.4) with $S = S_n$. As $n \rightarrow \infty$, we have $\|P_{S_n} \psi\| \rightarrow \|\psi\|$ and $\|P_{V \setminus S_n} \psi\| \rightarrow 0$, since $S_n \uparrow V$ and $\psi \in \ell^2(V)$. The right side of (3.4) is at most $M \|\psi\| \|P_{V \setminus S_n} \psi\|$, which tends to zero. We conclude that $|\operatorname{Im} z| \|\psi\|^2 = 0$ and $\psi = 0$. $\square$

A bounded-boundary exhaustion of the tree

For $u \in \mathbb{U}$, set

$$R_u^2 = \frac{1}{p} \sum_{j=1}^{N_u} |w_{uj}|^2.$$

For $u \in V$, this is $\sum_{j \leq N_u} |h_{u,uj}|^2$, the sum of the squared weights of the edges from u to its children. Since $R_o < \infty$ almost surely and $N_o \mathbf{1}_{\{R_o > M\}} \leq N_o$, which is integrable, by dominated convergence we have $\mathbb{E}[N_o \mathbf{1}_{\{R_o > M\}}] \rightarrow 0$ as $M \rightarrow \infty$. We fix a deterministic $M = M(p) < \infty$ such that

$$m_M = \mathbb{E}[N_o \mathbf{1}_{\{R_o > M\}}] < 1. \quad (3.5)$$

We call $u \in \mathbb{U}$ *good* if $R_u \leq M$ and *bad* otherwise. For $v \in V$, let $\lambda(v)$ be the number of good vertices among the strict ancestors of v , and set

$$S_n = \{v \in V : \lambda(v) \leq n\}, \quad n \geq 0.$$

Lemma 3.8. *Let $p > 0$. There exists an event Ω_{sa} with $\mathbb{P}(\Omega_{\text{sa}}) = 1$ on which the following holds. Every S_n is finite, $o \in S_n \subset S_{n+1}$, every ancestor of a vertex of S_n belongs to S_n , and $\bigcup_n S_n = V$. Further, for every n , every vertex outside S_n has at most one neighbor in S_n , and*

$$\sum_{u \notin S_n, u \sim v} |h_{vu}|^2 \leq M^2 \quad \text{for every } v \in S_n. \quad (3.6)$$

In particular, the boundary operators of the network (V, h) satisfy $\|B_{S_n}\| \leq M$.

Proof. The proof shows that the vertices reachable from a given vertex through bad vertices form a subcritical Galton–Watson tree, and that the sets S_n are finite unions of such trees.

STEP 1. *The bad clusters are subcritical Galton–Watson trees.* For $u \in \mathbb{U}$, let $\chi_u = N_u \mathbf{1}_{\{R_u > M\}}$. This is a function of N_u and of $(w_{uj})_{j \in \mathbb{N}}$. Since for distinct u the variables N_u are distinct and the families $(w_{uj})_{j \in \mathbb{N}}$ are indexed by disjoint subsets of $\mathbb{U}$, the family $(\chi_u)_{u \in \mathbb{U}}$ is independent and identically distributed, with mean $m_M < 1$. For $x \in \mathbb{U}$, define the *bad cluster*

$$K_x = \{xj_1 \cdots j_k : k \geq 0, j_i \leq \chi_{xj_1 \cdots j_{i-1}} \text{ for } 1 \leq i \leq k\}.$$

Then K_x is a Galton–Watson tree whose offspring numbers $(\chi_{xu})_{u \in \mathbb{U}}$ are independent with mean m_M . Let Z_k^x be the number of words of length $|x| + k$ in K_x . The event $\{u \in K_x\}$ is determined by the variables χ at the strict ancestors of u , which are independent of χ_u . Summing over the countably many words u of length $|x| + k$, we obtain

$$\mathbb{E}Z_{k+1}^x = \sum_{|u|=|x|+k} \mathbb{E}[\mathbf{1}_{\{u \in K_x\}} \chi_u] = m_M \mathbb{E}Z_k^x,$$

so $\mathbb{E}Z_k^x = m_M^k$ and $\mathbb{E}|K_x| = (1 - m_M)^{-1} < \infty$. Let Ω_{sa} be the event that K_x is finite for every $x \in \mathbb{U}$; as a countable intersection of events of probability one, it satisfies $\mathbb{P}(\Omega_{\text{sa}}) = 1$.

Since $j \leq \chi_u$ holds exactly when u is bad and $j \leq N_u$, we have, for $x \in V$,

$$K_x = \{y \in V(\mathcal{T}_x) : \text{every vertex } u \neq y \text{ on the path from } x \text{ to } y \text{ is bad}\}. \quad (3.7)$$

STEP 2. *The sets S_n are finite and exhaust V .* For $v \in V$ and $j \leq N_v$, we have $\lambda(vj) = \lambda(v) + \mathbf{1}_{\{v \text{ good}\}}$. This implies that λ is nondecreasing along every path away from the root, and S_n contains every ancestor of its vertices, and $o \in S_n$ since $\lambda(o) = 0$. Also $\lambda(v) \leq |v|$, which gives $\bigcup_n S_n = V$. We show by induction that S_n is finite on Ω_{sa} . By (3.7), we have $S_0 = K_o$. Let $n \geq 0$ and let $v \in V$ satisfy $\lambda(v) = n + 1$. Let g be the strict ancestor of v of largest generation which is good, and let t be the child of g on the path from g to v (possibly $t = v$). Since the good strict ancestors of v are g and the good strict ancestors of g , we have $\lambda(g) = n$. All vertices on the path from t to v , other than v , are bad, and (3.7) implies that $v \in K_t$. We conclude that

$$\{v \in V : \lambda(v) = n + 1\} \subset \bigcup_{g \in S_n} \bigcup_{j=1}^{N_g} K_{gj},$$

which is a finite union of finite sets on Ω_{sa} when S_n is finite.

STEP 3. *The boundary of S_n .* Let $v \in S_n$ and let $u \sim v$ with $u \notin S_n$. Since the parent of v belongs to S_n , the vertex u is a child of v , and $\lambda(u) = \lambda(v) + \mathbf{1}_{\{v \text{ good}\}} > n$ forces v to be good. It follows that the left side of (3.6) vanishes if v is bad, and is at most $R_v^2 \leq M^2$ if v is good. If $u \notin S_n$, then the children of u satisfy $\lambda \geq \lambda(u) > n$, and the only possible neighbor of u in S_n is its parent.

The boundary operator $B = B_{S_n}$ has matrix entries $B_{vu} = h_{vu}$ for $v \in S_n$ and $u \notin S_n$, and all its other entries vanish. Since each column of B has at most one nonzero entry, the matrix BB^* , which acts on $\ell^2(S_n)$ and has entries $\sum_{u \notin S_n} h_{vu} \overline{h_{v'u'}}$, is diagonal. Its diagonal entries are the left side of (3.6), which gives $\|B\|^2 = \|BB^*\| \leq M^2$. $\square$

Dirichlet truncations

Lemma 3.9. *Let T and $T_1, T_2, \dots$ be self-adjoint operators on a Hilbert space $\mathfrak{h}$. Let $\mathcal{D}$ be a core for T with $\mathcal{D} \subset \text{Dom}(T_n)$ for every n , and suppose that $T_n \varphi \rightarrow T \varphi$ for every $\varphi \in \mathcal{D}$. Then*

$(T_n - z)^{-1}f \rightarrow (T - z)^{-1}f$ for every $f \in \mathfrak{h}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Further, for all $f, g \in \mathfrak{h}$, we have $\langle f, (T_n - z)^{-1}g \rangle \rightarrow \langle f, (T - z)^{-1}g \rangle$ uniformly for z in compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Proof. The first assertion is a variant of [113, Theorem VIII.25(a)], where $\mathcal{D}$ is assumed to be a common core for T and all T_n ; the following argument does not use this assumption. Fix $z \in \mathbb{C} \setminus \mathbb{R}$. For $\varphi \in \mathcal{D}$ and $f = (T - z)\varphi$, we have

$$(T_n - z)^{-1}f - (T - z)^{-1}f = (T_n - z)^{-1}((T - z)\varphi - (T_n - z)\varphi) = (T_n - z)^{-1}(T - T_n)\varphi,$$

whose norm is at most $|\operatorname{Im} z|^{-1}\|(T - T_n)\varphi\| \rightarrow 0$. The set $(T - z)\mathcal{D}$ is dense in $\mathfrak{h}$: every $f \in \mathfrak{h}$ equals $(T - z)\psi$ for some $\psi \in \operatorname{Dom}(T)$, and if $\varphi_k \in \mathcal{D}$ converge to ψ in the graph norm of T , then $(T - z)\varphi_k \rightarrow f$. Since all resolvents have norm at most $|\operatorname{Im} z|^{-1}$, the convergence extends from this dense set to all f .

For the second assertion, let $K \subset \mathbb{C} \setminus \mathbb{R}$ be compact and $d = \operatorname{dist}(K, \mathbb{R}) > 0$. For every self-adjoint S and $z, z' \in K$, the resolvent identity $(S - z)^{-1} - (S - z')^{-1} = (z - z')(S - z)^{-1}(S - z')^{-1}$ gives

$$|\langle f, (S - z)^{-1}g \rangle - \langle f, (S - z')^{-1}g \rangle| \leq \frac{|z - z'| \|f\| \|g\|}{d^2}.$$

Then the functions $z \mapsto \langle f, (T_n - z)^{-1}g \rangle$ and $z \mapsto \langle f, (T - z)^{-1}g \rangle$ are Lipschitz on K with a common constant. Given $\varepsilon > 0$, we cover K by finitely many discs of radius $\varepsilon d^2 / (3\|f\|\|g\| + 1)$ centered in K , and choose n_0 such that the pointwise convergence holds within $\varepsilon/3$ at the finitely many centers for $n \geq n_0$. Then the difference is at most ε on K for $n \geq n_0$. $\square$

For a component C of a finite removal and a finite set $\Lambda \subset C$, the *Dirichlet truncation* $\mathcal{H}_C^{[\Lambda]} = P_\Lambda \mathcal{H}_{C,0} P_\Lambda$ is the bounded self-adjoint operator on $\ell^2(C)$ which acts on $\ell^2(\Lambda)$ by the finite matrix $(h_{uv})_{u,v \in \Lambda}$ and vanishes on $\ell^2(C \setminus \Lambda)$. When $C = V$, we write $\mathcal{H}^{[\Lambda]}$.

The self-adjoint operators

For a finite removal (F, D) , define the bounded self-adjoint operator of finite rank

$$W_{F,D} = \sum_{\{u,v\} \text{ deleted}} (h_{uv}|\delta_u\rangle\langle\delta_v| + h_{vu}|\delta_v\rangle\langle\delta_u|)$$

on $\ell^2(V)$, where the sum runs over the deleted edges of (F, D) , and $|f\rangle\langle g|$ denotes the operator $\varphi \mapsto \langle g, \varphi \rangle f$.

Theorem 3.10. *Let $p > 0$, and let Ω_{sa} be the event of Lemma 3.8; we have $\mathbb{P}(\Omega_{\text{sa}}) = 1$, and $\mathbb{P}^\infty(\Omega_{\text{sa}}) = 1$ when $p > 1$. On Ω_{sa} , the following statements hold simultaneously for every finite removal (F, D) of $\mathcal{T}$ and every component C of (F, D) .*

(i) The operator $\mathcal{H}_{C,0}$ is essentially self-adjoint on $c_c(C)$. We write $\mathcal{H}_C$ for its closure, $\mathcal{H} = \mathcal{H}_V$ for the operator of the full tree, and $\mathcal{H}_v = \mathcal{H}_{V(\tau_v)}$ for the operator of the forward subtree at $v \in V$.

(ii) The domain of $\mathcal{H}_C$ is the maximal domain:

$$\text{Dom}(\mathcal{H}_C) = \{f \in \ell^2(C) : \mathcal{H}_{C,\text{form}}f \in \ell^2(C)\},$$

and $\mathcal{H}_C f = \mathcal{H}_{C,\text{form}}f$ for $f \in \text{Dom}(\mathcal{H}_C)$.

(iii) Let $\Lambda_1 \subset \Lambda_2 \subset \dots$ be finite sets with $\bigcup_n \Lambda_n = C$. Then $(\mathcal{H}_C^{[\Lambda_n]} - z)^{-1}f \rightarrow (\mathcal{H}_C - z)^{-1}f$ for every $f \in \ell^2(C)$ and $z \in \mathbb{C} \setminus \mathbb{R}$, and for all $u, v \in C$,

$$\langle \delta_u, (\mathcal{H}_C^{[\Lambda_n]} - z)^{-1} \delta_v \rangle \rightarrow \langle \delta_u, (\mathcal{H}_C - z)^{-1} \delta_v \rangle$$

uniformly for z in compact subsets of $\mathbb{C} \setminus \mathbb{R}$. This applies to the balls $\Lambda_n = \{v \in C : \text{dist}_C(v, v_0) \leq n\}$ around any $v_0 \in C$, and for $C = V$ to the balls $\{v \in V : |v| \leq n\}$.

(iv) Let $C_1, \dots, C_m$ be the components of (F, D) . Then

$$\mathcal{H} - W_{F,D} = \mathcal{H}_{C_1} \oplus \dots \oplus \mathcal{H}_{C_m} \oplus 0_F$$

with respect to $\ell^2(V) = \ell^2(C_1) \oplus \dots \oplus \ell^2(C_m) \oplus \ell^2(F)$, where 0_F is the zero operator on $\ell^2(F)$. In particular $\text{Dom}(\mathcal{H}) = \text{Dom}(\mathcal{H}_{C_1}) \oplus \dots \oplus \text{Dom}(\mathcal{H}_{C_m}) \oplus \ell^2(F)$.

We write $\mathcal{H}^{F,D} = \mathcal{H}_{C_1} \oplus \dots \oplus \mathcal{H}_{C_m}$ for the reduced operator on $\ell^2(V \setminus F)$, and $\mathcal{H}^{(y)} = \mathcal{H}^{\{y\}, \emptyset}$ for the operator with the vertex y deleted.

Proof. We fix a realization in Ω_{sa} ; all statements below are deterministic consequences of the conclusions of Lemma 3.8, which do not depend on (F, D) .

STEP 1. *The full tree.* By Lemma 3.8, the network (V, h) satisfies the hypotheses of Lemma 3.7, which implies that $\mathcal{H}_0$ is essentially self-adjoint. Lemma 3.6 identifies the domain of its closure $\mathcal{H}$ with the maximal domain.

STEP 2. *Finite removals.* Let (F, D) be a finite removal and $W = W_{F,D}$. Since W is bounded and self-adjoint, the Kato–Rellich theorem [112, Theorem X.12] (with relative bound zero) shows that $\mathcal{H} - W$ is self-adjoint on $\text{Dom}(\mathcal{H})$ and essentially self-adjoint on every core of $\mathcal{H}$, in particular on $c_c(V)$. On $c_c(V)$, the operator $\mathcal{H}_0 - W$ is the weighted adjacency operator of the graph on V with the retained edges. By Lemma 3.5(i), this graph has the components $C_1, \dots, C_m$ and the isolated vertices of F . Hence

$$(\mathcal{H}_0 - W)|_{c_c(V)} = \mathcal{H}_{C_1,0} \oplus \dots \oplus \mathcal{H}_{C_m,0} \oplus 0 \quad (3.8)$$

with respect to $c_c(V) = c_c(C_1) \oplus \dots \oplus c_c(C_m) \oplus c_c(F)$.

The adjoint of a finite orthogonal sum of densely defined operators is the orthogonal sum of their adjoints. Indeed, a vector $(\varphi_j)_j$ is in the domain of the adjoint of $\bigoplus_j T_j$ exactly

when $(f_j)_j \mapsto \sum_j \langle \varphi_j, T_j f_j \rangle$ is bounded, and taking f supported in a single summand shows that this holds exactly when $\varphi_j \in \text{Dom}(T_j^*)$ for every j . For $z = \pm i$, it follows that the kernel of $T^* - z$, where T is the operator in (3.8), is the orthogonal sum of the kernels of $\mathcal{H}_{C_j,0}^* - z$ and of the kernel of $-z$ on $\ell^2(F)$, which is trivial. Since T is essentially self-adjoint, the kernel of $T^* - z$ is trivial, and so are the kernels of $\mathcal{H}_{C_j,0}^* - z$. Then [113, Theorem VIII.3 and its corollary] shows that every $\mathcal{H}_{C_j,0}$ is essentially self-adjoint. The closure of a finite orthogonal sum is the orthogonal sum of the closures, since the graph of the sum is the orthogonal sum of the graphs. Taking closures in (3.8) proves (iv), and (i) holds for every component. Statement (ii) follows from Lemma 3.6 applied to the network (C, h) .

STEP 3. Dirichlet truncations. Let Λ_n be as in (iii) and $f \in c_c(C)$. The vector $\mathcal{H}_{C,0}f$ is supported on the union of $\text{supp } f$ and its neighbors in C , which is finite. When Λ_n contains this union, we have $\mathcal{H}_C^{[\Lambda_n]}f = P_{\Lambda_n}\mathcal{H}_{C,0}f = \mathcal{H}_{C,0}f$. Since $c_c(C)$ is a core for $\mathcal{H}_C$, Lemma 3.9 implies (iii). Balls in the locally finite graph C are finite and exhaust C . $\square$

Remark 3.11. Statement (ii) is used in the following form. Let C be as in Theorem 3.10, let $E \in \mathbb{R}$, and let $g \in \ell^2(C)$. If $F \in \ell^2(C)$ satisfies $(\mathcal{H}_{C,\text{form}}F)(v) - EF(v) = g(v)$ for every $v \in C$, then $\mathcal{H}_{C,\text{form}}F = EF + g \in \ell^2(C)$, and (ii) implies that $F \in \text{Dom}(\mathcal{H}_C)$ and $(\mathcal{H}_C - E)F = g$. In particular, a vector $F \in \ell^2(V)$ satisfying $(\mathcal{H}_{\text{form}} - E)F = \delta_o$ pointwise lies in $\text{Dom}(\mathcal{H})$ and satisfies $(\mathcal{H} - E)F = \delta_o$; in particular, $\delta_o \in \text{Ran}(\mathcal{H} - E)$.

Remark 3.12. The exhaustion of Lemma 3.8 also applies directly to a component C of a finite removal: the sets $S_n \cap C$ exhaust C , each vertex of $C \setminus S_n$ has at most one neighbor in $S_n \cap C$, and the boundary sums in C are bounded by the left side of (3.6). Then Lemma 3.7 yields (i) without the perturbation argument. Statement (iv), which identifies the domains, is used repeatedly below.

The following consequence of (iv) decouples a component along finitely many further deleted edges.

Lemma 3.13. *On Ω_{sa} , let C be a component of a finite removal (F, D) , and let D' be a finite set of edges of $\mathcal{T}$ with both endpoints in C . Let $C'_1, \dots, C'_k$ be the components of $(F, D \cup D')$ contained in C . Then*

$$\mathcal{H}_C = (\mathcal{H}_{C'_1} \oplus \dots \oplus \mathcal{H}_{C'_k}) + W_{D'}, \quad \text{Dom}(\mathcal{H}_C) = \text{Dom}(\mathcal{H}_{C'_1}) \oplus \dots \oplus \text{Dom}(\mathcal{H}_{C'_k}),$$

where $W_{D'} = \sum_{\{u,v\} \in D'} (h_{uv}|\delta_u\rangle\langle\delta_v| + h_{vu}|\delta_v\rangle\langle\delta_u|)$ acts on $\ell^2(C)$. In particular, if $y \in C$ has the neighbors $x_1, \dots, x_d$ in C , then with respect to $\ell^2(C) = \mathbb{C}\delta_y \oplus \ell^2(C_{x_1 \rightarrow y}) \oplus \dots \oplus \ell^2(C_{x_d \rightarrow y})$,

$$\mathcal{H}_C = (0 \oplus \mathcal{H}_{C_{x_1 \rightarrow y}} \oplus \dots \oplus \mathcal{H}_{C_{x_d \rightarrow y}}) + \sum_{i=1}^d (h_{yx_i}|\delta_y\rangle\langle\delta_{x_i}| + h_{x_i y}|\delta_{x_i}\rangle\langle\delta_y|). \quad (3.9)$$

Proof. Since the edges of D' are retained by (F, D) , we have $W_{F, D \cup D'} = W_{F, D} + W_{D'}$. The components of $(F, D \cup D')$ are the components of (F, D) other than C , together with $C'_1, \dots, C'_k$. Applying Theorem 3.10(iv) to both finite removals, we find that $\mathcal{H} - W_{F, D} - W_{D'}$ equals the orthogonal sum of the operators $\mathcal{H}_{C''}$ over the components $C'' \neq C$ of (F, D) , of $\mathcal{H}_C - W_{D'}$, and of 0_F , and that it also equals the same sum with $\mathcal{H}_C - W_{D'}$ replaced by $\mathcal{H}_{C'_1} \oplus \dots \oplus \mathcal{H}_{C'_k}$. Both decompositions are orthogonal with respect to the same splitting of $\ell^2(V)$, and their parts acting in $\ell^2(C)$ agree, including their domains. Since $W_{D'}$ is bounded, $\mathcal{H}_C - W_{D'}$ has the domain $\text{Dom}(\mathcal{H}_C)$.

For (3.9), take D' to be the set of edges of C at y . By Lemma 3.5(ii), the components of $(F, D \cup D')$ inside C are $\{y\}$ and the branches $C_{x_i \rightarrow y}$, and the operator of the isolated vertex y is zero. $\square$

3.4 Resolvents and the cavity recursion

Definitions

We begin with the elementary properties of resolvent entries of a self-adjoint operator.

Lemma 3.14. *Let T be a self-adjoint operator on a Hilbert space $\mathfrak{h}$, let $f, g \in \mathfrak{h}$, and let $z = E + i\eta \in \mathbb{C} \setminus \mathbb{R}$.*

- (i) *The function $z \mapsto \langle f, (T - z)^{-1}g \rangle$ is analytic on $\mathbb{C} \setminus \mathbb{R}$, bounded by $\|f\| \|g\| / |\eta|$, and satisfies $\langle f, (T - z)^{-1}g \rangle = \overline{\langle g, (T - \bar{z})^{-1}f \rangle}$.*
- (ii) *The function $m(z) = \langle f, (T - z)^{-1}f \rangle$ satisfies*

$$m(z) = \int_{\mathbb{R}} \frac{\mu_f^T(d\lambda)}{\lambda - z}, \quad \text{Im } m(z) = \eta \|(T - z)^{-1}f\|^2 = \eta \int_{\mathbb{R}} \frac{\mu_f^T(d\lambda)}{|\lambda - z|^2},$$

and $m(\bar{z}) = \overline{m(z)}$. In particular, if $f \neq 0$ and $\eta > 0$, then $\text{Im } m(z) > 0$.

Proof. The resolvent is an analytic operator-valued function on $\mathbb{C} \setminus \mathbb{R}$ with norm at most $|\eta|^{-1}$, and $((T - z)^{-1})^* = (T - \bar{z})^{-1}$; this gives (i). The integral formula is the spectral theorem. By the resolvent identity, $(T - z)^{-1} - (T - \bar{z})^{-1} = (z - \bar{z})(T - \bar{z})^{-1}(T - z)^{-1}$, and we have

$$\begin{aligned} 2i \text{Im } m(z) &= m(z) - \overline{m(z)} = \langle f, ((T - z)^{-1} - (T - \bar{z})^{-1})f \rangle \\ &= 2i\eta \langle (T - z)^{-1}f, (T - z)^{-1}f \rangle, \end{aligned}$$

where we used (i) in the form $\overline{m(z)} = m(\bar{z})$. Since the resolvent is injective, $\text{Im } m(z) > 0$ when $f \neq 0$ and $\eta > 0$. $\square$

Definition 3.15. On Ω_{sa} , let (F, D) be a finite removal, let C be one of its components, and let $z \in \mathbb{C} \setminus \mathbb{R}$.

(i) The *resolvent entries* are

$$G_{uv}(z) = \langle \delta_u, (\mathcal{H} - z)^{-1} \delta_v \rangle, \quad G_{uv}^C(z) = \langle \delta_u, (\mathcal{H}_C - z)^{-1} \delta_v \rangle,$$

$$G_{uv}^{F,D}(z) = \langle \delta_u, (\mathcal{H}^{F,D} - z)^{-1} \delta_v \rangle,$$

for $u, v \in V$, $u, v \in C$, and $u, v \in V \setminus F$, respectively. We write $G^{(y)}$ for $G^{\{y\}, \emptyset}$. Since $\mathcal{H}^{F,D}$ is the orthogonal sum of the operators of its components, $G_{uv}^{F,D} = G_{uv}^C$ if $u, v \in C$, and $G_{uv}^{F,D} = 0$ if u and v lie in different components.

(ii) For adjacent $u, w \in C$, the *cavity resolvent* is the diagonal entry at u of the operator of the branch $C_{u \rightarrow w}$:

$$\Gamma_{u \rightarrow w}^C(z) = G_{uu}^{C_{u \rightarrow w}}(z).$$

For $C = V$ we write $\Gamma_{u \rightarrow w}$. The *forward cavity* at $v \in V \setminus \{o\}$ is $\Gamma_v = \Gamma_{v \rightarrow v^-}$; since $V_{v \rightarrow v^-} = V(\mathcal{T}_v)$, it equals $\langle \delta_v, (\mathcal{H}_v - z)^{-1} \delta_v \rangle$. At the root we set $\Gamma_o = G_{oo}$, the diagonal entry at the root of the operator of its forward subtree $\mathcal{T}_o = \mathcal{T}$.

(iii) The *spectral measure* of $v \in V$ is $\mu_v = \mu_{\delta_v}^{\mathcal{H}}$, and $G_{vv}(z) = \int \mu_v(d\lambda) / (\lambda - z)$.

By Lemma 3.14, all these entries are analytic in $z \in \mathbb{C} \setminus \mathbb{R}$ and bounded by $1/|\operatorname{Im} z|$, and the diagonal entries and cavity resolvents map $\mathbb{H}$ to $\mathbb{H}$. They are also measurable functions of the marked tree. Indeed, for a fixed finite removal and fixed words u, v , Theorem 3.10(iii) with $\Lambda_n = \{x \in C : |x| \leq n\}$ expresses $G_{uv}^C(z)$ as the limit of entries of the inverses of finite matrices, which are measurable functions of $(N_x)_{|x| \leq n}$ and $(w_x)_{|x| \leq n}$. For a random finite removal, we decompose according to its countably many values. In particular, let Φ_z be the measurable function of the families $((N_u)_{u \in \mathbb{U}}, (w_u)_{u \neq o})$ defined as the limit of $\langle \delta_o, (\mathcal{H}^{\{v \leq n\}} - z)^{-1} \delta_o \rangle$ when this limit exists, and as zero otherwise. We write $\Phi(\mathcal{T})$ for the function $z \mapsto \Phi_z(\mathcal{T})$ on $\mathbb{C} \setminus \mathbb{R}$. On Ω_{sa} we have $\Gamma_o(z) = \Phi_z(\mathcal{T})$, and for $j \leq N_o$ the relabelling $ju \mapsto u$ identifies the truncations of $\mathcal{H}_j$ to balls around j with the truncations of the operator of $\theta_j \mathcal{T}$ to balls around o . We conclude that

$$\Gamma_j(z) = \Phi_z(\theta_j \mathcal{T}), \quad 1 \leq j \leq N_o. \quad (3.10)$$

The Schur complement formula

Proposition 3.16. *Let $p > 0$. On Ω_{sa} , the following statements hold for every $z \in \mathbb{C} \setminus \mathbb{R}$; statements (i) and (ii) hold for every finite removal (F, D) and every component C of (F, D) .*

(i) For every $u \in C$,

$$G_{uu}^C(z) = - \left(z + \sum_{x \in C, x \sim u} |h_{ux}|^2 \Gamma_{x \rightarrow u}^C(z) \right)^{-1}. \quad (3.11)$$

(ii) For all adjacent $u, w \in C$,

$$\Gamma_{u \rightarrow w}^C(z) = - \left(z + \sum_{x \in C, x \sim u, x \neq w} |h_{ux}|^2 \Gamma_{x \rightarrow u}^C(z) \right)^{-1}. \quad (3.12)$$

(iii) Let $N = N_o$, $Y_j = |w_j|^2$, and $\Gamma_j(z) = \Phi_z(\theta_j \mathcal{T})$ for $j \geq 1$. Then

$$\Gamma_o(z) = -\left(z + \frac{1}{p} \sum_{j=1}^N Y_j \Gamma_j(z)\right)^{-1}, \quad (3.13)$$

where N has the law $\text{Pois}(p)$, the Y_j are independent with the law $\text{Exp}(1)$, the random analytic functions Γ_j on $\mathbb{C} \setminus \mathbb{R}$ are independent copies of Γ_o , and the three families N , $(Y_j)_{j \geq 1}$, $(\Gamma_j)_{j \geq 1}$ are independent. In particular, $\Gamma_o(z) \stackrel{d}{=} -\left(z + p^{-1} \sum_{j=1}^N Y_j \Gamma_j(z)\right)^{-1}$. For $j \leq N$, the function Γ_j is the forward cavity at the child j of the root, and $\Gamma_o(z)$ is the limit of the root entries of the resolvents of the Dirichlet truncations to the balls $\{v \in V : |v| \leq n\}$.

In (i) and (ii), the imaginary part of the expression in parentheses has the sign of $\text{Im } z$ and absolute value at least $|\text{Im } z|$. For the full tree, these formulas read

$$\Gamma_v(z) = -\left(z + \sum_{j=1}^{N_v} |h_{v,vj}|^2 \Gamma_{vj}(z)\right)^{-1}, \quad v \in V, \quad (3.14)$$

$$G_{vv}(z) = -\left(z + |h_{vv-}|^2 \Gamma_{v- \rightarrow v}(z) + \sum_{j=1}^{N_v} |h_{v,vj}|^2 \Gamma_{vj}(z)\right)^{-1}, \quad v \in V \setminus \{o\}, \quad (3.15)$$

$$\Gamma_{u \rightarrow v}(z) = -\left(z + \mathbf{1}_{\{u \neq o\}} |h_{uu-}|^2 \Gamma_{u- \rightarrow u}(z) + \sum_{j \leq N_u, uj \neq v} |h_{u,uj}|^2 \Gamma_{uj}(z)\right)^{-1}, \quad (3.16)$$

where in (3.16), $u \in V$ and v is a child of u .

The cavity $\Gamma_{u \rightarrow v}$ in (3.16) is a *rootward* cavity: it is the diagonal entry at u of the operator on $V \setminus V(\mathcal{T}_v)$. Formula (3.16) expresses it through the rootward cavity one generation closer to the root and through forward cavities; along a path from the root, the rootward cavities can be computed recursively from the root downwards. Note that (3.14) with $v = o$ is the formula for $\Gamma_o = G_{oo}$.

Proof. Fix C , $u \in C$ and z , and let $x_1, \dots, x_d$ be the neighbors of u in C . By Lemma 3.13, with respect to $\ell^2(C) = \mathbb{C}\delta_u \oplus \mathfrak{k}$, where $\mathfrak{k} = \bigoplus_i \ell^2(C_{x_i \rightarrow u})$, we have $\text{Dom}(\mathcal{H}_C) = \mathbb{C}\delta_u \oplus \text{Dom}(\mathcal{D})$ with $\mathcal{D} = \bigoplus_i \mathcal{H}_{C_{x_i \rightarrow u}}$, and

$$\mathcal{H}_C(\alpha\delta_u + g) = \langle a, g \rangle \delta_u + \alpha a + \mathcal{D}g, \quad \alpha \in \mathbb{C}, g \in \text{Dom}(\mathcal{D}),$$

where $a = \sum_i h_{x_i u} \delta_{x_i} \in \mathfrak{k}$. Here we used that $\langle a, g \rangle = \sum_i \overline{h_{x_i u}} g(x_i) = \sum_i h_{ux_i} g(x_i)$. The operator $\mathcal{D}$ is self-adjoint on $\mathfrak{k}$, and since it is block diagonal,

$$\langle a, (\mathcal{D} - z)^{-1} a \rangle = \sum_{i=1}^d |h_{ux_i}|^2 \langle \delta_{x_i}, (\mathcal{H}_{C_{x_i \rightarrow u}} - z)^{-1} \delta_{x_i} \rangle = \sum_{i=1}^d |h_{ux_i}|^2 \Gamma_{x_i \rightarrow u}^C(z).$$

By Lemma 3.14(ii), the imaginary part of this quantity has the sign of $\text{Im } z$ (or vanishes), and $s = -z - \langle a, (\mathcal{D} - z)^{-1}a \rangle$ satisfies $|\text{Im } s| \geq |\text{Im } z| > 0$. Set $\alpha = s^{-1}$ and $g = -\alpha(\mathcal{D} - z)^{-1}a \in \text{Dom}(\mathcal{D})$. Then $f = \alpha\delta_u + g \in \text{Dom}(\mathcal{H}_C)$, and

$$(\mathcal{H}_C - z)f = (\langle a, g \rangle - z\alpha)\delta_u + \alpha a + (\mathcal{D} - z)g = \alpha(-\langle a, (\mathcal{D} - z)^{-1}a \rangle - z)\delta_u = \delta_u.$$

This shows that $f = (\mathcal{H}_C - z)^{-1}\delta_u$, and $G_{uu}^C(z) = \langle \delta_u, f \rangle = \alpha = s^{-1}$, which is (3.11). Reading off the other coordinates of f , we also obtain, for every i and every $v \in C_{x_i \rightarrow u}$,

$$G_{vu}^C(z) = -G_{uu}^C(z) h_{x_i u} G_{vx_i}^{C_{x_i \rightarrow u}}(z). \quad (3.17)$$

For (ii), let $C' = C_{u \rightarrow w}$, which is a component of the finite removal $(F, D \cup \{\{u, w\}\})$ by Lemma 3.5(ii). The neighbors of u in C' are its neighbors in C other than w . For such a neighbor x , every vertex of $C_{x \rightarrow u}$ is joined to w by a path through x and u . This gives $C_{x \rightarrow u} \subset C'$, and Lemma 3.5(iii) implies that $C'_{x \rightarrow u} = C_{x \rightarrow u}$. Applying (3.11) to C' at u , and using $G_{uu}^{C'} = \Gamma_{u \rightarrow w}^C$ and $\Gamma_{x \rightarrow u}^{C'} = \Gamma_{x \rightarrow u}^C$, we obtain (3.12).

For the full tree, the branch $V_{vj \rightarrow v}$ is $V(\mathcal{T}_{vj})$ and $\Gamma_{vj \rightarrow v} = \Gamma_{vj}$, and the neighbors of v are its children and, if $v \neq o$, its parent. With these identifications, (3.14) and (3.15) are (3.12) with $(u, w) = (v, v^-)$ (or (3.11) at $u = o$) and (3.11) at $u = v$. Formula (3.16) is (3.12) with $(u, w) = (u, v)$, since the neighbors of u other than v are its parent (if $u \neq o$) and its other children.

For (iii), on Ω_{sa} , (3.14) at $v = o$, the identity $|h_{oj}|^2 = Y_j/p$, and (3.10) give (3.13); the functions Γ_j with $j > N$ do not enter it. By (3.1), the marked trees $\theta_j \mathcal{T}$ are independent copies of $\mathcal{T}$, independent of N_o and $(w_j)_{j \geq 1}$. The functions $\Gamma_j = \Phi(\theta_j \mathcal{T})$ are independent copies of $\Phi(\mathcal{T})$, which equals Γ_o almost surely, and they are independent of N and $(Y_j)_{j \geq 1}$. The variables N_o and $(w_j)_{j \geq 1}$ are independent by construction, and $Y_j = |w_j|^2$ has the law $\text{Exp}(1)$. The last statement is Theorem 3.10(iii). $\square$

Remark 3.17. Let $x \in V$ and let $o = v_0, v_1, \dots, v_k = x$ be the path from the root to x . When $\mathcal{T}$ is rerooted at x , the parent of a vertex $v \neq x$ is the next vertex on the path from v to x . If v is not on the path, this is v^- , since the path from v to x first climbs to the last common ancestor of v and x . The forward cavities of the rerooted tree are Γ_v for v off the path, and $\Gamma_{v_j \rightarrow v_{j+1}}$ for $0 \leq j < k$; the latter are rootward cavities, and (3.12) computes them recursively from v_0 to v_{k-1} in terms of the forward cavities of the subtrees hanging off the path. The same holds for every component of a finite removal, rerooted at any of its vertices.

The path product formula

Proposition 3.18. *On Ω_{sa} , let C be a component of a finite removal, let $u = v_0, v_1, \dots, v_L = v$ be a path in C with $L \geq 1$, and let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$G_{uv}^C(z) = (-1)^L G_{uu}^C(z) \prod_{j=1}^L h_{v_{j-1} v_j} \Gamma_{v_j \rightarrow v_{j-1}}^C(z), \quad (3.18)$$

$$G_{vu}^C(z) = (-1)^L G_{uu}^C(z) \prod_{j=1}^L h_{v_j v_{j-1}} \Gamma_{v_j \rightarrow v_{j-1}}^C(z). \quad (3.19)$$

If u and v lie in different components of a finite removal (F, D) , then $G_{uv}^{F,D}(z) = 0$. Every finite connected set $\Lambda \subset V$ is a component of the finite removal that deletes the edges between Λ and $V \setminus \Lambda$; in particular, these identities also hold for the Dirichlet truncation $\mathcal{H}^{[\Lambda]}$, for every finite connected Λ .

Proof. We prove (3.19) by induction on L , simultaneously for all components of all finite removals. For $L = 1$, it is (3.17) with $x_i = v = v_1$, since $G_{v_1 v_1}^{C_{v_1 \rightarrow u}} = \Gamma_{v_1 \rightarrow u}^C$. Let $L \geq 2$ and let $C' = C_{v_1 \rightarrow u}$, which is a component of a finite removal. The vertices $v_1, \dots, v_L$ belong to C' , because their paths to u pass through v_1 . By (3.17) and the induction hypothesis applied in C' to the path $v_1, \dots, v_L$, we have

$$G_{vu}^C = -G_{uu}^C h_{v_1 u} G_{v v_1}^{C'} = -G_{uu}^C h_{v_1 u} (-1)^{L-1} G_{v_1 v_1}^{C'} \prod_{j=2}^L h_{v_j v_{j-1}} \Gamma_{v_j \rightarrow v_{j-1}}^{C'}.$$

We have $G_{v_1 v_1}^{C'} = \Gamma_{v_1 \rightarrow u}^C$. For $j \geq 2$, the path from a vertex of $C_{v_j \rightarrow v_{j-1}}$ to u passes through $v_j, v_{j-1}, \dots, v_1$. Then $C_{v_j \rightarrow v_{j-1}} \subset C'$, and Lemma 3.5(iii) implies that $\Gamma_{v_j \rightarrow v_{j-1}}^{C'} = \Gamma_{v_j \rightarrow v_{j-1}}^C$. This proves (3.19).

By Lemma 3.14(i), we have $G_{uv}^C(z) = \overline{G_{vu}^C(\bar{z})}$. Applying (3.19) at $\bar{z}$, and using $\overline{G_{uu}^C(\bar{z})} = G_{uu}^C(z)$, $\overline{\Gamma_{v_j \rightarrow v_{j-1}}^C(\bar{z})} = \Gamma_{v_j \rightarrow v_{j-1}}^C(z)$ and $\overline{h_{v_j v_{j-1}}} = h_{v_{j-1} v_j}$, we obtain (3.18). The statement for different components holds because $\mathcal{H}^{F,D}$ is an orthogonal sum over components. $\square$

The Ward identity

Lemma 3.19. On Ω_{sa} , let C be a component of a finite removal and let $z = E + i\eta \in \mathbb{C} \setminus \mathbb{R}$.

(i) For every $u \in C$,

$$\sum_{v \in C} |G_{uv}^C(z)|^2 = \sum_{v \in C} |G_{vu}^C(z)|^2 = \frac{\text{Im } G_{uu}^C(z)}{\eta}.$$

In particular, if $\eta > 0$, then $\text{Im } G_{uu}^C(z) > 0$ and $\text{Im } \Gamma_{u \rightarrow w}^C(z) > 0$ for all u and all neighbors w of u in C .

(ii) For all adjacent $u, w \in C$,

$$\begin{aligned} \text{Im } \Gamma_{u \rightarrow w}^C(z) &= |\Gamma_{u \rightarrow w}^C(z)|^2 \left(\eta + \sum_{x \in C, x \sim u, x \neq w} |h_{ux}|^2 \text{Im } \Gamma_{x \rightarrow u}^C(z) \right), \\ \text{Im } G_{uu}^C(z) &= |G_{uu}^C(z)|^2 \left(\eta + \sum_{x \in C, x \sim u} |h_{ux}|^2 \text{Im } \Gamma_{x \rightarrow u}^C(z) \right). \end{aligned}$$

(iii) Let $u, x \in C$, and let y be a neighbor of x in C which does not lie on the path from u to x . If $\eta > 0$, then

$$\operatorname{Im} G_{uu}^C(z) \geq |G_{ux}^C(z)|^2 |h_{xy}|^2 \operatorname{Im} \Gamma_{y \rightarrow x}^C(z).$$

In particular, $\operatorname{Im} G_{oo}(z) \geq |G_{ox}(z)|^2 |h_{xy}|^2 \operatorname{Im} \Gamma_y(z)$ for every $x \in V$ and every child y of x .

Proof. For (i), we have $\sum_v |G_{vu}^C(z)|^2 = \|(\mathcal{H}_C - z)^{-1} \delta_u\|^2$, which equals $\operatorname{Im} G_{uu}^C(z)/\eta$ by Lemma 3.14(ii). By Lemma 3.14(i), we have $|G_{uv}^C(z)| = |G_{vu}^C(\bar{z})|$, and the first sum equals $\operatorname{Im} G_{uu}^C(\bar{z})/(-\eta) = \operatorname{Im} G_{uu}^C(z)/\eta$. The positivity statements follow from Lemma 3.14(ii), since $\Gamma_{u \rightarrow w}^C$ is a diagonal entry of the operator of $C_{u \rightarrow w}$.

For (ii), write (3.12) as $\Gamma = -1/s$. Then $\operatorname{Im} \Gamma = \operatorname{Im} s/|s|^2 = |\Gamma|^2 \operatorname{Im} s$, and $\operatorname{Im} s$ is the expression in parentheses. The same computation applies to (3.11).

For (iii), let $B = C_{y \rightarrow x}$. Let $u = v_0, \dots, v_k = x$ be the path from u to x (with $k = 0$ if $u = x$). No vertex of this path lies in B : otherwise, by Lemma 3.5(ii), its path to x , which is part of the path from u to x , would contain y . Let $v \in B$ and let $y = v_{k+1}, \dots, v_L = v$ be the path from y to v , which lies in B . Then $v_0, \dots, v_L$ is the path from u to v . Applying (3.18) in C to the paths from u to x and from u to v , and in B to the path from y to v , and using $G_{yy}^B = \Gamma_{y \rightarrow x}^C$ and $\Gamma_{v_j \rightarrow v_{j-1}}^B = \Gamma_{v_j \rightarrow v_{j-1}}^C$ for $j \geq k+2$ (by Lemma 3.5(iii), as in the proof of Proposition 3.18), we obtain

$$G_{uv}^C(z) = -G_{ux}^C(z) h_{xy} G_{yv}^B(z), \quad v \in B.$$

Using (i) in C and in B , we find

$$\frac{\operatorname{Im} G_{uu}^C(z)}{\eta} \geq \sum_{v \in B} |G_{uv}^C(z)|^2 = |G_{ux}^C(z)|^2 |h_{xy}|^2 \sum_{v \in B} |G_{yv}^B(z)|^2 = |G_{ux}^C(z)|^2 |h_{xy}|^2 \frac{\operatorname{Im} \Gamma_{y \rightarrow x}^C(z)}{\eta}.$$

For the last statement, a child y of x is never on the path from o to x , and $\Gamma_{y \rightarrow x} = \Gamma_y$. $\square$

3.5 Gauge and parity symmetries

The phases of the edge weights can be removed by a diagonal unitary transformation, because a tree has no cycles. For $\zeta \in \mathbb{C}$, let $\sigma(\zeta) = \zeta/|\zeta|$ if $\zeta \neq 0$ and $\sigma(0) = 1$.

Lemma 3.20. *On Ω_{sa} , let C be a component of a finite removal and fix $r \in C$. For $v \in C$, let $r = v_0, v_1, \dots, v_k = v$ be the path from r to v and set $\varphi_v = \prod_{j=1}^k \sigma(h_{v_{j-1}v_j})$, and let U be the unitary operator $(Uf)(v) = \varphi_v f(v)$ on $\ell^2(C)$. Then the weighted adjacency operator of C with the nonnegative weights $|h_{uv}|$ is essentially self-adjoint on $c_c(C)$, and its closure $\mathcal{H}_C^{\text{abs}}$ satisfies $U\mathcal{H}_C U^* = \mathcal{H}_C^{\text{abs}}$. Further, for all $u, v \in C$ and $z \in \mathbb{C} \setminus \mathbb{R}$, writing G^{abs} for the resolvent entries of $\mathcal{H}_C^{\text{abs}}$,*

$$G_{uv}^C(z) = \overline{\varphi_u} \varphi_v G_{uv}^{\text{abs}}(z), \quad G_{uv}^{\text{abs}}(z) = G_{vu}^{\text{abs}}(z), \quad |G_{uv}^C(z)| = |G_{vu}^C(z)|.$$

The same identities hold for the entries of $G^{F,D}$, which vanish between different components.

Proof. Let $u, v \in C$ be adjacent, and suppose that u lies on the path from r to v . Then $\varphi_v = \varphi_u \sigma(h_{uv})$, and

$$\varphi_u h_{uv} \overline{\varphi_v} = h_{uv} \overline{\sigma(h_{uv})} = |h_{uv}|, \quad \varphi_v h_{vu} \overline{\varphi_u} = \sigma(h_{uv}) \overline{h_{uv}} = |h_{uv}|,$$

where we used $|\varphi_u| = 1$ and $h_{vu} = \overline{h_{uv}}$. Every pair of adjacent vertices is of this form after possibly exchanging u and v . Since U is diagonal, it maps $c_c(C)$ onto itself and $U\mathcal{H}_{C,0}U^*$ is the weighted adjacency operator with the weights $|h_{uv}|$ on $c_c(C)$. Since conjugation by a unitary operator commutes with taking closures and adjoints, this operator is essentially self-adjoint with closure $U\mathcal{H}_CU^*$. Then $U(\mathcal{H}_C - z)^{-1}U^* = (\mathcal{H}_C^{\text{abs}} - z)^{-1}$, and taking the (u, v) entry gives the first identity.

For the second identity, let K be complex conjugation, $(Kf)(v) = \overline{f(v)}$. It is antiunitary, maps $c_c(C)$ onto itself, and commutes with the minimal operator with real weights. Taking closures, we have $K\mathcal{H}_C^{\text{abs}}K = \mathcal{H}_C^{\text{abs}}$. If $f = (\mathcal{H}_C^{\text{abs}} - z)^{-1}Kg$, then $Kf \in \text{Dom}(\mathcal{H}_C^{\text{abs}})$ and $(\mathcal{H}_C^{\text{abs}} - \bar{z})Kf = K(\mathcal{H}_C^{\text{abs}} - z)f = Kg$. This proves that $K(\mathcal{H}_C^{\text{abs}} - z)^{-1}K = (\mathcal{H}_C^{\text{abs}} - \bar{z})^{-1}$. Using Lemma 3.14(i) and $\langle \delta_u, Kf \rangle = \overline{\langle \delta_u, f \rangle}$, we have

$$G_{vu}^{\text{abs}}(z) = \overline{\langle \delta_u, (\mathcal{H}_C^{\text{abs}} - \bar{z})^{-1}\delta_v \rangle} = \overline{\langle \delta_u, K(\mathcal{H}_C^{\text{abs}} - z)^{-1}K\delta_v \rangle} = G_{uv}^{\text{abs}}(z).$$

The third identity follows from the first two, since $|\varphi_u| = |\varphi_v| = 1$. The statement for $G^{F,D}$ follows by applying the lemma to each component. $\square$

Recall that V_0 and V_1 are the sets of vertices of even and odd generation. Every edge of $\mathcal{T}$ joins V_0 and V_1 .

Proposition 3.21.

- (i) On Ω_{sa} , let C be a component of a finite removal, and let J be the unitary operator $(Jf)(v) = (-1)^{|v|}f(v)$ on $\ell^2(C)$. Then $J\mathcal{H}_CJ = -\mathcal{H}_C$. Further, $J\mathbf{1}_B(\mathcal{H}_C)J = \mathbf{1}_{-B}(\mathcal{H}_C)$ for every Borel set $B \subset \mathbb{R}$; the spectral measure of every δ_v , $v \in C$, is invariant under $\lambda \mapsto -\lambda$;

$$G_{uv}^C(-z) = -(-1)^{|u|+|v|}G_{uv}^C(z), \quad u, v \in C, z \in \mathbb{C} \setminus \mathbb{R};$$

and for every Borel set $I \subset \mathbb{R}$, the operator $\mathcal{H}_C$ has pure point spectrum in I if and only if it has pure point spectrum in $-I$, and it is purely absolutely continuous in I if and only if it is purely absolutely continuous in $-I$. In particular, this holds for $\mathcal{H}$ under $\mathbb{P}$ and under $\mathbb{P}^\infty$.

- (ii) For the finite model of Definition 3.1, the pairs (G_n, H_n) and $(G_n, -H_n)$ have the same law. In particular $H_n \stackrel{d}{=} -H_n$ and $H_n^{\text{gc}} \stackrel{d}{=} -H_n^{\text{gc}}$.

Proof. For (i), the operator J is a self-adjoint unitary which maps $c_c(C)$ onto itself. For $f \in c_c(C)$ and $v \in C$, we have

$$(J\mathcal{H}_{C,0}Jf)(v) = (-1)^{|v|} \sum_{u \sim v} h_{vu} (-1)^{|u|} f(u) = -(\mathcal{H}_{C,0}f)(v),$$

since adjacent vertices have generations differing by one. Taking closures, we find $J\mathcal{H}_C J = -\mathcal{H}_C$. Since the functional calculus is invariant under unitary conjugation, we have $J\phi(\mathcal{H}_C)J = \phi(-\mathcal{H}_C)$ for every bounded Borel function ϕ . With $\phi = \mathbf{1}_B$, this gives $J\mathbf{1}_B(\mathcal{H}_C)J = \mathbf{1}_{-B}(\mathcal{H}_C)$, and with $\phi(\lambda) = (\lambda - z)^{-1}$ it gives $J(\mathcal{H}_C - z)^{-1}J = -(\mathcal{H}_C + z)^{-1}$, whose (u, v) entry is the displayed identity. For every $f \in \ell^2(C)$ and Borel B , we find

$$\mu_{Jf}^{\mathcal{H}_C}(B) = \langle f, J\mathbf{1}_B(\mathcal{H}_C)Jf \rangle = \mu_f^{\mathcal{H}_C}(-B).$$

Since $J\delta_v = \pm\delta_v$, the spectral measure of δ_v is reflection invariant. If $\mathcal{H}_C$ has pure point spectrum in I , then for every f the restriction of $\mu_{Jf}^{\mathcal{H}_C}$ to I is purely atomic, and by the last display the restriction of $\mu_f^{\mathcal{H}_C}$ to $-I$ is purely atomic, and $\mathcal{H}_C$ has pure point spectrum in $-I$. The same argument applies to absolute continuity, since reflection preserves Lebesgue null sets.

For (ii), since the variables $-w_{ij}$ are independent standard complex Gaussians, independent of G_n , the pairs $(G_n, (w_{ij}))$ and $(G_n, (-w_{ij}))$ have the same law. The matrix H_n is a function of $(G_n, (w_{ij}))$, and replacing (w_{ij}) by $(-w_{ij})$ replaces H_n by $-H_n$. The set gc is a function of G_n . $\square$

3.6 The bipartite factorization

Since every edge joins V_0 and V_1 , the operator $\mathcal{H}$ maps functions on V_0 to functions on V_1 and conversely, and its square splits into two nonnegative blocks. In this section, we make this precise for the unbounded operator, and we write each block as a sum of rank-one terms, one for each vertex of the opposite parity, with independent Gamma-distributed squared norms. We begin with an abstract lemma.

Lemma 3.22. *Let $\mathfrak{h}_0, \mathfrak{h}_1$ be Hilbert spaces and let A be a closed, densely defined operator from $\text{Dom}(A) \subset \mathfrak{h}_0$ to $\mathfrak{h}_1$. Define the operator T_A on $\mathfrak{h}_0 \oplus \mathfrak{h}_1$ by*

$$\text{Dom}(T_A) = \text{Dom}(A) \oplus \text{Dom}(A^*), \quad T_A(f, g) = (A^*g, Af).$$

- (i) *The operator T_A is self-adjoint, $\text{Dom}(T_A^2) = \text{Dom}(A^*A) \oplus \text{Dom}(AA^*)$, and $T_A^2 = A^*A \oplus AA^*$. The operators A^*A and AA^* are nonnegative and self-adjoint. Further, $\text{Dom}((A^*A)^{1/2}) = \text{Dom}(A)$ and $\|(A^*A)^{1/2}f\| = \|Af\|$ for $f \in \text{Dom}(A)$.*
- (ii) *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then $z^2 \notin [0, \infty)$, and with $Q_0 = (A^*A - z^2)^{-1}$ and $Q_1 = (AA^* - z^2)^{-1}$,*

$$(T_A - z)^{-1}(f, g) = (zQ_0f + A^*Q_1g, AQ_0f + zQ_1g), \quad f \in \mathfrak{h}_0, g \in \mathfrak{h}_1.$$

In particular, $P_0(T_A - z)^{-1}P_0 = zQ_0$ and $P_1(T_A - z)^{-1}P_1 = zQ_1$, where P_0, P_1 are the projections onto $\mathfrak{h}_0, \mathfrak{h}_1$.

- (iii) *Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $g \in \text{Dom}(A^*)$. Then $Q_0A^*g = A^*Q_1g$, and*

$$\langle A^*g, (A^*A - z^2)^{-1}A^*g \rangle = \|g\|^2 + z \langle g, (T_A - z)^{-1}g \rangle.$$

(iv) Let B be a densely defined operator from $\text{Dom}(B) \subset \mathfrak{h}_1$ to $\mathfrak{h}_0$. If the operator $(f, g) \mapsto (Bg, Af)$ with domain $\text{Dom}(A) \oplus \text{Dom}(B)$ is self-adjoint, then $B = A^*$, including domains.

Proof. We begin with (iv) and a computation of adjoints. Let T' be the operator in (iv). A pair (φ, ψ) is in $\text{Dom}(T'^*)$ exactly when the map $(f, g) \mapsto \langle (\varphi, \psi), T'(f, g) \rangle = \langle \varphi, Bg \rangle + \langle \psi, Af \rangle$ is bounded on $\text{Dom}(A) \oplus \text{Dom}(B)$. Taking $g = 0$ and $f = 0$ separately, we see that this holds exactly when $\psi \in \text{Dom}(A^*)$ and $\varphi \in \text{Dom}(B^*)$, and then $T'^*(\varphi, \psi) = (A^*\psi, B^*\varphi)$. If $T' = T'^*$, then comparing domains, we find $\text{Dom}(B) = \text{Dom}(A^*)$, and comparing first components, we find $B\psi = A^*\psi$ for $\psi \in \text{Dom}(B)$. This proves (iv).

For (i), the same computation with $B = A^*$ gives $\text{Dom}(T_A^*) = \text{Dom}(A^{**}) \oplus \text{Dom}(A^*)$ and $T_A^*(\varphi, \psi) = (A^*\psi, A^{**}\varphi)$. Since A is closed, we have $A^{**} = A$ and $T_A^* = T_A$. A pair $(f, g) \in \text{Dom}(T_A)$ belongs to $\text{Dom}(T_A^2)$ exactly when $(A^*g, Af) \in \text{Dom}(A) \oplus \text{Dom}(A^*)$, that is, when $f \in \text{Dom}(A^*A)$ and $g \in \text{Dom}(AA^*)$, and then $T_A^2(f, g) = (A^*Af, AA^*g)$. The operator T_A^2 is self-adjoint by the functional calculus. As shown in the proof of Theorem 3.10, the adjoint of an orthogonal sum is the orthogonal sum of the adjoints. Hence A^*A and AA^* are self-adjoint; they are nonnegative because $\langle f, A^*Af \rangle = \|Af\|^2$. By the functional calculus, $|T_A| = (T_A^2)^{1/2} = (A^*A)^{1/2} \oplus (AA^*)^{1/2}$, and $\text{Dom}(|T_A|) = \text{Dom}(T_A)$ with $\||T_A|\varphi\| = \|T_A\varphi\|$, since both sides equal $(\int \lambda^2 d\mu_\varphi^{T_A})^{1/2}$. Applying this to $\varphi = (f, 0)$ gives the last assertion of (i).

For (ii), since $z^2 \in [0, \infty)$ only if z is real, the operators Q_0 and Q_1 are bounded. Let $f \in \mathfrak{h}_0$ and $\phi = Q_0f \in \text{Dom}(A^*A)$. Then $\phi \in \text{Dom}(A)$ and $A\phi \in \text{Dom}(A^*)$, which gives $(z\phi, A\phi) \in \text{Dom}(T_A)$ and

$$(T_A - z)(z\phi, A\phi) = (A^*A\phi - z^2\phi, zA\phi - zA\phi) = (f, 0).$$

Similarly, for $g \in \mathfrak{h}_1$ and $\psi = Q_1g \in \text{Dom}(AA^*)$, we have $(A^*\psi, z\psi) \in \text{Dom}(T_A)$ and $(T_A - z)(A^*\psi, z\psi) = (0, g)$. Adding these gives (ii).

For (iii), let $\psi = Q_1g$. Then $A^*\psi \in \text{Dom}(A)$, and $AA^*\psi = g + z^2\psi$ belongs to $\text{Dom}(A^*)$, because g and ψ do. We conclude that $A^*\psi \in \text{Dom}(A^*A)$ and

$$(A^*A - z^2)A^*\psi = A^*(AA^*\psi) - z^2A^*\psi = A^*(g + z^2\psi) - z^2A^*\psi = A^*g,$$

so $Q_0A^*g = A^*\psi$. Since $A^*\psi \in \text{Dom}(A)$ and $g \in \text{Dom}(A^*)$, we obtain

$$\begin{aligned} \langle A^*g, Q_0A^*g \rangle &= \langle A^*g, A^*\psi \rangle = \langle g, AA^*\psi \rangle = \|g\|^2 + z^2\langle g, Q_1g \rangle \\ &= \|g\|^2 + z\langle g, (T_A - z)^{-1}g \rangle, \end{aligned}$$

where in the last step we used (ii). □

We now apply this to the tree. For a component C of a finite removal, set $C_0 = C \cap V_0$ and $C_1 = C \cap V_1$, identify $\ell^2(C) = \ell^2(C_0) \oplus \ell^2(C_1)$, and write P_0 and P_1 for the

corresponding projections. For $y \in C_1$, let d_y be the number of neighbors of y in C . For $y \in C_1$, the row vector, the squared row norm and, when $U_y > 0$, the row direction are

$$a_y = \sum_{x \in C, x \sim y} h_{xy} \delta_x \in \ell^2(C_0), \quad U_y = \|a_y\|^2 = \sum_{x \in C, x \sim y} |h_{xy}|^2, \quad v_y = \frac{a_y}{\sqrt{U_y}}. \quad (3.20)$$

These depend on C ; when $C = V$ we have $d_y = \deg(y) = N_y + 1$. Almost surely all edge weights are nonzero, and then $U_y > 0$ exactly when $d_y \geq 1$. For $y \in C_1$, let $\mathcal{H}_C^{(y)} = \bigoplus_{x \in C, x \sim y} \mathcal{H}_{C_{x \rightarrow y}}$ be the operator on $\ell^2(C \setminus \{y\})$ with the vertex y deleted; by Lemma 3.5(ii), the branches $C_{x \rightarrow y}$ are the components of $C \setminus \{y\}$. For $C = V$, this is $\mathcal{H}^{(y)}$.

Proposition 3.23. *On Ω_{sa} , let C be a component of a finite removal (F, D) ; for instance $C = V$. Write $\mathcal{H}_C$ and G^C for its operator and resolvent.*

- (i) *There exists a unique closed, densely defined operator A from $\text{Dom}(A) \subset \ell^2(C_0)$ to $\ell^2(C_1)$ such that $\mathcal{H}_C = T_A$, that is,*

$$\mathcal{H}_C = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad \text{Dom}(\mathcal{H}_C) = \text{Dom}(A) \oplus \text{Dom}(A^*).$$

Its domain is $\text{Dom}(A) = \{f \in \ell^2(C_0) : \mathcal{H}_{C, \text{form}} f \in \ell^2(C_1)\}$. The space $c_c(C_0)$ is a core for A , and $c_c(C_1)$ is a core for A^* . Further, $\mathcal{H}_C^2 = A^*A \oplus AA^*$, with $\text{Dom}(\mathcal{H}_C^2) = \text{Dom}(A^*A) \oplus \text{Dom}(AA^*)$.

- (ii) *For $y \in C_1$, we have $\delta_y \in \text{Dom}(A^*)$ and $A^*\delta_y = a_y$. For $f \in \text{Dom}(A)$ and $y \in C_1$, we have $(Af)(y) = \langle a_y, f \rangle$. Further,*

$$\text{Dom}(A) = \left\{ f \in \ell^2(C_0) : \sum_{y \in C_1: U_y > 0} U_y |\langle v_y, f \rangle|^2 < \infty \right\},$$

and $\|Af\|^2 = \sum_{y \in C_1: U_y > 0} U_y |\langle v_y, f \rangle|^2$ for $f \in \text{Dom}(A)$, so the closed quadratic form $f \mapsto \|(A^*A)^{1/2}f\|^2$ of A^*A , with form domain $\text{Dom}(A)$, is the sum of the rank-one forms $f \mapsto U_y |\langle v_y, f \rangle|^2$. The closed linear span of $\{v_y : y \in C_1, U_y > 0\}$ is $\overline{\text{Ran } A^*} = (\ker A)^\perp$.

- (iii) *Suppose that (F, D) is selected without inspecting the retained marks, with inspected set Q , and that C is determined by $\mathcal{G}_Q$ (for instance, the component containing a fixed word, on the event that this word is in $V \setminus F$). Conditionally on $\mathcal{G}_Q$, the pairs (U_y, v_y) , for $y \in C_1$ with $d_y \geq 1$, are independent; the direction v_y is uniformly distributed on the unit sphere of the d_y -dimensional space $\text{span}\{\delta_x : x \in C, x \sim y\}$ and is independent of U_y ; and U_y has the density*

$$\frac{p^d}{\Gamma(d)} u^{d-1} e^{-pu} \mathbf{1}_{(0, \infty)}(u), \quad d = d_y.$$

The same holds under $\mathbb{P}^\infty$. For $C = V$, the hypothesis holds with $Q = \emptyset$ and $\mathcal{G}_Q = \mathcal{G}_0$.

- (iv) Let $y \in C_1$ with $U_y > 0$. The operator $A^{(y)} = A - |\delta_y\rangle\langle a_y|$ with domain $\text{Dom}(A)$ is closed, and $T_{A^{(y)}} = 0 \oplus \mathcal{H}_C^{(y)}$ with respect to $\ell^2(C) = \mathbb{C}\delta_y \oplus \ell^2(C \setminus \{y\})$. The square block

$$K_0^{(y)} = (A^{(y)})^* A^{(y)} = A^* A - U_y |v_y\rangle\langle v_y|, \quad \text{Dom}(K_0^{(y)}) = \text{Dom}(A^* A),$$

is nonnegative and self-adjoint, and for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$P_0(\mathcal{H}_C^{(y)} - z)^{-1} P_0 = z(K_0^{(y)} - z^2)^{-1}, \quad (3.21)$$

and

$$z\langle v_y, (K_0^{(y)} - z^2)^{-1} v_y \rangle = \sum_{x \in C, x \sim y} |v_y(x)|^2 \Gamma_{x \rightarrow y}^C(z). \quad (3.22)$$

Under the hypotheses of (iii), let y be determined by $\mathcal{G}_Q$, let R_y be the set of retained edges which are not incident to y , and let $\mathcal{G}^y$ be the σ -algebra generated by $\mathcal{G}_Q$, by v_y , and by the variables $w_e \mathbf{1}_{\{e \in R_y\}}$, $e \in \mathbb{U}_*$. Then v_y , the operator $\mathcal{H}_C^{(y)}$ and the operator $K_0^{(y)}$ are determined by $\mathcal{G}^y$, and conditionally on $\mathcal{G}^y$ the squared row norm U_y has the density in (iii) with $d = d_y$, under $\mathbb{P}$ and under $\mathbb{P}^\infty$.

- (v) For $z \in \mathbb{C} \setminus \mathbb{R}$, we have $P_0(\mathcal{H}_C - z)^{-1} P_0 = z(A^* A - z^2)^{-1}$ and $P_1(\mathcal{H}_C - z)^{-1} P_1 = z(AA^* - z^2)^{-1}$, and for every $y \in C_1$ with $U_y > 0$,

$$U_y \langle v_y, (A^* A - z^2)^{-1} v_y \rangle = 1 + z G_{yy}^C(z). \quad (3.23)$$

- (vi) For every Borel set $B \subset (0, \infty)$, with $B^2 = \{b^2 : b \in B\}$,

$$\mathbf{1}_B(\mathcal{H}_C) = \frac{1}{2}(I + \text{sgn}(\mathcal{H}_C))\mathbf{1}_{B^2}(\mathcal{H}_C^2), \quad \mathbf{1}_{-B}(\mathcal{H}_C) = \frac{1}{2}(I - \text{sgn}(\mathcal{H}_C))\mathbf{1}_{B^2}(\mathcal{H}_C^2),$$

and $\mathbf{1}_{B^2}(\mathcal{H}_C^2) = \mathbf{1}_{B^2}(A^* A) \oplus \mathbf{1}_{B^2}(AA^*)$. Further, $\mathcal{H}_C$ has pure point spectrum in $B \cup (-B)$ if and only if both $A^* A$ and AA^* have pure point spectrum in B^2 , and the same holds with “pure point” replaced by “purely absolutely continuous”.

- (vii) All statements hold with the roles of C_0 and C_1 exchanged and A replaced by A^* ; the rows are then indexed by $x \in C_0$, with $a_x = A\delta_x$.

Proof. The proof proceeds part by part.

Proof of (i). By Proposition 3.21(i), we have $J\mathcal{H}_C J = -\mathcal{H}_C$. It follows that J maps $\text{Dom}(\mathcal{H}_C)$ onto itself, and so do $P_0 = (I + J)/2$ and $P_1 = (I - J)/2$, which gives $\text{Dom}(\mathcal{H}_C) = \mathcal{D}_0 \oplus \mathcal{D}_1$ with $\mathcal{D}_i = \text{Dom}(\mathcal{H}_C) \cap \ell^2(C_i)$. For $f \in \mathcal{D}_0$, the vector $\mathcal{H}_C f = \mathcal{H}_{C, \text{form}} f$ is supported on C_1 , because every edge joins the two parity classes; similarly, $\mathcal{H}_C$ maps $\mathcal{D}_1$ into $\ell^2(C_0)$. Define A on $\text{Dom}(A) = \mathcal{D}_0$ and $\tilde{A}$ on $\mathcal{D}_1$ as the restrictions of $\mathcal{H}_C$. Then $\mathcal{H}_C(f, g) = (\tilde{A}g, Af)$ on $\text{Dom}(A) \oplus \text{Dom}(\tilde{A})$. Both operators are densely defined, since they are defined on the finitely supported vectors, and A is closed: if $f_n \rightarrow f$ and $Af_n \rightarrow g$, then $\mathcal{H}_C f_n \rightarrow g$, and since $\mathcal{H}_C$ is closed, $f \in \text{Dom}(\mathcal{H}_C) \cap \ell^2(C_0)$ and

$Af = g$. By Lemma 3.22(iv), we have $\tilde{A} = A^*$, that is, $\mathcal{H}_C = T_A$. If $\mathcal{H}_C = T_{A'}$ for another closed operator A' , then $\text{Dom}(A') = \{f : (f, 0) \in \text{Dom}(\mathcal{H}_C)\} = \text{Dom}(A)$ and $A'f = P_1\mathcal{H}_C(f, 0) = Af$, which proves uniqueness. The description of $\text{Dom}(A)$ follows from Theorem 3.10(ii).

The graph norm of $\mathcal{H}_C$ splits as $\|(f, g)\|^2 + \|\mathcal{H}_C(f, g)\|^2 = \|f\|^2 + \|Af\|^2 + \|g\|^2 + \|A^*g\|^2$. Let $f \in \text{Dom}(A)$. Since $c_c(C)$ is a core for $\mathcal{H}_C$, there exist $\varphi_n \in c_c(C)$ converging to $(f, 0)$ in the graph norm of $\mathcal{H}_C$. Then $P_0\varphi_n \in c_c(C_0)$ converges to f in the graph norm of A , by the splitting. This shows that $c_c(C_0)$ is a core for A , and the same argument applied to $(0, g)$ proves that $c_c(C_1)$ is a core for A^* . The statement about $\mathcal{H}_C^2$ is Lemma 3.22(i).

Proof of (ii). Since $\delta_y \in c_c(C_1) \subset \text{Dom}(A^*)$, we have $A^*\delta_y = \mathcal{H}_{C,\text{form}}\delta_y = \sum_{x \sim y} h_{xy}\delta_x = a_y$. For $f \in \text{Dom}(A)$, by Theorem 3.10(ii),

$$(Af)(y) = (\mathcal{H}_{C,\text{form}}f)(y) = \sum_{x \sim y} h_{yx}f(x) = \sum_{x \sim y} \overline{h_{xy}}f(x) = \langle a_y, f \rangle.$$

The same computation applies to every $f \in \ell^2(C_0)$, with $\mathcal{H}_{C,\text{form}}$ in place of A . Together with Theorem 3.10(ii), this implies that $f \in \text{Dom}(A)$ exactly when $\sum_y |\langle a_y, f \rangle|^2 < \infty$, and then $\|Af\|^2 = \sum_y |\langle a_y, f \rangle|^2$. The terms with $U_y = 0$ vanish, and the others equal $U_y |\langle v_y, f \rangle|^2$. By Lemma 3.22(i), the form domain of A^*A is $\text{Dom}(A)$ and $\|(A^*A)^{1/2}f\|^2 = \|Af\|^2$, which gives the form identity.

For the span, $f \perp \text{Ran } A^*$ means that $\langle A^*g, f \rangle = 0$ for all $g \in \text{Dom}(A^*)$, which holds exactly when $f \in \text{Dom}(A^{**}) = \text{Dom}(A)$ and $Af = 0$. We conclude that $(\text{Ran } A^*)^\perp = \ker A$ and $\overline{\text{Ran } A^*} = (\ker A)^\perp$. Since $c_c(C_1)$ is a core for A^* , the set $A^*c_c(C_1) = \text{span}\{a_y : y \in C_1\}$ is dense in $\text{Ran } A^*$. The nonzero vectors among the a_y are multiples of the v_y with $U_y > 0$.

Proof of (iii). By Lemma 3.4, conditionally on $\mathcal{G}_Q$, the marks of the retained edges are independent standard complex Gaussians. Fix $y \in C_1$ with $d_y = d \geq 1$. The coordinates of $\sqrt{p}a_y$ are $h_{xy}\sqrt{p}$ for the d neighbors x of y in C ; each of them is the mark of a retained edge or its complex conjugate. Since complex conjugation preserves the standard complex Gaussian law, the vector $\zeta = \sqrt{p}a_y \in \mathbb{C}^d$ has, conditionally on $\mathcal{G}_Q$, the density $\pi^{-d}e^{-|\zeta|^2}$ on $\mathbb{C}^d \cong \mathbb{R}^{2d}$. In polar coordinates $\zeta = r\vartheta$, with $r > 0$ and ϑ in the unit sphere $\mathbb{S}^{2d-1}$, Lebesgue measure is $r^{2d-1}dr\sigma(d\vartheta)$, where σ is the surface measure, of total mass $2\pi^d/\Gamma(d)$. Then (r, ϑ) has the product density $\pi^{-d}e^{-r^2}r^{2d-1}dr\sigma(d\vartheta)$; in particular, ϑ is uniform on the sphere and independent of r , and r has the density $2\Gamma(d)^{-1}r^{2d-1}e^{-r^2}$. The change of variables $u = r^2/p$ shows that $U_y = \|a_y\|^2 = r^2/p$ has the density in (iii), and $v_y = \zeta/|\zeta| = \vartheta$. Since each edge has exactly one endpoint in V_1 , the vectors a_y for distinct $y \in C_1$ are built from disjoint sets of retained marks, and they are independent conditionally on $\mathcal{G}_Q$. The statement under $\mathbb{P}^\infty$ follows in the same way from the $\mathbb{P}^\infty$ version of Lemma 3.4.

Proof of (iv). By (3.9), we have $\mathcal{H}_C = (0 \oplus \mathcal{H}_C^{(y)}) + W_y$, where

$$W_y = \sum_{x \in C, x \sim y} (h_{yx} |\delta_y\rangle \langle \delta_x| + h_{xy} |\delta_x\rangle \langle \delta_y|) = |\delta_y\rangle \langle a_y| + |a_y\rangle \langle \delta_y|,$$

and the domains agree. With respect to $\ell^2(C_0) \oplus \ell^2(C_1)$, the operator W_y maps (f, g) to $(g(y)a_y, \langle a_y, f \rangle \delta_y)$. Since $|\delta_y\rangle \langle a_y|$ is bounded, the operator $A^{(y)}$ is closed with domain $\text{Dom}(A)$, and its adjoint is $(A^{(y)})^* = A^* - |a_y\rangle \langle \delta_y|$ with domain $\text{Dom}(A^*)$; here we used that the adjoint of the sum of a densely defined operator and a bounded everywhere defined operator is the sum of the adjoints. We conclude that

$$0 \oplus \mathcal{H}_C^{(y)} = \mathcal{H}_C - W_y = \begin{pmatrix} 0 & A^* - |a_y\rangle \langle \delta_y| \\ A - |\delta_y\rangle \langle a_y| & 0 \end{pmatrix} = T_{A^{(y)}}.$$

By Lemma 3.22(i), $K_0^{(y)}$ is nonnegative and self-adjoint. A vector $f \in \text{Dom}(A)$ belongs to $\text{Dom}(K_0^{(y)})$ exactly when $A^{(y)}f = Af - \langle a_y, f \rangle \delta_y$ belongs to $\text{Dom}(A^*)$, and since $\delta_y \in \text{Dom}(A^*)$, this holds exactly when $Af \in \text{Dom}(A^*)$. This proves that $\text{Dom}(K_0^{(y)}) = \text{Dom}(A^*A)$, and for such f , using $\langle \delta_y, Af \rangle = \langle a_y, f \rangle$,

$$K_0^{(y)}f = (A^* - |a_y\rangle \langle \delta_y|)(Af - \langle a_y, f \rangle \delta_y) = A^*Af - \langle a_y, f \rangle a_y = A^*Af - U_y \langle v_y, f \rangle v_y.$$

By Lemma 3.22(ii) applied to $A^{(y)}$, we have $P_0(T_{A^{(y)}} - z)^{-1}P_0 = z(K_0^{(y)} - z^2)^{-1}$. Since $(T_{A^{(y)}} - z)^{-1} = (-z)^{-1} \oplus (\mathcal{H}_C^{(y)} - z)^{-1}$ and $\ell^2(C_0) \subset \ell^2(C \setminus \{y\})$, this is (3.21). For (3.22), the neighbors of y in C lie in distinct branches $C_{x \rightarrow y}$, which are distinct components of $C \setminus \{y\}$. We obtain

$$\langle v_y, (\mathcal{H}_C^{(y)} - z)^{-1}v_y \rangle = \sum_{x \sim y} |v_y(x)|^2 \langle \delta_x, (\mathcal{H}_{C_{x \rightarrow y}} - z)^{-1} \delta_x \rangle = \sum_{x \sim y} |v_y(x)|^2 \Gamma_{x \rightarrow y}^C(z).$$

For the last assertion, since the components $C_{x \rightarrow y}$ are determined by $\mathcal{G}_Q$ and the weights of their edges are the marks in R_y , the operator $\mathcal{H}_C^{(y)}$ is determined by $\mathcal{G}^y$; so is $K_0^{(y)}$, which by Lemma 3.22(i) is the restriction of $(0 \oplus \mathcal{H}_C^{(y)})^2$ to $\ell^2(C_0)$. Conditionally on $\mathcal{G}_Q$, the marks of the d_y retained edges at y are independent of the marks in R_y , and by the proof of (iii) the squared row norm U_y is independent of the direction v_y , with the stated density. Therefore, conditionally on $\mathcal{G}^y$, the squared row norm U_y has the stated density. Since the conditional statements given $\mathcal{G}_Q$ hold under $\mathbb{P}^\infty$ as well, by Lemma 3.4, the same argument applies under $\mathbb{P}^\infty$.

Proof of (v). The first two identities are Lemma 3.22(ii). For the third, we apply Lemma 3.22(iii) with $g = \delta_y$, using $A^*\delta_y = a_y = \sqrt{U_y}v_y$, $\|\delta_y\| = 1$ and $\langle \delta_y, (\mathcal{H}_C - z)^{-1}\delta_y \rangle = G_{yy}^C(z)$.

Proof of (vi). Since the square map $s(\lambda) = \lambda^2$ is a homeomorphism of $(0, \infty)$, the set B^2 is Borel. For every $\lambda \in \mathbb{R}$, with $\text{sgn}(0) = 0$, we have

$$\mathbf{1}_B(\lambda) = \mathbf{1}_{B^2}(\lambda^2)^{\frac{1}{2}}(1 + \text{sgn } \lambda), \quad \mathbf{1}_{-B}(\lambda) = \mathbf{1}_{B^2}(\lambda^2)^{\frac{1}{2}}(1 - \text{sgn } \lambda).$$

Indeed, for $\lambda > 0$ we have $\lambda^2 \in B^2$ exactly when $\lambda \in B$; for $\lambda < 0$ the first right side vanishes, and $\lambda^2 \in B^2$ exactly when $-\lambda \in B$; and $0 \notin B^2$. The functional calculus is multiplicative and satisfies $\phi(\mathcal{H}_C^2) = (\phi \circ s)(\mathcal{H}_C)$, which gives the two identities. Since $\mathcal{H}_C^2 = A^*A \oplus AA^*$ is an orthogonal sum of self-adjoint operators, its functional calculus acts blockwise.

For the spectral types, let $\varphi = (f_0, f_1) \in \ell^2(C)$. The same composition rule shows that the spectral measure of φ for $\mathcal{H}_C^2$ is the image of $\mu_\varphi^{\mathcal{H}_C}$ under s , and the block structure shows that it equals $\mu_{f_0}^{A^*A} + \mu_{f_1}^{AA^*}$. The restriction of $\mu_\varphi^{\mathcal{H}_C^2}$ to B^2 is the image under s of the restriction of $\mu_\varphi^{\mathcal{H}_C}$ to $B \cup (-B)$. The map s sends countable sets to countable sets, and the preimage under s of a countable subset of B^2 is countable. Further, s and its local inverses $\pm\sqrt{\cdot}$ are locally Lipschitz on $\mathbb{R} \setminus \{0\}$ and $(0, \infty)$, and therefore map Lebesgue null sets to Lebesgue null sets. We conclude that the restriction of $\mu_\varphi^{\mathcal{H}_C^2}$ to $B \cup (-B)$ is purely atomic (respectively absolutely continuous) if and only if the restriction of $\mu_\varphi^{\mathcal{H}_C}$ to B^2 is. Taking $\varphi = (f_0, 0)$ and $\varphi = (0, f_1)$, and using that a sum of two purely atomic (absolutely continuous) measures is purely atomic (absolutely continuous), we obtain the last assertion.

Proof of (vii). With respect to $\ell^2(C_1) \oplus \ell^2(C_0)$, we have $\mathcal{H}_C(g, f) = (Af, A^*g) = T_{A^*}(g, f)$, since $A^{**} = A$. All the arguments above apply to A^* in place of A . $\square$

3.7 Compactly supported eigenvectors

Let C be a component of a finite removal. A finitely supported vector $\psi \in c_c(C)$ lies in the domain of $\mathcal{H}_C$ and satisfies $\mathcal{H}_C\psi = \mathcal{H}_{C,\text{form}}\psi$. In particular, ψ is an eigenvector of $\mathcal{H}_C$ with eigenvalue E exactly when $(\mathcal{H}_{C,\text{form}}\psi)(v) = E\psi(v)$ for every $v \in C$. In this section, we show that such eigenvectors do not exist for $E \neq 0$ on infinite components. For $E = 0$, they occur infinitely often on the surviving tree. For a finite set T of vertices, we write $M_T = (h_{uv})_{u,v \in T}$ for the corresponding finite Hermitian matrix.

Proposition 3.24.

- (i) Let $\mathcal{T}'$ be a deterministic countable, infinite, connected, locally finite tree with vertex set V' . Let $(h_e)_e$ be independent complex random variables indexed by the edges of $\mathcal{T}'$, each with a law that is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$, and set $h_{vu} = \overline{h_{uv}}$, with an arbitrary fixed orientation of each edge (for example, independent nondegenerate complex Gaussian variables). Then almost surely there exist no $E \in \mathbb{R} \setminus \{0\}$ and no nonzero finitely supported $\psi: V' \rightarrow \mathbb{C}$ with $\sum_{u \sim v} h_{vu}\psi(u) = E\psi(v)$ for every $v \in V'$.

- (ii) Let $p > 0$. There exists an event Ω_{cm} with $\mathbb{P}(\Omega_{\text{cm}}) = 1$ such that on $\Omega_{\text{sa}} \cap \Omega_{\text{cm}}$, for every finite removal of $\mathcal{T}$ and every infinite component C of it, the operator $\mathcal{H}_C$ has no finitely supported eigenvector with a nonzero eigenvalue. In particular, if $p > 1$, then $\mathbb{P}^\infty$ -almost surely $\mathcal{H}$ has no finitely supported eigenvector with a nonzero eigenvalue.

The restriction to infinite components cannot be removed: on the extinction event $\mathcal{S}^c$, which has probability $1 - q_p > 0$, the operator $\mathcal{H}$ is a finite Hermitian matrix, and it has nonzero eigenvalues as soon as $\mathcal{T}$ has an edge. The conclusion for $\mathcal{H}$ holds $\mathbb{P}$ -almost surely on $\mathcal{S}$, but not $\mathbb{P}$ -almost surely.

Proof. The proof shows that a finitely supported eigenvector produces two disjoint finite subtrees whose matrices share a nonzero eigenvalue, and that this has probability zero for each fixed pair.

STEP 1. *Two disjoint subtrees with a common eigenvalue.* We work on the event that all weights are nonzero, which has probability one. Suppose that $E \neq 0$ and $\psi \neq 0$ are as in (i), and let $S = \text{supp } \psi$, a finite nonempty set. Since $\mathcal{T}'$ is infinite and connected, there exists an edge $\{a, b\}$ with $a \notin S$ and $b \in S$. The equation at a reads $0 = E\psi(a) = \sum_{u \sim a} h_{au}\psi(u)$. Since the term $h_{ab}\psi(b)$ is nonzero, there exists a second neighbor $b' \neq b$ of a with $\psi(b') \neq 0$. Let T_1 and T_2 be the vertex sets of the connected components of the subgraph of $\mathcal{T}'$ induced on S which contain b and b' , respectively. Both sets are connected and do not contain a , so in the graph obtained from $\mathcal{T}'$ by removing a , the set T_1 lies in the component of b and T_2 lies in the component of b' . These components are different, because the only path from b to b' in $\mathcal{T}'$ is b, a, b' . In particular, $T_1 \cap T_2 = \emptyset$. For $v \in T_i$, every neighbor u of v with $\psi(u) \neq 0$ lies in S , and therefore in T_i . It follows that

$$(M_{T_i}\psi|_{T_i})(v) = \sum_{u \in T_i} h_{vu}\psi(u) = \sum_{u \sim v} h_{vu}\psi(u) = E\psi(v), \quad v \in T_i,$$

and $\psi|_{T_i} \neq 0$. We conclude that E is a common nonzero eigenvalue of M_{T_1} and M_{T_2} , where T_1 and T_2 are disjoint finite sets which induce connected subgraphs.

STEP 2. *A fixed pair has no common nonzero eigenvalue.* Fix disjoint finite sets $T_1, T_2 \subset V'$ which induce connected subgraphs. We show that, almost surely, M_{T_1} and M_{T_2} have no common nonzero eigenvalue. (The event is Borel, because the eigenvalues of a Hermitian matrix, listed in increasing order, depend continuously on its entries.) If T_2 is a single vertex, then $M_{T_2} = 0$. Otherwise, let $e_1, \dots, e_k$, with $k \geq 1$, be the edges with both endpoints in T_2 , and let $\zeta = (h_{e_1}, \dots, h_{e_k}) \in \mathbb{C}^k$. Since $T_1 \cap T_2 = \emptyset$, the vector ζ is independent of the weights X_1 of the edges inside T_1 , and it has a density ρ on $\mathbb{C}^k \cong \mathbb{R}^{2k}$, the product of the densities of its coordinates. We write $\zeta = r\vartheta$ with $r = |\zeta|$ and $\vartheta \in \mathbb{S}^{2k-1}$. Then $M_{T_2} = rM(\vartheta)$, where $M(\vartheta)$ is the Hermitian matrix built in the same way from ϑ , and the nonzero eigenvalues of M_{T_2} are the numbers $r\lambda$ with λ a nonzero eigenvalue

of $M(\vartheta)$. Let $\Sigma(\vartheta, X_1)$ be the finite set of quotients κ/λ , with κ a nonzero eigenvalue of M_{T_1} and λ a nonzero eigenvalue of $M(\vartheta)$. A common nonzero eigenvalue exists only if $r \in \Sigma(\vartheta, X_1)$. In polar coordinates, the law of (r, ϑ) has the density $\rho(r\vartheta)r^{2k-1}$ with respect to $dr\sigma(d\vartheta)$. By Fubini's theorem and the independence of ζ and X_1 , we have

$$\mathbb{P}(r \in \Sigma(\vartheta, X_1)) = \mathbb{E} \left[\int_{\mathbb{S}^{2k-1}} \int_0^\infty \mathbf{1}_{\{r \in \Sigma(\vartheta, X_1)\}} \rho(r\vartheta) r^{2k-1} dr \sigma(d\vartheta) \right] = 0,$$

where the expectation is over X_1 . The inner integral vanishes because $\Sigma(\vartheta, X_1)$ is a finite set.

STEP 3. Conclusion. Since there are countably many pairs of finite subsets of V' , almost surely no pair as in Step 2 has a common nonzero eigenvalue. By Step 1, this proves (i).

For (ii), for disjoint finite sets $T_1, T_2 \subset \mathbb{U}$, let $A(T_1, T_2)$ be the event that $T_1, T_2 \subset V$, that both induce connected subgraphs of $\mathcal{T}$, and that M_{T_1} and M_{T_2} have a common nonzero eigenvalue. The first two conditions are $\mathcal{G}_0$ -measurable. Conditionally on $\mathcal{G}_0$, the marks are independent standard complex Gaussians. In particular, the weights of the edges inside T_1 and inside T_2 are independent and have densities, and Step 2 implies that $\mathbb{P}(A(T_1, T_2) \mid \mathcal{G}_0) = 0$. Let Ω_{cm} be the event that all marks are nonzero and that no event $A(T_1, T_2)$ occurs; then $\mathbb{P}(\Omega_{\text{cm}}) = 1$. On $\Omega_{\text{sa}} \cap \Omega_{\text{cm}}$, let C be an infinite component of a finite removal, and suppose that $\mathcal{H}_C$ has a finitely supported eigenvector with eigenvalue $E \neq 0$. The network (C, h) is an infinite, connected, locally finite tree with nonzero weights. Step 1 applies to it and produces disjoint finite sets $T_1, T_2 \subset C$ which induce connected subgraphs of C and whose matrices share the eigenvalue E . By Lemma 3.5(i), C is an induced subtree of $\mathcal{T}$, and T_1 and T_2 therefore induce connected subgraphs of $\mathcal{T}$ with the same matrices, and $A(T_1, T_2)$ occurs. This contradiction proves (ii). The last assertion is the case of the empty finite removal on $\mathcal{S}$, where $C = V$ is infinite. $\square$

At energy zero, the situation is different. Call a vertex $v \in V$ a *twin-leaf vertex* if $N_v = 2$ and $N_{v1} = N_{v2} = 0$, that is, if v has exactly two children and both are leaves.

Proposition 3.25.

- (i) On Ω_{sa} , let C be a component of a finite removal, let $v \in C$, and let $\ell_1 \neq \ell_2$ be neighbors of v in C whose only neighbor in C is v . Then

$$\psi = h_{v\ell_2} \delta_{\ell_1} - h_{v\ell_1} \delta_{\ell_2}$$

satisfies $\mathcal{H}_C \psi = 0$, and $\psi \neq 0$ if the weights are nonzero.

- (ii) Let $p > 1$. Then $\mathbb{P}^\infty$ -almost surely there are infinitely many twin-leaf vertices. Consequently, $\mathbb{P}^\infty$ -almost surely, the vectors $\psi_v = h_{v,v2} \delta_{v1} - h_{v,v1} \delta_{v2}$, for v a twin-leaf vertex, form an infinite family of nonzero, pairwise orthogonal, finitely supported eigenvectors of $\mathcal{H}$ with eigenvalue 0. In particular, 0 is an eigenvalue of $\mathcal{H}$ of infinite multiplicity.

Proof. For (i), since the vector ψ is finitely supported, it suffices to check the pointwise equation. At v , we have $(\mathcal{H}_{C,\text{form}}\psi)(v) = h_{v\ell_1}h_{v\ell_2} - h_{v\ell_2}h_{v\ell_1} = 0$. At ℓ_1 and ℓ_2 , the only neighbor is v , and $\psi(v) = 0$. Every other vertex u is not adjacent to ℓ_1 or ℓ_2 , since their only neighbor is v , and $(\mathcal{H}_{C,\text{form}}\psi)(u) = 0$.

For (ii), let $\mathcal{V}_n = \{v \in V : |v| = n\}$. We first recall that the generation sizes $|\mathcal{V}_n|$ tend to infinity almost surely on $\mathcal{S}$ (see [64, Section 4.3] for background on Galton–Watson processes); for completeness, we give the argument. Let $\mathcal{N}_n = \sigma(N_u : |u| < n)$. Then $|\mathcal{V}_n|$ is $\mathcal{N}_n$ -measurable and, conditionally on $\mathcal{N}_n$, the variable $|\mathcal{V}_{n+1}|$ is a sum of $|\mathcal{V}_n|$ independent $\text{Pois}(p)$ variables; hence $\mathbb{P}(|\mathcal{V}_{n+1}| = 0 \mid \mathcal{N}_n) = e^{-p|\mathcal{V}_n|}$. Fix $K \geq 1$, and let $\tau_1 < \tau_2 < \dots$ be the successive times n with $1 \leq |\mathcal{V}_n| \leq K$, which are stopping times for $(\mathcal{N}_n)$. On $\{\tau_j < \infty\}$, we have $\mathbb{P}(|\mathcal{V}_{\tau_j+1}| = 0 \mid \mathcal{N}_{\tau_j}) \geq e^{-pK}$, as one checks by decomposing according to the value of τ_j . If $|\mathcal{V}_{\tau_j+1}| = 0$, then $|\mathcal{V}_n| = 0$ for all $n > \tau_j$ and $\tau_{j+1} = \infty$. Therefore $\mathbb{P}(\tau_{j+1} < \infty) \leq (1 - e^{-pK})\mathbb{P}(\tau_j < \infty)$, and $\mathbb{P}(\tau_j < \infty) \rightarrow 0$ as $j \rightarrow \infty$, and almost surely $1 \leq |\mathcal{V}_n| \leq K$ for only finitely many n . On $\mathcal{S}$, we have $|\mathcal{V}_n| \geq 1$ for all n , and it follows that $|\mathcal{V}_n| > K$ for all large n . Since K is arbitrary, $|\mathcal{V}_n| \rightarrow \infty$ almost surely on $\mathcal{S}$.

For a generation- n vertex v , the event that v is a twin-leaf vertex depends on N_v, N_{v1}, N_{v2} , which are independent of $\mathcal{N}_n$, and these variables are distinct for distinct v . Conditionally on $\mathcal{N}_n$, these events are independent with probability

$$a_p = \mathbb{P}(N_o = 2) \mathbb{P}(N_o = 0)^2 = \frac{p^2 e^{-p}}{2} e^{-2p} = \frac{p^2 e^{-3p}}{2} > 0$$

each. Since $\mathcal{S} \subset \{|\mathcal{V}_n| \geq 1\}$, we obtain

$$\mathbb{P}(\mathcal{S} \cap \{\text{no twin-leaf vertex in generation } n\}) \leq \mathbb{E}[\mathbf{1}_{\{|\mathcal{V}_n| \geq 1\}}(1 - a_p)^{|\mathcal{V}_n|}].$$

As $n \rightarrow \infty$, the integrand tends to zero almost surely: on $\mathcal{S}$ because $|\mathcal{V}_n| \rightarrow \infty$, and on $\mathcal{S}^c$ because $|\mathcal{V}_n| = 0$ for all large n . It is bounded by one, so the right side tends to zero by dominated convergence. For every m , the event that $\mathcal{S}$ occurs and no generation $n \geq m$ contains a twin-leaf vertex is contained in each of the events on the left with $n \geq m$, and it has probability zero. Taking the union over m , we find that $\mathbb{P}$ -almost surely on $\mathcal{S}$ there are infinitely many twin-leaf vertices.

The two children of a twin-leaf vertex are leaves of $\mathcal{T}$, whose only neighbor is their parent. Then (i) applies with $C = V$ and shows that $\mathcal{H}\psi_v = 0$; on the event that all marks are nonzero, $\psi_v \neq 0$. Since the supports $\{v1, v2\}$ of the vectors ψ_v are pairwise disjoint for distinct v , these vectors are pairwise orthogonal, and in particular linearly independent. $\square$

Spine decompositions and fresh squared moduli

This section collects the probabilistic tools that we use to compute expectations of sums over the vertices of the tree. They are of three kinds.

The first are spine decompositions. If $F(v)$ is a nonnegative functional of the marked tree and of a vertex v of generation L , then the expectation of $\sum_{|v|=L} F(v)$ equals p^L times the expectation of F evaluated at the endpoint of a spine: a path of length L from the root whose vertices carry independent Poisson families of further children, and whose endpoint carries an independent copy of the whole tree. This identity is the many-to-one formula. We prove it, its analogue for ordered pairs of vertices, which is the many-to-two formula, and their versions under the survival-conditioned law $\mathbb{P}^\infty$. The second kind concerns endpoints: the law of an endpoint functional under $\mathbb{P}^\infty$, the children of an endpoint classified by the fate of their forward subtrees, and the growth of the set of vertices with infinite forward subtree. The third is an anti-concentration estimate for affine combinations of independent exponential variables, together with a definition of fresh squared moduli, which says when this estimate may be applied conditionally on a σ -field.

In Section 4.1, we realize the marked tree on the space of finite words and record its branching property. Section 4.2 recalls the Mecke formula and splits the children of the root according to the survival of their forward subtrees. Section 4.3 defines the spine laws and proves the many-to-one and many-to-two formulas. Section 4.4 compares endpoint laws under $\mathbb{P}$ and $\mathbb{P}^\infty$. Section 4.5 defines fresh squared moduli and proves the affine negative-moment lemma. Section 4.6 treats the children of a single vertex and the growth of the backbone.

Throughout this section, $p > 0$ is fixed. Whenever the survival event enters, we assume that $p > 1$, so that $\varrho_p > 0$.

4.1 The Ulam–Harris construction

Let $\mathbb{U}_0 = \{o\}$, where o denotes the empty word, let $\mathbb{U}_n = \{1, 2, \dots\}^n$ for $n \geq 1$, and set $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{U}_n$ and $\mathbb{U}_* = \mathbb{U} \setminus \{o\}$. For a word $u = i_1 \cdots i_n$ and a word v , we write $|u| = n$ for the length of u and uv for the concatenation of u and v . For $0 \leq l \leq n$, we let $u_l = i_1 \cdots i_l$ denote the prefix of u of length l , so that $u_0 = o$ and $u_n = u$, and for $u \neq o$ we write $u^- = u_{n-1}$. For $u, v \in \mathbb{U}$, we let $u \wedge v$ denote the longest common prefix of u

and v .

Let γ denote the standard complex Gaussian law, which has density $\pi^{-1}e^{-|w|^2}$ with respect to Lebesgue measure on $\mathbb{C}$. If w has law γ , then $|w|^2$ has law $\text{Exp}(1)$, and $w/|w|$ is uniform on the unit circle and independent of $|w|^2$. Let $Z = [0, 1]^{\{1, 2, \dots\}}$ with its product σ -field, and let κ be the product of the Lebesgue measures on the factors.

Definition 4.1. The configuration space is

$$\Omega = \{0, 1, 2, \dots\}^{\mathbb{U}} \times Z^{\mathbb{U}} \times \mathbb{C}^{\mathbb{U}_*},$$

with the product σ -field $\mathcal{F}$. Its elements are written

$$\omega = ((N_v)_{v \in \mathbb{U}}, (\zeta_v)_{v \in \mathbb{U}}, (w_v)_{v \in \mathbb{U}_*}).$$

We let $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ under which all coordinates are independent, each N_v has law $\text{Pois}(p)$, each ζ_v has law κ , and each w_v has law γ . The *rank* of $v \in \mathbb{U}$ is the first coordinate $\zeta_v(1)$ of $\zeta_v = (\zeta_v(1), \zeta_v(2), \dots)$; the variables ζ_v are called auxiliary marks.

(a) The vertex set is

$$V(\omega) = \{i_1 \cdots i_n \in \mathbb{U} : n \geq 0, i_l \leq N_{i_1 \cdots i_{l-1}} \text{ for } 1 \leq l \leq n\},$$

and $\mathcal{T} = \mathcal{T}(\omega)$ is the tree with vertex set $V(\omega)$, root o , and edges $\{v^-, v\}$ for $v \in V(\omega) \setminus \{o\}$. The children of $v \in V(\omega)$ are $v1, \dots, vN_v$.

(b) For $v \in V(\omega) \setminus \{o\}$, the edge $\{v^-, v\}$ carries the mark $w_{v-v} = w_v$. We set $h_{v-v} = p^{-1/2}w_v$, $h_{vv^-} = \overline{h_{v-v}}$, and $Y_v = Y_{v-v} = |w_v|^2$.

(c) For $u \in \mathbb{U}$, the shift $\theta_u : \Omega \rightarrow \Omega$ is defined by

$$\theta_u \omega = ((N_{uv})_{v \in \mathbb{U}}, (\zeta_{uv})_{v \in \mathbb{U}}, (w_{uv})_{v \in \mathbb{U}_*}).$$

(d) The survival event is $\mathcal{S} = \{\omega \in \Omega : |V(\omega)| = \infty\}$, and $\varrho_p = \mathbb{P}(\mathcal{S})$. For $p > 1$, we set $\mathbb{P}^\infty = \mathbb{P}(\cdot | \mathcal{S})$ and write $\mathbb{E}^\infty$ for the corresponding expectation.

(e) For $n \geq 0$, we set $\mathcal{V}_n = V(\omega) \cap \mathbb{U}_n$, $\mathcal{V}_n^\infty = \{v \in \mathcal{V}_n : \theta_v \omega \in \mathcal{S}\}$, and $Z_n = |\mathcal{V}_n^\infty|$. The *backbone* is the set $\bigcup_{n \geq 0} \mathcal{V}_n^\infty$ of vertices whose forward subtree is infinite.

Under $\mathbb{P}$, the offspring numbers N_v are independent with law $\text{Pois}(p)$, and $\mathcal{T}$ is a Poisson–Galton–Watson tree with mean offspring p . The marks w_v , $v \in \mathbb{U}_*$, are independent of $(N_v)_{v \in \mathbb{U}}$. This implies that, conditionally on $\mathcal{T}$, the marks of its edges are independent with law γ ; in particular, the squared moduli $Y_v = |w_v|^2$ are independent $\text{Exp}(1)$ variables, and they are independent of the phases $w_v/|w_v|$, which are uniform on the unit circle. We conclude that $\mathcal{T}$, together with the weights h_e , is a realization of the marked tree of Definition 3.2, with the auxiliary element $\mathfrak{a} = (\zeta_v)_{v \in \mathbb{U}}$ of the standard

Borel space $Z^{\mathbb{U}}$, which is independent of the marked tree. In particular, the σ -algebra $\mathcal{G}_0$ of Definition 3.2 is generated by the offspring numbers N_v and the auxiliary marks ζ_v , $v \in \mathbb{U}$. From now on we work with this realization: $\mathbb{P}$ and $\mathbb{E}$ refer to the law of ω , and $\mathcal{S}$, ϱ_p , $\mathbb{P}^\infty$ are the objects of Definition 3.2. The event $\mathcal{S}$ is measurable, since every vertex has finitely many children and therefore, by König's lemma, $\mathcal{S} = \bigcap_{n \geq 0} \{\mathcal{V}_n \neq \emptyset\}$. Recall that $0 < \varrho_p < 1$ for $p > 1$.

The coordinates of ω at labels outside $V(\omega)$ do not affect the marked tree. We keep them because they make the shifts measure preserving and the spine laws of Section 4.3 simple to write down. Note that each shift θ_u preserves $\mathbb{P}$, since it moves the coordinates at the labels uv to the labels v , and all coordinates of the same type have the same law.

If $u \in V(\omega)$, then $V(\theta_u \omega) = \{v \in \mathbb{U} : uv \in V(\omega)\}$, and the map $v \mapsto uv$ is an isomorphism from $\mathcal{T}(\theta_u \omega)$ onto the forward subtree $\mathcal{T}_u$. The edge $\{v^-, v\}$ of $\mathcal{T}(\theta_u \omega)$ carries the mark w_{uv} , which is also the mark of the corresponding edge $\{(uv)^-, uv\}$ of $\mathcal{T}_u$. We use this correspondence throughout: if a quantity $f(\omega)$ is defined in terms of the marked tree $\mathcal{T}(\omega)$ and its root o , then the same quantity defined in terms of the marked forward subtree $\mathcal{T}_u$ and its root u equals $f(\theta_u \omega)$. For example, $\{\mathcal{T}_u \text{ is infinite}\} = \{\theta_u \omega \in \mathcal{S}\}$, so that Z_n is the number of vertices of generation n whose forward subtree is infinite. Similarly, the forward cavity variable of Definition 3.15 satisfies $\Gamma_u(z) = G_{oo}(z) \circ \theta_u$ for every $u \in V(\omega) \setminus \{o\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Here, and in the rest of this section, a function such as $G_{oo}(z)$ which is defined only $\mathbb{P}$ -almost surely is extended by 0 to the exceptional null set; since every shift preserves $\mathbb{P}$, the identity $\Gamma_u(z) = G_{oo}(z) \circ \theta_u$ then holds for all u simultaneously, $\mathbb{P}$ -almost surely.

Lemma 4.2. *Let $p > 0$, let $M = \mathbb{C} \times \Omega$ with the product σ -field, and let $\mathfrak{m} = \gamma \otimes \mathbb{P}$.*

- (i) *For every $n \geq 0$, the configurations $\theta_v \omega$, $v \in \mathbb{U}_n$, are independent, each has law $\mathbb{P}$, and they are independent of the σ -field*

$$\mathcal{F}_n = \sigma(N_u, \zeta_u : u \in \mathbb{U}, |u| < n; w_u : u \in \mathbb{U}, 1 \leq |u| \leq n).$$

The set $\mathcal{V}_n$ is $\mathcal{F}_n$ -measurable.

- (ii) *The random elements N_o , ζ_o , and $(w_k, \theta_k \omega)$, $k = 1, 2, \dots$, are independent, and each $(w_k, \theta_k \omega)$ has law $\mathfrak{m}$. The map $\omega \mapsto (N_o, \zeta_o, ((w_k, \theta_k \omega))_{k \geq 1})$ is a bijection from Ω onto $\{0, 1, 2, \dots\} \times Z \times M^{\{1, 2, \dots\}}$, and both this map and its inverse are measurable.*
- (iii) *The child process $\Pi_o = \sum_{k=1}^{N_o} \delta_{(w_k, \theta_k \omega)}$ is a Poisson process on M with intensity measure $p \mathfrak{m}$, and it is independent of ζ_o . For $u \in \mathbb{U}$, the process $\Pi_u = \Pi_o \circ \theta_u = \sum_{k=1}^{N_u} \delta_{(w_{uk}, \theta_{uk} \omega)}$ lists the edge marks and the forward configurations of the children of u .*

Proof. The coordinates of $\theta_v \omega$ are the coordinates N_{vu} , ζ_{vu} with $u \in \mathbb{U}$ and w_{vu} with $u \in \mathbb{U}_*$. For distinct $v, v' \in \mathbb{U}_n$, no label of the form vu is of the form $v'u'$. Further, every label vu with $u \in \mathbb{U}$ has length at least n , and every label vu with $u \in \mathbb{U}_*$ has

length larger than n . The configurations $\theta_v \omega$, $v \in \mathbb{U}_n$, and the generators of $\mathcal{F}_n$ are therefore functions of pairwise disjoint sets of coordinates. Under the product measure $\mathbb{P}$, such functions are independent. Recall that θ_v preserves $\mathbb{P}$, so that each $\theta_v \omega$ has law $\mathbb{P}$. Finally, whether $v \in \mathbb{U}_n$ belongs to $V(\omega)$ depends only on $N_{v_0}, \dots, N_{v_{n-1}}$, which are $\mathcal{F}_n$ -measurable. This proves (i).

For (ii), we note that N_o , ζ_o , and the pairs $(w_k, \theta_k \omega)$, $k \geq 1$, are again functions of pairwise disjoint sets of coordinates. They are independent, and $(w_k, \theta_k \omega)$ has law $\gamma \otimes \mathbb{P}$ by (i). Every coordinate of ω appears exactly once in $(N_o, \zeta_o, ((w_k, \theta_k \omega))_{k \geq 1})$: the coordinates at a label ku appear in $\theta_k \omega$ at the label u , except for w_k , which appears separately. This gives the bijection; it is measurable in both directions because both directions act coordinatewise.

By (ii), conditionally on $N_o = n$, the points $(w_k, \theta_k \omega)$, $1 \leq k \leq n$, are independent with law $\mathfrak{m}$, and they are independent of ζ_o . A mixed binomial process whose number of points has law $\text{Pois}(p)$ is a Poisson process with intensity measure $p \mathfrak{m}$ [94, Proposition 3.5], which proves the first claim in (iii). The second follows from the definition of θ_u . $\square$

Remark 4.3. The auxiliary marks do not enter the marked tree. They serve two purposes. The rank $\zeta_v(1)$ is used to select children by a rule which does not depend on the labels, such as “the leaf child with the smallest rank”. The remaining coordinates of ζ_v may be used to attach further independent randomness to v , for instance a variable with a prescribed law on $\mathbb{R}$, obtained by applying the corresponding quantile function to the second coordinate of ζ_v . All statements below concern arbitrary measurable functionals of ω and apply to such constructions without change.

4.2 The Mecke formula and survival

Let $(X, \mathcal{X})$ be a measurable space. We let $\mathbf{N}(X)$ denote the set of finite counting measures $\pi = \sum_{i=1}^n \delta_{x_i}$, where $n \geq 0$ and $x_1, \dots, x_n \in X$, with the σ -field generated by the maps $\pi \mapsto \pi(B)$, $B \in \mathcal{X}$. For such π and $k \geq 1$, we let

$$\pi^{(k)} = \sum_{(i_1, \dots, i_k)} \delta_{(x_{i_1}, \dots, x_{i_k})},$$

where the sum runs over all ordered k -tuples of pairwise distinct indices in $\{1, \dots, n\}$; this measure on X^k does not depend on the enumeration of the atoms of π . Let μ be a probability measure on X . A Poisson process with intensity measure $p\mu$ is a random element Π of $\mathbf{N}(X)$ such that, for all pairwise disjoint $B_1, \dots, B_r \in \mathcal{X}$, the variables $\Pi(B_1), \dots, \Pi(B_r)$ are independent and $\Pi(B_i)$ has law $\text{Pois}(p\mu(B_i))$. Its law is determined by $p\mu$, and it coincides with the law of $\sum_{i=1}^N \delta_{X_i}$, where N has law $\text{Pois}(p)$ and $X_1, X_2, \dots$ are independent with law μ and independent of N ; see [94, Chapter 3].

Lemma 4.4. *Let $(X, \mathcal{X})$ be a measurable space, let μ be a probability measure on X , let $p > 0$, and let Π be a Poisson process on X with intensity measure $p\mu$.*

(i) *For every measurable $F: X \times \mathbf{N}(X) \rightarrow [0, \infty]$,*

$$\mathbb{E} \int F(x, \Pi - \delta_x) \Pi(dx) = p \int \mathbb{E} F(x, \Pi) \mu(dx). \quad (4.1)$$

(ii) *For every $k \geq 1$ and every measurable $F: X^k \times \mathbf{N}(X) \rightarrow [0, \infty]$,*

$$\begin{aligned} \mathbb{E} \int F\left(x_1, \dots, x_k, \Pi - \sum_{i=1}^k \delta_{x_i}\right) \Pi^{(k)}(dx_1 \cdots dx_k) \\ = p^k \int \mathbb{E} F(x_1, \dots, x_k, \Pi) \mu^{\otimes k}(dx_1 \cdots dx_k). \end{aligned} \quad (4.2)$$

Identity (4.1) is the Mecke equation [97]; see [94, Theorem 4.1], and (4.2) is its multivariate form [94, Theorem 4.4]. In words, (4.1) states that the measure $B \mapsto \mathbb{E} \int \mathbf{1}_B(x, \Pi - \delta_x) \Pi(dx)$ on $X \times \mathbf{N}(X)$ equals $p\mu \otimes \text{Law}(\Pi)$. Summing a functional over the points of Π amounts to drawing the selected point from μ , with total weight p , while the remaining points form a Poisson process with the original intensity measure which is independent of the selected point (Slivnyak's theorem). We include the short proof for finite intensity measures.

Proof of Lemma 4.4. Since both sides of (4.2) depend only on the law of Π , we may assume that $\Pi = \sum_{i=1}^N \delta_{X_i}$, where N has law $\text{Pois}(p)$ and $X_1, X_2, \dots$ are independent with law μ and independent of N . Then the left side of (4.2) equals

$$\mathbb{E} \sum_{(i_1, \dots, i_k)} F\left(X_{i_1}, \dots, X_{i_k}, \sum_{i \leq N, i \notin \{i_1, \dots, i_k\}} \delta_{X_i}\right),$$

where the sum runs over ordered k -tuples of pairwise distinct indices in $\{1, \dots, N\}$. Conditionally on $N = n \geq k$, there are $n!/(n-k)!$ such tuples, and by exchangeability of $(X_1, \dots, X_n)$, each term has the same conditional expectation as $F(X_1, \dots, X_k, \sum_{i=k+1}^n \delta_{X_i})$. Since $\mathbb{P}(N = n) n!/(n-k)! = p^k \mathbb{P}(N = n-k)$, we obtain

$$p^k \sum_{n \geq k} \mathbb{P}(N = n-k) \mathbb{E} F\left(X_1, \dots, X_k, \sum_{i=k+1}^n \delta_{X_i}\right) = p^k \mathbb{E} F\left(X_1, \dots, X_k, \sum_{i=k+1}^{k+N'} \delta_{X_i}\right),$$

where N' is a $\text{Pois}(p)$ variable independent of $(X_i)_{i \geq 1}$. The process $\sum_{i=k+1}^{k+N'} \delta_{X_i}$ is a Poisson process with intensity $p\mu$ which is independent of $(X_1, \dots, X_k)$. Then (4.2) follows from Tonelli's theorem. The case $k = 1$ is (4.1). $\square$

We now split the children of the root according to the survival of their forward subtrees. Recall the space $M = \mathbb{C} \times \Omega$, the law $m = \gamma \otimes \mathbb{P}$, and the child process Π_0 from Lemma 4.2.

Lemma 4.5. *Let $p > 1$. Set $M^\infty = \mathbb{C} \times \mathcal{S}$ and $M^f = \mathbb{C} \times (\Omega \setminus \mathcal{S})$, let $m^\infty = \gamma \otimes \mathbb{P}^\infty$ and $m^f = \gamma \otimes \mathbb{P}(\cdot \mid \Omega \setminus \mathcal{S})$, and let Π^∞ and Π^f be the restrictions of Π_o to M^∞ and M^f . Let $N^\infty = \Pi^\infty(M)$ and $N^f = \Pi^f(M)$ be the numbers of children of the root with infinite and with finite forward subtree.*

- (i) *Under $\mathbb{P}$, the random elements Π^∞ , Π^f , and ζ_o are independent, Π^∞ is a Poisson process with intensity measure $pq_p m^\infty$, and Π^f is a Poisson process with intensity measure $p(1 - q_p)m^f$. In particular, N^∞ and N^f are independent with laws $\text{Pois}(pq_p)$ and $\text{Pois}(p(1 - q_p))$. Moreover, $\mathcal{S} = \{N^\infty \geq 1\}$ and $q_p = 1 - e^{-pq_p}$.*
- (ii) *Under $\mathbb{P}^\infty$, the random elements Π^∞ , Π^f , and ζ_o are independent; Π^f and ζ_o have the same laws as under $\mathbb{P}$, and Π^∞ has the law of a Poisson process with intensity measure $pq_p m^\infty$ conditioned to have at least one point.*
- (iii) *For every measurable $Z: \Omega \rightarrow [0, \infty]$,*

$$\mathbb{E}^\infty Z = q_p^{-1} \mathbb{E}[Z \mathbf{1}_{\mathcal{S}}] \leq q_p^{-1} \mathbb{E} Z. \quad (4.3)$$

- (iv) *For every $k \geq 1$ and every measurable $F: M^k \times \mathbf{N}(M) \rightarrow [0, \infty]$,*

$$\begin{aligned} \mathbb{E}^\infty \int F(m_1, \dots, m_k, \Pi_o - \sum_{i=1}^k \delta_{m_i}) (\Pi^\infty)^{(k)}(dm_1 \cdots dm_k) \\ = p^k q_p^{k-1} \int \mathbb{E} F(m_1, \dots, m_k, \Pi_o) (m^\infty)^{\otimes k}(dm_1 \cdots dm_k). \end{aligned} \quad (4.4)$$

The left side of (4.4) is the sum, over ordered k -tuples of distinct children of the root with infinite forward subtrees, of F evaluated at their marks and at the process of all remaining children. On the right side, the selected marks are independent with law m^∞ , so that their edge marks have law γ and their forward configurations have law $\mathbb{P}^\infty$, while the residual process is the unconditioned child process Π_o under $\mathbb{P}$, independent of the selected marks. For $k = 1$ the factor is p .

Proof. We begin with (i). Let $u_1, u_2: M \rightarrow [0, \infty)$ be measurable and set $u = u_1 \mathbf{1}_{M^\infty} + u_2 \mathbf{1}_{M^f}$. Using the representation of Π_o from Lemma 4.2, we have

$$\begin{aligned} \mathbb{E} \exp(-\Pi^\infty(u_1) - \Pi^f(u_2)) &= \mathbb{E} \left(\int e^{-u} dm \right)^{N_o} = \exp \left(-p \int (1 - e^{-u}) dm \right) \\ &= \exp \left(-p \int_{M^\infty} (1 - e^{-u_1}) dm \right) \exp \left(-p \int_{M^f} (1 - e^{-u_2}) dm \right). \end{aligned}$$

The right side is the joint Laplace functional of two independent Poisson processes with intensity measures $p m(\cdot \cap M^\infty) = pq_p m^\infty$ and $p m(\cdot \cap M^f) = p(1 - q_p)m^f$. Since the joint Laplace functional determines the joint law of (Π^∞, Π^f) (see [94, Chapter 2]; alternatively, this is the restriction theorem [94, Theorem 5.2]), the first claim follows; ζ_o is independent of Π_o by Lemma 4.2. Since $|V(\omega)| = 1 + \sum_{k=1}^{N_o} |V(\theta_k \omega)|$ and $N_o < \infty$, the tree is infinite if

and only if the forward subtree of some child of the root is infinite, that is, $\mathcal{S} = \{N^\infty \geq 1\}$. Then $q_p = \mathbb{P}(N^\infty \geq 1) = 1 - e^{-pq_p}$.

For (ii), let A, B, C be measurable sets. Since $\mathcal{S} = \{\Pi^\infty \neq 0\}$, part (i) implies that

$$\mathbb{P}^\infty(\Pi^\infty \in A, \Pi^f \in B, \zeta_o \in C) = q_p^{-1} \mathbb{P}(\Pi^\infty \in A, \Pi^\infty \neq 0) \mathbb{P}(\Pi^f \in B) \mathbb{P}(\zeta_o \in C),$$

and $q_p = \mathbb{P}(\Pi^\infty \neq 0)$. This is (ii). Part (iii) is the definition of $\mathbb{P}^\infty$ together with $Z\mathbf{1}_{\mathcal{S}} \leq Z$.

For (iv), we note that the integrand on the left side vanishes unless $N^\infty \geq k \geq 1$, that is, unless $\omega \in \mathcal{S}$. By (iii), the left side equals q_p^{-1} times the corresponding expectation under $\mathbb{P}$. The measure $(\Pi^\infty)^{(k)}$ is the restriction of $\Pi_o^{(k)}$ to $(M^\infty)^k$. Applying (4.2) to Π_o and to the function $F(m_1, \dots, m_k, \pi) \prod_{i=1}^k \mathbf{1}_{M^\infty}(m_i)$, we find that the left side of (4.4) equals

$$q_p^{-1} p^k \int \mathbb{E} F(m_1, \dots, m_k, \Pi_o) \prod_{i=1}^k \mathbf{1}_{M^\infty}(m_i) m^{\otimes k}(dm_1 \cdots dm_k).$$

Since $\mathbf{1}_{M^\infty} m = q_p m^\infty$, this is the right side of (4.4). $\square$

4.3 Spine decompositions

We now define the laws of the spine trees and prove the many-to-one and many-to-two formulas. They are instances of the Palm (Mecke) formula for the Poisson process; see Remark 4.9. The idea is the classical size-biasing of the offspring numbers along a path. In a sum over the vertices v of generation L , the vertex v exists only if each of its ancestors has enough children; weighting by this event replaces the law $\text{Pois}(p)$ of each ancestor's offspring number by the law of $1 + \text{Pois}(p)$, at the cost of a factor p per generation, and makes the position of the next ancestor among its siblings uniform.

Definition 4.6. Let $L \geq 0$.

- (a) *One-spine law.* Let ω' be a random configuration with law $\mathbb{P}$, let $N'_0, \dots, N'_{L-1}$ be random variables with law $\text{Pois}(p)$, and let $I_1, \dots, I_L$ be random variables such that $\omega', N'_0, \dots, N'_{L-1}$ are independent and, conditionally on them, $I_1, \dots, I_L$ are independent and I_{l+1} is uniformly distributed on $\{1, \dots, N'_l + 1\}$. Set $v = I_1 I_2 \cdots I_L \in \mathbb{U}_L$, so that $v_l = I_1 \cdots I_l$, and let ω be the configuration obtained from ω' by setting $N_{v_l} = N'_l + 1$ for $0 \leq l < L$, all other coordinates of ω being equal to those of ω' . The one-spine law $\mathbb{Q}_L$ is the law of (ω, v) on $\Omega \times \mathbb{U}_L$.
- (b) *Two-spine law.* Let $0 \leq j < L$. Let ω' be a random configuration with law $\mathbb{P}$, and let N'_l ($0 \leq l \leq j$) and $N_l^{(1)}, N_l^{(2)}$ ($j < l < L$) be random variables with law $\text{Pois}(p)$, all of them independent. Conditionally on these, let the following random variables be independent: for $0 \leq l < j$, a variable I_{l+1} uniform on $\{1, \dots, N'_l + 1\}$; a pair (J_1, J_2)

uniform on the set of ordered pairs of distinct elements of $\{1, \dots, N'_j + 2\}$; and for $j < l < L$ and $i \in \{1, 2\}$, a variable $I_{l+1}^{(i)}$ uniform on $\{1, \dots, N_l^{(i)} + 1\}$. Set

$$u = I_1 \cdots I_j J_1 I_{j+2}^{(1)} \cdots I_L^{(1)}, \quad v = I_1 \cdots I_j J_2 I_{j+2}^{(2)} \cdots I_L^{(2)},$$

and let ω be obtained from ω' by setting $N_{v_l} = N'_l + 1$ for $0 \leq l < j$, $N_{v_j} = N'_j + 2$, and $N_{u_l} = N_l^{(1)} + 1$, $N_{v_l} = N_l^{(2)} + 1$ for $j < l < L$, all other coordinates being equal to those of ω' . The two-spine law $\mathbb{Q}_{L,j}$ is the law of (ω, u, v) on $\Omega \times \mathbb{U}_L \times \mathbb{U}_L$.

(c) *Backbone laws.* Let $p > 1$. We set

$$\mathbb{Q}_L^\infty = \mathbb{Q}_L(\cdot \mid \theta_v \omega \in \mathcal{S}), \quad \mathbb{Q}_{L,j}^\infty = \mathbb{Q}_{L,j}(\cdot \mid \theta_u \omega \in \mathcal{S}, \theta_v \omega \in \mathcal{S}).$$

We write $\mathbb{E}_Q$ for the expectation with respect to one of these laws Q , and we use (ω, v) , respectively (ω, u, v) , for the canonical random elements; the labels $I_l, J_i, I_l^{(i)}$ are the letters of u and v .

Note that under $\mathbb{Q}_L$ the vertex v belongs to $V(\omega)$, because $I_{l+1} \leq N_{v_l}$ for every $l < L$. We call the path $o = v_0, v_1, \dots, v_L = v$ the spine, the edges $\{v_{l-1}, v_l\}$ the spine edges, v the endpoint, and $\mathcal{T}(\omega)$ under $\mathbb{Q}_L$ the spine tree of length L . For $l < L$, the children of v_l other than v_{l+1} are the side children of v_l ; there are N'_l of them. Under $\mathbb{Q}_{L,j}$, both u and v belong to $V(\omega)$, and their last common ancestor $u \wedge v = v_j = u_j$, the fork, has generation j , because $J_1 \neq J_2$. The fork has N'_j side children besides u_{j+1} and v_{j+1} . The well-definedness of the backbone laws follows from the next lemma, which shows that the endpoint configurations have law $\mathbb{P}$, so that $\theta_v \omega \in \mathcal{S}$ has probability $\varrho_p > 0$.

Lemma 4.7. *Let $L \geq 0$, and let M and $\mathfrak{m} = \gamma \otimes \mathbb{P}$ be as in Lemma 4.2. Under $\mathbb{Q}_L$, for $0 \leq l < L$, let S_l be the finite sequence of the pairs $(w_{v_l k}, \theta_{v_l k} \omega)$, where k runs in increasing order through $\{1, \dots, N_{v_l}\} \setminus \{I_{l+1}\}$, and let $\Sigma_l \in \mathbf{N}(M)$ be the counting measure whose atoms are the entries of S_l .*

(i) *The random elements*

$$w_{v_1}, \dots, w_{v_L}, \quad \zeta_{v_0}, \dots, \zeta_{v_{L-1}}, \quad S_0, \dots, S_{L-1}, \quad \theta_v \omega$$

are independent. Each w_{v_l} has law γ , each ζ_{v_l} has law κ , and $\theta_v \omega$ has law $\mathbb{P}$. Each S_l has length N'_l , with law $\text{Pois}(p)$, and conditionally on its length its entries are independent with law $\mathfrak{m}$. In particular, $\Sigma_0, \dots, \Sigma_{L-1}$ are independent Poisson processes on M with intensity measure $p \mathfrak{m}$.

- (ii) *Conditionally on the random elements in (i), the labels $I_1, \dots, I_L$ are independent, and I_{l+1} is uniform on $\{1, \dots, |S_l| + 1\}$, where $|S_l|$ is the length of S_l .*
- (iii) *Let $p > 1$. Under $\mathbb{Q}_L^\infty$, assertions (i) and (ii) hold with the single change that $\theta_v \omega$ has law $\mathbb{P}^\infty$.*

- (iv) Let $0 \leq j < L$. Under $\mathbf{Q}_{L,j}$, define sequences S as above at the vertices v_l with $l < j$ (omitting the child v_{l+1}), at the fork v_j (omitting the two children u_{j+1} and v_{j+1}), and at the vertices u_l and v_l with $j < l < L$ (omitting u_{l+1} , respectively v_{l+1}). Then the $2L - j$ spine edge marks w_{v_l} ($1 \leq l \leq L$) and w_{u_l} ($j < l \leq L$), the auxiliary marks of the $2L - j - 1$ spine vertices other than u and v , the $2L - j - 1$ sequences S , and the two endpoint configurations $\theta_u \omega$ and $\theta_v \omega$ are independent, with the laws described in (i). Under $\mathbf{Q}_{L,j}^\infty$ ($p > 1$), the same holds, except that the two endpoint configurations have law $\mathbb{P}^\infty$.

The marked spine tree is assembled from the random elements in (i) and the labels: the spine edges carry the marks w_{v_l} , the side children of v_l carry the entries of S_l (edge mark and forward configuration), and the endpoint carries the forward configuration $\theta_v \omega$. In particular, under $\mathbf{Q}_L$ the squared moduli $Y_{v_1}, \dots, Y_{v_L}$ of the spine marks are independent $\text{Exp}(1)$ variables, every side child has a forward subtree which is an independent Poisson–Galton–Watson tree with independent marks, and the endpoint has an independent forward subtree with the law of the whole tree.

Proof. Fix $n = (n_0, \dots, n_{L-1}) \in \{0, 1, \dots\}^L$ and $i = (i_1, \dots, i_L)$ with $1 \leq i_{l+1} \leq n_l + 1$, and let $v = i_1 \cdots i_L$. On the event $E_{n,i} = \{N' = n, I = i\}$, which has probability $\prod_{l < L} \mathbb{P}(N = n_l)(n_l + 1)^{-1}$ for a $\text{Pois}(p)$ variable N , we have $v = v$, and ω is obtained from ω' by setting $N_{v_l} = n_l + 1$ for $l < L$. Since the pair (N', I) is independent of ω' , the configuration ω' still has law $\mathbb{P}$ conditionally on $E_{n,i}$.

We note that the modified coordinates $N_{v_0}, \dots, N_{v_{L-1}}$ are not coordinates of $\theta_{v_l k} \omega$ for $k \neq i_{l+1}$, because no $v_{l'}$ is a descendant of $v_l k$; they are not coordinates of $\theta_v \omega$ either. We conclude that, on $E_{n,i}$, the random elements in (i) are equal to

$$w'_{v_l} \ (1 \leq l \leq L), \quad \zeta'_{v_l} \ (0 \leq l < L), \quad \theta_v \omega',$$

and, for $0 \leq l < L$, the sequence of pairs $(w'_{v_l k}, \theta_{v_l k} \omega')$ with k running in increasing order through $\{1, \dots, n_l + 1\} \setminus \{i_{l+1}\}$. These are functions of pairwise disjoint sets of coordinates of ω' . Indeed, the forward configurations of distinct side children and of v use the coordinates at pairwise disjoint sets of labels, none of which is one of $v_0, \dots, v_{L-1}$, and they do not use the edge marks w_{v_l} and $w_{v_l k}$, which appear separately. Since the coordinates of ω' are independent, these random elements are, conditionally on $E_{n,i}$, independent with laws γ , κ , $m^{\otimes n_l}$, and $\mathbb{P}$, respectively. The conditional joint law depends on (n, i) only through the lengths n_l . Summing over i and then over n , and using that $N'_0, \dots, N'_{L-1}$ are independent with law $\text{Pois}(p)$, we find (i), except for its last sentence. Further, the conditional probability of $\{I = i\}$ given $N' = n$ and the random elements in (i) equals $\prod_{l < L} (n_l + 1)^{-1}$, and this is (ii), since $n_l = |S_l|$. The process Σ_l is a mixed binomial process with a $\text{Pois}(p)$ number of independent m -distributed points, hence a Poisson process with intensity measure $p m$ [94, Proposition 3.5], and the processes Σ_l are independent because the sequences S_l are.

For (iii), let g be a nonnegative measurable function of the random elements in (i) other than $\theta_v\omega$, together with the labels, and let $h: \Omega \rightarrow [0, \infty]$ be measurable. By (i), (ii), and $\mathbb{Q}_L(\theta_v\omega \in \mathcal{S}) = \mathbb{P}(\mathcal{S}) = \varrho_p$, we have

$$\mathbb{E}_{\mathbb{Q}_L^\infty}[g h(\theta_v\omega)] = \varrho_p^{-1} \mathbb{E}_{\mathbb{Q}_L}[g h(\theta_v\omega) \mathbf{1}_{\mathcal{S}}(\theta_v\omega)] = \mathbb{E}_{\mathbb{Q}_L}[g] \mathbb{E}^\infty h,$$

which is (iii). The proof of (iv) is the same. The only change is at the fork: on the corresponding event E , we have $N_{v_j} = n_j + 2$, the two omitted children are $v_j J_1$ and $v_j J_2$, and (J_1, J_2) takes each of the $(n_j + 2)(n_j + 1)$ possible values with the same conditional probability. The backbone version follows as in (iii), since the two endpoint configurations are independent with law $\mathbb{P}$ under $\mathbb{Q}_{L,j}$. $\square$

Proposition 4.8. *Let $p > 0$ and $L \geq 0$. All functions below are measurable; a function on $\Omega \times \mathbb{U}$ is measurable if it is measurable in ω for each fixed word.*

(i) *For every $F: \Omega \times \mathbb{U} \rightarrow [0, \infty]$,*

$$\mathbb{E} \sum_{v \in \mathcal{V}_L} F(\omega, v) = p^L \mathbb{E}_{\mathbb{Q}_L} F(\omega, v). \quad (4.5)$$

(ii) *Let $0 \leq j < L$. For every $F: \Omega \times \mathbb{U} \times \mathbb{U} \rightarrow [0, \infty]$,*

$$\mathbb{E} \sum_{\substack{u, v \in \mathcal{V}_L \\ |u \wedge v| = j}} F(\omega, u, v) = p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}} F(\omega, u, v). \quad (4.6)$$

(iii) *Let $p > 1$. For every F as in (i),*

$$\mathbb{E}^\infty \sum_{v \in \mathcal{V}_L} F(\omega, v) = \varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L}[F(\omega, v) \mathbf{1}_{\mathcal{S}}(\omega)] \leq \varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L} F(\omega, v), \quad (4.7)$$

$$\mathbb{E}^\infty \sum_{v \in \mathcal{V}_L^\infty} F(\omega, v) = p^L \mathbb{E}_{\mathbb{Q}_L^\infty} F(\omega, v) \leq \varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L} F(\omega, v). \quad (4.8)$$

(iv) *Let $p > 1$ and $0 \leq j < L$. For every F as in (ii),*

$$\begin{aligned} \mathbb{E}^\infty \sum_{\substack{u, v \in \mathcal{V}_L \\ |u \wedge v| = j}} F(\omega, u, v) &= \varrho_p^{-1} p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}}[F(\omega, u, v) \mathbf{1}_{\mathcal{S}}(\omega)] \\ &\leq \varrho_p^{-1} p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}} F(\omega, u, v), \end{aligned} \quad (4.9)$$

$$\begin{aligned} \mathbb{E}^\infty \sum_{\substack{u, v \in \mathcal{V}_L^\infty \\ |u \wedge v| = j}} F(\omega, u, v) &= \varrho_p p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}^\infty} F(\omega, u, v) \\ &\leq \varrho_p^{-1} p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}} F(\omega, u, v). \end{aligned} \quad (4.10)$$

The sum in (4.6) runs over ordered pairs of distinct vertices of generation L whose last common ancestor has generation j ; summing over $0 \leq j < L$ gives all ordered pairs

of distinct vertices of generation L , and the diagonal pairs are covered by (4.5). Under the survival-conditioned law, the backbone sums in (4.8) and (4.10) have exact representations by the backbone spine laws, in which only the endpoint configurations are conditioned on survival, while the side processes keep their unconditioned Poisson laws (Lemma 4.7). For the full sums, (4.7) and (4.9) are exact as well, but the indicator $\mathbf{1}_S(\omega)$ depends on the side subtrees; in applications we use only the resulting upper bounds.

Proof. We prove the four parts in order.

STEP 1. One spine. Fix $v = i_1 \cdots i_L \in \mathbb{U}_L$. Then $v \in V(\omega)$ if and only if $N_{v_l} \geq i_{l+1}$ for $0 \leq l < L$. For $n = (n_0, \dots, n_{L-1}) \in \{0, 1, \dots\}^L$, let $\omega^{v,n}$ denote the configuration obtained from ω by setting $N_{v_l} = n_l$ for $0 \leq l < L$. Under $\mathbb{P}$, the coordinates $N_{v_0}, \dots, N_{v_{L-1}}$ are independent of each other and of the remaining coordinates, and $\omega^{v,n}$ does not depend on them. Let N denote a $\text{Pois}(p)$ variable. By Tonelli's theorem, we have

$$\mathbb{E}[F(\omega, v) \mathbf{1}\{v \in V(\omega)\}] = \sum_{n: n_l \geq i_{l+1} \forall l} \prod_{l=0}^{L-1} \mathbb{P}(N = n_l) \mathbb{E}F(\omega^{v,n}, v).$$

Since $\mathbb{P}(N = n) = p n^{-1} \mathbb{P}(N = n - 1)$ for $n \geq 1$, the substitution $n_l = n'_l + 1$ yields

$$\mathbb{E}[F(\omega, v) \mathbf{1}\{v \in V(\omega)\}] = p^L \sum_{n' \in \{0, 1, \dots\}^L} \prod_{l=0}^{L-1} \frac{\mathbb{P}(N = n'_l) \mathbf{1}\{i_{l+1} \leq n'_l + 1\}}{n'_l + 1} \mathbb{E}F(\omega^{v, n'+1}, v),$$

where $n' + 1 = (n'_0 + 1, \dots, n'_{L-1} + 1)$. In the construction of $\mathbb{Q}_L$, the event $\{N' = n', I = (i_1, \dots, i_L)\}$ has probability equal to the product in the last display, and on this event $(\omega, v) = ((\omega')^{v, n'+1}, v)$, where ω' has law $\mathbb{P}$ and is independent of (N', I) . Summing the last display over $v \in \mathbb{U}_L$, that is, over $(i_1, \dots, i_L)$, we obtain

$$\mathbb{E} \sum_{v \in \mathbb{V}_L} F(\omega, v) = \sum_{v \in \mathbb{U}_L} \mathbb{E}[F(\omega, v) \mathbf{1}\{v \in V(\omega)\}] = p^L \mathbb{E}_{\mathbb{Q}_L} F(\omega, v),$$

which is (4.5). In the first equality we used Tonelli's theorem to exchange the sum over the countable set $\mathbb{U}_L$ with the expectation.

STEP 2. Two spines. Fix $u, v \in \mathbb{U}_L$ with $|u \wedge v| = j$, and write $a = u \wedge v$, so that $u_{j+1} \neq v_{j+1}$. Then $u, v \in V(\omega)$ if and only if the following constraints hold: N_{a_l} is at least the $(l+1)$ -th letter of a for $0 \leq l < j$; N_a is at least the last letters of both u_{j+1} and v_{j+1} ; and for $j < l < L$, N_{u_l} and N_{v_l} are at least the last letters of u_{l+1} and v_{l+1} , respectively. These constraints involve the offspring numbers at $2L - j - 1$ distinct labels. We expand in these offspring numbers as in Step 1. At the $j + 2(L - j - 1)$ labels other than the fork, we use $\mathbb{P}(N = n) = p n^{-1} \mathbb{P}(N = n - 1)$. At the fork, the two letters are distinct, which forces $N_a \geq 2$, and we use $\mathbb{P}(N = n) = p^2(n(n-1))^{-1} \mathbb{P}(N = n - 2)$ for $n \geq 2$. For a fixed value n of N_a , there are $n(n-1)$ ordered pairs of distinct letters in $\{1, \dots, n\}$, which

matches the uniform law of (J_1, J_2) in Definition 4.6. Summing over (u, v) as in Step 1 and collecting the powers of p , namely $p^j \cdot p^2 \cdot p^{2(L-j-1)} = p^{2L-j}$, we arrive at (4.6).

STEP 3. Conditioned sums. Let $p > 1$. Applying (4.3) and then (4.5) to $F(\omega, v)\mathbf{1}_S(\omega)$, we have the identity in (4.7); the inequality follows from $\mathbf{1}_S \leq 1$. If $v \in \mathcal{V}_L^\infty$, then the forward subtree of v is infinite, and in particular $\omega \in \mathcal{S}$. By (4.3) and (4.5), we have

$$\mathbb{E}^\infty \sum_{v \in \mathcal{V}_L^\infty} F(\omega, v) = \varrho_p^{-1} \mathbb{E} \sum_{v \in \mathcal{V}_L} F(\omega, v) \mathbf{1}_S(\theta_v \omega) = \varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L} [F(\omega, v) \mathbf{1}_S(\theta_v \omega)].$$

By Lemma 4.7, we have $\mathbb{Q}_L(\theta_v \omega \in \mathcal{S}) = \varrho_p$. Then the right side equals $p^L \mathbb{E}_{\mathbb{Q}_L^\infty} F(\omega, v)$, and it is at most $\varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L} F(\omega, v)$. This proves (4.8).

STEP 4. Conditioned pairs. The identity and inequality in (4.9) follow from (4.6) exactly as in Step 3. For (4.10), we argue as in Step 3 and obtain

$$\mathbb{E}^\infty \sum_{\substack{u, v \in \mathcal{V}_L^\infty \\ |u \wedge v| = j}} F(\omega, u, v) = \varrho_p^{-1} p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}} [F(\omega, u, v) \mathbf{1}_S(\theta_u \omega) \mathbf{1}_S(\theta_v \omega)].$$

By Lemma 4.7(iv), the two endpoint configurations are independent with law $\mathbb{P}$ under $\mathbb{Q}_{L,j}$, and the probability that both survive is ϱ_p^2 . Then the right side equals $\varrho_p p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}^\infty} F(\omega, u, v)$, and it is at most $\varrho_p^{-1} p^{2L-j} \mathbb{E}_{\mathbb{Q}_{L,j}} F(\omega, u, v)$. This completes the proof. $\square$

Remark 4.9. The identity $n \mathbb{P}(N = n) = p \mathbb{P}(N = n - 1)$, used at each ancestor in Step 1, is the Mecke formula (4.1) for the number of points of a Poisson process. If $F(\omega, v)$ depends on ω and v only through the marked tree and the auxiliary marks, and is invariant under relabelings of siblings, then (4.5) can also be obtained by applying (4.1) to the child process at each of the L ancestors, by induction on L ; the labels I_l then play no role, and only Lemma 4.7(i) is needed to evaluate the right side. We have stated the formulas for labelled configurations so that they apply without a separate invariance check, for example to functionals that select children by their labels.

The spine laws are absolutely continuous with respect to $\mathbb{P}$. In particular, every almost sure property of the tree holds along spines.

Corollary 4.10. *Let $p > 0$, $L \geq 0$, and $0 \leq j < L$. The law of ω under $\mathbb{Q}_L$ has density $p^{-L} |\mathcal{V}_L|$ with respect to $\mathbb{P}$, and the law of ω under $\mathbb{Q}_{L,j}$ has density $p^{-(2L-j)} |\{(u, v) \in \mathcal{V}_L^2 : |u \wedge v| = j\}|$ with respect to $\mathbb{P}$. If $p > 1$, the laws of ω under $\mathbb{Q}_L^\infty$ and $\mathbb{Q}_{L,j}^\infty$ are absolutely continuous with respect to $\mathbb{P}^\infty$. Further, if $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, then almost surely under each of the laws $\mathbb{P}$, $\mathbb{P}^\infty$, $\mathbb{Q}_L$, $\mathbb{Q}_{L,j}$, $\mathbb{Q}_L^\infty$, and $\mathbb{Q}_{L,j}^\infty$, we have $\theta_u \omega \in A$ for all $u \in \mathbb{U}$ simultaneously.*

Proof. Applying (4.5) to $F(\omega, v) = \mathbf{1}_B(\omega)$, we find that $\mathbb{Q}_L(\omega \in B) = p^{-L} \mathbb{E}[|\mathcal{V}_L| \mathbf{1}_B]$ for $B \in \mathcal{F}$, which is the first claim. The second follows in the same way from (4.6). For

$p > 1$, (4.8) implies that $\mathbb{Q}_L^\infty(\omega \in B) = p^{-L} \mathbb{E}^\infty[|\mathcal{V}_L^\infty| \mathbf{1}_B]$, and (4.10) gives the analogous formula for $\mathbb{Q}_{L,j}^\infty$. Finally, $\mathbb{P}(\theta_u^{-1}A) = \mathbb{P}(A) = 1$ for every u , because θ_u preserves $\mathbb{P}$. The countable intersection $\bigcap_{u \in \mathbb{U}} \theta_u^{-1}A$ therefore has $\mathbb{P}$ -probability one, and by absolute continuity it has probability one under the other laws. $\square$

For instance, the conclusions of Theorem 3.10, Proposition 3.16, and Proposition 3.18 hold almost surely under all spine laws.

We next describe recursively defined variables along a spine. This is the form in which the many-to-one formula enters the computation of path moments.

Corollary 4.11. *Let $p > 0$ and $L \geq 1$. Let $(X, \mathcal{X})$ be a measurable space, let $f_0, \dots, f_L: \Omega \rightarrow X$ be measurable, and let $\Psi_0, \dots, \Psi_{L-1}: Z \times \mathbf{N}(\mathbb{C} \times X) \rightarrow X$ be measurable maps such that, for $\mathbb{P}$ -almost every ω and every $0 \leq l < L$,*

$$f_l(\omega) = \Psi_l\left(\zeta_0, \sum_{k=1}^{N_0} \delta_{(w_k, f_{l+1}(\theta_k \omega))}\right). \quad (4.11)$$

Under $\mathbb{Q}_L$, set $X_l = f_l(\theta_{v_l} \omega)$ for $0 \leq l \leq L$, and

$$\Sigma_l^f = \sum_{k \in \{1, \dots, N_{v_l}\} \setminus \{I_{l+1}\}} \delta_{(w_{v_l k}, f_{l+1}(\theta_{v_l k} \omega))}, \quad 0 \leq l < L.$$

- (i) *The random elements $X_L, w_{v_1}, \dots, w_{v_L}, \zeta_{v_0}, \dots, \zeta_{v_{L-1}}$, and $\Sigma_0^f, \dots, \Sigma_{L-1}^f$ are independent. The variable X_L has law $\mathbb{P} \circ f_L^{-1}$, and Σ_l^f is a Poisson process on $\mathbb{C} \times X$ with intensity measure $p \gamma \otimes (\mathbb{P} \circ f_{l+1}^{-1})$.*
 - (ii) *Almost surely, $X_l = \Psi_l(\zeta_{v_l}, \Sigma_l^f + \delta_{(w_{v_{l+1}}, X_{l+1})})$ for $0 \leq l < L$.*
 - (iii) *Let $p > 1$. Under $\mathbb{Q}_L^\infty$, (i) and (ii) hold with the single change that X_L has law $\mathbb{P}^\infty \circ f_L^{-1}$.*
- Further, for every measurable $\Phi: X^{L+1} \times \mathbb{C}^L \rightarrow [0, \infty]$,*

$$\mathbb{E} \sum_{v \in \mathcal{V}_L} \Phi\left((f_l(\theta_{v_l} \omega))_{l=0}^L, (w_{v_l})_{l=1}^L\right) = p^L \mathbb{E}_{\mathbb{Q}_L} \Phi\left((X_l)_{l=0}^L, (w_{v_l})_{l=1}^L\right), \quad (4.12)$$

and the right side is the expectation of a function of the independent random elements in (i), computed through the backward recursion in (ii). The same holds under $\mathbb{P}^\infty$, with the sum over $\mathcal{V}_L^\infty$ and $\mathbb{Q}_L^\infty$ in place of $\mathbb{Q}_L$.

Proof. The process Σ_l^f is the image of the process Σ_l of Lemma 4.7 under the measurable map $(w, \omega'') \mapsto (w, f_{l+1}(\omega''))$, and $X_L = f_L(\theta_v \omega)$. Then (i) follows from Lemma 4.7(i) and the fact that the image of a Poisson process under a measurable map is a Poisson process with the image intensity measure [94, Theorem 5.1]; for mixed binomial processes this is immediate, because the images of independent points are independent.

For (ii), let A be the set of ω for which (4.11) holds for every l . By Corollary 4.10, almost surely under $\mathbb{Q}_L$, we have $\theta_{v_l}\omega \in A$ for every l . The configuration $\theta_{v_l}\omega$ has root offspring number N_{v_l} , root auxiliary mark ζ_{v_l} , and child points $(w_{v_l k}, \theta_{v_l k}\omega)$, $1 \leq k \leq N_{v_l}$. The child with $k = l+1$ is v_{l+1} , and its point contributes $\delta_{(w_{v_{l+1}}, X_{l+1})}$ in (4.11); the other children contribute Σ_l^f . This proves (ii). Part (iii) follows in the same way from Lemma 4.7(iii). Finally, (4.12) is (4.5) applied to $F(\omega, v) = \Phi((f_l(\theta_{v_l}\omega))_l, (w_{v_l})_l)$, and its conditioned version is (4.8). $\square$

The same argument applies to the two-spine laws. Under $\mathbb{Q}_{L,j}$, the variables along the two arms are computed from the two endpoint variables $f_L(\theta_u\omega)$ and $f_L(\theta_v\omega)$, which are independent with law $\mathbb{P} \circ f_L^{-1}$, by the recursion of Corollary 4.11(ii). At the fork, Ψ_j receives the side process of the fork together with the two points $(w_{u_{j+1}}, f_{j+1}(\theta_{u_{j+1}}\omega))$ and $(w_{v_{j+1}}, f_{j+1}(\theta_{v_{j+1}}\omega))$, and along the common part the recursion is that of (ii). All spine edge marks, auxiliary marks, and side processes are independent, by Lemma 4.7(iv).

Remark 4.12. As an example, fix $z \in \mathbb{C} \setminus \mathbb{R}$ and $s > 0$, let $f_l = G_{oo}(z)$ for every l , and let $\Psi_l(\zeta, \pi) = -(z + p^{-1} \int |w|^2 x \pi(dw dx))^{-1}$. By Proposition 3.16 at the root, (4.11) holds $\mathbb{P}$ -almost surely, and $X_0 = G_{oo}(z)$ and $X_l = \Gamma_{v_l}(z)$ for $1 \leq l \leq L$. By Proposition 3.18, for $v \in \mathcal{V}_L$ we have

$$|G_{ov}(z)|^s = |G_{oo}(z)|^s \prod_{l=1}^L p^{-s/2} Y_{v_l}^{s/2} |\Gamma_{v_l}(z)|^s.$$

By (4.12), we have

$$\mathbb{E} \sum_{v \in \mathcal{V}_L} |G_{ov}(z)|^s = p^{L(1-s/2)} \mathbb{E}_{\mathbb{Q}_L} \left[|X_0|^s \prod_{l=1}^L Y_{v_l}^{s/2} |X_l|^s \right].$$

On the right, X_L has the law of $G_{oo}(z)$, the squared moduli Y_{v_l} of the spine marks are independent $\text{Exp}(1)$ variables, and $X_l = -(z + p^{-1} Y_{v_{l+1}} X_{l+1} + \Xi_l)^{-1}$ for $l < L$, where $\Xi_l = p^{-1} \int |w|^2 x \Sigma_l^f(dw dx)$ are independent copies of the side sum $p^{-1} \sum_{k=1}^N Y_k \Gamma_k$ in the cavity equation, independent of the squared moduli of the spine marks and of X_L .

Remark 4.13. Corollary 4.11 also applies to recursions which are not driven by the tree alone. Let $E > 0$ and let π be a real solution at energy E in the sense of Definition 5.1. Let q_π be a measurable map from $[0, 1]$ to $\mathbb{R} \setminus \{0\}$ which pushes Lebesgue measure forward to π , and define $f^{(0)}(\omega) = q_\pi(\zeta_o(2))$, where $\zeta_o(2)$ is the second coordinate of ζ_o , and, for $m \geq 1$,

$$f^{(m)}(\omega) = E - p^{-1} \sum_{k=1}^{N_o} \frac{Y_k}{f^{(m-1)}(\theta_k \omega)}$$

(we set $f^{(m)}(\omega) = E$ if some denominator vanishes, and we use the same convention for Ψ_l below). By induction on m , $f^{(m)}$ has law π under $\mathbb{P}$ for every $m \geq 0$. Indeed,

by Lemma 4.2(ii), the pairs $(Y_k, f^{(m-1)}(\theta_k \omega))$ are independent with law $\text{Exp}(1) \otimes \pi$ and independent of N_0 . Then $f^{(m)}$ has the law of $E - p^{-1} \sum_{k \leq N} Y_k / D_k$ with independent $D_k \sim \pi$, which is π ; in particular the denominators vanish with probability zero. Fix $m \geq L$ and apply Corollary 4.11 with $f_l = f^{(m-l)}$ and $\Psi_l(\zeta, \pi') = E - p^{-1} \int |w|^2 x^{-1} \pi'(dw dx)$. In this case, X_L has law π and the side processes are Poisson with intensity measure $p \gamma \otimes \pi$, for every $m \geq L$. The family $D_v = f^{(m-|v|)}(\theta_v \omega)$, $|v| \leq m$, satisfies $D_v = E - p^{-1} \sum_k Y_{vk} / D_{vk}$ for $|v| < m$, and its marks at generation m are independent with law π and independent of $(N_u)_{|u| < m}$ and $(w_u)_{1 \leq |u| \leq m}$. Every family $(D_v)_{|v| \leq m}$ with these two properties has the same joint law with $(N_u)_{|u| < m}$ and $(w_u)_{1 \leq |u| \leq m}$, because the marks at generation m determine the others through the recursion. This applies in particular to a recursive tree process as in Lemma 5.13, restricted to generations at most m , provided its marks at generation m are independent with law π and independent of $(N_u)_{|u| < m}$ and $(w_u)_{1 \leq |u| \leq m}$; this is part of the definition of a recursive tree process [17]. The many-to-one and many-to-two formulas of this section, and Corollary 4.11 in particular, therefore apply to functionals of $(N_u)_{|u| < m}$, $(w_u)_{1 \leq |u| \leq m}$, and $(D_v)_{|v| \leq m}$ for such a process. The offspring numbers at generation m must not enter these functionals, since a mark D_v of a recursive tree process with $|v| = m$ depends on N_v .

4.4 Conditioned endpoint laws

The following proposition compares laws of endpoint functionals under $\mathbb{P}$ and $\mathbb{P}^\infty$, and it transfers limits from $\mathbb{P}$ to $\mathbb{P}^\infty$ and to the backbone spine laws. Here an endpoint functional is any measurable function of the root configuration ω , or of an endpoint configuration $\theta_v \omega$ under a spine law; it may involve the forest obtained from the forward subtree by removing finitely many vertices and edges, or a finite decoration, since these are again measurable functions of the configuration.

Proposition 4.14. *Let $p > 1$, let $(X, \mathcal{X})$ be a measurable space, and let $X: \Omega \rightarrow X$ be measurable.*

- (i) *As measures on X , $\mathbb{P}^\infty \circ X^{-1} \leq \varrho_p^{-1} \mathbb{P} \circ X^{-1}$. Equivalently, $\mathbb{E}^\infty \varphi(X) \leq \varrho_p^{-1} \mathbb{E} \varphi(X)$ for every measurable $\varphi: X \rightarrow [0, \infty]$. In particular, if $C \in \mathcal{X}$ and $\mathbb{P}(X \notin C) \leq \delta$, then $\mathbb{P}^\infty(X \in C) \geq 1 - \varrho_p^{-1} \delta$.*
- (ii) *Let $L \geq 0$ and $0 \leq j < L$. Under $\mathbb{Q}_L^\infty$, the endpoint variable $X(\theta_v \omega)$ has law $\mathbb{P}^\infty \circ X^{-1}$, and it is independent of the spine edge marks, the auxiliary marks of the spine vertices, and the side sequences. Under $\mathbb{Q}_{L,j}^\infty$, the pair $(X(\theta_u \omega), X(\theta_v \omega))$ has law $(\mathbb{P}^\infty \circ X^{-1})^{\otimes 2}$, and it is independent of the remaining random elements of Lemma 4.7(iv). In particular, these laws are bounded by $\varrho_p^{-1} \mathbb{P} \circ X^{-1}$ and $\varrho_p^{-2} (\mathbb{P} \circ X^{-1})^{\otimes 2}$, respectively.*
- (iii) *For $0 < \eta \leq 1$, let Z_η and Z_0 be real random variables on Ω such that $Z_\eta \rightarrow Z_0$ $\mathbb{P}$ -almost surely as $\eta \downarrow 0$, and such that the family $\{Z_\eta : 0 < \eta \leq 1\}$ is uniformly integrable under $\mathbb{P}$.*

Then Z_0 is $\mathbb{P}$ -integrable, $Z_\eta \rightarrow Z_0$ in $L^1(\mathbb{P}^\infty)$, and in particular $\mathbb{E}^\infty Z_\eta \rightarrow \mathbb{E}^\infty Z_0$. Further, $\mathbb{E}_{Q_L^\infty} Z_\eta(\theta_\nu \omega) \rightarrow \mathbb{E}_{Q_L^\infty} Z_0(\theta_\nu \omega)$ for every $L \geq 0$.

- (iv) For $0 \leq \eta \leq 1$, let $X_\eta: \Omega \rightarrow \mathbb{R}^k$ be measurable with $X_\eta \rightarrow X_0$ $\mathbb{P}$ -almost surely as $\eta \downarrow 0$, and let $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}$ be a bounded Borel function which is continuous at every point outside a Borel set D with $\mathbb{P}(X_0 \in D) = 0$. Then $\mathbb{E}^\infty \varphi(X_\eta) \rightarrow \mathbb{E}^\infty \varphi(X_0)$ as $\eta \downarrow 0$.

Proof. For $B \in \mathcal{X}$, we have by (4.3) that $\mathbb{P}^\infty(X \in B) = \varrho_p^{-1} \mathbb{P}(X \in B, \mathcal{S}) \leq \varrho_p^{-1} \mathbb{P}(X \in B)$, which is (i); the last assertion of (i) follows by taking $B = X \setminus C$. Part (ii) follows from Lemma 4.7(iii) and (iv), together with (i).

For (iii), it suffices to consider an arbitrary sequence $\eta_n \downarrow 0$. Since $\mathbb{P}^\infty \leq \varrho_p^{-1} \mathbb{P}$, the convergence $Z_{\eta_n} \rightarrow Z_0$ holds $\mathbb{P}^\infty$ -almost surely. For $K > 0$, we have

$$\sup_n \mathbb{E}^\infty [|Z_{\eta_n}| \mathbf{1}_{\{|Z_{\eta_n}| > K\}}] \leq \varrho_p^{-1} \sup_n \mathbb{E} [|Z_{\eta_n}| \mathbf{1}_{\{|Z_{\eta_n}| > K\}}],$$

and the right side tends to zero as $K \rightarrow \infty$. This shows that the sequence is uniformly integrable under $\mathbb{P}^\infty$ as well. By Fatou's lemma, Z_0 is integrable under $\mathbb{P}$. Then Vitali's convergence theorem (see [64, Section 4.6]), applied under $\mathbb{P}^\infty$, implies that $Z_{\eta_n} \rightarrow Z_0$ in $L^1(\mathbb{P}^\infty)$. The last statement follows from (ii), since $\mathbb{E}_{Q_L^\infty} Z(\theta_\nu \omega) = \mathbb{E}^\infty Z$ for every integrable Z .

For (iv), $\varphi(X_\eta) \rightarrow \varphi(X_0)$ on the event $\{X_\eta \rightarrow X_0\} \cap \{X_0 \notin D\}$, which has $\mathbb{P}$ -probability one and therefore $\mathbb{P}^\infty$ -probability one. The bounded convergence theorem concludes the proof. $\square$

Remark 4.15. The domination in Proposition 4.14(i) cannot be reversed with a constant independent of X . For instance, $\mathbb{P}(N_0 = 0) = e^{-p}$, while $\mathbb{P}^\infty(N_0 = 0) = 0$, because a root without children has a finite tree. If a functional X equals a fixed value x_0 on $\{N_0 = 0\}$, as the real boundary denominator does (it equals E there), then the part of the atom of $\mathbb{P} \circ X^{-1}$ at x_0 coming from the event $\{N_0 = 0\}$ is absent from $\mathbb{P}^\infty \circ X^{-1}$. In particular, conditioned endpoint laws are in general not equal to the corresponding unconditioned laws, and only upper bounds transfer from $\mathbb{P}$ to $\mathbb{P}^\infty$ through (i).

4.5 Fresh squared moduli

Many estimates in this paper bound negative moments of quantities of the form $c + \sum_j a_j Y_j$, where the Y_j are squared moduli of edge marks and c and the a_j depend on the rest of the tree. The following definition makes precise when such an estimate may be applied conditionally.

Definition 4.16. Let $(\hat{\Omega}, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{A}$ be a σ -field. Let M be a $\mathcal{G}$ -measurable random variable with values in $\{0, 1, 2, \dots\}$, and let $Y_1, Y_2, \dots$ be random

variables with values in $[0, \infty)$. We say that $Y_1, \dots, Y_M$ are *fresh squared moduli given $\mathcal{G}$* , or *fresh given $\mathcal{G}$* , if for every $m \geq 0$ and every bounded Borel function $\varphi: [0, \infty)^m \rightarrow \mathbb{R}$,

$$\mathbb{E}[\varphi(Y_1, \dots, Y_m) \mathbf{1}\{M = m\} \mid \mathcal{G}] = \mathbf{1}\{M = m\} \int_{[0, \infty)^m} \varphi(y) e^{-(y_1 + \dots + y_m)} dy \quad (4.13)$$

almost surely. Random variables that are $\mathcal{G}$ -measurable are called revealed.

In words, on the event $\{M = m\}$, the variables $Y_1, \dots, Y_m$ are independent $\text{Exp}(1)$ variables conditionally on $\mathcal{G}$. The number M of fresh variables is revealed, while their values are not; the variables Y_j with $j > M$ play no role. In the applications, $\widehat{\Omega}$ is Ω with one of the laws $\mathbb{P}, \mathbb{P}^\infty, \mathbb{Q}_L, \mathbb{Q}_L^\infty, \mathbb{Q}_{L,j}, \mathbb{Q}_{L,j}^\infty$, and the Y_j are the squared moduli of the marks of a random finite set of edges, listed in a revealed order. We record three elementary consequences of the definition.

- (a) If $Y_1, \dots, Y_M$ are fresh given $\mathcal{G}$, and $\mathcal{G}' \subset \mathcal{G}$ is a σ -field with respect to which M is measurable, then $Y_1, \dots, Y_M$ are fresh given $\mathcal{G}'$. This follows from (4.13) by the tower property of conditional expectation.
- (b) Revealing some of the fresh variables leaves the others fresh. Precisely, let $M' \leq M$ be $\mathcal{G}$ -measurable, and let $\mathcal{G}'$ be generated by $\mathcal{G}$ and the variables $Y_j \mathbf{1}\{M' < j \leq M\}$, $j \geq 1$. Then $Y_1, \dots, Y_{M'}$ are fresh given $\mathcal{G}'$, because on $\{M = m, M' = m'\}$ the variables $Y_1, \dots, Y_{m'}$ are independent of $Y_{m'+1}, \dots, Y_m$ conditionally on $\mathcal{G}$. In a sum $c + \sum_{j \leq M} a_j Y_j$, the revealed terms $a_j Y_j$, $M' < j \leq M$, are then absorbed into the constant, and the required lower bound must be checked for the new constant $c + \sum_{M' < j \leq M} a_j Y_j$.
- (c) Revealed variables may be treated as constants. Let $Y_1, \dots, Y_M$ be fresh given $\mathcal{G}$, let $k \geq 1$, and let X be a $\mathcal{G}$ -measurable random variable with values in $\mathbb{C}^k$. Then for every $m \geq 0$ and every Borel function $\Phi: \mathbb{C}^k \times [0, \infty)^m \rightarrow [0, \infty]$,

$$\mathbb{E}[\Phi(X, Y_1, \dots, Y_m) \mathbf{1}\{M = m\} \mid \mathcal{G}] = \mathbf{1}\{M = m\} \int_{[0, \infty)^m} \Phi(X, y) e^{-(y_1 + \dots + y_m)} dy \quad (4.14)$$

almost surely. Indeed, if $\Phi = \mathbf{1}_{D_1 \times D_2}$ with Borel sets $D_1 \subset \mathbb{C}^k$ and $D_2 \subset [0, \infty)^m$, then $\mathbf{1}_{D_1}(X)$ is $\mathcal{G}$ -measurable and may be taken out of the conditional expectation, and (4.14) reduces to (4.13) with $\varphi = \mathbf{1}_{D_2}$. The Borel sets $D \subset \mathbb{C}^k \times [0, \infty)^m$ for which (4.14) holds with $\Phi = \mathbf{1}_D$ form a Dynkin system, by linearity and the monotone convergence theorem for conditional expectations, and this system contains the π -system of measurable rectangles. By the π - λ theorem [81, Chapter 1], it contains all Borel sets. Approximating Φ from below by simple functions and using monotone convergence once more, we obtain (4.14). We refer to (4.14) as the substitution identity.

The main source of fresh variables is the following lemma: edge marks of edges that are selected without looking at their marks are independent standard complex Gaussian

variables given everything else. It complements Lemma 3.4: there, a selection may inspect some marks, which are then removed, and the retained marks are fresh; here, a selection may inspect every mark outside the selected set. For a finite set $a \subset \mathbb{U}_*$, let

$$\mathcal{G}_a = \sigma(N_v, \zeta_v : v \in \mathbb{U}; w_v : v \in \mathbb{U}_* \setminus a),$$

and for $x = (x_v)_{v \in a} \in \mathbb{C}^a$, let $\omega^{a \leftarrow x}$ denote the configuration obtained from ω by replacing w_v by x_v for every $v \in a$.

Lemma 4.17. *Let $p > 0$, and let A be a random finite subset of $\mathbb{U}_*$ such that $\{A = a\} \in \mathcal{G}_a$ for every finite set $a \subset \mathbb{U}_*$.*

(i) *For every measurable $F: \Omega \rightarrow [0, \infty]$,*

$$\mathbb{E}F(\omega) = \mathbb{E} \int_{\mathbb{C}^A} F(\omega^{A \leftarrow x}) \gamma^{\otimes A}(\mathrm{d}x). \quad (4.15)$$

If $p > 1$, the same identity holds with $\mathbb{E}^\infty$ in place of $\mathbb{E}$ on both sides.

(ii) *Let $L \geq 0$, $0 \leq j < L$, and let $A: \Omega \times \mathbb{U}_L \rightarrow \{\text{finite subsets of } \mathbb{U}_*\}$ be such that $A(\cdot, v)$ satisfies the hypothesis above for every $v \in \mathbb{U}_L$. Then for every measurable $F: \Omega \times \mathbb{U}_L \rightarrow [0, \infty]$, with $A = A(\omega, v)$,*

$$\mathbb{E}_{Q_L} F(\omega, v) = \mathbb{E}_{Q_L} \int_{\mathbb{C}^A} F(\omega^{A \leftarrow x}, v) \gamma^{\otimes A}(\mathrm{d}x),$$

and the same identity holds under Q_L^∞ if $p > 1$. The analogous identities hold under $Q_{L,j}$ and $Q_{L,j}^\infty$ for $A = A(\omega, u, v)$ such that $A(\cdot, u, v)$ satisfies the hypothesis above for all $u, v \in \mathbb{U}_L$.

(iii) *In the setting of (i) or (ii), list the elements of A as $v^{(1)}, \dots, v^{(M)}$, where $M = |A|$, in a fixed total order of $\mathbb{U}$. Let $\mathcal{H}_A$ be the σ -field generated by A , by the configuration $\omega^{A \leftarrow 1}$ in which all selected marks are replaced by 1, by the phases $w_v/|w_v|$, $v \in A$ (set to 1 if $w_v = 0$), and, in the setting of (ii), by the endpoint v , respectively (u, v) . Then $Y_{v^{(1)}}, \dots, Y_{v^{(M)}}$ are fresh squared moduli given $\mathcal{H}_A$.*

Proof. For (i), fix a finite set $a \subset \mathbb{U}_*$. The indicator $\mathbf{1}\{A = a\}$ and the configuration $\omega^{a \leftarrow x}$, for fixed x , are functions of the coordinates of ω other than $(w_v)_{v \in a}$, and under $\mathbb{P}$ the coordinates $(w_v)_{v \in a}$ are independent of the other coordinates and have law $\gamma^{\otimes a}$. Since $\omega = \omega^{a \leftarrow (w_v)_{v \in a}}$, we have by Fubini's theorem that

$$\mathbb{E}[\mathbf{1}\{A = a\}F(\omega)] = \mathbb{E}\left[\mathbf{1}\{A = a\} \int_{\mathbb{C}^a} F(\omega^{a \leftarrow x}) \gamma^{\otimes a}(\mathrm{d}x)\right].$$

Summing over the countably many finite sets a gives (4.15). Under $\mathbb{P}^\infty$, we apply (4.15) to $F\mathbf{1}_S$ and use that $\mathbf{1}_S(\omega^{A \leftarrow x}) = \mathbf{1}_S(\omega)$, since S depends only on the offspring numbers.

For (ii), we use (4.5) to write $\mathbb{E}_{Q_L} F(\omega, v) = p^{-L} \sum_{v \in \mathbb{U}_L} \mathbb{E}[\mathbf{1}\{v \in V(\omega)\}F(\omega, v)]$. For each fixed v , we apply (i) with the random set $A(\cdot, v)$ to the function $\omega \mapsto \mathbf{1}\{v \in$

$V(\omega)\}F(\omega, v)$, and we use that $V(\omega^{a \leftarrow x}) = V(\omega)$. Then we apply (4.5) backwards to the function $(\omega, v) \mapsto \int F(\omega^{A(\omega, v) \leftarrow x}, v) \gamma^{\otimes A(\omega, v)}(dx)$. This proves the identity under $\mathbb{Q}_L$. Under $\mathbb{Q}_L^\infty$, we argue in the same way with (4.8) and the second part of (i), using that $\mathbf{1}_S(\theta_v \omega)$ depends only on the offspring numbers. The two-spine identities follow in the same way from (4.6) and (4.10).

For (iii), we first note that changing the selected marks does not change A : if $\omega \in \{A = a\}$, then $\omega^{a \leftarrow x} \in \{A = a\}$ for every $x \in \mathbb{C}^a$, since $\{A = a\} \in \mathcal{G}_a$. Fix $m \geq 0$, a bounded Borel function $\varphi \geq 0$ on $[0, \infty)^m$, and a bounded measurable function $g \geq 0$ of the generators of $\mathcal{H}_A$. We write $x_v = y_v^{1/2} e^{i\phi_v}$; under γ , y_v has law $\text{Exp}(1)$, $e^{i\phi_v}$ is uniform on the unit circle, and the two are independent. Applying (i) to the function $F = g \cdot \mathbf{1}\{M = m\} \varphi(Y_{v(1)}, \dots, Y_{v(m)})$, we obtain

$$\begin{aligned} \mathbb{E}[g \mathbf{1}\{M = m\} \varphi(Y_{v(1)}, \dots, Y_{v(m)})] \\ = \mathbb{E}\left[\mathbf{1}\{M = m\} \int g(\omega^{A \leftarrow x}) \gamma^{\otimes A}(dx)\right] \int \varphi(y) e^{-(y_1 + \dots + y_m)} dy. \end{aligned}$$

Here we used that $g(\omega^{A \leftarrow x})$ depends on x only through the phases of the x_v , while φ depends only on their moduli. Applying (i) once more, to the function $g \mathbf{1}\{M = m\}$, shows that the first factor on the right equals $\mathbb{E}[g \mathbf{1}\{M = m\}]$. This is (4.13) for $\mathcal{G} = \mathcal{H}_A$. The same argument applies in the setting of (ii). $\square$

For example, take $A = \{1, \dots, N_o\}$, the set of children of the root. We have $\{A = a\} \in \sigma(N_o) \subset \mathcal{G}_a$, and $\mathcal{H}_A$ contains N_o, ζ_o , all forward configurations $\theta_k \omega$, $k \geq 1$, and the phases of the w_k . By Lemma 4.17(iii) and consequence (a) after Definition 4.16, the squared moduli $Y_1, \dots, Y_{N_o}$ of the root edge marks are fresh given the σ -field generated by N_o, ζ_o , the forward configurations of the children, and the phases. This is the situation of the cavity equation. Similarly, under $\mathbb{Q}_L$, the choice $A(\omega, v) = \{v_1, \dots, v_L\}$ in Lemma 4.17(ii) shows that the squared moduli $Y_{v_1}, \dots, Y_{v_L}$ of the spine marks are fresh given the σ -field generated by the endpoint v , the phases of the spine edge marks, and all coordinates of ω other than the spine edge marks.

We now state the affine negative-moment lemma.

Lemma 4.18. *Let $0 < r < 1$ and $d > 0$, and set $C_{r,d} = (1 - r)^{-1}(2/d)^r$. Let $m \geq 0$, let $Y_1, \dots, Y_m$ be independent random variables with law $\text{Exp}(1)$, let $a_1, \dots, a_m \in \mathbb{C}$ and $c \in \mathbb{C}$, and set $S = c + \sum_{j=1}^m a_j Y_j$.*

(i) *If $c \neq 0$, then for every $\varepsilon > 0$,*

$$\mathbb{P}(|S| \leq \varepsilon) \leq \min\left\{1, \frac{2\varepsilon\sqrt{m+1}}{|c|}\right\}. \quad (4.16)$$

(ii) *If $c \neq 0$, then*

$$\mathbb{E}|S|^{-r} \leq \frac{1}{1-r} \left(\frac{2\sqrt{m+1}}{|c|}\right)^r. \quad (4.17)$$

In particular, if $|c| \geq d$, then $\mathbb{E}|S|^{-r} \leq C_{r,d}(m+1)^{r/2}$.

- (iii) Suppose that $c, a_1, \dots, a_m$ are real. If $a_1 = \dots = a_m = 0$, then the law of S is δ_c . Otherwise, the law of S is absolutely continuous, and it has a density ρ such that

$$\rho(x) \leq \frac{\sqrt{m}}{\sqrt{2\pi}|x-c|} \quad \text{for every } x \in \mathbb{R} \setminus \{c\}. \quad (4.18)$$

In either case, the absolutely continuous part of the law of S has a density which is at most $\sqrt{m+1}/(2d)$ at every x with $|x-c| \geq d$.

- (iv) Let $\mathcal{G}$, M , and $Y_1, Y_2, \dots$ be as in Definition 4.16, with $Y_1, \dots, Y_M$ fresh given $\mathcal{G}$. Let $c, a_1, a_2, \dots$ be $\mathcal{G}$ -measurable complex random variables, and set $S = c + \sum_{j=1}^M a_j Y_j$. Then, for every $\varepsilon > 0$, almost surely on the event $\{c \neq 0\}$,

$$\mathbb{P}(|S| \leq \varepsilon \mid \mathcal{G}) \leq \min\left\{1, \frac{2\varepsilon\sqrt{M+1}}{|c|}\right\}, \quad \mathbb{E}[|S|^{-r} \mid \mathcal{G}] \leq \frac{1}{1-r} \left(\frac{2\sqrt{M+1}}{|c|}\right)^r.$$

If $|c| \geq d$ almost surely, then $\mathbb{E}|S|^{-r} \leq C_{r,d} \mathbb{E}(M+1)^{r/2} \leq C_{r,d}(1 + \mathbb{E}M)^{r/2}$. If c and all a_j are real, then for every Borel set $B \subset \mathbb{R}$, almost surely,

$$\mathbb{P}(S \in B, |S-c| \geq d \mid \mathcal{G}) \leq \frac{\sqrt{M+1}}{2d} |B|,$$

where $|B|$ denotes the Lebesgue measure of B .

The bounds in (i)–(iii) depend on the coefficients a_j only through their number. This is what makes the lemma applicable when the coefficients are cavity variables of subtrees, about which nothing is known except that they are revealed.

Proof. We first prove the density bound (iii) for real coefficients using the Gamma–Dirichlet decomposition of $(Y_1, \dots, Y_m)$. Then we deduce the small-ball bound (i) for complex coefficients by rotating c to the positive real axis and projecting onto the real axis, and we integrate it to obtain (ii). Finally, we derive the conditional version (iv) by treating the revealed variables as constants.

STEP 1. The Gamma–Dirichlet decomposition. Let $m \geq 1$, set $T = Y_1 + \dots + Y_m$, and set $U_j = Y_j/T$ for $1 \leq j \leq m$. Then T has the Gamma density

$$f_m(t) = \frac{t^{m-1}e^{-t}}{(m-1)!}, \quad t > 0,$$

and the vector $(U_1, \dots, U_m)$ is independent of T (it is uniformly distributed on the simplex). Indeed, the joint density of $(Y_1, \dots, Y_m)$ is $e^{-(y_1+\dots+y_m)} = e^{-t}$, and the change of variables $(y_1, \dots, y_m) \mapsto (t, u_1, \dots, u_{m-1})$ has Jacobian t^{m-1} ; it follows that the joint

density of $(T, U_1, \dots, U_{m-1})$ is a product of a function of t and a function of $(u_1, \dots, u_{m-1})$. We also need the bound

$$\sigma_m = \sup_{t>0} t f_m(t) = \frac{m^m e^{-m}}{(m-1)!} = \frac{m^{m+1} e^{-m}}{m!} \leq \frac{\sqrt{m}}{\sqrt{2\pi}}. \quad (4.19)$$

The supremum is attained at $t = m$, since $t \mapsto t^m e^{-t}$ is maximal there, and in the last inequality we used Stirling's formula in the form $m! \geq \sqrt{2\pi m} (m/e)^m$, valid for every $m \geq 1$.

STEP 2. Real coefficients. Let $c, a_1, \dots, a_m$ be real. If all a_j vanish (including the case $m = 0$), then $S = c$. Otherwise $m \geq 1$, and we may assume that $a_1 \neq 0$. Then $S = a_1 Y_1 + (c + \sum_{j \geq 2} a_j Y_j)$ is the sum of an absolutely continuous variable and an independent variable, and its law is therefore absolutely continuous. With the notation of Step 1, set $A = \sum_{j=1}^m a_j U_j$, so that $S = c + AT$, where A is independent of T . Since $\{A = 0\} \subset \{S = c\}$ and $\mathbb{P}(S = c) = 0$, we have $\mathbb{P}(A = 0) = 0$. For a Borel set $B \subset \mathbb{R}$, Tonelli's theorem and the change of variables $x = c + At$ give

$$\mathbb{P}(S \in B) = \mathbb{E} \left[\int_0^\infty \mathbf{1}_B(c + At) f_m(t) dt \right] = \int_B \rho(x) dx,$$

where

$$\rho(x) = \mathbb{E} \left[\mathbf{1} \left\{ \frac{x-c}{A} > 0 \right\} \frac{1}{|A|} f_m \left(\frac{x-c}{A} \right) \right],$$

and where the integrand in the definition of ρ is set to zero on the null event $\{A = 0\}$. The function ρ is Borel by Tonelli's theorem. Let $x \neq c$, and suppose that $t = (x - c)/A > 0$. Then $|A| = |x - c|/t$, so that

$$\frac{1}{|A|} f_m \left(\frac{x-c}{A} \right) = \frac{t f_m(t)}{|x-c|} \leq \frac{\sigma_m}{|x-c|}.$$

Taking expectations and using (4.19), we arrive at (4.18). The last assertion of (iii) follows because $\sqrt{m}/\sqrt{2\pi} \leq \sqrt{m+1}/2$, and because the absolutely continuous part of δ_c is zero.

STEP 3. The small-ball bound. Let $c \neq 0$, let θ be an argument of c , and set

$$Q = \operatorname{Re}(e^{-i\theta} S) = |c| + \sum_{j=1}^m b_j Y_j, \quad b_j = \operatorname{Re}(e^{-i\theta} a_j) \in \mathbb{R}.$$

Then $|S| = |e^{-i\theta} S| \geq |Q|$, which implies that $\mathbb{P}(|S| \leq \varepsilon) \leq \mathbb{P}(|Q| \leq \varepsilon)$. If $\varepsilon > |c|/2$, then the right side of (4.16) equals 1 and there is nothing to prove. Let $0 < \varepsilon \leq |c|/2$. If all b_j vanish, then $Q = |c| > \varepsilon$, and $\mathbb{P}(|Q| \leq \varepsilon) = 0$. Otherwise, by Step 2 applied to Q , whose constant is $|c|$, the law of Q has a density bounded by $\sigma_m/|x - |c||$. Every $x \in [-\varepsilon, \varepsilon]$ satisfies $|x - |c|| \geq |c|/2$, so

$$\mathbb{P}(|Q| \leq \varepsilon) \leq 2\varepsilon \cdot \frac{2\sigma_m}{|c|} \leq \frac{4}{\sqrt{2\pi}} \cdot \frac{\varepsilon \sqrt{m}}{|c|} \leq \frac{2\varepsilon \sqrt{m+1}}{|c|}.$$

This proves (i).

STEP 4. *The negative moment.* By (i), $\mathbb{P}(S = 0) = 0$. For $t > 0$, we have $t^{-r} = \int_t^\infty r\varepsilon^{-r-1} d\varepsilon$, and Tonelli's theorem implies that

$$\mathbb{E}|S|^{-r} = \int_0^\infty r\varepsilon^{-r-1} \mathbb{P}(|S| \leq \varepsilon) d\varepsilon.$$

Set $K = 2\sqrt{m+1}/|c|$. By (i), the right side is at most

$$\int_0^{1/K} rK\varepsilon^{-r} d\varepsilon + \int_{1/K}^\infty r\varepsilon^{-r-1} d\varepsilon = \frac{r}{1-r}K^r + K^r = \frac{K^r}{1-r},$$

which is (4.17). If $|c| \geq d$, then $K^r \leq (2/d)^r(m+1)^{r/2}$, which gives the second assertion of (ii).

STEP 5. *The conditional version.* Fix $m \geq 0$. By consequence (c) after Definition 4.16, applied with $X = (c, a_1, \dots, a_m)$, the substitution identity (4.14) holds for every Borel function $\Phi: \mathbb{C}^{1+m} \times [0, \infty)^m \rightarrow [0, \infty]$.

We now apply (4.14) with $\Phi(c, a, y) = \mathbf{1}\{|c + \sum_j a_j y_j| \leq \varepsilon\} \mathbf{1}\{c \neq 0\}$. By (i), applied for fixed values of $c \neq 0$ and $a_1, \dots, a_m$, the right side of (4.14) is at most $\mathbf{1}\{M = m\} \mathbf{1}\{c \neq 0\} \min\{1, 2\varepsilon\sqrt{m+1}/|c|\}$. Summing over m gives the small-ball bound in (iv). The bound on $\mathbb{E}[|S|^{-r} \mid \mathcal{G}]$ follows in the same way from (ii), with $\Phi(c, a, y) = |c + \sum_j a_j y_j|^{-r} \mathbf{1}\{c \neq 0\}$. If $|c| \geq d$ almost surely, then taking expectations, we have $\mathbb{E}|S|^{-r} \leq C_{r,d} \mathbb{E}(M+1)^{r/2}$, and Jensen's inequality for the concave function $t \mapsto t^{r/2}$ gives $\mathbb{E}(M+1)^{r/2} \leq (1 + \mathbb{E}M)^{r/2}$. Finally, if c and the a_j are real, we apply (4.14) with $\Phi(c, a, y) = \mathbf{1}\{c + \sum_j a_j y_j \in B, |\sum_j a_j y_j| \geq d\}$. For fixed real $c, a_1, \dots, a_m$, the integral of Φ vanishes if all a_j vanish, and otherwise it is at most $\int_{B \cap \{|x-c| \geq d\}} \rho(x) dx \leq \sqrt{m+1} |B| / (2d)$ by (iii). Summing over m completes the proof. $\square$

Remark 4.19. Freshness is a property of the variables together with the σ -field, and it is lost as soon as the σ -field carries information about their values. We give an example. Let Y_1 have law $\text{Exp}(1)$, let $0 < \delta \leq 1/2$, and let $\mathcal{G} = \sigma(A_\delta)$, where $A_\delta = \{|Y_1 - 1| \leq \delta\}$ is a truncation event for Y_1 . The variable Y_1 is not fresh given $\mathcal{G}$: on A_δ , its conditional law is supported in $[1 - \delta, 1 + \delta]$. The conclusion of Lemma 4.18(iv) fails as well. Let $M = 1$, $c = 1$, and $a_1 = -1$, which are constants and hence $\mathcal{G}$ -measurable, so that $S = 1 - Y_1$. On A_δ , we have

$$\mathbb{E}[|S|^{-r} \mid \mathcal{G}] = \frac{\int_{1-\delta}^{1+\delta} |1-y|^{-r} e^{-y} dy}{\int_{1-\delta}^{1+\delta} e^{-y} dy} \geq \frac{e^{-1-\delta} \cdot 2\delta^{1-r}(1-r)^{-1}}{2\delta e^{-1+\delta}} = \frac{e^{-2\delta} \delta^{-r}}{1-r},$$

which tends to infinity as $\delta \rightarrow 0$, while the bound $(1-r)^{-1}2^{3r/2}$ of Lemma 4.18(iv) does not depend on δ . The same failure occurs if $\mathcal{G}$ contains $\sigma(Y_1)$, that is, if Y_1 is conditioned

on its own value. In this case Y_1 is revealed, the term $a_1 Y_1$ must be absorbed into the constant as in (b) after Definition 4.16, and the new constant $1 - Y_1$ is not bounded away from zero. In particular, whenever an event defined in terms of a squared modulus is included in the conditioning σ -field, that squared modulus can no longer be treated as fresh.

As a first application, we bound fractional moments of the diagonal resolvent at the root, uniformly in the imaginary part of the spectral parameter.

Corollary 4.20. *Let $p > 0$, $0 < r < 1$, and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathbb{E}|G_{oo}(z)|^r \leq \frac{1}{1-r} \left(\frac{2}{|z|} \right)^r (p+1)^{r/2}.$$

If $p > 1$, then $\mathbb{E}^\infty |G_{oo}(z)|^r$ is at most ϱ_p^{-1} times the same bound.

Proof. By Proposition 3.16 at the root and the identity $\Gamma_k(z) = G_{oo}(z) \circ \theta_k$, we have, $\mathbb{P}$ -almost surely,

$$G_{oo}(z) = - \left(z + \frac{1}{p} \sum_{k=1}^{N_o} Y_k G_{oo}(z) \circ \theta_k \right)^{-1}.$$

Let $\mathcal{G}$ be the σ -field generated by N_o , ζ_o , the configurations $\theta_k \omega$ for $k \geq 1$, and the phases $w_k/|w_k|$ for $k \geq 1$. By the example following the proof of Lemma 4.17, together with consequence (a) after Definition 4.16, the squared moduli $Y_1, \dots, Y_{N_o}$ are fresh given $\mathcal{G}$. The coefficients $a_k = p^{-1} G_{oo}(z) \circ \theta_k$ are $\mathcal{G}$ -measurable, and the constant $c = z$ is deterministic and nonzero. Then Lemma 4.18(iv) implies that $\mathbb{E}[|G_{oo}(z)|^r \mid \mathcal{G}] \leq (1-r)^{-1} (2\sqrt{N_o+1}/|z|)^r$. Taking expectations and using Jensen's inequality, $\mathbb{E}(N_o+1)^{r/2} \leq (p+1)^{r/2}$, we find the first claim. The second follows from (4.3). $\square$

Since $|z| \geq |\operatorname{Re} z|$, the bound is uniform over all z with real part bounded away from zero, which is the uniformity required at energies $E \neq 0$.

4.6 Selections at a vertex and the backbone

We now classify the children of a vertex by the shape of their forward subtrees. For a vertex $x \in V(\omega)$, a child xk of x is a *leaf child* if $N_{xk} = 0$, so that its forward subtree consists of xk alone; it is a *surviving child* if $\theta_{xk} \omega \in \mathcal{S}$; and it is an *other child* if neither holds. We write N_x^{leaf} , N_x^{surv} , and N_x^{oth} for the numbers of these children. For a vertex x with $N_x^{\text{leaf}} \geq 1$, we let $\ell(x)$ denote the leaf child xk of x for which the pair $(\zeta_{xk}(1), k)$ is lexicographically smallest; thus $\ell(x)$ is the leaf child with the smallest rank, ties (which have probability zero) being broken by the label. Recall $\mathbf{M} = \mathbb{C} \times \Omega$ and $\mathbf{m} = \gamma \otimes \mathbb{P}$, and set

$$\mathbf{m}^{\text{leaf}} = \gamma \otimes \mathbb{P}(\cdot \mid N_o = 0), \quad \mathbf{m}^\infty = \gamma \otimes \mathbb{P}^\infty, \quad \mathbf{m}^{\text{oth}} = \gamma \otimes \mathbb{P}(\cdot \mid \Omega \setminus (\mathcal{S} \cup \{N_o = 0\})).$$

The last conditioning event has probability $1 - \varrho_p - e^{-p} \geq \mathbb{P}(N_0 = 1, N_1 = 0) = pe^{-2p} > 0$. Finally, let

$$\mathcal{G}^{\text{leaf}} = \sigma(N_v, \zeta_v : v \in \mathbb{U}; w_v \mathbf{1}\{N_v \geq 1\} : v \in \mathbb{U}_*)$$

be the σ -field generated by the configuration in which the marks of all edges leading to leaves have been erased.

Lemma 4.21. *Let $p > 1$.*

- (i) *Under $\mathbb{P}$, the processes Π^{leaf} , Π^{surv} , and Π^{oth} formed by the points $(w_k, \theta_k \omega)$ of the leaf children, the surviving children, and the other children of the root are independent Poisson processes with intensity measures $pe^{-p}\mathfrak{m}^{\text{leaf}}$, $p\varrho_p\mathfrak{m}^\infty$, and $p(1 - \varrho_p - e^{-p})\mathfrak{m}^{\text{oth}}$, and they are independent of ζ_0 . In particular, N_0^{leaf} , N_0^{surv} , and N_0^{oth} are independent Poisson variables with means pe^{-p} , $p\varrho_p$, and $p(1 - \varrho_p - e^{-p})$. Under $\mathbb{P}^\infty$, the processes Π^{leaf} , Π^{oth} , and the mark ζ_0 have the same laws as under $\mathbb{P}$ and are independent of Π^{surv} , which has the law of a Poisson process with intensity measure $p\varrho_p\mathfrak{m}^\infty$ conditioned to have at least one point. In particular, for every measurable set $B \subset \mathbb{M}^\infty = \mathbb{C} \times \mathcal{S}$,*

$$\mathbb{P}^\infty(N_0^{\text{leaf}} \geq 1, \Pi^{\text{surv}}(B) \geq 1) = \varrho_p^{-1}(1 - e^{-pe^{-p}})(1 - e^{-p\mathfrak{m}(B)}). \quad (4.20)$$

- (ii) *Let $\mathcal{R} = \{N_0^{\text{leaf}} = 1, N_0^{\text{surv}} = 1, N_0^{\text{oth}} = 0\}$ be the event that the root has exactly one leaf child, exactly one surviving child, and no other child, and on $\mathcal{R}$ let $\ell, y \in \{1, 2\}$ be the labels of the leaf child and of the surviving child. Then $\mathcal{R} \subset \mathcal{S}$, $\mathbb{P}(\mathcal{R}) = p^2\varrho_pe^{-2p}$, and $\mathbb{P}^\infty(\mathcal{R}) = p^2e^{-2p}$. Moreover, for all measurable $F_1, F_2: \mathbb{M} \rightarrow [0, \infty]$ and $F_3: \mathcal{Z} \rightarrow [0, \infty]$,*

$$\mathbb{E}^\infty[\mathbf{1}_{\mathcal{R}} F_1(w_y, \theta_y \omega) F_2(w_\ell, \theta_\ell \omega) F_3(\zeta_0)] = p^2e^{-2p} \int F_1 d\mathfrak{m}^\infty \int F_2 d\mathfrak{m}^{\text{leaf}} \int F_3 d\kappa. \quad (4.21)$$

In particular, conditionally on $\mathcal{R}$ under $\mathbb{P}^\infty$, the squared moduli Y_ℓ and Y_y are independent $\text{Exp}(1)$ variables, the forward configuration $\theta_y \omega$ of the surviving child has law $\mathbb{P}^\infty$, and these three random elements are independent of each other and of the phases $w_\ell/|w_\ell|$, $w_y/|w_y|$.

- (iii) *The root has exactly one child with probability $\mathbb{P}(N_0 = 1) = \mathbb{P}^\infty(N_0 = 1) = pe^{-p}$. Conditionally on $\{N_0 = 1\}$, the point $(w_1, \theta_1 \omega)$ has law $\mathfrak{m}$ under $\mathbb{P}$ and law $\mathfrak{m}^\infty$ under $\mathbb{P}^\infty$. Along a spine, let $L \geq 1$, let $J \subset \{0, \dots, L-1\}$, and let $\mathcal{C}_J = \{N_{v_l} = 1 \text{ for all } l \in J\}$ be the event that the spine vertices v_l , $l \in J$, have no side children. Then $\mathbb{Q}_L(\mathcal{C}_J) = \mathbb{Q}_L^\infty(\mathcal{C}_J) = e^{-p|J|}$; conditionally on $\mathcal{C}_J$, the conclusions of Lemma 4.7 hold with S_l empty for $l \in J$; and for every measurable $F: \Omega \times \mathbb{U} \rightarrow [0, \infty]$,*

$$\mathbb{E} \sum_{v \in \mathcal{V}_L} F(\omega, v) \prod_{l \in J} \mathbf{1}\{N_{v_l} = 1\} = p^L e^{-p|J|} \mathbb{E}_{\mathbb{Q}_L}[F(\omega, v) \mid \mathcal{C}_J].$$

The same identity holds with $\mathbb{E}^\infty$, $\mathcal{V}_L^\infty$, and $\mathbb{Q}_L^\infty$ in place of $\mathbb{E}$, $\mathcal{V}_L$, and $\mathbb{Q}_L$.

- (iv) *Let X be a random finite subset of $\mathbb{U}$ which is $\mathcal{G}^{\text{leaf}}$ -measurable, and let $A = \{\ell(x) : x \in X \cap V(\omega), N_x^{\text{leaf}} \geq 1\}$ be the set of selected leaf children of the vertices in X . Then*

$\{A = a\} \in \mathcal{G}_a$ for every finite $a \subset \mathbb{U}_*$, and the conclusions of Lemma 4.17 hold for A under $\mathbb{P}$ and $\mathbb{P}^\infty$. In particular, the selected squared moduli $(Y_v)_{v \in A}$ are fresh given $\mathcal{H}_A$: conditionally on $\mathcal{H}_A$, they are independent $\text{Exp}(1)$ variables. The same holds under the spine laws if $X = X(\omega, v)$, respectively $X(\omega, u, v)$, is $\mathcal{G}^{\text{leaf}}$ -measurable for each fixed value of the endpoints; for example, $X = \{v\}$ or $X = \{u, v\}$.

- (v) Under $\mathbb{Q}_L$, respectively $\mathbb{Q}_L^\infty$, the endpoint configuration $\theta_v \omega$ has law $\mathbb{P}$, respectively $\mathbb{P}^\infty$, and is independent of the spine edge marks, the auxiliary marks of the spine vertices, and the side sequences. In particular, (i)–(iii) hold for the children of the endpoint v , with $\theta_v \omega$ in place of ω ; for example, $\mathbb{Q}_L^\infty(\mathcal{R} \circ \theta_v) = p^2 e^{-2p}$, and (4.21) holds conditionally on the spine data. Under $\mathbb{Q}_{L,j}^\infty$, the events $\mathcal{R} \circ \theta_u$ and $\mathcal{R} \circ \theta_v$ are independent, each has probability $p^2 e^{-2p}$, and (4.21) holds jointly at the two endpoints, with independent factors.

Proof. For (i), the sets $M^{\text{leaf}} = \mathbb{C} \times \{N_o = 0\}$, $M^\infty = \mathbb{C} \times \mathcal{S}$, and $M^{\text{oth}} = \mathbb{C} \times (\Omega \setminus (\mathcal{S} \cup \{N_o = 0\}))$ partition M , and their m -masses are e^{-p} , q_p , and $1 - q_p - e^{-p}$. For measurable $u_1, u_2, u_3: M \rightarrow [0, \infty)$, the computation in the proof of Lemma 4.5(i) gives

$$\mathbb{E} \exp(-\Pi^{\text{leaf}}(u_1) - \Pi^{\text{surv}}(u_2) - \Pi^{\text{oth}}(u_3)) = \prod_{i=1}^3 \exp\left(-p \int_{M_i} (1 - e^{-u_i}) \, dm\right),$$

where $(M_1, M_2, M_3) = (M^{\text{leaf}}, M^\infty, M^{\text{oth}})$. As there, this shows that the three processes are independent Poisson processes with intensity measures $p \, m(\cdot \cap M_i)$, which are the stated ones; they are independent of ζ_o by Lemma 4.2. The statement under $\mathbb{P}^\infty$ follows as in Lemma 4.5(ii), since $\mathcal{S} = \{N_o^{\text{surv}} \geq 1\}$. For (4.20), we note that $\{\Pi^{\text{surv}}(B) \geq 1\} \subset \mathcal{S}$, that $\Pi^{\text{surv}}(B) = \Pi_o(B)$ has law $\text{Pois}(p \, m(B))$, and that it is independent of N_o^{leaf} , which has law $\text{Pois}(p e^{-p})$; dividing by q_p gives (4.20).

For (ii), write $M_k = (w_k, \theta_k \omega)$. On $\mathcal{R}$, we have $N_o = 2$, and exactly one of M_1, M_2 lies in M^∞ while the other lies in M^{leaf} . By Lemma 4.2(ii), N_o, ζ_o, M_1, M_2 are independent, and M_1, M_2 have law m . Using also $\mathbb{P}(N_o = 2) = p^2 e^{-p}/2$, we obtain

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\mathcal{R}} F_1(M_y) F_2(M_\ell) F_3(\zeta_o)] &= \mathbb{P}(N_o = 2) \cdot 2 \int_{M^\infty} F_1 \, dm \int_{M^{\text{leaf}}} F_2 \, dm \int F_3 \, d\kappa \\ &= p^2 e^{-p} \cdot q_p \int F_1 \, dm^\infty \cdot e^{-p} \int F_2 \, dm^{\text{leaf}} \int F_3 \, d\kappa, \end{aligned}$$

where the factor 2 accounts for the two possible values of (y, ℓ) . The surviving child has an infinite forward subtree, which gives $\mathcal{R} \subset \mathcal{S}$. Dividing by q_p gives (4.21), and taking $F_1 = F_2 = F_3 = 1$ gives the two probabilities. For the last assertion, we take $F_1(w, \omega'') = g_1(|w|^2) g_2(w/|w|) g_3(\omega'')$ and $F_2(w, \omega'') = g_4(|w|^2) g_5(w/|w|)$, and use that under γ the modulus squared and the phase are independent with laws $\text{Exp}(1)$ and uniform.

For (iii), $\mathbb{P}(N_o = 1) = p e^{-p}$, and $\{N_o = 1\} \cap \mathcal{S} = \{N_o = 1, \theta_1 \omega \in \mathcal{S}\}$ has probability $p e^{-p} q_p$ by Lemma 4.2(ii); dividing by q_p gives $\mathbb{P}^\infty(N_o = 1) = p e^{-p}$. The conditional

laws follow in the same way. Along a spine, we have $N_{v_l} = N'_l + 1$ and therefore $\mathcal{C}_J = \{|S_l| = 0 \text{ for } l \in J\}$, which has $\mathbb{Q}_L$ -probability $e^{-p|J|}$ by Lemma 4.7(i). Since $\mathcal{C}_J$ is an event in terms of the sequences S_l , conditioning on it preserves the independence in Lemma 4.7 and only changes the laws of S_l , $l \in J$, to the point mass at the empty sequence. Under $\mathbb{Q}_L^\infty$ the event $\mathcal{C}_J$ has the same probability, because it is independent of $\theta_v \omega$. The displayed identity is (4.5) applied to $F(\omega, v) \prod_{l \in J} \mathbf{1}\{N_{v_l} = 1\}$, and its conditioned version is (4.8).

For (iv), fix a finite set $a \subset \mathbb{U}_*$, and let $\tilde{\omega} = \omega^{a \leftarrow 0}$ be the configuration in which the marks w_v , $v \in a$, are replaced by 0. Every element of A is a leaf, and therefore $\{A = a\} \subset \{N_v = 0 \text{ for all } v \in a\}$. On the latter event, the generators $w_v \mathbf{1}\{N_v \geq 1\}$ of $\mathcal{G}^{\text{leaf}}$ take the same values at ω and at $\tilde{\omega}$, and so do all offspring numbers and auxiliary marks. Since X is $\mathcal{G}^{\text{leaf}}$ -measurable and the selection $x \mapsto \ell(x)$ depends only on the offspring numbers and the ranks, we have $\mathbf{1}\{A = a\}(\omega) = \mathbf{1}\{A = a\}(\tilde{\omega})$ on this event. Off this event, both sides vanish, because ω and $\tilde{\omega}$ have the same offspring numbers. Since the right side does not depend on $(w_v)_{v \in a}$, we conclude that $\{A = a\} \in \mathcal{G}_a$. The remaining assertions are Lemma 4.17; the independence of the selected leaf marks at distinct vertices of X is part of (4.15), which identifies their joint conditional law as $\gamma^{\otimes A}$. Under the spine laws, we argue in the same way for each fixed value of the endpoints and apply Lemma 4.17(ii).

Part (v) follows from Lemma 4.7(i), (iii), and (iv). $\square$

We emphasize that the structural events in Lemma 4.21 are defined in terms of offspring numbers and survival, and that the selections use only offspring numbers and ranks; neither inspects an edge mark. This is why the selected squared moduli remain fresh; compare Remark 4.19, where the conditioning event is defined in terms of the squared modulus itself.

We conclude with the growth of the backbone. By Lemma 4.5(ii), the probability under $\mathbb{P}^\infty$ that the root has exactly one surviving child is

$$q = \frac{pq_p e^{-pq_p}}{1 - e^{-pq_p}} = pe^{-pq_p},$$

where we used $q_p = 1 - e^{-pq_p}$. With $\lambda = pq_p > 0$, we have $q = \lambda/(e^\lambda - 1) < 1$, since $e^\lambda - 1 > \lambda$.

Lemma 4.22. *Let $p > 1$ and $q = pe^{-pq_p}$.*

- (i) *On $\mathcal{S}$, we have $Z_0 = 1$ and $Z_{n+1} \geq Z_n$ for every $n \geq 0$.*
- (ii) *For all $n \geq 0$, $m \geq 1$, and $K \geq 0$, $\mathbb{P}^\infty(Z_n = m, Z_{n+K} = m) = q^{mK} \mathbb{P}^\infty(Z_n = m)$.*
- (iii) *$Z_n \rightarrow \infty$ as $n \rightarrow \infty$, $\mathbb{P}^\infty$ -almost surely.*

Proof. The proof proceeds in four steps. After the elementary monotonicity, we compute the probability that the backbone of a surviving tree consists of a single line for K

generations, using the Mecke formula. Then we use the branching property at generation n to obtain (ii), from which (iii) follows.

STEP 1. *Monotonicity.* On $\mathcal{S}$, the root has an infinite forward subtree, and $Z_0 = 1$. If $v \in \mathcal{V}_n^\infty$, then the forward subtree of v is infinite and v has finitely many children, so at least one child of v has an infinite forward subtree and belongs to $\mathcal{V}_{n+1}^\infty$. Since distinct vertices have distinct children, we have $Z_{n+1} \geq Z_n$. This proves (i). Further, for all $n, K \geq 0$ and $\omega \in \Omega$, we have

$$Z_{n+K}(\omega) = \sum_{v \in \mathcal{V}_n^\infty} Z_K(\theta_v \omega). \quad (4.22)$$

Indeed, for $v \in \mathcal{V}_n$, the vertices of $\mathcal{V}_{n+K}^\infty$ that descend from v are exactly the vertices vu with $u \in \mathcal{V}_K^\infty(\theta_v \omega)$, and there are none unless the forward subtree of v is infinite, that is, unless $v \in \mathcal{V}_n^\infty$.

STEP 2. *A single line.* For $K \geq 0$, let $\mathcal{A}_K = \mathcal{S} \cap \{Z_K = 1\}$. We claim that $\mathbb{P}^\infty(\mathcal{A}_K) = q^K$. For $K = 0$ this holds since $\mathcal{A}_0 = \mathcal{S}$. Let $K \geq 1$. By Step 1, we have $Z_1 = 1$ on $\mathcal{A}_K$, that is, the root has exactly one child k with $\theta_k \omega \in \mathcal{S}$, and (4.22) with $n = 1$ gives $Z_{K-1}(\theta_k \omega) = 1$. Conversely, if the root has exactly one surviving child k and $\theta_k \omega \in \mathcal{A}_{K-1}$, then $\omega \in \mathcal{A}_K$. In terms of the child process, this reads

$$\mathbf{1}_{\mathcal{A}_K}(\omega) = \sum_{k=1}^{N_0} \mathbf{1}_{\mathcal{A}_{K-1}}(\theta_k \omega) \prod_{k' \leq N_0, k' \neq k} \mathbf{1}_{\Omega \setminus \mathcal{S}}(\theta_{k'} \omega) = \int F(m, \Pi_0 - \delta_m) \Pi_0(\mathrm{d}m),$$

where $F((\omega, \omega''), \pi) = \mathbf{1}_{\mathcal{A}_{K-1}}(\omega'') \mathbf{1}\{\pi(M^\infty) = 0\}$ and $M^\infty = \mathbb{C} \times \mathcal{S}$. In the first equality we used $\mathcal{A}_{K-1} \subset \mathcal{S}$. By Lemma 4.2(iii) and (4.1), we obtain

$$\mathbb{P}(\mathcal{A}_K) = p \mathbb{P}(\mathcal{A}_{K-1}) \mathbb{P}(\Pi_0(M^\infty) = 0) = p e^{-p q_p} \mathbb{P}(\mathcal{A}_{K-1}).$$

Dividing by q_p , we find $\mathbb{P}^\infty(\mathcal{A}_K) = q \mathbb{P}^\infty(\mathcal{A}_{K-1})$, and the claim follows by induction.

STEP 3. *Proof of (ii).* Let $n \geq 0$, $m \geq 1$, and $K \geq 0$. On $\{Z_n = m\}$, each term in (4.22) is at least 1 by Step 1, and $Z_{n+K} = m$ holds if and only if $\theta_v \omega \in \mathcal{A}_K$ for every $v \in \mathcal{V}_n^\infty$. We obtain

$$\mathbf{1}\{Z_n = m, Z_{n+K} = m\} = \sum_{\substack{a \subset \mathbb{U}_n \\ |a|=m}} \mathbf{1}\{a \subset \mathcal{V}_n\} \prod_{v \in a} \mathbf{1}_{\mathcal{A}_K}(\theta_v \omega) \prod_{v \in \mathcal{V}_n \setminus a} \mathbf{1}_{\Omega \setminus \mathcal{S}}(\theta_v \omega),$$

since only the term with $a = \mathcal{V}_n^\infty$ can be nonzero. By Lemma 4.2(i), the configurations $\theta_v \omega$, $v \in \mathbb{U}_n$, are independent with law $\mathbb{P}$ and independent of $\mathcal{F}_n$, and the finite set $\mathcal{V}_n$ is $\mathcal{F}_n$ -measurable. Conditioning on $\mathcal{F}_n$, we find that

$$\mathbb{P}(Z_n = m, Z_{n+K} = m \mid \mathcal{F}_n) = \sum_{\substack{a \subset \mathbb{U}_n \\ |a|=m}} \mathbf{1}\{a \subset \mathcal{V}_n\} \mathbb{P}(\mathcal{A}_K)^m (1 - q_p)^{|\mathcal{V}_n| - m}.$$

The same computation with $\mathcal{S}$ in place of $\mathcal{A}_K$ gives the conditional probability of $\{Z_n = m\}$, with q_p^m in place of $\mathbb{P}(\mathcal{A}_K)^m$. By Step 2, we have $\mathbb{P}(\mathcal{A}_K) = q_p q^K$, and the first conditional probability is therefore q^{mK} times the second. Taking expectations, and using that both events are contained in $\mathcal{S}$ because $m \geq 1$, we deduce (ii) after division by q_p .

STEP 4. *Proof of (iii).* By (i), the limit $Z_\infty = \lim_n Z_n \in \{1, 2, \dots\} \cup \{\infty\}$ exists on $\mathcal{S}$. For $m \geq 1$, we have

$$\{Z_\infty = m\} \subset \bigcup_{n \geq 0} \bigcap_{K \geq 0} \{Z_n = m, Z_{n+K} = m\}.$$

By (ii), each event in the union has $\mathbb{P}^\infty$ -probability at most $\inf_K q^{mK} = 0$, since $q < 1$. We conclude that $\mathbb{P}^\infty(Z_\infty < \infty) = 0$, which is (iii). $\square$

Real solutions of the cavity equation

At an energy $E > 0$ where the imaginary parts of the cavity variables vanish at the real axis, the cavity equation of Proposition 3.16 becomes an equation for a probability measure on the real line. This section studies that equation and its relation with the tree.

In Section 5.1, we define real solutions and the normalizations used in the rest of the paper. In Section 5.2, we prove properties shared by all real solutions: the structure of their atoms, a density bound near zero, fractional moments of both signs, full support, a fractional cancellation estimate, and a bound on the probability that the denominator is small on the scale $p^{-1/2}$. These properties hold for every $p > 1$, every $E > 0$ and every real solution. The atom structure, the moments, the full support and the fractional cancellation estimate are the hypotheses under which the free energy is defined and identified in Section 6.

The rest of the section connects real solutions with the tree. In Section 5.3, we define a deterministic set $\mathcal{E}_p$ of regular energies, of full Lebesgue measure, at which all boundary values used in this paper exist, and we prove a zero–one law for the imaginary part of the root cavity variable. If this imaginary part vanishes under $\mathbb{P}^\infty$, then the denominators at the real axis are finite, real and nonzero, and the law of the root denominator is a real solution, which we call the real boundary law. In Section 5.4, we construct, for an arbitrary real solution, a recursive tree process with that solution as its marginal. Section 5.5 records fractional moment bounds at $z = E + i\eta$ with $\eta > 0$ that are uniform in η . Finally, Section 5.6 proves that when the boundary is real, every cavity variable in a fixed neighborhood of the root, of a spine, or of two spines converges as the imaginary part η tends to zero, and identifies the limit with the real recursive tree process.

Throughout this section, $p > 1$ is fixed. Dependence of constants on parameters is indicated explicitly.

5.1 Real solutions and their normalizations

By Proposition 3.16, the root cavity variable $\Gamma_o(z)$ satisfies the cavity equation

$$\Gamma_o(z) \stackrel{\text{d}}{=} -\left(z + \frac{1}{p} \sum_{j=1}^N Y_j \Gamma_j(z)\right)^{-1},$$

where $N \sim \text{Pois}(p)$, the Y_j are $\text{Exp}(1)$ variables, the $\Gamma_j(z)$ are copies of $\Gamma_o(z)$, and all these variables are independent. If the variables are real at $z = E$, then the inverse cavity variable $D = -1/\Gamma$, which we call the denominator, satisfies the equation in the following definition.

Definition 5.1. Let $E > 0$. A *real solution at energy E* is a Borel probability measure π on $\mathbb{R} \setminus \{0\}$ with the following property. Let N , $(Y_j)_{j \geq 1}$ and $(D_j)_{j \geq 1}$ be independent, with $N \sim \text{Pois}(p)$, $Y_j \sim \text{Exp}(1)$ and $D_j \sim \pi$ for every j . Then the random variable

$$E - \frac{1}{p} \sum_{j=1}^N \frac{Y_j}{D_j} \quad (5.1)$$

has law π . We abbreviate this property by

$$D \stackrel{\text{d}}{=} E - \frac{1}{p} \sum_{j=1}^N \frac{Y_j}{D_j},$$

where $D \sim \pi$ and the variables on the right are as above, and we call D the *denominator*.

Since $D_j \neq 0$ almost surely and N is finite, the sum in (5.1) is a real number almost surely, and the definition makes sense. Since π is carried by $\mathbb{R} \setminus \{0\}$, the definition includes the requirement that the variable (5.1) is nonzero almost surely. We only consider positive energies. By Proposition 3.21, spectral statements at negative energies follow from those at positive energies, and if π is a real solution at energy E , then the law of $-D$ satisfies the same equation with E replaced by $-E$.

The following normalizations are used throughout the paper.

Definition 5.2. Let $E > 0$, let π be a real solution at energy E , and let $D \sim \pi$. Let $Y \sim \text{Exp}(1)$ be independent of D .

(i) For a *scaling* $g > 0$, set $B = gD$ and $\beta = g^2 Y / p$. The *transfer matrix* is

$$M(B, \beta) = \beta^{-1/2} \begin{pmatrix} 0 & 1 \\ -\beta & B \end{pmatrix}.$$

(ii) For $E > 2$, set

$$g(E) = \frac{E - \sqrt{E^2 - 4}}{2}, \quad \theta = g(E)^2, \quad K = \frac{p}{\theta},$$

and, with the scaling $g = g(E)$, set $B = g(E)D$ and $X = 1/B$. We call this the *B-normalization*.

(iii) The *scaled energy* is $\xi = E / \sqrt{p}$. For $\xi > 2 / \sqrt{p}$, set

$$a_p(\xi) = \frac{1}{\sqrt{p} g(\xi \sqrt{p})}.$$

- (iv) The scaling $g = 1/E$ gives $\tilde{B} = D/E$, in which the atom of the law sits at 1 (see Proposition 5.5(i)). We use the parameter $\tau = 1/(pE^2)$.

We record the identities behind these normalizations.

Lemma 5.3. *Let $E > 0$, let π be a real solution at energy E , and let $N, (Y_j), (D_j)$ be as in Definition 5.1.*

- (i) *For every $g > 0$, the matrix $M(B, \beta)$ has determinant one, and with $B_j = gD_j$ and $\beta_j = g^2Y_j/p$ we have*

$$B \stackrel{d}{=} gE - \sum_{j=1}^N \frac{\beta_j}{B_j}.$$

- (ii) *Let $E > 2$. Then $Eg(E) = 1 + \theta$, $\beta = Y/K$, $X = -\Gamma/g(E)$ when $D = -1/\Gamma$, and*

$$B \stackrel{d}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j}.$$

Further, $E^{-1} < g(E) \leq 2E^{-1}$; in particular $g(E) < 1$ and $E^{-2} < \theta \leq 4E^{-2}$.

- (iii) *Let $E = \xi\sqrt{p} > 2$. Then*

$$a_p(\xi) = \frac{\xi + \sqrt{\xi^2 - 4/p}}{2}, \quad K = p^2 a_p(\xi)^2, \quad \frac{\xi}{2} \leq a_p(\xi) < \xi.$$

- (iv) *With $\tilde{B}_j = D_j/E$, we have*

$$\tilde{B} \stackrel{d}{=} 1 - \tau \sum_{j=1}^N \frac{Y_j}{\tilde{B}_j}.$$

If $E > 2$, then $\tilde{B} = B/(1 + \theta)$ and $\tau = \theta/(p(1 + \theta)^2)$.

Proof. The determinant of $M(B, \beta)$ is $\beta^{-1}(0 \cdot B + \beta) = 1$. Multiplying (5.1) by g , we obtain

$$gD \stackrel{d}{=} gE - \frac{g}{p} \sum_{j=1}^N \frac{Y_j}{D_j} = gE - \sum_{j=1}^N \frac{g^2 Y_j / p}{gD_j},$$

which is (i).

For (ii), the numbers $(E \pm \sqrt{E^2 - 4})/2$ are the roots of $t^2 - Et + 1 = 0$. In particular, $g(E)^2 - Eg(E) + 1 = 0$, that is, $Eg(E) = 1 + \theta$, and since the product of the two roots is one, we have

$$\frac{1}{g(E)} = \frac{E + \sqrt{E^2 - 4}}{2}. \quad (5.2)$$

Inserting $gE = 1 + \theta$ and $g^2/p = \theta/p$ into (i), we obtain the equation for B . The identities $\beta = g(E)^2 Y/p = Y/K$ and $X = 1/(g(E)D) = -\Gamma/g(E)$ follow from the definitions.

Since $E \leq E + \sqrt{E^2 - 4} < 2E$, (5.2) implies that $E^{-1} < g(E) \leq 2E^{-1}$, and the remaining bounds follow because $E > 2$.

For (iii), using (5.2), we find

$$a_p(\xi) = \frac{E + \sqrt{E^2 - 4}}{2\sqrt{p}} = \frac{\xi\sqrt{p} + \sqrt{\xi^2 p - 4}}{2\sqrt{p}} = \frac{\xi + \sqrt{\xi^2 - 4/p}}{2},$$

and $K = p/g(E)^2 = p \cdot pa_p(\xi)^2$. The bounds on $a_p(\xi)$ follow from $0 \leq \sqrt{\xi^2 - 4/p} < \xi$.

For (iv), we divide (5.1) by E and use $Y_j/(pED_j) = \tau Y_j/\tilde{B}_j$. If $E > 2$, then $B/(1+\theta) = g(E)D/(g(E)E) = \tilde{B}$, and $\theta/(p(1+\theta)^2) = g(E)^2/(pg(E)^2E^2) = \tau$. $\square$

The following table summarizes the normalizations. The last column is the atom produced by the event $\{N = 0\}$, on which the denominator equals the constant term of its equation; by Proposition 5.5, this is the only atom of a real solution.

variable	equation in law	atom
$D = -1/\Gamma$	$D \stackrel{d}{=} E - p^{-1} \sum_{j \leq N} Y_j/D_j$	$e^{-p} \delta_E$
$B = gD$, scaling $g > 0$	$B \stackrel{d}{=} gE - \sum_{j \leq N} \beta_j/B_j$, $\beta_j = g^2 Y_j/p$	$e^{-p} \delta_{gE}$
$B = g(E)D$, $X = 1/B$, $E > 2$	$B \stackrel{d}{=} 1 + \theta - (\theta/p) \sum_{j \leq N} Y_j/B_j$	$e^{-p} \delta_{1+\theta}$
$\tilde{B} = D/E$	$\tilde{B} \stackrel{d}{=} 1 - \tau \sum_{j \leq N} Y_j/\tilde{B}_j$	$e^{-p} \delta_1$

5.2 A priori properties of real solutions

We use the density bound of Lemma 4.18(iii) for affine combinations of independent exponential variables, which is proved there by the Gamma–Dirichlet decomposition, and the following bound on translated Gamma integrals.

Lemma 5.4. *Let $q \in (0, 1)$, and let T_q be a random variable with density*

$$\omega_q(t) = \frac{t^{q/2} e^{-t}}{\Gamma(1 + q/2)}, \quad t > 0.$$

Then for every $u \in \mathbb{R}$,

$$\mathbb{E}|T_q - u|^{-q} \leq \frac{70}{1 - q} (1 + |u|)^{-q}.$$

Proof. Let $I(u) = \mathbb{E}|T_q - u|^{-q}$. Since $t^{q/2} e^{-t/2} \leq (q/e)^{q/2} \leq 1$ for $t > 0$ and $\Gamma(1 + q/2) \geq 1/2$, we have

$$\omega_q(t) \leq 2e^{-t/2} \leq 2, \quad t > 0. \quad (5.3)$$

We consider three cases.

Let $|u| \leq 2$. Splitting the integral according to whether $|t - u| \leq 1$ and using (5.3) on the first part, we have

$$I(u) \leq 2 \int_{|t-u| \leq 1} |t - u|^{-q} dt + \int_0^\infty \omega_q(t) dt = \frac{4}{1-q} + 1 \leq \frac{5}{1-q}.$$

Since $(1 + |u|)^{-q} \geq 3^{-q} \geq 1/3$, this gives $I(u) \leq 15(1 - q)^{-1}(1 + |u|)^{-q}$.

Let $u < -2$. Since $|t - u| = t + |u| \geq |u|$ for $t > 0$, we have $I(u) \leq |u|^{-q} \leq 2(1 + |u|)^{-q}$, where we used $1 + |u| \leq 2|u|$.

Let $u > 2$. We split $(0, \infty)$ into $(0, u/2]$, the set of $t > u/2$ with $|t - u| \leq 1$, and the set of $t > u/2$ with $|t - u| > 1$. On the first set, we have $|t - u| \geq u/2$, and its contribution is at most $(u/2)^{-q} \leq 2u^{-q}$. On the second set, we have $\omega_q(t) \leq 2e^{-u/4}$ by (5.3), and its contribution is at most $4e^{-u/4}/(1 - q)$. The contribution of the third set is at most $\int_{u/2}^\infty 2e^{-t/2} dt = 4e^{-u/4}$. Since the maximum of $u^q e^{-u/4}$ over $u > 0$ is $(4q/e)^q \leq 4$, we have $e^{-u/4} \leq 4u^{-q}$. Combining these bounds, we obtain

$$I(u) \leq \left(2 + \frac{16}{1-q} + 16\right)u^{-q} \leq \frac{34}{1-q}u^{-q} \leq \frac{68}{1-q}(1 + u)^{-q},$$

where in the last step we used $1 + u \leq 2u$. The three cases prove the lemma. $\square$

Proposition 5.5. *Let $E > 0$, let π be a real solution at energy E , and let $D \sim \pi$.*

- (i) *We have $\pi = e^{-p}\delta_E + \pi_{ac}$, where π_{ac} is absolutely continuous with respect to Lebesgue measure. In particular, π has an atom of mass e^{-p} at E and no other atom.*
- (ii) *The density f of π_{ac} can be chosen so that*

$$f(x) \leq \sqrt{\frac{p}{2\pi}} \frac{1}{|x - E|} \quad \text{for all } x \neq E.$$

In particular, $f(x) \leq \sqrt{p}/E$ for $|x| \leq E/2$, and $\mathbb{P}(|D| \leq t) \leq 2\sqrt{p}t/E$ for $0 \leq t \leq E/2$.

- (iii) *For every $q \in (0, 1)$,*

$$\mathbb{E}|D|^{-q} \leq 2\left(1 + \frac{\sqrt{p}}{1-q}\right)E^{-q}, \quad \mathbb{E}|D|^q \leq E^q + p^{1-q}\Gamma(1+q)\mathbb{E}|D|^{-q}.$$

- (iv) *The support of π is $\mathbb{R}$.*

- (v) *Let $q \in (0, 1)$ and $g > 0$, set $B = gD$, and let T_q be as in Lemma 5.4 and independent of D . Then for every $a \geq 0$ and $\sigma \in \{-1, 1\}$,*

$$\mathbb{E}|B - \sigma a T_q|^{-q} \leq \frac{70}{1-q} \mathbb{E}(a + |B|)^{-q} \leq C \min\{1, a^{-q}\},$$

where $C = 70(1 - q)^{-1} \max\{1, g^{-q}\mathbb{E}|D|^{-q}\}$ depends only on p , E , g and q , and $\min\{1, a^{-q}\} = 1$ for $a = 0$.

By (i), (iii) and (iv), for every $g > 0$, the law of $B = gD$ has no atom at zero, has full support, and satisfies $\mathbb{E}|B|^q < \infty$ for every $q \in (0, 1)$. These are the hypotheses under which the free energy is defined in Section 6.

Proof. Throughout the proof, N , (Y_j) and (D_j) are as in Definition 5.1, and D has the law of

$$E - R, \quad R = \frac{1}{p} \sum_{j=1}^N \frac{Y_j}{D_j}. \quad (5.4)$$

STEP 1. *Atoms and density.* On the event $\{N = 0\}$, which has probability e^{-p} , the variable (5.4) equals E . Fix $n \geq 1$ and condition on $N = n$ and on $D_1, \dots, D_n$. Then $E - R = E + \sum_{j \leq n} a_j Y_j$ with real nonzero coefficients $a_j = -1/(pD_j)$, and $Y_1, \dots, Y_n$ are independent $\text{Exp}(1)$ variables independent of the conditioning. By Lemma 4.18(iii), the conditional law of $E - R$ has a density $\rho_n(x) = \rho_n(x; D_1, \dots, D_n)$ with

$$\rho_n(x) \leq \sqrt{\frac{n}{2\pi}} \frac{1}{|x - E|}, \quad x \neq E, \quad (5.5)$$

and the formula for ρ_n in the proof of that lemma is jointly measurable in $(x, D_1, \dots, D_n)$. Using Tonelli's theorem, we find that for every Borel set $A \subset \mathbb{R}$,

$$\begin{aligned} \pi(A) &= e^{-p} \mathbf{1}_A(E) + \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{E} \int_A \rho_n(x; D_1, \dots, D_n) dx \\ &= e^{-p} \mathbf{1}_A(E) + \int_A f(x) dx, \end{aligned}$$

with

$$f(x) = \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{E} \rho_n(x; D_1, \dots, D_n).$$

This proves (i). For $x \neq E$, (5.5) implies that

$$f(x) \leq \frac{\mathbb{E} \sqrt{N}}{\sqrt{2\pi} |x - E|} \leq \sqrt{\frac{p}{2\pi}} \frac{1}{|x - E|},$$

where we used Jensen's inequality $\mathbb{E} \sqrt{N} \leq \sqrt{\mathbb{E} N} = \sqrt{p}$. If $|x| \leq E/2$, then $|x - E| \geq E/2$ and $f(x) \leq 2\sqrt{p/(2\pi)}/E \leq \sqrt{p}/E$. Since the interval $[-t, t]$ does not contain the atom E for $0 \leq t \leq E/2$, we have $\mathbb{P}(|D| \leq t) = \int_{-t}^t f \leq 2\sqrt{p} t/E$. This proves (ii).

STEP 2. *Moments.* By (ii),

$$\mathbb{E}|D|^{-q} \leq \left(\frac{E}{2}\right)^{-q} + \int_{-E/2}^{E/2} |x|^{-q} f(x) dx \leq \left(\frac{2}{E}\right)^q + \frac{\sqrt{p}}{E} \cdot \frac{2(E/2)^{1-q}}{1-q} \leq 2E^{-q} + \frac{2\sqrt{p}}{1-q} E^{-q}.$$

In the first inequality, we bounded $|D|^{-q}$ by $(E/2)^{-q}$ on the event $\{|D| > E/2\}$. For the positive moment, we use (5.4) and the inequality $|a + b|^q \leq |a|^q + |b|^q$, valid for $q \in (0, 1]$, to obtain

$$|D|^q \leq E^q + p^{-q} \sum_{j=1}^N Y_j^q |D_j|^{-q}.$$

Taking expectations and using the independence of N , (Y_j) and (D_j) , we find

$$\mathbb{E}|D|^q \leq E^q + p^{-q} \mathbb{E}N \mathbb{E}Y_1^q \mathbb{E}|D|^{-q} = E^q + p^{1-q} \Gamma(1+q) \mathbb{E}|D|^{-q},$$

which proves (iii).

STEP 3. Full support. It suffices to show that every open interval I has $\pi(I) > 0$. By (i), the event $\{N = 1, D_1 = E\}$ has probability $pe^{-p}\pi(\{E\}) = pe^{-2p} > 0$. On this event, the variable (5.4) equals $E - Y_1/(pE)$, which has the density $pEe^{-pE(E-x)}$ on $(-\infty, E)$. This density is strictly positive on $(-\infty, E)$, and we conclude that $\pi(I) > 0$ for every open interval I that meets $(-\infty, E)$. Since $E > 0$, this applies to $I = (-\infty, 0)$, and the event $\{N = 1, D_1 < 0\}$ has probability $pe^{-p}\pi((-\infty, 0)) > 0$. On this event, the variable (5.4) equals $E + Y_1/(p|D_1|)$, and conditionally on D_1 it has the strictly positive density $p|D_1|e^{-p|D_1|(x-E)}$ on (E, ∞) . As before, we conclude that $\pi(I) > 0$ for every open interval I that meets (E, ∞) . Every open interval meets $(-\infty, E)$ or (E, ∞) , which proves (iv).

STEP 4. Fractional cancellation. Fix $b \in \mathbb{R} \setminus \{0\}$. For $a > 0$, we have $|b - \sigma a T_q| = a|T_q - \sigma b/a|$ since $|\sigma| = 1$, and Lemma 5.4 implies that

$$\mathbb{E}|b - \sigma a T_q|^{-q} = a^{-q} \mathbb{E}\left|T_q - \frac{\sigma b}{a}\right|^{-q} \leq \frac{70}{1-q} a^{-q} \left(1 + \frac{|b|}{a}\right)^{-q} = \frac{70}{1-q} (a + |b|)^{-q}.$$

For $a = 0$, the left side equals $|b|^{-q} = (a + |b|)^{-q}$. Since T_q is independent of B and $B \neq 0$ almost surely, we may condition on B and obtain the first inequality in (v). For the second, we have $\mathbb{E}(a + |B|)^{-q} \leq \min\{a^{-q}, \mathbb{E}|B|^{-q}\}$, and $\mathbb{E}|B|^{-q} = g^{-q} \mathbb{E}|D|^{-q}$ is finite by (iii). Finally, $\min\{a^{-q}, M\} \leq \max\{1, M\} \min\{1, a^{-q}\}$ for $M \geq 0$, as one sees by treating $a \geq 1$ and $a < 1$ separately. This proves (v). $\square$

The next lemma bounds the probability that the denominator is small on the scale $p^{-1/2}$, uniformly in the energy and in the real solution.

Lemma 5.6. *Let $E > 0$, let π be a real solution at energy E , let $D \sim \pi$, and for $a > 0$ set $q_a = \mathbb{P}(|D| \leq a/\sqrt{p})$. Then*

$$q_a \leq e^{-pq_a} + 2a^2.$$

In particular, let $\varepsilon \in (0, 1)$, and set $a(\varepsilon) = \sqrt{\varepsilon}/2$ and $p(\varepsilon) = \varepsilon^{-1} \log(2/\varepsilon)$. Then $q_a < \varepsilon$ for every $p > \max\{1, p(\varepsilon)\}$, every $E > 0$, every real solution at energy E , and every $a \in (0, a(\varepsilon)]$.

Proof. Let N , (Y_j) and (D_j) be as in Definition 5.1, and recall that D has the law of (5.4). We call an index $j \leq N$ *small* if $|D_j| \leq a/\sqrt{p}$, and we let K be the number of small indices. Since the D_j are independent of N , the variable K has, conditionally on N , the binomial law with parameters N and q_a . Its generating function is $\mathbb{E}s^K = \mathbb{E}(1 - q_a + q_a s)^N = e^{-pq_a(1-s)}$, $s \in [0, 1]$, which is that of $\text{Pois}(pq_a)$; in particular $\mathbb{P}(K = 0) = e^{-pq_a}$.

On the event $\{K \geq 1\}$, let j_* be the smallest small index. Then $\{K \geq 1\}$ and j_* are measurable with respect to $\sigma(N, (D_j)_{j \geq 1})$. Let $\mathcal{G}$ be the σ -algebra generated by N , $(D_j)_{j \geq 1}$ and the variables $Y_j \mathbf{1}_{\{j \neq j_*\}}$, $j \geq 1$. For each $k \geq 1$, the variable Y_k is independent of N , $(D_j)_{j \geq 1}$ and $(Y_j)_{j \neq k}$, and the event $\{j_* = k\}$ is determined by N and (D_j) . Then on $\{K \geq 1\}$, conditionally on $\mathcal{G}$, the variable Y_{j_*} has the $\text{Exp}(1)$ law. Further,

$$E - R = c - \frac{Y_{j_*}}{pD_{j_*}}, \quad c = E - \frac{1}{p} \sum_{j \leq N, j \neq j_*} \frac{Y_j}{D_j},$$

where c is $\mathcal{G}$ -measurable. For a real constant c and $\kappa \neq 0$, the variable $c + \kappa Y_{j_*}$ has a density bounded by $|\kappa|^{-1}$. Here $|\kappa|^{-1} = p|D_{j_*}| \leq a\sqrt{p}$, and on $\{K \geq 1\}$ we have

$$\mathbb{P}(|E - R| \leq a/\sqrt{p} \mid \mathcal{G}) \leq \frac{2a}{\sqrt{p}} \cdot a\sqrt{p} = 2a^2.$$

Since $E - R$ has the law of D , we conclude that

$$q_a \leq \mathbb{P}(K = 0) + \mathbb{E}[\mathbb{P}(|E - R| \leq a/\sqrt{p} \mid \mathcal{G}) \mathbf{1}_{\{K \geq 1\}}] \leq e^{-pq_a} + 2a^2.$$

For the second assertion, let $a \leq a(\varepsilon)$ and $p > p(\varepsilon)$; then $2a^2 \leq \varepsilon/2$ and $e^{-p\varepsilon} < \varepsilon/2$. If $q_a \geq \varepsilon$, then the first assertion gives $q_a \leq e^{-p\varepsilon} + \varepsilon/2 < \varepsilon$, which is impossible. This proves the second assertion. $\square$

5.3 Regular energies and boundary values

In this section, we define the regular energies and study the boundary values of resolvent entries at a regular energy. The main result is Proposition 5.11.

Notation

We use the realization of the marked tree on the configuration space Ω of Definition 4.1. Thus $\omega = ((N_v)_{v \in \mathbb{U}}, (\zeta_v)_{v \in \mathbb{U}}, (w_v)_{v \in \mathbb{U}_*})$ has law $\mathbb{P}$, the children of $v \in V = V(\omega)$ are $v1, \dots, vN_v$, the edge $\{v^-, v\}$ carries the mark w_v , and $Y_v = |w_v|^2 = p|h_{v^-v}|^2$. We write θ_u for the shifts, $\mathbb{U}_{\leq m} = \bigcup_{n \leq m} \mathbb{U}_n$, $u\mathbb{U} = \{uv : v \in \mathbb{U}\}$, and $\mathcal{V}_n = V \cap \mathbb{U}_n$ for the set of vertices of generation n . For $u \in \mathbb{U}$, we set $\mathcal{S}_u = \{\theta_u \omega \in \mathcal{S}\}$, and for $u \in V$ we set $\zeta_u = \mathbf{1}_{\mathcal{S}_u}$, so that $\zeta_u = 1$ if and only if the forward subtree $\mathcal{T}_u$ is infinite. We recall from

Lemma 4.2(i) that for every $n \geq 0$, the configurations $\theta_u \omega$, $u \in \mathbb{U}_n$, are independent with law $\mathbb{P}$ and independent of the σ -field

$$\mathcal{F}_n = \sigma(N_u, \zeta_u : |u| < n; w_u : 1 \leq |u| \leq n),$$

which determines $\mathcal{V}_n$. We also recall from the discussion after Definition 4.1 that, $\mathbb{P}$ -almost surely, $\Gamma_u(z) = G_{00}(z) \circ \theta_u$ for every $u \in V \setminus \{o\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

A *removed set* is a pair $\Delta = (\Delta_V, \Delta_E)$ of finite sets $\Delta_V \subset \mathbb{U}$ and $\Delta_E \subset \mathbb{U}_*$, where a word $u \in \Delta_E$ stands for the edge $\{u^-, u\}$. It determines the finite removal of Definition 3.3 in which the vertices of $\Delta_V \cap V$ and the edges $\{u^-, u\}$ with $u \in \Delta_E \cap V$ are removed, and every finite removal arises in this way. Its *depth* $d(\Delta)$ is the maximal length of a word in $\Delta_V \cup \Delta_E$, and $d(\Delta) = 0$ if Δ is empty. We write $\mathcal{T}^\Delta$ for the reduced forest, $\mathcal{H}^\Delta$ for the reduced operator, and $G_{xy}^\Delta(z)$, $G_{xy}^C(z)$, $C_{x \rightarrow w}$ and $\Gamma_{x \rightarrow w}^C(z)$ for the resolvent entries, components, branches and cavity resolvents of Definition 3.15 and Lemma 3.5. These objects are defined for all removed sets simultaneously on the almost sure event Ω_{sa} of Theorem 3.10. If Δ is empty, then $C = V$, and we recover G_{xy} , $\Gamma_{x \rightarrow w}$, the forward cavities $\Gamma_v = \Gamma_{v \rightarrow v^-}$ for $v \neq o$, and $\Gamma_o = G_{00}$. Since there are countably many removed sets, a statement that holds almost surely for each fixed removed set holds almost surely for all removed sets simultaneously; this covers finite removals selected at random, as in Definition 3.3.

Let $\mathcal{K}$ be the countable set of quadruples $\kappa = (\Delta, x, y, k)$, where Δ is a removed set, $x, y \in \mathbb{U}$ and $k \in \{0, 1, 2, 3\}$. For $\kappa \in \mathcal{K}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we set $\phi_\kappa = \delta_x + i^k \delta_y$ and

$$F_\kappa(z) = \langle \phi_\kappa, (\mathcal{H}^\Delta - z)^{-1} \phi_\kappa \rangle \quad \text{if } x, y \in V \setminus \Delta_V,$$

and $F_\kappa(z) = 0$ otherwise. By the polarization identity, we have

$$G_{xy}^\Delta(z) = \frac{1}{4} \sum_{k=0}^3 i^{-k} F_{(\Delta, x, y, k)}(z), \quad (5.6)$$

so every resolvent entry of every reduced operator is a fixed linear combination of four functions F_κ . For instance, $\Gamma_v = G_{vv}^\Delta$ for the removed set with $\Delta_V = \emptyset$ and $\Delta_E = \{v\}$, and $\Gamma_{x \rightarrow w}^C = G_{xx'}^{\Delta'}$, where Δ' is obtained from the removed set defining C by adding the edge $\{x, w\}$.

If $x, y \in V \setminus \Delta_V$, then Lemma 3.14(ii) gives $F_\kappa(z) = \int \mu_\kappa(d\lambda) / (\lambda - z)$, where μ_κ is the spectral measure of ϕ_κ for $\mathcal{H}^\Delta$, a positive measure of total mass $\|\phi_\kappa\|^2 \leq 4$. Thus F_κ is the Borel transform of a finite measure. For such a function, the limit $\lim_{\eta \downarrow 0} F_\kappa(E + i\eta)$ exists and is finite for Lebesgue-almost every $E \in \mathbb{R}$; see, for example, [123] or [13].

Regular energies

Definition 5.7. The set $\mathcal{E}_p$ of *regular energies* is the set of $E \in \mathbb{R} \setminus \{0\}$ such that, $\mathbb{P}$ -almost surely, the following two conditions hold.

- (a) For every $\kappa \in \mathcal{K}$, the limit $F_\kappa(E + i0) = \lim_{\eta \downarrow 0} F_\kappa(E + i\eta)$ exists in $\mathbb{C}$.
- (b) For every finite, nonempty, connected set $\Lambda \subset V$, the energy E is not an eigenvalue of the matrix $\mathcal{H}_\Lambda = (\langle \delta_x, \mathcal{H} \delta_y \rangle)_{x,y \in \Lambda}$.

For $E \in \mathcal{E}_p$, we write $G_{xy}^\Lambda(E + i0)$, $\Gamma_{x \rightarrow w}^C(E + i0)$, $\Gamma_v(E + i0)$, and so on, for the corresponding limits, which exist and are finite almost surely by (a) and (5.6).

By Lemma 5.8 below, $\mathcal{E}_p$ is a Borel set whose complement is Lebesgue-null. Since $\mathbb{P}^\infty \leq \varrho_p^{-1} \mathbb{P}$, every $\mathbb{P}$ -almost sure statement is also a $\mathbb{P}^\infty$ -almost sure statement. Condition (b) states that every finite subtree of $\mathcal{T}$, with its edge weights, has E outside its spectrum. Every finite component of a reduced forest is such a subtree, and so is every finite forward subtree $\mathcal{T}_v$ together with all its sub-bushes. For such a finite component C and $x, y \in C$, the function $G_{xy}^C(z)$ is a rational function of z with poles in the spectrum of $\mathcal{H}_C$. At $E \in \mathcal{E}_p$ it is analytic near E , and its boundary value $G_{xy}^C(E + i0) = G_{xy}^C(E)$ is the corresponding entry of $(\mathcal{H}_C - E)^{-1}$. In particular, the diagonal entries $G_{xx}^C(E + i0)$ are real.

Lemma 5.8. *The set $\mathcal{E}_p$ is a Borel set, its complement in $\mathbb{R}$ has Lebesgue measure zero, and $\mathcal{E}_p = -\mathcal{E}_p$.*

Proof. **STEP 1. Joint measurability.** Fix $\kappa \in \mathcal{K}$. By Theorem 3.10(iii) and the discussion after Definition 3.15, on Ω_{sa} the function $F_\kappa(z)$ is the limit, uniformly for z in compact subsets of $\mathbb{C} \setminus \mathbb{R}$, of the corresponding quantities for the restrictions of $\mathcal{H}^\Delta$ to finite balls. These are rational functions of z whose coefficients are measurable functions of ω . Then $(\omega, E, \eta) \mapsto F_\kappa(E + i\eta)$ is jointly measurable on $\Omega \times \mathbb{R} \times (0, \infty)$.

Since $\eta \mapsto F_\kappa(E + i\eta)$ is continuous, the limit as $\eta \downarrow 0$ exists in $\mathbb{C}$ if and only if for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $|F_\kappa(E + i\eta) - F_\kappa(E + i\eta')| \leq 1/k$ for all rational $\eta, \eta' \in (0, 1/n)$. By this criterion, the set $\mathcal{B}_a$ of pairs (ω, E) for which (a) fails is measurable. The set $\mathcal{B}_b$ of pairs for which (b) fails is the union over finite sets $\Lambda \subset \mathbb{U}$ of the sets $\{\Lambda \subset V, \Lambda \text{ connected, } \det(\mathcal{H}_\Lambda - E) = 0\}$, and it is measurable as well. Let $\mathcal{B} = \mathcal{B}_a \cup \mathcal{B}_b \cup ((\Omega \setminus \Omega_{\text{sa}}) \times \mathbb{R})$.

STEP 2. Full measure. Fix $\omega \in \Omega_{\text{sa}}$. For each κ , the function F_κ is either zero or the Borel transform of a finite measure; in both cases its boundary value exists in $\mathbb{C}$ for Lebesgue-almost every E . Since $\mathcal{K}$ is countable, the section $\{E : (\omega, E) \in \mathcal{B}_a\}$ is Lebesgue-null. The section $\{E : (\omega, E) \in \mathcal{B}_b\}$ is contained in the union of the spectra of the countably many finite matrices $\mathcal{H}_\Lambda$, which is a countable set. By Tonelli's theorem,

$$\int_{\mathbb{R}} \mathbb{P}(\{\omega : (\omega, E) \in \mathcal{B}\}) dE = \mathbb{E}|\{E : (\omega, E) \in \mathcal{B}\}| = 0,$$

where $|\cdot|$ denotes Lebesgue measure. The function $E \mapsto \mathbb{P}(\{\omega : (\omega, E) \in \mathcal{B}\})$ is Borel by Tonelli's theorem, and $\mathcal{E}_p = \{E \neq 0 : \mathbb{P}(\{\omega : (\omega, E) \in \mathcal{B}\}) = 0\}$. We conclude that $\mathcal{E}_p$ is a Borel set of full measure.

STEP 3. Symmetry. Let U be the diagonal unitary $U\delta_v = (-1)^{|v|}\delta_v$ on $\ell^2(V \setminus \Delta_V)$. Since the edges of $\mathcal{T}^\Delta$ join consecutive generations, $U\mathcal{H}^\Delta U = -\mathcal{H}^\Delta$ on finitely supported vectors, and hence on the closure. For $\kappa = (\Delta, x, y, k)$ with $x, y \in V \setminus \Delta_V$, we have $U\phi_\kappa = (-1)^{|x|}\phi_{\kappa'}$ with $\kappa' = (\Delta, x, y, k')$, where $k' = k$ if $|x| + |y|$ is even and $k' \equiv k + 2 \pmod{4}$ otherwise. We have, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$F_\kappa(z) = \langle U\phi_\kappa, (U\mathcal{H}^\Delta U - z)^{-1}U\phi_\kappa \rangle = -\langle \phi_{\kappa'}, (\mathcal{H}^\Delta + z)^{-1}\phi_{\kappa'} \rangle.$$

Taking $z = -E + i\eta$ and using Lemma 3.14(i), we obtain

$$F_\kappa(-E + i\eta) = -\overline{F_{\kappa'}(E + i\eta)}.$$

This shows that condition (a) holds at $-E$ if and only if it holds at E . Similarly, $U\mathcal{H}_\Lambda U = -\mathcal{H}_\Lambda$ for every finite connected Λ , which shows that the spectrum of $\mathcal{H}_\Lambda$ is symmetric and that condition (b) holds at $-E$ if and only if it holds at E . Hence $\mathcal{E}_p = -\mathcal{E}_p$. $\square$

Boundary values of the Schur recursion

We next pass the Schur formulas of Proposition 3.16 to the real axis. For a component C of $\mathcal{T}^\Delta$ and $x \in C$, we write $y \sim_C x$ if y is a neighbor of x in C .

Lemma 5.9. *Let $E \in \mathcal{E}_p$. Almost surely, all edge weights of $\mathcal{T}$ are nonzero, and the following holds for every removed set Δ , every component C of $\mathcal{T}^\Delta$, every $x \in C$, and every neighbor w of x in C . The numbers*

$$D_x^C = E + \sum_{y \sim_C x} |h_{xy}|^2 \Gamma_{y \rightarrow x}^C(E + i0), \quad D_{x \rightarrow w}^C = E + \sum_{y \sim_C x, y \neq w} |h_{xy}|^2 \Gamma_{y \rightarrow x}^C(E + i0)$$

are finite and nonzero, $G_{xx}^C(E + i0) = -1/D_x^C$, and $\Gamma_{x \rightarrow w}^C(E + i0) = -1/D_{x \rightarrow w}^C$. In particular, $G_{xx}^C(E + i0)$ and $\Gamma_{x \rightarrow w}^C(E + i0)$ are nonzero, and

$$\operatorname{Im} G_{xx}^C(E + i0) = |G_{xx}^C(E + i0)|^2 \sum_{y \sim_C x} |h_{xy}|^2 \operatorname{Im} \Gamma_{y \rightarrow x}^C(E + i0), \quad (5.7)$$

$$\operatorname{Im} \Gamma_{x \rightarrow w}^C(E + i0) = |\Gamma_{x \rightarrow w}^C(E + i0)|^2 \sum_{y \sim_C x, y \neq w} |h_{xy}|^2 \operatorname{Im} \Gamma_{y \rightarrow x}^C(E + i0). \quad (5.8)$$

Proof. Each mark w_v is nonzero almost surely, and since there are countably many marks, all edge weights are nonzero almost surely. By Proposition 3.16(i), almost surely, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$G_{xx}^C(z) = -\frac{1}{D_x^C(z)}, \quad D_x^C(z) = z + \sum_{y \sim_C x} |h_{xy}|^2 \Gamma_{y \rightarrow x}^C(z).$$

The sum has finitely many terms, and each $\Gamma_{y \rightarrow x}^C(E + i\eta)$ converges to a finite limit by condition (a) of Definition 5.7. Then $D_x^C(E + i\eta) \rightarrow D_x^C$, which is finite. If $D_x^C = 0$, then

$|G_{xx}^C(E + i\eta)| = 1/|D_x^C(E + i\eta)| \rightarrow \infty$, which contradicts the existence of a finite limit of $G_{xx}^C(E + i\eta)$. We conclude that $D_x^C \neq 0$ and $G_{xx}^C(E + i0) = -1/D_x^C$, which is nonzero because D_x^C is finite. For $\text{Im } z > 0$, we have $\text{Im}(-1/D) = \text{Im } D/|D|^2$, and

$$\text{Im } G_{xx}^C(z) = |G_{xx}^C(z)|^2 \left(\eta + \sum_{y \sim_C x} |h_{xy}|^2 \text{Im } \Gamma_{y \rightarrow x}^C(z) \right).$$

Letting $\eta \downarrow 0$ gives (5.7). The assertions about $\Gamma_{x \rightarrow w}^C$ follow in the same way from Proposition 3.16(ii). Since $\mathcal{K}$ is countable, all these statements hold simultaneously on one almost sure event. $\square$

All imaginary parts in (5.7) and (5.8) are nonnegative, since they are limits of imaginary parts of Borel transforms of positive measures. The next lemma shows that a positive imaginary part at the real axis can only be created by an infinite forward subtree.

Lemma 5.10. *Let $E \in \mathcal{E}_p$. Almost surely, the following holds for every removed set Δ , every component C of $\mathcal{T}^\Delta$, every $x \in C$, and every neighbor w of x in C .*

- (i) *If $\text{Im } G_{xx}^C(E + i0) > 0$ or $\text{Im } \Gamma_{x \rightarrow w}^C(E + i0) > 0$, then there exists $y \in V$ with $\text{Im } \Gamma_y(E + i0) > 0$.*
- (ii) *On the event that $\text{Im } \Gamma_y(E + i0) > 0$ for every $y \in V$ such that $\mathcal{T}_y$ is infinite, we have $\text{Im } G_{xx}^C(E + i0) > 0$ if and only if C is infinite, and $\text{Im } \Gamma_{x \rightarrow w}^C(E + i0) > 0$ if and only if $C_{x \rightarrow w}$ is infinite.*

Proof. We work on the intersection of the almost sure events of Definition 5.7 and Lemma 5.9, and we use two facts about paths in the tree. First, a *non-backtracking path* $x_0, x_1, x_2, \dots$, that is, a path with $x_{k+1} \sim x_k$ and $x_{k+1} \neq x_{k-1}$, visits distinct vertices. If it is infinite, then there exists $k_0 \leq |x_0|$ such that x_{k+1} is a child of x_k for all $k \geq k_0$. Indeed, a step to the parent cannot follow a step to a child without returning to the previous vertex; the path takes at most $|x_0|$ steps toward the root, and then it moves away from the root forever. Second, if $|y| > d(\Delta)$, then the forward subtree $\mathcal{T}_y$ contains no vertex of Δ_V and no edge of Δ_E , since all its vertices and all lower endpoints of its edges have length at least $|y|$. If moreover $y \in C$ and $y^- \in C$, then by Lemma 3.5(ii) the branch $C_{y \rightarrow y^-}$ is the vertex set of $\mathcal{T}_y$, and $\Gamma_{y \rightarrow y^-}^C = \Gamma_y$.

Proof of (i). Suppose that $\text{Im } \Gamma_{x \rightarrow w}^C(E + i0) > 0$, and set $x_0 = x$ and $x_{-1} = w$. By (5.8), there exists a neighbor $x_1 \neq x_{-1}$ of x_0 in C with $\text{Im } \Gamma_{x_1 \rightarrow x_0}^C(E + i0) > 0$. Applying (5.8) again to the oriented edge $x_1 \rightarrow x_0$, and repeating, we obtain an infinite non-backtracking path $(x_k)_{k \geq 0}$ in C with $\text{Im } \Gamma_{x_k \rightarrow x_{k-1}}^C(E + i0) > 0$ for all $k \geq 0$. Let $k > k_0$ satisfy $|x_k| > d(\Delta)$. Then x_k is a child of x_{k-1} , and by the second fact, $\Gamma_{x_k \rightarrow x_{k-1}}^C = \Gamma_{x_k}$. The vertex $y = x_k$ satisfies $\text{Im } \Gamma_y(E + i0) > 0$. If instead $\text{Im } G_{xx}^C(E + i0) > 0$, then (5.7) provides a neighbor w' of x in C with $\text{Im } \Gamma_{w' \rightarrow x}^C(E + i0) > 0$, and we apply the previous argument to the oriented edge $w' \rightarrow x$.

Proof of (ii). If $C_{x \rightarrow w}$ is finite, then condition (b) of Definition 5.7 and the discussion after it show that $\Gamma_{x \rightarrow w}^C(E + i0)$ is real. Suppose that $C_{x \rightarrow w}$ is infinite. It induces an infinite, connected, locally finite subtree, and König's lemma provides an infinite non-backtracking path $x = x_0, x_1, x_2, \dots$ in $C_{x \rightarrow w}$; its first step is not w , since $w \notin C_{x \rightarrow w}$. Let $k > k_0$ satisfy $|x_k| > \max\{d(\Delta), |x|, |w|\}$. Then x_k is a child of x_{k-1} , and by the second fact, $\Gamma_{x_k \rightarrow x_{k-1}}^C = \Gamma_{x_k}$. The forward subtree $\mathcal{T}_{x_k}$ is infinite, since it contains $x_k, x_{k+1}, \dots$, and by assumption $\text{Im } \Gamma_{x_k}(E + i0) > 0$. For $0 \leq i \leq k-1$, we keep only the term $y = x_{i+1}$ in (5.8) for the oriented edge $x_i \rightarrow x_{i-1}$, where $x_{-1} = w$, and obtain

$$\text{Im } \Gamma_{x_i \rightarrow x_{i-1}}^C(E + i0) \geq |\Gamma_{x_i \rightarrow x_{i-1}}^C(E + i0)|^2 |h_{x_i x_{i+1}}|^2 \text{Im } \Gamma_{x_{i+1} \rightarrow x_i}^C(E + i0).$$

The first two factors on the right are positive by Lemma 5.9. By induction from $i = k-1$ down to $i = 0$, we obtain $\text{Im } \Gamma_{x \rightarrow w}^C(E + i0) > 0$.

For the diagonal entries, if C is finite, then $G_{xx}^C(E + i0)$ is real. If C is infinite, then one of the finitely many branches $C_{y \rightarrow x}$, $y \sim_C x$, is infinite, since by Lemma 3.5(ii) these branches together with $\{x\}$ make up C . For this y , the previous paragraph gives $\text{Im } \Gamma_{y \rightarrow x}^C(E + i0) > 0$, and (5.7) gives $\text{Im } G_{xx}^C(E + i0) > 0$. $\square$

The zero-one law and the real boundary law

Recall from Definition 3.2 that, for $p > 1$, the survival probability ϱ_p is the unique solution in $(0, 1)$ of $1 - \varrho = e^{-p\varrho}$.

Proposition 5.11. *Let $E \in \mathcal{E}_p$ with $E > 0$, and set $u = \mathbb{P}(\text{Im } \Gamma_o(E + i0) > 0)$.*

(i) *We have $u = 1 - e^{-pu}$ and $u \in \{0, \varrho_p\}$, and*

$$\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1 - \frac{u}{\varrho_p} \in \{0, 1\}.$$

If $u = \varrho_p$, then almost surely (under $\mathbb{P}$, and hence under $\mathbb{P}^\infty$), the following holds for every vertex $v \in V$, every removed set Δ , every component C of $\mathcal{T}^\Delta$ and every $x \in C$: $\text{Im } \Gamma_v(E + i0) > 0$ if and only if $\mathcal{T}_v$ is infinite, and $\text{Im } G_{xx}^C(E + i0) > 0$ if and only if C is infinite. In particular, $\mathbb{P}^\infty$ -almost surely, $\text{Im } G_{vv}(E + i0) > 0$ for every $v \in V$.

(ii) *Suppose that $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Then $u = 0$, and almost surely (under $\mathbb{P}$, and hence under $\mathbb{P}^\infty$) the following hold. For every $v \in V$, the boundary value $\Gamma_v(E + i0)$ is finite, real and nonzero, and the denominators $D_v = -1/\Gamma_v(E + i0)$ satisfy*

$$D_v = E - \frac{1}{p} \sum_{k=1}^{N_v} \frac{Y_{vk}}{D_{vk}}. \quad (5.9)$$

For every removed set Δ , every component C of $\mathcal{T}^\Delta$, every $x \in C$ and every neighbor w of x in C , the boundary values $G_{xx}^C(E + i0)$ and $\Gamma_{x \rightarrow w}^C(E + i0)$ are real and nonzero. Finally, the law of D_o under $\mathbb{P}$ is a real solution at energy E .

Proof. We work on the almost sure events of Definition 5.7 and Lemmas 5.9 and 5.10, and on the event where $\Gamma_v(z) = G_{oo}(z) \circ \theta_v$ for all $v \in V \setminus \{o\}$. Let $\mathfrak{A}$ be the event that $\lim_{k \rightarrow \infty} G_{oo}(E + i/k)$ exists and has positive imaginary part. For $v \in V$, the event $A_v = \{\text{Im } \Gamma_v(E + i0) > 0\}$ then coincides with $\theta_v^{-1}\mathfrak{A}$ up to a null set.

STEP 1. Finite trees. On the event $\mathcal{S}^c$, the tree $\mathcal{T}$ is finite, and condition (b) of Definition 5.7 with $\Lambda = V$ shows that $\Gamma_o(E + i0) = G_{oo}(E)$ is real. In other words, $A_o \subset \mathcal{S}$ up to a null set. Since θ_v preserves $\mathbb{P}$, we also have $\theta_v^{-1}\mathfrak{A} \subset \mathcal{S}_v$ up to a null set, for every $v \in \mathbb{U}$.

STEP 2. The fixed point equation. By (5.8), applied with $C = V$ and the oriented edge $v \rightarrow v^-$, and by (5.7) at the root, we have

$$\text{Im } \Gamma_v(E + i0) = |\Gamma_v(E + i0)|^2 \sum_{k=1}^{N_v} |h_{v,vk}|^2 \text{Im } \Gamma_{vk}(E + i0).$$

Since $\Gamma_v(E + i0) \neq 0$ and the edge weights are nonzero, we obtain $A_v = \bigcup_{k \leq N_v} A_{vk}$ almost surely. At the root, $A_o = \bigcup_{k \leq N_o} \theta_k^{-1}\mathfrak{A}$ up to a null set. By Lemma 4.2(ii), the configurations $\theta_k \omega$, $k \geq 1$, are independent with law $\mathbb{P}$ and independent of N_o , and the events $\theta_k^{-1}\mathfrak{A}$ are independent of each other and of N_o , with probability u each. This gives

$$1 - u = \mathbb{P}(A_o^c) = \mathbb{E}(1 - u)^{N_o} = e^{-pu}.$$

The function $u \mapsto 1 - e^{-pu} - u$ is strictly concave on $[0, 1]$, vanishes at 0, has derivative $p - 1 > 0$ at 0, and equals $-e^{-p} < 0$ at 1; it has exactly two zeros in $[0, 1]$, namely 0 and ϱ_p . By Step 1, we have $\mathbb{P}^\infty(A_o) = \mathbb{P}(A_o \cap \mathcal{S})/\varrho_p = u/\varrho_p$, which is 0 or 1.

STEP 3. Propagation when $u = \varrho_p$. Suppose $u = \varrho_p$. By Step 1, $\mathbb{P}(\mathfrak{A} \triangle \mathcal{S}) = \mathbb{P}(\mathcal{S}) - \mathbb{P}(\mathfrak{A}) = 0$. Since θ_v preserves $\mathbb{P}$, the event $\theta_v^{-1}\mathfrak{A} \triangle \mathcal{S}_v$ is null for every $v \in \mathbb{U}$. Taking the union over the countably many v , we find that almost surely $A_v = \mathcal{S}_v$ for every $v \in V$. This is the first assertion about Γ_v , and on this event Lemma 5.10(ii) gives the assertion about G_{xx}^C . Under $\mathbb{P}^\infty$, the tree is infinite, and taking Δ empty and $C = V$, we find that $\text{Im } G_{vv}(E + i0) > 0$ for all $v \in V$. This proves (i).

STEP 4. The case $\mathbb{P}^\infty(A_o) = 0$. Suppose $\mathbb{P}^\infty(A_o) = 0$. Decomposing according to survival and extinction, and using that $\mathbb{P}(A_o \cap \mathcal{S}^c) = 0$ by Step 1, we have

$$u = \mathbb{P}(A_o) = \varrho_p \mathbb{P}^\infty(A_o) + \mathbb{P}(A_o \cap \mathcal{S}^c) = 0.$$

Since θ_v preserves $\mathbb{P}$, we have $\mathbb{P}(\theta_v^{-1}\mathfrak{A}) = u = 0$ for every $v \in \mathbb{U}$, and almost surely $\Gamma_v(E + i0)$ is real for every $v \in V$. It is finite by condition (a) and nonzero by Lemma 5.9. By Lemma 5.10(i), the boundary values $G_{xx}^C(E + i0)$ and $\Gamma_{x \rightarrow w}^C(E + i0)$ are real for every removed set, component and vertex, and they are nonzero by Lemma 5.9. Further,

Lemma 5.9 with $C = V$, applied to the oriented edge $v \rightarrow v^-$ if $v \neq o$ and to the diagonal entry $G_{oo} = \Gamma_o$ if $v = o$, gives

$$D_v = -\frac{1}{\Gamma_v(E + i0)} = E + \sum_{k=1}^{N_v} |h_{v,vk}|^2 \Gamma_{vk}(E + i0) = E - \frac{1}{p} \sum_{k=1}^{N_v} \frac{Y_{vk}}{D_{vk}},$$

which is (5.9).

STEP 5. *The law of D_o .* Let $\mathfrak{D}: \Omega \rightarrow \mathbb{R}$ be the measurable map defined by $\mathfrak{D} = -1/\lim_{k \rightarrow \infty} G_{oo}(E + i/k)$ if this limit exists and is real and nonzero, and $\mathfrak{D} = E$ otherwise. By Step 4 and the identity $\Gamma_v = G_{oo} \circ \theta_v$, almost surely $D_v = \mathfrak{D}(\theta_v \omega)$ for every $v \in V$. By (5.9) at the root,

$$D_o = E - \frac{1}{p} \sum_{k=1}^{N_o} \frac{Y_k}{\mathfrak{D}(\theta_k \omega)}.$$

By Lemma 4.2(ii), the variables N_o , $(Y_k)_{k \geq 1}$ and $(\theta_k \omega)_{k \geq 1}$ are independent, and the $\theta_k \omega$ have law $\mathbb{P}$. It follows that the variables $\mathfrak{D}(\theta_k \omega)$, $k \geq 1$, are independent with the law of D_o , and independent of N_o and of the squared moduli of the root edge marks. Since $D_o \in \mathbb{R} \setminus \{0\}$ almost surely, the law of D_o is a real solution at energy E . $\square$

The nonvanishing of the boundary denominators in Proposition 5.11(ii) is a consequence of the finiteness of the boundary values, which is part of the definition of $\mathcal{E}_p$. We also record the structure of the law of D_o given the children. Under the no-ac hypothesis, by Step 5 of the proof of Proposition 5.11, D_o is either the constant E (if $N_o = 0$) or an affine function of $Y_1, \dots, Y_{N_o}$ with the nonzero real coefficients $-1/(p\mathfrak{D}(\theta_k \omega))$ (if $N_o \geq 1$). These coefficients are measurable with respect to the σ -field generated by N_o and $(\theta_k \omega)_{k \geq 1}$, and the squared moduli $Y_1, \dots, Y_{N_o}$ are fresh given this σ -field, by the example after Lemma 4.17 and consequence (a) after Definition 4.16. By Lemma 4.18(iii) and (4.14), the conditional law of D_o on the event $\{N_o \geq 1\}$ has no atoms. The same argument is used in Section 5.6 for the real recursive tree process.

Definition 5.12. Let $E \in \mathcal{E}_p$ with $E > 0$. Following [12], we say that the *no-ac hypothesis* holds at E if $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. In this case, with $D_o = -1/\Gamma_o(E + i0)$, the *real boundary law* at E is $\pi_E = \text{Law}_{\mathbb{P}}(D_o)$, and the *conditioned laws* are

$$\pi_E^\infty = \text{Law}_{\mathbb{P}^\infty}(D_o), \quad \pi_E^f = \text{Law}_{\mathbb{P}}(D_o \mid \mathcal{S}^c).$$

In the B -normalization ($E > 2$), the real boundary law is the law of $g(E)D_o$ under $\mathbb{P}$.

By Proposition 5.11(ii), π_E is a real solution at energy E , and all the conclusions of Proposition 5.5 apply to it. Further,

$$\pi_E = \varrho_p \pi_E^\infty + (1 - \varrho_p) \pi_E^f, \quad \pi_E^\infty \leq \varrho_p^{-1} \pi_E \quad \text{as measures,}$$

and π_E^∞ has no atoms. To see the last claim, note that on $\mathcal{S}$ the root has at least one child, and by the previous paragraph the law of D_o restricted to the event $\{N_o \geq 1\}$ has no atoms. Then $\mathbb{P}^\infty(D_o = c) \leq \varrho_p^{-1} \mathbb{P}(D_o = c, N_o \geq 1) = 0$ for every $c \in \mathbb{R}$. In Remark 5.15 below, we note that the boundary denominators form a recursive tree process with marginal π_E which is a function of the tree.

5.4 Recursive tree processes

For an arbitrary real solution, a family of denominators satisfying the recursion need not be a function of the tree. The following lemma constructs such a family on an extended probability space. It is used in Section 8, where the real solution need not be the real boundary law.

Lemma 5.13. *Let $E > 0$ and let π be a real solution at energy E . There exist a probability space and, on it, a random configuration ω with law $\mathbb{P}$ (Definition 4.1) and real random variables $(D_v)_{v \in \mathbb{U}}$ with the following properties.*

- (i) *For every $v \in \mathbb{U}$, we have $D_v \sim \pi$, and almost surely $D_v \in \mathbb{R} \setminus \{0\}$ and*

$$D_v = E - \frac{1}{p} \sum_{k=1}^{N_v} \frac{Y_{vk}}{D_{vk}}.$$

- (ii) *For every $n \geq 0$, the variables D_v , $v \in \mathbb{U}_n$, are independent with law π , and they are independent of the σ -field $\mathcal{F}_n$ of Section 5.3.*
- (iii) *Let $v_1, \dots, v_k \in \mathbb{U}$ be distinct words none of which is a prefix of another, and let $\mathcal{F}(v_1, \dots, v_k)$ be the σ -field generated by the coordinates N_u, ζ_u with $u \notin \bigcup_i v_i \mathbb{U}$ and w_u with $u \notin \bigcup_i v_i \mathbb{U}_*$, where $v \mathbb{U}_* = v \mathbb{U} \setminus \{v\}$. Then the pairs $(\theta_{v_i} \omega, (D_{v_i u})_{u \in \mathbb{U}})$, $1 \leq i \leq k$, are independent of each other and of $\mathcal{F}(v_1, \dots, v_k)$, and each has the law of $(\omega, (D_u)_{u \in \mathbb{U}})$. In particular, for every $n \geq 0$ this holds for the pairs indexed by $v \in \mathbb{U}_n$, with $\mathcal{F}_n$ in place of $\mathcal{F}(v_1, \dots, v_k)$.*

We call $(\omega, (D_v)_{v \in \mathbb{U}})$ a *recursive tree process* with marginal π , in the sense of [17]. By (i), the restriction of (D_v) to V is a family of real denominators on the tree that satisfies the real recursion at every vertex.

Remark 5.14. We comment on the measurability structure of the process.

- (i) For $|v| = n$, the variable D_v is not independent of $\theta_v \omega$; for instance, $D_v = E$ on the event $\{N_v = 0\}$. Property (ii) asserts independence from $\mathcal{F}_n$ only, and $\mathcal{F}_n$ contains the offspring numbers N_u with $|u| < n$, but not N_v with $|v| = n$. By (i), the family $(D_v)_{v \in \mathbb{U}_{\leq n}}$ is determined by $(D_v)_{v \in \mathbb{U}_n}$ and $\mathcal{F}_n$. Accordingly, for a measurable functional $F \geq 0$ of $\mathcal{F}_n$, of $(D_v)_{v \in \mathbb{U}_{\leq n}}$ and of a word $v \in \mathbb{U}_L$ with $L \leq n$, property (ii) allows one to integrate out $(D_v)_{v \in \mathbb{U}_n}$ as independent π -distributed variables;

the result is a measurable functional of ω and v , to which the many-to-one and many-to-two formulas of Proposition 4.8 apply. This is the setting of Remark 4.13.

- (ii) The variables D_v need not be measurable with respect to ω ; that is, the process need not be endogenous in the sense of [17]. Under the no-ac hypothesis at a regular energy, the boundary denominators of Proposition 5.11(ii) form an endogenous recursive tree process with marginal π_E (Remark 5.15).

Proof of Lemma 5.13. For $u \in \mathbb{U}$, let $\xi_u = (N_u, \zeta_u, (w_{uk})_{k \geq 1})$ be the *innovation* at u [17], with values in the Polish space $\Xi = \mathbb{N}_0 \times [0, 1]^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}}$, and let ν_{ξ} be its law under $\mathbb{P}$. Every coordinate of ω appears in exactly one innovation, and ω and $(\xi_u)_{u \in \mathbb{U}}$ determine each other, and under $\mathbb{P}$ the innovations are independent with law ν_{ξ} . In terms of innovations, $\mathcal{F}_n = \sigma(\xi_u : |u| < n)$, and $\theta_v \omega$ corresponds to $(\xi_{vu})_{u \in \mathbb{U}}$. For an innovation $\xi = (n, \zeta, (w_k)_{k \geq 1})$ and a sequence $(d_k)_{k \geq 1}$ in $\mathbb{R} \setminus \{0\}$, we set

$$\Psi(\xi, (d_k)_{k \geq 1}) = E - \frac{1}{p} \sum_{k=1}^n \frac{|w_k|^2}{d_k}.$$

STEP 1. Finite-depth processes. Fix $m \geq 0$. On some probability space, let $(\xi_u)_{u \in \mathbb{U}}$ and $(\Theta_v)_{v \in \mathbb{U}_m}$ be independent, with $\xi_u \sim \nu_{\xi}$ and $\Theta_v \sim \pi$. We define $D_v^{(m)} = \Theta_v$ for $|v| = m$, and for $|v| < m$, by backward induction on $|v|$,

$$D_v^{(m)} = \Psi(\xi_v, (D_{vk}^{(m)})_{k \geq 1}).$$

We claim that for $0 \leq l \leq m$, the family $((\xi_u)_{|u| < l}, (D_v^{(m)})_{|v| = l})$ has independent coordinates, with $\xi_u \sim \nu_{\xi}$ and $D_v^{(m)} \sim \pi$; in particular $D_v^{(m)} \neq 0$ almost surely, and the recursion is well defined. For $l = m$ this holds by construction. Suppose that it holds for $l + 1 \leq m$. Then the pairs $(\xi_v, (D_{vk}^{(m)})_{k \geq 1})$, $|v| = l$, are independent of each other and of $(\xi_u)_{|u| < l}$, and in each pair the variables $\xi_v, D_{v1}^{(m)}, D_{v2}^{(m)}, \dots$ are independent with laws $\nu_{\xi}, \pi, \pi, \dots$. Since π is a real solution, $D_v^{(m)} = \Psi(\xi_v, (D_{vk}^{(m)})_k)$ has law π . This proves the claim for l .

STEP 2. Consistency. For $l \leq m$, the family $((\xi_u)_{|u| < l}, (D_v^{(m)})_{|v| \leq l})$ is the image of $((\xi_u)_{|u| < l}, (D_v^{(m)})_{|v| = l})$ under a map that does not depend on m , namely the recursion from generation l to the root. By Step 1, the law of the latter family does not depend on $m \geq l$. It follows that the law μ_l of $((\xi_u)_{|u| < l}, (D_v^{(l)})_{|v| \leq l})$ on $\Omega_l = \Xi^{\{|u| < l\}} \times \mathbb{R}^{\{|v| \leq l\}}$ is the image of μ_{l+1} under the coordinate projection $\Omega_{l+1} \rightarrow \Omega_l$. The spaces Ω_l are countable products of Polish spaces, and the Kolmogorov extension theorem [81] provides a probability measure μ on $\Xi^{\mathbb{U}} \times \mathbb{R}^{\mathbb{U}}$ whose image under the projection onto every Ω_l is μ_l . We let (ξ_u) and (D_v) be the coordinates under μ , and ω the configuration corresponding to (ξ_u) .

STEP 3. *Properties (i) and (ii).* For every l , the coordinates $(\xi_u)_{|u|<l}$ are independent with law ν_ξ under μ_l , which shows that ω has law $\mathbb{P}$. By Step 1 with $m = l$, the variables $(D_v)_{|v|=l}$ are independent with law π and independent of $(\xi_u)_{|u|<l}$, which proves (ii) and the first part of (i); in particular $D_v \neq 0$ almost surely. The event $\{D_v = \Psi(\xi_v, (D_{vk})_{k \geq 1})\}$ depends only on coordinates in $\Omega_{|v|+1}$, and it has probability one under $\mu_{|v|+1}$ by construction. This proves (i).

STEP 4. *Property (iii).* In terms of innovations, $\theta_{v_i}\omega$ corresponds to $(\xi_{v_i u})_{u \in \mathbb{U}}$, and $\mathcal{F}(v_1, \dots, v_k)$ is generated by the innovations ξ_u with $u \notin \bigcup_i v_i \mathbb{U}$; indeed, the mark w_{v_i} of the edge into v_i belongs to the innovation at the parent of v_i . Let $m \geq \max_i |v_i|$. Under μ_m , the family $(\xi_{v_i u}, D_{v_i u}), |v_i u| \leq m$ (with $\xi_{v_i u}$ only for $|v_i u| < m$), is the image of $(\xi_{v_i u})_{|v_i u| < m}$ and $(D_{v_i u})_{|v_i u| = m}$ under the recursion. By Step 1, these inputs for $i = 1, \dots, k$ and the family $(\xi_u)_{u \notin \bigcup_i v_i \mathbb{U}, |u| < m}$ are independent, since they consist of distinct coordinates of a family with independent coordinates. This shows that the restrictions of the pairs in (iii) to words of length at most m are independent of each other and of the restriction of $\mathcal{F}(v_1, \dots, v_k)$ to innovations at words of length less than m . Further, the inputs for v_i , read in the coordinates $v_i u \mapsto u$, have the same law as the inputs of the process of depth $m - |v_i|$ rooted at o , and the recursion is the same map. It follows that the restriction of $(\xi_{v_i u}, D_{v_i u})_{u \in \mathbb{U}}$ to $|v_i u| \leq m$ has the law of the restriction of $(\xi_u, D_u)_{u \in \mathbb{U}}$ to $|u| \leq m - |v_i|$. Since m is arbitrary, and laws on countable products are determined by their finite-dimensional marginals, (iii) follows. For $\{v_1, \dots, v_k\} = \mathbb{U}_n$ (which is infinite, but the argument is the same), the σ -field $\mathcal{F}(v_1, \dots, v_k)$ is $\mathcal{F}_n$. $\square$

Remark 5.15. Let $E \in \mathcal{E}_p$ with $E > 0$, suppose that the no-ac hypothesis holds at E , let $\mathfrak{D}$ be the map of Step 5 of the proof of Proposition 5.11, and set $D_v = \mathfrak{D}(\theta_v \omega)$ for $v \in \mathbb{U}$, where ω has law $\mathbb{P}$. Then $(\omega, (D_v)_{v \in \mathbb{U}})$ satisfies the conclusions of Lemma 5.13 with $\pi = \pi_E$, every D_v is a measurable function of ω , and almost surely $D_v = -1/\Gamma_v(E + i0)$ for every $v \in V$. In the terminology of [17], this is an endogenous recursive tree process with marginal π_E . In particular, when the real solution in Lemma 5.13 is the real boundary law, the process may be taken to be this one.

To verify the claim, note that $\mathfrak{D}$ takes values in $\mathbb{R} \setminus \{0\}$ by its definition, and that the last assertion is Step 5 of the proof of Proposition 5.11. Let A be the event that $\mathfrak{D}(\omega) = E - p^{-1} \sum_{k \leq N_0} Y_k / \mathfrak{D}(\theta_k \omega)$; it has probability one by Step 5 of that proof. Since $\theta_{vk} = \theta_k \circ \theta_v$, and since the offspring number and the root edge marks of $\theta_v \omega$ are N_v and w_{vk} , $k \geq 1$, the recursion of Lemma 5.13(i) at the word v is the event $\theta_v^{-1} A$. This event has probability one because θ_v preserves $\mathbb{P}$, and D_v has the law of D_o , which is π_E . This proves (i). For (ii) and (iii), we note that the pair $(\theta_{v_i} \omega, (D_{v_i u})_{u \in \mathbb{U}})$ is a measurable function of $\theta_{v_i} \omega$, since $D_{v_i u} = \mathfrak{D}(\theta_u(\theta_{v_i} \omega))$. The configurations $\theta_{v_1} \omega, \dots, \theta_{v_k} \omega$ and the generators of $\mathcal{F}(v_1, \dots, v_k)$ are functions of pairwise disjoint sets of the independent coordinates of ω , and each $\theta_{v_i} \omega$ has law $\mathbb{P}$; this is the argument of the proof of Lemma 4.2(i). This gives

(iii), and (ii) follows in the same way from Lemma 4.2(i).

5.5 Fractional moments at positive imaginary part

The following lemma integrates one step of the cavity recursion, at a single vertex. It extends Lemma 4.18 to products of several negative powers sharing the same fresh variables, with additional weights given by powers of these variables. We write $C_{r,d} = (1-r)^{-1}(2/d)^r$ for the constant of Lemma 4.18.

Lemma 5.16. *Let $r \in (0,1)$, $d > 0$, $b \geq 0$ and $k \in \mathbb{N}_0$, and set $c_b = \max\{1, (2b/e)^b\}$. Let $n \geq 0$, let $Y_1, \dots, Y_n$ be independent $\text{Exp}(1)$ variables, let $c \in \mathbb{C}$ with $|c| \geq d$ and $\gamma_1, \dots, \gamma_n \in \mathbb{C}$, let $J_1, \dots, J_l \subset \{1, \dots, n\}$, let $a_1, \dots, a_l \geq 0$ with $a_1 + \dots + a_l \leq r$, and let $K \subset \{1, \dots, n\}$ with $|K| \leq k$ and $b_j \in [0, b]$ for $j \in K$. Then*

$$\mathbb{E} \left[\prod_{i=1}^l \left| c + \sum_{j \in J_i} \gamma_j Y_j \right|^{-a_i} \prod_{j \in K} Y_j^{b_j} \right] \leq (2c_b)^k (1 + C_{r,d}(n+1)^{r/2}).$$

The same bound holds for the conditional expectation given a σ -field $\mathcal{G}$ if $Y_1, \dots, Y_n$ are fresh given $\mathcal{G}$ (Definition 4.16), and n, c , the γ_j , the J_i , the a_i , K and the b_j are $\mathcal{G}$ -measurable, with $|c| \geq d$ almost surely.

Proof. For $j \in K$ and $y > 0$, we have $y^{b_j} e^{-y} \leq c_b e^{-y/2}$, since the maximum of $y^{b_j} e^{-y/2}$ is $(2b_j/e)^{b_j} \leq c_b$. For every measurable $\Phi \geq 0$, this gives

$$\mathbb{E} [Y_j^{b_j} \Phi(Y_j)] \leq c_b \int_0^\infty e^{-y/2} \Phi(y) dy = 2c_b \mathbb{E} \Phi(2Y_j).$$

Applying this to each $j \in K$ in turn, using the independence of the Y_j and Tonelli's theorem, we find that the left side of the claimed bound is at most $(2c_b)^{|K|}$ times $\mathbb{E} \prod_i |X_i|^{-a_i}$, where $X_i = c + \sum_{j \in J_i} \gamma'_j Y_j$ with $\gamma'_j = 2\gamma_j$ for $j \in K$ and $\gamma'_j = \gamma_j$ otherwise. Let $A = a_1 + \dots + a_l$. If $A = 0$, there is nothing more to prove. Otherwise, Hölder's inequality with the exponents A/a_i gives

$$\mathbb{E} \prod_{i=1}^l |X_i|^{-a_i} \leq \prod_{i=1}^l (\mathbb{E} |X_i|^{-A})^{a_i/A} \leq \max_i \mathbb{E} |X_i|^{-A}.$$

Since $|x|^{-A} \leq 1 + |x|^{-r}$ for $0 < A \leq r$, Lemma 4.18(ii) with $m = |J_i| \leq n$ gives $\mathbb{E} |X_i|^{-A} \leq 1 + C_{r,d}(n+1)^{r/2}$. For the conditional version, we use the substitution identity (4.14) (consequence (c) after Definition 4.16), with X a finite-dimensional vector encoding the $\mathcal{G}$ -measurable parameters c, γ_j, J_i, a_i, K and b_j . On each event $\{n = n_0\}$ these are finitely many numbers and finite sets of indices, so such an encoding exists. The identity reduces the claim on this event to the unconditional bound for fixed values of the revealed variables. $\square$

For $v \in V$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we write $D_v(z) = -1/\Gamma_v(z)$ for the denominator at z . By Proposition 3.16,

$$D_v(z) = z + \frac{1}{p} \sum_{k=1}^{N_v} Y_{vk} \Gamma_{vk}(z). \quad (5.10)$$

Lemma 5.17. *Let $r \in (0, 1)$ and $d > 0$. Then the following hold for every $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| \geq d$.*

- (i) $\mathbb{E}|\Gamma_o(z)|^r \leq C_{r,d}(p+1)^{r/2}$ and $\mathbb{E}^\infty|\Gamma_o(z)|^r \leq \varrho_p^{-1}C_{r,d}(p+1)^{r/2}$.
- (ii) *Let J be a random set of children of the root that is a function of the offspring numbers and the auxiliary marks, and let*

$$\Gamma_o^J(z) = -\left(z + \frac{1}{p} \sum_{k \in J} Y_k \Gamma_k(z)\right)^{-1}$$

be the diagonal entry at the root of the resolvent after the edges from the root to the children outside J are deleted. Then $\mathbb{E}|\Gamma_o^J(z)|^r \leq C_{r,d}(p+1)^{r/2}$. If $\mathbb{Q}$ is a probability measure on Ω with $\mathbb{Q} \leq K\mathbb{P}$ for a constant K , then all bounds of (i) and (ii) hold under $\mathbb{Q}$ with an extra factor K .

- (iii) *For $q \in (0, 1)$ and $g > 0$,*

$$\mathbb{E}|D_o(z)|^q \leq |z|^q + p^{1-q}\Gamma(1+q)C_{q,d}(p+1)^{q/2}, \quad \mathbb{E}|gD_o(z)|^{-q} \leq g^{-q}C_{q,d}(p+1)^{q/2}.$$

- (iv) *Let $E > 2$ and $X_\eta = -\Gamma_o(E + i\eta)/g(E)$. Then $\sup_{\eta > 0} \mathbb{E}|X_\eta|^r \leq C_{r,1}(p+1)^{r/2}$.*

In particular, for every compact interval $I \subset (0, \infty)$ and $r \in (0, 1)$, there exists $C = C(p, I, r)$ such that

$$\sup_{E \in I, 0 < \eta \leq 1} \left(\mathbb{E}|\Gamma_o(E + i\eta)|^r + \mathbb{E}^\infty|\Gamma_o(E + i\eta)|^r + \mathbb{E}|D_o(E + i\eta)|^r + \mathbb{E}|D_o(E + i\eta)|^{-r} \right) \leq C.$$

Proof. Part (i) is Corollary 4.20, since $\Gamma_o = G_{oo}$ and $|z| \geq d$. For (ii), let $\mathcal{G}$ be the σ -field generated by N_o , the auxiliary marks, the configurations $\theta_k\omega$, $k \geq 1$, and the phases of the root edge marks. As shown after Lemma 4.17, the squared moduli $Y_1, \dots, Y_{N_o}$ are fresh given $\mathcal{G}$, and the set J and the coefficients $\mathbf{1}_{\{k \in J\}}\Gamma_k(z)/p$ are $\mathcal{G}$ -measurable. Lemma 4.18(iv), with constant z and these coefficients, gives $\mathbb{E}[|\Gamma_o^J(z)|^r | \mathcal{G}] \leq C_{r,d}(N_o + 1)^{r/2}$, and taking expectations and using Jensen's inequality, we find the bound. For $\mathbb{Q} \leq K\mathbb{P}$, we have $\mathbb{E}_{\mathbb{Q}}Z \leq K\mathbb{E}Z$ for every $Z \geq 0$.

For (iii), we use (5.10) and $|a + b|^q \leq |a|^q + |b|^q$ to obtain

$$\mathbb{E}|D_o(z)|^q \leq |z|^q + p^{-q} \mathbb{E} \sum_{k=1}^{N_o} Y_k^q |\Gamma_k(z)|^q = |z|^q + p^{1-q}\Gamma(1+q) \mathbb{E}|\Gamma_o(z)|^q,$$

where we used that N_o , the Y_k and the $\Gamma_k(z) = G_{oo}(z) \circ \theta_k$ are independent and that the $\Gamma_k(z)$ have the law of $\Gamma_o(z)$, by Lemma 4.2(ii). Combined with (i), this gives the first bound in (iii). The second bound is (i) with $r = q$, since $|gD_o(z)|^{-q} = g^{-q}|\Gamma_o(z)|^q$.

For (iv), let $z = E + i\eta$ and $X_k = -\Gamma_k(z)/g(E)$; then $X_\eta = 1/(g(E)D_o(z))$. We multiply (5.10) at the root by $g(E)$ and use $g(E)E = 1 + \theta$ from Lemma 5.3 and $g(E)\Gamma_k(z) = -\theta X_k$. We obtain

$$X_\eta = \left(1 + \theta + ig(E)\eta - \frac{\theta}{p} \sum_{k=1}^{N_o} Y_k X_k\right)^{-1}.$$

The constant term has modulus at least $1 + \theta \geq 1$, and the argument of (ii) with $d = 1$ applies.

The final assertion follows from (i) and (iii), with $d = \min I$ and $|z| \leq \max I + 1$. $\square$

The bounds of Lemma 5.17(ii) also apply when structural decorations are prescribed at the root. For example, under the law conditioned on the event that the root has exactly one child, which has $\mathbb{P}$ -probability pe^{-p} , all bounds hold with the factor $K = e^p/p$. If a vertex is required to have no children other than a selected one, then its denominator without the selected child is the deterministic number z .

5.6 Fixed-depth convergence

In Sections 9 and 10, the resonance argument compares, at a fixed depth, expectations of functionals of the cavity variables at $E + i\eta$, $\eta > 0$, with the corresponding expectations for real denominators, and then lets η tend to zero with the depth fixed. This section provides the convergence statement for this step. The functionals are sums over one or two vertices of a fixed generation, which the many-to-one and many-to-two formulas of Proposition 4.8 express through the spine laws. We first record the properties of the spine laws that we use, then define the depth- m data and the real recursive tree process, and finally prove the convergence.

Spine laws

We use the spine laws of Definition 4.6. These are the one-spine laws Q_L and Q_L^∞ , with canonical random elements (ω, v) , and the two-spine laws $Q_{L,j}$ and $Q_{L,j}^\infty$, with canonical random elements (ω, u, v) ; here $L \geq 0$ and $0 \leq j < L$. We call L the length of the spine law. For $L = 0$, the law Q_0 is the law of (ω, o) with $\omega \sim \mathbb{P}$, and Q_0^∞ is the law of (ω, o) with $\omega \sim \mathbb{P}^\infty$; in this way $\mathbb{P}$ and $\mathbb{P}^\infty$ are spine laws of length zero. The *spine vertices* are the prefixes v_l , $0 \leq l \leq L$, of v , together with the prefixes u_l , $j < l \leq L$, of u for a two-spine law. The *endpoints* are v , and u for a two-spine law; the *endpoint law* is $\mathbb{P}$ for Q_L and $Q_{L,j}$, and $\mathbb{P}^\infty$ for Q_L^∞ and $Q_{L,j}^\infty$. For a spine vertex x which is not an endpoint, the

spine children of x are its children that are spine vertices (two at the fork v_j of a two-spine law, one otherwise), and its other children are its *side children*.

Lemma 5.18. *Let $\mathbf{Q}$ be a spine law of length L .*

- (i) *If $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, then $\mathbf{Q}$ -almost surely, $\theta_u \omega \in A$ for every $u \in \mathbb{U}$.*
- (ii) *Let $\mathcal{A}$ be a random finite subset of $\mathbb{U}_*$ which is a function of the offspring numbers, the auxiliary marks, and the distinguished vertices. Then, under $\mathbf{Q}$, the squared moduli $Y_u = |w_u|^2$, $u \in \mathcal{A}$, listed in a fixed order of $\mathbb{U}$, are fresh given the σ -field generated by the offspring numbers, the auxiliary marks, the distinguished vertices, the phases $w_u/|w_u|$ of all marks, and the marks w_u with $u \notin \mathcal{A}$.*
- (iii) *Let $L \geq 1$. If $\mathbf{Q}$ is $\mathbf{Q}_L$ or $\mathbf{Q}_L^\infty$, write $\mathbf{v} = I_1 \mathbf{v}'$. Then $N_0 - 1$ has law $\text{Pois}(p)$, and the pair $(\theta_{I_1} \omega, \mathbf{v}')$ is independent of N_0 and has law $\mathbf{Q}_{L-1}$, respectively $\mathbf{Q}_{L-1}^\infty$. If $\mathbf{Q}$ is $\mathbf{Q}_{L,j}$ or $\mathbf{Q}_{L,j}^\infty$ with $j \geq 1$, the same holds for the triple $(\theta_{I_1} \omega, \mathbf{u}', \mathbf{v}')$ with $\mathbf{u} = I_1 \mathbf{u}'$ and $\mathbf{v} = I_1 \mathbf{v}'$, whose law is $\mathbf{Q}_{L-1,j-1}$, respectively $\mathbf{Q}_{L-1,j-1}^\infty$. If $j = 0$, write $\mathbf{u} = J_1 \mathbf{u}'$ and $\mathbf{v} = J_2 \mathbf{v}'$. Then $N_0 - 2$ has law $\text{Pois}(p)$, and the pairs $(\theta_{J_1} \omega, \mathbf{u}')$ and $(\theta_{J_2} \omega, \mathbf{v}')$ are independent of each other and of N_0 , each with law $\mathbf{Q}_{L-1}$, respectively $\mathbf{Q}_{L-1}^\infty$.*

Proof. Part (i) is Corollary 4.10. Part (ii) follows from Lemma 4.17(ii) and (iii) (and from Lemma 4.17(i) and (iii) for $L = 0$). Indeed, for fixed values of the distinguished vertices, the event $\{\mathcal{A} = a\}$ is determined by the offspring numbers and the auxiliary marks; it belongs to the σ -field $\mathcal{G}_a$ of that lemma, and the σ -field $\mathcal{H}_{\mathcal{A}}$ of Lemma 4.17(iii) is generated by the variables listed in (ii).

For (iii), we use the construction of $\mathbf{Q}_L$ in Definition 4.6, with ω' , $N'_0, \dots, N'_{L-1}$ and $I_1, \dots, I_L$. Set $\omega'' = \theta_{I_1} \omega'$. Since ω' has law $\mathbb{P}$ and is independent of (N', I) , and since every shift preserves $\mathbb{P}$, the configuration ω'' has law $\mathbb{P}$ conditionally on (N', I) ; that is, ω'' has law $\mathbb{P}$ and is independent of (N', I) . The label $\mathbf{v}_l = I_1 \cdots I_l$ of ω corresponds to the label $I_2 \cdots I_l$ of $\theta_{I_1} \omega$, and $\theta_{I_1} \omega$ is obtained from ω'' by setting $N_{I_2 \cdots I_l} = N'_l + 1$ for $1 \leq l < L$, all other coordinates being unchanged. This is the construction of $\mathbf{Q}_{L-1}$ from ω'' , $(N'_1, \dots, N'_{L-1})$ and $(I_2, \dots, I_L)$. These inputs have the joint law required in Definition 4.6, and they are independent of $N'_0 = N_0 - 1$, because the N'_l are independent and I_{l+1} depends only on N'_l and on independent randomness. This proves the claim for $\mathbf{Q}_L$. Under $\mathbf{Q}_L^\infty$, we condition on $\{\theta_{\mathbf{v}} \omega \in \mathcal{S}\} = \{\theta_{\mathbf{v}'}(\theta_{I_1} \omega) \in \mathcal{S}\}$, which is an event of the pair $(\theta_{I_1} \omega, \mathbf{v}')$; since this pair is independent of N_0 , conditioning on the event keeps the independence and replaces $\mathbf{Q}_{L-1}$ by $\mathbf{Q}_{L-1}^\infty$. The two-spine laws with $j \geq 1$ are treated in the same way. For $j = 0$, the fork is the root, and $N_0 = N'_0 + 2$. The configurations $\theta_{J_1} \omega'$ and $\theta_{J_2} \omega'$ are independent with law $\mathbb{P}$, and independent of (N', J, I) , because $J_1 \neq J_2$ and the two configurations use disjoint sets of coordinates of ω' . Each of $(\theta_{J_1} \omega, \mathbf{u}')$ and $(\theta_{J_2} \omega, \mathbf{v}')$ is obtained by the construction of $\mathbf{Q}_{L-1}$ from one of these configurations and from the variables $N_l^{(i)}$ and $I_l^{(i)}$ of the corresponding arm, and the

two sets of inputs are independent of each other and of N'_0 . Under $\mathbb{Q}_{L,0}^\infty$ we condition on the intersection of an event of the first pair and an event of the second pair, which preserves the independence. $\square$

Depth- m data and side denominators

For $m \geq 0$, let $\mathcal{K}_m$ be the set of $\kappa = (\Delta, x, y, k) \in \mathcal{K}$ such that x, y and the words in $\Delta_V \cup \Delta_E$ have length at most m , and let $Z = [0, 1]^\mathbb{N}$ be the space of auxiliary marks.

Definition 5.19. Let $m \geq 0$. The *space of depth- m data* is the Polish space

$$\mathcal{X}_m = \mathbb{N}_0^{\mathbb{U}_{<m}} \times Z^{\mathbb{U}_{<m}} \times \mathbb{C}^{\mathbb{U}_{\leq m} \setminus \{o\}} \times \{0, 1\}^{\mathbb{U}_{\leq m}} \times \mathbb{C}^{\mathcal{K}_m},$$

with the product topology, where $\mathbb{U}_{<m} = \mathbb{U}_{\leq m-1}$. For $z \in \mathbb{C} \setminus \mathbb{R}$, the variables of the first m generations, which we call the *depth- m data*, form the element of $\mathcal{X}_m$

$$\mathcal{L}_m(z) = \left((N_v)_{|v|<m}, (\zeta_v)_{|v|<m}, (w_v)_{1 \leq |v| \leq m}, (\varsigma_v)_{|v| \leq m}, (F_\kappa(z))_{\kappa \in \mathcal{K}_m} \right),$$

where $\varsigma_v = 0$ if $v \notin V$. If $E \in \mathcal{E}_p$, the depth- m data $\mathcal{L}_m(E)$ is defined in the same way, with $F_\kappa(z)$ replaced by $F_\kappa(E + i0)$.

The first three components determine the marked ball of radius m around the root and the auxiliary marks at the vertices of generation less than m ; they generate the σ -field $\mathcal{F}_m$. By (5.6), the last component contains every resolvent entry $G_{xy}^\Delta(z)$ with x, y and Δ inside the ball, in particular the forward cavities $\Gamma_x(z)$ for $|x| \leq m$, the cavity resolvents $\Gamma_{x \rightarrow w}^C(z)$, and the path entries $G_{ox}^\Delta(z)$. Functionals of the depth- m data therefore include the quantities that the resonance argument uses at a fixed depth: the marks and the genealogy near one or two spines, side denominators, selections by auxiliary ranks, survival indicators, and resolvent entries of the forests obtained by removing finitely many vertices and edges near the spines.

For $x \in V$, a set J of children of x , and $z \in \mathbb{C} \setminus \mathbb{R}$, we let

$$\Gamma_x^J(z) = - \left(z + \frac{1}{p} \sum_{y \in J} Y_y \Gamma_y(z) \right)^{-1} \quad (5.11)$$

be the diagonal entry at x of the resolvent of the forward subtree $\mathcal{T}_x$ after the edges from x to its children outside J are deleted; the identity is Proposition 3.16(ii) for this finite removal. If $|x| < m$, then $\Gamma_x^J(z)$ is one of the resolvent entries contained in $\mathcal{L}_m(z)$, selected by a function of x, J and N_x . For a spine vertex x which is not an endpoint, the *side denominator* at x is

$$D_x^{\text{side}}(z) = z + \frac{1}{p} \sum_{y \text{ side child of } x} Y_y \Gamma_y(z) = - \frac{1}{\Gamma_x^J(z)}, \quad J = \{\text{side children of } x\},$$

so that, by (5.10),

$$D_x(z) = D_x^{\text{side}}(z) + \frac{1}{p} \sum_{y \text{ spine child of } x} Y_y \Gamma_y(z). \quad (5.12)$$

For a spine law of length L and $m \geq L + 1$, the side denominators are functions of the depth- m data and of the distinguished vertices.

The real recursive tree process

Definition 5.20. Let $E \in \mathcal{E}_p$ with $E > 0$, suppose that the no-ac hypothesis holds at E , let $L \geq 0$, and let $\pi' \in \{\pi_E, \pi_E^\infty\}$.

- (i) The *one-spine real recursive tree process* of length L with endpoint law π' consists of independent random variables $D_l^{\text{side}}, 0 \leq l < L$, with law $\pi_E, Y_l, 1 \leq l \leq L$, with law $\text{Exp}(1)$, and D_L with law π' , together with the variables $D_l, 0 \leq l < L$, defined by backward induction by

$$D_l = D_l^{\text{side}} - \frac{Y_{l+1}}{pD_{l+1}}. \quad (5.13)$$

- (ii) Let $0 \leq j < L$. The *two-spine real recursive tree process* with fork at generation j and endpoint law π' consists of independent random variables $D_l^{\text{side}} (0 \leq l \leq j)$ and $D_l^{\text{side},a} (j < l < L, a \in \{1,2\})$ with law $\pi_E, Y_l (1 \leq l \leq j)$ and $Y_l^a (j < l \leq L, a \in \{1,2\})$ with law $\text{Exp}(1)$, and D_L^1, D_L^2 with law π' , together with the variables defined by backward induction by

$$D_l^a = D_l^{\text{side},a} - \frac{Y_{l+1}^a}{pD_{l+1}^a} \quad (j < l < L, a \in \{1,2\}),$$

$$D_j = D_j^{\text{side}} - \sum_{a=1}^2 \frac{Y_{j+1}^a}{pD_{j+1}^a}, \quad D_l = D_l^{\text{side}} - \frac{Y_{l+1}}{pD_{l+1}} \quad (l < j).$$

The recursions are well defined almost surely. Indeed, by induction from the endpoints, each denominator is almost surely nonzero: the endpoint laws are carried by $\mathbb{R} \setminus \{0\}$, and each other denominator is, conditionally on the side denominator and on the denominators of the next generation, an affine function of one or two independent $\text{Exp}(1)$ variables with nonzero coefficients; such a variable has no atoms. In words, the one-spine real recursive tree process is the sequence of real denominators along a path. The side environments of the path carry independent denominators with the real boundary law, and its endpoint carries the real boundary law or the conditioned law.

Lemma 5.21. Let $E \in \mathcal{E}_p$ with $E > 0$, suppose that the no-ac hypothesis holds at E , and let $\mathbf{Q}$ be a spine law of length L .

- (i) $\mathbf{Q}$ -almost surely, all conclusions of Definition 5.7, Lemma 5.9, Lemma 5.10 and Proposition 5.11(ii) hold. In particular, for every $v \in V$, we have $D_v(E + i\eta) \rightarrow D_v = \mathfrak{D}(\theta_v \omega) \in$

$\mathbb{R} \setminus \{0\}$, and for every spine vertex x which is not an endpoint, $D_x^{\text{side}}(E + i\eta) \rightarrow D_x^{\text{side}}(E) \in \mathbb{R} \setminus \{0\}$, where

$$D_x^{\text{side}}(E) = E - \frac{1}{p} \sum_{y \text{ side child of } x} \frac{Y_y}{D_y}.$$

- (ii) Under $\mathbb{Q}_L$, respectively $\mathbb{Q}_L^\infty$, the variables $D_{v_l}^{\text{side}}(E)$, $0 \leq l < L$, Y_{v_l} , $1 \leq l \leq L$, and D_v have the joint law of the one-spine real recursive tree process of length L with endpoint law π_E , respectively π_E^∞ , and D_{v_l} is the corresponding variable D_l of this process. The analogous statement holds for the two-spine laws and the two-spine real recursive tree process.
- (iii) Let $m \geq L$. Under $\mathbb{Q}$, conditionally on $\mathcal{F}_m$, on the distinguished vertices, and on $(\zeta_v)_{v \in \mathcal{V}_m}$, the variables D_v , $v \in \mathcal{V}_m$, are independent, with law π_E^∞ if $\zeta_v = 1$ and π_E^f if $\zeta_v = 0$. The denominators D_v with $|v| < m$ are determined by these and by $\mathcal{F}_m$ through (5.9).

Proof. The conclusions in (i) hold $\mathbb{P}$ -almost surely, by Definition 5.7, Lemmas 5.9 and 5.10, and Proposition 5.11(ii) together with Step 5 of its proof, and by Lemma 5.18(i) they hold $\mathbb{Q}$ -almost surely. For a spine vertex x which is not an endpoint, the side denominator equals $-1/\Gamma_x^J(z)$, where J is the set of side children of x , and $\Gamma_x^J(z)$ is a diagonal entry of the resolvent of a reduced operator. By Lemma 5.9, its boundary value is finite and nonzero, and equals $-1/(E + \sum_{y \in J} |h_{xy}|^2 \Gamma_y(E + i0))$. Then $D_x^{\text{side}}(E + i\eta)$ converges to the finite nonzero number $D_x^{\text{side}}(E)$. This proves (i).

For (ii), we consider $\mathbb{Q}_L$ and use the notation of Lemma 4.7. Let Ψ be the map on finite counting measures on $\mathbb{C} \times \Omega$ defined by $\Psi(\sum_i \delta_{(w_i, \omega_i)}) = E - p^{-1} \sum_i |w_i|^2 / \mathfrak{D}(\omega_i)$. By (i), almost surely $D_{v_l}^{\text{side}}(E) = \Psi(\Sigma_l)$ for $0 \leq l < L$, where Σ_l is the side process at v_l . By Lemma 4.7(i), the processes Σ_l are independent Poisson processes with intensity measure $p \gamma \otimes \mathbb{P}$, and they are independent of the spine edge marks w_{v_l} and of the endpoint configuration $\theta_v \omega$, which has law $\mathbb{P}$. On the other hand, under $\mathbb{P}$, Proposition 5.11(ii) and Step 5 of its proof give $D_o = \Psi(\Pi_o)$ almost surely, where Π_o is the child process of the root, a Poisson process with intensity measure $p \gamma \otimes \mathbb{P}$ by Lemma 4.2(iii). Hence each $D_{v_l}^{\text{side}}(E)$ has law π_E . Further, $Y_{v_l} = |w_{v_l}|^2$ has law $\text{Exp}(1)$, and $D_v = \mathfrak{D}(\theta_v \omega)$ has law π_E . These variables are independent, since they are functions of the independent random elements of Lemma 4.7(i). Finally, (5.9) at v_l , split as in (5.12), gives (5.13). Under $\mathbb{Q}_L^\infty$, Lemma 4.7(iii) gives the same conclusions, except that $\theta_v \omega$ has law $\mathbb{P}^\infty$ and D_v has law π_E^∞ . The two-spine laws are treated in the same way, using Lemma 4.7(iv).

For (iii), we first consider $\mathbb{Q} = \mathbb{Q}_L$. Let $g \geq 0$ be a measurable function of the generators of $\mathcal{F}_m$ and of a word in $\mathbb{U}_L$, and let $h \geq 0$ be a measurable function of $(\theta_u \omega)_{u \in \mathbb{U}_m}$. By (4.5) and Lemma 4.2(i), since $\mathcal{V}_L$ is $\mathcal{F}_m$ -measurable for $L \leq m$, we have

$$\mathbb{E}_{\mathbb{Q}_L}[g(\omega, \mathbf{v})h] = p^{-L} \mathbb{E} \sum_{v \in \mathcal{V}_L} g(\omega, v) h = p^{-L} \mathbb{E} \sum_{v \in \mathcal{V}_L} g(\omega, v) \cdot \mathbb{E} h = \mathbb{E}_{\mathbb{Q}_L}[g(\omega, \mathbf{v})] \mathbb{E} h.$$

Under $\mathbb{Q}_L$, the configurations $\theta_u \omega$, $u \in \mathbb{U}_m$, are therefore independent with law $\mathbb{P}$, and independent of $\mathcal{F}_m$ and $\mathbf{v}$. Conditioning further on the survival indicators $\zeta_u = \mathbf{1}_{\mathcal{S}}(\theta_u \omega)$,

$u \in \mathcal{V}_m$, we find that these configurations are conditionally independent with law $\mathbb{P}(\cdot \mid \mathcal{S})$ or $\mathbb{P}(\cdot \mid \mathcal{S}^c)$. Since $D_u = \mathfrak{D}(\theta_u \omega)$, this proves the claim under $\mathbb{Q}_L$. Under $\mathbb{Q}_L^\infty$, the law is $\mathbb{Q}_L$ conditioned on $\{\theta_v \omega \in \mathcal{S}\}$, and this event occurs if and only if $\zeta_u = 1$ for some $u \in \mathcal{V}_m$ with prefix v ; it is therefore measurable with respect to the conditioning σ -field, and conditioning on it does not change the conditional laws. The two-spine laws are treated in the same way, using (4.6). The last sentence of (iii) is (5.9). $\square$

The next lemma helps to verify that the boundaries of the events used in Section 10, which are defined by finitely many inequalities on denominators and squared moduli, are null sets for the limiting law.

Lemma 5.22. *In the setting of Lemma 5.21, the following hold for every $c \in \mathbb{R}$. For every word $v \in \mathbb{U}$, we have $\mathbb{Q}(v \in V, N_v \geq 1, D_v = c) = 0$; in particular, $\mathbb{Q}(D_x = c) = 0$ for every spine vertex x which is not an endpoint. For every such x , we have $\mathbb{Q}(D_x^{\text{side}}(E) = c) = 0$ if $c \neq E$. For every endpoint t , we have $\mathbb{Q}(D_t = c) = 0$ if $c \neq E$ or if the endpoint law is $\mathbb{P}^\infty$. Finally, $\mathbb{Q}(Y_u = c) = 0$ for every $u \in \mathbb{U}_*$.*

Proof. Fix $v \in \mathbb{U}$ and let $\mathcal{A} = \{vk : 1 \leq k \leq N_v\}$ if $v \in V$, and $\mathcal{A} = \emptyset$ otherwise; this is a function of the offspring numbers. By Lemma 5.18(ii), the squared moduli Y_{vk} , $k \leq N_v$, are fresh given a σ -field with respect to which the denominators $D_{vk} = \mathfrak{D}(\theta_{vk} \omega)$ are measurable, since these depend only on coordinates of $\theta_{vk} \omega$. On $\{v \in V, N_v \geq 1\}$, (5.9) expresses D_v as an affine function of these fresh variables with nonzero coefficients $-1/(pD_{vk})$. By Lemma 4.18(iii) and (4.14), its conditional law has no atoms. A spine vertex which is not an endpoint has at least one child. The claims about $D_x^{\text{side}}(E)$ and D_t follow from Lemma 5.21(ii), Proposition 5.5(i) applied to π_E , and the absence of atoms of π_E^∞ . The squared moduli have continuous laws. $\square$

In particular, let Φ be a function of finitely many of the variables in Lemma 5.22, evaluated at $E + i\eta$ through the real parts of the corresponding coordinates of the depth- m data, and suppose that Φ is continuous outside a finite union of level sets of these variables, at levels c for which Lemma 5.22 gives probability zero. Since the denominators are continuous functions of the nonvanishing cavity coordinates, the discontinuity set of Φ is then $\mathbb{Q}$ -null for the limiting depth- m data.

Fractional products

For a word $v \in \mathbb{U}_L$, the spine vertices of v are its prefixes $v_0, \dots, v_L$, and the spine child of v_l is v_{l+1} for $l < L$. For a pair $(u, v) \in \mathbb{U}_L^2$ with $|u \wedge v| = j < L$, the spine vertices are the prefixes of u and of v , and the spine children of a spine vertex are those of its children that are spine vertices. This agrees with Section 5.6 when $v = \mathbf{v}$ or $(u, v) = (u, \mathbf{v})$.

Definition 5.23. Let $L \geq 0$, $m \geq L + 1$, $s \in (0, 1)$, $b \geq 0$ and $k, l_0 \in \mathbb{N}$, and let $\mathbf{v}$ denote a word $v \in \mathbb{U}_L$ or a pair $(u, v) \in \mathbb{U}_L^2$ with $|u \wedge v| = j$ for a fixed $j < L$. A *fractional product* of depth L with cavity exponents at most s and exponents of the squared moduli at most b is a function on $\Omega \times \{\mathbf{v}\} \times (\mathbb{C} \setminus \mathbb{R})$ of the form

$$W(\omega, \mathbf{v}; z) = \prod_x \left(\prod_{i=1}^{l_0} |\Gamma_x^{J_{x,i}}(z)|^{a_{x,i}} \prod_{y \in K_x} Y_y^{b_{x,y}} \right),$$

where the first product runs over the spine vertices x of $\mathbf{v}$ in V (and $W = 0$ if some spine vertex is not in V), and where, for each spine vertex x :

- (i) the sets $J_{x,1}, \dots, J_{x,l_0}$ and K_x of children of x , with $|K_x| \leq k$, and the numbers $b_{x,y} \in [0, b]$ are given by measurable rules depending only on the position of x among the spine vertices, on the offspring numbers and auxiliary marks of $\theta_x \omega$ at words of length less than $m - |x|$, and on the labels of the spine children of x ;
- (ii) Γ_x^J is defined in (5.11);
- (iii) the exponents $a_{x,i} \geq 0$, which we call the cavity exponents, are deterministic, depend only on the position of x among the spine vertices, and satisfy $\sum_i a_{x,i} \leq s$.

For $E \in \mathcal{E}_p$, we define $W(\omega, \mathbf{v}; E)$ by the same formula with $\Gamma_x^J(E + i0)$ in place of $\Gamma_x^J(z)$.

For example, by Proposition 3.18, we have

$$|G_{ov}(z)|^s = p^{-sL/2} |\Gamma_o(z)|^s \prod_{l=1}^L Y_{v_l}^{s/2} |\Gamma_{v_l}(z)|^s$$

for $v \in \mathcal{V}_L$, and this is $p^{-sL/2}$ times a fractional product with cavity exponents s and exponents $s/2$ of the squared moduli, in which $J_{x,1}$ is the set of all children of x , $K_{v_l} = \{v_{l+1}\}$ for $0 \leq l < L$, and $K_{v_L} = \emptyset$. A product $|G_{ou}(z)|^{s_1} |G_{ov}(z)|^{s_2}$ along two spines is, up to a constant, a fractional product with cavity exponent $s_1 + s_2$ on the common segment of the two spines; the definition requires $s_1 + s_2 < 1$ there.

Lemma 5.24. Let $\mathbf{Q}$ be a spine law of length L , let W be a fractional product of depth L of the corresponding type, with cavity exponents at most $r \in (0, 1)$, exponents of the squared moduli at most b , and $|K_x| \leq k$, and let $d > 0$. Then for every $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| \geq d$,

$$\mathbb{E}_{\mathbf{Q}} W(\omega, \mathbf{v}; z) \leq \varrho_p^{-2} \left((2c_b)^k (1 + C_{r,d}(p+3)^{r/2}) \right)^n,$$

where $\mathbf{v} = \mathbf{v}$ or $(u, \mathbf{v})$, n is the number of spine vertices, and c_b and $C_{r,d}$ are as in Lemma 5.16.

Proof. We argue by induction on L , integrating the factor of the root first. Let $R_o(z)$ be the factor of W indexed by $x = o$, and let $W'(z)$ be the product of the other factors; then $W = R_o W'$. Let $\mathcal{A}$ be the set of children of the root, and let $\mathcal{G}$ be the σ -field of

Lemma 5.18(ii) for this set. Then the squared moduli $Y_1, \dots, Y_{N_o}$ of the root edge marks are fresh given $\mathcal{G}$. The forward cavities $\Gamma_k(z) = G_{oo}(z) \circ \theta_k$, the sets $J_{o,i}$ and K_o , the numbers $b_{o,y}$, and the factor $W'(z)$ are $\mathcal{G}$ -measurable, since they depend only on the offspring numbers, the auxiliary marks, the distinguished vertices, and the configurations $\theta_k \omega$, which do not contain the marks w_k . The conditional version of Lemma 5.16, with $n = N_o$, $c = z$ and $\gamma_k = \Gamma_k(z)/p$, gives

$$\mathbb{E}_{\mathbb{Q}}[R_o(z) \mid \mathcal{G}] \leq (2c_b)^k (1 + C_{r,d}(N_o + 1)^{r/2}).$$

Taking expectations, we find

$$\mathbb{E}_{\mathbb{Q}} W(z) \leq (2c_b)^k \mathbb{E}_{\mathbb{Q}} \left[(1 + C_{r,d}(N_o + 1)^{r/2}) W'(z) \right]. \quad (5.14)$$

If $L = 0$, then $W' = 1$, the law of N_o is $\text{Pois}(p)$ under $\mathbb{P}$, and $\mathbb{E}^\infty Z \leq \varrho_p^{-1} \mathbb{E} Z$ for $Z \geq 0$; Jensen's inequality and (5.14) give the claim. Let $L \geq 1$, and consider a one-spine law. By Lemma 5.18(iii), $N_o - 1$ has law $\text{Pois}(p)$ and is independent of the pair $(\theta_{I_1} \omega, \mathbf{v}')$, which has law $\mathbb{Q}_{L-1}$ or $\mathbb{Q}_{L-1}^\infty$. The factor W' is a function of this pair, and in the coordinates of $\theta_{I_1} \omega$ it is a fractional product of depth $L - 1$ with the same parameters, by Definition 5.23(i) and (iii) and since $m - 1 \geq L$. Using this independence in (5.14) and Jensen's inequality $\mathbb{E}(N_o + 1)^{r/2} \leq (p + 2)^{r/2}$, we find

$$\mathbb{E}_{\mathbb{Q}} W(z) \leq (2c_b)^k (1 + C_{r,d}(p + 2)^{r/2}) \mathbb{E}_{\mathbb{Q}'} W'(z),$$

where $\mathbb{Q}'$ is the law of the pair, and the induction hypothesis applies to the last factor. For two-spine laws with $j \geq 1$, the same argument applies with the triple of Lemma 5.18(iii). For $j = 0$, the factor W' is the product of two fractional products of depth $L - 1$, one for each of the two independent pairs of Lemma 5.18(iii), and $N_o - 2$ has law $\text{Pois}(p)$; the expectation factorizes, and $\mathbb{E}(N_o + 1)^{r/2} \leq (p + 3)^{r/2}$. In each case, the factor ϱ_p^{-1} appears once for each endpoint with endpoint law $\mathbb{P}^\infty$, and there are at most two endpoints. $\square$

Convergence at fixed depth

Proposition 5.25. *Let $E \in \mathcal{E}_p$ with $E > 0$, and suppose that the no-ac hypothesis holds at E , that is, $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Let $\mathbb{Q}$ be a spine law of length L , write $\mathbf{v} = \mathbf{v}$ or $\mathbf{v} = (\mathbf{u}, \mathbf{v})$ for its distinguished vertices, and let $m \geq L + 1$.*

- (i) *$\mathbb{Q}$ -almost surely, $\mathcal{L}_m(E + i\eta) \rightarrow \mathcal{L}_m(E)$ in $\mathcal{X}_m$ as $\eta \downarrow 0$. The side denominators and the endpoint denominators converge almost surely, and their limits, together with the squared moduli of the spine edge marks, have the joint law of the real recursive tree process of Lemma 5.21(ii); the denominators at the spine vertices converge to the corresponding denominators of this process. In particular, the joint law of these variables at $E + i\eta$ converges weakly to that of the real recursive tree process.*

- (ii) Let Φ be a bounded measurable function of $\mathcal{L}_m$ and $\mathbf{v}$, and suppose that $\mathbb{Q}$ -almost surely $\Phi(\cdot, \mathbf{v})$ is continuous at $\mathcal{L}_m(E)$. Then

$$\lim_{\eta \downarrow 0} \mathbb{E}_{\mathbb{Q}} \Phi(\mathcal{L}_m(E + i\eta), \mathbf{v}) = \mathbb{E}_{\mathbb{Q}} \Phi(\mathcal{L}_m(E), \mathbf{v}).$$

- (iii) Let $0 < s < r < 1$, and let W be a fractional product of depth L of the type of $\mathbb{Q}$ with cavity exponents at most s . Then $\sup_{0 < \eta \leq 1} \mathbb{E}_{\mathbb{Q}} W(\omega, \mathbf{v}; E + i\eta)^{r/s} < \infty$. In particular, the family $\{W(\omega, \mathbf{v}; E + i\eta)\}_{0 < \eta \leq 1}$ is uniformly integrable under $\mathbb{Q}$, and for every Φ as in (ii),

$$\lim_{\eta \downarrow 0} \mathbb{E}_{\mathbb{Q}} [W(\omega, \mathbf{v}; E + i\eta) \Phi(\mathcal{L}_m(E + i\eta), \mathbf{v})] = \mathbb{E}_{\mathbb{Q}} [W(\omega, \mathbf{v}; E) \Phi(\mathcal{L}_m(E), \mathbf{v})].$$

- (iv) Let W and Φ be as in (iii) for $\mathbb{Q} = \mathbb{Q}_L$, and set $F_{\eta}(\omega, v) = W(\omega, v; E + i\eta) \Phi(\mathcal{L}_m(E + i\eta), v)$ for $0 \leq \eta \leq 1$, with the convention that $\eta = 0$ refers to the boundary values. Then

$$\lim_{\eta \downarrow 0} \mathbb{E} \sum_{v \in \mathcal{V}_L} F_{\eta}(\omega, v) = \mathbb{E} \sum_{v \in \mathcal{V}_L} F_0(\omega, v), \quad \lim_{\eta \downarrow 0} \mathbb{E}^{\infty} \sum_{v \in \mathcal{V}_L} F_{\eta}(\omega, v) = \mathbb{E}^{\infty} \sum_{v \in \mathcal{V}_L} F_0(\omega, v).$$

The same holds for the backbone sums $\mathbb{E}^{\infty} \sum_{v \in \mathcal{V}_L^{\infty}} F_{\eta}(\omega, v)$ when the hypotheses of (iii) hold for $\mathbb{Q}_L^{\infty}$, and for the sums over pairs in Proposition 4.8(ii) and (iv) when they hold for $\mathbb{Q}_{L,j}$, respectively $\mathbb{Q}_{L,j}^{\infty}$. Each limit equals the right side of the corresponding identity in Proposition 4.8 with $F = F_0$.

- (v) Let finitely many triples (Q_k, W_k, Φ_k) , $1 \leq k \leq K$, be given at a common depth m , where each W_k is either the constant 1 or a fractional product with cavity exponents at most some $s_k < 1$, and each Φ_k satisfies the hypothesis of (ii) for Q_k . For every $\varepsilon > 0$ there exists $\eta_0 > 0$ such that, for all $0 < \eta \leq \eta_0$ and all k ,

$$\left| \mathbb{E}_{Q_k} [W_k(\omega, \mathbf{v}; E + i\eta) \Phi_k(\mathcal{L}_m(E + i\eta), \mathbf{v})] - \mathbb{E}_{Q_k} [W_k(\omega, \mathbf{v}; E) \Phi_k(\mathcal{L}_m(E), \mathbf{v})] \right| \leq \varepsilon.$$

The same holds for finitely many sums as in (iv).

Proof. STEP 1. *Almost sure convergence.* By Definition 5.7(a) and Lemma 5.18(i), $\mathbb{Q}$ -almost surely $F_{\kappa}(E + i\eta) \rightarrow F_{\kappa}(E + i0)$ for every $\kappa \in \mathcal{K}_m$. The other components of $\mathcal{L}_m(z)$ do not depend on z . Since convergence in the product topology is coordinatewise convergence, $\mathcal{L}_m(E + i\eta) \rightarrow \mathcal{L}_m(E)$ almost surely. The convergence of the side denominators and the law of the limit are Lemma 5.21(i) and (ii), and almost sure convergence implies weak convergence of the joint law. This proves (i).

STEP 2. *Bounded functionals.* By Step 1 and the continuity assumption, $\Phi(\mathcal{L}_m(E + i\eta), \mathbf{v}) \rightarrow \Phi(\mathcal{L}_m(E), \mathbf{v})$ almost surely. Since Φ is bounded, the dominated convergence theorem implies (ii).

STEP 3. *Fractional products.* Let $0 < \eta \leq 1$ and $z = E + i\eta$; note that $|z| \geq E$. Raising W to the power r/s multiplies the exponents $a_{x,i}$ and $b_{x,y}$ by r/s , which shows that $W^{r/s}$

is a fractional product with cavity exponents at most r and exponents of the squared moduli at most br/s . By Lemma 5.24 with $d = E$, we have a bound on $\mathbb{E}_{\mathbb{Q}} W(\omega, \mathbf{v}; z)^{r/s}$ that does not depend on η . Since $r/s > 1$, the family $\{W(\omega, \mathbf{v}; E + i\eta)\}_{0 < \eta \leq 1}$ is uniformly integrable. Since each $\Gamma_x^J(z)$ appearing in W has $|x| \leq L < m$, it is one of the resolvent entries contained in $\mathcal{L}_m(z)$, selected by a function of quantities which do not depend on η . By Step 1, it converges almost surely to $\Gamma_x^J(E + i0)$, which implies that $W(\omega, \mathbf{v}; E + i\eta) \rightarrow W(\omega, \mathbf{v}; E)$ almost surely. The products $W\Phi$ converge almost surely by Step 2, and they are uniformly integrable, because their absolute values are bounded by $\sup |\Phi|$ times W . By Vitali's convergence theorem [64, Section 4.6], we conclude (iii).

STEP 4. Sums over vertices. We write $\Phi = \Phi^+ - \Phi^-$ with $\Phi^\pm \geq 0$, and we apply the identities of Proposition 4.8 to $W\Phi^+$ and $W\Phi^-$ separately, at $E + i\eta$ and at the boundary. By (4.5), we have $\mathbb{E} \sum_{v \in \mathcal{V}_L} F_\eta(\omega, v) = p^L \mathbb{E}_{\mathbb{Q}_L} F_\eta(\omega, v)$, and both sides are finite by (iii). By (iii) applied to $\Phi^\pm$, the right side converges to $p^L \mathbb{E}_{\mathbb{Q}_L} F_0(\omega, v) = \mathbb{E} \sum_{v \in \mathcal{V}_L} F_0(\omega, v)$. Under $\mathbb{P}^\infty$, we have $\mathbb{E}^\infty \sum_{v \in \mathcal{V}_L} F_\eta(\omega, v) = \varrho_p^{-1} p^L \mathbb{E}_{\mathbb{Q}_L} [F_\eta(\omega, v) \zeta_o]$ by (4.7), and Φ_{ζ_o} satisfies the hypothesis of (ii) because ζ_o is a coordinate of the depth- m data taking the values 0 and 1. The backbone sums and the sums over pairs are treated in the same way, using (4.8), (4.6), (4.9) and (4.10). This proves (iv).

STEP 5. Uniformity over finite families. For each k , (ii) or (iii) implies that the difference in (v) tends to zero as $\eta \downarrow 0$; in particular, it is at most ε for $\eta \leq \eta_k$, for some $\eta_k > 0$. We take $\eta_0 = \min_k \eta_k$, and we treat finitely many sums as in (iv) in the same way. $\square$

Remark 5.26. We comment on the use of Proposition 5.25.

- (i) The order of choices is: the energy E and the depth m are fixed first, then the finite family of spine laws and functionals, then the tolerance ε , and finally the imaginary part η_0 . The proposition does not interchange the limits $m \rightarrow \infty$ and $\eta \downarrow 0$.
- (ii) In the backbone version, by Lemma 5.21(ii), the side denominators of the limiting real recursive tree process have the unconditioned real boundary law π_E , and only the endpoint denominators have the conditioned law π_E^∞ . Functionals that look beyond the endpoints are described by Lemma 5.21(iii).
- (iii) Conditioning on a structural event of positive probability, such as a selection of decorations at an endpoint as in Lemma 4.21, amounts to multiplying Φ by the indicator of an event depending only on the genealogy, the survival indicators and the auxiliary marks in the ball, and dividing by its probability. If the event depends on the auxiliary ranks only through their relative order, then its indicator is continuous at every point of $\mathcal{X}_m$ at which the ranks in the ball are distinct, since the genealogy and survival coordinates are discrete; this holds almost surely. Indicators of events defined by finitely many inequalities on the real parts of side denominators,

spine denominators and endpoint denominators, and on squared moduli, satisfy the hypothesis of (ii) when the levels are those allowed by Lemma 5.22.

The transfer operator and the free energy

In this section, we study the positive operator that governs the growth of fractional moments of products of transfer matrices, and we define the free energy. We work with a general law of the rescaled denominator. Throughout this section, we fix $p > 1$ and $K > 0$, and we let ν be a Borel probability measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \quad \text{supp } \nu = \mathbb{R}, \quad \int_{\mathbb{R}} |b|^q \nu(db) < \infty \quad \text{for every } q \in (0, 1). \quad (6.1)$$

We let B be a random variable with law ν , and we let β be a random variable independent of B with the exponential law of rate K , that is, with density Ke^{-Kt} on $(0, \infty)$. If π is a real solution at energy $E > 0$ (Definition 5.1), D has law π , and $g > 0$, then the law of $B = gD$ satisfies (6.1) by Proposition 5.5, and $\beta = g^2 Y/p$ with $Y \sim \text{Exp}(1)$ has the exponential law of rate $K = p/g^2$. This is the situation in which the results of this section are applied.

We now describe the contents of this section. In Section 6.1, we define the transfer matrices, their action on the projective line, and the transfer operator $\mathcal{Q}_s$, and we record some elementary facts about angular integrals of matrices. In Section 6.2, we prove Theorem 6.11, which states that $\mathcal{Q}_s$ has a simple dominant eigenvalue $\lambda(s)$ with a strictly positive eigenfunction, that $\lambda(s)$ is real-analytic in s , and that $\lambda(s)$ determines the growth of angular moments of products of independent transfer matrices. In Section 6.3, we define the free energy $\varphi(s) = \log \lambda(s)$ and establish its basic properties: independence of the scaling, convexity, the symmetry $\varphi(s) = \varphi(2 - s)$, monotonicity, and the strict inequality $\varphi(1) < \log p$. In Section 6.4, we prove inverse moment bounds for products of three independent transfer matrices, and we use them to compare the free energy with moments of norms of products. Finally, in Section 6.5, we identify the free energy with the exponential growth rate of the path moments $\Phi_L(s)$, which are the quantities controlled in Section 8.

6.1 Transfer matrices and projective coordinates

We identify the real projective line $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$, where $x \in \mathbb{R}$ corresponds to the line through $(x, 1)$ and ∞ corresponds to the line through $(1, 0)$; this is a homeomorphism when $\mathbb{R} \cup \{\infty\}$ carries the topology of the one-point compactification. We write $C(\mathbb{RP}^1)$ for the Banach space of continuous functions $F: \mathbb{RP}^1 \rightarrow \mathbb{C}$ with the norm $\|F\|_\infty =$

$\max_{x \in \mathbb{RP}^1} |F(x)|$. We say that a real-valued function F is nonnegative, and write $F \geq 0$, if $F(x) \geq 0$ for all x . Vectors in $\mathbb{C}^2$ carry the Hermitian norm $\|\cdot\|$, and matrices carry the corresponding operator norm.

Definition 6.1. (i) For $b \in \mathbb{R}$ and $\beta > 0$, define

$$M(b, \beta) = \beta^{-1/2} \begin{pmatrix} 0 & 1 \\ -\beta & b \end{pmatrix}. \quad (6.2)$$

We have $\det M(b, \beta) = 1$.

- (ii) For $x \in \mathbb{R}$, set $v_x = (x, 1) / \sqrt{1 + x^2}$, and set $v_\infty = (1, 0)$. For an invertible real 2×2 matrix A and $x \in \mathbb{RP}^1$, let $A \cdot x \in \mathbb{RP}^1$ denote the point corresponding to the line through Av_x . The number $\|Av_x\|$ does not depend on the choice of sign of the unit vector representing x .
- (iii) For $b \in \mathbb{R}$ and $\beta > 0$, define the projective map $f_{b, \beta}(x) = M(b, \beta) \cdot x$. In coordinates, we have

$$f_{b, \beta}(x) = (b - \beta x)^{-1} \quad (x \in \mathbb{R}, x \neq b/\beta), \quad f_{b, \beta}(b/\beta) = \infty, \quad f_{b, \beta}(\infty) = 0.$$

- (iv) Let m be the Borel probability measure on $\mathbb{RP}^1$ given by $m(dx) = \pi^{-1}(1 + x^2)^{-1} dx$ on $\mathbb{R}$ and $m(\{\infty\}) = 0$, where $\pi = 3.14 \dots$ (elsewhere, π denotes a real solution); it is the image of the uniform probability measure on the unit circle under the map sending a unit vector to the line it spans. For an invertible real 2×2 matrix A and $s \in [0, 2]$, define

$$J_s(A) = \int_{\mathbb{RP}^1} \|Av_x\|^{-s} m(dx).$$

- (v) For $s \in (0, 2)$, define $e_s(x) = |x|^s$ for $x \in \mathbb{R}$, and define $h_s: \mathbb{RP}^1 \rightarrow [0, 1]$ by

$$h_s(x) = \frac{|x|^s}{(1 + x^2)^{s/2}} \quad (x \in \mathbb{R}), \quad h_s(\infty) = 1.$$

The function h_s is continuous on $\mathbb{RP}^1$, vanishes only at 0, and $e_s = h_s(1 + x^2)^{s/2}$ on $\mathbb{R}$.

- (vi) For $s \in (0, 2)$ and a Borel function $F: \mathbb{R} \rightarrow [0, \infty]$, define the transfer operator in the x -coordinate

$$(\mathcal{A}_s F)(x) = p \mathbb{E} \left[\beta^{s/2} |x|^s F(f_{B, \beta}(x)) \right], \quad x \in \mathbb{R}.$$

This is well defined, because $f_{B, \beta}(x) \in \mathbb{R}$ almost surely for every $x \in \mathbb{R}$: for $x = 0$ we have $f_{B, \beta}(0) = 1/B$ and $\mathbb{P}(B = 0) = 0$, and for $x \neq 0$ we have $f_{B, \beta}(x) = \infty$ only if $\beta = B/x$, an event of probability zero since β has a density and is independent of B .

(vii) For $s \in (0, 2)$, a bounded Borel function $F: \mathbb{RP}^1 \rightarrow \mathbb{C}$, and $x \in \mathbb{RP}^1$, define the transfer operator

$$(\mathcal{Q}_s F)(x) = p \mathbb{E} \left[\|M(B, \beta)v_x\|^{-s} F(f_{B, \beta}(x)) \right].$$

The expectation converges absolutely by Lemma 6.5 below.

For $x \in \mathbb{R}$, we have $M(b, \beta)(x, 1)^\top = \beta^{-1/2}(1, b - \beta x)^\top$ and $M(b, \beta)(1, 0)^\top = \beta^{-1/2}(0, -\beta)^\top$. Taking norms, we find

$$\|M(b, \beta)v_x\|^{-s} = \frac{\beta^{s/2}(1 + x^2)^{s/2}}{(1 + (b - \beta x)^2)^{s/2}} \quad (x \in \mathbb{R}), \quad \|M(b, \beta)v_\infty\|^{-s} = \beta^{-s/2}. \quad (6.3)$$

For fixed (b, β) , the functions $x \mapsto \|M(b, \beta)v_x\|^{-s}$ and $x \mapsto f_{b, \beta}(x)$ are continuous on $\mathbb{RP}^1$.

The next lemma collects two elementary facts about the projective action. Part (i) is the identity that makes the norm multipliers in $\mathcal{Q}_s$ telescope along products, and part (ii) computes the projective Jacobian.

Lemma 6.2. *Let A and A' be real 2×2 matrices with determinant one.*

(i) *For every $x \in \mathbb{RP}^1$, we have $(A'A) \cdot x = A' \cdot (A \cdot x)$ and*

$$\|A'Av_x\| = \|A'v_{A \cdot x}\| \|Av_x\|. \quad (6.4)$$

(ii) *For every Borel function $F: \mathbb{RP}^1 \rightarrow [0, \infty]$, we have*

$$\int_{\mathbb{RP}^1} F(A \cdot x) \|Av_x\|^{-2} m(dx) = \int_{\mathbb{RP}^1} F(y) m(dy). \quad (6.5)$$

Proof. For (i), since the vector Av_x spans the line $A \cdot x$, we have $Av_x = \pm \|Av_x\| v_{A \cdot x}$. Applying A' , we obtain $A'Av_x = \pm \|Av_x\| A'v_{A \cdot x}$. This vector spans the line $A' \cdot (A \cdot x)$, and taking norms gives (6.4).

For (ii), we parametrize $\mathbb{RP}^1$ by angles. For $\theta \in \mathbb{R}$, let $u_\theta = (\cos \theta, \sin \theta)$. The map $\theta \mapsto [u_\theta]$ from $[0, \pi)$ to $\mathbb{RP}^1$ is a bijection, and the image of $\pi^{-1} d\theta$ under it is m ; indeed, u_θ spans the line of $x = \cot \theta$, and $dx = -(1 + x^2) d\theta$. The map $u \mapsto Au / \|Au\|$ is a homeomorphism of the unit circle, with inverse $u \mapsto A^{-1}u / \|A^{-1}u\|$. Let $\Theta: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous lift, so that $Au_\theta = \|Au_\theta\| u_{\Theta(\theta)}$. Writing $w(\theta) = Au_\theta$, we have

$$\Theta'(\theta) = \frac{\det(w(\theta), w'(\theta))}{\|w(\theta)\|^2} = \frac{\det(A) \det(u_\theta, u'_\theta)}{\|Au_\theta\|^2} = \|Au_\theta\|^{-2} > 0,$$

where we used that $w'(\theta) = Au'_\theta$ and $\det(u_\theta, u'_\theta) = 1$. Since a homeomorphism of the circle has degree ± 1 and Θ is increasing, we have $\Theta(\theta + 2\pi) = \Theta(\theta) + 2\pi$. Further, since

$Au_{\theta+\pi} = -Au_{\theta}$, the continuous function $\theta \mapsto \Theta(\theta + \pi) - \Theta(\theta)$ takes values in $\pi + 2\pi\mathbb{Z}$, and it is therefore constant; since twice this constant equals $\Theta(\theta + 2\pi) - \Theta(\theta) = 2\pi$, it equals π . We conclude that Θ is an increasing C^1 bijection of $\mathbb{R}$ with $\Theta(\theta + \pi) = \Theta(\theta) + \pi$, and it induces the projective action of A on $\mathbb{R}/\pi\mathbb{Z} \simeq \mathbb{RP}^1$. For $G(\theta) = F([u_{\theta}])$, which is π -periodic, the change of variables $\vartheta = \Theta(\theta)$ gives

$$\int_0^\pi G(\Theta(\theta)) \|Au_{\theta}\|^{-2} d\theta = \int_{\Theta(0)}^{\Theta(0)+\pi} G(\vartheta) d\vartheta = \int_0^\pi G(\vartheta) d\vartheta.$$

Dividing by π gives (6.5). □

Lemma 6.3. *Let A be a real 2×2 matrix with determinant one. Then $J_0(A) = J_2(A) = 1$, and for every $s \in [0, 2]$, we have*

$$J_s(A) = J_{2-s}(A^{-1}) = J_{2-s}(A). \quad (6.6)$$

Further, $s \mapsto \log J_s(A)$ is convex on $[0, 2]$, and $J_s(A) \leq 1$ for $s \in [0, 2]$.

Proof. We have $J_0(A) = m(\mathbb{RP}^1) = 1$, and $J_2(A) = 1$ by (6.5) with $F = 1$. Fix $s \in [0, 2]$ and $x \in \mathbb{RP}^1$, and set $y = A \cdot x$. By (6.4) applied to $A^{-1}A = I$, we have $1 = \|A^{-1}v_y\| \|Av_x\|$, and we can write

$$\|Av_x\|^{-s} = \|Av_x\|^{-2} \|Av_x\|^{2-s} = \|Av_x\|^{-2} \|A^{-1}v_{A \cdot x}\|^{-(2-s)}.$$

Integrating against m and using (6.5) with $F(y) = \|A^{-1}v_y\|^{-(2-s)}$, we find $J_s(A) = J_{2-s}(A^{-1})$.

To prove that $J_t(A^{-1}) = J_t(A)$ for $t \in [0, 2]$, we use the singular value decomposition. There exist orthogonal matrices U, V and $\sigma \geq 1$ such that $A = U \operatorname{diag}(\sigma, \sigma^{-1}) V^T$; the two singular values are σ and σ^{-1} because $\det A = 1$. Then $A^{-1} = V \operatorname{diag}(\sigma^{-1}, \sigma) U^T$. The quantity J_t is invariant under multiplication by orthogonal matrices on either side: on the left because $\|UAv\| = \|Av\|$, and on the right because $\|AVv_x\| = \|Av_{V \cdot x}\|$ for orthogonal V and the projective action of an orthogonal matrix preserves m (it is induced by a rotation or a reflection of the unit circle). It follows that $J_t(A) = J_t(D_\sigma)$ and $J_t(A^{-1}) = J_t(D_\sigma^{-1})$, where $D_\sigma = \operatorname{diag}(\sigma, \sigma^{-1})$. Let R be the rotation by $\pi/2$. Then $D_\sigma^{-1} = RD_\sigma R^{-1}$, and the same invariance implies $J_t(D_\sigma^{-1}) = J_t(D_\sigma)$. This proves (6.6).

For $s_0, s_1 \in [0, 2]$ and $\tau \in (0, 1)$, Hölder's inequality with exponents $1/(1-\tau)$ and $1/\tau$ gives

$$J_{(1-\tau)s_0 + \tau s_1}(A) = \int (\|Av_x\|^{-s_0})^{1-\tau} (\|Av_x\|^{-s_1})^\tau m(dx) \leq J_{s_0}(A)^{1-\tau} J_{s_1}(A)^\tau,$$

which is the convexity of $\log J_s(A)$. Taking $s_0 = 0$ and $s_1 = 2$, we find $J_s(A) \leq J_0(A)^{1-s/2} J_2(A)^{s/2} = 1$. □

For exponents below one, the angular integral $J_s(A)$ is comparable to a power of the norm of A .

Lemma 6.4. *For every $s \in (0, 1)$, there exists $C_s < \infty$ such that for every real 2×2 matrix A with determinant one, setting $t = \|A\|$, we have*

$$t^{-s} \leq J_s(A) \leq C_s t^{-s}.$$

One can take $C_s = \pi^{-1} \int_0^\pi |\cos \theta|^{-s} d\theta$.

Proof. The lower bound follows from $\|Av_x\| \leq t$ for every x . For the upper bound, write $A = U \operatorname{diag}(t, t^{-1}) V^\top$ with U, V orthogonal. As in the proof of Lemma 6.3, we have $J_s(A) = J_s(\operatorname{diag}(t, t^{-1}))$. In the angular parametrization $u_\theta = (\cos \theta, \sin \theta)$, we have $\|\operatorname{diag}(t, t^{-1})u_\theta\| \geq t|\cos \theta|$, and therefore

$$J_s(A) \leq \frac{t^{-s}}{\pi} \int_0^\pi |\cos \theta|^{-s} d\theta = C_s t^{-s}.$$

The constant C_s is finite because $s < 1$. □

6.2 The Perron–Frobenius eigenvalue of the transfer operator

Boundedness and holomorphy

We begin with a bound on the expected norm multiplier of one factor $M(b, \beta)$, uniform over the input direction.

Lemma 6.5. *For $s \in (0, 2)$ and $K > 0$, set*

$$C_{s,K} = 2^{s/2} K^{-s/2} \Gamma(1 + s/2) + 2^s K^{s/2} \Gamma(1 - s/2).$$

Then for every $b \in \mathbb{R}$, we have

$$\sup_{x \in \mathbb{RP}^1} \int_0^\infty K e^{-Kt} \|M(b, t)v_x\|^{-s} dt \leq C_{s,K} (1 + |b|^{s/2}). \quad (6.7)$$

The constant $C_{s,K}$ is continuous in $(s, K) \in (0, 2) \times (0, \infty)$.

Proof. Let $r = s/2 \in (0, 1)$, and let $I(x)$ denote the integral on the left side of (6.7). By (6.3), we have $I(\infty) = K \int_0^\infty t^{-r} e^{-Kt} dt = K^r \Gamma(1 - r)$. For $x \in \mathbb{R}$ with $|x| \leq 1$, we discard the denominator in (6.3) and obtain $I(x) \leq 2^r \int_0^\infty K t^r e^{-Kt} dt = 2^r K^{-r} \Gamma(1 + r)$.

Now let $|x| = a \geq 1$ and $\sigma = \operatorname{sgn} x$. Substituting $u = at$ in the integral defining $I(x)$ and using (6.3), we have

$$I(x) = K \frac{(1 + a^2)^r}{a^{1+r}} \int_0^\infty e^{-Ku/a} \frac{u^r}{(1 + (b - \sigma u)^2)^r} du. \quad (6.8)$$

Since $u = |\sigma u| \leq |b| + |b - \sigma u|$ and $r < 1$, we have $u^r \leq |b|^r + |b - \sigma u|^r$. Using $|b - \sigma u|^r \leq (1 + (b - \sigma u)^2)^{r/2}$ and $(1 + (b - \sigma u)^2)^{-r} \leq (1 + (b - \sigma u)^2)^{-r/2}$, we have

$$\frac{u^r}{(1 + (b - \sigma u)^2)^r} \leq (1 + |b|^r) H(b - \sigma u), \quad H(v) = (1 + v^2)^{-r/2}.$$

The function H is even and nonincreasing on $[0, \infty)$. Hence $\int_{J'} H \leq \int_{-l/2}^{l/2} H$ for every interval $J' \subset \mathbb{R}$ of length l . (Indeed, if J denotes the centered interval of length l , then $J' \setminus J$ and $J \setminus J'$ have the same length, and $H \leq H(l/2)$ on $J' \setminus J$ while $H \geq H(l/2)$ on $J \setminus J'$.) We write $e^{-Ku/a} = \int_u^\infty (K/a) e^{-Kt/a} dt$ and use Tonelli's theorem. Since $\{b - \sigma u : 0 \leq u \leq t\}$ is an interval of length t , this gives

$$\begin{aligned} \int_0^\infty e^{-Ku/a} H(b - \sigma u) du &= \int_0^\infty \frac{K}{a} e^{-Kt/a} \int_0^t H(b - \sigma u) du dt \\ &\leq \int_0^\infty \frac{K}{a} e^{-Kt/a} 2 \int_0^{t/2} H(v) dv dt \\ &= 2 \int_0^\infty H(v) e^{-2Kv/a} dv \\ &\leq 2 \int_0^\infty v^{-r} e^{-2Kv/a} dv = 2 \Gamma(1-r) \left(\frac{a}{2K}\right)^{1-r}. \end{aligned}$$

Inserting this bound into (6.8) and using $(1 + a^2)^r \leq 2^r a^{2r}$, we find

$$I(x) \leq K 2^r a^{r-1} \cdot 2 \Gamma(1-r) \left(\frac{a}{2K}\right)^{1-r} (1 + |b|^r) = 4^r K^r \Gamma(1-r) (1 + |b|^r).$$

Combining the three cases gives (6.7). The continuity of $C_{s,K}$ is evident from its formula. $\square$

To prove analyticity in s , we extend $\mathcal{Q}_s$ to complex exponents. Let $\Sigma = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2\}$. For $z \in \Sigma$, $x \in \mathbb{RP}^1$, $b \in \mathbb{R}$, and $\beta > 0$, set

$$R_z(x; b, \beta) = \exp(-z \log \|M(b, \beta)v_x\|),$$

so that $|R_z| = R_{\operatorname{Re} z}$ and $R_s(x; b, \beta) = \|M(b, \beta)v_x\|^{-s}$ for real s . For a bounded Borel function $F: \mathbb{RP}^1 \rightarrow \mathbb{C}$, define

$$(\mathcal{Q}_z F)(x) = p \mathbb{E} \left[R_z(x; B, \beta) F(f_{B,\beta}(x)) \right], \quad x \in \mathbb{RP}^1.$$

For real $z = s \in (0, 2)$, this is the operator $\mathcal{Q}_s$ of Definition 6.1.

Lemma 6.6. *Suppose that ν satisfies (6.1).*

(i) *For every $z \in \Sigma$, the operator $\mathcal{Q}_z$ maps $C(\mathbb{RP}^1)$ into $C(\mathbb{RP}^1)$, and*

$$\|\mathcal{Q}_z\| \leq p C_{\operatorname{Re} z, K} (1 + \mathbb{E}|B|^{\operatorname{Re} z/2}).$$

The right side is bounded on every compact subset of Σ .

- (ii) For real $s \in (0, 2)$, the operator $\mathcal{Q}_s$ is positive, that is, $\mathcal{Q}_s F \geq 0$ for $F \geq 0$, and $\|\mathcal{Q}_s\| = \|\mathcal{Q}_s \mathbf{1}\|_\infty$. Further, $\mathcal{Q}_s \mathbf{1}(x) > 0$ for every $x \in \mathbb{RP}^1$.
- (iii) The map $z \mapsto \mathcal{Q}_z$ from Σ to the bounded operators on $C(\mathbb{RP}^1)$ is holomorphic in the operator norm.

Proof. For $\sigma \in (0, 2)$, conditioning on B and applying Lemma 6.5, we have

$$\sup_{x \in \mathbb{RP}^1} \mathbb{E}[R_\sigma(x; B, \beta)] \leq C_{\sigma, K}(1 + \mathbb{E}|B|^{\sigma/2}) < \infty, \quad (6.9)$$

where the finiteness follows from (6.1) because $\sigma/2 < 1$. Since $\mathbb{E}|B|^{\sigma/2}$ is a finite continuous function of $\sigma \in (0, 2)$ (by dominated convergence, with dominating function $1 + |B|^{\sigma_+/2}$ on $\sigma \leq \sigma_+ < 2$), the right side of (6.9) is bounded on compact subsets of $(0, 2)$.

We now prove (i). Fix $z \in \Sigma$ and $F \in C(\mathbb{RP}^1)$, and write $\sigma = \operatorname{Re} z$. Since $|R_z| = R_\sigma$, (6.9) implies that $|\mathcal{Q}_z F(x)| \leq p C_{\sigma, K}(1 + \mathbb{E}|B|^{\sigma/2})\|F\|_\infty$ for every x . It remains to show that $\mathcal{Q}_z F$ is continuous. Let $x_n \rightarrow x$ in $\mathbb{RP}^1$. For each fixed (b, β) , the maps $y \mapsto R_z(y; b, \beta)$ and $y \mapsto f_{b, \beta}(y)$ are continuous on $\mathbb{RP}^1$ by (6.3). Then the random variables $Z_n = R_z(x_n; B, \beta)F(f_{B, \beta}(x_n))$ converge almost surely to $Z = R_z(x; B, \beta)F(f_{B, \beta}(x))$. Fix $t \in (\sigma, 2)$. Then $|Z_n|^{t/\sigma} \leq \|F\|_\infty^{t/\sigma} R_t(x_n; B, \beta)$, and (6.9) with σ replaced by t shows that $\sup_n \mathbb{E}|Z_n|^{t/\sigma} < \infty$. Since $t/\sigma > 1$, the sequence (Z_n) is uniformly integrable, and by Vitali's convergence theorem, we have $\mathbb{E}Z_n \rightarrow \mathbb{E}Z$. This proves (i).

For (ii), positivity is immediate from the definition, and for real F we have $|\mathcal{Q}_s F| \leq \mathcal{Q}_s |F| \leq \|F\|_\infty \mathcal{Q}_s \mathbf{1}$, which gives $\|\mathcal{Q}_s\| = \|\mathcal{Q}_s \mathbf{1}\|_\infty$; the same bound holds for complex F since $|\mathcal{Q}_s F| \leq \mathcal{Q}_s |F|$. Since the integrand $R_s(x; B, \beta)$ in $\mathcal{Q}_s \mathbf{1}(x)$ is strictly positive, we have $\mathcal{Q}_s \mathbf{1}(x) > 0$.

For (iii), fix $z \in \Sigma$, let $\sigma = \operatorname{Re} z$, and choose $\varepsilon > 0$ with $[\sigma - 2\varepsilon, \sigma + 2\varepsilon] \subset (0, 2)$. We write $\ell = \log \|M(B, \beta)v_x\|$, and we define the operator

$$(\mathcal{D}_z F)(x) = -p \mathbb{E}[\ell R_z(x; B, \beta) F(f_{B, \beta}(x))].$$

For every $w \in \mathbb{C}$, we have $|e^w - 1 - w| \leq |w|^2 e^{|w|}/2$. Applying this with $w = -h\ell$ for $h \in \mathbb{C}$ with $|h| \leq \varepsilon/2$, we obtain

$$\begin{aligned} |R_{z+h} - R_z + h\ell R_z| &= |R_z| |e^{-h\ell} - 1 + h\ell| \leq \frac{|h|^2}{2} \ell^2 e^{\varepsilon|\ell|/2} R_\sigma \\ &\leq \frac{8|h|^2}{\varepsilon^2} e^{\varepsilon|\ell|} R_\sigma \leq \frac{8|h|^2}{\varepsilon^2} (R_{\sigma-\varepsilon} + R_{\sigma+\varepsilon}). \end{aligned}$$

In the second inequality, we used $\ell^2 e^{-\varepsilon|\ell|/2} \leq 16\varepsilon^{-2} e^{-2} \leq 16\varepsilon^{-2}$, and in the third, we used $e^{\varepsilon|\ell|} e^{-\sigma\ell} \leq e^{-(\sigma-\varepsilon)\ell} + e^{-(\sigma+\varepsilon)\ell}$. By (6.9), we conclude that

$$\|\mathcal{Q}_{z+h} - \mathcal{Q}_z - h\mathcal{D}_z\| \leq \frac{8p|h|^2}{\varepsilon^2} \sup_{x \in \mathbb{RP}^1} \mathbb{E}[R_{\sigma-\varepsilon}(x; B, \beta) + R_{\sigma+\varepsilon}(x; B, \beta)] \leq C|h|^2,$$

with C independent of h . Since $|\ell|R_\sigma \leq \varepsilon^{-1}(R_{\sigma-\varepsilon} + R_{\sigma+\varepsilon})$, the same reasoning as in (i) shows that $\mathcal{D}_z$ is a bounded operator on $C(\mathbb{RP}^1)$. We conclude that $z \mapsto \mathcal{Q}_z$ is complex differentiable in operator norm at every point of Σ , with derivative $\mathcal{D}_z$. $\square$

Kernel measures and total variation

For $s \in (0, 2)$ and $x \in \mathbb{RP}^1$, we let $\kappa_s(x, \cdot)$ be the finite Borel measure on $\mathbb{RP}^1$ defined by

$$\kappa_s(x, A) = p \mathbb{E} \left[\|M(B, \beta)v_x\|^{-s} \mathbf{1}_A(f_{B, \beta}(x)) \right],$$

so that $\mathcal{Q}_s F(x) = \int F(y) \kappa_s(x, dy)$ for bounded Borel F . We call $\kappa_s(x, \cdot)$ the kernel measure of $\mathcal{Q}_s$ at x . By Tonelli's theorem, $x \mapsto \kappa_s(x, A)$ is Borel for each Borel set A ; that is, κ_s is a finite kernel. We define its iterates by $\kappa_s^{(1)} = \kappa_s$ and $\kappa_s^{(n+1)}(x, A) = \int \kappa_s(x, dy) \kappa_s^{(n)}(y, A)$. By induction and Tonelli's theorem, we have $\mathcal{Q}_s^n F(x) = \int F d\kappa_s^{(n)}(x, \cdot)$ for bounded Borel F , the Chapman–Kolmogorov relations $\kappa_s^{(m+n)}(x, \cdot) = \int \kappa_s^{(m)}(x, dy) \kappa_s^{(n)}(y, \cdot)$ hold, and

$$\kappa_s^{(n)}(x, \mathbb{RP}^1) = \mathcal{Q}_s^n \mathbf{1}(x) \leq \|\mathcal{Q}_s\|^n. \quad (6.10)$$

Directly from the definition, we also have

$$\kappa_s^{(n+1)}(x, \cdot) = p \mathbb{E} \left[\|M(B, \beta)v_x\|^{-s} \kappa_s^{(n)}(f_{B, \beta}(x), \cdot) \right]. \quad (6.11)$$

For a finite signed Borel measure μ on $\mathbb{RP}^1$, we write $\|\mu\|_{\text{TV}} = |\mu|(\mathbb{RP}^1)$ for its total variation norm. We say that the n -step kernel measure is continuous in total variation at $x_0 \in \mathbb{RP}^1$ if $\|\kappa_s^{(n)}(x, \cdot) - \kappa_s^{(n)}(x_0, \cdot)\|_{\text{TV}} \rightarrow 0$ as $x \rightarrow x_0$.

The kernel measures at the two special points are explicit. Since $f_{b, \beta}(\infty) = 0$ and $f_{b, \beta}(0) = 1/b$, (6.3) yields

$$\kappa_s(\infty, \cdot) = p \mathbb{E}[\beta^{-s/2}] \delta_0, \quad \kappa_s(0, A) = p \mathbb{E}[\beta^{s/2}] \mathbb{E}[(1 + B^2)^{-s/2} \mathbf{1}_A(1/B)]. \quad (6.12)$$

At a finite nonzero input, the kernel measure has a density. To describe it, let $\iota: \mathbb{R} \rightarrow \mathbb{RP}^1$ be the map $\iota(w) = 1/w$, with $\iota(0) = \infty$; it is a Borel bijection onto $\mathbb{RP}^1 \setminus \{0\}$ with Borel inverse. Set $\phi(t) = t^{s/2} e^{-Kt} \mathbf{1}_{\{t > 0\}}$, which is a bounded continuous function on $\mathbb{R}$ because $s > 0$. For $x \in \mathbb{R} \setminus \{0\}$ and $w \in \mathbb{R}$, define

$$k_x(w) = \frac{pK}{|x|} (1 + x^2)^{s/2} (1 + w^2)^{-s/2} \int \phi\left(\frac{b-w}{x}\right) \nu(dw). \quad (6.13)$$

Lemma 6.7. *Suppose that ν satisfies (6.1), and let $s \in (0, 2)$.*

- (i) *For $x \in \mathbb{R} \setminus \{0\}$, the kernel measure $\kappa_s(x, \cdot)$ is the image of the measure $k_x(w) dw$ under ι . Further, $k_x(w) > 0$ for every $w \in \mathbb{R}$.*

- (ii) The one-step kernel measure is continuous in total variation at every $x \in \mathbb{R} \setminus \{0\}$.
- (iii) The two-step kernel measure is continuous in total variation at every $x \in \mathbb{R}$.
- (iv) The three-step kernel measure is continuous in total variation at every $x \in \mathbb{RP}^1$.

Further, $\mathcal{Q}_s^3$ is a compact operator on $C(\mathbb{RP}^1)$.

Proof. We begin with (i). Fix $x \in \mathbb{R} \setminus \{0\}$ and a Borel set $A \subset \mathbb{RP}^1$. For fixed b , we substitute $w = b - \beta x$ in the integral over β , so that $\beta = (b - w)/x$, $d\beta = dw/|x|$, and the constraint $\beta > 0$ becomes $(b - w)/x > 0$. By (6.3), the integrand $\|M(b, \beta)v_x\|^{-s}$ equals $\beta^{s/2}(1 + x^2)^{s/2}(1 + w^2)^{-s/2}$, and $f_{b,\beta}(x) = \iota(w)$. Using Tonelli's theorem to integrate over b last, we obtain

$$\kappa_s(x, A) = \int_{\mathbb{R}} k_x(w) \mathbf{1}_A(\iota(w)) dw,$$

which is the first claim. For the second claim, note that $\phi(t) > 0$ for $t > 0$. If $x > 0$, the integral in (6.13) is positive because $\nu((w, \infty)) > 0$, and if $x < 0$, it is positive because $\nu((-\infty, w)) > 0$; both follow from $\text{supp } \nu = \mathbb{R}$.

For (ii), let $x_n \rightarrow x$ in $\mathbb{R} \setminus \{0\}$. Since ϕ is bounded and continuous, dominated convergence gives $\int \phi((b - w)/x_n) \nu(db) \rightarrow \int \phi((b - w)/x) \nu(db)$ for every w . The prefactors in (6.13) are continuous in x , so $k_{x_n}(w) \rightarrow k_x(w)$ for every w . Further, $\int k_{x_n} = \mathcal{Q}_s \mathbf{1}(x_n) \rightarrow \mathcal{Q}_s \mathbf{1}(x) = \int k_x$ by Lemma 6.6(i). We now use Scheffé's lemma: since $(k_x - k_{x_n})^+ \leq k_x$ and $(k_x - k_{x_n})^+ \rightarrow 0$ pointwise, dominated convergence gives $\int (k_x - k_{x_n})^+ \rightarrow 0$, and then

$$\int |k_x - k_{x_n}| = 2 \int (k_x - k_{x_n})^+ - \int (k_x - k_{x_n}) \rightarrow 0.$$

Since ι is a Borel bijection with Borel inverse, the total variation norm of the difference of the image measures equals $\int |k_x - k_{x_n}|$. This proves (ii).

The remaining parts rest on the following claim.

Claim. Let $n \geq 1$ and $x_0 \in \mathbb{RP}^1$, and suppose that for $\nu \otimes \text{Exp}(K)$ -almost every (b, β) , the n -step kernel measure is continuous in total variation at $f_{b,\beta}(x_0)$. Then the $(n + 1)$ -step kernel measure is continuous in total variation at x_0 .

To prove the claim, let $x_k \rightarrow x_0$, and write $R^{(k)} = \|M(B, \beta)v_{x_k}\|^{-s}$, $R = \|M(B, \beta)v_{x_0}\|^{-s}$, $y_k = f_{B,\beta}(x_k)$, and $y = f_{B,\beta}(x_0)$. By (6.11) and (6.10), we have

$$\begin{aligned} & \|\kappa_s^{(n+1)}(x_k, \cdot) - \kappa_s^{(n+1)}(x_0, \cdot)\|_{\text{TV}} \\ & \leq p \|\mathcal{Q}_s\|^n \mathbb{E} |R^{(k)} - R| + p \mathbb{E} \left[R \|\kappa_s^{(n)}(y_k, \cdot) - \kappa_s^{(n)}(y, \cdot)\|_{\text{TV}} \right]. \end{aligned}$$

(The total variation distance is a Borel function of (y_k, y) , because it equals the supremum of $|\mathcal{Q}_s^n F(y_k) - \mathcal{Q}_s^n F(y)|$ over a countable dense subset of the unit ball of $C(\mathbb{RP}^1; \mathbb{R})$, and each $\mathcal{Q}_s^n F$ is continuous.) By continuity of the projective action, $R^{(k)} \rightarrow R$ and $y_k \rightarrow y$ almost surely. The family $(R^{(k)})$ is uniformly integrable by the argument in the proof of Lemma 6.6(i), and the first term tends to zero by Vitali's convergence theorem. In the

second term, the integrand tends to zero almost surely by hypothesis, and it is bounded by $2\|\mathcal{Q}_s\|^n R$, which is integrable. By dominated convergence, the second term tends to zero. This proves the claim.

We now prove (iii). Let $x_0 \in \mathbb{R} \setminus \{0\}$. Almost surely, $f_{B,\beta}(x_0)$ is finite (as noted in Definition 6.1(vi)) and nonzero (since $f_{b,\beta}(x_0) = 0$ only if $x_0 = \infty$). By (ii) and the claim with $n = 1$, the two-step kernel measure is continuous at x_0 . For $x_0 = 0$, we have $f_{B,\beta}(0) = 1/B$, which lies in $\mathbb{R} \setminus \{0\}$ almost surely because $\mathbb{P}(B = 0) = 0$. Again (ii) and the claim imply continuity at 0.

For (iv), let $x_0 \in \mathbb{R}$. By (iii), the two-step kernel measure is continuous at every point of $\mathbb{R}$, and $f_{B,\beta}(x_0) \in \mathbb{R}$ almost surely. The claim with $n = 2$ implies continuity of the three-step kernel measure at x_0 . For $x_0 = \infty$, we have $f_{b,\beta}(\infty) = 0$ for all (b, β) , and the two-step kernel measure is continuous at 0 by (iii). The claim with $n = 2$ implies continuity at ∞ .

Finally, we deduce compactness. By (iv) and the compactness of $\mathbb{RP}^1$, the map $x \mapsto \kappa_s^{(3)}(x, \cdot)$ is uniformly continuous in total variation. For $F \in C(\mathbb{RP}^1)$ with $\|F\|_\infty \leq 1$ and $x, x' \in \mathbb{RP}^1$, we have

$$|\mathcal{Q}_s^3 F(x) - \mathcal{Q}_s^3 F(x')| \leq \|\kappa_s^{(3)}(x, \cdot) - \kappa_s^{(3)}(x', \cdot)\|_{\text{TV}}.$$

The image of the unit ball under $\mathcal{Q}_s^3$ is therefore equicontinuous, and it is bounded by $\|\mathcal{Q}_s\|^3$. By the Arzelà–Ascoli theorem, it is relatively compact in $C(\mathbb{RP}^1)$. $\square$

Remark 6.8. The argument above does not apply to $\mathcal{Q}_s$ or $\mathcal{Q}_s^2$. By (6.12), the kernel measure at ∞ is a multiple of the point mass δ_0 , while by Lemma 6.7(i) the kernel measures at finite nonzero inputs are absolutely continuous with respect to m ; since such measures form a closed set in total variation, the one-step kernel measure is not continuous in total variation at ∞ . For the second power, (6.12) gives $\kappa_s^{(2)}(\infty, \cdot) = p \mathbb{E}[\beta^{-s/2}] \kappa_s(0, \cdot)$, and $\kappa_s(0, \cdot)$ has an atom at $1/b_0$ whenever $\nu(\{b_0\}) > 0$. A real solution always has such an atom: by Proposition 5.5, the law of D has an atom of mass e^{-p} at E , and the law of $B = gD$ has an atom at gE . On the other hand, $\kappa_s^{(2)}(x, \cdot)$ is absolutely continuous for $x \in \mathbb{R} \setminus \{0\}$, since it is a mixture of the absolutely continuous kernel measures $\kappa_s(y, \cdot)$, $y \in \mathbb{R} \setminus \{0\}$. It follows that the two-step kernel measure is not continuous in total variation at ∞ in this case. The third step from ∞ passes through 0 and then through the almost surely finite nonzero point $1/B$, where the kernel measures are absolutely continuous; this is why the third power is the first one to which the argument applies.

Lemma 6.9. *Suppose that ν satisfies (6.1), and let $s \in (0, 2)$. For every $n \geq 2$ and $x \in \mathbb{RP}^1$, the measure $\kappa_s^{(n)}(x, \cdot)$ has full support, that is, $\kappa_s^{(n)}(x, O) > 0$ for every nonempty open set $O \subset \mathbb{RP}^1$. In particular, for every $n \geq 2$ and every real-valued $F \in C(\mathbb{RP}^1)$ with $F \geq 0$ and $F \neq 0$, we have $\min_{x \in \mathbb{RP}^1} \mathcal{Q}_s^n F(x) > 0$.*

Proof. Let $O \subset \mathbb{RP}^1$ be open and nonempty. We first show that $\kappa_s(y, O) > 0$ for every $y \in \mathbb{R}$. For $y \in \mathbb{R} \setminus \{0\}$, this follows from Lemma 6.7(i), since $\iota^{-1}(O \setminus \{0\})$ is a nonempty open subset of $\mathbb{R}$ and $k_y > 0$ everywhere. For $y = 0$, the set $\{b \in \mathbb{R} \setminus \{0\} : 1/b \in O\}$ is open and nonempty (because $O \setminus \{0, \infty\}$ is open and nonempty), and it has positive ν -measure since $\text{supp } \nu = \mathbb{R}$. By (6.12), we conclude that $\kappa_s(0, O) > 0$.

Next, let $x \in \mathbb{RP}^1$. If $x \in \mathbb{R}$, then $\kappa_s(x, \cdot)$ has positive total mass and gives no mass to $\{\infty\}$ (by Lemma 6.7(i) for $x \neq 0$, and by $\mathbb{P}(B = 0) = 0$ for $x = 0$), which implies that $\kappa_s^{(2)}(x, O) = \int \kappa_s(x, dy) \kappa_s(y, O) > 0$. If $x = \infty$, then $\kappa_s^{(2)}(\infty, O) = p \mathbb{E}[\beta^{-s/2}] \kappa_s(0, O) > 0$ by (6.12). This proves the claim for $n = 2$. For $n \geq 3$, we have $\kappa_s^{(n)}(x, O) = \int \kappa_s^{(n-2)}(x, dy) \kappa_s^{(2)}(y, O) > 0$, because $\kappa_s^{(n-2)}(x, \mathbb{RP}^1) = \mathcal{Q}_s^{n-2} \mathbf{1}(x) > 0$ by Lemma 6.6(ii) and induction.

Finally, if $F \geq 0$ is continuous and not identically zero, then $\{F > 0\}$ is open and nonempty, and we have $\mathcal{Q}_s^n F(x) = \int F d\kappa_s^{(n)}(x, \cdot) > 0$ for every x . Since $\mathcal{Q}_s^n F$ is continuous and $\mathbb{RP}^1$ is compact, its minimum is positive. $\square$

The Perron–Frobenius eigenvalue

We use the following form of the stability of an isolated simple eigenvalue under small perturbations, both for analyticity in s and for continuity in the law of B . For a bounded operator T on a complex Banach space, we write $\sigma(T)$ for its spectrum.

Lemma 6.10. *Let X be a complex Banach space and let T_0 be a bounded operator on X . Suppose that $\mu_0 \in \sigma(T_0)$ is an isolated point of $\sigma(T_0)$ whose Riesz projection has rank one, and that there exists $r_0 < |\mu_0|$ such that $|w| \leq r_0$ for every $w \in \sigma(T_0) \setminus \{\mu_0\}$. Fix $\delta \in (0, (|\mu_0| - r_0)/3)$, let Γ be the circle $\{|w - \mu_0| = \delta\}$, and let $\psi \in X$ and $\ell \in X^*$ satisfy $\ell(\Pi_0 \psi) \neq 0$, where Π_0 is the Riesz projection of T_0 at μ_0 . Then there exists $\varepsilon > 0$ such that every bounded operator T with $\|T - T_0\| < \varepsilon$ has the following properties.*

- (i) *We have $\sigma(T) \subset \{|w| < r_0 + \delta\} \cup \{|w - \mu_0| < \delta\}$, and $\sigma(T) \cap \{|w - \mu_0| < \delta\}$ consists of a single point $\mu(T)$, which is an eigenvalue of T whose Riesz projection $\Pi_T = (2\pi i)^{-1} \oint_{\Gamma} (w - T)^{-1} dw$ has rank one. In particular, $\mu(T)$ is the unique point of $\sigma(T)$ of maximal modulus.*
- (ii) *We have $\ell(\Pi_T \psi) \neq 0$ and $\mu(T) = \ell(T \Pi_T \psi) / \ell(\Pi_T \psi)$. The maps $T \mapsto \Pi_T$ and $T \mapsto \mu(T)$ are continuous on $\{\|T - T_0\| < \varepsilon\}$.*
- (iii) *If $U \subset \mathbb{C}$ is open and $z \mapsto T(z)$ is a holomorphic map from U into the bounded operators on X (in operator norm) with $\|T(z) - T_0\| < \varepsilon$ for all $z \in U$, then $z \mapsto \mu(T(z))$ is holomorphic on U .*

Proof. Let $\Omega = \{w \in \mathbb{C} : |w| \geq r_0 + \delta, |w - \mu_0| \geq \delta\}$. Then Ω is closed and contained in the resolvent set of T_0 . Since the map $w \mapsto \|(w - T_0)^{-1}\|$ is continuous on the resolvent set and bounded by $(|w| - \|T_0\|)^{-1}$ for $|w| > \|T_0\|$, we have $M = \sup_{w \in \Omega} \|(w - T_0)^{-1}\| < \infty$.

Suppose that $\|T - T_0\| < 1/(2M)$. For $w \in \Omega$, we write $w - T = (w - T_0)(I - (w - T_0)^{-1}(T - T_0))$; the second factor is invertible by the Neumann series, with inverse of norm at most 2. Then Ω lies in the resolvent set of T , $\|(w - T)^{-1}\| \leq 2M$, and the resolvent identity gives

$$\|(w - T)^{-1} - (w - T_0)^{-1}\| = \|(w - T)^{-1}(T - T_0)(w - T_0)^{-1}\| \leq 2M^2\|T - T_0\|$$

for $w \in \Omega$. This proves the first inclusion in (i). Since $|\mu_0| - \delta > r_0 + 2\delta$, the circle Γ lies in Ω , and integrating the last bound over Γ gives $\|\Pi_T - \Pi_0\| \leq 2\delta M^2\|T - T_0\|$. We now choose $\varepsilon \leq 1/(2M)$ so small that $2\delta M^2\varepsilon < 1$ and $|\ell(\Pi_T\psi) - \ell(\Pi_0\psi)| < |\ell(\Pi_0\psi)|$ whenever $\|T - T_0\| < \varepsilon$. Since $\|\Pi_T - \Pi_0\| < 1$, the projection Π_T has rank one. Indeed, if f lies in the range of Π_T and $\Pi_0 f = 0$, then $\|f\| = \|(\Pi_T - \Pi_0)f\| < \|f\|$ unless $f = 0$, so Π_0 is injective on the range of Π_T ; in the same way, Π_T is injective on the range of Π_0 , which is one-dimensional (see also [82, Section I.4.6]). By the properties of Riesz projections [82, Section III.6.4], the range of Π_T is invariant under T , and the spectrum of the restriction of T to it equals $\sigma(T) \cap \{|w - \mu_0| < \delta\}$. Since the range is one-dimensional, T acts on it as multiplication by a scalar $\mu(T)$, and this scalar is the only point of $\sigma(T)$ in the disc. Every other point of $\sigma(T)$ has modulus less than $r_0 + \delta < |\mu_0| - \delta < |\mu(T)|$. This proves (i).

For (ii), we have $T\Pi_T = \mu(T)\Pi_T$, since T acts on the range of Π_T as multiplication by $\mu(T)$. Applying ℓ to $T\Pi_T\psi = \mu(T)\Pi_T\psi$ gives the formula for $\mu(T)$, and its denominator is nonzero by the choice of ε . For T and T_1 in the ball $\{\|T - T_0\| < \varepsilon\}$ and $w \in \Gamma$, both resolvents are bounded by $2M$, and the resolvent identity gives $\|(w - T)^{-1} - (w - T_1)^{-1}\| \leq 4M^2\|T - T_1\|$. Integrating over Γ , we obtain $\|\Pi_T - \Pi_{T_1}\| \leq 4\delta M^2\|T - T_1\|$. This proves continuity of Π_T , and continuity of $\mu(T)$ follows from the formula.

For (iii), fix $w \in \Gamma$. The map $z \mapsto (w - T(z))^{-1}$ is holomorphic on U , being the composition of a holomorphic map with inversion, which is holomorphic on the invertible operators; its derivative $(w - T(z))^{-1}T'(z)(w - T(z))^{-1}$ is jointly continuous in $(w, z) \in \Gamma \times U$. It follows that $z \mapsto \Pi_{T(z)}$ is holomorphic in operator norm (see also [82, Section VII.1.3]), and $z \mapsto \mu(T(z)) = \ell(T(z)\Pi_{T(z)}\psi)/\ell(\Pi_{T(z)}\psi)$ is a quotient of holomorphic functions with nonvanishing denominator. $\square$

We now state and prove the main result of this section.

Theorem 6.11. *Suppose that v satisfies (6.1), and let $s \in (0, 2)$. Let $\lambda(s)$ be the spectral radius of $\mathcal{Q}_s$ on $C(\mathbb{RP}^1)$.*

- (i) *The operator $\mathcal{Q}_s$ is bounded and positive on $C(\mathbb{RP}^1)$. For every $n \geq 2$, the operator $\mathcal{Q}_s^n$ is strongly positive: $\min_{x \in \mathbb{RP}^1} \mathcal{Q}_s^n F(x) > 0$ for every real-valued $F \in C(\mathbb{RP}^1)$ with $F \geq 0$ and $F \neq 0$. Further, $\mathcal{Q}_s^3$ is compact.*

- (ii) We have $\lambda(s) > 0$, and $\lambda(s)$ is an eigenvalue of $\mathcal{Q}_s$ whose Riesz projection has rank one. There exists a real-valued $\psi_s \in C(\mathbb{RP}^1)$ with $\min_x \psi_s(x) > 0$ and $\mathcal{Q}_s \psi_s = \lambda(s) \psi_s$, and there exists $r(s) < \lambda(s)$ such that $|z| \leq r(s)$ for every $z \in \sigma(\mathcal{Q}_s) \setminus \{\lambda(s)\}$.
- (iii) There exists a finite positive Borel measure ℓ_s on $\mathbb{RP}^1$ with full support such that $\int \psi_s \, d\ell_s = 1$, $\int \mathcal{Q}_s F \, d\ell_s = \lambda(s) \int F \, d\ell_s$ for $F \in C(\mathbb{RP}^1)$, and

$$\lim_{L \rightarrow \infty} \sup_{\|F\|_\infty \leq 1} \left\| \lambda(s)^{-L} \mathcal{Q}_s^L F - \left(\int F \, d\ell_s \right) \psi_s \right\|_\infty = 0.$$

- (iv) The function $s \mapsto \lambda(s)$ is real-analytic on $(0, 2)$.
- (v) Let $M_1, M_2, \dots$ be independent copies of $M(B, \beta)$. For every $L \geq 1$, every bounded Borel function F , and every $x \in \mathbb{RP}^1$, we have

$$\mathcal{Q}_s^L F(x) = p^L \mathbb{E} \left[\left\| M_L \cdots M_1 v_x \right\|^{-s} F((M_L \cdots M_1) \cdot x) \right]. \quad (6.14)$$

Further, there exists $C \geq 1$, depending on s, p, K , and v , such that for every $L \geq 1$,

$$C^{-1} \lambda(s)^L \leq p^L \mathbb{E} J_s(M_L \cdots M_1) \leq C \lambda(s)^L. \quad (6.15)$$

In particular, the limit below exists and

$$\log \lambda(s) = \log p + \lim_{L \rightarrow \infty} \frac{1}{L} \log \mathbb{E} J_s(M_L \cdots M_1).$$

Proof. The proof proceeds as follows. Part (i) collects Lemmas 6.6, 6.7, and 6.9. We apply the Krein–Rutman theorem to the compact strongly positive operator $T = \mathcal{Q}_s^3$, and we transfer the conclusions to $\mathcal{Q}_s$ using the Riesz projection of T , which commutes with $\mathcal{Q}_s$. Analyticity then follows from Lemma 6.10 and the holomorphy of $z \mapsto \mathcal{Q}_z$, and (6.15) follows from two-sided bounds on ψ_s .

STEP 1. *Part (i).* Boundedness and positivity are Lemma 6.6(i) and (ii), strong positivity is Lemma 6.9, and compactness of $\mathcal{Q}_s^3$ is Lemma 6.7.

STEP 2. *The Perron–Frobenius eigenvector of $T = \mathcal{Q}_s^3$.* Let $\mathcal{K} = \{F \in C(\mathbb{RP}^1; \mathbb{R}) : F \geq 0\}$. This is a closed cone in the real Banach space $C(\mathbb{RP}^1; \mathbb{R})$, and its interior $\{F : \min F > 0\}$ is nonempty. By Step 1, the operator T is compact, and it maps $\mathcal{K} \setminus \{0\}$ into the interior of $\mathcal{K}$. By the Krein–Rutman theorem [90], in the form for compact strongly positive operators [60, Theorem 19.3], the spectral radius r_T of T is positive, and there exists ψ in the interior of $\mathcal{K}$ with $T\psi = r_T\psi$. Since $|T^n F| \leq T^n |F|$ for complex F , we have $\|T^n\| = \|T^n \mathbf{1}\|_\infty$ both on $C(\mathbb{RP}^1; \mathbb{R})$ and on $C(\mathbb{RP}^1)$, and by Gelfand’s formula, r_T is also the spectral radius of T on the complex space $C(\mathbb{RP}^1)$. The cited theorem also asserts that r_T is simple and is the only eigenvalue of modulus r_T . Since we work on the complex space $C(\mathbb{RP}^1)$ and use algebraic simplicity below, we verify these two assertions directly.

First, let $w \in C(\mathbb{RP}^1)$ be nonzero with $Tw = \mu w$ and $|\mu| = r_T$. Since $|Tw| \leq T|w|$ pointwise, we have $r_T|w| \leq T|w|$. The function $\psi/|w|$ on $\{w \neq 0\}$ tends to infinity at the boundary of this set, because $\min \psi > 0$, and it attains its infimum $t_* \in (0, \infty)$ at some x_0 . The function $\chi = \psi - t_*|w|$ is nonnegative and vanishes at x_0 , and

$$T\chi = r_T\psi - t_*T|w| \leq r_T\psi - t_*r_T|w| = r_T\chi.$$

If $\chi \neq 0$, then $T\chi(x_0) > 0$ by strong positivity, contradicting $T\chi(x_0) \leq r_T\chi(x_0) = 0$. We conclude that $|w| = \psi/t_*$, and then $T|w| = r_T|w| = |Tw|$. Fix $x \in \mathbb{RP}^1$. Writing $T|w|(x) = \int |w| d\kappa_s^{(3)}(x, \cdot)$ and $|Tw(x)| = |\int w d\kappa_s^{(3)}(x, \cdot)|$, equality in the triangle inequality for integrals shows that $w = c|w|$ holds $\kappa_s^{(3)}(x, \cdot)$ -almost everywhere for some constant c with $|c| = 1$. Since both sides are continuous and $\kappa_s^{(3)}(x, \cdot)$ has full support by Lemma 6.9, we have $w = c|w| = (c/t_*)\psi$ everywhere. We conclude that every eigenvector for an eigenvalue of modulus r_T is a multiple of ψ , and that r_T is the only eigenvalue of modulus r_T .

Second, we show that r_T is algebraically simple. Since T is compact, r_T is an isolated point of $\sigma(T)$ and its Riesz projection Π has finite rank, equal to the dimension of the generalized eigenspace $\bigcup_k \ker(T - r_T)^k$ [82, Theorem III.6.26]. If this dimension were larger than one, then, since $\ker(T - r_T)$ is spanned by ψ , there would exist w with $(T - r_T)w = \psi$, and taking real parts we could assume that w is real. By induction, $T^L w = r_T^L w + Lr_T^{L-1}\psi$ for $L \geq 1$. Let $a = \|w\|_\infty / \min \psi$, so that $|w| \leq a\psi$. Positivity gives $|T^L w| \leq T^L|w| \leq aT^L\psi = ar_T^L\psi$, and therefore

$$Lr_T^{L-1}\psi \leq |T^L w| + r_T^L|w| \leq 2ar_T^L\psi.$$

This gives $L \leq 2ar_T$ for every L , which is impossible. This contradiction shows that Π has rank one, and $\Pi F = \ell'(F)\psi$ for some $\ell' \in C(\mathbb{RP}^1)^*$ with $\ell'(\psi) = 1$.

Finally, by [82, Theorem III.6.26], the nonzero points of $\sigma(T)$ are eigenvalues and can accumulate only at 0. By the first assertion above, every point of $\sigma(T) \setminus \{r_T\}$ has modulus less than r_T , and only finitely many of these points have modulus at least $r_T/2$. Then there exists $r' < r_T$ with $|z| \leq r'$ for every $z \in \sigma(T) \setminus \{r_T\}$.

STEP 3. Transfer to $\mathcal{Q}_s$; part (ii). Since $\mathcal{Q}_s$ commutes with T , it commutes with the resolvent of T and with Π . Then $T(\mathcal{Q}_s\psi) = \mathcal{Q}_sT\psi = r_T\mathcal{Q}_s\psi$, and since $\ker(T - r_T)$ is spanned by ψ , we have $\mathcal{Q}_s\psi = \lambda\psi$ for some scalar λ . Since $\mathcal{Q}_s\psi \geq (\min \psi)\mathcal{Q}_s\mathbf{1}$ and $\mathcal{Q}_s\mathbf{1} > 0$, we have $\lambda > 0$, and $\lambda^3 = r_T$. Let $X_1 = \ker \Pi$. The decomposition $C(\mathbb{RP}^1) = \text{Ran } \Pi \oplus X_1$ is invariant under $\mathcal{Q}_s$, and $\mathcal{Q}_s$ acts on $\text{Ran } \Pi = \text{span}\{\psi\}$ as multiplication by λ . By the properties of Riesz projections [82, Section III.6.4], we have $\sigma(T|_{X_1}) = \sigma(T) \setminus \{r_T\} \subset \{|z| \leq r'\}$. By the spectral mapping theorem for polynomials, $\sigma(T|_{X_1}) = \{z^3 : z \in \sigma(\mathcal{Q}_s|_{X_1})\}$, and we obtain $\sigma(\mathcal{Q}_s|_{X_1}) \subset \{|z| \leq (r')^{1/3}\}$. We set $r(s) = (r')^{1/3} < \lambda$. Then $\sigma(\mathcal{Q}_s) = \{\lambda\} \cup \sigma(\mathcal{Q}_s|_{X_1})$, the point λ is isolated in $\sigma(\mathcal{Q}_s)$, and its Riesz projection is the

projection onto $\text{Ran } \Pi$ along X_1 , which is Π ; this has rank one. The spectral radius of $\mathcal{Q}_s$ is therefore λ , that is, $\lambda = \lambda(s)$. Setting $\psi_s = \psi$ proves (ii).

STEP 4. *Part (iii).* Let $N = \mathcal{Q}_s(I - \Pi)$. Then $\mathcal{Q}_s = \lambda(s)\Pi + N$ and $\Pi N = N\Pi = 0$, which give $\mathcal{Q}_s^L = \lambda(s)^L\Pi + N^L$ for $L \geq 1$. The spectral radius of N equals that of $\mathcal{Q}_s|_{X_1}$, which is at most $r(s)$. By Gelfand's formula, we have $\|N^L\|^{1/L} \rightarrow r(N) \leq r(s) < \lambda(s)$. This gives $\|\lambda(s)^{-L}\mathcal{Q}_s^L - \Pi\| = \|(N/\lambda(s))^L\| \rightarrow 0$. In particular, Π is positive, being a norm limit of positive operators. For real $F \geq 0$, we have $\ell'(F)\psi = \Pi F \geq 0$. Then ℓ' is a positive linear functional, and by the Riesz–Markov theorem it is given by integration against a finite positive Borel measure ℓ_s . We have $\int \psi_s d\ell_s = \ell'(\psi) = 1$, and $\Pi\mathcal{Q}_s = \lambda(s)\Pi$ gives $\int \mathcal{Q}_s F d\ell_s = \lambda(s) \int F d\ell_s$. For full support, let $F \geq 0$ be real, continuous, and nonzero. By Step 1, there exists $c > 0$ with $\mathcal{Q}_s^3 F \geq c\psi_s$, and positivity of Π gives

$$\lambda(s)^3 \left(\int F d\ell_s \right) \psi_s = \Pi \mathcal{Q}_s^3 F \geq c \Pi \psi_s = c \psi_s.$$

Since $\psi_s > 0$, we have $\int F d\ell_s > 0$. Applying this to a nonzero continuous $F \geq 0$ supported in a given nonempty open set shows that ℓ_s has full support.

STEP 5. *Part (iv).* Fix $s_0 \in (0, 2)$. By Step 3 and Step 4, the operator $T_0 = \mathcal{Q}_{s_0}$ satisfies the hypotheses of Lemma 6.10 with $\mu_0 = \lambda(s_0)$, $r_0 = r(s_0)$, $\psi = \psi_{s_0}$, and $\ell = \ell_{s_0}$; here $\ell(\Pi_0\psi) = \int \psi_{s_0} d\ell_{s_0} = 1$, since the Riesz projection Π_0 of $\mathcal{Q}_{s_0}$ at $\lambda(s_0)$ is $F \mapsto (\int F d\ell_{s_0})\psi_{s_0}$ by Steps 3 and 4. Let $\varepsilon > 0$ be as in that lemma. By Lemma 6.6(iii), $z \mapsto \mathcal{Q}_z$ is holomorphic, in particular continuous, and there exists an open disc $U \subset \Sigma$ centered at s_0 with $\|\mathcal{Q}_z - \mathcal{Q}_{s_0}\| < \varepsilon$ for $z \in U$. By Lemma 6.10(iii), $z \mapsto \mu(\mathcal{Q}_z)$ is holomorphic on U . For real $s \in U$, the spectral radius $\lambda(s)$ belongs to $\sigma(\mathcal{Q}_s)$ by (ii) and has maximal modulus in $\sigma(\mathcal{Q}_s)$, and Lemma 6.10(i) implies that $\lambda(s) = \mu(\mathcal{Q}_s)$. Then λ coincides on $U \cap \mathbb{R}$ with a holomorphic function, which proves that λ is real-analytic near s_0 .

STEP 6. *Part (v).* We prove (6.14) by induction on L ; the case $L = 1$ is the definition of $\mathcal{Q}_s$. Suppose (6.14) holds for L , and let $M_1, \dots, M_{L+1}$ be independent copies of $M(B, \beta)$. Then $\mathcal{Q}_s^{L+1}F(x) = p \mathbb{E}[\|M_1 v_x\|^{-s} (\mathcal{Q}_s^L F)(M_1 \cdot x)]$. We apply the induction hypothesis to $\mathcal{Q}_s^L F$ with the matrices $M_2, \dots, M_{L+1}$, which are independent of M_1 , and use Fubini's theorem. Fubini's theorem applies because, by Tonelli's theorem and the same computation with $|F|$ in place of F , the absolute value of the integrand has expectation at most $p^{-L-1} \mathcal{Q}_s^{L+1}|F|(x) \leq p^{-L-1} \|\mathcal{Q}_s\|^{L+1} \|F\|_\infty$. We obtain

$$\mathcal{Q}_s^{L+1}F(x) = p^{L+1} \mathbb{E} \left[\|M_1 v_x\|^{-s} \|M_{L+1} \cdots M_2 v_{M_1 \cdot x}\|^{-s} F((M_{L+1} \cdots M_2) \cdot (M_1 \cdot x)) \right].$$

By Lemma 6.2(i), we have $\|M_{L+1} \cdots M_2 v_{M_1 \cdot x}\| \|M_1 v_x\| = \|M_{L+1} \cdots M_1 v_x\|$ and $(M_{L+1} \cdots M_2) \cdot (M_1 \cdot x) = (M_{L+1} \cdots M_1) \cdot x$. This completes the induction.

Taking $F = \mathbf{1}$ in (6.14), integrating against m , and using Tonelli's theorem, we obtain $\int \mathcal{Q}_s^L \mathbf{1} \, dm = p^L \mathbb{E} J_s(M_L \cdots M_1)$. Let $a = \min \psi_s > 0$ and $A = \max \psi_s$. Since $\mathcal{Q}_s^L$ is positive and $a \leq \psi_s \leq A$, we have

$$\frac{a}{A} \lambda(s)^L \leq \frac{1}{A} \mathcal{Q}_s^L \psi_s \leq \mathcal{Q}_s^L \mathbf{1} \leq \frac{1}{a} \mathcal{Q}_s^L \psi_s \leq \frac{A}{a} \lambda(s)^L$$

pointwise on $\mathbb{RP}^1$. Integrating against the probability measure m gives (6.15) with $C = A/a$. Taking logarithms, dividing by L , and letting $L \rightarrow \infty$ gives the last statement. $\square$

Continuity in the law of the denominator

We now show that the Perron–Frobenius eigenvalue depends continuously on the law of B and on the rate of β , under weak convergence of the laws and a uniform moment bound. Since atoms of ν produce atoms in the kernel measures of $\mathcal{Q}_s$ (see Remark 6.8), the operator $\mathcal{Q}_s$ itself is not norm-continuous in this sense in general, and we work with the third power.

Proposition 6.12. *Let $I \subset \mathbb{R}$ be an interval. For each $E \in I$, let ν_E be a Borel probability measure on $\mathbb{R}$ satisfying (6.1), and let $K(E) > 0$. Suppose that $E \mapsto K(E)$ is continuous, that $E \mapsto \nu_E$ is continuous for the topology of weak convergence, and that there exists $q_0 \in (0, 1)$ such that*

$$\sup_{E \in I'} \int |b|^{q_0} \nu_E(\mathrm{d}b) < \infty \quad \text{for every compact } I' \subset I.$$

For $E \in I$ and $s \in (0, 2)$, let $\mathcal{Q}_{E,s}$ and $\lambda_E(s)$ be the transfer operator and its Perron–Frobenius eigenvalue from Theorem 6.11, defined with the law ν_E and the rate $K(E)$. Then for every $s \in (0, 2q_0)$, the map $E \mapsto \mathcal{Q}_{E,s}^3$ is continuous from I to the bounded operators on $C(\mathbb{RP}^1)$ in operator norm, and $E \mapsto \lambda_E(s)$ is continuous on I . In particular, if the moment bound holds for some $q_0 > 1/2$, then $E \mapsto \lambda_E(1)$ is continuous on I , and if it holds for every $q_0 \in (0, 1)$, then $E \mapsto \lambda_E(s)$ is continuous on I for every $s \in (0, 2)$.

Proof. The proof repeats the total variation argument of Lemma 6.7, with the law and the input varying together, on a probability space on which the denominators converge almost surely. Fix $s \in (0, 2q_0)$, $E_0 \in I$, and a sequence $E_n \rightarrow E_0$ in I . The set $I' = \{E_0\} \cup \{E_n : n \geq 1\}$ is a compact subset of I . We write $K_n = K(E_n)$ for $n \geq 0$, and let $k_- = \inf_n K_n > 0$ and $k_+ = \sup_n K_n < \infty$. Fix $t \in (s, 2q_0)$. In the following, n ranges over $\{0, 1, 2, \dots\}$, and the index 0 refers to E_0 .

STEP 1. Coupling and uniform bounds. By the Skorokhod representation theorem [37, Theorem 6.7], there exist random variables B_n with law ν_{E_n} , defined on one probability space, such that $B_n \rightarrow B_0$ almost surely. Enlarging the space if necessary, let Y be an $\text{Exp}(1)$ random variable independent of $(B_n)_{n \geq 0}$, and set $\beta_n = Y/K_n$. Then (B_n, β_n) has law

$\nu_{E_n} \otimes \text{Exp}(K_n)$ for each n , and $\beta_n \rightarrow \beta_0$ almost surely. For $\sigma \in \{s, t\}$, by Lemma 6.5, we have

$$m_\sigma = \sup_{n \geq 0} \sup_{x \in \mathbb{RP}^1} \mathbb{E}[R_\sigma(x; B_n, \beta_n)] \leq \sup_{n \geq 0} C_{\sigma, K_n} (2 + \mathbb{E}|B_n|^{q_0}) < \infty, \quad (6.16)$$

where we used $|b|^{\sigma/2} \leq 1 + |b|^{q_0}$ (since $\sigma/2 < q_0$), the continuity of $C_{\sigma, K}$ in K on $[k_-, k_+]$, and the moment bound on I' . In particular, $\sup_n \|\mathcal{Q}_{E_n, s}\| \leq p m_s$. Further, since $R_s^{t/s} = R_t$ and $t/s > 1$, every family of random variables of the form $R_s(x_n; B_n, \beta_n)$ with $x_n \in \mathbb{RP}^1$ is bounded in $L^{t/s}$, and in particular uniformly integrable.

Let $\kappa_n^{(k)}(x, \cdot)$ denote the k -step kernel measure of $\mathcal{Q}_{E_n, s}$ at x . By (6.10), we have $\kappa_n^{(k)}(x, \mathbb{RP}^1) \leq (p m_s)^k$ for all n, k, x . We say that the k -step kernel measures are jointly continuous at $y \in \mathbb{RP}^1$ if $\|\kappa_n^{(k)}(y_n, \cdot) - \kappa_0^{(k)}(y, \cdot)\|_{\text{TV}} \rightarrow 0$ for every sequence $y_n \rightarrow y$ in $\mathbb{RP}^1$. The maps $(b, \beta, x) \mapsto \|M(b, \beta)v_x\|^{-s}$ and $(b, \beta, x) \mapsto f_{b, \beta}(x)$ are continuous on $\mathbb{R} \times (0, \infty) \times \mathbb{RP}^1$, because $M(b, \beta)$ depends continuously on (b, β) and $x \mapsto v_x$ is continuous up to sign.

STEP 2. *The one-step kernel measures are jointly continuous at every $y \in \mathbb{R} \setminus \{0\}$.* For $K > 0$, let $\phi_K(u) = u^{s/2} e^{-Ku} \mathbf{1}_{\{u > 0\}}$. The map $(K, u) \mapsto \phi_K(u)$ is continuous on $(0, \infty) \times \mathbb{R}$, and $\phi_K \leq \sup_{u > 0} u^{s/2} e^{-k_- u} < \infty$ for $K \geq k_-$. Let $y_n \rightarrow y \in \mathbb{R} \setminus \{0\}$; then $y_n \in \mathbb{R} \setminus \{0\}$ for large n . By Lemma 6.7(i), applied to ν_{E_n} and K_n , the kernel measure $\kappa_n^{(1)}(y_n, \cdot)$ is the image under ι of the measure with density

$$k_n(w) = \frac{p K_n}{|y_n|} (1 + y_n^2)^{s/2} (1 + w^2)^{-s/2} \mathbb{E} \left[\phi_{K_n} \left(\frac{B_n - w}{y_n} \right) \right].$$

For each w , the random variables $\phi_{K_n}((B_n - w)/y_n)$ converge almost surely to $\phi_{K_0}((B_0 - w)/y)$ and are uniformly bounded. By dominated convergence, $k_n(w)$ converges to the density $k_\infty(w)$ of $\kappa_0^{(1)}(y, \cdot)$. Further, $\int k_n = p \mathbb{E} R_s(y_n; B_n, \beta_n) \rightarrow p \mathbb{E} R_s(y; B_0, \beta_0) = \int k_\infty$, by almost sure convergence and uniform integrability (Step 1). Scheffé's lemma, in the form proved in Lemma 6.7(ii), gives $\int |k_n - k_\infty| \rightarrow 0$, which is the claim.

STEP 3. Propagation. We claim that if $k \geq 1$, $x \in \mathbb{RP}^1$, and the k -step kernel measures are jointly continuous at $f_{B_0, \beta_0}(x)$ almost surely, then the $(k+1)$ -step kernel measures are jointly continuous at x . Let $x_n \rightarrow x$, and write $R^{(n)} = R_s(x_n; B_n, \beta_n)$ and $y_n = f_{B_n, \beta_n}(x_n)$ for $n \geq 1$, and $R = R_s(x; B_0, \beta_0)$ and $y = f_{B_0, \beta_0}(x)$. Since (6.11) is an identity between expectations over the law of (B_n, β_n) , it holds on the coupled space, and we obtain

$$\begin{aligned} & \|\kappa_n^{(k+1)}(x_n, \cdot) - \kappa_0^{(k+1)}(x, \cdot)\|_{\text{TV}} \\ & \leq p (p m_s)^k \mathbb{E} |R^{(n)} - R| + p \mathbb{E} \left[R \|\kappa_n^{(k)}(y_n, \cdot) - \kappa_0^{(k)}(y, \cdot)\|_{\text{TV}} \right]. \end{aligned}$$

By Step 1, $R^{(n)} \rightarrow R$ and $y_n \rightarrow y$ almost surely. The first term tends to zero by uniform integrability. The integrand in the second term is measurable, by the countable supremum

used in the proof of Lemma 6.7. For almost every outcome, the hypothesis applies to the sequence $y_n \rightarrow y$, and this integrand tends to zero; it is bounded by $2(p m_s)^k R$, which is integrable. Dominated convergence completes the proof of the claim.

STEP 4. *The three-step kernel measures are jointly continuous at every $x \in \mathbb{RP}^1$.* As in the proof of Lemma 6.7, for $x \in \mathbb{R}$ the point $f_{B_0, \beta_0}(x)$ lies in $\mathbb{R} \setminus \{0\}$ almost surely. By Step 2 and Step 3 with $k = 1$, the two-step kernel measures are jointly continuous at every $x \in \mathbb{R}$. For $x \in \mathbb{R}$, the point $f_{B_0, \beta_0}(x)$ lies in $\mathbb{R}$ almost surely, and $f_{B_0, \beta_0}(\infty) = 0$. Step 3 with $k = 2$ implies the claim.

STEP 5. *Norm continuity of the third power.* We show that $\sup_{x \in \mathbb{RP}^1} \|\kappa_n^{(3)}(x, \cdot) - \kappa_0^{(3)}(x, \cdot)\|_{\text{TV}} \rightarrow 0$. If not, then there exist $\eta > 0$, a subsequence (which we do not relabel), and points x_n with $\|\kappa_n^{(3)}(x_n, \cdot) - \kappa_0^{(3)}(x_n, \cdot)\|_{\text{TV}} \geq \eta$. By compactness of $\mathbb{RP}^1$, we may pass to a further subsequence with $x_n \rightarrow x$. Steps 1–4 apply to every sequence converging to E_0 , in particular to this subsequence, and they give $\|\kappa_n^{(3)}(x_n, \cdot) - \kappa_0^{(3)}(x, \cdot)\|_{\text{TV}} \rightarrow 0$. By Lemma 6.7(iv) for ν_{E_0} , we also have $\|\kappa_0^{(3)}(x_n, \cdot) - \kappa_0^{(3)}(x, \cdot)\|_{\text{TV}} \rightarrow 0$. This contradicts the choice of x_n . Since $|\mathcal{Q}_{E_n, s}^3 F(x) - \mathcal{Q}_{E_0, s}^3 F(x)| \leq \|F\|_\infty \|\kappa_n^{(3)}(x, \cdot) - \kappa_0^{(3)}(x, \cdot)\|_{\text{TV}}$, we conclude that $\|\mathcal{Q}_{E_n, s}^3 - \mathcal{Q}_{E_0, s}^3\| \rightarrow 0$.

STEP 6. *Continuity of the eigenvalue.* Let $T_n = \mathcal{Q}_{E_n, s}^3$. By Step 2 of the proof of Theorem 6.11, applied to ν_{E_0} and $K(E_0)$, the point $\mu_0 = \lambda_{E_0}(s)^3$ is isolated in $\sigma(T_0)$, its Riesz projection Π has rank one, there exists $r' < \mu_0$ with $|z| \leq r'$ for every $z \in \sigma(T_0) \setminus \{\mu_0\}$, and $\Pi\psi = \psi$ for $\psi = \psi_{E_0, s}$. Let $\ell \in C(\mathbb{RP}^1)^*$ be the functional with $\Pi F = \ell(F)\psi$, so that $\ell(\Pi\psi) = 1$. Fix $\delta \in (0, (\mu_0 - r')/3)$ and let ε be as in Lemma 6.10. By Step 5, $\|T_n - T_0\| < \varepsilon$ for all large n . For such n , Lemma 6.10(i) shows that $\sigma(T_n)$ has a unique point of maximal modulus, and that this point lies within distance δ of μ_0 . By Theorem 6.11(ii) for ν_{E_n} and the spectral mapping theorem, the spectral radius of T_n is $\lambda_{E_n}(s)^3$ and belongs to $\sigma(T_n)$; it is therefore this point, and $|\lambda_{E_n}(s)^3 - \lambda_{E_0}(s)^3| < \delta$ for all large n . Since δ can be taken arbitrarily small, $\lambda_{E_n}(s) \rightarrow \lambda_{E_0}(s)$. This completes the proof. $\square$

6.3 The free energy

Independence of the scaling and the definition of the free energy

The transfer matrices of a real solution depend on the choice of scaling $g > 0$. The next lemma shows that different scalings give similar transfer operators.

Lemma 6.13 (Change of scaling). *Let $E > 0$ and let π be a real solution at energy E . For $g > 0$, let ν_g be the law of gD , where D has law π , and let $K_g = p/g^2$.*

(i) *For every $g > 0$, the law ν_g satisfies (6.1), and $\int |b|^{-q} \nu_g(\mathrm{d}b) < \infty$ for every $q \in (0, 1)$.*

- (ii) Let $s \in (0, 2)$, and for $g > 0$ let $\mathcal{Q}_s^{(g)}$ be the transfer operator defined with ν_g and K_g . For $g, g' > 0$, set $S = \text{diag}(c^{-1/2}, c^{1/2})$ with $c = g'/g$, and define $(\Theta F)(y) = \|Sv_y\|^{-s} F(S \cdot y)$. Then Θ is a bounded invertible operator on $C(\mathbb{RP}^1)$, both Θ and Θ^{-1} are positive, and

$$\mathcal{Q}_s^{(g')} = \Theta^{-1} \mathcal{Q}_s^{(g)} \Theta.$$

In particular, the spectral radius of $\mathcal{Q}_s^{(g)}$ does not depend on g .

Proof. For (i), Proposition 5.5 states that π has full support, that its only atom is at $E \neq 0$, and that $\mathbb{E}|D|^{\pm q} < \infty$ for $q \in (0, 1)$; these properties are preserved by the scaling $D \mapsto gD$.

For (ii), let $Y \sim \text{Exp}(1)$ be independent of D and set $\alpha = Y/p$. Then $(gD, g^2\alpha)$ has law $\nu_g \otimes \text{Exp}(K_g)$, and $\mathcal{Q}_s^{(g)}$ is defined by the random matrix $M = M(gD, g^2\alpha)$, and similarly $\mathcal{Q}_s^{(g')}$ is defined by $\tilde{M} = M(g'D, g'^2\alpha)$. A direct computation from (6.2) shows that for every $d \in \mathbb{R}$ and $a, \gamma > 0$, we have

$$M(\gamma d, \gamma^2 a) = S_\gamma M(d, a) S_\gamma^{-1}, \quad S_\gamma = \text{diag}(\gamma^{-1/2}, \gamma^{1/2}). \quad (6.17)$$

Indeed, the $(1, 2)$, $(2, 1)$, and $(2, 2)$ entries of both sides are $\gamma^{-1}a^{-1/2}$, $-\gamma a^{1/2}$, and $a^{-1/2}d$, respectively, and the $(1, 1)$ entries vanish. Since $S_{g'} = S_c S_g$, (6.17) gives $\tilde{M} = S M S^{-1}$ with $S = S_c$.

The map $y \mapsto S \cdot y$ is a homeomorphism of $\mathbb{RP}^1$ and $y \mapsto \|Sv_y\|^{-s}$ is continuous and positive. Then Θ is bounded and positive, with positive inverse $(\Theta^{-1}G)(x) = \|Sv_{S^{-1} \cdot x}\|^s G(S^{-1} \cdot x)$. Fix $y \in \mathbb{RP}^1$ and a realization of M . By Lemma 6.2(i), we have $\tilde{M} \cdot (S \cdot y) = S \cdot (M \cdot y)$ and

$$\|\tilde{M}v_{S \cdot y}\| = \|SMv_y\| \|S^{-1}v_{S \cdot y}\| = \|Sv_{M \cdot y}\| \|Mv_y\| \|Sv_y\|^{-1},$$

where in the last step we used $\|S^{-1}v_{S \cdot y}\| \|Sv_y\| = \|S^{-1}Sv_y\| = 1$. For $F \in C(\mathbb{RP}^1)$, we obtain

$$(\mathcal{Q}_s^{(g')} F)(S \cdot y) = \|Sv_y\|^s p \mathbb{E} \left[\|Mv_y\|^{-s} \|Sv_{M \cdot y}\|^{-s} F(S \cdot (M \cdot y)) \right] = \|Sv_y\|^s (\mathcal{Q}_s^{(g)} \Theta F)(y).$$

Multiplying by $\|Sv_y\|^{-s}$, we find $\Theta \mathcal{Q}_s^{(g')} F = \mathcal{Q}_s^{(g)} \Theta F$, which is the claimed identity. Since similar operators have the same spectrum, this implies the last statement. $\square$

Definition 6.14. Suppose that ν satisfies (6.1), and let $\lambda(s)$ be the Perron–Frobenius eigenvalue of $\mathcal{Q}_s$ from Theorem 6.11.

- (i) The free energy is $\varphi(s) = \log \lambda(s)$ for $s \in (0, 2)$. The name follows [4, 12], where the same quantity, up to normalization, is called the free energy.
- (ii) Let $M_1, M_2, \dots$ be independent copies of $M(B, \beta)$. For $s \in (0, 2)$ and $L \geq 0$, we set $f_L(s) = \mathbb{E} \|M_L \cdots M_1\|^{-s}$, with $f_0(s) = 1$.

- (iii) Let π be a real solution at energy $E > 0$. The free energy of π is $\varphi_\pi(s) = \log \lambda(s)$, where $\lambda(s)$ is computed with the law ν_g and the rate $K_g = p/g^2$ of Lemma 6.13 for some $g > 0$; by Lemma 6.13(ii), it does not depend on g . When the real solution is determined by the energy, we also write $\varphi_E(s)$. We define $f_L(s)$ for π as in (ii), with a fixed scaling.

Properties of the free energy

Proposition 6.15. *Suppose that ν satisfies (6.1), and let φ be its free energy.*

- (i) *The function φ is real-analytic and convex on $(0, 2)$, and $\varphi(s) = \varphi(2 - s)$ for $s \in (0, 2)$.*
- (ii) *We have $\varphi'(1) = 0$, and φ is nonincreasing on $(0, 1]$ and nondecreasing on $[1, 2)$.*
- (iii) *We have $0 \leq \mathbb{E} \log \|M(B, \beta)\| < \infty$, and for every $s \in (0, 2)$,*

$$\log p - s \mathbb{E} \log \|M(B, \beta)\| \leq \varphi(s) \leq \log p.$$

In particular, $\lim_{s \downarrow 0} \varphi(s) = \log p$.

- (iv) *We have $\varphi(1) < \log p$.*

- (v) *We have $\varphi'(s) < 0$ for every $s \in (0, 1)$. In particular, φ is strictly decreasing on $(0, 1]$, and $\varphi(s) < \log p$ for every $s \in (0, 2)$.*

If π is a real solution at energy $E > 0$, then φ_π is independent of the scaling, and (i)–(v) hold for φ_π .

Proof. Throughout the proof, $M_1, M_2, \dots$ are independent copies of $M = M(B, \beta)$, and we abbreviate $W_L = M_L \cdots M_1$ and $j_L(s) = \mathbb{E} J_s(W_L)$. By Theorem 6.11(v), we have

$$\varphi(s) = \log p + \lim_{L \rightarrow \infty} \frac{1}{L} \log j_L(s), \quad s \in (0, 2). \quad (6.18)$$

Proof of (i). Real-analyticity is Theorem 6.11(iv). By Lemma 6.3, we have $J_s(W_L) = J_{2-s}(W_L)$ for every realization. Then $j_L(s) = j_L(2 - s)$, and (6.18) implies that $\varphi(s) = \varphi(2 - s)$. For convexity, fix $L, s_0, s_1 \in (0, 2)$, and $\tau \in (0, 1)$. Applying Hölder's inequality with exponents $1/(1 - \tau)$ and $1/\tau$ to the integral over $m \otimes \mathbb{P}$ of $\|W_L v_x\|^{-(1-\tau)s_0} \|W_L v_x\|^{-\tau s_1}$, we obtain $j_L((1 - \tau)s_0 + \tau s_1) \leq j_L(s_0)^{1-\tau} j_L(s_1)^\tau$. Then $s \mapsto L^{-1} \log j_L(s)$ is convex, and so is its pointwise limit $\varphi(s) - \log p$.

Proof of (ii). Differentiating $\varphi(s) = \varphi(2 - s)$ gives $\varphi'(s) = -\varphi'(2 - s)$, and taking $s = 1$ gives $\varphi'(1) = 0$. Since φ is convex and differentiable, φ' is nondecreasing. Then $\varphi' \leq 0$ on $(0, 1]$ and $\varphi' \geq 0$ on $[1, 2)$.

Proof of (iii). Since $\det M = 1$, the product of the two singular values of M is one; in particular, $\|M\| \geq 1$ and $\log \|M\| \geq 0$. Bounding the operator norm by the Hilbert–Schmidt norm, we have $\|M(b, \beta)\|^2 \leq \beta^{-1}(1 + \beta^2 + b^2)$, and therefore

$$\log \|M(b, \beta)\| \leq \frac{1}{2} |\log \beta| + \frac{1}{2} \log(1 + \beta^2) + \frac{1}{2} \log(1 + b^2).$$

The first two terms have finite expectation since β is exponential. For the third, fix $q \in (0, 1)$; since $\log y \leq (2/q)y^{q/2}$ for $y \geq 1$, we have $\log(1 + B^2) \leq (2/q)(1 + B^2)^{q/2} \leq (2/q)(1 + |B|^q)$, which is integrable by (6.1). This proves $\mathbb{E} \log \|M\| < \infty$.

By Lemma 6.3, we have $J_s(W_L) \leq 1$. Then $j_L(s) \leq 1$, which implies $\varphi(s) \leq \log p$ by (6.18). For the lower bound, by Jensen's inequality for the probability measure m , we have

$$J_s(A) = \int \exp(-s \log \|Av_x\|) m(dx) \geq \exp\left(-s \int \log \|Av_x\| m(dx)\right) \geq \|A\|^{-s}$$

for every A with determinant one. Using Jensen's inequality again, and then $\log \|W_L\| \leq \sum_{j=1}^L \log \|M_j\|$, we obtain

$$j_L(s) \geq \mathbb{E} \exp(-s \log \|W_L\|) \geq \exp(-s \mathbb{E} \log \|W_L\|) \geq \exp(-sL \mathbb{E} \log \|M\|).$$

Inserting this into (6.18) gives the lower bound. Letting $s \downarrow 0$ gives $\lim_{s \downarrow 0} \varphi(s) = \log p$.

Proof of (iv). We work in the real Hilbert space $L^2(m)$ of real-valued functions, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|_2$. For a real 2×2 matrix A with determinant one, define $(U_A F)(x) = \|Av_x\|^{-1} F(A \cdot x)$. By (6.5) applied to $|F|^2$, U_A is an isometry of $L^2(m)$. By (6.4), we have $U_A U_{A'} = U_{A'A}$; indeed,

$$(U_A U_{A'} F)(x) = \|Av_x\|^{-1} \|A'v_{A \cdot x}\|^{-1} F(A' \cdot (A \cdot x)) = \|A'Av_x\|^{-1} F((A'A) \cdot x).$$

Taking $A' = A^{-1}$ and then $A = A'^{-1}$, we have $U_{A^{-1}} U_A = U_A U_{A^{-1}} = U_I = I$, and U_A is unitary with $U_A^{-1} = U_{A^{-1}}$.

Let $\psi = \psi_1$ be the eigenfunction from Theorem 6.11(ii) with $s = 1$; it is continuous, real-valued, and bounded below by a positive constant. By definition, $\mathcal{Q}_1 \psi(x) = p \mathbb{E}[(U_M \psi)(x)]$. By Tonelli's theorem (all integrands are nonnegative), the Cauchy-Schwarz inequality, and unitarity, we have

$$\lambda(1) \|\psi\|_2^2 = \langle \psi, \mathcal{Q}_1 \psi \rangle = p \mathbb{E} \langle \psi, U_M \psi \rangle \leq p \mathbb{E} [\|\psi\|_2 \|U_M \psi\|_2] = p \|\psi\|_2^2.$$

We have shown that $\lambda(1) \leq p$. Suppose, for the sake of contradiction, that $\lambda(1) = p$. Then the display implies $\mathbb{E} [\|\psi\|_2^2 - \langle \psi, U_M \psi \rangle] = 0$. Since $\|U_M \psi\|_2 = \|\psi\|_2$, we have $\|\psi\|_2^2 - \langle \psi, U_M \psi \rangle = \|U_M \psi - \psi\|_2^2 / 2 \geq 0$, and therefore $\mathbb{E} \|U_M \psi - \psi\|_2^2 = 0$. The function $(b, \beta) \mapsto \|U_{M(b, \beta)} \psi - \psi\|_2^2$ is continuous on $\mathbb{R} \times (0, \infty)$, by dominated convergence: the integrand is continuous in (b, β) for each x , and $\|M(b, \beta)v_x\|^{-1} \leq \|M(b, \beta)^{-1}\|$ is bounded uniformly in x and in (b, β) in compact sets. This function vanishes $\nu \otimes \text{Exp}(K)$ -almost everywhere, and this measure charges every nonempty open subset of $\mathbb{R} \times (0, \infty)$ because $\text{supp } \nu = \mathbb{R}$. Since the function is continuous, it vanishes identically. In particular, $U_{M(0,1)} \psi = \psi$ and $U_{M(0,4)} \psi = \psi$.

A direct computation from (6.2) gives

$$M(0,4)^{-1}M(0,1) = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} = \Delta.$$

Then $U_\Delta = U_{M(0,1)}U_{M(0,4)^{-1}} = U_{M(0,1)}U_{M(0,4)}^{-1}$, and $U_\Delta\psi = \psi$ in $L^2(m)$. Since both sides are continuous and m has full support, we have $\psi(x) = \|\Delta v_x\|^{-1}\psi(\Delta \cdot x)$ for every x . Let μ be the finite measure $\psi^2 m$. For every Borel $F \geq 0$, (6.5) with $A = \Delta$ gives

$$\int F(\Delta \cdot x) \mu(dx) = \int F(\Delta \cdot x) \psi(\Delta \cdot x)^2 \|\Delta v_x\|^{-2} m(dx) = \int F(y) \mu(dy),$$

so μ is invariant under the map $x \mapsto \Delta \cdot x$, which is $x \mapsto x/4$ in coordinates. For $k \in \mathbb{Z}$, let $I_k = [4^k, 4^{k+1})$. The preimage of I_k under $x \mapsto x/4$ is I_{k+1} , and invariance gives $\mu(I_{k+1}) = \mu(I_k)$ for every k . The intervals I_k are disjoint with union $(0, \infty)$, and $\mu((0, \infty)) \leq \mu(\mathbb{RP}^1) < \infty$, which forces $\mu(I_k) = 0$ for every k and $\mu((0, \infty)) = 0$. This contradicts $\mu((0, \infty)) = \int_{(0, \infty)} \psi^2 dm > 0$, which holds because $\psi > 0$ and $m((0, \infty)) > 0$. We conclude that $\lambda(1) < p$, that is, $\varphi(1) < \log p$.

Proof of (v). By (ii), $\varphi' \leq 0$ on $(0, 1)$. Suppose that $\varphi'(s_0) = 0$ for some $s_0 \in (0, 1)$. Since φ' is nondecreasing and $\varphi'(1) = 0$, we have $\varphi' = 0$ on $[s_0, 1]$, and φ is constant on $[s_0, 1]$. Since φ is real-analytic on the interval $(0, 2)$, it is constant on $(0, 2)$. Then (iii) implies $\varphi(1) = \lim_{s \downarrow 0} \varphi(s) = \log p$, contradicting (iv). We conclude that $\varphi' < 0$ on $(0, 1)$, and φ is strictly decreasing on $(0, 1]$. For $s \in (0, 1]$, this implies $\varphi(s) < \varphi(s/2) \leq \log p$, and for $s \in [1, 2)$ we use $\varphi(s) = \varphi(2 - s)$.

Finally, if π is a real solution, then ν_g satisfies (6.1) for every $g > 0$ by Lemma 6.13(i), and φ_π does not depend on g by Lemma 6.13(ii). Applying (i)–(v) to ν_g , we obtain the last statement. $\square$

The symmetry $\varphi(s) = \varphi(2 - s)$ is inherited from the identity $J_s(A) = J_{2-s}(A)$ of Lemma 6.3. A symmetry of this kind for the linearized cavity equation goes back to Abou-Chacra, Anderson, and Thouless [2]; see [109] for a discussion. The proof of Proposition 6.15(iv) uses only (6.1), and in particular it does not use atoms of ν . The next lemma selects an exponent below one at which the free energy has a useful tangent line.

Lemma 6.16. *Suppose that ν satisfies (6.1) and that $\varphi(1) > 0$. Then there exists $s \in (1/2, 1)$ such that*

$$\gamma = -\varphi'(s) > 0, \quad F = \varphi(s) - (1 - s)\gamma > 0, \quad \varphi(s) < \log p.$$

In particular, this holds for the free energy φ_π of a real solution π with $0 < \varphi_\pi(1)$.

Proof. By Proposition 6.15(v), we have $\varphi'(s) < 0$ and $\varphi(s) < \log p$ for every $s \in (0, 1)$. Since φ is continuously differentiable on $(0, 2)$ and $\varphi'(1) = 0$, we have

$$\lim_{s \uparrow 1} (\varphi(s) - (1-s)(-\varphi'(s))) = \lim_{s \uparrow 1} (\varphi(s) + (1-s)\varphi'(s)) = \varphi(1) > 0.$$

Choosing $s \in (1/2, 1)$ sufficiently close to one gives all three inequalities. $\square$

6.4 Products of three independent factors

In this section, we show that the product of three independent transfer matrices has bounded inverse fractional moments of its bilinear forms, uniformly in the vectors. Here and below, a factor is a random matrix $M(B, \beta)$ in which β has an exponential law and is independent of B , as in Section 6.2. The smoothing comes from the exponential variables β_i alone; the variables B_i enter only through their moments. We then use this bound to compare the free energy with moments of norms of products.

Inverse moments of multiaffine polynomials

A multiaffine polynomial in n variables is a polynomial of the form $\mathfrak{P}(t) = \sum_{\alpha \subset \{1, \dots, n\}} c_\alpha \prod_{i \in \alpha} t_i$ with coefficients $c_\alpha \in \mathbb{C}$, that is, a polynomial of degree at most one in each variable. For $s \in (0, 1)$ and $K > 0$, let $\omega_{s,K}$ be the measure on $\mathbb{R}$ given by $\omega_{s,K}(dt) = Kt^{s/2}e^{-Kt}\mathbf{1}_{\{t>0\}} dt$. Its total mass is $\omega_{s,K}(\mathbb{R}) = K^{-s/2}\Gamma(1+s/2)$.

Lemma 6.17. *Let $s \in (0, 1)$, $K > 0$, and $n \geq 1$. There exists a constant $C_{s,K,n} < \infty$ such that for every nonzero multiaffine polynomial $\mathfrak{P}$ in n variables with coefficients c_α ,*

$$\int_{(0,\infty)^n} |\mathfrak{P}(t)|^{-s} \omega_{s,K}(dt_1) \cdots \omega_{s,K}(dt_n) \leq C_{s,K,n} \left(\max_\alpha |c_\alpha| \right)^{-s}.$$

One can take $C_{s,K,n} = C_{s,K,1}^n$, and $C_{s,K,1}$ is bounded for (s, K) in compact subsets of $(0, 1) \times (0, \infty)$.

Proof. We argue by induction on n .

STEP 1. *A translated singularity.* Let

$$A_{s,K} = \sup_{x_0 \in \mathbb{R}} \int_0^\infty |t - x_0|^{-s} \omega_{s,K}(dt).$$

Let $w_* = \sup_{t>0} Kt^{s/2}e^{-Kt} < \infty$. On $\{|t - x_0| < 1\}$, we bound the density of $\omega_{s,K}$ by w_* , and $\int_{|t-x_0|<1} |t - x_0|^{-s} dt = 2/(1-s)$. On $\{|t - x_0| \geq 1\}$, we bound $|t - x_0|^{-s}$ by one. This gives $A_{s,K} \leq 2w_*/(1-s) + K^{-s/2}\Gamma(1+s/2) < \infty$.

STEP 2. *The case $n = 1$.* Let $\mathfrak{P}(t) = at + b$ with $(a, b) \in \mathbb{C}^2 \setminus \{0\}$. Replacing $\mathfrak{P}$ by $\mathfrak{P}/\max(|a|, |b|)$ multiplies both sides of the claimed inequality by $\max(|a|, |b|)^s$, so we may assume that $\max(|a|, |b|) = 1$.

Suppose first that $|a| \geq 1/2$. Then $|at + b| = |a| |t - z|$ with $z = -b/a$, and $|t - z| \geq |t - \operatorname{Re} z|$ for real t , and integrating gives $\int |\mathfrak{P}|^{-s} d\omega_{s,K} \leq |a|^{-s} A_{s,K} \leq 2^s A_{s,K}$.

Suppose next that $|a| < 1/2$, so that $|b| = 1$. Let $R = 1/(2|a|)$, with $R = \infty$ if $a = 0$. For $0 < t \leq R$, we have $|at + b| \geq 1 - |a|t \geq 1/2$, and this range contributes at most $2^s \omega_{s,K}(\mathbb{R})$. If $a \neq 0$, then for $t > R$, we write $|at + b|^{-s} = |a|^{-s} |t - z|^{-s} \leq |a|^{-s} |t - \operatorname{Re} z|^{-s}$ and $e^{-Kt} \leq e^{-KR/2} e^{-Kt/2}$. Since $Kt^{s/2} e^{-Kt/2} dt = 2\omega_{s,K/2}(dt)$, the range $t > R$ contributes at most

$$|a|^{-s} e^{-KR/2} \cdot 2A_{s,K/2} = 2A_{s,K/2} |a|^{-s} e^{-K/(4|a|)} \leq 2A_{s,K/2} \left(\frac{4s}{eK} \right)^s,$$

where we used $\sup_{y>0} y^{-s} e^{-K/(4y)} = (4s/(eK))^s$. We conclude that the lemma holds for $n = 1$ with

$$C_{s,K,1} = \max \left\{ 2^s A_{s,K}, 2^s K^{-s/2} \Gamma(1 + s/2) + 2A_{s,K/2} \left(\frac{4s}{eK} \right)^s \right\}.$$

The bounds on $A_{s,K}$ in Step 1 are bounded for (s, K) in compact subsets of $(0, 1) \times (0, \infty)$, and hence so is $C_{s,K,1}$.

STEP 3. The induction step. Let $n \geq 2$ and suppose that the lemma holds for $n - 1$ variables with the constant $C_{s,K,1}^{n-1}$. We write $t = (t', t_n)$ with $t' \in \mathbb{R}^{n-1}$. Every multiaffine polynomial $\mathfrak{P}$ in n variables can be written uniquely as $\mathfrak{P}(t) = t_n \mathfrak{P}_1(t') + \mathfrak{P}_0(t')$, where $\mathfrak{P}_1$ and $\mathfrak{P}_0$ are multiaffine in $n - 1$ variables; the coefficients of $\mathfrak{P}_1$ are the c_α with $n \in \alpha$, and those of $\mathfrak{P}_0$ are the c_α with $n \notin \alpha$. Let $\Omega \in \{\mathfrak{P}_1, \mathfrak{P}_0\}$ be a polynomial that has a coefficient of maximal modulus among all coefficients of $\mathfrak{P}$, so that $\Omega \neq 0$ and $\max_\alpha |c_\alpha(\Omega)| = \max_\alpha |c_\alpha(\mathfrak{P})|$. The zero set of the nonzero polynomial Ω in $\mathbb{R}^{n-1}$ has Lebesgue measure zero (its real or imaginary part is a nonzero real polynomial); in particular, it is null for $\omega_{s,K}^{\otimes(n-1)}$. For t' with $\Omega(t') \neq 0$, we have $(\mathfrak{P}_1(t'), \mathfrak{P}_0(t')) \neq (0, 0)$, and the case $n = 1$ applied to $t_n \mapsto t_n \mathfrak{P}_1(t') + \mathfrak{P}_0(t')$ gives

$$\int |\mathfrak{P}(t', t_n)|^{-s} \omega_{s,K}(dt_n) \leq C_{s,K,1} \max(|\mathfrak{P}_1(t')|, |\mathfrak{P}_0(t')|)^{-s} \leq C_{s,K,1} |\Omega(t')|^{-s}.$$

Integrating over t' , and using Tonelli's theorem and the induction hypothesis for Ω , we obtain

$$\int |\mathfrak{P}|^{-s} d\omega_{s,K}^{\otimes n} \leq C_{s,K,1} \int |\Omega|^{-s} d\omega_{s,K}^{\otimes(n-1)} \leq C_{s,K,1}^n \left(\max_\alpha |c_\alpha(\mathfrak{P})| \right)^{-s}.$$

This completes the induction. □

Three independent factors

For $b \in \mathbb{C}$ and $\beta > 0$, we define $M(b, \beta)$ by the formula (6.2); it is a complex 2×2 matrix with determinant one. For vectors $u, v \in \mathbb{C}^2$ and a matrix G , we write $u^T G v = \sum_{i,j} u_i G_{ij} v_j$ for the bilinear (not sesquilinear) pairing.

Proposition 6.18. *Let $s \in (0, 1)$, $K > 0$, and $a_+, a_- \geq 0$. Let B_1, B_2, B_3 be independent real random variables such that*

$$\mathbb{E}|B_i|^s \leq a_+, \quad \mathbb{E}|B_i|^{-s} \leq a_-, \quad i \in \{1, 2, 3\},$$

and let $\beta_1, \beta_2, \beta_3$ be independent random variables with the exponential law of rate K , independent of (B_1, B_2, B_3) . Set $M_i = M(B_i, \beta_i)$ and $G = M_3 M_2 M_1$, and let

$$C_* = 2^s C_{s,K,3} (1 + a_+)(1 + 2a_-),$$

where $C_{s,K,3}$ is the constant of Lemma 6.17. Then the following hold.

(i) *For all nonzero $u, v \in \mathbb{R}^2$, we have $\mathbb{E}|u^\top G v|^{-s} \leq C_* \|u\|^{-s} \|v\|^{-s}$.*

(ii) *For all nonzero real 2×2 matrices A and C , we have $\mathbb{E}\|C G A\|^{-s} \leq C_* \|C\|^{-s} \|A\|^{-s}$.*

The laws of B_1, B_2, B_3 need not coincide, and each may be any law with the stated moment bounds, including a law with atoms or a point mass at a nonzero constant. The same bounds hold for complex factors, in the sense of Proposition 6.19.

Proof. Both sides of (i) are homogeneous of degree $-s$ in u and in v , so we may assume that $\|u\| = \|v\| = 1$. We write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, and we let $N_i = \beta_i^{1/2} M_i = \begin{pmatrix} 0 & 1 \\ -\beta_i & B_i \end{pmatrix}$. Multiplying out, we have

$$\begin{aligned} N_2 N_1 &= \begin{pmatrix} -\beta_1 & B_1 \\ -B_2 \beta_1 & B_1 B_2 - \beta_2 \end{pmatrix}, \\ N_3 N_2 N_1 &= \begin{pmatrix} -B_2 \beta_1 & B_1 B_2 - \beta_2 \\ \beta_1 \beta_3 - B_2 B_3 \beta_1 & B_1 B_2 B_3 - B_1 \beta_3 - B_3 \beta_2 \end{pmatrix}. \end{aligned}$$

We see that $\mathfrak{P} = u^\top N_3 N_2 N_1 v$ is a multiaffine polynomial in $(\beta_1, \beta_2, \beta_3)$, with coefficients depending on B_1, B_2, B_3, u, v . Set

$$a = u_1 + B_3 u_2, \quad b = u_2.$$

Collecting monomials, we find that the coefficients of $\beta_1 \beta_3, \beta_1, \beta_2, \beta_3$, and 1 are

$$c_{13} = b v_1, \quad c_1 = -B_2 a v_1, \quad c_2 = -a v_2, \quad c_3 = -B_1 b v_2, \quad c_\emptyset = B_1 B_2 a v_2,$$

and the coefficients of $\beta_1 \beta_2, \beta_2 \beta_3$, and $\beta_1 \beta_2 \beta_3$ vanish. Let $d = \min(1, |B_1|, |B_2|)$. The four coefficients c_{13}, c_1, c_2, c_3 are the products $b v_1, a v_1, a v_2, b v_2$, multiplied by $1, -B_2, -1, -B_1$, respectively. Since $d \leq 1$, $d \leq |B_1|$, and $d \leq |B_2|$, we find

$$\begin{aligned} \max_\alpha |c_\alpha| &\geq d \max(|b v_1|, |a v_1|, |a v_2|, |b v_2|) \\ &\geq \frac{d}{2} \left(|b v_1|^2 + |a v_1|^2 + |a v_2|^2 + |b v_2|^2 \right)^{1/2} = \frac{d}{2} (|a|^2 + |b|^2)^{1/2}, \end{aligned}$$

where in the second inequality we bounded the maximum of four numbers below by half of their Euclidean norm, and in the last step we used $|v_1|^2 + |v_2|^2 = 1$. Further, $(a, b)^\top = Su$ with $S = \begin{pmatrix} 1 & B_3 \\ 0 & 1 \end{pmatrix}$, and $S^{-1} = \begin{pmatrix} 1 & -B_3 \\ 0 & 1 \end{pmatrix}$ has norm at most $1 + |B_3|$. Since $1 = \|u\| \leq \|S^{-1}\| \|Su\| \leq (1 + |B_3|) \|Su\|$, we have $(|a|^2 + |b|^2)^{1/2} \geq (1 + |B_3|)^{-1}$. We conclude that

$$\max_{\alpha} |c_{\alpha}| \geq \frac{d}{2(1 + |B_3|)}. \quad (6.19)$$

Since $u^\top Gv = (\beta_1 \beta_2 \beta_3)^{-1/2} \mathfrak{P}$, we have $|u^\top Gv|^{-s} = (\beta_1 \beta_2 \beta_3)^{s/2} |\mathfrak{P}|^{-s}$. The variables $\beta_1, \beta_2, \beta_3$ are independent of (B_1, B_2, B_3) and have joint density $\prod_i K e^{-K\beta_i}$. By Fubini's theorem, the conditional expectation given (B_1, B_2, B_3) is

$$\begin{aligned} \mathbb{E} \left[|u^\top Gv|^{-s} \mid B_1, B_2, B_3 \right] &= \int_{(0, \infty)^3} |\mathfrak{P}(\beta)|^{-s} \omega_{s,K}(\mathrm{d}\beta_1) \omega_{s,K}(\mathrm{d}\beta_2) \omega_{s,K}(\mathrm{d}\beta_3) \\ &\leq 2^s C_{s,K,3} (1 + |B_3|)^s d^{-s}, \end{aligned}$$

where we used Lemma 6.17 and (6.19). We have $(1 + |B_3|)^s \leq 1 + |B_3|^s$ and $d^{-s} \leq 1 + |B_1|^{-s} + |B_2|^{-s}$. Taking expectations and using the independence of B_3 and (B_1, B_2) , we obtain

$$\mathbb{E} |u^\top Gv|^{-s} \leq 2^s C_{s,K,3} (1 + \mathbb{E} |B_3|^s) (1 + \mathbb{E} |B_1|^{-s} + \mathbb{E} |B_2|^{-s}) \leq C_*.$$

This proves (i).

For (ii), choose a unit vector v_0 with $\|Av_0\| = \|A\|$ and a unit vector w_0 with $\|C^\top w_0\| = \|C^\top\| = \|C\|$, and set $v = Av_0/\|A\|$ and $u = C^\top w_0/\|C\|$. These are unit vectors depending only on A and C , and

$$\|CGA\| \geq |w_0^\top CGAv_0| = |(C^\top w_0)^\top G(Av_0)| = \|C\| \|A\| |u^\top Gv|.$$

Raising to the power $-s$, taking expectations, and applying (i), we obtain (ii). $\square$

The same argument applies to complex denominators, which arise when the spectral parameter has positive imaginary part.

Proposition 6.19. *Let $s \in (0, 1)$, $K > 0$, and $a_+, a_- \geq 0$. Let B_1, B_2, B_3 be independent complex random variables with $\mathbb{E}|B_i|^s \leq a_+$ and $\mathbb{E}|B_i|^{-s} \leq a_-$ for $i \in \{1, 2, 3\}$, let $\beta_1, \beta_2, \beta_3$ be as in Proposition 6.18, and set $M_i = M(B_i, \beta_i)$ and $G = M_3 M_2 M_1$. Let C_* be the constant of Proposition 6.18. Then the following hold.*

(i) *For all nonzero $u, v \in \mathbb{C}^2$, we have $\mathbb{E} |u^\top Gv|^{-s} \leq C_* \|u\|^{-s} \|v\|^{-s}$.*

(ii) *For all nonzero complex 2×2 matrices A and C , we have $\mathbb{E} \|CGA\|^{-s} \leq C_* \|C\|^{-s} \|A\|^{-s}$.*

In particular, the bounds hold with the same constant C_* for every member of a family of triples (B_1, B_2, B_3) of independent complex random variables satisfying the moment bounds with the same a_+ and a_- .

Proof. For (i), the proof of Proposition 6.18(i) applies with the following changes. The expansion of $u^\top N_3 N_2 N_1 v$ is a polynomial identity and is unchanged, and $\mathfrak{P}$ is a multiaffine polynomial in $(\beta_1, \beta_2, \beta_3)$ with the complex coefficients displayed there. The identity $|bv_1|^2 + |av_1|^2 + |av_2|^2 + |bv_2|^2 = (|a|^2 + |b|^2)(|v_1|^2 + |v_2|^2)$ holds for complex numbers, the bound $\|S^{-1}\| \leq 1 + |B_3|$ holds for the Hermitian operator norm, and Lemma 6.17 allows complex coefficients. Then (6.19) and the rest of the argument hold verbatim.

For (ii), choose unit vectors $v_0, w_0 \in \mathbb{C}^2$ with $\|Av_0\| = \|A\|$ and $\|C^*w_0\| = \|C\|$, where C^* is the conjugate transpose. Set $v = Av_0/\|A\|$, and let $u = \overline{C^*w_0}/\|C\|$, the entrywise complex conjugate of C^*w_0 divided by $\|C\|$. Then $(C^*w_0)^* = \|C\|u^\top$, and

$$\|CGA\| \geq |w_0^* CGAv_0| = |(C^*w_0)^* G(Av_0)| = \|C\| \|A\| |u^\top Gv|.$$

Part (i) completes the proof. The last statement holds because C_* depends only on s, K, a_+ , and a_- . $\square$

Remark 6.20. Proposition 6.19 applies to the denominators of the tree at the spectral parameter $E + i\eta$ with $\eta > 0$. Let $E > 0, g > 0$, and $0 < \eta \leq 1$, and let $B_\eta = gD_\eta$ with $D_\eta = -1/\Gamma_o(E + i\eta)$. By Lemma 5.17, both $\sup_{0 < \eta \leq 1} \mathbb{E}|B_\eta|^{-s} = g^{-s} \sup_{0 < \eta \leq 1} \mathbb{E}|\Gamma_o(E + i\eta)|^s$ and $\sup_{0 < \eta \leq 1} \mathbb{E}|B_\eta|^s = g^s \sup_{0 < \eta \leq 1} \mathbb{E}|D_\eta|^s$ are finite for $s \in (0, 1)$. Then the constant C_* can be chosen uniformly in $\eta \in (0, 1]$.

Corollary 6.21. *Let $s \in (0, 1)$, $K > 0$, $a_+, a_- \geq 0$, and let C_* be as in Proposition 6.18. Let G and G' be independent random matrices, each a product of three factors as in Proposition 6.19 (with possibly different laws of the B_i , all satisfying the moment bounds with a_+ and a_-).*

(i) *For all nonzero $u, v \in \mathbb{C}^2$ and every nonzero complex 2×2 matrix A , we have*

$$\mathbb{E}|u^\top GAG'v|^{-s} \leq C_*^2 \|u\|^{-s} \|A\|^{-s} \|v\|^{-s}.$$

(ii) *For all nonzero complex 2×2 matrices A, C , and F , we have*

$$\mathbb{E}\|CGFG'A\|^{-s} \leq C_*^2 \|C\|^{-s} \|F\|^{-s} \|A\|^{-s}.$$

If $\|F\| \geq 1$, which holds in particular if $|\det F| \geq 1$, the right side is at most $C_*^2 \|C\|^{-s} \|A\|^{-s}$.

(iii) *Let $\mathcal{G}$ be a σ -algebra independent of (G, G') , and let u, v, A, C, F be $\mathcal{G}$ -measurable random vectors and matrices that are almost surely nonzero. Then the bounds in (i) and (ii) hold almost surely with $\mathbb{E}$ replaced by $\mathbb{E}[\cdot | \mathcal{G}]$.*

Proof. For a nonzero complex matrix A , choose a unit vector w_0 with $\|A^*w_0\| = \|A\|$, and let $u_A = \overline{A^*w_0}/\|A\|$. As in the proof of Proposition 6.19(ii), we have

$$\|Ax\| \geq |w_0^* Ax| = \|A\| |u_A^\top x| \quad \text{for every } x \in \mathbb{C}^2. \quad (6.20)$$

For (i), we condition on G' ; since G is independent of G' , Proposition 6.19(i) applied to the vectors u and $AG'v$ gives $\mathbb{E}[|u^T GAG'v|^{-s} | G'] \leq C_* \|u\|^{-s} \|AG'v\|^{-s}$ (both sides are infinite if $AG'v = 0$). By (6.20), we have $\|AG'v\|^{-s} \leq \|A\|^{-s} |u_A^T G'v|^{-s}$, and Proposition 6.19(i) implies $\mathbb{E}|u_A^T G'v|^{-s} \leq C_* \|v\|^{-s}$. Combining these bounds proves (i).

For (ii), we condition on G' and apply Proposition 6.19(ii) to G with the matrices C and $FG'A$, which gives $\mathbb{E}[\|CGFG'A\|^{-s} | G'] \leq C_* \|C\|^{-s} \|FG'A\|^{-s}$. Applying Proposition 6.19(ii) to G' with the matrices F and A gives $\mathbb{E}\|FG'A\|^{-s} \leq C_* \|F\|^{-s} \|A\|^{-s}$. If $|\det F| \geq 1$, then $\|F\|^2 \geq |\det F| \geq 1$, since $|\det F|$ is the product of the two singular values of F .

For (iii), since (G, G') is independent of $\mathcal{G}$, the conditional expectation given $\mathcal{G}$ of a nonnegative measurable function of (G, G') and of $\mathcal{G}$ -measurable variables is obtained by integrating over (G, G') with the $\mathcal{G}$ -measurable variables held fixed [81, Chapter 6]. Applying (i) and (ii) for fixed values of u, v, A, C, F gives (iii). $\square$

Moments of norms of products

Corollary 6.22. *Let $s \in (0, 1)$. Suppose that ν satisfies (6.1) and $\int |b|^{-s} \nu(db) < \infty$, and let $f_L(s) = \mathbb{E}\|M_L \cdots M_1\|^{-s}$ be as in Definition 6.14. Then the limit $\varrho_s = \lim_{L \rightarrow \infty} f_L(s)^{1/L}$ exists, and $\varrho_s = \lambda(s)/p = e^{\varphi(s) - \log p} \in (0, 1)$. Further, let C_* be the constant of Proposition 6.18 with $a_+ = \mathbb{E}|B|^s$ and $a_- = \mathbb{E}|B|^{-s}$. Then for every $L \geq 1$,*

$$C_*^{-1} \varrho_s^{L+3} \leq f_L(s) \leq \varrho_s^L.$$

Proof. We abbreviate $f_L = f_L(s)$ and $W_L = M_L \cdots M_1$. Since $\det M_j = 1$, every product has norm at least one, and $0 < f_L \leq 1$. For $L, N \geq 0$, we have $\|W_{L+N}\| \leq \|M_{L+N} \cdots M_{L+1}\| \|W_L\|$, and the two factors on the right are independent, the first having the law of W_N . This implies that $f_{L+N} \geq f_N f_L$, that is, the sequence $(\log f_L)$ is superadditive. By Fekete's lemma, we have $\lim_{L \rightarrow \infty} L^{-1} \log f_L = \sup_{L \geq 1} L^{-1} \log f_L \in [\log f_1, 0]$. This proves that ϱ_s exists, that $\varrho_s > 0$, and that $f_L \leq \varrho_s^L$.

By Lemma 6.4, we have $f_L \leq \mathbb{E}J_s(W_L) \leq C_s f_L$, with C_s independent of L . Then Theorem 6.11(v) implies that $\log \varrho_s = \log \lambda(s) - \log p = \varphi(s) - \log p$, and $\varphi(s) < \log p$ by Proposition 6.15(v).

For the lower bound, let $L, N \geq 1$, and write $W_{L+N+3} = C'GA$ with $A = W_L$, $G = M_{L+3}M_{L+2}M_{L+1}$, and $C' = M_{L+N+3} \cdots M_{L+4}$. The matrix G is a product of three independent factors as in Proposition 6.18, with $a_+ = \mathbb{E}|B|^s$ and $a_- = \mathbb{E}|B|^{-s}$, and it is independent of (A, C') . Since G is independent of (A, C') , we may apply Proposition 6.18(ii) conditionally on (A, C') , integrating over G with A and C' held fixed (as in the proof of Corollary 6.21(iii)), and we obtain $\mathbb{E}[\|C'GA\|^{-s} | A, C'] \leq C_* \|C'\|^{-s} \|A\|^{-s}$. Taking expectations and using the independence of A and C' , where C' has the law of W_N , we obtain

$$f_{L+N+3} \leq C_* f_N f_L.$$

By induction on $k \geq 1$, this gives $f_{k(L+3)-3} \leq C_*^{k-1} f_L^k$. Since $n^{-1} \log f_n \rightarrow \log \varrho_s$, taking logarithms and dividing by k gives

$$\frac{k(L+3)-3}{k} \cdot \frac{\log f_{k(L+3)-3}}{k(L+3)-3} \leq \frac{k-1}{k} \log C_* + \log f_L.$$

Letting $k \rightarrow \infty$, we find $(L+3) \log \varrho_s \leq \log C_* + \log f_L$, which is the lower bound. $\square$

6.5 Path moments

In this section, we identify the free energy with the growth rate of the path moments $\Phi_L(s)$, which are iterates of the transfer operator $\mathcal{A}_s$ in the x -coordinate. We begin by writing the iterates of $\mathcal{A}_s$ along a chain of independent factors.

Lemma 6.23. *Suppose that ν satisfies (6.1), and let $s \in (0, 2)$ and $L \geq 1$.*

- (i) *Let $B'_0, \dots, B'_{L-1}$ have law ν and $\beta_1, \dots, \beta_L$ have the exponential law of rate K , all independent. For $x \in \mathbb{R}$, set $X_L = x$ and $X_{j-1} = f_{B'_{j-1}, \beta_j}(X_j)$ for $j = L, \dots, 1$. Then almost surely $X_j \in \mathbb{R}$ for every j , and for every Borel $F: \mathbb{R} \rightarrow [0, \infty]$,*

$$(\mathcal{A}_s^L F)(x) = p^L \mathbb{E} \left[\prod_{j=1}^L \beta_j^{s/2} |X_j|^s F(X_0) \right]. \quad (6.21)$$

- (ii) *Let π be a real solution at energy $E > 0$, and for $g > 0$ let $\mathcal{A}_s^{(g)}$ be the transfer operator in the x -coordinate defined with ν_g and $K_g = p/g^2$, and let π_g^x be the law of $1/(gD)$, where D has law π . Let $D_L, D'_0, \dots, D'_{L-1}$ have law π and $Y_1, \dots, Y_L$ have law $\text{Exp}(1)$, all independent, and set $D_{j-1} = D'_{j-1} - Y_j/(pD_j)$ for $j = L, \dots, 1$. Then almost surely $D_j \in \mathbb{R} \setminus \{0\}$ for every j , and for every $g > 0$,*

$$g^s \int \mathcal{A}_s^{(g)L} e_s \, d\pi_g^x = p^L \mathbb{E} \left[|D_0|^{-s} \prod_{j=1}^L \left(\frac{Y_j}{p} \right)^{s/2} |D_j|^{-s} \right]. \quad (6.22)$$

In particular, the left side does not depend on g .

Proof. For (i), we first show that $X_j \in \mathbb{R}$ almost surely. Suppose that $X_j \in \mathbb{R}$ almost surely. The variables (B'_{j-1}, β_j) are independent of X_j , which is a function of $B'_j, \dots, B'_{L-1}, \beta_{j+1}, \dots, \beta_L$. By Definition 6.1(vi), for every fixed $y \in \mathbb{R}$ we have $f_{B'_{j-1}, \beta_j}(y) \in \mathbb{R}$ almost surely, and Fubini's theorem implies $X_{j-1} \in \mathbb{R}$ almost surely. Formula (6.21) for $L = 1$ is the definition of $\mathcal{A}_s$. Suppose it holds for L . Then

$$(\mathcal{A}_s^{L+1} F)(x) = p \mathbb{E} \left[\beta_{L+1}^{s/2} |x|^s (\mathcal{A}_s^L F)(f_{B'_L, \beta_{L+1}}(x)) \right].$$

Inserting (6.21) for $\mathcal{A}_s^L F$, evaluated at the point $f_{B'_L, \beta_{L+1}}(x)$ with the independent variables $B'_0, \dots, B'_{L-1}, \beta_1, \dots, \beta_L$, and using Tonelli's theorem, we obtain (6.21) for $L+1$, with $X_{L+1} = x$ and $X_L = f_{B'_L, \beta_{L+1}}(x)$.

For (ii), the variables D_j are almost surely finite and nonzero by the same argument: given $D_j \in \mathbb{R} \setminus \{0\}$, the variable D_{j-1} is finite, and it vanishes only if $D'_{j-1} = Y_j/(pD_j)$, which has probability zero because Y_j has a density and is independent of (D'_{j-1}, D_j) . Set $B'_j = gD'_j$, $\beta_j = g^2 Y_j/p$, and $X_j = 1/(gD_j)$. Then B'_j has law ν_g , β_j has the exponential law of rate K_g , X_L has law π_g^x , all are independent, and

$$f_{B'_{j-1}, \beta_j}(X_j) = \left(gD'_{j-1} - \frac{g^2 Y_j}{p} \cdot \frac{1}{gD_j} \right)^{-1} = \frac{1}{gD_{j-1}} = X_{j-1}.$$

By (i) with $F = e_s$, applied conditionally on X_L and integrated against π_g^x using Tonelli's theorem, we obtain

$$\begin{aligned} g^s \int \mathcal{A}_s^{(g)L} e_s \, d\pi_g^x &= g^s p^L \mathbb{E} \left[|X_0|^s \prod_{j=1}^L \beta_j^{s/2} |X_j|^s \right] \\ &= g^s p^L \mathbb{E} \left[g^{-s} |D_0|^{-s} \prod_{j=1}^L g^s \left(\frac{Y_j}{p} \right)^{s/2} g^{-s} |D_j|^{-s} \right], \end{aligned}$$

which is (6.22). $\square$

Definition 6.24. Let π be a real solution at energy $E > 0$, let $s \in (0, 1)$, and let $L \geq 0$. The path moment is

$$\Phi_L(s) = \Phi_L(s; E) = g^s \int \mathcal{A}_s^{(g)L} e_s \, d\pi_g^x \in [0, \infty],$$

where $g > 0$, and $\mathcal{A}_s^{(g)}$ and π_g^x are as in Lemma 6.23(ii); for $L = 0$, we set $\mathcal{A}_s^0 e_s = e_s$. By Lemma 6.23(ii), $\Phi_L(s)$ does not depend on g and is given by the right side of (6.22).

Remark 6.25. We explain the name. Let π be a real solution at E , and let (D_v) be the variables of the recursive tree process of Lemma 5.13, which have law π and satisfy $D_v = E - p^{-1} \sum_u Y_u/D_u$, the sum running over the children u of v . (When the no-ac hypothesis of Proposition 5.11 holds and π is the real boundary law, the boundary denominators $-1/\Gamma_v(E + i0)$ satisfy the same recursion.) For a vertex v at generation L with path $o = v_0, \dots, v_L = v$, the formula of Proposition 3.18 with $\Gamma_{v_j} = -1/D_{v_j}$ and $|h_{v_{j-1}v_j}|^2 = Y_{v_j}/p$ gives the formal column entry

$$|\Gamma_o \prod_{j=1}^L h_{v_{j-1}v_j} \Gamma_{v_j}|^s = |D_o|^{-s} \prod_{j=1}^L \left(\frac{Y_{v_j}}{p} \right)^{s/2} |D_{v_j}|^{-s}.$$

Summing over v and applying the many-to-one formula (Proposition 4.8), the sum becomes p^L times an expectation along a single path, on which $D_{v_{j-1}} = D'_{j-1} - Y_{v_j}/(pD_{v_j})$,

where D'_{j-1} is the contribution of the other children of v_{j-1} ; since these children form an independent Poisson family, the cavity equation shows that D'_{j-1} has law π . Comparing with (6.22), the expected generation- L sum is $\Phi_L(s; E)$. This identity is stated and proved in Proposition 8.3.

The operators $\mathcal{A}_s$ and $\mathcal{Q}_s$ are related by the weight function h_s of Definition 6.1(v). For the identification, we use the following conjugacy. Here, for a Borel function $G: \mathbb{RP}^1 \rightarrow [0, \infty]$, we define $\mathcal{Q}_s^L G(x) = \int G \, d\kappa_s^{(L)}(x, \cdot) \in [0, \infty]$; for bounded G , this agrees with the operator iterate.

Lemma 6.26. *Suppose that v satisfies (6.1), and let $s \in (0, 2)$. For $x \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$, and $\beta > 0$, we have*

$$\beta^{s/2} |x|^s h_s(f_{b,\beta}(x)) = h_s(x) \|M(b, \beta)v_x\|^{-s}. \quad (6.23)$$

Further, for every Borel $G: \mathbb{RP}^1 \rightarrow [0, \infty]$, every $L \geq 1$, and every $x \in \mathbb{R} \setminus \{0\}$,

$$\mathcal{A}_s^L(h_s G)(x) = h_s(x) \mathcal{Q}_s^L G(x). \quad (6.24)$$

Proof. Let $w = b - \beta x$. If $w \neq 0$, then $f_{b,\beta}(x) = 1/w$ and $h_s(1/w) = (1 + w^2)^{-s/2}$, and (6.3) gives

$$h_s(x) \|M(b, \beta)v_x\|^{-s} = \frac{|x|^s}{(1 + x^2)^{s/2}} \cdot \frac{\beta^{s/2}(1 + x^2)^{s/2}}{(1 + w^2)^{s/2}} = \beta^{s/2} |x|^s h_s(1/w).$$

If $w = 0$, then $f_{b,\beta}(x) = \infty$, $h_s(\infty) = 1$, and both sides of (6.23) equal $\beta^{s/2} |x|^s$. This proves (6.23).

We prove (6.24) by induction on L . For $x \in \mathbb{R} \setminus \{0\}$, the point $f_{b,\beta}(x)$ lies in $\mathbb{R} \setminus \{0\}$ almost surely (it equals ∞ only on a null event, and it equals 0 only if $x = \infty$). For $L = 1$, by (6.23), we have

$$\begin{aligned} \mathcal{A}_s(h_s G)(x) &= p \mathbb{E} \left[\beta^{s/2} |x|^s h_s(f_{B,\beta}(x)) G(f_{B,\beta}(x)) \right] \\ &= h_s(x) p \mathbb{E} \left[\|M(B, \beta)v_x\|^{-s} G(f_{B,\beta}(x)) \right] = h_s(x) \mathcal{Q}_s G(x). \end{aligned}$$

Suppose that (6.24) holds for L . Since $f_{B,\beta}(x) \in \mathbb{R} \setminus \{0\}$ almost surely, we may apply the induction hypothesis at this point, and (6.23) gives

$$\mathcal{A}_s^{L+1}(h_s G)(x) = p \mathbb{E} \left[\beta^{s/2} |x|^s h_s(f_{B,\beta}(x)) (\mathcal{Q}_s^L G)(f_{B,\beta}(x)) \right] = h_s(x) \mathcal{Q}_s^{L+1} G(x).$$

In the last step, we used (6.11) to write $\mathcal{Q}_s^{L+1} G(x) = p \mathbb{E} [\|M(B, \beta)v_x\|^{-s} (\mathcal{Q}_s^L G)(f_{B,\beta}(x))]$. $\square$

The identification of the free energy with the path moments requires a fractional moment bound for the denominator perturbed by a tilted squared modulus. For $q \in (0, 1)$, let T_q be a random variable with density $t^{q/2}e^{-t}/\Gamma(1+q/2)$ on $(0, \infty)$, independent of B . We consider the bound

$$\mathbb{E}|B - \sigma a T_q|^{-q} \leq C_q \min\{1, a^{-q}\} \quad \text{for all } \sigma \in \{-1, 1\} \text{ and } a \geq 0. \quad (6.25)$$

By Proposition 5.5, if $B = gD$ with D distributed according to a real solution at energy $E > 0$, then (6.25) holds for every $q \in (0, 1)$, with a constant C_q depending on p, E, g , and q .

Proposition 6.27. *Let $s \in (0, 1)$. Suppose that v satisfies (6.1), and that there exist $q \in (s, 1)$ and $C_q < \infty$ such that (6.25) holds. Let v^x be the law of $1/B$.*

(i) *There exists $C \geq 1$, depending on s, p, K, v, q , and C_q , such that for every $L \geq 1$,*

$$C^{-1}\lambda(s)^L \leq \int \mathcal{A}_s^L e_s \, dv^x \leq C\lambda(s)^L.$$

In particular, $\int \mathcal{A}_s^L e_s \, dv^x$ is finite for every L , and $\lim_{L \rightarrow \infty} L^{-1} \log \int \mathcal{A}_s^L e_s \, dv^x = \varphi(s)$.

(ii) *We have $\mathbb{E}|B|^{-s} < \infty$, and the norm moments satisfy*

$$C_*^{-1} e^{(L+3)(\varphi(s) - \log p)} \leq f_L(s) \leq e^{L(\varphi(s) - \log p)}, \quad L \geq 1,$$

where C_* is as in Corollary 6.22. In particular, $\lim_{L \rightarrow \infty} L^{-1} \log f_L(s) = \varphi(s) - \log p$.

(iii) *Let π be a real solution at energy $E > 0$. Then the hypotheses above hold for v_g and $K_g = p/g^2$, for every $g > 0$ and every $s \in (0, 1)$. In particular, there exists $C \geq 1$, depending on s, p, E , and π , such that*

$$C^{-1} e^{L\varphi_\pi(s)} \leq \Phi_L(s; E) \leq C e^{L\varphi_\pi(s)}, \quad L \geq 1,$$

and therefore

$$\lim_{L \rightarrow \infty} \frac{1}{L} \log \Phi_L(s; E) = \varphi_\pi(s).$$

Further, $\varphi_\pi(1) = \lim_{s \uparrow 1} \lim_{L \rightarrow \infty} L^{-1} \log \Phi_L(s; E)$, and the bounds in (ii) hold for the norm moments of π with every scaling.

Proof. Since the weight function h_s vanishes at 0, the initial function $e_s = |x|^s$ of the path moment corresponds under (6.24) to the function $u_s(y) = (1 + y^2)^{s/2}$ (with $u_s(\infty) = \infty$), which is unbounded on $\mathbb{RP}^1$. We show that one application of $\mathcal{Q}_s$ turns u_s into a bounded continuous positive function, using (6.25); the uniform integrability needed for continuity comes from (6.25) at the larger exponent q . After this, the bounds follow from Theorem 6.11.

STEP 1. *A tilted moment bound.* For $r \in (0, 1)$, we have $y^{r/2} e^{-y} dy = \Gamma(1 + r/2) \mathbb{P}(T_r \in dy)$. Writing $\beta = Y/K$ with $Y \sim \text{Exp}(1)$ independent of B , we therefore have, for $x \in \mathbb{R}$,

$$\mathbb{E}[\beta^{r/2} |B - \beta x|^{-r}] = K^{-r/2} \mathbb{E}\left[Y^{r/2} \left|B - \frac{x}{K} Y\right|^{-r}\right] = K^{-r/2} \Gamma(1 + r/2) \mathbb{E}\left|B - \sigma \frac{|x|}{K} T_r\right|^{-r}, \quad (6.26)$$

where $\sigma = \text{sgn } x$ (and $\sigma = 1$ if $x = 0$). Applying (6.26) with $r = q$ and (6.25), we obtain

$$(1 + x^2)^{q/2} \mathbb{E}[\beta^{q/2} |B - \beta x|^{-q}] \leq K^{-q/2} \Gamma(1 + q/2) C_q (1 + x^2)^{q/2} \min\{1, K^q |x|^{-q}\}.$$

The right side is at most $(1 + K^2)^{q/2}$ times the prefactor, uniformly in $x \in \mathbb{R}$: for $|x| \leq K$ this is immediate, and for $|x| \geq K$ we have $(1 + x^2)^{q/2} K^q |x|^{-q} = K^q (1 + x^{-2})^{q/2} \leq (K^2 + 1)^{q/2}$. Then there exists $C_1 < \infty$, depending on q, K , and C_q , with

$$\sup_{x \in \mathbb{R}} (1 + x^2)^{q/2} \mathbb{E}[\beta^{q/2} |B - \beta x|^{-q}] \leq C_1. \quad (6.27)$$

STEP 2. *The function $\mathcal{Q}_s u_s$ is continuous and positive on $\mathbb{RP}^1$.* Let $x \in \mathbb{R}$. Almost surely $w = B - \beta x \neq 0$, and then $f_{B, \beta}(x) = 1/w$ and $u_s(1/w) = (1 + w^{-2})^{s/2} = (1 + w^2)^{s/2} |w|^{-s}$. By (6.3), we have

$$\mathcal{Q}_s u_s(x) = p \mathbb{E} Z_x, \quad Z_x = \beta^{s/2} (1 + x^2)^{s/2} |B - \beta x|^{-s}.$$

At ∞ , we have $f_{B, \beta}(\infty) = 0$ and $u_s(0) = 1$, which give $\mathcal{Q}_s u_s(\infty) = p \mathbb{E} \beta^{-s/2} = p K^{s/2} \Gamma(1 - s/2)$. Since $Z_x^{q/s} = \beta^{q/2} (1 + x^2)^{q/2} |B - \beta x|^{-q}$ and $q/s > 1$, (6.27) implies that the family $\{Z_x : x \in \mathbb{R}\}$ is bounded in $L^{q/s}$; in particular, it is uniformly integrable and bounded in L^1 . In particular, $\sup_{x \in \mathbb{R}} \mathcal{Q}_s u_s(x) < \infty$.

Let $x_n \rightarrow x_0 \in \mathbb{R}$. The event $\{B \neq \beta x_0\}$ has probability one: for $x_0 = 0$ because $\mathbb{P}(B = 0) = 0$, and for $x_0 \neq 0$ because β has a density and is independent of B . On this event, $Z_{x_n} \rightarrow Z_{x_0}$, and by Vitali's convergence theorem, we have $\mathcal{Q}_s u_s(x_n) \rightarrow \mathcal{Q}_s u_s(x_0)$. Now let $x_n \in \mathbb{R}$ with $|x_n| \rightarrow \infty$. Since $|B - \beta x_n| (1 + x_n^2)^{-1/2} \rightarrow \beta$ almost surely, we have $Z_{x_n} \rightarrow \beta^{s/2} \beta^{-s} = \beta^{-s/2}$ almost surely, and Vitali's theorem gives $\mathcal{Q}_s u_s(x_n) \rightarrow p \mathbb{E} \beta^{-s/2} = \mathcal{Q}_s u_s(\infty)$. This proves that $\mathcal{Q}_s u_s \in C(\mathbb{RP}^1)$. Since $Z_x > 0$ almost surely for every $x \in \mathbb{R}$ and $\mathbb{E} \beta^{-s/2} > 0$, the function $\mathcal{Q}_s u_s$ is strictly positive, and by compactness $c_1 = \min \mathcal{Q}_s u_s > 0$. Let $c_2 = \max \mathcal{Q}_s u_s$.

STEP 3. *Proof of (i).* Since $e_s = h_s u_s$ on $\mathbb{R}$, Lemma 6.26 with $G = u_s$ implies $\mathcal{A}_s^L e_s(x) = h_s(x) \mathcal{Q}_s^L u_s(x)$ for $x \in \mathbb{R} \setminus \{0\}$. By the Chapman–Kolmogorov relations and Tonelli's theorem, $\mathcal{Q}_s^L u_s = \mathcal{Q}_s^{L-1}(\mathcal{Q}_s u_s)$, where on the right $\mathcal{Q}_s^{L-1}$ acts on the continuous function $\mathcal{Q}_s u_s$. The law ν^x is concentrated on $\mathbb{R} \setminus \{0\}$, because B is finite and $\mathbb{P}(B = 0) = 0$. We conclude that

$$\int \mathcal{A}_s^L e_s d\nu^x = \int h_s(x) \mathcal{Q}_s^{L-1}(\mathcal{Q}_s u_s)(x) \nu^x(dx), \quad L \geq 1. \quad (6.28)$$

Let ψ_s be the eigenfunction of Theorem 6.11(ii), and let $a = \min \psi_s$ and $A = \max \psi_s$. As in Step 6 of the proof of Theorem 6.11, positivity of $\mathcal{Q}_s$ and $c_1 \leq \mathcal{Q}_s u_s \leq c_2$ give

$$\frac{c_1 a}{A} \lambda(s)^{L-1} \leq \mathcal{Q}_s^{L-1}(\mathcal{Q}_s u_s) \leq \frac{c_2 A}{a} \lambda(s)^{L-1}$$

pointwise on $\mathbb{RP}^1$. Further, since $0 < h_s \leq 1$ on $\mathbb{R} \setminus \{0\}$, we have $0 < \int h_s d\nu^x \leq 1$. Inserting these bounds into (6.28) proves (i) with $C = \max\{c_2 A / (a \lambda(s)), A \lambda(s) / (c_1 a \int h_s d\nu^x)\}$, enlarged to be at least one. The limit statement follows by taking logarithms.

STEP 4. *Proof of (ii).* Taking $a = 0$ in (6.25), we obtain $\mathbb{E}|B|^{-q} \leq C_q$ and $\mathbb{E}|B|^{-s} \leq 1 + C_q < \infty$. Then Corollary 6.22 applies, and we have the bounds with $\varrho_s = e^{\varphi(s) - \log p}$.

STEP 5. *Proof of (iii).* By Lemma 6.13(i) and Proposition 5.5, the law ν_g satisfies (6.1) and (6.25) for every $g \in (0, 1)$, and the hypotheses hold for every $s \in (0, 1)$, with any $q \in (s, 1)$. By Definition 6.24, $\Phi_L(s; E) = g^s \int \mathcal{A}_s^{(g)L} e_s d\pi_g^x$, and π_g^x is the law of $1/B$ for $B = gD$. Part (i) and $\lambda(s) = e^{\varphi_\pi(s)}$ imply the bounds on $\Phi_L(s; E)$, after multiplying C by $\max(g^s, g^{-s})$. Since φ_π is continuous at 1 by Proposition 6.15(i), we have $\varphi_\pi(1) = \lim_{s \uparrow 1} \varphi_\pi(s)$, which implies the formula for $\varphi_\pi(1)$. The last statement is (ii) applied with each scaling. $\square$

PART II

Spectral type from the sign of the free energy

Rank-one perturbations of the squared row norms

In this section, we prove two criteria for the spectral type of $\mathcal{H}$ on a set of energies of the form $I \cup (-I)$, where $I \subset (0, \infty)$. Both criteria rest on the bipartite structure recorded in Proposition 3.23. The square block A^*A is a sum of the rank-one terms $U_y|v_y\rangle\langle v_y|$ over the rows $y \in V_1$, and, conditional on all other variables, the squared row norm U_y has a Lebesgue density. Removing the term of one row leaves an operator $K_0^{(y)}$ that does not depend on U_y , and the spectral measure of v_y for A^*A is the member with coupling $u = U_y$ of the rank-one family $K_0^{(y)} + u|v_y\rangle\langle v_y|$. For Lebesgue-almost every coupling, the spectral type of such a family is determined by the boundary values of the Borel transform of the spectral measure of v_y for $K_0^{(y)}$. This is the mechanism behind the Simon–Wolff criterion [124] and spectral averaging; see [123] and [13] for expositions.

The first criterion, Proposition 7.9, deduces pure point spectrum from the finiteness of reciprocal-square integrals of spectral measures of the vertex-deleted operators; it is used in Section 8. The second, Proposition 7.11, deduces purely absolutely continuous spectrum from positivity, with positive probability, of the imaginary part of the boundary value of the root cavity resolvent; it is used in Section 10.

Section 7.1 collects facts about Borel transforms and rank-one perturbations, the main one being Lemma 7.3. Section 7.2 records the measurability of resolvent entries of the tree operators. Section 7.3 relates the row-deleted square blocks to the vertex-deleted operators. Section 7.4 and Section 7.5 prove Proposition 7.9 and Proposition 7.11, respectively.

Throughout this section, $p > 1$ is fixed. For a self-adjoint operator K on a separable Hilbert space $\mathfrak{h}$ and a vector $\varphi \in \mathfrak{h}$, we write μ_φ^K for the spectral measure of φ , so that $\mu_\varphi^K(B) = \langle \varphi, \mathbf{1}_B(K)\varphi \rangle$ for Borel sets $B \subset \mathbb{R}$. We let $\mathfrak{h}_{\text{pp}}(K)$ and $\mathfrak{h}_{\text{ac}}(K)$ denote the closed subspaces of vectors whose spectral measures are pure point and absolutely continuous, respectively. For a Borel set $B \subset \mathbb{R}$, we say that K has pure point spectrum in B if $\text{Ran } \mathbf{1}_B(K) \subset \mathfrak{h}_{\text{pp}}(K)$, and that K is purely absolutely continuous in B if $\text{Ran } \mathbf{1}_B(K) \subset \mathfrak{h}_{\text{ac}}(K)$. Since $\mu_{\mathbf{1}_B(K)\varphi}^K$ is the restriction of μ_φ^K to B , the first property holds if and only if the restriction of μ_φ^K to B is pure point for every $\varphi \in \mathfrak{h}$, and similarly for the second. We write $|B|$ for the Lebesgue measure of B , and $B^2 = \{t^2 : t \in B\}$.

7.1 Borel transforms and rank-one perturbations

For a finite positive Borel measure μ on $\mathbb{R}$, its Borel transform is

$$F_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dt)}{t - z}, \quad z \in \mathbb{H}.$$

It is holomorphic on $\mathbb{H}$, and for $\lambda \in \mathbb{R}$ and $\eta > 0$ we have

$$\operatorname{Im} F_\mu(\lambda + i\eta) = \int_{\mathbb{R}} \frac{\eta \mu(dt)}{(t - \lambda)^2 + \eta^2}. \quad (7.1)$$

If μ is not the zero measure, then $\operatorname{Im} F_\mu > 0$ on $\mathbb{H}$. We say that the vertical boundary value $F_\mu(\lambda + i0)$ exists if $\lim_{\eta \downarrow 0} F_\mu(\lambda + i\eta)$ exists in $\mathbb{C}$; in that case $F_\mu(\lambda + i0)$ denotes the limit. We also define the reciprocal-square integral

$$R_\mu(\lambda) = \int_{\mathbb{R}} \frac{\mu(dt)}{(t - \lambda)^2} \in [0, \infty], \quad (7.2)$$

with the convention that the integrand equals $+\infty$ at $t = \lambda$, so that $R_\mu(\lambda) = \infty$ whenever $\mu(\{\lambda\}) > 0$. By (7.1), the function $\eta \mapsto \eta^{-1} \operatorname{Im} F_\mu(\lambda + i\eta)$ is the integral of $((t - \lambda)^2 + \eta^2)^{-1}$ against μ , which increases as $\eta \downarrow 0$. By the monotone convergence theorem, we have

$$R_\mu(\lambda) = \lim_{\eta \downarrow 0} \frac{\operatorname{Im} F_\mu(\lambda + i\eta)}{\eta} = \sup_{n \in \mathbb{N}} n \operatorname{Im} F_\mu(\lambda + i/n). \quad (7.3)$$

Lemma 7.1. *Let μ be a finite positive Borel measure on $\mathbb{R}$, and let $\mu = \mu_{\text{ac}} + \mu_{\text{sing}}$ be its decomposition into parts that are absolutely continuous and singular with respect to Lebesgue measure.*

- (i) *For Lebesgue-almost every $\lambda \in \mathbb{R}$, the vertical boundary value $F_\mu(\lambda + i0)$ exists.*
- (ii) *The measure μ_{sing} is carried by the set of λ such that $\lim_{\eta \downarrow 0} \operatorname{Im} F_\mu(\lambda + i\eta) = \infty$. In particular, $\mu_{\text{sing}}(\mathbb{R} \setminus N_\mu) = 0$, where N_μ is the set of λ at which $F_\mu(\lambda + i0)$ does not exist.*
- (iii) *We have $\mu(\{\lambda \in \mathbb{R} : \lim_{\eta \downarrow 0} \operatorname{Im} F_\mu(\lambda + i\eta) = 0\}) = 0$.*
- (iv) *If $R_\mu(\lambda) < \infty$, then $\mu(\{\lambda\}) = 0$, the boundary value $F_\mu(\lambda + i0)$ exists, and it is real and equal to $\int_{\mathbb{R}} (t - \lambda)^{-1} \mu(dt)$.*

Proof. Part (i) is the classical theorem on the existence of finite boundary values of Borel transforms of finite measures; see [123] and [13]. (One proof applies Fatou's theorem on boundary values of bounded holomorphic functions to $(F_\mu - i)/(F_\mu + i)$; see [118, Chapter 11].)

For (ii), we first note that if $|t - \lambda| < \eta$, then $(t - \lambda)^2 + \eta^2 \leq 2\eta^2$. Restricting the integral in (7.1) to such t , we obtain the Poisson-kernel lower bound

$$\operatorname{Im} F_\mu(\lambda + i\eta) \geq \frac{\mu((\lambda - \eta, \lambda + \eta))}{2\eta}. \quad (7.4)$$

By the differentiation theorem for positive measures that are singular with respect to Lebesgue measure [118, Chapter 7], for μ_{sing} -almost every λ we have

$$\lim_{\eta \downarrow 0} \frac{\mu_{\text{sing}}((\lambda - \eta, \lambda + \eta))}{2\eta} = \infty.$$

Since $\mu \geq \mu_{\text{sing}}$, (7.4) implies that $\text{Im } F_\mu(\lambda + i\eta) \rightarrow \infty$ for μ_{sing} -almost every λ . At such λ the boundary value $F_\mu(\lambda + i0)$ does not exist, which proves (ii).

For (iii), let $\mathcal{A}$ be the set in (iii); it is a Borel set, because the limit of the continuous function $\eta \mapsto \text{Im } F_\mu(\lambda + i\eta)$ may be taken along rational η . By (ii), we have $\mu_{\text{sing}}(\mathcal{A}) = 0$. For $\lambda \in \mathcal{A}$, the bound (7.4) implies that $\lim_{\eta \downarrow 0} \mu((\lambda - \eta, \lambda + \eta)) / (2\eta) = 0$. By the differentiation theorem for measures [118, Chapter 7], at Lebesgue-almost every λ this limit exists and equals the density $f = d\mu_{\text{ac}} / d\lambda$; the singular part contributes zero Lebesgue-almost everywhere. We conclude that $f = 0$ almost everywhere on $\mathcal{A}$, and $\mu_{\text{ac}}(\mathcal{A}) = \int_{\mathcal{A}} f(\lambda) d\lambda = 0$.

For (iv), the convention in (7.2) shows that $\mu(\{\lambda\}) = 0$. For $t \neq \lambda$ and $\eta > 0$, we have $|t - \lambda - i\eta|^{-1} \leq |t - \lambda|^{-1}$, and by the Cauchy–Schwarz inequality,

$$\int_{\mathbb{R}} \frac{\mu(dt)}{|t - \lambda|} \leq \mu(\mathbb{R})^{1/2} R_\mu(\lambda)^{1/2} < \infty.$$

By the dominated convergence theorem, we have $F_\mu(\lambda + i\eta) \rightarrow \int_{\mathbb{R}} (t - \lambda)^{-1} \mu(dt) \in \mathbb{R}$ as $\eta \downarrow 0$. $\square$

The next lemma compares vertical boundary values at E^2 with limits along the curve $\eta \mapsto (E + i\eta)^2$, which is the curve that appears when the resolvent of $\mathcal{H}$ at $E + i\eta$ is expressed through resolvents of $\mathcal{H}^2$.

Lemma 7.2. *Let μ be a finite positive Borel measure on $\mathbb{R}$ and let $E > 0$. Then $\lim_{\eta \downarrow 0} F_\mu((E + i\eta)^2)$ exists in $\mathbb{C}$ if and only if $F_\mu(E^2 + i0)$ exists, and in that case the two limits coincide.*

Proof. For $\eta > 0$, set $w_\eta = (E + i\eta)^2 = E^2 - \eta^2 + 2iE\eta$ and $w'_\eta = E^2 + 2iE\eta$. Both lie in $\mathbb{H}$, they have the same imaginary part $2E\eta$, and $|w_\eta - w'_\eta| = \eta^2$. For $w \in \mathbb{H}$, we have $\int_{\mathbb{R}} |t - w|^{-2} \mu(dt) = \text{Im } F_\mu(w) / \text{Im } w$ by (7.1). Using the Cauchy–Schwarz inequality, we find

$$\begin{aligned} |F_\mu(w_\eta) - F_\mu(w'_\eta)| &= \left| \int_{\mathbb{R}} \frac{(w_\eta - w'_\eta) \mu(dt)}{(t - w_\eta)(t - w'_\eta)} \right| \leq \eta^2 \left(\int_{\mathbb{R}} \frac{\mu(dt)}{|t - w_\eta|^2} \right)^{1/2} \left(\int_{\mathbb{R}} \frac{\mu(dt)}{|t - w'_\eta|^2} \right)^{1/2} \\ &= \frac{\eta}{2E} (\text{Im } F_\mu(w_\eta) \text{Im } F_\mu(w'_\eta))^{1/2}. \end{aligned} \quad (7.5)$$

Suppose that $F_\mu(E^2 + i0)$ exists. Since $w'_\eta = E^2 + i(2E\eta)$, we have $F_\mu(w'_\eta) \rightarrow F_\mu(E^2 + i0)$, and there exist $M < \infty$ and $\eta_0 \in (0, E]$ such that $\text{Im } F_\mu(w'_\eta) \leq M$ for $\eta \leq \eta_0$.

For such η , write $a = \operatorname{Im} F_\mu(w_\eta)$. By (7.5) and the inequality $\sqrt{aM} \leq (a + M)/2$, we have $a \leq M + (a + M)/4$, which implies that $a \leq 2M$. Inserting this into (7.5), we obtain $|F_\mu(w_\eta) - F_\mu(w'_\eta)| \leq \sqrt{2M\eta}/(2E) \rightarrow 0$, and therefore $F_\mu(w_\eta) \rightarrow F_\mu(E^2 + i0)$. The converse implication follows by exchanging the roles of w_η and w'_η in the same argument. $\square$

We now turn to rank-one perturbations. Let L be a self-adjoint operator on a separable Hilbert space $\mathfrak{h}$, and let $v \in \mathfrak{h}$ with $\|v\| = 1$. For $u \in \mathbb{R}$, set

$$L_u = L + u|v\rangle\langle v|, \quad \operatorname{Dom}(L_u) = \operatorname{Dom}(L), \quad \mu_u = \mu_v^{L_u}, \quad F_u = F_{\mu_u}. \quad (7.6)$$

Each L_u is self-adjoint, since it is a bounded symmetric perturbation of L , and each μ_u is a probability measure. Let $N \subset \mathbb{R}$ be the set of λ at which $F_0(\lambda + i0)$ does not exist, and for $u \in \mathbb{R}$ set

$$Z_u = \{\lambda \in \mathbb{R} \setminus N : F_0(\lambda + i0) \in \mathbb{R}, 1 + uF_0(\lambda + i0) = 0\}. \quad (7.7)$$

Note that $Z_0 = \emptyset$.

Lemma 7.3. *In the setting of (7.6) and (7.7), the following statements hold.*

(i) *For every $u \in \mathbb{R}$ and $z \in \mathbb{H}$, we have $1 + uF_0(z) \neq 0$ and*

$$F_u(z) = \frac{F_0(z)}{1 + uF_0(z)}. \quad (7.8)$$

(ii) *The set N is a Lebesgue-null Borel set, and for every $u \in \mathbb{R}$ the singular part of μ_u is carried by $N \cup Z_u$:*

$$(\mu_u)_{\operatorname{sing}}(\mathbb{R} \setminus (N \cup Z_u)) = 0.$$

(iii) *For every Borel set $B \subset \mathbb{R}$, the map $u \mapsto \mu_u(B)$ is Borel and*

$$\int_{\mathbb{R}} \mu_u(B) \, du = |B|. \quad (7.9)$$

(iv) *Let $(\Omega', \mathcal{A})$ be a measurable space, and suppose that for each $\omega \in \Omega'$ we are given $L = L(\omega)$ and $v = v(\omega)$ as above, possibly on a Hilbert space depending on ω , such that $\omega \mapsto F_0(\omega, z)$ is $\mathcal{A}$ -measurable for every $z \in \mathbb{H}$. Write $N(\omega)$, $Z_u(\omega)$, $\mu_u(\omega)$ for the corresponding objects. Then the sets*

$$\{(\omega, \lambda) : \lambda \in N(\omega)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}), \quad \{(\omega, u, \lambda) : \lambda \in Z_u(\omega)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$$

are measurable, the function $(\omega, \lambda) \mapsto R_{\mu_0(\omega)}(\lambda)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ -measurable, and for every $T \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ the map $(\omega, u) \mapsto \mu_u(\omega)(T_\omega)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ -measurable, where $T_\omega = \{\lambda : (\omega, \lambda) \in T\}$.

The identity (7.8) is the Aronszajn–Krein formula (see [27]), and (7.9) is spectral averaging; see [123]. Part (iv) is what allows the null sets in (ii) and (iii) to be used inside conditional Fubini arguments in which L and v are random.

Proof. We prove the four parts in the order (i), (iv), (ii), (iii).

Proof of (i). Fix $z \in \mathbb{H}$. Since $\text{Ran}(L - z)^{-1} = \text{Dom}(L) = \text{Dom}(L_u)$, the second resolvent identity yields

$$(L - z)^{-1} - (L_u - z)^{-1} = (L_u - z)^{-1}(L_u - L)(L - z)^{-1} = u(L_u - z)^{-1}|v\rangle\langle v|(L - z)^{-1}.$$

Applying both sides to v and taking the inner product with v , we obtain $F_0(z) - F_u(z) = uF_u(z)F_0(z)$, that is, $F_u(z)(1 + uF_0(z)) = F_0(z)$. If $u \neq 0$ and $1 + uF_0(z) = 0$, then $F_0(z) = -1/u$ would be real, which is impossible because $\text{Im } F_0(z) > 0$. This proves (i).

Proof of (iv). We first show that $(\omega, z) \mapsto F_0(\omega, z)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{H})$ -measurable. For $n \in \mathbb{N}$, partition $\mathbb{H}$ into countably many Borel sets $Q_{n,k}$ of diameter at most $1/n$, choose $z_{n,k} \in Q_{n,k}$, and set $F^{(n)}(\omega, z) = F_0(\omega, z_{n,k})$ for $z \in Q_{n,k}$. Each $F^{(n)}$ is jointly measurable, and $F^{(n)} \rightarrow F_0$ pointwise because $F_0(\omega, \cdot)$ is continuous. By (7.8), the function $(\omega, u, z) \mapsto F_u(\omega, z)$ is then jointly measurable as well.

Let $\mathcal{C}$ be the set of (ω, λ) satisfying the rational Cauchy criterion

$$\forall k \in \mathbb{N} \exists m \in \mathbb{N} \forall \eta, \eta' \in \mathbb{Q} \cap (0, 1/m) : |F_0(\omega, \lambda + i\eta) - F_0(\omega, \lambda + i\eta')| < 1/k. \quad (7.10)$$

It is obtained from jointly measurable sets by countable unions and intersections, so $\mathcal{C} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$. If $F_0(\omega, \lambda + i0)$ exists, then (7.10) holds. Conversely, if (7.10) holds, then by continuity of $\eta \mapsto F_0(\omega, \lambda + i\eta)$ the same inequality, with $\leq$ in place of $<$, holds for all real $\eta, \eta' \in (0, 1/m)$, and completeness of $\mathbb{C}$ shows that the limit exists. It follows that $\{(\omega, \lambda) : \lambda \in N(\omega)\}$ is the complement of $\mathcal{C}$. On $\mathcal{C}$ we set $F_0^*(\omega, \lambda) = \lim_{n \rightarrow \infty} F_0(\omega, \lambda + i/n)$, and we set $F_0^* = 0$ off $\mathcal{C}$; this function is jointly measurable, as a pointwise limit of measurable functions multiplied by $1_{\mathcal{C}}$. Then

$$\{(\omega, u, \lambda) : \lambda \in Z_u(\omega)\} = \{(\omega, u, \lambda) : (\omega, \lambda) \in \mathcal{C}, \text{Im } F_0^*(\omega, \lambda) = 0, 1 + uF_0^*(\omega, \lambda) = 0\}$$

is measurable. The measurability of $R_{\mu_0(\omega)}(\lambda)$ follows from (7.3).

For the last assertion, let $P_\eta(x) = \pi^{-1}\eta/(x^2 + \eta^2)$, so that $\pi^{-1}\text{Im } F_u(\lambda + i\eta) = \int P_\eta(\lambda - t)\mu_u(dt)$ by (7.1). For a continuous function f with compact support, we have by Tonelli's theorem

$$\frac{1}{\pi} \int_{\mathbb{R}} f(\lambda) \text{Im } F_u(\lambda + i\eta) d\lambda = \int_{\mathbb{R}} (P_\eta * f)(t) \mu_u(dt).$$

Since $P_\eta * f \rightarrow f$ uniformly as $\eta \downarrow 0$ and μ_u is a probability measure, the right side converges to $\int f d\mu_u$. The left side, with $\eta = 1/n$, is a measurable function of (ω, u) by

Fubini's theorem, because its integrand is jointly measurable and bounded by $n|f(\lambda)|$. This shows that $(\omega, u) \mapsto \int f d\mu_u(\omega)$ is measurable. Approximating the indicator of an open interval from below by such f and applying the π - λ theorem, we find that $(\omega, u) \mapsto \mu_u(\omega)(B)$ is measurable for every Borel set B . Finally, the class of $T \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ for which $(\omega, u) \mapsto \mu_u(\omega)(T_\omega)$ is measurable contains the rectangles $A_0 \times B$, for which the map is $\mathbf{1}_{A_0}(\omega)\mu_u(\omega)(B)$, and it is closed under complements and increasing unions because each $\mu_u(\omega)$ is a probability measure. By the π - λ theorem it is all of $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$; see also [81, Chapter 1].

Proof of (ii). By (iv), applied with a one-point space Ω' , the set N is Borel, and it is Lebesgue-null by Lemma 7.1(i). We now fix $u \in \mathbb{R}$ and $\lambda \notin N \cup Z_u$, and write $a = F_0(\lambda + i0)$. If $1 + ua \neq 0$, then (7.8) implies that $F_u(\lambda + i\eta) \rightarrow a/(1 + ua) \in \mathbb{C}$; in particular, $\operatorname{Im} F_u(\lambda + i\eta)$ stays bounded as $\eta \downarrow 0$. If $1 + ua = 0$, then $u \neq 0$ and $a = -1/u$ is real, and $\lambda \in Z_u$, which was excluded. Hence $\mathbb{R} \setminus (N \cup Z_u)$ does not meet the set on which $\operatorname{Im} F_u(\lambda + i\eta) \rightarrow \infty$, and Lemma 7.1(ii), applied to μ_u , gives $(\mu_u)_{\text{sing}}(\mathbb{R} \setminus (N \cup Z_u)) = 0$.

Proof of (iii). The measurability of $u \mapsto \mu_u(B)$ is the case of (iv) with a one-point space Ω' . Fix $z \in \mathbb{H}$ and write $F_0(z) = a + ib$ with $b > 0$. By (7.8),

$$\operatorname{Im} F_u(z) = \frac{\operatorname{Im}(F_0(z) \overline{(1 + uF_0(z))})}{|1 + uF_0(z)|^2} = \frac{b}{(1 + ua)^2 + u^2b^2}.$$

With $c = a^2 + b^2$, we have $(1 + ua)^2 + u^2b^2 = c(u + a/c)^2 + b^2/c$, and therefore

$$\int_{\mathbb{R}} \operatorname{Im} F_u(z) du = b \int_{\mathbb{R}} \frac{du}{c(u + a/c)^2 + b^2/c} = b \cdot \frac{\pi}{b} = \pi. \quad (7.11)$$

Define the measure $\nu(B) = \int_{\mathbb{R}} \mu_u(B) du$ on Borel sets $B \subset \mathbb{R}$. For $z = x + iy \in \mathbb{H}$, Tonelli's theorem and (7.11) give

$$\int_{\mathbb{R}} \frac{y \nu(dt)}{(t - x)^2 + y^2} = \int_{\mathbb{R}} \operatorname{Im} F_u(x + iy) du = \pi = \int_{\mathbb{R}} \frac{y dt}{(t - x)^2 + y^2}. \quad (7.12)$$

The measure ν is locally finite: for $t \in [x - y, x + y]$ we have $y/((t - x)^2 + y^2) \geq 1/(2y)$, and (7.12) implies that $\nu([x - y, x + y]) \leq 2\pi y$. Taking $z = i$ in (7.12), we also obtain $\int_{\mathbb{R}} (1 + t^2)^{-1} \nu(dt) = \pi$.

It remains to show that a locally finite measure ν with $\int_{\mathbb{R}} (1 + t^2)^{-1} \nu(dt) < \infty$ is determined by its Poisson transform $(P_y * \nu)(x) = \pi^{-1} \int y((t - x)^2 + y^2)^{-1} \nu(dt)$. Let f be continuous with support in $[-r, r]$. For $0 < y \leq 1$ and $|t| \geq 2r$, we have $|(P_y * f)(t)| \leq \|f\|_{\infty} \cdot 2r \cdot \pi^{-1} y(t/2)^{-2}$, and $|(P_y * f)(t)| \leq \|f\|_{\infty}$ for all t ; hence $|(P_y * f)(t)| \leq C_f(1 + t^2)^{-1}$ for some $C_f < \infty$. By Fubini's theorem,

$$\int_{\mathbb{R}} f(x)(P_y * \nu)(x) dx = \int_{\mathbb{R}} (P_y * f)(t) \nu(dt).$$

As $y \downarrow 0$, we have $P_y * f \rightarrow f$ pointwise, and the dominated convergence theorem (with the dominating function $C_f(1+t^2)^{-1}$) shows that the right side tends to $\int f \, d\nu$. The same holds with ν replaced by Lebesgue measure. Since the two Poisson transforms agree by (7.12), we conclude that $\int f \, d\nu = \int f(t) \, dt$ for every continuous f with compact support. By the uniqueness part of the Riesz representation theorem, ν is Lebesgue measure, which is (7.9). $\square$

The following lemma gives the two spectral criteria in the form needed below. Part (i) is the criterion of Simon and Wolff [124], and part (ii) is its absolutely continuous counterpart [123]. We state both for an individual coupling u , under a condition that holds for Lebesgue-almost every u by spectral averaging; this is the form that combines with conditional Fubini arguments.

Lemma 7.4. *In the setting of (7.6) and (7.7), let $S \subset \mathbb{R}$ be a Borel set.*

- (i) *Let $T = \{\lambda \in S : R_{\mu_0}(\lambda) = \infty\}$. For every $u \in \mathbb{R}$, the restriction of μ_u to $S \setminus T$ is carried by the set of eigenvalues of L_u , which is countable. In particular, the restriction of μ_u to S is pure point for every u with $\mu_u(T) = 0$. If $|T| = 0$, this holds for Lebesgue-almost every $u \in \mathbb{R}$.*
- (ii) *Let T' be the set of $\lambda \in S$ such that $F_0(\lambda + i0)$ does not exist or $\operatorname{Im} F_0(\lambda + i0) = 0$. For every $u \in \mathbb{R}$, we have $(\mu_u)_{\text{sing}}(S \setminus T') = 0$. In particular, the restriction of μ_u to S is absolutely continuous for every u with $\mu_u(T') = 0$. If $|T'| = 0$, this holds for Lebesgue-almost every $u \in \mathbb{R}$.*

Proof. We begin with (i). Fix $u \in \mathbb{R}$ and $\lambda \in S \setminus T$. By Lemma 7.1(iv), $F_0(\lambda + i0) = a$ exists and is real. Suppose first that $1 + ua \neq 0$. Then (7.8) implies that $F_u(\lambda + i\eta) \rightarrow a/(1 + ua) \in \mathbb{R}$. Then $\operatorname{Im} F_u(\lambda + i\eta) \rightarrow 0$, and λ belongs to the set $\mathcal{A}_u = \{\lambda : \lim_{\eta \downarrow 0} \operatorname{Im} F_u(\lambda + i\eta) = 0\}$. Suppose next that $1 + ua = 0$. We claim that λ is an eigenvalue of L_u . To that end, set $\psi_\eta = (L - \lambda - i\eta)^{-1}v$ for $\eta > 0$. By the spectral theorem,

$$\|\psi_\eta - \psi_{\eta'}\|^2 = \int_{\mathbb{R}} \left| \frac{1}{t - \lambda - i\eta} - \frac{1}{t - \lambda - i\eta'} \right|^2 \mu_0(dt).$$

The integrand is bounded by $4(t - \lambda)^{-2}$, which is μ_0 -integrable because $R_{\mu_0}(\lambda) < \infty$, and it tends to zero for $t \neq \lambda$ as $\eta, \eta' \downarrow 0$; since $\mu_0(\{\lambda\}) = 0$, the dominated convergence theorem shows that (ψ_η) is Cauchy as $\eta \downarrow 0$, with a limit ψ . We have $(L - \lambda)\psi_\eta = v + i\eta\psi_\eta \rightarrow v$. Since L is closed, we obtain $\psi \in \operatorname{Dom}(L)$ and $(L - \lambda)\psi = v$. In particular $\psi \neq 0$. Moreover, $\langle v, \psi \rangle = \lim_{\eta \downarrow 0} F_0(\lambda + i\eta) = a$. Then

$$(L_u - \lambda)\psi = v + u\langle v, \psi \rangle v = (1 + ua)v = 0,$$

which proves the claim. We conclude that $S \setminus T \subset \mathcal{A}_u \cup \sigma_p(L_u)$, where $\sigma_p(L_u)$ is the set of eigenvalues of L_u . By Lemma 7.1(iii), $\mu_u(\mathcal{A}_u) = 0$. This implies that the restriction of

μ_u to $S \setminus T$ is carried by $\sigma_p(L_u)$. This set is countable, because eigenvectors of distinct eigenvalues are orthogonal and $\mathfrak{h}$ is separable. If moreover $\mu_u(T) = 0$, the restriction of μ_u to S is carried by a countable set, that is, it is pure point. Finally, if $|T| = 0$, then (7.9) gives $\int_{\mathbb{R}} \mu_u(T) du = |T| = 0$, and $\mu_u(T) = 0$ for Lebesgue-almost every u .

For (ii), let $\lambda \in S \setminus T'$. Then $\lambda \notin N$ and $\operatorname{Im} F_0(\lambda + i0) > 0$, and $\lambda \notin Z_u$ for every u . By Lemma 7.3(ii), we have $(\mu_u)_{\text{sing}}(S \setminus T') = 0$. If $\mu_u(T') = 0$, then $(\mu_u)_{\text{sing}}(S) \leq \mu_u(T') = 0$. The last assertion follows from (7.9) as in (i). $\square$

7.2 Measurability of resolvent entries

We use the realization of the marked tree in Definition 3.2: vertices are words in the Ulam–Harris set $\mathbb{U}$, so that every vertex is a deterministic word, and the marked tree is built from the independent families $(N_u)_{u \in \mathbb{U}}$ and $(w_u)_{u \in \mathbb{U}_*}$. A quantity indexed by words that are not vertices of the relevant graph is set equal to zero; with this convention, events such as $\{x \in V\}$ and $\{x \sim y\}$ are measurable. For a vertex y , we write $\mathcal{H}^{(y)}$ for the operator with the vertex y deleted and $\mu_x^{(y)}$ for the spectral measure of δ_x for $\mathcal{H}^{(y)}$, as in Theorem 3.10 and Definition 3.15. By Lemma 3.5(ii), for a neighbor x of y , the component of x in $V \setminus \{y\}$ is the branch $V_{x \rightarrow y}$, so that

$$G_{xx}^{(y)}(z) = \Gamma_{x \rightarrow y}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \sim y. \quad (7.13)$$

Lemma 7.5. *Let (F, D) be a finite removal given by a fixed finite set F of words and a fixed finite set D of edges, identified with words as in Definition 3.2 (on the event that these are vertices and edges of $\mathcal{T}$), and let u, x be words. Then the map $(\omega, z) \mapsto G_{ux}^{F,D}(z)$ is measurable on $\Omega \times \mathbb{H}$. In particular, for all words u, w , the map $(\omega, z) \mapsto \Gamma_{u \rightarrow w}(z)$ is measurable.*

Proof. For fixed z , the map $\omega \mapsto G_{ux}^{F,D}(z)$ is measurable, as explained after Definition 3.15: by Theorem 3.10(iii), it is the limit of entries of inverses of finite matrices whose entries are measurable functions of finitely many of the variables N_v and w_v . It is also continuous in $z \in \mathbb{H}$. For $n \in \mathbb{N}$, partition $\mathbb{H}$ into countably many Borel sets $Q_{n,k}$ of diameter at most $1/n$, choose $z_{n,k} \in Q_{n,k}$, and set $G^{(n)}(\omega, z) = G_{ux}^{F,D}(z_{n,k})$ for $z \in Q_{n,k}$. Each $G^{(n)}$ is jointly measurable, and $G^{(n)} \rightarrow G_{ux}^{F,D}$ pointwise by continuity. The last assertion holds because $\Gamma_{u \rightarrow w} = G_{uu}^{\mathcal{O}, \{e\}}$, where e is the edge between u and w . $\square$

7.3 Row deletion and the square blocks

We now recall the row structure of Proposition 3.23, for $C = V$, in a form that treats both parities at once. Let y be a vertex with $\deg(y) \geq 1$, and let $\epsilon \in \{0, 1\}$ be the parity of its neighbors, so that $y \in V_{1-\epsilon}$. Let P_ϵ be the orthogonal projection onto $\ell^2(V_\epsilon)$, and let

S_ϵ denote the square block of $\mathcal{H}^2$ on $\ell^2(V_\epsilon)$, that is, $S_0 = A^*A$ and $S_1 = AA^*$. The row vector, squared row norm and row direction of y are

$$a_y = \sum_{x \sim y} h_{xy} \delta_x \in \ell^2(V_\epsilon), \quad U_y = \|a_y\|^2 = \sum_{x \sim y} |h_{xy}|^2, \quad v_y = U_y^{-1/2} a_y. \quad (7.14)$$

Almost surely every edge weight is nonzero, and then $U_y > 0$. Set

$$K^{(y)} = S_\epsilon - U_y |v_y\rangle \langle v_y|. \quad (7.15)$$

By Proposition 3.23(iv) and (vii), $K^{(y)}$ is a nonnegative self-adjoint operator with $\text{Dom}(K^{(y)}) = \text{Dom}(S_\epsilon)$; for $y \in V_1$ it is the operator $K_0^{(y)}$ of Proposition 3.23. Further, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$P_\epsilon(\mathcal{H}^{(y)} - z)^{-1} P_\epsilon = z (K^{(y)} - z^2)^{-1}, \quad (7.16)$$

where both sides are regarded as operators on $\ell^2(V_\epsilon)$; this is meaningful because $y \notin V_\epsilon$. We set

$$\nu_y = \mu_{v_y}^{K^{(y)}}, \quad m_y(w) = F_{v_y}(w) = \langle v_y, (K^{(y)} - w)^{-1} v_y \rangle, \quad w \in \mathbb{H}. \quad (7.17)$$

Lemma 7.6. *Almost surely, the following holds for every vertex y with $\deg(y) \geq 1$.*

(i) *For every $z \in \mathbb{C}$ with $\text{Re } z > 0$ and $\text{Im } z > 0$,*

$$m_y(z^2) = \frac{1}{z U_y} \sum_{x \sim y} |h_{xy}|^2 \Gamma_{x \rightarrow y}(z). \quad (7.18)$$

(ii) *For every neighbor x of y , the measure $\mu_x^{(y)}$ is invariant under $\lambda \mapsto -\lambda$. Moreover, ν_y is the image of the measure $U_y^{-1} \sum_{x \sim y} |h_{xy}|^2 \mu_x^{(y)}$ under the map $\lambda \mapsto \lambda^2$.*

(iii) *For every $E > 0$,*

$$R_{\nu_y}(E^2) \leq \frac{2}{E^2 U_y} \sum_{x \sim y} |h_{xy}|^2 R_{\mu_x^{(y)}}(E). \quad (7.19)$$

Proof. We work on the probability-one event on which the conclusions of Theorem 3.10 and Proposition 3.23 hold and all edge weights are nonzero. Part (i) is the identity (3.22) (for $y \in V_1$; for $y \in V_0$ we use Proposition 3.23(vii)), divided by z , since $|v_y(x)|^2 = |h_{xy}|^2 / U_y$.

For (ii), write $\mu_{v_y}^{(y)}$ for the spectral measure of v_y for $\mathcal{H}^{(y)}$. By Lemma 3.5(ii), the neighbors of y lie in distinct components of the graph $V \setminus \{y\}$, and $\mathcal{H}^{(y)}$ is the orthogonal sum of the operators of these components (Theorem 3.10(iv)). It follows that $\langle \delta_x, f(\mathcal{H}^{(y)}) \delta_{x'} \rangle = 0$ for distinct neighbors x, x' of y and every bounded Borel function f , and by (7.14),

$$\mu_{v_y}^{(y)} = \frac{1}{U_y} \sum_{x \sim y} |h_{xy}|^2 \mu_x^{(y)}. \quad (7.20)$$

By Proposition 3.21(i), applied to the components of $V \setminus \{y\}$, each $\mu_x^{(y)}$ is invariant under $\lambda \mapsto -\lambda$, and by (7.20) so is $\mu_{v_y}^{(y)}$. Let ν' be the image of $\mu_{v_y}^{(y)}$ under $\lambda \mapsto \lambda^2$. For z with $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$, the symmetry and the identity $(\lambda - z)^{-1} + (-\lambda - z)^{-1} = 2z(\lambda^2 - z^2)^{-1}$ give

$$\int_{\mathbb{R}} \frac{\mu_{v_y}^{(y)}(d\lambda)}{\lambda - z} = \int_{\mathbb{R}} \frac{z \mu_{v_y}^{(y)}(d\lambda)}{\lambda^2 - z^2} = z \int_{\mathbb{R}} \frac{\nu'(dt)}{t - z^2}.$$

By (7.16), the left side equals $z m_y(z^2) = z \int (t - z^2)^{-1} \nu_y(dt)$. Since $z \mapsto z^2$ maps the open first quadrant onto $\mathbb{H}$, the Borel transforms of ν' and ν_y agree on $\mathbb{H}$. By the Stieltjes inversion formula (or by the uniqueness argument in the proof of Lemma 7.3(iii)), $\nu' = \nu_y$. This proves (ii).

For (iii), we use (ii) and the factorization $\lambda^2 - E^2 = (\lambda - E)(\lambda + E)$:

$$R_{\nu_y}(E^2) = \int_{\mathbb{R}} \frac{\mu_{v_y}^{(y)}(d\lambda)}{(\lambda - E)^2(\lambda + E)^2} \leq 2 \int_{[0, \infty)} \frac{\mu_{v_y}^{(y)}(d\lambda)}{(\lambda - E)^2(\lambda + E)^2} \leq \frac{2}{E^2} R_{\mu_{v_y}^{(y)}}(E).$$

In the first inequality, we used that the integrand is even in λ and that $\mu_{v_y}^{(y)}$ is symmetric; in the second, we used that $(\lambda + E)^2 \geq E^2$ for $\lambda \geq 0$. Inserting (7.20), we obtain (7.19). $\square$

The next lemma passes from the spectral measures of the row directions to a square block. Together with Proposition 3.23(vi), which passes from the two square blocks to $\mathcal{H}$, it reduces the spectral type of $\mathcal{H}$ in $I \cup (-I)$ to that of the row measures in I^2 .

Lemma 7.7. *Let K be a self-adjoint operator on a separable Hilbert space $\mathfrak{h}$, let $B \subset \mathbb{R}$ be a Borel set with $0 \notin B$, and let $(\varphi_j)_{j \in \mathbb{N}}$ be vectors whose closed linear span contains $(\ker K)^\perp$. If the restriction of $\mu_{\varphi_j}^K$ to B is pure point (respectively absolutely continuous) for every j , then K has pure point spectrum (respectively is purely absolutely continuous) in B .*

Proof. We treat the pure point case; the other is identical. Since $0 \notin B$, we have $\mathbf{1}_B(K)\psi = \mathbf{1}_B(0)\psi = 0$ for $\psi \in \ker K$. For each j , the spectral measure of $\mathbf{1}_B(K)\varphi_j$ is the restriction of $\mu_{\varphi_j}^K$ to B , so $\mathbf{1}_B(K)\varphi_j \in \mathfrak{h}_{\text{pp}}(K)$. Since $\mathbf{1}_B(K)$ is bounded and $\mathfrak{h}_{\text{pp}}(K)$ is closed, $\mathbf{1}_B(K)$ maps the closed span of the φ_j , and hence $(\ker K)^\perp$, into $\mathfrak{h}_{\text{pp}}(K)$. Together with $\mathbf{1}_B(K) \ker K = \{0\}$, this gives $\operatorname{Ran} \mathbf{1}_B(K) \subset \mathfrak{h}_{\text{pp}}(K)$. $\square$

Finally, we record the conditional structure of a single row. For a word y , let $\mathcal{G}^y$ be the σ -algebra of Proposition 3.23(iv) (with $C = V$, and with the parities exchanged as in Proposition 3.23(vii) if $|y|$ is even). It is generated by the genealogy and auxiliary variables $\mathcal{G}_0$, by the row direction v_y , and by the marks of all edges not incident to y . We write $\Omega_y = \{y \in V, \deg(y) \geq 1\}$, which belongs to $\mathcal{G}_0$. A probability kernel from

$(\Omega, \mathcal{G}^y)$ to $\mathbb{R}$ is a map $\omega \mapsto \kappa(\omega)$ into the Borel probability measures on $\mathbb{R}$ such that $\omega \mapsto \kappa(\omega)(B)$ is $\mathcal{G}^y$ -measurable for every Borel set $B \subset \mathbb{R}$. For $d \geq 1$, let

$$g_d(u) = \frac{p^d}{\Gamma(d)} u^{d-1} e^{-pu} \mathbf{1}_{(0, \infty)}(u). \quad (7.21)$$

Lemma 7.8. *Let $Q \in \{\mathbb{P}, \mathbb{P}^\infty\}$ and let y be a word.*

- (i) *There exists a probability kernel ν'_y from $(\Omega, \mathcal{G}^y)$ to $\mathbb{R}$ such that, Q -almost surely on Ω_y , we have $\nu'_y = \nu_y$.*
- (ii) *For every $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{R})$ -measurable function $\Phi: \Omega \times \mathbb{R} \rightarrow [0, \infty]$,*

$$\mathbb{E}_Q[\Phi(\cdot, U_y) \mathbf{1}_{\Omega_y}] = \mathbb{E}_Q \left[\mathbf{1}_{\Omega_y} \int_0^\infty \Phi(\cdot, u) g_{\deg(y)}(u) du \right]. \quad (7.22)$$

Proof. We begin with (i). The operator $\mathcal{H}^{(y)}$ depends only on the genealogy and on the marks of edges not incident to y . For a neighbor x of y , let $\widehat{G}_x(z)$ be the limit of the diagonal entries at x of the resolvents at z of the Dirichlet truncations of $\mathcal{H}^{(y)}$ to the balls of radius n around x in the component of x , when this limit exists, and zero otherwise. The argument of Lemma 7.5 shows that $\widehat{G}_x$ is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{H})$ -measurable, and $\widehat{G}_x = G_{xx}^{(y)}$ on Ω_{sa} by Theorem 3.10(iii). For $w \in \mathbb{H}$, let $\sqrt{w}$ be the square root in the open first quadrant, and set

$$M_y(w) = \frac{1}{\sqrt{w}} \sum_{x \sim y} |v_y(x)|^2 \widehat{G}_x(\sqrt{w})$$

on Ω_y , and $M_y(w) = -1/w$ off Ω_y . This function is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{H})$ -measurable, and by (7.18) and (7.13), almost surely on Ω_y we have $M_y = m_y$ on $\mathbb{H}$.

We recover ν_y from M_y by the Stieltjes inversion formula: for a finite measure ν and $a < b$, the limit as $\eta \downarrow 0$ of $\pi^{-1} \int_a^b \operatorname{Im} F_\nu(t + i\eta) dt$ is $(\nu((a, b)) + \nu([a, b]))/2$ [123]. Taking $a = -m$ and $b = q + 1/m$ and letting $m \rightarrow \infty$, we obtain $\nu((-\infty, q])$. For rational q , set

$$f_q = \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\pi} \int_{-m}^{q+1/m} \operatorname{Im} M_y(t + i/n) dt.$$

By Fubini's theorem, f_q is $\mathcal{G}^y$ -measurable (the integrand is jointly measurable and bounded on $[-m, q + 1/m]$, since $|\widehat{G}_x(z)| \leq 1/\operatorname{Im} z$), and almost surely on Ω_y we have $f_q = \nu_y((-\infty, q])$ for every rational q . Let A be the set of $\omega \in \Omega_y$ such that $q \mapsto f_q(\omega)$ is nondecreasing on $\mathbb{Q}$, with limits 0 at $-\infty$ and 1 at $+\infty$. Then $A \in \mathcal{G}^y$, and $Q(\Omega_y \setminus A) = 0$. For $\omega \in A$, let $\nu'_y(\omega)$ be the probability measure with distribution function $\lambda \mapsto \inf\{f_q(\omega) : q \in \mathbb{Q}, q > \lambda\}$, and for $\omega \notin A$ let $\nu'_y(\omega) = \delta_0$. Then $\omega \mapsto \nu'_y(\omega)((-\infty, \lambda])$ is $\mathcal{G}^y$ -measurable for every λ , and since the Borel sets B for which $\omega \mapsto \nu'_y(\omega)(B)$ is measurable form a Dynkin system, ν'_y is a probability kernel. Almost surely on Ω_y , the distribution functions of ν'_y and ν_y agree at every λ , by right continuity, and so $\nu'_y = \nu_y$.

For (ii), Proposition 3.23(iv) (and (vii)) states that, under $\mathbb{P}$ and under $\mathbb{P}^\infty$, conditionally on $\mathcal{G}^y$, the squared row norm U_y has the density $g_{\deg(y)}$ on Ω_y ; recall that the number of neighbors of y in V is $\deg(y)$, which is $\mathcal{G}^y$ -measurable. The identity (7.22) is the disintegration formula for this conditional law [81, Theorem 6.4], applied with the $\mathcal{G}^y$ -measurable random element $\omega \mapsto \omega$ of $(\Omega, \mathcal{G}^y)$. $\square$

We now attach a rank-one family to the kernel ν'_y . For $\omega \in \Omega$, let $L^y(\omega)$ be the operator of multiplication by the variable on $L^2(\nu'_y(\omega))$, and let $\mathbf{1}$ be the constant function, whose spectral measure for L^y is ν'_y . Let $(\mu_u^y)_{u \in \mathbb{R}}$ be the family (7.6) with $L = L^y$ and $v = \mathbf{1}$, so that $\mu_0^y = \nu'_y$. The Borel transform of ν'_y is $\mathcal{G}^y$ -measurable, so Lemma 7.3(iv) applies with $(\Omega', \mathcal{A}) = (\Omega, \mathcal{G}^y)$. By (7.8), the Borel transform of μ_u^y is determined by u and the Borel transform of μ_0^y , and the same holds for the spectral measures of v_y for the operators $K^{(y)} + u|v_y\rangle\langle v_y|$, whose member with $u = 0$ is ν_y . Since a finite measure is determined by its Borel transform, we conclude from Lemma 7.8(i) and (7.15) that, $\mathbb{Q}$ -almost surely on Ω_y ,

$$\mu_u^y = \mu_{v_y}^{K^{(y)} + u|v_y\rangle\langle v_y|} \quad \text{for every } u \in \mathbb{R}, \quad \text{in particular} \quad \mu_{U_y}^y = \mu_{v_y}^{S_\epsilon}, \quad (7.23)$$

where ϵ is the parity of the neighbors of y .

7.4 The pure point criterion

Proposition 7.9. *Let $\mathbb{Q} \in \{\mathbb{P}, \mathbb{P}^\infty\}$ and let $I \subset (0, \infty)$ be a Borel set (for example, a compact interval). Suppose that $\mathbb{Q}$ -almost surely, for Lebesgue-almost every $E \in I$, for every vertex y and every neighbor x of y ,*

$$\int_{\mathbb{R}} \frac{\mu_x^{(y)}(d\lambda)}{(\lambda - E)^2} < \infty. \quad (7.24)$$

Then $\mathbb{Q}$ -almost surely, $\mathcal{H}$ has pure point spectrum in $I \cup (-I)$. If the hypothesis holds $\mathbb{P}$ -almost surely, then it holds $\mathbb{P}^\infty$ -almost surely.

The order of quantifiers in the hypothesis is the following: there exists an event of $\mathbb{Q}$ -probability one on which, outside a Lebesgue-null set of energies $E \in I$ that may depend on the realization, (7.24) holds simultaneously for all pairs of adjacent vertices. The integral in (7.24) is $R_{\mu_x^{(y)}}(E)$ in the notation of (7.2); in particular, it is infinite if E is an eigenvalue of the component of x in $V \setminus \{y\}$ with $\mu_x^{(y)}(\{E\}) > 0$.

Proof. The last assertion holds because $\mathbb{P}^\infty = \mathbb{P}(\cdot \mid \mathcal{S})$ is absolutely continuous with respect to $\mathbb{P}$. We now fix $\mathbb{Q}$ and assume the hypothesis under $\mathbb{Q}$.

The proof isolates one row y at a time. Conditional on all variables except the squared row norm U_y , the spectral measure of v_y for the square block is the member at coupling

U_y of a rank-one family, and Lemma 7.6 converts the hypothesis into the finiteness of the reciprocal-square integral of the unperturbed member at almost every point of I^2 . The Simon–Wolff criterion and spectral averaging then show that the perturbed measure is pure point on I^2 for almost every coupling, and the density of U_y transfers this to the realized coupling. The row directions span the orthogonal complement of the kernel of each square block, which gives the conclusion for the square blocks and then for $\mathcal{H}$.

STEP 1. Reduction to a single row. We claim that it suffices to prove the following statement for every word y : Q-almost surely on $\{y \in V, \deg(y) \geq 1\}$, the restriction of $\mu_{v_y}^{S_\epsilon}$ to I^2 is pure point, where ϵ is the parity of the neighbors of y . Indeed, the set of words is countable, and this then holds Q-almost surely for all such y simultaneously. Fix $\epsilon \in \{0, 1\}$. The vertices whose neighbors lie in V_ϵ are those in $V_{1-\epsilon}$, and $A^*\delta_y = a_y$ for $y \in V_1$, while $A\delta_x = a_x$ for $x \in V_0$. By Proposition 3.23(ii) and (vii), the row directions v_y , $y \in V_{1-\epsilon}$ with $\deg(y) \geq 1$, span $(\ker A)^\perp$ if $\epsilon = 0$ and $(\ker A^*)^\perp$ if $\epsilon = 1$. Since $\ker A^*A = \ker A$ and $\ker AA^* = \ker A^*$, they span $(\ker S_\epsilon)^\perp$. Since $0 \notin I^2$, Lemma 7.7 shows that S_ϵ has pure point spectrum in I^2 . By Proposition 3.23(vi), $\mathcal{H}$ has pure point spectrum in $I \cup (-I)$.

STEP 2. The exceptional set of the unperturbed row measure. Fix a word y , let ν'_y be the kernel of Lemma 7.8(i), and set

$$T_y = \{\lambda \in I^2 : R_{\nu'_y}(\lambda) = \infty\}.$$

We claim that $\{(\omega, \lambda) : \lambda \in T_y\}$ is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{R})$ -measurable and that $|T_y| = 0$, Q-almost surely on Ω_y . The measurability follows from Lemma 7.3(iv), applied with $(\Omega', \mathcal{A}) = (\Omega, \mathcal{G}^y)$, $L = L^y$ and $v = \mathbf{1}$. For the second assertion, consider a realization in Ω_y and in the probability-one event of the hypothesis, of Lemma 7.6 and of Lemma 7.8(i), so that $\nu'_y = \nu_y$. Let $E \in I$ be an energy at which (7.24) holds for all $x \sim y$. Then (7.19) implies that $R_{\nu_y}(E^2) < \infty$. We conclude that $T_y \subset q(\mathcal{N})$, where $q(E) = E^2$ and $\mathcal{N} \subset I$ is the Lebesgue-null set of energies at which the hypothesis fails. Since q is locally Lipschitz, $|T_y| = 0$.

STEP 3. Spectral averaging in the squared row norm. Let $(\mu_u^y)_{u \in \mathbb{R}}$ be the rank-one family of (7.23), so that $\mu_0^y = \nu'_y$ and, almost surely on Ω_y , $\mu_{U_y}^y = \mu_{v_y}^{S_\epsilon}$. By Lemma 7.4(i) with $L = L^y$ and $S = I^2$, the restriction of μ_u^y to I^2 is pure point whenever $\mu_u^y(T_y) = 0$. Set

$$\Phi(\omega, u) = \mathbf{1}\{\mu_u^y(T_y) > 0\}.$$

By Lemma 7.3(iv), with $T = \{(\omega, \lambda) : \lambda \in T_y\}$, the function Φ is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{R})$ -measurable. For every ω with $|T_y| = 0$, spectral averaging (7.9) gives $\int_{\mathbb{R}} \mu_u^y(T_y) du = |T_y| = 0$, and

$\Phi(\omega, u) = 0$ for Lebesgue-almost every u . Using (7.22) and Step 2, we obtain

$$\begin{aligned} \mathbb{Q}(\{\mu_{U_y}^y(T_y) > 0\} \cap \Omega_y) &\leq \mathbb{Q}(\{|T_y| > 0\} \cap \Omega_y) \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\Omega_y} \mathbf{1}_{\{|T_y| = 0\}} \int_0^\infty \Phi(\cdot, u) g_{\deg(y)}(u) du \right] = 0. \end{aligned}$$

Then, $\mathbb{Q}$ -almost surely on Ω_y , the restriction of $\mu_{v_y}^{S_\epsilon} = \mu_{U_y}^y$ to I^2 is pure point. By Step 1, this completes the proof. $\square$

Remark 7.10. The hypothesis of Proposition 7.9 concerns spectral measures of the vertex-deleted operators at the real axis. In Section 8, it is verified by constructing, for almost every energy E , a solution $f \in \ell^2$ of the equation $(\mathcal{H}^{(y)} - E)f = \delta_x$ (see Lemma 8.10).

7.5 The absolutely continuous criterion

Proposition 7.11. *Let $I \subset (0, \infty)$ be a Borel set (for example, an interval). Suppose that for Lebesgue-almost every $E \in I$,*

$$\mathbb{P}(\text{Im } \Gamma_o(E + i0) > 0) > 0, \quad (7.25)$$

where the event is that the vertical boundary value $\Gamma_o(E + i0)$ exists and has positive imaginary part. Then $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ is purely absolutely continuous in $I \cup (-I)$.

By Proposition 5.11(i), at every $E \in \mathcal{E}_p$ with $E > 0$ the probability in (7.25) is either 0 or ϱ_p . Before the proof, we record two deterministic lemmas. The first propagates positivity of imaginary parts along paths, and the second describes the infinite components left after deleting a vertex. They are the pathwise form, at all energies outside a null set, of statements proved at a fixed regular energy in Section 5.

Lemma 7.12. *Fix a realization in Ω_{sa} with all edge weights nonzero, and fix $E \in \mathbb{R}$ such that $\Gamma_{u \rightarrow w}(E + i0)$ exists for every oriented edge (u, w) . If (u, w) is an oriented edge and $\text{Im } \Gamma_{x \rightarrow u}(E + i0) > 0$ for some neighbor $x \neq w$ of u , then $\text{Im } \Gamma_{u \rightarrow w}(E + i0) > 0$.*

Proof. By (3.12) with $C = V$, we have $\Gamma_{u \rightarrow w}(z) = -\Delta(z)^{-1}$ with

$$\Delta(z) = z + \sum_{x' \sim u, x' \neq w} |h_{ux'}|^2 \Gamma_{x' \rightarrow u}(z).$$

Since the sum is finite, the boundary value $\Delta(E + i0)$ exists. Every $\Gamma_{x' \rightarrow u}$ maps $\mathbb{H}$ to $\mathbb{H}$, and we have $\text{Im } \Gamma_{x' \rightarrow u}(E + i0) \geq 0$ and $\text{Im } \Delta(E + i0) \geq |h_{ux}|^2 \text{Im } \Gamma_{x \rightarrow u}(E + i0) > 0$. In particular $\Delta(E + i0) \neq 0$, so $\Gamma_{u \rightarrow w}(E + i0) = -\Delta(E + i0)^{-1}$, and $\text{Im } \Gamma_{u \rightarrow w}(E + i0) = \text{Im } \Delta(E + i0) / |\Delta(E + i0)|^2 > 0$. $\square$

Lemma 7.13. *Fix a realization as in Lemma 7.12, and fix $E \in \mathbb{R}$ such that $\Gamma_{u \rightarrow w}(E + i0)$ exists for every oriented edge (u, w) and $\text{Im } \Gamma_v(E + i0) > 0$ for every vertex v whose forward subtree $\mathcal{T}_v$ is infinite. Let y be a vertex and x a neighbor of y whose component in $V \setminus \{y\}$ is infinite. Then $\text{Im } \Gamma_{x \rightarrow y}(E + i0) > 0$.*

Proof. If x is a child of y , then the component of x is $V(\mathcal{T}_x)$ and $\Gamma_{x \rightarrow y} = \Gamma_x$; the claim is then part of the hypothesis. Otherwise $y \neq o$ and $x = y^-$. Let $o = a_0, a_1, \dots, a_k = y^-$ be the path from the root to y^- , and set $a_{k+1} = y$. The component of y^- is $V \setminus V(\mathcal{T}_y)$, and it is the union of the path $\{a_0, \dots, a_k\}$ and the sets $V(\mathcal{T}_w)$, where w ranges over the children of the a_j ($0 \leq j \leq k$) with $w \neq a_{j+1}$. There are finitely many such w , since the path is finite and degrees are finite. As the component is infinite, one of these forward subtrees, say $\mathcal{T}_w$ with w a child of a_j and $w \neq a_{j+1}$, is infinite, and $\text{Im } \Gamma_w(E + i0) > 0$ by hypothesis.

We show by induction on $i = j, \dots, k$ that $\text{Im } \Gamma_{a_i \rightarrow a_{i+1}}(E + i0) > 0$. For $i = j$, the vertex w is a neighbor of a_j different from a_{j+1} , and $\Gamma_{w \rightarrow a_j} = \Gamma_w$, so Lemma 7.12 applies. For $j \leq i < k$, the vertex a_i is a neighbor of a_{i+1} different from a_{i+2} , and Lemma 7.12 together with the induction hypothesis gives the claim for $i + 1$. The case $i = k$ is $\text{Im } \Gamma_{y^- \rightarrow y}(E + i0) > 0$. $\square$

Proof of Proposition 7.11. The proof establishes, at almost every energy of I , that the row measures ν_y have Borel transforms with positive imaginary boundary values, and then applies the absolutely continuous criterion of Lemma 7.4(ii) with spectral averaging in U_y . Positivity on every infinite forward subtree comes from (7.25) and the zero-one law of Proposition 5.11; Lemma 7.13 transfers it to the attachment points of the infinite components of the vertex-deleted tree; and Lemma 7.6(i) together with Lemma 7.2 transfers it to the Borel transform of ν_y at the point E^2 .

STEP 1. Positivity on infinite forward subtrees. Let $\mathcal{B}_1$ be the set of $(\omega, E) \in \Omega \times I$ such that, for every vertex v with $\mathcal{T}_v$ infinite, the boundary value $\Gamma_v(E + i0)$ exists and has positive imaginary part. By Lemma 7.5 and the rational Cauchy criterion (7.10), the set of (ω, E) at which $\Gamma_v(E + i0)$ exists is measurable and the boundary value is a measurable function on it. Since the event that $\mathcal{T}_v$ is infinite is measurable and there are countably many words v , the set $\mathcal{B}_1$ is jointly measurable. Let $E \in I \cap \mathcal{E}_p$ be an energy at which (7.25) holds; this is the case for Lebesgue-almost every $E \in I$, since the complement of $\mathcal{E}_p$ is Lebesgue-null (Definition 5.7). By Proposition 5.11(i), the probability in (7.25) equals q_p , and $\mathbb{P}^\infty((\omega, E) \in \mathcal{B}_1) = 1$. By Tonelli's theorem,

$$\mathbb{E}^\infty |\{E \in I : (\omega, E) \notin \mathcal{B}_1\}| = \int_I \mathbb{P}^\infty((\omega, E) \notin \mathcal{B}_1) dE = 0.$$

Further, each $\Gamma_{u \rightarrow w}$ is the Borel transform of a spectral measure. By Lemma 7.1(i), for every realization and every oriented edge, $\Gamma_{u \rightarrow w}(E + i0)$ exists for Lebesgue-almost every

E ; there are countably many oriented edges. We conclude that there exists an event Ω_1 with $\mathbb{P}^\infty(\Omega_1) = 1$, contained in Ω_{sa} , in the event of Lemma 7.6, and in the event that all edge weights are nonzero and $\mathcal{T}$ is infinite, with the following property. For every $\omega \in \Omega_1$ there exists a Lebesgue-null set $\mathcal{N}(\omega) \subset I$ such that for every $E \in I \setminus \mathcal{N}(\omega)$, all boundary values $\Gamma_{u \rightarrow w}(E + i0)$ exist, and $\text{Im } \Gamma_v(E + i0) > 0$ for every v with $\mathcal{T}_v$ infinite.

STEP 2. Positivity for the row measures. Fix a word y , and let ν'_y be the kernel of Lemma 7.8(i). Let T'_y be the set of $\lambda \in I^2$ such that the boundary value $F_{\nu'_y}(\lambda + i0)$ does not exist or has imaginary part zero. By Lemma 7.3(iv), applied with $(\Omega', \mathcal{A}) = (\Omega, \mathcal{G}^y)$, $L = L^y$ and $v = \mathbf{1}$, the set $\{(\omega, \lambda) : \lambda \in T'_y\}$ is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{R})$ -measurable. We claim that $|T'_y| = 0$, $\mathbb{P}^\infty$ -almost surely on Ω_y .

To that end, fix $\omega \in \Omega_1 \cap \Omega_y$ such that $\nu'_y = \nu_y$, so that $F_{\nu'_y} = m_y$, and fix $E \in I \setminus \mathcal{N}(\omega)$. The set $V \setminus \{y\}$ is the union of the components of the $\deg(y) < \infty$ neighbors of y , and V is infinite; hence some neighbor x_0 of y has an infinite component. By Lemma 7.13, $\text{Im } \Gamma_{x_0 \rightarrow y}(E + i0) > 0$. By (7.18) with $z = E + i\eta$, the limit

$$\ell = \lim_{\eta \downarrow 0} m_y((E + i\eta)^2) = \frac{1}{EU_y} \sum_{x \sim y} |h_{xy}|^2 \Gamma_{x \rightarrow y}(E + i0)$$

exists, and $\text{Im } \ell \geq (EU_y)^{-1} |h_{x_0 y}|^2 \text{Im } \Gamma_{x_0 \rightarrow y}(E + i0) > 0$, because all terms have non-negative imaginary part and $h_{x_0 y} \neq 0$. By Lemma 7.2, applied to ν_y , the boundary value $m_y(E^2 + i0)$ exists and equals ℓ , and $E^2 \notin T'_y$. In particular, T'_y is contained in $\{E^2 : E \in \mathcal{N}(\omega)\}$, which is Lebesgue-null. Since $\mathbb{P}^\infty(\Omega_1) = 1$ and $\nu'_y = \nu_y$ almost surely on Ω_y , this proves the claim.

STEP 3. Spectral averaging. Fix a word y , let ϵ be the parity of its neighbors, and let (μ_u^y) be the rank-one family of (7.23), so that $\mu_0^y = \nu'_y$ and, almost surely on Ω_y , $\mu_{v_y}^{S_\epsilon} = \mu_{v_y}^y$. By Lemma 7.4(ii) with $L = L^y$ and $S = I^2$, the restriction of μ_u^y to I^2 is absolutely continuous whenever $\mu_u^y(T'_y) = 0$. As in Step 3 of the proof of Proposition 7.9, the function $\Phi(\omega, u) = \mathbf{1}\{\mu_u^y(T'_y) > 0\}$ is $\mathcal{G}^y \otimes \mathcal{B}(\mathbb{R})$ -measurable, it vanishes for Lebesgue-almost every u when $|T'_y| = 0$, and (7.22) with $Q = \mathbb{P}^\infty$ gives $\mathbb{P}^\infty(\{\mu_{U_y}^y(T'_y) > 0\} \cap \Omega_y) = 0$. It follows that, $\mathbb{P}^\infty$ -almost surely, the restriction of $\mu_{v_y}^{S_\epsilon}$ to I^2 is absolutely continuous for every vertex y with $\deg(y) \geq 1$.

STEP 4. Conclusion. As in Step 1 of the proof of Proposition 7.9, the row directions of the vertices in $V_{1-\epsilon}$ span $(\ker S_\epsilon)^\perp$, and $0 \notin I^2$. By Lemma 7.7, each square block S_ϵ is purely absolutely continuous in I^2 , and by Proposition 3.23(vi), $\mathcal{H}$ is purely absolutely continuous in $I \cup (-I)$, $\mathbb{P}^\infty$ -almost surely. $\square$

Remark 7.14. The hypothesis of Proposition 7.11 is a statement about the unconditioned law at almost every energy of I , and the conclusion is a statement about the survival-conditioned operator. On the extinction event the tree is finite and the spectrum is pure

point, and the conclusion does not hold under $\mathbb{P}$. The energy 0 is excluded: $\mathcal{H}$ has finitely supported eigenvectors with eigenvalue zero (Proposition 3.25).

Negative free energy implies localization

In this section, we prove that a real solution of the cavity equation with negative free energy at $s = 1$ forces pure point spectrum. The precise statement is Theorem 8.4: if for almost every energy E in a set $U \subset (0, \infty)$ there exists a real solution π at E with $\varphi_\pi(1) < 0$, then $\mathcal{H}$ has pure point spectrum in $U \cup (-U)$, almost surely under $\mathbb{P}$ and under $\mathbb{P}^\infty$. The real solution is not assumed to be related to the boundary values of the resolvent of $\mathcal{H}$, and the conclusion concerns only the operator.

We now describe the argument. Fix $E > 0$ and a real solution π at E . By Lemma 5.13, there exists a family of real denominators (D_v) on the tree, with marginal law π , satisfying the cavity recursion $D_v = E - p^{-1} \sum_i Y_{v,vi}/D_{vi}$ at every vertex. It is defined on an extension of the probability space of the marked tree. Setting $\gamma_v = -1/D_v$, the products

$$F(v) = \gamma_o \prod_{j=1}^L (-h_{v_j v_{j-1}} \gamma_{v_j}), \quad o = v_0, v_1, \dots, v_L = v,$$

define a function F on the vertices that solves $(\mathcal{H} - E)F = \delta_o$ pointwise. We call it the formal inverse column. By the many-to-one formula, the expected generation- L sum of $|F(v)|^s$ is the path moment $\Phi_L(s; E)$ of Definition 6.24, whose exponential growth rate is the free energy $\varphi_\pi(s)$ by Proposition 6.27. If $\varphi_\pi(1) < 0$, we may choose $s < 1$ with $\varphi_\pi(s) < 0$, and then $F \in \ell^s \subset \ell^2$ almost surely. Using the maximal domain characterization of Theorem 3.10, we then obtain $\delta_o = (\mathcal{H} - E)F$, and this implies that $\int \mu_o(d\lambda)/(\lambda - E)^2 < \infty$. The Simon–Wolff criterion of Proposition 7.9 requires the same statement for every neighbor of every removed vertex. We obtain it by rerooting: on a component of the forest obtained by removing a vertex or an edge, rerooted at any of its vertices, the inverse column is computed from finitely many real cavity marks along an ancestral path together with finitely many forward columns hanging off that path.

In Section 8.1, we define the formal inverse columns and state the main results, Proposition 8.3 and Theorem 8.4. Section 8.2 proves the many-to-one identity. Section 8.3 proves summability of the forward columns, and Section 8.4 constructs the rerooted columns. Section 8.5 completes the proofs.

Throughout this section, $p > 1$ is fixed.

8.1 The formal inverse column

We begin with a deterministic lemma, which is the real-energy analogue of the Schur complement formula of Proposition 3.16 read as an equation for a column of the resolvent.

Lemma 8.1. *Let C be a countable, connected, locally finite tree with edge weights $h_{xy} = \overline{h_{yx}}$, let $r \in C$, and let $E \in \mathbb{R}$. For $x \in C \setminus \{r\}$, let $b(x)$ be the neighbor of x on the path from x to r , and let $\text{ch}(x)$ be the set of neighbors x' of x with $x' \neq b(x)$; let $\text{ch}(r)$ be the set of all neighbors of r . Suppose that real numbers $(\gamma_x)_{x \in C}$ satisfy, for every $x \in C$,*

$$\Delta_x = E + \sum_{x' \in \text{ch}(x)} |h_{xx'}|^2 \gamma_{x'} \neq 0, \quad \gamma_x = -\frac{1}{\Delta_x}. \quad (8.1)$$

Define $f(r) = \gamma_r$, and $f(x) = -\gamma_x h_{xb(x)} f(b(x))$ for $x \neq r$, recursively along the path from r to x . Then for every $x \in C$,

$$\sum_{x' \sim x} h_{xx'} f(x') - E f(x) = \delta_r(x). \quad (8.2)$$

Proof. Fix $x \in C$. For $x' \in \text{ch}(x)$ we have $b(x') = x$, hence $f(x') = -\gamma_{x'} h_{x'x} f(x)$ and $h_{xx'} f(x') = -|h_{xx'}|^2 \gamma_{x'} f(x)$. The set $\text{ch}(x)$ is finite, and by (8.1),

$$\sum_{x' \in \text{ch}(x)} h_{xx'} f(x') - E f(x) = -\Delta_x f(x) = \frac{f(x)}{\gamma_x}.$$

If $x = r$, then $\text{ch}(r)$ contains all neighbors of r , and the right side equals $\gamma_r / \gamma_r = 1$. If $x \neq r$, the only neighbor of x outside $\text{ch}(x)$ is $b(x)$, and $f(x) / \gamma_x = -h_{xb(x)} f(b(x))$; the left side of (8.2) then equals $-h_{xb(x)} f(b(x)) + h_{xb(x)} f(b(x)) = 0$. $\square$

We now fix $E > 0$ and a real solution π at energy E (Definition 5.1), and we let $(\omega, (D_v)_{v \in \mathbb{U}})$ be a recursive tree process with marginal π , as in Lemma 5.13. We use the realization of the marked tree of Definition 4.1 and Section 5.3: the edge joining v and vi carries the mark $w_{v,vi} = w_{vi}$, the weight $h_{v,vi} = w_{v,vi} / \sqrt{p}$ and the squared modulus $Y_{v,vi} = |w_{v,vi}|^2$. As in the proof of Lemma 5.13, the innovation at a word v is $\xi_v = (N_v, \zeta_v, (w_{v,vi})_{i \geq 1})$, and for every word w the marked tree $\mathcal{T}_w$ rooted at w is built from the innovations at the words in $w\mathbb{U} = \{wu : u \in \mathbb{U}\}$. For $w \in V$, it is the forward subtree of w . By Lemma 5.13, the configuration ω has the law $\mathbb{P}$, so the marked tree $\mathcal{T}$ has the law of Definition 3.2. We keep writing $\mathbb{P}$ and $\mathbb{E}$ for the probability measure and the expectation on the space of Lemma 5.13. We may do so because events concerning only the marked tree have the same probability there as under $\mathbb{P}$. For a word w , we write

$$\Theta_w = (\theta_w \omega, (D_{wu})_{u \in \mathbb{U}})$$

for the process seen from w , where θ_w is the shift of Definition 4.1; it is in measurable bijection with $(\xi_{wu}, D_{wu})_{u \in \mathbb{U}}$, and $\Theta_o = (\omega, (D_u)_{u \in \mathbb{U}})$ is the process itself. We use the following three properties of the process, which are the conclusions of Lemma 5.13.

- (R1) Almost surely, every D_v is a nonzero real number, and the recursion $D_v = E - p^{-1} \sum_{i \leq N_v} Y_{v,vi} / D_{vi}$ holds at every word v .
- (R2) For every $n \geq 0$, the variables D_v , $|v| = n$, are independent with law π , and they are independent of $\mathcal{F}_n = \sigma(\xi_u : |u| < n)$.
- (R3) If $v_1, \dots, v_k$ are words none of which is a prefix of another, then $\Theta_{v_1}, \dots, \Theta_{v_k}$ are independent of each other and of the coordinates N_u, ξ_u with $u \notin \bigcup_i v_i \mathbb{U}$ and w_u with $u \notin \bigcup_i v_i \mathbb{U}_*$, where $v \mathbb{U}_* = v \mathbb{U} \setminus \{v\}$, and each Θ_{v_i} has the law of Θ_o .

We set

$$\gamma_v = -\frac{1}{D_v}, \quad v \in \mathbb{U}. \quad (8.3)$$

Definition 8.2. For a word w , the *forward formal column* F_w is the function on $V(\mathcal{T}_w)$ defined by $F_w(w) = \gamma_w$ and

$$F_w(vi) = -\gamma_{vi} h_{vi,v} F_w(v), \quad vi \in V(\mathcal{T}_w), v \in V(\mathcal{T}_w),$$

and extended by zero to the other words. Equivalently, if $w = u_0, u_1, \dots, u_m = u$ is the path from w to $u \in V(\mathcal{T}_w)$, then

$$F_w(u) = \gamma_w \prod_{j=1}^m (-h_{u_j u_{j-1}} \gamma_{u_j}). \quad (8.4)$$

The *formal inverse column* is $F = F_o$.

The function F_w is obtained from Θ_w by the same rule by which F is obtained from Θ_o . By the recursion (R1), for every $x \in V(\mathcal{T}_w)$ we have

$$E + \sum_{i=1}^{N_x} |h_{x,xi}|^2 \gamma_{xi} = E - \frac{1}{p} \sum_{i=1}^{N_x} \frac{Y_{x,xi}}{D_{xi}} = D_x = -\frac{1}{\gamma_x}.$$

This means that the numbers γ_x , $x \in V(\mathcal{T}_w)$, satisfy (8.1) on the tree $\mathcal{T}_w$ with root $r = w$, for which $b(x) = x^-$ and $\text{ch}(x)$ is the set of children of x . By Lemma 8.1, almost surely, for every word w ,

$$(\mathcal{H}_{w,\text{form}} F_w)(x) - E F_w(x) = \delta_w(x), \quad x \in V(\mathcal{T}_w), \quad (8.5)$$

where $\mathcal{H}_{w,\text{form}}$ is the formal weighted adjacency operator of $\mathcal{T}_w$. In general F_w need not belong to ℓ^2 . The point of the next proposition is that it does when $\varphi_\pi(1) < 0$, that is, when the free energy of π at $s = 1$ is negative.

Recall the path moments of Definition 6.24: for $s \in (0, 1)$, $L \geq 0$ and a scaling $g > 0$,

$$\Phi_L(s; E) = g^s \int_{\mathbb{R}} (\mathcal{A}_s^L e_s)(x) \pi^x(dx), \quad (8.6)$$

where $e_s(x) = |x|^s$, π^x is the law of $X = 1/(gD)$ with $D \sim \pi$, and $\mathcal{A}_s = \mathcal{A}_s^{(g)}$ is the operator of Definition 6.1 for the law of $B = gD$ and the rate $K = p/g^2$, so that $\beta = g^2 Y/p$ with $Y \sim \text{Exp}(1)$. By Lemma 6.23(ii), $\Phi_L(s; E)$ does not depend on g . For a self-adjoint operator T and a vector ψ , we use the reciprocal-square integral $R_{\mu_\psi^T}(E) = \int \mu_\psi^T(d\lambda)/(\lambda - E)^2$ of (7.2), which is infinite if $\mu_\psi^T(\{E\}) > 0$. For a component C of a forest obtained by a finite removal and $r \in C$, we write μ_r^C for the spectral measure of δ_r for $\mathcal{H}_C$ (Theorem 3.10).

Proposition 8.3. *Let $E > 0$, let π be a real solution at energy E , let $(\xi_v, D_v)_{v \in \mathbb{U}}$ be a recursive tree process with marginal π (Lemma 5.13), and let F be the formal inverse column of Definition 8.2.*

(i) *For every $s \in (0, 1)$ and $L \geq 0$,*

$$\mathbb{E} \sum_{v \in V: |v|=L} |F(v)|^s = \Phi_L(s; E). \quad (8.7)$$

Suppose in addition that $\varphi_\pi(1) < 0$. Then the following hold.

- (ii) *There exists $s_0 \in (0, 1)$ such that $\varphi_\pi(s) < 0$ for $s \in [s_0, 1)$. For every such s , almost surely $F \in \ell^s(V)$; moreover $F \in \text{Dom}(\mathcal{H})$ and $(\mathcal{H} - E)F = \delta_o$.*
- (iii) *Almost surely under $\mathbb{P}$ and under $\mathbb{P}^\infty$, the following holds. For every vertex v , we have $\delta_v \in \text{Ran}(\mathcal{H} - E)$ and $\int \mu_v(d\lambda)/(\lambda - E)^2 < \infty$. For every vertex y and every edge e of $\mathcal{T}$, every component C of the reduced forest obtained by removing y (or e), and every $r \in C$, we have $\delta_r \in \text{Ran}(\mathcal{H}_C - E)$ and $\int \mu_r^C(d\lambda)/(\lambda - E)^2 < \infty$. In particular, for every vertex y and every neighbor x of y ,*

$$\int_{\mathbb{R}} \frac{\mu_x^{(y)}(d\lambda)}{(\lambda - E)^2} < \infty. \quad (8.8)$$

We note that the assertions in (iii) concern only the marked tree and do not refer to π . Only their proof uses the recursive tree process. Combining Proposition 8.3 with the Simon–Wolff criterion of Proposition 7.9, we obtain the main result of this section.

Theorem 8.4. *Let $U \subset (0, \infty)$ be a Borel set (for example, an open set). Suppose that for Lebesgue-almost every $E \in U$ there exists a real solution π_E at energy E with $\varphi_{\pi_E}(1) < 0$. Then, almost surely under $\mathbb{P}$ and under $\mathbb{P}^\infty$, the operator $\mathcal{H}$ has pure point spectrum in $U \cup (-U)$.*

8.2 The many-to-one identity

We prove Proposition 8.3(i). For a vertex v with $|v| = L$ and path $o = v_0, \dots, v_L = v$, and for $s \in (0, 1)$, set

$$W_L(v) = \prod_{j=1}^L \left(\frac{Y_{v_{j-1}, v_j}}{p} \right)^{s/2} |D_{v_j}|^{-s}, \quad W_0(o) = 1. \quad (8.9)$$

Since $|h_{v_j v_{j-1}}|^s = (Y_{v_{j-1}, v_j} / p)^{s/2}$ and $|\gamma_v| = |D_v|^{-1}$, (8.4) gives

$$|F(v)|^s = |D_o|^{-s} W_L(v). \quad (8.10)$$

Lemma 8.5. *Let $s \in (0, 1)$ and $g > 0$, and set $X_o = 1/(gD_o)$. For every $L \geq 0$ and every Borel function $k: \mathbb{R} \rightarrow [0, \infty]$,*

$$\mathbb{E} \sum_{v \in V: |v|=L} k(X_o) W_L(v) = \int_{\mathbb{R}} (\mathcal{A}_s^L k)(x) \pi^x(dx). \quad (8.11)$$

The identity (8.11) is the many-to-one formula of Proposition 4.8 for the recursive tree process; compare Corollary 4.11 and Remark 4.13. We give a direct proof by induction on L , using the Mecke formula at the root, which works with the consistent process of Lemma 5.13 on the whole tree and so requires no truncation. (Alternatively, the left side of (8.11) is a function of $(\xi_u)_{|u|<L}$ and $(D_u)_{|u|=L}$ only, by the recursion (R1). By (R2), the variables $(D_u)_{|u|=L}$ are independent with law π and independent of $(\xi_u)_{|u|<L}$, and after integrating them out, Proposition 4.8 applies to the resulting function of the first L generations. The offspring numbers N_u with $|u| = L$ do not enter.)

Proof. We prove (8.11) by induction on L , simultaneously for all k . For $L = 0$, the left side is $\mathbb{E} k(X_o) = \int k d\pi^x$, since $D_o \sim \pi$.

Let $L \geq 1$ and assume the claim for $L - 1$. For a process $\Theta = (\omega', (D'_u)_{u \in \mathbb{U}})$ with the law of Θ_o , let $d(\Theta) = D'_o$ be its root mark and

$$S_{L-1}(\Theta) = \sum_{u \in V(\Theta): |u|=L-1} W_{L-1}(u)(\Theta),$$

where $V(\Theta)$ and $W_{L-1}(u)(\Theta)$ are the vertex set and the weight (8.9) computed from Θ . The vertices of generation L are the words iu with $1 \leq i \leq N_o$ and $u \in V(\Theta_i)$, $|u| = L - 1$, and $W_L(iu) = (Y_{o,i}/p)^{s/2} |D_i|^{-s} W_{L-1}(u)(\Theta_i)$. We obtain

$$\sum_{v \in V: |v|=L} k(X_o) W_L(v) = \sum_{i=1}^{N_o} k(X_o) \left(\frac{Y_{o,i}}{p} \right)^{s/2} |D_i|^{-s} S_{L-1}(\Theta_i), \quad (8.12)$$

where $D_i = d(\Theta_i)$ and, by (R1),

$$X_o = \frac{1}{gD_o}, \quad D_o = E - \frac{1}{p} \sum_{j=1}^{N_o} \frac{Y_{o,j}}{d(\Theta_j)}.$$

By (R3), applied to the words $1, \dots, n$ for every n , and since the coordinates of ω are independent under $\mathbb{P}$, the variables N_o , $(w_{o,i})_{i \geq 1}$ and $(\Theta_i)_{i \geq 1}$ are independent, the $w_{o,i}$ are standard complex Gaussians, and each Θ_i has the law of Θ_o . The pairs $m_i = (w_{o,i}, \Theta_i)$ are independent with law $\mu = \text{Law}(w_{o,1}) \otimes \text{Law}(\Theta_o)$, and they are independent of

$N_o \sim \text{Pois}(p)$. Then $\Pi = \sum_{i=1}^{N_o} \delta_{m_i}$ is a Poisson process with intensity measure $p\mu$ [94, Proposition 3.5]. The right side of (8.12) equals $\int \Psi(m, \Pi - \delta_m) \Pi(\mathrm{d}m)$, where for $m = (w, \Theta)$ and a finite counting measure ϱ ,

$$\Psi(m, \varrho) = k \left(\left(gE - \frac{g|w|^2}{p d(\Theta)} - \frac{g}{p} \int \frac{|w'|^2}{d(\Theta')} \varrho(\mathrm{d}w' \mathrm{d}\Theta') \right)^{-1} \right) \left(\frac{|w|^2}{p} \right)^{s/2} |d(\Theta)|^{-s} S_{L-1}(\Theta),$$

with $\Psi = 0$ if a denominator vanishes, which happens with probability zero. By the Mecke formula (4.1) (Lemma 4.4), we have

$$\mathbb{E} \sum_{v \in V: |v|=L} k(X_o) W_L(v) = p \int \mathbb{E} \Psi(m, \Pi) \mu(\mathrm{d}m). \quad (8.13)$$

We now evaluate the right side. Fix $m = (w, \Theta)$ with $d = d(\Theta) \neq 0$. The points of Π carry independent pairs $(|w'|^2, d(\Theta'))$ with law $\text{Exp}(1) \otimes \pi$, and their number is $\text{Pois}(p)$, so by Definition 5.1 the variable $D' = E - p^{-1} \int |w'|^2 d(\Theta')^{-1} \Pi(\mathrm{d}w' \mathrm{d}\Theta')$ has law π . Set $x = 1/(gd)$ and $\beta = g^2|w|^2/p$. Then $g|w|^2/(pd) = \beta x$ and $(|w|^2/p)^{s/2} |d|^{-s} = \beta^{s/2} |x|^s$, and therefore

$$\mathbb{E} \Psi(m, \Pi) = \beta^{s/2} |x|^s \mathbb{E} \left[k((gD' - \beta x)^{-1}) \right] S_{L-1}(\Theta).$$

Under μ , the variable $|w|^2$ has law $\text{Exp}(1)$ and is independent of Θ , and β has the exponential law of rate p/g^2 and is independent of gD' , whose law is that of gD . Integrating over w and recalling the definition of $\mathcal{A}_s$ and of $f_{B,\beta}(x) = (B - \beta x)^{-1}$, we obtain

$$\begin{aligned} p \int \mathbb{E} \Psi(m, \Pi) \mu(\mathrm{d}m) &= \mathbb{E} \left[(\mathcal{A}_s k)(1/(g d(\Theta_o))) S_{L-1}(\Theta_o) \right] \\ &= \mathbb{E} \sum_{u \in V: |u|=L-1} (\mathcal{A}_s k)(X_o) W_{L-1}(u). \end{aligned}$$

By the induction hypothesis applied to the Borel function $\mathcal{A}_s k: \mathbb{R} \rightarrow [0, \infty]$, the right side equals $\int (\mathcal{A}_s^{L-1} \mathcal{A}_s k) \mathrm{d}\pi^x$. Together with (8.13), this proves (8.11) for L . $\square$

Proof of Proposition 8.3(i). By (8.10), we have $|F(v)|^s = |D_o|^{-s} W_L(v) = g^s |X_o|^s W_L(v)$. Applying Lemma 8.5 with $k = g^s e_s$, we obtain $\mathbb{E} \sum_{|v|=L} |F(v)|^s = g^s \int \mathcal{A}_s^L e_s \mathrm{d}\pi^x = \Phi_L(s; E)$, by (8.6). (The right side is also the spine expectation (6.22) of Lemma 6.23, which is the form in which the many-to-one formula would produce it.) $\square$

8.3 Summability of the forward columns

Lemma 8.6. *Let $E > 0$ and let π be a real solution at energy E with $\varphi_\pi(1) < 0$. There exists $s_0 \in (0, 1)$ such that $\varphi_\pi(s) < 0$ for every $s \in [s_0, 1)$. For every such s , almost surely, for every word w ,*

$$\sum_{u \in V(\mathcal{T}_w)} |F_w(u)|^s < \infty.$$

Proof. By Proposition 6.15(i), the free energy φ_π is real-analytic, and in particular continuous, on $(0, 2)$. Since $\varphi_\pi(1) < 0$, there exists $s_0 \in (0, 1)$ with $\varphi_\pi < 0$ on $[s_0, 1)$. (By Proposition 6.15(v), φ_π is strictly decreasing on $(0, 1]$, so the set of $s \in (0, 1)$ with $\varphi_\pi(s) < 0$ is an interval with right endpoint 1.)

Fix $s \in [s_0, 1)$. By Proposition 6.27(iii), there exists $C < \infty$ such that $\Phi_L(s; E) \leq C e^{L\varphi_\pi(s)}$ for every $L \geq 1$, and $\Phi_0(s; E) = \mathbb{E}|D|^{-s} < \infty$ by Proposition 5.5. By Proposition 8.3(i) and Tonelli's theorem,

$$\mathbb{E} \sum_{v \in V} |F(v)|^s = \sum_{L \geq 0} \Phi_L(s; E) \leq \mathbb{E}|D|^{-s} + C \sum_{L \geq 1} e^{L\varphi_\pi(s)} < \infty,$$

since $\varphi_\pi(s) < 0$. We conclude that $\sum_v |F(v)|^s < \infty$ almost surely. For a word w , the column F_w is the same function of Θ_w as F is of Θ_o , and Θ_w has the law of Θ_o by (R3). Then $\sum_u |F_w(u)|^s < \infty$ almost surely, and a countable intersection over words w completes the proof. $\square$

8.4 Rerooted columns

We now construct inverse columns at arbitrary vertices of the full tree and of the components left after removing one vertex or one edge. A *removed set* is one of the following: the empty set; a word y , standing for the vertex y ; or a word $y \neq o$, standing for the edge $\{y^-, y\}$. Fix a removed set and a word r . On the event that $r \in V$, that $y \in V$ (if the removed set is not empty), and that $r \neq y$ (for a removed vertex), let C be the component containing r of the reduced forest obtained from $\mathcal{T}$ by the removal (with $C = V$ for the empty removed set). For a word x , let $\text{anc}(x)$ be the set of its prefixes, including o and x . We define

$$S = \text{anc}(r) \cup \text{anc}(y)$$

(with $S = \text{anc}(r)$ for the empty removed set), $S_C = S \cap C$, and

$$W_C = \{w \in C \setminus S : w^- \in S\}.$$

The set S is a finite set of words which contains o and all prefixes of its elements, and it is determined by the removed set and r . For $x \in C \setminus \{r\}$, let $b(x)$ be the neighbor of x on the path in C from x to r , and let $\text{ch}_C(x)$ be the set of neighbors $x' \neq b(x)$ of x in C ; let $\text{ch}_C(r)$ be the set of all neighbors of r in C .

Lemma 8.7. *On the event above, the following hold.*

- (i) *The set S_C contains r and induces a finite connected subtree of C .*
- (ii) *The set W_C is finite. For every $w \in W_C$, the set $V(\mathcal{T}_w)$ is contained in $C \setminus S$, and $C \setminus S_C$ is the disjoint union of the sets $V(\mathcal{T}_w)$, $w \in W_C$.*
- (iii) *Let $w \in W_C$ and $x \in V(\mathcal{T}_w)$. Then $b(x) = x^-$, and $\text{ch}_C(x)$ is the set of children of x in $\mathcal{T}$.*

(iv) If $x \in S_C$ and $x' \in \text{ch}_C(x) \setminus S_C$, then x' is a child of x and $x' \in W_C$.

Proof. For (i), we have $r \in S \cap C$. The set S is a union of ancestral paths starting at o , and it induces a subtree of $\mathcal{T}$. If $x, x' \in S_C$, then the path in $\mathcal{T}$ between them lies in S , and it lies in C because C induces a subtree of $\mathcal{T}$ (Lemma 3.5(i)). This shows that S_C is connected. It is finite because S is.

We next prove (ii). Let $w \in W_C$. If some $u \in V(\mathcal{T}_w)$ belonged to S , then its prefix w would belong to S , which is false. We find that $V(\mathcal{T}_w) \cap S = \emptyset$. In particular, $V(\mathcal{T}_w)$ contains neither the removed vertex y nor an endpoint of the removed edge, since both y and y^- lie in S . Every edge of $\mathcal{T}_w$ is therefore retained, and $V(\mathcal{T}_w)$ is connected in the reduced forest. Since it contains $w \in C$, we obtain $V(\mathcal{T}_w) \subset C \setminus S$. Conversely, let $u \in C \setminus S_C$, so that $u \notin S$. Since $o \in S$, there exists a shortest prefix w of u with $w \notin S$. Then $w \neq o$, $w^- \in S$, and $u \in V(\mathcal{T}_w)$. By the previous argument, $V(\mathcal{T}_w)$ lies in one component, which is C because it contains u . In particular $w \in C$. It remains to check that $w^- \in C$, that is, that the edge $\{w^-, w\}$ is retained and $w^- \neq y$. If the edge $\{w^-, w\}$ were the removed edge, then $w = y \in S$. If $w^- = y$ were the removed vertex, then the component of w would be $V(\mathcal{T}_w)$, which contains r ; then w would be a prefix of r and would lie in S . Both are impossible, and we conclude that $w^- \in S_C$ and $w \in W_C$. If $w, w' \in W_C$ are distinct and w is a prefix of w' , then $w'^- \in V(\mathcal{T}_w) \cap S$, which is impossible. We conclude that the sets $V(\mathcal{T}_w)$, $w \in W_C$, are disjoint. Finally, W_C is finite, since its elements are children of the finitely many vertices in S .

For (iii), we have $r \in S$ and $r \notin V(\mathcal{T}_w)$. The path from x to r therefore leaves $V(\mathcal{T}_w)$, and since $V(\mathcal{T}_w)$ is joined to the rest of $\mathcal{T}$ only by the edge $\{w^-, w\}$, the path first moves from x to x^- . All children of x belong to $V(\mathcal{T}_w) \subset C$, and the edges to them are retained. The neighbors of x in C other than x^- are its children.

For (iv), the parent of x belongs to S (if $x \neq o$), since S is closed under prefixes. It follows that x' is a child of x , and $x' \in C \setminus S$ with $x'^- = x \in S$, that is, $x' \in W_C$. $\square$

We now define real cavity marks on C , adapted to the root r . For $x \in C \setminus S_C$, we set $\gamma_x^C = \gamma_x$. For $x \in S_C$, we define, recursively from the leaves of the finite tree S_C rooted at r toward r ,

$$\Delta_x^C = E + \sum_{x' \in \text{ch}_C(x)} |h_{xx'}|^2 \gamma_{x'}^C, \quad \gamma_x^C = -\frac{1}{\Delta_x^C}. \quad (8.14)$$

This is well defined as long as the denominators Δ_x^C do not vanish: the elements of $\text{ch}_C(x)$ are either in S_C , where they are closer to the leaves, or in W_C by Lemma 8.7(iv). For the empty removed set and $r = o$, we have $S_C = \{o\}$, W_C is the set of children of o , and $\gamma_o^C = \gamma_o$. In general, the numbers γ_x^C with $x \in S_C$ are real counterparts of the cavity resolvents $\Gamma_{x \rightarrow b(x)}^C$ of the rerooted component (compare Remark 3.17).

Lemma 8.8. *Fix a removed set and a word r . Almost surely, on the event above, every denominator Δ_x^C , $x \in S_C$, is finite and nonzero; hence every γ_x^C , $x \in C$, is finite and nonzero.*

Proof. The proof uses that the squared modulus of the mark of each edge of the skeleton S_C is a fresh exponential variable for the denominator at its endpoint farther from the leaves. Let $W(S) = \{si : s \in S, i \in \mathbb{N}\} \setminus S$. Its elements are not prefixes of each other: if $w \in W(S)$ were a proper prefix of $w' = si \in W(S)$, then w would be a prefix of $s \in S$ and would belong to S . By (R3), applied to finite subsets of $W(S)$, and since the coordinates of ω are independent under $\mathbb{P}$, we see that the following random elements are independent: the variables N_s and $w_{s,si}$ for $s \in S$ and $i \in \mathbb{N}$, and the processes Θ_w for $w \in W(S)$. The event in question, the sets S_C and $W_C \subset W(S)$, and the maps b and ch_C are functions of $(N_s)_{s \in S}$. The weights of the edges with an endpoint in S_C are functions of the variables $w_{s,si}$, and γ_w is a function of Θ_w .

For $x \in S_C$ and $x' \in \text{ch}_C(x)$, let $C_{x' \rightarrow x}$ be the branch of C at x' away from x . By induction from the leaves, $\gamma_{x'}^C$ is a function of $(N_s)_{s \in S}$, of the marks of the edges with both endpoints in $C_{x' \rightarrow x}$, and of the processes Θ_w with $w \in W_C \cap C_{x' \rightarrow x}$.

We now prove by induction from the leaves of S_C toward r that, almost surely, Δ_x^C is finite and nonzero for every $x \in S_C$. Fix $x \in S_C$ and suppose the claim holds for the elements of $\text{ch}_C(x) \cap S_C$. Then $\gamma_{x'}^C$ is finite and nonzero for every $x' \in \text{ch}_C(x)$: for $x' \in S_C$ by the induction hypothesis, and for $x' \in W_C$ because $\gamma_{x'}^C = -1/D_{x'}$ with $D_{x'} \in \mathbb{R} \setminus \{0\}$. In particular Δ_x^C is finite. If $\text{ch}_C(x)$ is empty, then $\Delta_x^C = E \neq 0$. Otherwise, let x_* be the element of $\text{ch}_C(x)$ that comes first in a fixed enumeration of $\mathbb{U}$, which is a function of $(N_s)_{s \in S}$. Since there are countably many words, it suffices to show that $\Delta_x^C \neq 0$ almost surely on the event $\{x_* = x'\}$, for a fixed word x' . On this event, the edge $e = \{x, x'\}$ joins two vertices of $S \cup W(S)$, one of which is the parent of the other and lies in S . Its mark is therefore one of the variables $w_{s,si}$, with s and i determined by x and x' ; we denote it by ω_e . We write

$$\Delta_x^C = A + \frac{|\omega_e|^2}{p} \gamma_{x'}^C, \quad A = E + \sum_{x'' \in \text{ch}_C(x) \setminus \{x'\}} |h_{xx''}|^2 \gamma_{x''}^C.$$

Let $\mathcal{R}_e$ be the σ -algebra generated by all the random elements listed in the first paragraph except ω_e . Since the edge e has an endpoint at x , it is not an edge of any branch $C_{x'' \rightarrow x}$. By the previous paragraph, the event $\{x_* = x'\}$ and the variables A and $\gamma_{x'}^C$ are $\mathcal{R}_e$ -measurable. The variable ω_e is independent of $\mathcal{R}_e$, and $|\omega_e|^2$ has, conditionally on $\mathcal{R}_e$, the $\text{Exp}(1)$ law, which has no atoms. Since $\gamma_{x'}^C \neq 0$ almost surely on $\{x_* = x'\}$, we have

$$\mathbb{P}(\{\Delta_x^C = 0\} \cap \{x_* = x'\} \mid \mathcal{R}_e) = \mathbf{1}_{\{x_* = x'\}} \mathbb{P}(|\omega_e|^2 = -pA/\gamma_{x'}^C \mid \mathcal{R}_e) = 0$$

almost surely. This completes the induction, and the last assertion follows from (8.14) and $\gamma_x^C = \gamma_x$ for $x \notin S_C$. $\square$

Lemma 8.9. *Let $s \in (0, 1)$ be such that the conclusion of Lemma 8.6 holds. Fix a removed set and a word r , and work on the intersection of the almost sure events of Lemma 8.6, Lemma 8.8 and (R1)*

with the event above. Define $F^{C,r}$ on C by $F^{C,r}(r) = \gamma_r^C$ and $F^{C,r}(x) = -\gamma_x^C h_{xb(x)} F^{C,r}(b(x))$ for $x \neq r$. Then

$$(\mathcal{H}_{C,\text{form}} F^{C,r})(x) - EF^{C,r}(x) = \delta_r(x), \quad x \in C, \quad (8.15)$$

and $\sum_{x \in C} |F^{C,r}(x)|^s < \infty$.

Proof. We verify the hypotheses of Lemma 8.1 on the tree C with root r and the numbers γ_x^C . The neighbors of x in C are $b(x)$ (if $x \neq r$) and the elements of $\text{ch}_C(x)$. For $x \in S_C$, (8.1) is the definition (8.14), with nonzero denominators by Lemma 8.8. For $x \in V(\mathcal{T}_w)$ with $w \in W_C$, Lemma 8.7(iii) shows that $\text{ch}_C(x)$ is the set of children of x , all of which lie in $V(\mathcal{T}_w)$, so that $\gamma_{x'}^C = \gamma_{x'}$ for $x' \in \text{ch}_C(x)$. The relation (8.1) then holds by the computation after Definition 8.2. By Lemma 8.7(ii), these two cases cover C . Then Lemma 8.1 implies (8.15).

For summability, let $w \in W_C$. For $x \in V(\mathcal{T}_w)$ we have $b(x) = x^-$ and $\gamma_x^C = \gamma_x$, so $F^{C,r}$ and F_w satisfy the same recursion $f(x) = -\gamma_x h_{xx^-} f(x^-)$ on $V(\mathcal{T}_w) \setminus \{w\}$. Comparing their values at w , we find

$$F^{C,r}(x) = \frac{F^{C,r}(w)}{\gamma_w} F_w(x), \quad x \in V(\mathcal{T}_w).$$

By Lemma 8.7(ii),

$$\sum_{x \in C} |F^{C,r}(x)|^s = \sum_{x \in S_C} |F^{C,r}(x)|^s + \sum_{w \in W_C} \left| \frac{F^{C,r}(w)}{\gamma_w} \right|^s \sum_{x \in V(\mathcal{T}_w)} |F_w(x)|^s.$$

The sets S_C and W_C are finite, all values of $F^{C,r}$ are finite, $\gamma_w \neq 0$, and each inner sum is finite by Lemma 8.6. This proves that the left side is finite. $\square$

8.5 Proofs of the main results

We use the following characterization of the range of $T - E$ for a self-adjoint operator T .

Lemma 8.10. *Let T be a self-adjoint operator on a Hilbert space, let ψ be a vector, and let $E \in \mathbb{R}$. If $f \in \text{Dom}(T)$ and $(T - E)f = \psi$, then $\mu_\psi^T(\{E\}) = 0$ and $R_{\mu_\psi^T}(E) \leq \|f\|^2$. Conversely, if $R_{\mu_\psi^T}(E) < \infty$, then $\psi \in \text{Ran}(T - E)$.*

Proof. Suppose $(T - E)f = \psi$. By the spectral theorem, for every bounded Borel function k ,

$$\langle \psi, k(T)\psi \rangle = \int_{\mathbb{R}} k(\lambda)(\lambda - E)^2 \mu_f^T(d\lambda),$$

that is, $\mu_\psi^T(d\lambda) = (\lambda - E)^2 \mu_f^T(d\lambda)$. In particular, $\mu_\psi^T(\{E\}) = 0$ and $R_{\mu_\psi^T}(E) = \mu_f^T(\mathbb{R} \setminus \{E\}) \leq \|f\|^2$. Conversely, suppose $R_{\mu_\psi^T}(E) < \infty$. Then $\mu_\psi^T(\{E\}) = 0$. Let $k(\lambda) =$

$(\lambda - E)^{-1}$ for $\lambda \neq E$ and $k(E) = 0$. Then $\int |k|^2 d\mu_\psi^T = R_{\mu_\psi^T}(E) < \infty$, and since $|\lambda k(\lambda)| \leq 1 + |E||k(\lambda)|$, also $\int |\lambda k(\lambda)|^2 d\mu_\psi^T < \infty$. By the functional calculus, $f = k(T)\psi$ belongs to $\text{Dom}(T)$, and $(T - E)f = \mathbf{1}_{\mathbb{R} \setminus \{E\}}(T)\psi = \psi$, since $\|\mathbf{1}_{\{E\}}(T)\psi\|^2 = \mu_\psi^T(\{E\}) = 0$. $\square$

Proof of Proposition 8.3(ii) and (iii). Part (i) was proved in Section 8.2. We assume that $\varphi_\pi(1) < 0$, let s_0 be as in Lemma 8.6, and fix $s \in [s_0, 1)$.

STEP 1. The root column. For the empty removed set and $r = o$, we have $S_C = \{o\}$, $\gamma_o^C = \gamma_o$ and $\gamma_x^C = \gamma_x$ for $x \neq o$, and $F^{V,o} = F$. By Lemma 8.9, almost surely $F \in \ell^s(V)$, and $F \in \ell^2(V)$ because $\|f\|_2 \leq \|f\|_s$ for $s \leq 2$. By (8.15), $\mathcal{H}_{\text{form}}F = EF + \delta_o \in \ell^2(V)$. By the maximal domain characterization Theorem 3.10(ii), in the form of Remark 3.11, $F \in \text{Dom}(\mathcal{H})$ and $(\mathcal{H} - E)F = \delta_o$. This proves (ii).

STEP 2. All columns. Fix a removed set and a word r . On the intersection of the almost sure events of Lemma 8.9 and Theorem 3.10 with the event on which C is defined (see Section 8.4), Lemma 8.9 yields $F^{C,r} \in \ell^s(C) \subset \ell^2(C)$ and $\mathcal{H}_{C,\text{form}}F^{C,r} = EF^{C,r} + \delta_r \in \ell^2(C)$. By Remark 3.11, $F^{C,r} \in \text{Dom}(\mathcal{H}_C)$ and $(\mathcal{H}_C - E)F^{C,r} = \delta_r$. In particular $\delta_r \in \text{Ran}(\mathcal{H}_C - E)$, and Lemma 8.10 gives $R_{\mu_r^C}(E) \leq \|F^{C,r}\|^2 < \infty$. There are countably many removed sets and words r , so these conclusions hold almost surely for all of them simultaneously. For the empty removed set, $C = V$ and $\mu_r^C = \mu_r$. For the removal of a vertex y , the operator $\mathcal{H}^{(y)}$ is the orthogonal sum of the operators of the components (Theorem 3.10(iv)), and $\mu_x^{(y)} = \mu_x^C$ for $x \in C$; this gives (8.8).

STEP 3. Transfer to $\mathbb{P}$ and $\mathbb{P}^\infty$. Let $\mathcal{A}_E$ be the event that $R_{\mu_r^C}(E) < \infty$ for every removed set, every word r and the corresponding component C (whenever these are defined). By (7.3), $R_{\mu_r^C}(E) = \sup_n n \text{Im } G_{rr}^C(E + i/n)$, and by Lemma 7.5 this is a measurable function of the marked tree. It follows that $\mathcal{A}_E$ is an event of the marked tree. By Step 2, it has probability one on the space of Lemma 5.13, where the marked tree has the law $\mathbb{P}$, and we have $\mathbb{P}(\mathcal{A}_E) = 1$. Since $\mathbb{P}^\infty \leq \varrho_p^{-1}\mathbb{P}$, also $\mathbb{P}^\infty(\mathcal{A}_E) = 1$. By Lemma 8.10, on $\mathcal{A}_E$ every δ_r belongs to $\text{Ran}(\mathcal{H}_C - E)$. This proves (iii). $\square$

Proof of Theorem 8.4. Let $\mathcal{N}_0 \subset U$ be a Lebesgue-null set such that a real solution π with $\varphi_\pi(1) < 0$ exists at every $E \in U \setminus \mathcal{N}_0$. Let $\mathcal{B}$ be the set of pairs $(\omega, E) \in \Omega \times U$ such that, for every vertex y and every neighbor x of y , $R_{\mu_x^{(y)}}(E) < \infty$, where Ω is the probability space of the marked tree. For fixed words y, x , we have $R_{\mu_x^{(y)}}(E) = \sup_n n \text{Im } G_{xx}^{(y)}(E + i/n)$ by (7.3), which is jointly measurable in (ω, E) by Lemma 7.5. Since there are countably many pairs of words, $\mathcal{B}$ is jointly measurable.

For $E \in U \setminus \mathcal{N}_0$, Proposition 8.3(iii), applied with a real solution π at E with $\varphi_\pi(1) < 0$, gives $\mathbb{P}((\omega, E) \in \mathcal{B}) = 1$. The event $\mathcal{B}$ does not involve the real solution, and no

measurable dependence of the solution on E is needed. By Tonelli's theorem,

$$\mathbb{E}|\{E \in U : (\omega, E) \notin \mathcal{B}\}| = \int_U \mathbb{P}((\omega, E) \notin \mathcal{B}) \, dE = 0,$$

since the integrand vanishes outside the null set $\mathcal{N}_0$. This implies that, $\mathbb{P}$ -almost surely, for Lebesgue-almost every $E \in U$, we have $R_{\mu_x^{(y)}}(E) < \infty$ for every vertex y and every neighbor x of y . This is the hypothesis of Proposition 7.9 with $I = U$ and $\mathbb{Q} = \mathbb{P}$, and it also holds $\mathbb{P}^\infty$ -almost surely. By Proposition 7.9, $\mathcal{H}$ has pure point spectrum in $U \cup (-U)$, almost surely under $\mathbb{P}$ and under $\mathbb{P}^\infty$. $\square$

Remark 8.11. The energy 0 is excluded in Theorem 8.4, and indeed $\mathcal{H}$ has finitely supported eigenvectors with eigenvalue zero (Proposition 3.25). In Proposition 8.3, the real solution π is arbitrary, and the recursive tree process need not be endogenous, that is, the denominators D_v need not be functions of the marked tree. This causes no difficulty, because the assertions of Proposition 8.3(iii) are events of the marked tree; see Step 3 of the proof. If $E \in \mathcal{E}_p$, the no-ac hypothesis holds at E , and π is the real boundary law, then the boundary denominators of Proposition 5.11(ii) form an endogenous recursive tree process with marginal π (Remark 5.15). For this process, (3.19) shows that $F(v)$ is the limit of $G_{v0}(E + i\eta)$ as $\eta \downarrow 0$.

Resonances along paths

This section and the next prove that the boundary values of the cavity variables cannot all be real at an energy where the real boundary law has positive free energy at $s = 1$. The argument follows the resonance strategy of Aizenman and Warzel [12]; variants of it appear in [4, 7]. We look for vertices x at a large depth whose Green function entry G_{ox} is exponentially large, and we use the Ward identity to convert such a vertex into a large imaginary part at the root. In this section, we collect the deterministic identities and the estimates along one or two paths that this argument uses. The counting argument itself is given in Section 10.

In Section 9.1, we express the Green function between the endpoints of a path through a product of transfer matrices, and we record the Schur complement formula for two vertices. In Section 9.2, we study the end of a path: we show that two vertices with a single child each bring the rootward cavity variable into a fixed disc, uniformly in its incoming value, and we estimate the probability that a pendant leaf creates a large diagonal entry of the Green function, at one vertex and at two vertices simultaneously. Section 9.3 selects blocks of transfer matrices whose truncated free energy approximates the free energy and estimates the probability of a regular growth event, on which the growth of the transfer products along a path stays close to its typical value. Section 9.4 averages bilinear forms of products of transfer matrices over four groups of three independent factors. Finally, Section 9.5 records a semicontinuity property of probabilities used to pass from the real axis to a small positive imaginary part.

The results of Sections 9.1 and 9.2 are deterministic, or concern one or two squared moduli with an explicit density, and they hold under the hypotheses on p and z stated in each lemma; in particular, they do not use a real solution. From Section 9.3 on, we fix $p > 1$, an energy $E > 0$, and a real solution π at energy E ; in Section 10, π is the real boundary law. Whenever an energy E is fixed, we write $z = E + i\eta$ with $\eta \geq 0$. Statements about the tree hold on the almost sure event Ω_{sa} of Theorem 3.10, intersected with the almost sure event on which all edge weights are nonzero.

9.1 Transfer matrices along a path

In this section, we prove two deterministic identities for the resolvent of a component of a forest obtained by a finite removal. The first, Lemma 9.3, expresses the Green function

between the endpoints of a path, and its ratio to the diagonal entry at the far endpoint, through the product of the transfer matrices along the path. In Section 10, it is applied in Lemma 10.8 to the path from the root to a resonant vertex, where the ratio converts a large diagonal entry into a large entry G_{ox} , and in Lemma 10.11 to the path between the parents of two resonant vertices, where the Green function measures the interaction between them. The second, Lemma 9.4, is the Schur complement formula for the 2×2 block of the resolvent at two vertices; it is used in Lemma 10.12 to write the two diagonal entries at a pair of resonant vertices as functions of the weights of their pendant leaves.

We use the transfer matrices of Definition 6.1 with the scaling $g = 1$, in which the denominator is $B = D$ and $\beta = Y/p$ is the squared weight, and we allow complex denominators, as in Proposition 6.19.

Definition 9.1. (i) For $d \in \mathbb{C}$ and $t > 0$, the *factor* with denominator d and squared weight t is the matrix

$$M(d, t) = t^{-1/2} \begin{pmatrix} 0 & 1 \\ -t & d \end{pmatrix}.$$

For real d , this is the matrix $M(b, \beta)$ of Definition 6.1 with $b = d$ and $\beta = t$. We write $e_2 = (0, 1)^\top$.

(ii) Let C be a component of a forest obtained by a finite removal, let $w_0, w_1, \dots, w_r$ be a path in C with $r \geq 0$, and let $z \in \mathbb{C} \setminus \mathbb{R}$. For $0 \leq k \leq r$, the *side denominator* at w_k is

$$d_k = z + \sum_a |h_{w_k a}|^2 \Gamma_{a \rightarrow w_k}^C(z),$$

where the sum runs over the neighbors a of w_k in C other than w_{k-1} and w_{k+1} (only those of them that exist). For $1 \leq k \leq r$, we set $t_k = |h_{w_{k-1} w_k}|^2$. The *rootward denominators* are defined by $R_0 = d_0$ and $R_k = d_k - t_k/R_{k-1}$ for $1 \leq k \leq r$, and the *transfer product* of the path is

$$A = M(d_0, t_1) M(d_1, t_2) \cdots M(d_{r-1}, t_r),$$

with $A = I$ if $r = 0$.

In words, the k -th factor of a path pairs the side denominator at w_k with the squared weight of the edge leaving w_k towards w_{k+1} . Every factor has determinant one, since $\det M(d, t) = t^{-1}(0 \cdot d + t) = 1$, and so does every product of factors. Let A be a complex 2×2 matrix with $\det A = 1$, and let $\sigma_1 \geq \sigma_2 \geq 0$ be its singular values. Then $\sigma_1 \sigma_2 = |\det A| = 1$, so that $\sigma_2 > 0$, $\sigma_1 = \sigma_2^{-1}$, and $\sigma_1 \geq 1$. Since $\|A\| = \sigma_1$ and $\|A^{-1}\| = \sigma_2^{-1}$, we have

$$\|A^{-1}\| = \|A\| \geq 1. \quad (9.1)$$

Bounding the operator norm by the Hilbert–Schmidt norm, we have

$$\|M(d, t)\| \leq t^{-1/2} (1 + t^2 + |d|^2)^{1/2}. \quad (9.2)$$

The following algebraic identity relates products of factors to continuants.

Lemma 9.2. *Let $r \geq 1$, let $d_0, \dots, d_r \in \mathbb{C}$, and let $t_1, \dots, t_r > 0$. Define $K_{-2} = 0$, $K_{-1} = 1$, $K_0 = d_0$, and $K_k = d_k K_{k-1} - t_k K_{k-2}$ for $1 \leq k \leq r$. Let $A = M(d_0, t_1) \cdots M(d_{r-1}, t_r)$ and $\tau = (t_1 \cdots t_r)^{1/2}$. Then*

$$\tau e_2^\top A = (-t_r K_{r-2}, K_{r-1}).$$

In particular, $\tau e_2^\top A e_2 = K_{r-1}$ and $\tau e_2^\top A(1, d_r)^\top = K_r$. If the numbers $R_0 = d_0$ and $R_k = d_k - t_k/R_{k-1}$, $1 \leq k \leq r$, are well defined and nonzero, then $K_k = R_0 R_1 \cdots R_k$ for $0 \leq k \leq r$.

Proof. Let $N_k = t_{k+1}^{1/2} M(d_k, t_{k+1})$, which has the rows $(0, 1)$ and $(-t_{k+1}, d_k)$. We show by induction on $l \in \{1, \dots, r\}$ that $e_2^\top N_0 \cdots N_{l-1} = (-t_l K_{l-2}, K_{l-1})$. For $l = 1$, we have $e_2^\top N_0 = (-t_1, d_0) = (-t_1 K_{-1}, K_0)$. If the claim holds for $l < r$, then

$$e_2^\top N_0 \cdots N_l = (-t_l K_{l-2}, K_{l-1}) \begin{pmatrix} 0 & 1 \\ -t_{l+1} & d_l \end{pmatrix} = (-t_{l+1} K_{l-1}, d_l K_{l-1} - t_l K_{l-2}),$$

which is $(-t_{l+1} K_{l-1}, K_l)$. Since $N_0 \cdots N_{r-1} = \tau A$, the case $l = r$ is the first claim. Multiplying it by e_2 and by $(1, d_r)^\top$ gives K_{r-1} and $-t_r K_{r-2} + d_r K_{r-1} = K_r$. For the last claim, we have $K_0 = R_0$, and if $K_{k-1} = R_0 \cdots R_{k-1}$ and $K_{k-2} = R_0 \cdots R_{k-2}$ (with the empty product equal to 1), then $K_k = R_0 \cdots R_{k-1}(d_k - t_k/R_{k-1}) = R_0 \cdots R_k$. $\square$

We now identify the rootward denominators with cavity variables and obtain the path formulas.

Lemma 9.3. *In the setting of Definition 9.1(ii), suppose that $r \geq 1$. Then the rootward denominators are well defined and nonzero, and*

$$R_k = -\frac{1}{\Gamma_{w_k \rightarrow w_{k+1}}^C(z)} \quad (0 \leq k < r), \quad R_r = -\frac{1}{G_{w_r w_r}^C(z)}. \quad (9.3)$$

Further,

$$|G_{w_0 w_r}^C(z)| = |e_2^\top A(1, d_r)^\top|^{-1}, \quad (9.4)$$

$$\left| \frac{G_{w_0 w_r}^C(z)}{G_{w_r w_r}^C(z)} \right| = |e_2^\top A e_2|^{-1} \geq \|A\|^{-1}. \quad (9.5)$$

Proof. We prove (9.3) by induction on k . By (3.12) for the oriented edge from w_0 to w_1 , the number $-1/\Gamma_{w_0 \rightarrow w_1}^C(z)$ equals $z + \sum_a |h_{w_0 a}|^2 \Gamma_{a \rightarrow w_0}^C(z)$, where a runs over the neighbors of w_0 in C other than w_1 ; this is $d_0 = R_0$. Let $1 \leq k < r$, and suppose that $R_{k-1} = -1/\Gamma_{w_{k-1} \rightarrow w_k}^C(z)$. The neighbors of w_k in C other than w_{k+1} are w_{k-1} and the neighbors entering the definition of d_k . Then (3.12) implies that

$$-\frac{1}{\Gamma_{w_k \rightarrow w_{k+1}}^C(z)} = d_k + |h_{w_k w_{k-1}}|^2 \Gamma_{w_{k-1} \rightarrow w_k}^C(z) = d_k - \frac{t_k}{R_{k-1}} = R_k.$$

The same computation with (3.11) at w_r , whose neighbors are w_{r-1} and the neighbors entering d_r , gives the second identity in (9.3). All these numbers are finite and nonzero, since by Proposition 3.16 the expressions in parentheses in (3.11) and (3.12) have imaginary parts of absolute value at least $|\operatorname{Im} z| > 0$.

By (3.19), applied to the path from $u = w_r$ to $v = w_0$, that is, to the vertices $v_l = w_{r-l}$, we have

$$G_{w_0 w_r}^C(z) = (-1)^r G_{w_r w_r}^C(z) \prod_{k=0}^{r-1} h_{w_k w_{k+1}} \Gamma_{w_k \rightarrow w_{k+1}}^C(z).$$

Taking absolute values and using (9.3), we find

$$|G_{w_0 w_r}^C(z)| = \frac{(t_1 \cdots t_r)^{1/2}}{|R_0 R_1 \cdots R_r|}, \quad \left| \frac{G_{w_0 w_r}^C(z)}{G_{w_r w_r}^C(z)} \right| = \frac{(t_1 \cdots t_r)^{1/2}}{|R_0 \cdots R_{r-1}|}.$$

By Lemma 9.2, $R_0 \cdots R_r = K_r = \tau e_2^\top A(1, d_r)^\top$ and $R_0 \cdots R_{r-1} = K_{r-1} = \tau e_2^\top A e_2$, where $\tau = (t_1 \cdots t_r)^{1/2}$; here the weights t_k are positive, since the edge weights are nonzero. This proves (9.4) and the identity in (9.5); in particular, $e_2^\top A e_2 = \tau^{-1} R_0 \cdots R_{r-1} \neq 0$. The inequality in (9.5) follows from the Cauchy–Schwarz inequality $|e_2^\top A e_2| = |\langle e_2, A e_2 \rangle| \leq \|A e_2\| \leq \|A\|$, which holds for every complex 2×2 matrix (the condition $\det A = 1$ is not needed here). $\square$

In (9.5), the side denominator d_r at the endpoint does not enter; the ratio depends only on the side denominators $d_0, \dots, d_{r-1}$ and on the weights of the path. In Section 10, we apply (9.5) to a path from the root to a vertex x of the full tree, and (9.4) to the path between the parents of two vertices x and y , in the tree from which x and y have been deleted.

The next lemma is the Schur complement formula for the 2×2 block of the resolvent at two vertices. For a component C of the forest obtained by a finite removal (F, D) and $x, y \in C$, we write $G_{ab}^{C \setminus \{x, y\}}(z)$ for the entries of the resolvent of the reduced operator of the finite removal $(F \cup \{x, y\}, D)$ at vertices $a, b \in C \setminus \{x, y\}$. These vanish if a and b lie in different components of the corresponding reduced forest.

Lemma 9.4. *Let C be a component of a forest obtained by a finite removal, let $x, y \in C$ be distinct and not adjacent, and let $z \in \mathbb{C} \setminus \mathbb{R}$. For $u, v \in \{x, y\}$, set*

$$S_{uv}(z) = -z \mathbf{1}_{\{u=v\}} - \sum_{a \sim_C u} \sum_{b \sim_C v} h_{ua} G_{ab}^{C \setminus \{x, y\}}(z) h_{bv},$$

where $a \sim_C u$ means that a is a neighbor of u in C . Then the matrix $S(z) = (S_{uv}(z))_{u, v \in \{x, y\}}$ is invertible, and

$$\begin{pmatrix} G_{xx}^C(z) & G_{xy}^C(z) \\ G_{yx}^C(z) & G_{yy}^C(z) \end{pmatrix} = S(z)^{-1}. \quad (9.6)$$

Let x' and y' be the neighbors of x and y on the path from x to y in C . Then

$$S_{xy}(z) = -h_{xx'} G_{x'y'}^{C \setminus \{x,y\}}(z) h_{y'y'}, \quad |S_{xy}(z) S_{yx}(z)| = |h_{xx'}|^2 |h_{y'y'}|^2 |G_{x'y'}^{C \setminus \{x,y\}}(z)|^2, \quad (9.7)$$

and $S_{xx}(z) = -z - \sum_{a \sim_C x} |h_{xa}|^2 G_{aa}^{C \setminus \{x,y\}}(z)$; the analogous formula holds for $S_{yy}(z)$. Finally, if $\text{Im } z > 0$, then the Hermitian matrix $\text{Im } S(z) = (S(z) - S(z)^*) / (2i)$ satisfies $\text{Im } S(z) \leq -(\text{Im } z)I$.

Proof. Let D' be the set of edges of C with an endpoint in $\{x, y\}$. By Lemma 3.13, the components contained in C of the reduced forest obtained by adding D' to the removed edges are $\{x\}$, $\{y\}$, and the components of $C \setminus \{x, y\}$. With respect to $\ell^2(C) = \mathbb{C}^{\{x,y\}} \oplus \mathfrak{h}$, where $\mathfrak{h} = \ell^2(C \setminus \{x, y\})$, the same lemma gives $\mathcal{H}_C = (0 \oplus K) + W_{D'}$ and $\text{Dom}(\mathcal{H}_C) = \mathbb{C}^{\{x,y\}} \oplus \text{Dom}(K)$, where K is the direct sum of the operators of the components of $C \setminus \{x, y\}$. These components are also the components contained in C of the reduced forest of the finite removal $(F \cup \{x, y\}, D)$, and in both reduced forests the operator of such a component is the closure of the weighted adjacency operator of the same induced subtree (Lemma 3.5(i)). Then $\langle \delta_a, (K - z)^{-1} \delta_b \rangle = G_{ab}^{C \setminus \{x,y\}}(z)$ for $a, b \in C \setminus \{x, y\}$. Define the linear map $J: \mathbb{C}^{\{x,y\}} \rightarrow \mathfrak{h}$ by $J\delta_u = \sum_{a \sim_C u} h_{au} \delta_a$; it is bounded, since x and y have finitely many neighbors. Since x and y are not adjacent, we have, for $f \in \mathbb{C}^{\{x,y\}}$ and $g \in \text{Dom}(K)$,

$$\mathcal{H}_C(f \oplus g) = J^*g \oplus (Jf + Kg).$$

Indeed, the coordinate of $W_{D'}g$ at $u \in \{x, y\}$ is $\sum_{a \sim_C u} h_{ua}g(a) = \langle J\delta_u, g \rangle$, because $h_{ua} = \overline{h_{au}}$.

Fix $f \in \mathbb{C}^{\{x,y\}}$ and set $g = -(K - z)^{-1}Jf \in \text{Dom}(K)$. Then $f \oplus g \in \text{Dom}(\mathcal{H}_C)$ and

$$(\mathcal{H}_C - z)(f \oplus g) = (-zf - J^*(K - z)^{-1}Jf) \oplus 0 = (S(z)f) \oplus 0,$$

where we used that the matrix of $J^*(K - z)^{-1}J$ has the entries $\langle J\delta_u, (K - z)^{-1}J\delta_v \rangle = \sum_{a,b} h_{ua} G_{ab}^{C \setminus \{x,y\}}(z) h_{bv}$. Since $\mathcal{H}_C$ is self-adjoint, $\mathcal{H}_C - z$ is a bijection from $\text{Dom}(\mathcal{H}_C)$ onto $\ell^2(C)$ with inverse $(\mathcal{H}_C - z)^{-1}$. Then $(\mathcal{H}_C - z)^{-1}((S(z)f) \oplus 0) = f \oplus g$. Projecting onto $\mathbb{C}^{\{x,y\}}$, we find $\Pi(\mathcal{H}_C - z)^{-1}\Pi S(z)f = f$, where Π is the orthogonal projection onto $\mathbb{C}^{\{x,y\}}$. In particular, $S(z)$ is injective, hence invertible, and its inverse is the matrix on the left side of (9.6).

For (9.7), let $a \sim_C x$ and $b \sim_C y$ lie in the same component of $C \setminus \{x, y\}$. A path from a to b in this component, preceded by x and followed by y , is a path from x to y in the tree C . Since paths in a tree are unique, we have $a = x'$ and $b = y'$. All other terms in the sum defining $S_{xy}(z)$ vanish, which gives the first identity. The same argument shows that $S_{yx}(z) = -h_{yy'} G_{y'x'}^{C \setminus \{x,y\}}(z) h_{x'x}$, and Lemma 3.20, applied to the component of $C \setminus \{x, y\}$ containing x' and y' , gives $|G_{y'x'}^{C \setminus \{x,y\}}(z)| = |G_{x'y'}^{C \setminus \{x,y\}}(z)|$. This proves the second

identity. Finally, two distinct neighbors of x lie in different components of $(F \cup \{x\}, D)$, by Lemma 3.5(ii), and a fortiori in different components of $C \setminus \{x, y\}$, since deleting y as well only splits components further; the cross terms in the sum defining $S_{xx}(z)$ vanish, and we obtain the formula for $S_{xx}(z)$.

For the last claim, let $\operatorname{Im} z > 0$ and $f \in \mathbb{C}^{\{x, y\}}$, and set $g = Jf$. Since K is self-adjoint, we have $\operatorname{Im} \langle g, (K - z)^{-1} g \rangle = \operatorname{Im} z \|(K - z)^{-1} g\|^2 \geq 0$. We have $\langle f, \operatorname{Im} S(z) f \rangle = \operatorname{Im} \langle f, S(z) f \rangle = -\operatorname{Im} z \|f\|^2 - \operatorname{Im} \langle g, (K - z)^{-1} g \rangle \leq -\operatorname{Im} z \|f\|^2$. $\square$

9.2 Endpoint estimates

In this section, we study the last few vertices of a path. The path to a resonant vertex of Section 10 ends with a single-child segment, that is, a sequence of vertices with exactly one child each, and its endpoint has a pendant leaf and a surviving child. Lemmas 9.5 and 9.6 show that, with a probability bounded below uniformly in the rest of the tree, the single-child segment places the denominator with the leaf removed at the endpoint in a fixed region; they are used in Lemmas 10.8 and 10.9. Lemma 9.7 bounds the probability that the weight of the pendant leaf makes the diagonal resolvent entry at the endpoint large, from above and from below; the lower bound is used in the first moment (Lemma 10.9). Lemma 9.8 bounds the probability that this happens at two vertices simultaneously; it is used in the second moment (Lemma 10.12). Finally, Lemma 9.9 records the properties of the quantile $\zeta_\alpha(\eta)$ of $\operatorname{Im} \Gamma_o$ on which the contradiction argument of Section 10.6 is based.

We fix $E > 0$ and write $z = E + i\eta$. We first show that two consecutive vertices without side children bring the rootward cavity variable into a fixed disc, uniformly in its incoming value. Consider a vertex w with a single neighbor w' other than the next vertex of a path, and let $X = -\Gamma_{w' \rightarrow w}(z)$ be the incoming rootward variable. If the squared weight of the edge ww' is t , then by (3.12) the outgoing variable is $(z - tX)^{-1}$. We set

$$\rho_1 = e^{-1} - e^{-2}, \quad \varepsilon_E = \frac{\rho_1 E}{16}, \quad b_2 = \frac{\varepsilon_E E}{8}, \quad b_3 = \frac{3E^2}{32}, \quad (9.8)$$

and we let $\mathfrak{D}_E = \{X \in \mathbb{C} : |X - E^{-1}| \leq (3E)^{-1}\}$. Every $X \in \mathfrak{D}_E$ satisfies $|X| \leq 4/(3E)$.

Lemma 9.5. *Let $E > 0$ and $0 \leq \eta \leq E/8$.*

(i) *Let $X \in \mathbb{C}$, and let t be a random variable with density pe^{-pt} on $(0, \infty)$. Then*

$$\mathbb{P}\left(\frac{1}{p} < t < \frac{2}{p}, |z - tX| > \varepsilon_E\right) \geq \frac{\rho_1}{2}.$$

- (ii) *If $|X| < \varepsilon_E^{-1}$ and $0 < t < b_2$, then $z - tX \neq 0$ and $(z - tX)^{-1} \in \mathfrak{D}_E$.*
 (iii) *If $X \in \mathfrak{D}_E$ and $0 < t \leq b_3$, then $z - tX \neq 0$ and $(z - tX)^{-1} \in \mathfrak{D}_E$.*

Proof. We first note that if $w \in \mathbb{C}$ satisfies $|w - E| \leq E/4$, then $|w| \geq 3E/4$, and

$$\left| \frac{1}{w} - \frac{1}{E} \right| = \frac{|E - w|}{|w|E} \leq \frac{E/4}{3E^2/4} = \frac{1}{3E}, \quad (9.9)$$

so that $w^{-1} \in \mathfrak{D}_E$. For (ii), we have $|z - tX - E| \leq \eta + t|X| < E/8 + b_2/\varepsilon_E = E/4$, and (9.9) applies. For (iii), we have $t|X| \leq b_3 \cdot 4/(3E) = E/8$, so that $|z - tX - E| \leq E/4$, and (9.9) applies again.

For (i), we have $\mathbb{P}(1/p < t < 2/p) = e^{-1} - e^{-2} = \rho_1$, and it suffices to show that the event $\{1/p < t < 2/p, |z - tX| \leq \varepsilon_E\}$ has probability at most $\rho_1/2$. If $|X| \leq pE/4$, then for $t < 2/p$ we have $|z - tX| \geq \operatorname{Re} z - t|X| > E - E/2 = E/2 > \varepsilon_E$, and the event is empty. If $|X| > pE/4$, then $|z - tX| = |X| |t - z/X|$, and the set of real t with $|z - tX| \leq \varepsilon_E$ is contained in an interval of length $2\varepsilon_E/|X| < 8\varepsilon_E/(pE)$. Since the density of t is at most p , the event has probability at most $8\varepsilon_E/E = \rho_1/2$. $\square$

In applications, X is random and $t = Y/p$, where Y is the squared modulus of an edge mark. Part (i) then holds conditionally, in the following sense. Let $\mathcal{G}$ be a σ -field, let X be a $\mathcal{G}$ -measurable complex random variable, and suppose that Y is fresh given $\mathcal{G}$ in the sense of Definition 4.16, with $M = 1$; that is, Y has the law $\operatorname{Exp}(1)$ conditionally on $\mathcal{G}$. We apply the substitution identity (4.14) with $m = 1$, $c = X$, $a_1 = 0$, and

$$\Phi(c, a_1, y) = \mathbf{1}\left\{1 < y < 2, \left|z - \frac{y}{p}c\right| > \varepsilon_E\right\}.$$

For each fixed value of X , the integral of $\Phi(X, 0, \cdot)$ against $\operatorname{Exp}(1)$ is the probability in (i) with $t = Y/p$, which has density pe^{-pt} . We conclude that

$$\mathbb{P}\left(\frac{1}{p} < t < \frac{2}{p}, |z - tX| > \varepsilon_E \mid \mathcal{G}\right) \geq \frac{\rho_1}{2} \quad \text{almost surely.}$$

In the same way, Lemmas 9.7 and 9.8 below hold conditionally on a σ -field $\mathcal{G}$ given which the squared moduli of the marks of the leaf edges are fresh and the parameters are revealed: the parameter p in Lemma 9.7, and in Lemma 9.8 the three numbers B_{xx} , B_{yy} , and $B_{xy}B_{yx}$, through which alone the event and the bound depend. In each case, (4.14) is applied with c and $a_1, \dots, a_m$ equal to these revealed numbers.

The next lemma describes the end of a path. In Section 10, the endpoint x of a path has three neighbors: its parent, whose rootward variable X has been brought into $\mathfrak{D}_E$ by the single-child segment, as in Lemma 9.5; a child y with a bounded cavity variable; and a pendant leaf. The *denominator with the leaf removed* at x is the quantity Δ below.

Lemma 9.6. *Let $E > 0$, $0 \leq \eta \leq E/8$, and $R' > 0$. Let $X \in \mathfrak{D}_E$, $0 < t \leq b_3$, $X' \in \mathbb{C}$ with $|X'| \leq R'$, and $0 < t' \leq E/(8R')$, and set $\Delta = z - tX - t'X'$. Then $|\Delta - E| \leq 3E/8$ and*

$$\frac{E^2}{2} \leq \operatorname{Re}(z\Delta) \leq \frac{3E^2}{2}.$$

Proof. We have $|\Delta - E| \leq \eta + t|X| + t'|X'| \leq E/8 + E/8 + E/8 = 3E/8$, where we used $t|X| \leq b_3 \cdot 4/(3E) = E/8$. In particular, $5E/8 \leq \operatorname{Re} \Delta \leq 11E/8$ and $|\operatorname{Im} \Delta| \leq 3E/8$. Since $\operatorname{Re}(z\Delta) = E \operatorname{Re} \Delta - \eta \operatorname{Im} \Delta$ and $\eta \leq E/8$, we have

$$\operatorname{Re}(z\Delta) \geq \frac{5E^2}{8} - \frac{3E^2}{64} \geq \frac{E^2}{2}, \quad \operatorname{Re}(z\Delta) \leq \frac{11E^2}{8} + \frac{3E^2}{64} \leq \frac{3E^2}{2}. \quad \square$$

We next estimate the probability that a pendant leaf creates a large diagonal entry. If x has a leaf child ℓ with squared weight $t = |h_{x\ell}|^2$, then $\Gamma_{\ell \rightarrow x}(z) = -1/z$, and (3.11) gives

$$G_{xx}(z) = -\left(\Delta - \frac{t}{z}\right)^{-1} = \frac{z}{t - z\Delta}, \quad (9.10)$$

where $\Delta = -1/G_{xx}^{\{\ell\}}(z)$ is the denominator with the leaf removed, computed in the tree from which ℓ is deleted. In particular, $G_{xx}(z)$ is a Möbius function of t , with a pole at the pole parameter $\mathfrak{p} = z\Delta$. If $\eta > 0$, then $\operatorname{Im} \Delta \geq \eta > 0$ and $\operatorname{Im} \mathfrak{p} = E \operatorname{Im} \Delta + \eta \operatorname{Re} \Delta$.

Lemma 9.7. *Let $p > 0$, $E > 0$, $0 < \eta \leq 1$, and $T \geq 4(E+1)/E^2$, and let t be a random variable with density pe^{-pt} on $(0, \infty)$. Let $\mathfrak{p} \in \mathbb{C} \setminus \mathbb{R}$.*

(i) *We have $\mathbb{P}(|z| \geq T|t - \mathfrak{p}|) \leq 2p|z|/T$.*

(ii) *If $E^2/2 \leq \operatorname{Re} \mathfrak{p} \leq 3E^2/2$ and $|\operatorname{Im} \mathfrak{p}| \leq |z|/(2T)$, then*

$$\mathbb{P}(|z| \geq T|t - \mathfrak{p}|) \geq \frac{c_{\text{tune}}}{T}, \quad c_{\text{tune}} = pEe^{-2pE^2}.$$

Proof. The set of real t with $|t - \mathfrak{p}| \leq |z|/T$ is empty if $|\operatorname{Im} \mathfrak{p}| > |z|/T$, and otherwise it is the interval of center $\operatorname{Re} \mathfrak{p}$ and half-length $\varrho = ((|z|/T)^2 - (\operatorname{Im} \mathfrak{p})^2)^{1/2} \leq |z|/T$. Since the density of t is at most p , this proves (i). Under the assumptions of (ii), we have $\varrho \geq (\sqrt{3}/2)|z|/T \geq (\sqrt{3}/2)E/T$. Further, $\varrho \leq |z|/T \leq (E+1)/T \leq E^2/4$, so that the interval is contained in $[E^2/4, 7E^2/4]$, where the density of t is at least $pe^{-7pE^2/4}$. The probability in (ii) is therefore at least $2\varrho pe^{-7pE^2/4} \geq \sqrt{3} pEe^{-7pE^2/4}/T \geq c_{\text{tune}}/T$. $\square$

By (9.10), the event in Lemma 9.7 with $\mathfrak{p} = z\Delta$ is the event $|G_{xx}(z)| \geq T$.

For two vertices x and y with pendant leaves, Lemma 9.4 expresses the 2×2 block of the resolvent at $\{x, y\}$ as a function of the squared weights of the two leaf edges. The following lemma bounds the probability that both diagonal entries are large.

Lemma 9.8. *Let $p \geq 1$, $z \in \mathbb{C} \setminus \mathbb{R}$, and $T > 0$, and let $B = (B_{uv})_{u,v \in \{x,y\}}$ be a complex 2×2 matrix such that $\operatorname{diag}(t_x, t_y) - B$ is invertible for all $t_x, t_y > 0$. Set*

$$R(t_x, t_y) = z(\operatorname{diag}(t_x, t_y) - B)^{-1},$$

and let t_x, t_y be independent random variables with density pe^{-pt} on $(0, \infty)$. Then

$$\mathbb{P}(|R_{xx}| \geq T, |R_{yy}| \geq T) \leq \frac{8p^2|z|^2}{T^2} + \frac{8p^2|z|}{T} \min\{1, |B_{xy}B_{yx}|^{1/2}\}.$$

Proof. Set $\theta = |z|/T$, $X = t_x - B_{xx}$, $Y = t_y - B_{yy}$, and $c = B_{xy}B_{yx}$. The determinant $XY - c$ is nonzero, and

$$R_{xx} = \frac{zY}{XY - c}, \quad R_{yy} = \frac{zX}{XY - c}.$$

Let $\mathcal{E}$ be the event in the statement. We first bound the probability of $\{|R_{xx}| \geq T\}$, which contains $\mathcal{E}$. Fix t_y . If $Y = 0$, then $R_{xx} = 0$. If $Y \neq 0$, then $R_{xx} = z/(X - c/Y)$, and $|R_{xx}| \geq T$ holds if and only if $|t_x - B_{xx} - c/Y| \leq \theta$, which confines t_x to an interval of length at most 2θ . By Tonelli's theorem and the bound p on the density of t_x , we obtain $\mathbb{P}(\mathcal{E}) \leq 2p\theta$.

Next, $|R_{xx}| \geq T$ and $|R_{yy}| \geq T$ mean that $|XY - c| \leq \theta|Y|$ and $|XY - c| \leq \theta|X|$, so on $\mathcal{E}$ we have $|XY - c| \leq \theta \min\{|X|, |Y|\}$. Consider the part of $\mathcal{E}$ on which $|X| \leq |Y|$. There $X \neq 0$: otherwise the last inequality would give $|XY - c| \leq 0$, contradicting $XY - c \neq 0$. Dividing by $|X|$, we find $|Y - c/X| \leq \theta$. Then $|X| \leq |Y| \leq |c|/|X| + \theta$, that is, $|X|^2 - \theta|X| \leq |c|$, which implies $|X| \leq \theta + |c|^{1/2}$; indeed, if $|X| > \theta + |c|^{1/2}$, then $|X|(|X| - \theta) > (\theta + |c|^{1/2})|c|^{1/2} \geq |c|$. Thus t_x lies in an interval of length at most $2(\theta + |c|^{1/2})$, and for each such t_x , the condition $|t_y - B_{yy} - c/X| \leq \theta$ confines t_y to an interval of length at most 2θ . The part of $\mathcal{E}$ on which $|X| \leq |Y|$ is therefore contained in a set of area at most $4\theta(\theta + |c|^{1/2})$, and by symmetry the same holds for the part on which $|Y| \leq |X|$. Since the joint density of (t_x, t_y) is at most p^2 , we find $\mathbb{P}(\mathcal{E}) \leq 8p^2\theta(\theta + |c|^{1/2})$.

Combining the two bounds, and using $2p\theta \leq 8p^2\theta$ for $p \geq 1$ when $|c| \geq 1$, we find

$$\mathbb{P}(\mathcal{E}) \leq \min\{2p\theta, 8p^2\theta^2 + 8p^2\theta|c|^{1/2}\} \leq 8p^2\theta^2 + 8p^2\theta \min\{1, |c|^{1/2}\},$$

which is the claim. $\square$

Finally, we record the properties of the quantiles of the imaginary part of the root cavity variable that we use. For $p > 1$, $E > 0$, $\eta > 0$, and $\alpha \in (0, 1)$, we define

$$\zeta_\alpha(\eta) = \sup\{t \geq 0 : \mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta) \geq t) \geq \alpha\}. \quad (9.11)$$

Lemma 9.9. *Let $p > 1$, $E > 0$, and $\alpha \in (0, 1)$.*

- (i) *For every $\eta > 0$, we have $0 < \zeta_\alpha(\eta) < \infty$, $\mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta) \geq \zeta_\alpha(\eta)) \geq \alpha$, and $\mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta) \geq t) < \alpha$ for every $t > \zeta_\alpha(\eta)$.*
- (ii) *Let $r \in (0, 1)$ and $C_r = \sup_{0 < \eta \leq 1} \mathbb{E}^\infty |\Gamma_o(E + i\eta)|^r$, which is finite by Lemma 5.17. Set $R_{\text{aux}} = (4C_r/\alpha)^{1/r}$. Then for every $\eta \in (0, 1]$,*

$$\mathbb{P}^\infty(|\Gamma_o(E + i\eta)| < R_{\text{aux}}, \operatorname{Im} \Gamma_o(E + i\eta) \geq \zeta_\alpha(\eta)/2) \geq \frac{3\alpha}{4}.$$

Proof. Fix $\eta > 0$, write $I = \operatorname{Im} \Gamma_o(E + i\eta)$, and let $\psi(t) = \mathbb{P}^\infty(I \geq t)$. By Lemma 3.19(i), $I > 0$ almost surely, so $\psi(t) \rightarrow 1 > \alpha$ as $t \downarrow 0$, and the set in (9.11) contains some $t > 0$.

Since $\psi(t) \leq \mathbb{P}^\infty(|\Gamma_o(E + i\eta)| \geq t) \rightarrow 0$ as $t \rightarrow \infty$, the set is bounded. We conclude that $0 < \zeta_\alpha(\eta) < \infty$. The function ψ is nonincreasing, and it is left-continuous because $\{I \geq t\} = \bigcap_{t' < t} \{I \geq t'\}$. Then the set in (9.11) is the interval $[0, \zeta_\alpha(\eta)]$. This proves (i). For (ii), Markov's inequality implies that $\mathbb{P}^\infty(|\Gamma_o(E + i\eta)| \geq R_{\text{aux}}) \leq C_r R_{\text{aux}}^{-r} = \alpha/4$, and by (i), $\psi(\zeta_\alpha(\eta)/2) \geq \psi(\zeta_\alpha(\eta)) \geq \alpha$. The claim follows. $\square$

9.3 Blocks and regular growth

The path from the root to a resonant vertex is required to have a transfer product whose norm grows at a prescribed exponential rate $\hat{\gamma}$, with fluctuations of order $b_n = n^{3/4}$. In this section, we estimate the probability of this requirement for independent factors. To control the norm of a product by the norms of its parts, and to have increments that are bounded, we group the factors into blocks of a fixed length m and keep only blocks whose denominators and weights lie in a fixed compact set. Lemma 9.11 chooses m and the compact set so that the free energy of the truncated blocks is close to the free energy φ , together with its derivative at the exponent s of Lemma 6.16. Lemma 9.12 then shows, by an exponential change of measure, that the *regular growth event*, on which the i -th partial sum of the block growths stays within distance b of $im\hat{\gamma}$ for every i , has probability $e^{-imk \pm sb}$. The lower bound enters the first moment (Lemma 10.9), and the upper bound enters the second moment (Lemma 10.11).

We fix $p > 1$, $E > 0$, and a real solution π at energy E , and we write $\varphi = \varphi_\pi$ for its free energy. The *factor law* ν_{fac} is the law of (d, t) , where d has law π , $t = Y/p$ with $Y \sim \text{Exp}(1)$, and d and Y are independent. Factors $M(d_i, t_i)$ with independent $(d_i, t_i) \sim \nu_{\text{fac}}$ are independent copies of the matrix $M(B, \beta)$ of Section 6 with the scaling $g = 1$, in which $B = D$ has law π and β has the exponential law of rate $K = p$. For $u \in (0, 1)$ and $r \geq 0$, we let

$$f_r(u) = \mathbb{E} \|M(d_1, t_1) \cdots M(d_r, t_r)\|^{-u}$$

be the norm moments of Definition 6.14, computed with independent factors of law ν_{fac} ; the order of the factors does not affect the law of the product. By Proposition 5.5(iii), $\mathbb{E}|d|^{-u} < \infty$, and Corollary 6.22 gives

$$f_r(u) \leq e^{r(\varphi(u) - \log p)}, \quad r \geq 0, u \in (0, 1). \quad (9.12)$$

Definition 9.10. Let $m \geq 1$ and $R > 1$.

- (i) A *block* of length m is a family $(d_i, t_i)_{i=1}^m$ with $d_i \in \mathbb{C}$ and $t_i > 0$. Its *growth* is

$$Z = \log \|M(d_1, t_1)M(d_2, t_2) \cdots M(d_m, t_m)\| \geq 0.$$

The block is *retained* if $R^{-1} \leq |d_i| \leq R$ and $R^{-1} \leq t_i \leq R$ for $1 \leq i \leq m$, and *strictly retained* if these inequalities are strict. We write H_R and H_R° for the corresponding events.

(ii) For independent factors $(d_i, t_i) \sim \nu_{\text{fac}}$, the block free energy is

$$\varphi_{m,R}(u) = \log p + \frac{1}{m} \log \mathbb{E}[\mathbf{1}_{H_R} e^{-uZ}], \quad u \in \mathbb{R}.$$

The growth is nonnegative by (9.1). By (9.2), a factor with $R^{-1} \leq |d| \leq R$ and $R^{-1} \leq t \leq R$ has norm at most $C_R = R^{1/2}(1 + 2R^2)^{1/2}$, and a retained block satisfies

$$0 \leq Z \leq m \log C_R. \quad (9.13)$$

The probability $\mathbb{P}(H_R)$ is positive. Indeed, since π has full support (Proposition 5.5(iv)), the set $\{R^{-1} \leq |d| \leq R\}$, which contains the open interval (R^{-1}, R) , has positive π -probability; the event $\{R^{-1} \leq t \leq R\}$ has positive probability since $R > 1$; and H_R is an intersection of $2m$ independent such events. Let $\Lambda_{m,R}(u) = \log \mathbb{E}[\mathbf{1}_{H_R} e^{-uZ}]$, so that $\varphi_{m,R}(u) = \log p + m^{-1} \Lambda_{m,R}(u)$. Since $0 \leq Z \leq m \log C_R$ on H_R and $\mathbb{P}(H_R) > 0$, the function $u \mapsto \mathbb{E}[\mathbf{1}_{H_R} e^{-uZ}]$ is finite and positive, and it is smooth on $\mathbb{R}$, with derivatives $\mathbb{E}[\mathbf{1}_{H_R} (-Z)^k e^{-uZ}]$ obtained by differentiating under the expectation (the integrands are bounded uniformly for u in compact sets). Then $\varphi_{m,R}$ is finite and smooth on $\mathbb{R}$. It is convex by Hölder's inequality: for $u_0, u_1 \in \mathbb{R}$ and $\tau \in (0, 1)$, applying Hölder's inequality with the exponents $1/(1-\tau)$ and $1/\tau$ to $\mathbf{1}_{H_R} e^{-(1-\tau)u_0 Z} \cdot \mathbf{1}_{H_R} e^{-\tau u_1 Z}$, we find

$$\Lambda_{m,R}((1-\tau)u_0 + \tau u_1) \leq (1-\tau)\Lambda_{m,R}(u_0) + \tau\Lambda_{m,R}(u_1).$$

The next lemma replaces the free energy by a block free energy, with a small loss in the quantities of Lemma 6.16.

Lemma 9.11. *Suppose that $\varphi(1) > 0$, and let $s \in (1/2, 1)$, $\gamma = -\varphi'(s) > 0$, and $F = \varphi(s) - (1-s)\gamma > 0$ be as in Lemma 6.16. There exist $m \geq 1$ and $R > \max\{E, E^{-1}\}$ with the following properties. Write $\widehat{\varphi} = \varphi_{m,R}$, and set*

$$\widehat{\gamma} = -\widehat{\varphi}'(s), \quad \widehat{F} = \widehat{\varphi}(s) - (1-s)\widehat{\gamma}, \quad \iota = \log p - \widehat{\varphi}(s) - s\widehat{\gamma}.$$

- (i) $\widehat{\varphi}(s) \leq \varphi(s)$.
- (ii) $\widehat{\gamma} \geq \gamma/2$, $\widehat{F} \geq F/2$, and $(2s-1)\widehat{\gamma} - 2(\varphi(s) - \widehat{\varphi}(s)) \geq (2s-1)\gamma/2$.
- (iii) $\iota \geq 0$ and $\log p - \iota = \widehat{\varphi}(s) + s\widehat{\gamma} > 0$.
- (iv) $\mathbb{P}(H_R \setminus H_R^\circ) = 0$.

Proof. The proof has three steps. We first let $m \rightarrow \infty$ without truncation, then let $R \rightarrow \infty$ at fixed m , and finally verify the inequalities.

STEP 1. Long blocks. Let Z be the growth of a block of m independent factors, and set $\varphi^{(m)}(u) = \log p + m^{-1} \log \mathbb{E} e^{-uZ} = \log p + m^{-1} \log f_m(u)$ for $u \in (0, 1)$. By Corollary 6.22, $m^{-1} \log f_m(u) \rightarrow \varphi(u) - \log p$ as $m \rightarrow \infty$, that is, $\varphi^{(m)}(u) \rightarrow \varphi(u)$ for every $u \in (0, 1)$; by (9.12), $\varphi^{(m)}(u) \leq \varphi(u)$. By Hölder's inequality, as in the discussion after (9.13), $\varphi^{(m)}$ is convex on $(0, 1)$. It is differentiable, and the derivative of

$u \mapsto \mathbb{E}e^{-uZ}$ is $-\mathbb{E}[Ze^{-uZ}]$. Indeed, for $u, u+h \in (0,1)$, the mean value theorem gives $|e^{-(u+h)Z} - e^{-uZ}| \leq |h|Z$, since $Z \geq 0$, and we may differentiate under the expectation by dominated convergence with the dominating function Z . This function is integrable, since $Z \leq \sum_{i=1}^m \log \|M(d_i, t_i)\|$ by submultiplicativity, and $\mathbb{E} \log \|M(d_1, t_1)\| < \infty$ by Proposition 6.15(iii). For $0 < h < \min\{s, 1-s\}$, convexity gives

$$\frac{\varphi^{(m)}(s) - \varphi^{(m)}(s-h)}{h} \leq (\varphi^{(m)})'(s) \leq \frac{\varphi^{(m)}(s+h) - \varphi^{(m)}(s)}{h}.$$

Letting $m \rightarrow \infty$, we find that $\liminf_m (\varphi^{(m)})'(s)$ and $\limsup_m (\varphi^{(m)})'(s)$ lie between $(\varphi(s) - \varphi(s-h))/h$ and $(\varphi(s+h) - \varphi(s))/h$. Letting $h \downarrow 0$, and using that φ is differentiable at s (Proposition 6.15(i)), we conclude that $(\varphi^{(m)})'(s) \rightarrow \varphi'(s)$.

STEP 2. Truncation. Fix m . Since $d_i \neq 0$ and $t_i \in (0, \infty)$ almost surely, we have $\mathbf{1}_{H_R} \uparrow 1$ almost surely as $R \rightarrow \infty$. By monotone convergence, $\mathbb{E}[\mathbf{1}_{H_R} e^{-sZ}] \uparrow \mathbb{E}e^{-sZ}$ and $\mathbb{E}[\mathbf{1}_{H_R} Z e^{-sZ}] \uparrow \mathbb{E}[Z e^{-sZ}]$, and both limits are finite and the first is positive. Since $\varphi'_{m,R}(s) = -m^{-1} \mathbb{E}[\mathbf{1}_{H_R} Z e^{-sZ}] / \mathbb{E}[\mathbf{1}_{H_R} e^{-sZ}]$, we obtain $\varphi_{m,R}(s) \rightarrow \varphi^{(m)}(s)$ and $\varphi'_{m,R}(s) \rightarrow (\varphi^{(m)})'(s)$ as $R \rightarrow \infty$. Further, $\varphi_{m,R}(s) \leq \varphi^{(m)}(s) \leq \varphi(s)$, because $\mathbf{1}_{H_R} \leq 1$.

STEP 3. Choice of m and R . Let $\varepsilon = \min\{\gamma/2, F/4, (2s-1)\gamma/8\}$. By Steps 1 and 2, we may choose m and then $R > \max\{E, E^{-1}\}$ such that $|\widehat{\varphi}(s) - \varphi(s)| \leq \varepsilon$ and $|\widehat{\varphi}'(s) - \varphi'(s)| \leq \varepsilon$. Then (i) holds by Step 2, and

$$\widehat{\gamma} \geq \gamma - \varepsilon \geq \frac{\gamma}{2}, \quad \widehat{F} \geq \varphi(s) - \varepsilon - (1-s)(\gamma + \varepsilon) \geq F - 2\varepsilon \geq \frac{F}{2},$$

$$(2s-1)\widehat{\gamma} - 2(\varphi(s) - \widehat{\varphi}(s)) \geq (2s-1)(\gamma - \varepsilon) - 2\varepsilon \geq (2s-1)\gamma - 3\varepsilon \geq \frac{(2s-1)\gamma}{2},$$

where we used $(2s-1)\varepsilon \leq \varepsilon$. This proves (ii). For (iii), let $\Lambda = \Lambda_{m,R}$ for the chosen m and R , so that $\Lambda(u) = m(\widehat{\varphi}(u) - \log p)$. It is convex and differentiable on $\mathbb{R}$, with $\Lambda'(s) = -m\widehat{\gamma}$, and

$$m\iota = m \log p - m\widehat{\varphi}(s) - sm\widehat{\gamma} = -\Lambda(s) + s\Lambda'(s).$$

A convex differentiable function lies above its tangent lines; at the point 0, the tangent line at s gives $\Lambda(0) \geq \Lambda(s) + \Lambda'(s)(0-s)$. Then $m\iota \geq -\Lambda(0) = -\log \mathbb{P}(H_R) \geq 0$. (Equivalently, $m\iota$ is the relative entropy of the tilted block law $\widehat{\mu}$ of the proof of Lemma 9.12 with respect to the law of a block, and $-\log \mathbb{P}(H_R)$ is a lower bound for it.) Further, $\log p - \iota = \widehat{\varphi}(s) + s\widehat{\gamma} = \widehat{F} + \widehat{\gamma} > 0$. For (iv), the only atom of π is at E (Proposition 5.5(i)), and $R, R^{-1} \neq E$; thus $\mathbb{P}(|d_i| \in \{R, R^{-1}\}) = 0$, and $\mathbb{P}(t_i \in \{R, R^{-1}\}) = 0$ since t_i has a density. $\square$

From now on in this section, $m, R, \widehat{\varphi}, \widehat{\gamma}, \widehat{F}$, and ι are as in Lemma 9.11. The number ι is the exponential cost per factor of the regular growth events of the next lemma, on

which the partial sums of the block growths stay close to $m\hat{\gamma}$ per block, their typical value under the exponentially tilted block law.

Lemma 9.12. *Let $(d_i, t_i)_{i \geq 1}$ be independent factors with law ν_{fac} , and for $l \geq 1$ let Z_l , H_l , and H_l° be the growth and the retention events of the block $(d_i, t_i)_{(l-1)m < i \leq lm}$. Let $S_k = Z_1 + \cdots + Z_k$, let $k \geq 1$, and let $b > 0$.*

(i) *We have*

$$\mathbb{P}(H_1 \cap \cdots \cap H_k \cap \{S_k \leq km\hat{\gamma} + b\}) \leq e^{-\imath mk + sb}.$$

(ii) *We have*

$$\mathbb{P}(H_1^\circ \cap \cdots \cap H_k^\circ \cap \{|S_i - \imath m\hat{\gamma}| < b \text{ for } 1 \leq i \leq k\}) \geq e^{-\imath mk - sb} \left(1 - \frac{k(m \log C_R)^2}{b^2}\right).$$

Proof. Let μ be the law of a block, that is, the law of $(d_i, t_i)_{i=1}^m$ on $(\mathbb{C} \times (0, \infty))^m$, let $\Lambda = \Lambda_{m,R}$ be as in the proof of Lemma 9.11, and let $\hat{\mu}$ be the tilted law $\hat{\mu}(\mathbf{db}) = e^{-\Lambda(s)} \mathbf{1}_{H_R}(\mathbf{b}) e^{-sZ(\mathbf{b})} \mu(\mathbf{db})$, where $Z(\mathbf{b})$ is the growth of the block $\mathbf{b}$. It is a probability measure by the definition of $\Lambda(s)$. Let $\hat{\mathbb{P}}$ be a probability measure under which the blocks $\mathbf{b}_1, \mathbf{b}_2, \dots$ are independent with law $\hat{\mu}$, and write $\hat{\mathbb{E}}$ for its expectation. For every measurable $\Phi \geq 0$ of the first k blocks, we have

$$\mathbb{E} \left[\Phi \prod_{l=1}^k \mathbf{1}_{H_l} \right] = e^{k\Lambda(s)} \hat{\mathbb{E}} [\Phi e^{sS_k}]. \quad (9.14)$$

Indeed, since the blocks are independent with law μ under $\mathbb{P}$, the right side equals

$$\begin{aligned} & e^{k\Lambda(s)} \int \Phi(\mathbf{b}_1, \dots, \mathbf{b}_k) e^{s \sum_{l=1}^k Z(\mathbf{b}_l)} \prod_{l=1}^k e^{-\Lambda(s)} \mathbf{1}_{H_R}(\mathbf{b}_l) e^{-sZ(\mathbf{b}_l)} \mu(\mathbf{db}_l) \\ &= \int \Phi(\mathbf{b}_1, \dots, \mathbf{b}_k) \prod_{l=1}^k \mathbf{1}_{H_R}(\mathbf{b}_l) \mu(\mathbf{db}_l), \end{aligned}$$

which is the left side. Note that $k\Lambda(s) + skm\hat{\gamma} = km(\hat{\varphi}(s) - \log p + s\hat{\gamma}) = -\imath mk$. Applying (9.14) with $\Phi = \mathbf{1}_{\{S_k \leq km\hat{\gamma} + b\}}$ and bounding $e^{sS_k} \leq e^{s(km\hat{\gamma} + b)}$ on this event, we conclude (i).

For (ii), let $\mathcal{A}$ be the event in (ii). Since $\hat{\mu} \ll \mu$ and $\mu(H_R \setminus H_R^\circ) = 0$ by Lemma 9.11(iv), under $\hat{\mathbb{E}}$ every block lies in H_R° almost surely. Applying (9.14) with $\Phi = \mathbf{1}_{\mathcal{A}}$, and using $e^{sS_k} \geq e^{s(km\hat{\gamma} - b)}$ on $\mathcal{A}$, we find

$$\mathbb{P}(\mathcal{A}) \geq e^{-\imath mk - sb} \hat{\mathbb{P}}(|S_i - \imath m\hat{\gamma}| < b \text{ for } 1 \leq i \leq k).$$

Under $\hat{\mathbb{P}}$, the variables Z_l are independent and identically distributed, and they take values in $[0, m \log C_R]$ by (9.13). Their mean is

$$\hat{\mathbb{E}} Z_1 = e^{-\Lambda(s)} \hat{\mathbb{E}} [\mathbf{1}_{H_R} Z e^{-sZ}] = -\Lambda'(s) = m\hat{\gamma},$$

where we used the formula for the derivative of Λ from the discussion after (9.13). In particular, their variance satisfies $\widehat{\text{Var}}(Z_1) \leq \widehat{\mathbb{E}}Z_1^2 \leq (m \log C_R)^2$. It remains to show that

$$\widehat{\mathbb{P}}\left(\max_{1 \leq i \leq k} |S'_i| \geq b\right) \leq \frac{k \widehat{\text{Var}}(Z_1)}{b^2}, \quad S'_i = S_i - im\widehat{\gamma}, \quad (9.15)$$

which is Kolmogorov's maximal inequality (see [64, Section 2.5]); since the right side is at most $k(m \log C_R)^2/b^2$, this proves (ii). For completeness, we include the proof. Let σ be the smallest $i \in \{1, \dots, k\}$ with $|S'_i| \geq b$, and $\sigma = \infty$ if there is none. For $1 \leq i \leq k$, the event $\{\sigma = i\}$ and the variable S'_i are functions of the first i blocks, while $S'_k - S'_i = \sum_{l=i+1}^k (Z_l - m\widehat{\gamma})$ is a function of the blocks $i+1, \dots, k$, which are independent of the first i blocks; further, $\widehat{\mathbb{E}}[S'_k - S'_i] = 0$. All these variables are bounded. Expanding the square, we find

$$\begin{aligned} \widehat{\mathbb{E}}[(S'_k)^2 \mathbf{1}\{\sigma = i\}] &= \widehat{\mathbb{E}}[(S'_i)^2 \mathbf{1}\{\sigma = i\}] + 2 \widehat{\mathbb{E}}[S'_i \mathbf{1}\{\sigma = i\}] \widehat{\mathbb{E}}[S'_k - S'_i] \\ &\quad + \widehat{\mathbb{E}}[(S'_k - S'_i)^2 \mathbf{1}\{\sigma = i\}] \geq b^2 \widehat{\mathbb{P}}(\sigma = i), \end{aligned}$$

where we used the independence in the middle term, which therefore vanishes, that the third term is nonnegative, and that $|S'_i| \geq b$ on $\{\sigma = i\}$ in the first term. Summing over $1 \leq i \leq k$, and using that the events $\{\sigma = i\}$ are disjoint with union $\{\max_{i \leq k} |S'_i| \geq b\}$, we obtain

$$b^2 \widehat{\mathbb{P}}\left(\max_{1 \leq i \leq k} |S'_i| \geq b\right) \leq \widehat{\mathbb{E}}[(S'_k)^2] = \sum_{l=1}^k \widehat{\text{Var}}(Z_l) = k \widehat{\text{Var}}(Z_1),$$

where in the first equality we used that the centered variables $Z_l - m\widehat{\gamma}$ are independent. This proves (9.15). $\square$

The upper bound in (i) requires only an upper bound on the final partial sum. In Section 10, it is applied to the blocks of the common part of two paths and to the blocks of each of their two arms separately.

9.4 Averaging over fresh factors

In the second moment, the interaction between two resonant vertices is measured by the Green function between their parents, in the tree from which both vertices are deleted. By (9.4), its inverse is a bilinear form of the transfer product along the path between the parents. This path runs from the first parent up to the last common ancestor, which we call the fork, and down to the second parent. Its transfer product consists of the factors of the two arms and the factor at the fork. In this section, we bound negative moments of such bilinear forms by the norms of the products along the two arms. The factor at the fork depends on the environment of the common part of the two paths. This environment is constrained by the regular growth condition, so we cannot average over it, and we

treat the factor at the fork as an arbitrary matrix of determinant one. To separate the two arms across this factor, and to remove the two endpoint vectors, we average over four groups of three independent factors, which we call *averaging triples*. Two of them are next to the endpoints of the path, and two are next to the fork. The remaining factors are arbitrary, provided they are independent of the averaging triples. Lemmas 9.13 and 9.14 are applied in the proof of Lemma 10.11. Recall the constant C_* of Proposition 6.18.

Lemma 9.13. *Let $s \in (0, 1)$, $K > 0$, and $a_+, a_- \geq 0$, and let C_* be the constant of Proposition 6.18. Let $\mathcal{G}$ be a σ -algebra, and let G_1, G_2, G_3, G_4 be random matrices such that $\mathcal{G}, G_1, G_2, G_3, G_4$ are independent, and each G_i is a product of three factors as in Proposition 6.19, with denominators satisfying the moment bounds with a_+ and a_- , and with parameters β of the exponential law of rate K . Let A_1, A_2, A_3 be $\mathcal{G}$ -measurable random complex 2×2 matrices with determinant one, and let w be a $\mathcal{G}$ -measurable random vector in $\mathbb{C}^2 \setminus \{0\}$. Then, almost surely,*

$$\mathbb{E} \left[|e_2^T G_1 A_1 G_2 A_2 G_3 A_3 G_4 w|^{-s} \mid \mathcal{G} \right] \leq C_*^4 \|A_1\|^{-s} \|A_3\|^{-s} \|w\|^{-s}.$$

Proof. The proof averages first over the averaging triples G_1 and G_4 next to the endpoints, with everything else fixed, and then over the averaging triples G_2 and G_3 next to the fork.

Let $\mathcal{G}'$ be the σ -algebra generated by $\mathcal{G}$, G_2 , and G_3 . Since $\mathcal{G}, G_1, \dots, G_4$ are independent, the σ -algebra $\mathcal{G}'$ and the matrices G_1 and G_4 are independent (by grouping independent families); in particular, $\mathcal{G}'$ is independent of (G_1, G_4) , and G_1 and G_4 are independent. The matrix $A = A_1 G_2 A_2 G_3 A_3$ is $\mathcal{G}'$ -measurable and has determinant one, so it is nonzero, and the vector w is $\mathcal{G}'$ -measurable and nonzero. By Corollary 6.21(i) and (iii), applied with the σ -algebra $\mathcal{G}'$, with $u = e_2$, $v = w$, and with the averaging triples $G = G_1$ and $G' = G_4$, we have

$$\mathbb{E} \left[|e_2^T G_1 A G_4 w|^{-s} \mid \mathcal{G}' \right] \leq C_*^2 \|A\|^{-s} \|w\|^{-s}.$$

Here we used $\|e_2\| = 1$. Next, $\mathcal{G}$ is independent of (G_2, G_3) , and G_2 and G_3 are independent. By Corollary 6.21(ii) and (iii), applied with the σ -algebra $\mathcal{G}$, with $C = A_1$, $G = G_2$, $F = A_2$, $G' = G_3$, and $A = A_3$, and since $\|A_2\| \geq 1$ by (9.1), we have

$$\mathbb{E} [\|A_1 G_2 A_2 G_3 A_3\|^{-s} \mid \mathcal{G}] \leq C_*^2 \|A_1\|^{-s} \|A_3\|^{-s}.$$

Since $\mathcal{G} \subset \mathcal{G}'$, the tower property gives

$$\mathbb{E} \left[|e_2^T G_1 A G_4 w|^{-s} \mid \mathcal{G} \right] = \mathbb{E} \left[\mathbb{E} \left[|e_2^T G_1 A G_4 w|^{-s} \mid \mathcal{G}' \right] \mid \mathcal{G} \right] \leq C_*^2 \|w\|^{-s} \mathbb{E} [\|A\|^{-s} \mid \mathcal{G}],$$

where we used that w is $\mathcal{G}$ -measurable. Inserting the second bound, we obtain the claim. $\square$

The factors A_1 and A_3 will be products of factors along the two arms of a pair of paths. Some of these factors have a deterministic denominator, and the next lemma removes them at a fixed cost.

Lemma 9.14. *Let $u \in (0, 1)$, $r, k \geq 0$, and let $c_1, \dots, c_k \in \mathbb{C}$ with $0 < |c_i| \leq E + 1$. Let V be a product of r independent factors of law ν_{fac} (with $V = I$ if $r = 0$), and let $H = M(c_1, t'_1) \cdots M(c_k, t'_k)$, where $t'_1, \dots, t'_k$ are independent with density pe^{-pt} and independent of V . Set $C_{\text{arm}} = \mathbb{E}[(t'_1)^{-u/2}(1 + (t'_1)^2 + (E + 1)^2)^{u/2}]$, which is finite and at least one. Then*

$$\mathbb{E}\|HV\|^{-u} \leq C_{\text{arm}}^k e^{r(\varphi(u) - \log p)}, \quad \mathbb{E}\|VH\|^{-u} \leq C_{\text{arm}}^k e^{r(\varphi(u) - \log p)}.$$

Proof. By (9.1), $\|V\| = \|H^{-1}HV\| \leq \|H\| \|HV\|$, which implies that $\|HV\|^{-u} \leq \|H\|^u \|V\|^{-u}$. By submultiplicativity and (9.2), $\|H\|^u \leq \prod_{i=1}^k (t'_i)^{-u/2}(1 + (t'_i)^2 + (E + 1)^2)^{u/2}$. Taking expectations, and using that $t'_1, \dots, t'_k$ and V are independent together with (9.12), we obtain the first bound. The constant C_{arm} is finite, since $(1 + (t'_1)^2 + (E + 1)^2)^{u/2} \leq 1 + (t'_1)^u + (E + 1)^u$ for $u/2 \leq 1$ and since $\mathbb{E}(t'_1)^{-u/2}$ and $\mathbb{E}(t'_1)^u$ are finite; it is at least one, since $(t'_1)^{-1}(1 + (t'_1)^2) \geq 2$. The second bound follows in the same way from $\|V\| \leq \|VH\| \|H^{-1}\|$. $\square$

9.5 Passage to the real axis

In Section 10, the events whose probabilities we bound are defined at $E + i\eta$ with a small $\eta > 0$, while the probabilities are computed for the real recursive tree process at $\eta = 0$. The passage between the two uses only almost sure convergence, through the following form of the portmanteau theorem. Part (i) gives lower bounds for events defined by strict inequalities, and it is used in Lemmas 10.4 and 10.5 and in the proof of Lemma 10.9. Part (ii) gives upper bounds for events defined by non-strict inequalities, weighted by a bounded factor, and it is used in Lemma 10.4 and in the proof of Lemma 10.12. Neither part requires the limiting law to give zero mass to the boundary of the event, and neither requires uniform integrability.

Lemma 9.15. *Let $(\Omega', \mathcal{A}, \mathbb{Q})$ be a probability space, let $k \geq 1$, and for $0 \leq \eta \leq 1$ let V_η be random vectors in $\mathbb{C}^k$ such that $V_\eta \rightarrow V_0$ almost surely as $\eta \downarrow 0$.*

- (i) *For every open set $O \subset \mathbb{C}^k$, we have $\liminf_{\eta \downarrow 0} \mathbb{Q}(V_\eta \in O) \geq \mathbb{Q}(V_0 \in O)$.*
- (ii) *Let $F \subset \mathbb{C}^k$ be closed, and let h_η , $0 \leq \eta \leq 1$, be random variables with $0 \leq h_\eta \leq C$ for a constant C and $h_\eta \rightarrow h_0$ almost surely. Then*

$$\limsup_{\eta \downarrow 0} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_F(V_\eta)h_\eta] \leq \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_F(V_0)h_0].$$

Proof. It suffices to consider a sequence $\eta_j \downarrow 0$. For (i), if $V_0 \in O$ and $V_{\eta_j} \rightarrow V_0$, then $V_{\eta_j} \in O$ for all large j , since O is open. This implies that $\liminf_j \mathbf{1}_O(V_{\eta_j}) \geq \mathbf{1}_O(V_0)$ almost surely, and (i) follows from Fatou's lemma. For (ii), if $V_0 \notin F$, then $V_{\eta_j} \notin F$ for all large j , since the complement of F is open. Hence $\limsup_j \mathbf{1}_F(V_{\eta_j})h_{\eta_j} \leq \mathbf{1}_F(V_0)h_0$ almost surely, and Fatou's lemma applied to $C - \mathbf{1}_F(V_{\eta_j})h_{\eta_j} \geq 0$ gives (ii). $\square$

We apply Lemma 9.15 under the spine laws of Definition 4.6, with V_η a finite collection of coordinates of the depth- m data $\mathcal{L}_m(E + i\eta)$ of Definition 5.19, or of continuous functions of them. Under the no-ac hypothesis, Proposition 5.25(i) provides the required almost sure convergence. Events defined by strict inequalities between continuous functions of the coordinates are of the form $\{V_\eta \in O\}$ with O open, and events defined by non-strict inequalities are of the form $\{V_\eta \in F\}$ with F closed. The growth of a block is a continuous function of the factors only where the weights t_i are positive. In Section 10, it appears only together with the retention conditions of Definition 9.10: a strict inequality for partial sums of growths together with strict retention defines an open set, since the set of strictly retained blocks is open and the growths are continuous on it; and a non-strict inequality together with retention defines a closed set, since the set of retained blocks is closed and contained in the open set $\{t_i > 0 \text{ for all } i\}$, on which the growths are continuous. Further, if A_η , $0 \leq \eta \leq 1$, are events with $\mathbf{1}_{A_\eta} \rightarrow \mathbf{1}_{A_0}$ almost surely as $\eta \downarrow 0$ (for instance, events that do not depend on η), we may add $\mathbf{1}_{A_\eta}$ as a coordinate of V_η . Since $\mathbf{1}_{A_\eta} \in \{0, 1\}$, the event $A_\eta \cap \{V_\eta \in O\}$ is then of the form $\{(V_\eta, \mathbf{1}_{A_\eta}) \in O \times \{w \in \mathbb{C} : \operatorname{Re} w > 1/2\}\}$, with an open set, and $A_\eta \cap \{V_\eta \in F\}$ is of the form $\{(V_\eta, \mathbf{1}_{A_\eta}) \in F \times \{w \in \mathbb{C} : \operatorname{Re} w \geq 1/2\}\}$, with a closed set. In Section 10, lower bounds are proved for strict inequalities and upper bounds for non-strict ones, and no condition on the boundaries of the events is needed.

Positive free energy implies delocalization

In this section, we prove that the boundary values of the cavity variables cannot all be real at a regular energy where the real boundary law has positive free energy at $s = 1$. Together with the spectral averaging criterion of Proposition 7.11, this gives purely absolutely continuous spectrum on intervals where, for almost every regular energy, all possible real boundary laws have positive free energy at $s = 1$. Recall from Definition 5.12 that, for $E \in \mathcal{E}_p$ with $E > 0$, the no-ac hypothesis holds at E if $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$; in this case, the law π_E of $D_o = -1/\Gamma_o(E + i0)$ under $\mathbb{P}$ is a real solution at energy E , by Proposition 5.11(ii), and we call it the real boundary law.

Theorem 10.1. *There exists a numerical constant $p_* > 1$ such that the following holds for every $p \geq p_*$. Let $E \in \mathcal{E}_p$ with $E > 0$, and suppose that $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Then the real boundary law π_E satisfies $\varphi_{\pi_E}(1) \leq 0$.*

Corollary 10.2. *Let $p \geq p_*$, where p_* is as in Theorem 10.1, and let $I \subset (0, \infty)$ be an interval. Suppose that for Lebesgue-almost every $E \in I \cap \mathcal{E}_p$, either $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) < 1$, or $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$ and the real boundary law π_E satisfies $\varphi_{\pi_E}(1) > 0$. Then $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ is purely absolutely continuous in $I \cup (-I)$.*

In particular, if $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) < 1$ for every $E \in I \cap \mathcal{E}_p$, then the conclusion of Corollary 10.2 holds.

Proof of Corollary 10.2, assuming Theorem 10.1. By Lemma 5.8, the set $\mathcal{E}_p$ has full Lebesgue measure. Let $E \in I \cap \mathcal{E}_p$ be an energy at which the hypothesis holds; this is the case for almost every $E \in I$. If $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$, then Theorem 10.1 implies that $\varphi_{\pi_E}(1) \leq 0$, contradicting the hypothesis. Thus $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) < 1$. By Proposition 5.11(i), this probability equals $1 - u/q_p \in \{0, 1\}$, where $u = \mathbb{P}(\text{Im } \Gamma_o(E + i0) > 0) \in \{0, q_p\}$. Hence it equals zero, and $\mathbb{P}(\text{Im } \Gamma_o(E + i0) > 0) = u = q_p > 0$. This verifies the hypothesis of Proposition 7.11 at almost every $E \in I$, and Proposition 7.11 gives the conclusion. $\square$

10.1 Outline of the proof

The proof of Theorem 10.1 follows the resonance strategy of Aizenman and Warzel, who used it in [12] to prove absolutely continuous spectrum for the Anderson model on

regular trees beyond the band of the hopping operator. Versions of this strategy for tree operators were developed by Aggarwal, Bordenave, and Lopatto for the local limits of Lévy matrices [7], and by Aggarwal and Lopatto for the Anderson model on the Bethe lattice [4]. We adapt it to the Poisson tree with complex Gaussian weights. The argument uses only the properties of real solutions established in Sections 5 and 6, so it applies to an arbitrary real boundary law. We now describe its steps.

The quantile. Fix $p \geq p_*$ and $E \in \mathcal{E}_p$ with $E > 0$, suppose that the no-ac hypothesis holds at E , and suppose, for a contradiction, that $\varphi(1) > 0$, where $\varphi = \varphi_{\pi_E}$. Let $\alpha = q_*/2$ be the numerical level fixed in (10.5) below. For $\eta > 0$, let $\zeta_\alpha(\eta)$ be the quantile of $\text{Im } \Gamma_o(E + i\eta)$ under $\mathbb{P}^\infty$ defined in (9.11). By Lemma 9.9(i), we have $0 < \zeta_\alpha(\eta) < \infty$, and $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i\eta) \geq t) < \alpha$ for every $t > \zeta_\alpha(\eta)$. Under the no-ac hypothesis, $\text{Im } \Gamma_o(E + i\eta) \rightarrow 0$ almost surely, so that $\zeta_\alpha(\eta) \rightarrow 0$ as $\eta \downarrow 0$, at an unknown rate. For this reason, we always compare $\text{Im } \Gamma_o(E + i\eta)$ with the quantile at the same imaginary part η . We show that there exist $\delta > 0$ and $C < \infty$ such that, for every large n , there exists $\eta_n > 0$ with

$$\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i\eta_n) \geq e^{2\delta n - Cn^{3/4}} \zeta_\alpha(\eta_n)) \geq q_* > \alpha. \quad (10.1)$$

For large n , the threshold in (10.1) is larger than $\zeta_\alpha(\eta_n)$, which contradicts Lemma 9.9(i).

Exponents. By Lemma 6.16, there exists $s \in (1/2, 1)$ such that $\gamma = -\varphi'(s) > 0$ and $F = \varphi(s) - (1-s)\gamma > 0$. The number F is the value at $u = 1$ of the tangent line to φ at s . Since $\varphi'(1) = 0$, it tends to $\varphi(1) > 0$ as $s \uparrow 1$, and this is how s is chosen. In Section 9.3, the free energy is replaced by the free energy $\hat{\varphi}$ of blocks of m factors with entries in a compact set, with the corresponding numbers $\hat{\gamma} \approx \gamma$ and $\hat{F} \approx F$ (Lemma 9.11). We fix $\delta > 0$ small in terms of F and $(2s-1)\gamma$, and we set $N = n + 10$, $b_n = n^{3/4}$, and $T_n = e^{(\hat{\gamma} + \delta)n}$.

The gain from a resonance. Let x be a vertex of generation N , and let A be the product of the transfer matrices (factors, in the sense of Definition 9.1) along the path from o to x , evaluated at $z = E + i\eta$. By Lemma 9.3, we have $|G_{ox}(z)| \geq |G_{xx}(z)|/\|A\|$. If y is a child of x , the Ward identity of Lemma 3.19(iii) gives $\text{Im } \Gamma_o(z) \geq |G_{ox}(z)|^2 |h_{xy}|^2 \text{Im } \Gamma_y(z)$. Suppose that $\|A\| \leq e^{\hat{\gamma}n + O(b_n)}$, that $|G_{xx}(z)| \geq T_n$, that $|h_{xy}|^2$ is bounded below, and that $\text{Im } \Gamma_y(z) \geq \zeta_\alpha(\eta)/2$. Then $\text{Im } \Gamma_o(z) \geq e^{2\delta n - O(b_n)} \zeta_\alpha(\eta)$. We call the factor $e^{2\delta n - O(b_n)}$ by which $\text{Im } \Gamma_o(z)$ exceeds the quantile the *gain*; the precise statement is Lemma 10.8.

Resonant vertices. We count the vertices x of generation N for which the conditions of the previous paragraph hold; they are called resonant (Definition 10.7). The conditions are the following. The factors of the first n generations satisfy a regular growth condition: the logarithms of the norms of the products of consecutive blocks of m factors have partial sums within b_n of their typical values (R3). The last ten vertices of the path have no side children (R1), and the squared weights of their edges lie in fixed intervals (R4).

By Lemma 9.5, this forces the rootward cavity variable at the parent of x into a fixed disc, whatever its value at generation n . The vertex x has exactly one leaf child and one surviving child y (R2), and the cavity variable of y is bounded and has imaginary part at least $\zeta_\alpha(\eta)/2$ (R5); we call y the *seed*, since its imaginary part is the one that the resonance amplifies. By (9.10), the diagonal entry $G_{xx}(z)$ is then a Möbius function of the squared weight t of the leaf edge, with a pole whose real part lies in $[E^2/2, 3E^2/2]$ (Lemma 9.6) and whose imaginary part, which we call the *pole width*, is small (R6). Since t has the density pe^{-pt} , the event $|G_{xx}(z)| \geq T_n$ (R7) has probability of order T_n^{-1} (Lemma 9.7).

The first moment. By the many-to-one formula for the backbone, the expected number of resonant vertices is p^N times the probability that the endpoint of a spine of length N is resonant. The structural conditions (R1), (R2) and the conditions (R4), (R5) cost a factor independent of n , the tuning (R7) costs a factor of order T_n^{-1} , and the regular growth condition (R3) costs a factor $e^{-\iota n + O(b_n)}$, where $\iota = \log p - \hat{\varphi}(s) - s\hat{\gamma}$ (Lemma 9.12, which computes this probability after an exponential tilt of the law of a block at the exponent s). Since $\log p - \iota - \hat{\gamma} - \delta = \hat{F} - \delta$, we obtain in Lemma 10.9 that

$$\mathbb{E}^\infty N_n(\eta) \geq e^{(\hat{F}-\delta)n - O(b_n)}.$$

The probability of the regular growth event is computed for the one-spine real recursive tree process, at $\eta = 0$, where the factors along a spine are independent with the law ν_{fac} determined by π_E (Lemma 5.21). The growth of their products is then governed by φ . The bound is transferred to a small imaginary part $\eta > 0$, with n fixed, by Lemma 9.15: lower bounds are proved for events defined by strict inequalities, which are open, and upper bounds for events defined by non-strict inequalities, which are closed.

The second moment. By the many-to-two formula, the expected number of ordered pairs (u, v) of resonant vertices whose paths split at generation j is $\varrho_p p^{2N-j}$ times a probability under a two-spine law. The two leaf weights are fresh, and by Lemma 9.4 the 2×2 block of the resolvent at $\{u, v\}$ is a function of them. By Lemma 9.8, the probability that both diagonal entries exceed T_n is at most of order $T_n^{-2} + T_n^{-1} \min\{1, \mathfrak{D}\}$, where $\mathfrak{D}$ measures the interaction of u and v through the path that joins their parents. The first term is the one for independent vertices. With the upper bound of Lemma 9.12(i) for regular growth events on the common part and on the two arms of the paths, it contributes $e^{O(b_n)} (\mathbb{E}^\infty N_n)^2$ after summation over j . For the second term, we bound $\mathbb{E} \mathfrak{D}^s$ by averaging over four groups of three fresh factors, placed next to the two ends and next to the fork of the joining path (Lemma 9.13). What remains are the negative moments of the norms of the products along the two arms, which decay like $e^{(n-j)(\varphi(s) - \log p)}$, where $\varphi(s) < \log p$. The resulting exponent is affine in j , and its values at $j = 0$ and $j = n$ are negative by the choice of δ . We obtain $\mathbb{E}^\infty [N_n(N_n - 1)] \leq e^{O(b_n)} (\mathbb{E}^\infty N_n)^2$

(Lemma 10.12), and the Paley–Zygmund inequality gives $\mathbb{P}^\infty(N_n \geq 1) \geq e^{-O(b_n)}$. On this event, $\operatorname{Im} \Gamma_o(E + i\eta) \geq e^{2\delta n - O(b_n)} \zeta_\alpha(\eta)$.

Amplification. It remains to raise the probability $e^{-O(b_n)}$ to the fixed level $q_* > \alpha$. In Section 10.2, we show that if $\operatorname{Im} \Gamma_o \geq y$ with probability at least e^{-a} , then $\operatorname{Im} \Gamma_o \geq c_3 e^{-C_3(1+a)} y$ with probability at least q_* (Corollary 10.6). The main step uses the Poisson structure of the children of the root. A random set of children, whose number has a Poisson law with mean about 64, carries independent copies of an event of probability q , while the other children keep $|D_o|$ in a fixed annulus. If one of the copies occurs, then so does the event at the root, with a fixed loss in the imaginary part, and this happens with probability at least $4q$. After $O(1+a)$ iterations, the probability reaches q_* . This is the only step that requires $p \geq p_*$, and p_* does not depend on E , since it uses only Lemma 5.6, which holds uniformly over all real solutions. With $a = O(b_n)$, we obtain (10.1).

Order of choices. The constants are fixed in the order (a)–(f) listed in Section 10.3. The numerical constants come first; then the constants that depend on p and E ; then the depth n ; and finally the imaginary part η_n , which depends on n . Each estimate at positive η is obtained from an estimate at $\eta = 0$ with n fixed, and the two limits $n \rightarrow \infty$ and $\eta \downarrow 0$ are never taken in the same estimate. The quantile $\zeta_\alpha(\eta)$ enters only through the seed condition (R5), whose probability is bounded at each fixed η by Lemma 9.9(ii), and through the gain.

Organization. Section 10.2 contains the amplification. Section 10.3 fixes the constants, defines the resonant vertices, and proves the gain. Sections 10.4 and 10.5 contain the first and second moments, and Section 10.6 completes the proof.

10.2 Amplification at the root

In this section, we show that an event of small probability on which $\operatorname{Im} \Gamma_o$ is large can be converted into an event of a fixed probability, at the cost of a moderate factor in the imaginary part. We begin with the numerical constants. Let

$$a_0 = \frac{1}{32\sqrt{2}}, \quad (10.2)$$

so that $4a_0 = a(1/32)$ in the notation of Lemma 5.6. Let $N_0 \in \mathbb{N}$ be such that a $\operatorname{Pois}(128)$ variable exceeds $N_0 - 1$ with probability at most $1/32$. Let $\lambda \in [32, 128]$ and $N \sim \operatorname{Pois}(\lambda)$. Since a $\operatorname{Pois}(128)$ variable is the sum of N and an independent $\operatorname{Pois}(128 - \lambda)$ variable, we have $\mathbb{P}(N \geq N_0) \leq 1/32$, and the identity $\mathbb{E}[N \mathbf{1}\{N > N_0\}] = \lambda \mathbb{P}(N \geq N_0)$ gives

$$\mathbb{P}(N > N_0) \leq \frac{1}{32}, \quad \mathbb{E}[N \mathbf{1}\{N \leq N_0\}] \geq \frac{\lambda}{2} \geq 16. \quad (10.3)$$

Set $y_+ = a_0^2/(4N_0)$, $y_- = y_+/2$, $J = [y_-, y_+]$, and $\rho_J = \mathbb{P}(Y \in J)$ for $Y \sim \text{Exp}(1)$. We let $p_* \geq 32 \log 64 + 1$ be the smallest number such that

$$\frac{7}{16} p_* \rho_J \geq 64, \quad e^{-p_*/2+128} \leq \frac{1}{16}, \quad (10.4)$$

and we set

$$q_0 = \frac{1}{2N_0}, \quad q_* = \frac{1}{4N_0}, \quad \alpha = \frac{q_*}{2}. \quad (10.5)$$

All these constants are numerical. Let $p \geq p_*$. Since $1 - e^{-p/2} > 1/2$, the concave function $\varrho \mapsto 1 - e^{-p\varrho} - \varrho$ is positive at $\varrho = 1/2$, and therefore $\varrho_p > 1/2$. By Lemma 5.6 with $\varepsilon = 1/32$, every real solution at every energy $E > 0$ satisfies $\mathbb{P}(|D| \leq 4a_0/\sqrt{p}) < 1/32$. By (10.4), we have $p(\varrho_p - 1/16)\rho_J \geq 64$, $e^{-(p\varrho_p-128)} \leq 1/16$, and $e^{-(p-1)\varrho_p} \leq 1/16$.

Standing assumptions. From now on until the end of this section, we fix $p \geq p_*$ and $E \in \mathcal{E}_p$ with $E > 0$, and we assume that the no-ac hypothesis holds at E . We write $\pi = \pi_E$ for the real boundary law, $D_o(z) = -1/\Gamma_o(z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$, and $D_o(E) = -1/\Gamma_o(E + i0)$. By Proposition 5.11(ii), $\mathbb{P}$ -almost surely $\Gamma_v(E + i\eta) \rightarrow \Gamma_v(E + i0) \in \mathbb{R} \setminus \{0\}$ as $\eta \downarrow 0$ for every vertex v . The law of $D_o(E)$ is π under $\mathbb{P}$ and π_E^∞ under $\mathbb{P}^\infty$, and π_E^∞ has no atoms (see the discussion after Definition 5.12). We set

$$d_- = \frac{a_0}{\sqrt{p}}, \quad h_0 = \frac{d_-^2}{4}, \quad (10.6)$$

we let $d_+ \geq 16d_-$ be such that $\mathbb{P}(|D_o(E)| > d_+/2) \leq 1/32$, and we set $c_0 = \min\{1, y_-/(pd_+^2)\}$. Then $\mathbb{P}(|D_o(E)| \leq 4d_-) < 1/32$ by the choice of a_0 , and $N_0 y_+/(pd_-) = d_-/4$.

Definition 10.3. For $\eta \in (0, 1]$ and $y > 0$, let $\mathcal{A}_y(\eta)$ be the event that

$$d_- \leq |D_o(E + i\eta)| \leq d_+ \quad \text{and} \quad y \leq \text{Im } \Gamma_o(E + i\eta) \leq 1.$$

Recall from Lemma 4.2 that the children of the root form the Poisson process $\Pi_o = \sum_k \delta_{(w_k, \theta_k \omega)}$ on $M = \mathbb{C} \times \Omega$ with intensity measure $p \mathfrak{m}$, where $\mathfrak{m} = \gamma \otimes \mathbb{P}$. The auxiliary mark ζ_k of the child k is the root coordinate of $\theta_k \omega$; its coordinates other than the rank $\zeta_k(1)$ are independent uniform variables, independent of the marked tree (Remark 4.3). We use them to color children.

Lemma 10.4. *There exists $\eta_{\text{ch}} \in (0, 1]$, depending on p and E , such that the following holds for every $\eta \in (0, \eta_{\text{ch}}]$, every $y > 0$, and every $q \in (0, q_0]$. If $\mathbb{P}^\infty(\mathcal{A}_y(\eta)) \geq q$, then $\mathbb{P}^\infty(\mathcal{A}_{c_0 y}(\eta)) \geq 4q$.*

Proof. We write $z = E + i\eta$. The proof splits the children of the root into a bounded number of *amplifying children*, which carry copies of the event $\mathcal{A}_y(\eta)$, and the remaining *reserve children*, which keep $|D_o|$ in the interval $[d_-, d_+]$ of Definition 10.3.

STEP 1. *Amplifying children.* Let

$$\tau = \frac{64}{p\rho_J \mathbb{P}(\mathcal{S} \cap \{d_- \leq |D_o(E)| \leq d_+\})}.$$

Since $\mathbb{P}(|D_o(E)| \notin [d_-, d_+]) \leq \mathbb{P}(|D_o(E)| \leq 4d_-) + \mathbb{P}(|D_o(E)| > d_+/2) \leq 1/16$, the probability in the denominator is at least $\varrho_p - 1/16$, and $\tau \leq 1$ by the choice of p_* . For $0 \leq \eta \leq 1$, let C_η be the set of $(w, \omega') \in M$ such that $\omega' \in \mathcal{S}$, $|w|^2 \in J$, $\zeta_o(\omega')(2) \leq \tau$, and

$$d_- \leq |D_o(E + i\eta)(\omega')| \leq d_+, \quad \text{Im } \Gamma_o(E + i\eta)(\omega') \leq 1,$$

where for $\eta = 0$ the second condition is void. A child k is an amplifying child at imaginary part η if $(w_k, \theta_k \omega) \in C_\eta$, and a reserve child otherwise. As in the proof of Lemma 4.5(i), the restrictions Π_η^{ch} and Π_η^{res} of Π_o to C_η and to its complement are independent Poisson processes under $\mathbb{P}$, with intensity measures $p \mathfrak{m}(\cdot \cap C_\eta)$ and $p \mathfrak{m}(\cdot \setminus C_\eta)$ [94, Theorem 5.2]. Since $|w|^2$, the color $\zeta_o(\omega')(2)$, and the remaining coordinates of ω' are independent under $\mathfrak{m}$, the number N_η^{ch} of amplifying children has the law $\text{Pois}(\lambda(\eta))$, where

$$\lambda(\eta) = p\tau\rho_J \mathbb{P}(\mathcal{S} \cap \{d_- \leq |D_o(E + i\eta)| \leq d_+, \text{Im } \Gamma_o(E + i\eta) \leq 1\}),$$

and $\lambda(0) = 64$ by the choice of τ . Under $\mathbb{P}^\infty$, we have $D_o(E + i\eta) \rightarrow D_o(E)$ and $\text{Im } \Gamma_o(E + i\eta) \rightarrow 0$ almost surely. We apply Lemma 9.15 under $\mathbb{P}^\infty$ to the vectors $V_\eta = (D_o(E + i\eta), \text{Im } \Gamma_o(E + i\eta))$ and $V_0 = (D_o(E), 0)$, with the closed set $\{(x, y) : d_- \leq |x| \leq d_+, \text{Re } y \leq 1\}$ in part (ii) (with $h_\eta = 1$) and the open set $\{(x, y) : d_- < |x| < d_+, \text{Re } y < 1\}$ in part (i). Since $\lambda(\eta) = p\tau\rho_J\varrho_p \mathbb{P}^\infty(d_- \leq |D_o(E + i\eta)| \leq d_+, \text{Im } \Gamma_o(E + i\eta) \leq 1)$, we find

$$\begin{aligned} \mathbb{P}^\infty(d_- < |D_o(E)| < d_+) &\leq \liminf_{\eta \downarrow 0} \frac{\lambda(\eta)}{p\tau\rho_J\varrho_p} \\ &\leq \limsup_{\eta \downarrow 0} \frac{\lambda(\eta)}{p\tau\rho_J\varrho_p} \leq \mathbb{P}^\infty(d_- \leq |D_o(E)| \leq d_+). \end{aligned}$$

The two outer probabilities coincide, since π_E^∞ has no atoms, and they equal $\lambda(0)/(p\tau\rho_J\varrho_p)$. Thus $\lambda(\eta) \rightarrow 64$ as $\eta \downarrow 0$.

STEP 2. *The reserve.* Set

$$D_\eta^{\text{res}} = z + \frac{1}{p} \sum_{k \text{ reserve}} Y_k \Gamma_k(z), \quad h_\eta^{\text{res}} = \eta + \frac{1}{p} \sum_{k \text{ reserve}} Y_k \text{Im } \Gamma_k(z),$$

where the reserve is taken at imaginary part η , and let $\mathcal{R}_\eta$ be the event that $2d_- < |D_\eta^{\text{res}}| < 3d_+/4$, that $h_\eta^{\text{res}} < h_0$, and that some reserve child at imaginary part η is surviving. This event is measurable with respect to Π_η^{res} , and it is contained in $\mathcal{S}$. We claim that $\liminf_{\eta \downarrow 0} \mathbb{P}(\mathcal{R}_\eta) \geq 13/16$.

By Proposition 5.11(ii), almost surely, for every child k , we have $D_k(E + i\eta) \rightarrow D_k(E)$ and $\text{Im } \Gamma_k(E + i\eta) \rightarrow 0$, where $D_k(z) = D_o(z) \circ \theta_k$. Further, almost surely no surviving child k satisfies $|D_k(E)| \in \{d_-, d_+\}$. Indeed, by the Mecke formula (4.1) applied to the child process Π_o , the expected number of such children is

$$p \mathbb{P}(\mathcal{S} \cap \{|D_o(E)| \in \{d_-, d_+\}\}) = p q_p \mathbb{P}^\infty(|D_o(E)| \in \{d_-, d_+\}) = 0,$$

since π_E^∞ has no atoms. Consider a child k with $|w_k|^2 \in J$ and $\zeta_k(2) \leq \tau$. If it is surviving, then $|D_k(E + i\eta)|$ converges to a limit which is different from d_- and d_+ , and $\text{Im } \Gamma_k(E + i\eta) \rightarrow 0 < 1$. Hence the indicator that k is an amplifying child at imaginary part η is constant for all small η , and equal to the indicator at $\eta = 0$. If it is not surviving, then it is not an amplifying child for any η . Since the root has finitely many children, it follows that almost surely the set of amplifying children at imaginary part η is the set of amplifying children at $\eta = 0$ for all small η . Then, almost surely, $h_\eta^{\text{res}} \rightarrow 0$, and $D_\eta^{\text{res}} \rightarrow D_0^{\text{res}} = E - p^{-1} \sum_k Y_k / D_k(E)$, where the sum runs over the reserve children at $\eta = 0$; here we used that $\Gamma_k(E + i0) = -1/D_k(E)$ is real for every child. Further, the indicator $\varsigma_\eta^{\text{res}}$ of the event that some reserve child at imaginary part η is surviving is equal to ς_0^{res} for all small η . We apply Lemma 9.15(i) to the random vector $V_\eta = (D_\eta^{\text{res}}, h_\eta^{\text{res}}, \varsigma_\eta^{\text{res}})$ and to the open set $\{(x, h, \varsigma) : 2d_- < |x| < 3d_+/4, \text{Re } h < h_0, \text{Re } \varsigma > 1/2\}$, with $V_0 = (D_0^{\text{res}}, 0, \varsigma_0^{\text{res}})$, so that $V_\eta \rightarrow V_0$ almost surely. Since $0 < h_0$, we find that $\liminf_{\eta \downarrow 0} \mathbb{P}(\mathcal{R}_\eta)$ is at least the probability that $2d_- < |D_0^{\text{res}}| < 3d_+/4$ and that some reserve child at $\eta = 0$ is surviving.

Let $\mathcal{W}$ be the event that $4d_- \leq |D_o(E)| \leq d_+/2$ and $N_0^{\text{ch}} \leq N_0$. Since $N_0^{\text{ch}} \sim \text{Pois}(64)$, we have $\mathbb{P}(\mathcal{W}^c) \leq 3/32$ by (10.3). By (5.9), $D_o(E) - D_0^{\text{res}} = -p^{-1} \sum_k Y_k / D_k(E)$, where the sum runs over the amplifying children at $\eta = 0$. On $\mathcal{W}$, each term satisfies $Y_k \leq y_+$ and $|D_k(E)| \geq d_-$, so that $|D_o(E) - D_0^{\text{res}}| \leq N_0 y_+ / (p d_-) = d_-/4$, and therefore

$$2d_- < \frac{15d_-}{4} \leq |D_0^{\text{res}}| \leq \frac{d_+}{2} + \frac{d_-}{4} < \frac{3d_+}{4}.$$

The surviving reserve children at $\eta = 0$ form a Poisson process with total mass $p q_p - 64$, so none exists with probability $e^{-(p q_p - 64)} \leq 1/16$. We conclude that the probability above is at least $1 - 3/32 - 1/16 \geq 13/16$, which proves the claim.

By Steps 1 and 2, there exists $\eta_{\text{ch}} \in (0, 1]$ such that $\lambda(\eta) \in [32, 128]$ and $\mathbb{P}(\mathcal{R}_\eta) \geq 1/2$ for every $\eta \in (0, \eta_{\text{ch}}]$. We fix such an η for the rest of the proof.

STEP 3. The parent. Let $\mathcal{B}_\eta$ be the event that $\mathcal{R}_\eta$ holds, that $N_\eta^{\text{ch}} \leq N_0$, and that some amplifying child k satisfies $\text{Im } \Gamma_k(z) \geq y$. We claim that $\mathcal{B}_\eta \subset \mathcal{A}_{c_0 y}(\eta)$. By (5.10), we have $D_o(z) = D_\eta^{\text{res}} + p^{-1} \sum_k Y_k \Gamma_k(z)$, where the sum runs over the amplifying children. Each amplifying child satisfies $Y_k \leq y_+$ and $|\Gamma_k(z)| = |D_k(z)|^{-1} \leq d_-^{-1}$, so on $\mathcal{B}_\eta$ the sum is at most $N_0 y_+ / (p d_-) = d_-/4$ in absolute value. Then

$$d_- \leq \frac{7d_-}{4} \leq |D_o(z)| \leq \frac{3d_+}{4} + \frac{d_-}{4} \leq d_+.$$

Since $\text{Im } \Gamma_o = \text{Im } D_o / |D_o|^2$ and $\text{Im } D_o(z) = h_\eta^{\text{res}} + p^{-1} \sum_k Y_k \text{Im } \Gamma_k(z)$, we have on $\mathcal{B}_\eta$

$$\text{Im } \Gamma_o(z) \leq d_-^{-2} \left(h_0 + \frac{N_0 y_+}{p} \right) = \frac{1}{4} + \frac{N_0 y_+}{a_0^2} = \frac{1}{2},$$

where we used $\text{Im } \Gamma_k(z) \leq 1$ for amplifying children. Keeping only the amplifying child with $\text{Im } \Gamma_k(z) \geq y$, we also have $\text{Im } \Gamma_o(z) \geq d_+^{-2} p^{-1} y - y \geq c_0 y$. This proves the claim.

STEP 4. Probability. By the Poisson property, conditionally on $N_\eta^{\text{ch}} = l$, the amplifying children are independent with law $\mathbf{m}(\cdot \mid C_\eta)$ [94, Chapter 3]. Each of them satisfies $\text{Im } \Gamma_k(z) \geq y$ with probability

$$q' = \frac{\mathbf{m}(C_\eta \cap \{\text{Im } \Gamma_o(z) \geq y\})}{\mathbf{m}(C_\eta)} = \frac{\mathbb{P}^\infty(\mathcal{A}_y(\eta))}{\mathbb{P}^\infty(d_- \leq |D_o(z)| \leq d_+, \text{Im } \Gamma_o(z) \leq 1)} \geq q.$$

Here we used that under $\mathbf{m}$ the squared modulus $|w|^2$ and the color are independent of the remaining coordinates of the configuration, so that the factors ρ_J , τ , and ϱ_p cancel, and that the denominator in the last expression is at most one. For $l \leq N_0$, we have $lq \leq 1/2$ since $q \leq q_0$, and therefore $1 - (1 - q')^l \geq 1 - e^{-lq} \geq lq/2$. Since Π_η^{ch} and Π_η^{res} are independent, (10.3) implies that

$$\mathbb{P}(\mathcal{B}_\eta) \geq \mathbb{P}(\mathcal{R}_\eta) \mathbb{E} \left[\frac{N_\eta^{\text{ch}} q}{2} \mathbf{1}_{\{N_\eta^{\text{ch}} \leq N_0\}} \right] \geq \frac{1}{2} \cdot \frac{q}{2} \cdot 16 = 4q.$$

Since $\mathcal{B}_\eta \subset \mathcal{R}_\eta \subset \mathcal{S}$, we have $\mathbb{P}^\infty(\mathcal{B}_\eta) = \varrho_p^{-1} \mathbb{P}(\mathcal{B}_\eta) \geq 4q$, and Step 3 completes the proof. $\square$

The input of Lemma 10.4 is an event of the form $\mathcal{A}_y(\eta)$, which constrains $|D_o|$ to the interval $[d_-, d_+]$ and bounds $\text{Im } \Gamma_o$ from above. The next lemma produces such an event from an arbitrary lower bound on $\text{Im } \Gamma_o$, using one more generation.

Lemma 10.5. *There exist $\eta_{\text{reg}} \in (0, \eta_{\text{ch}}]$ and $c_1, c_2 \in (0, 1]$, depending on p and E , such that the following holds for every $\eta \in (0, \eta_{\text{reg}}]$, every $y > 0$, and every $a \geq 0$. If $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i\eta) \geq y) \geq e^{-a}$, then $\mathbb{P}^\infty(\mathcal{A}_{c_2 e^{-2a} y}(\eta)) \geq c_1 e^{-3a}$.*

Proof. We write $z = E + i\eta$. Let $C_\natural = \sup_{0 < \eta' \leq 1} \mathbb{E}^\infty |\Gamma_o(E + i\eta')|^{1/2}$, which is finite by Lemma 5.17, and set

$$\lambda_r = \frac{1}{100}, \quad \kappa = \min \left\{ 1, \frac{p d_-}{8}, \frac{p d_-^2}{8} \right\}, \quad R_a = \max \{ 1, 4 C_\natural^2 e^{2a} \}.$$

By Markov's inequality, $\mathbb{P}^\infty(|\Gamma_o(z)| > R_a) \leq C_\natural R_a^{-1/2} \leq e^{-a}/2$.

STEP 1. Candidate and reserve. A child k of the root is a *candidate* if $\zeta_k(3) \leq \lambda_r/p$, and a *reserve* child otherwise. As in the proof of Lemma 10.4, the candidates and the reserve

children form independent Poisson processes Π^c and Π' with intensity measures $\lambda_r \mathfrak{m}$ and $(p - \lambda_r) \mathfrak{m}$. This coloring does not depend on η . Let $D'_\eta = z + p^{-1} \sum_k Y_k \Gamma_k(z)$ and $h'_\eta = \eta + p^{-1} \sum_k Y_k \text{Im } \Gamma_k(z)$, where the sums run over the reserve children, and let $\mathcal{R}'_\eta$ be the event that $2d_- < |D'_\eta| < 3d_+/4$, that $h'_\eta < h_0$, and that some reserve child is surviving. By Proposition 5.11(ii), almost surely $D'_\eta \rightarrow D'_0 = E - p^{-1} \sum_k Y_k / D_k(E)$ and $h'_\eta \rightarrow 0$, where the sum runs over the reserve children. The event that some reserve child is surviving does not depend on η . We apply Lemma 9.15(i) to the vector (D'_η, h'_η) , with the indicator of this event added as a coordinate, as in the discussion after Lemma 9.15, and we obtain

$$\liminf_{\eta \downarrow 0} \mathbb{P}(\mathcal{R}'_\eta) \geq \mathbb{P}(2d_- < |D'_0| < 3d_+/4, \text{ some reserve child is surviving}).$$

Let $O = \{x \in \mathbb{R} : 2d_- < |x| < 3d_+/4\}$. Since Π' and Π^c are independent and $D'_0 = D_o(E)$ on the event $\{\Pi^c = 0\}$, which has probability $e^{-\lambda_r}$, we have

$$\mathbb{P}(D'_0 \notin O) = \frac{\mathbb{P}(D'_0 \notin O, \Pi^c = 0)}{e^{-\lambda_r}} \leq e^{\lambda_r} \mathbb{P}(D_o(E) \notin O) \leq e^{\lambda_r} \left(\frac{1}{32} + \frac{1}{32} \right) \leq \frac{1}{8}.$$

The surviving reserve children form a Poisson process of total mass $(p - \lambda_r) \varrho_p$, and none exists with probability at most $e^{-(p-1)\varrho_p} \leq 1/16$. Thus the lower bound for $\liminf_{\eta \downarrow 0} \mathbb{P}(\mathcal{R}'_\eta)$ obtained above is at least $1 - 1/8 - 1/16 = 13/16$. We let $\eta_{\text{reg}} \in (0, \eta_{\text{ch}}]$ be such that $\mathbb{P}(\mathcal{R}'_\eta) \geq 1/2$ for every $\eta \in (0, \eta_{\text{reg}}]$, and we fix such an η .

STEP 2. The candidate. Let $I_a = [\kappa/(1 + R_a), 2\kappa/(1 + R_a)]$, and let $\mathcal{K}$ be the event that there is exactly one candidate k , that it is surviving, that $|w_k|^2 \in I_a$, and that $\text{Im } \Gamma_k(z) \geq y$ and $|\Gamma_k(z)| \leq R_a$. Since Π^c is a Poisson process with intensity measure $\lambda_r \mathfrak{m}$, and since $|w|^2$ is independent of the forward configuration under $\mathfrak{m}$,

$$\mathbb{P}(\mathcal{K}) = \lambda_r e^{-\lambda_r} \mathbb{P}(Y \in I_a) \varrho_p \mathbb{P}^\infty(\text{Im } \Gamma_o(z) \geq y, |\Gamma_o(z)| \leq R_a) \geq \lambda_r e^{-\lambda_r} \frac{\kappa e^{-2}}{1 + R_a} \cdot \frac{\varrho_p e^{-a}}{2},$$

where $Y \sim \text{Exp}(1)$, and we used $\mathbb{P}(Y \in I_a) \geq |I_a| e^{-2\kappa}$ and $\kappa \leq 1$.

STEP 3. The parent. On $\mathcal{R}'_\eta \cap \mathcal{K}$, we have $D_o(z) = D'_\eta + p^{-1} Y_k \Gamma_k(z)$ for the candidate k , and $|p^{-1} Y_k \Gamma_k(z)| \leq p^{-1} \cdot 2\kappa R_a / (1 + R_a) \leq 2\kappa/p \leq d_-/4$. As in Step 3 of the proof of Lemma 10.4, this gives $d_- \leq |D_o(z)| \leq d_+$. Further, $\text{Im } \Gamma_k(z) \leq |\Gamma_k(z)| \leq R_a$ and

$$\text{Im } \Gamma_o(z) = |\Gamma_o(z)|^2 \left(h'_\eta + \frac{Y_k}{p} \text{Im } \Gamma_k(z) \right) \leq d_-^2 \left(h_0 + \frac{2\kappa}{p} \right) \leq \frac{1}{4} + \frac{1}{4},$$

where we used $\kappa \leq p d_-^2/8$. Keeping only the candidate, we also have $\text{Im } \Gamma_o(z) \geq d_+^{-2} p^{-1} \kappa y / (1 + R_a)$. Since $1 + R_a \leq 2R_a \leq C' e^{2a}$ with $C' = 2 \max\{1, 4C_\frac{1}{4}^2\}$, we conclude that $\mathcal{R}'_\eta \cap \mathcal{K} \subset \mathcal{A}_{c_2 e^{-2a} y}(\eta)$, where $c_2 = \min\{1, \kappa / (p d_+^2 C')\}$.

STEP 4. *Conclusion.* The events $\mathcal{R}'_\eta$ and $\mathcal{K}$ are independent, since they depend on Π' and Π^c , respectively, and $\mathcal{R}'_\eta \subset \mathcal{S}$. By Steps 1 and 2,

$$\mathbb{P}^\infty(\mathcal{A}_{c_2 e^{-2a}y}(\eta)) \geq \varrho_p^{-1} \mathbb{P}(\mathcal{R}'_\eta) \mathbb{P}(\mathcal{K}) \geq \frac{\lambda_r e^{-\lambda_r \kappa}}{4e^2 C'} e^{-3a},$$

which is the claim with $c_1 = \min\{1, \lambda_r e^{-\lambda_r \kappa} / (4e^2 C')\}$. $\square$

Corollary 10.6. *There exist $c_3 \in (0, 1]$ and $C_3 \geq 1$, depending on p and E , such that the following holds for every $\eta \in (0, \eta_{\text{reg}}]$, every $y > 0$, and every $a \geq 0$. If $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i\eta) \geq y) \geq e^{-a}$, then*

$$\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i\eta) \geq c_3 e^{-C_3(1+a)} y) \geq q_*.$$

Proof. By Lemma 10.5, the event $\mathcal{A}_{y_1}(\eta)$ with $y_1 = c_2 e^{-2a} y$ has $\mathbb{P}^\infty$ -probability at least $q_1 = c_1 e^{-3a}$. We claim that $\mathbb{P}^\infty(\mathcal{A}_{c_0^i y_1}(\eta)) \geq \min\{4^i q_1, q_*\}$ for every $i \geq 0$. For $i = 0$ this holds. If it holds for i , we apply Lemma 10.4 with $q = \min\{4^i q_1, q_*\}$, which is at most q_0 , and obtain $\mathbb{P}^\infty(\mathcal{A}_{c_0^{i+1} y_1}(\eta)) \geq 4 \min\{4^i q_1, q_*\} \geq \min\{4^{i+1} q_1, q_*\}$. Let k be the smallest integer $i \geq 0$ with $4^i q_1 \geq q_*$. Since $q_* \leq 1$, we have $k \leq 1 + \max\{0, \log_4(q_*/q_1)\} \leq 1 + (3a + \log(1/c_1)) / \log 4$, and since $c_0 \leq 1$ and $\mathcal{A}_{c_0^k y_1}(\eta) \subset \{\text{Im } \Gamma_o(E + i\eta) \geq c_0^k y_1\}$, we obtain the claim with $c_3 = c_2 c_0^{1 + \log(1/c_1) / \log 4}$ and $C_3 = 2 + 3 \log(1/c_0) / \log 4$. $\square$

10.3 Resonant vertices

We keep the standing assumptions of Section 10.2, and we assume in addition that $\varphi(1) > 0$, where $\varphi = \varphi_\pi$ is the free energy of the real boundary law. We use the notation of Section 9.3 for this real solution: the factor law ν_{fac} and the norm moments $f_r(u)$. We fix the remaining constants in the following order.

- (a) The numerical constants $a_0, N_0, J, p_*, q_0, q_*$, and $\alpha = q_*/2$ of Section 10.2, and the constants $d_\pm, h_0, c_0, \eta_{\text{ch}}, \eta_{\text{reg}}, c_1, c_2, c_3$, and C_3 of Lemmas 10.4 and 10.5 and Corollary 10.6, which depend on p and E .
- (b) The cutoff R_{aux} of Lemma 9.9(ii) for the level α and $r = 1/2$, and $b_{\text{aux}} = E / (8R_{\text{aux}})$.
- (c) The exponent $s \in (1/2, 1)$ and the numbers $\gamma = -\varphi'(s) > 0$ and $F = \varphi(s) - (1 - s)\gamma > 0$ of Lemma 6.16; then $m, R, \hat{\varphi}, \hat{\gamma}, \hat{F}$, and ι of Lemma 9.11; and

$$\delta = \min \left\{ \frac{F}{4}, \frac{(2s-1)\gamma}{4} \right\}. \quad (10.7)$$

- (d) The length $c = 10$ of the single-child segment and the constants $\rho_1, \varepsilon_E, b_2$, and b_3 of (9.8).

- (e) A depth $n \in m\mathbb{N}$, large depending on all previous constants, together with

$$N = n + c, \quad L = \frac{n}{m}, \quad b_n = n^{3/4}, \quad T_n = e^{(\hat{\gamma} + \delta)n}. \quad (10.8)$$

(f) An imaginary part $\eta_n > 0$, small depending on n and all previous constants.

The constants in (a)–(d) depend only on p and E , through the real boundary law π ; they do not depend on n or on η , and those listed as numerical in (a) do not depend on E either. The quantile $\zeta_\alpha(\eta)$ is a function of η and is not one of the constants. By Lemma 9.11(ii) and (10.7), the numbers

$$\kappa_0 = (2s - 1)\hat{\gamma} - 2(\varphi(s) - \hat{\varphi}(s)) - \delta, \quad \hat{F} - \delta \quad (10.9)$$

satisfy $\kappa_0 \geq (2s - 1)\gamma/4 > 0$ and $\hat{F} - \delta \geq F/4 > 0$.

We now define the vertices that we count. For a vertex x of generation N , we write $o = x_0, x_1, \dots, x_N = x$ for its ancestors, and for $z = E + i\eta$ with $\eta > 0$, we set

$$t_k = \frac{Y_{x_k}}{p}, \quad d_{k-1}(z) = z + \frac{1}{p} \sum_u Y_u \Gamma_u(z), \quad X_{k-1}(z) = -\Gamma_{x_{k-1} \rightarrow x_k}(z), \quad 1 \leq k \leq N, \quad (10.10)$$

where the sum runs over the children u of x_{k-1} other than x_k . Thus t_k is the squared weight of the edge from x_{k-1} to x_k , $d_k(z)$ is the side denominator at x_k in the sense of Definition 9.1(ii) for the path from o to x in the whole tree, and $X_k(z)$ is the rootward variable at x_k . The *factors* of x are $(d_k(z), t_{k+1})$, $0 \leq k < N$, and for $1 \leq l \leq L$, the l -th block of x consists of the factors with $(l - 1)m \leq k < lm$. We write $Z_l(z)$ for its growth (Definition 9.10).

Recall from Section 4.6 that a child of a vertex is a leaf child if it has no children, and a surviving child if its forward subtree is infinite. We call the vertices $x_n, \dots, x_{N-1}$ the *single-child segment* of x .

Definition 10.7. Let $n \in m\mathbb{N}$, let $\eta \in (0, \min\{1, E/8\}]$, and set $z = E + i\eta$. A vertex $x \in \mathcal{V}_N$ is *resonant at $E + i\eta$* if the following conditions hold.

- (R1) *Single-child segment.* $N_{x_k} = 1$ for $n \leq k \leq N - 1$; that is, each vertex of the single-child segment of x has exactly one child.
- (R2) *Decoration.* x has exactly one leaf child ℓ_x , exactly one surviving child y_x , and no other child.
- (R3) *Regular growth.* The blocks $1, \dots, L$ of x are strictly retained, and $|Z_1(z) + \dots + Z_i(z) - im\hat{\gamma}| < b_n$ for $1 \leq i \leq L$.
- (R4) *Segment weights.* $1/p < t_{n+1} < 2/p$ and $|z - t_{n+1}X_n(z)| > \varepsilon_E$; $b_2/2 < t_{n+2} < b_2$; and $b_3/2 < t_k < b_3$ for $n + 3 \leq k \leq N$.
- (R5) *Seed.* $b_{\text{aux}}/2 < |h_{xy_x}|^2 < b_{\text{aux}}$, $|\Gamma_{y_x}(z)| < R_{\text{aux}}$, and $\text{Im } \Gamma_{y_x}(z) \geq \zeta_\alpha(\eta)/2$.
- (R6) *Pole width.* $|\text{Im}(z\Delta_x(z))| \leq |z|/(2T_n)$, where $\Delta_x(z) = -1/G_{xx}^{\{\ell_x\}}(z)$ is the denominator with the leaf removed at x , computed in the tree from which ℓ_x is deleted.
- (R7) *Resonance.* $|G_{xx}(z)| \geq T_n$.

We write $N_n(\eta)$ for the number of vertices $x \in \mathcal{V}_N$ that are resonant at $E + i\eta$.

Condition (R1) is the event $\mathcal{C}_J$, $J = \{n, \dots, N-1\}$, of Lemma 4.21(iii), and (R2) is the event $\mathcal{R} \circ \theta_x$ of Lemma 4.21(ii). A resonant vertex has a surviving child, so it belongs to $\mathcal{V}_N^\infty$. On (R1), the side denominators at the vertices of the single-child segment are $d_k(z) = z$ for $n \leq k < N$, and by (3.16),

$$X_k(z) = (z - t_k X_{k-1}(z))^{-1}, \quad n \leq k \leq N-1. \quad (10.11)$$

On (R2), by (3.11) for the tree from which ℓ_x is deleted, in which x has the two neighbors x_{N-1} and y_x ,

$$\Delta_x(z) = z - t_N X_{N-1}(z) - |h_{xy_x}|^2 X'_x(z), \quad X'_x(z) = -\Gamma_{y_x}(z); \quad (10.12)$$

here the cavity variables at x_{N-1} and y_x in the tree from which ℓ_x is deleted are $\Gamma_{x_{N-1} \rightarrow x}(z)$ and $\Gamma_{y_x}(z)$, by Lemma 3.5(iii), since the corresponding branches do not contain ℓ_x .

We record which variables each condition depends on, since this determines which squared moduli may be used as fresh variables in Sections 10.4 and 10.5. Conditions (R1) and (R2) are structural: they depend only on the offspring numbers of $x_n, \dots, x_N$ and of the children of x , and on the survival of these children. Condition (R3) depends only on the factors of x with index $k < n$, that is, on the squared moduli $Y_{x_1}, \dots, Y_{x_n}$ and on the side children of $x_0, \dots, x_{n-1}$ with their edge marks and forward subtrees. Condition (R4) depends on the squared moduli $Y_{x_{n+1}}, \dots, Y_{x_N}$ and on the rootward variable $X_n(z)$. The latter is computed in the component of x_n after the edge from x_n to x_{n+1} is deleted, and it does not depend on these squared moduli. Condition (R5) depends only on the mark of the edge from x to y_x and on the forward subtree of y_x . Conditions (R1)–(R6) do not depend on the mark of the edge from x to ℓ_x , and only (R7) does. In the first moment, we use as fresh variables, in this order, the squared modulus of the mark of the leaf edge and the squared moduli $Y_{x_{n+1}}, \dots, Y_{x_N}$; the squared moduli inspected by (R3) and (R5) are revealed and are never used as fresh variables.

Lemma 10.8. *There exists $c_{\text{gain}} > 0$, depending on p and E , such that the following holds for every $n \in m\mathbb{N}$ and $\eta \in (0, \min\{1, E/8\}]$. If $x \in \mathcal{V}_N$ is resonant at $E + i\eta$, then*

$$\text{Im } \Gamma_o(E + i\eta) \geq c_{\text{gain}} e^{2\delta n - 2b_n} \zeta_\alpha(\eta). \quad (10.13)$$

Further, on the event that (R1)–(R5) hold, the pole parameter $z\Delta_x(z)$ of (9.10) satisfies $E^2/2 \leq \text{Re}(z\Delta_x(z)) \leq 3E^2/2$.

Proof. Let $z = E + i\eta$. By (9.5) for the path $o = x_0, \dots, x_N = x$ in the whole tree, whose side denominators are the numbers $d_k(z)$ of (10.10), we have

$$\left| \frac{G_{ox}(z)}{G_{xx}(z)} \right| \geq \|A\|^{-1}, \quad A = \prod_{k=0}^{N-1} M(d_k(z), t_{k+1}).$$

By (R1), the last c factors are $M(z, t_{k+1})$, $n \leq k < N$, and by (R4) and (9.2), each of them has norm at most

$$C_{\text{conn}} = \left(\min \left\{ \frac{1}{p}, \frac{b_2}{2}, \frac{b_3}{2} \right\} \right)^{-1/2} \left(1 + \max \left\{ \frac{2}{p}, b_2, b_3 \right\}^2 + (E+1)^2 \right)^{1/2}.$$

The product of the first n factors is the product of the L blocks of x , and its norm is at most $e^{Z_1(z)+\dots+Z_L(z)} < e^{n\hat{\gamma}+b_n}$, by (R3). Thus $\|A\| \leq C_{\text{conn}}^c e^{n\hat{\gamma}+b_n}$. By (R7), we obtain $|G_{ox}(z)| \geq C_{\text{conn}}^{-c} e^{-n\hat{\gamma}-b_n} T_n = C_{\text{conn}}^{-c} e^{\delta n - b_n}$. The child y_x of x does not lie on the path from o to x , and Lemma 3.19(iii) and (R5) give

$$\text{Im } \Gamma_o(z) = \text{Im } G_{oo}(z) \geq |G_{ox}(z)|^2 |h_{xy_x}|^2 \text{Im } \Gamma_{y_x}(z) \geq C_{\text{conn}}^{-2c} e^{2\delta n - 2b_n} \cdot \frac{b_{\text{aux}}}{2} \cdot \frac{\zeta_\alpha(\eta)}{2}.$$

This proves (10.13) with $c_{\text{gain}} = b_{\text{aux}} C_{\text{conn}}^{-2c} / 4$.

For the last assertion, we have $X_{n+1}(z) = (z - t_{n+1} X_n(z))^{-1}$ by (10.11), and (R4) gives $|z - t_{n+1} X_n(z)| > \varepsilon_E$, so that $|X_{n+1}(z)| < \varepsilon_E^{-1}$. Since $t_{n+2} < b_2$, Lemma 9.5(ii) gives $X_{n+2}(z) \in \mathfrak{D}_E$. Since $t_k < b_3$ for $n+3 \leq k \leq N$, Lemma 9.5(iii) and induction on k give $X_k(z) \in \mathfrak{D}_E$ for $n+2 \leq k \leq N-1$; note that $N-1 = n+9 \geq n+2$. By (R4) and (R5), we have $t_N < b_3$, $|X'_x(z)| < R_{\text{aux}}$, and $|h_{xy_x}|^2 < E/(8R_{\text{aux}})$. Lemma 9.6, applied to (10.12), gives the claim. $\square$

10.4 The first moment

Under the backbone one-spine law $\mathbb{Q}_N^\infty$ of Definition 4.6, we write $v_0, \dots, v_N = v$ for the spine, and we apply the notation (10.10) and Definition 10.7 to $x = v$. We let $\mathcal{T}$ be the event that conditions (R1) and (R2) hold for v , and we set

$$Q_* = e^{-pc} p^2 e^{-2p}. \quad (10.14)$$

Lemma 10.9. *There exist $c_4 > 0$ and $n_1 \in \mathbb{N}$, depending on p , E , and the constants (a)–(d), such that for every $n \in m\mathbb{N}$ with $n \geq n_1$ there exists $\eta_1(n) \in (0, \min\{1, E/8\}]$ with*

$$\mathbb{E}^\infty N_n(\eta) \geq c_4 e^{(\hat{F}-\delta)n - sb_n} \quad \text{for every } \eta \in (0, \eta_1(n)].$$

Proof. Let n_1 be such that $T_n \geq 4(E+1)/E^2$ and $L(m \log C_R)^2 / b_n^2 = m(\log C_R)^2 n^{-1/2} \leq 1/2$ for $n \geq n_1$, where C_R is the constant of (9.13). Fix $n \in m\mathbb{N}$ with $n \geq n_1$, let $\eta \in (0, \min\{1, E/8\}]$, and write $z = E + i\eta$ and $\mathbb{Q} = \mathbb{Q}_N^\infty$. The proof first integrates the leaf weight, then the squared moduli $Y_{v_{n+1}}, \dots, Y_{v_N}$, and finally passes to the real axis in the regular growth condition.

STEP 1. Many-to-one formula and structure. Every resonant vertex belongs to $\mathcal{V}_N^\infty$. By (4.8), we have

$$\mathbb{E}^\infty N_n(\eta) = p^N \mathbb{Q}(v \text{ is resonant at } E + i\eta).$$

By Lemma 4.21(iii) and (v), we have $Q(\mathcal{T}) = Q_*$. Here $S_0, \dots, S_{N-1}$ denote the side sequences of Lemma 4.7; the sequences $S_n, \dots, S_{N-1}$ are empty on $\mathcal{T}$.

STEP 2. The leaf weight. Let $\mathcal{A} = \{\ell(v)\}$ if the endpoint v has a leaf child, and $\mathcal{A} = \emptyset$ otherwise, where $\ell(v)$ is the selected leaf child of Section 4.6; on $\mathcal{T}$, we have $\mathcal{A} = \{\ell_v\}$. By Lemma 4.21(iv) with $X = \{v\}$, applied under Q , the squared modulus $Y_{\ell(v)}$ is fresh given the σ -field $\mathcal{H}_{\mathcal{A}}$ of Lemma 4.17(iii), which is generated by $\mathcal{A}$, the endpoint v , the configuration $\omega^{\mathcal{A} \leftarrow 1}$, and the phase of $w_{\ell(v)}$.

We claim that the indicator F of the event that (R1)–(R6) hold for v is $\mathcal{H}_{\mathcal{A}}$ -measurable, and that so is $\Delta_v(z)$ on this event. Indeed, by the discussion after Definition 10.7, F is a function of the configuration which does not change when the mark of the edge from v to ℓ_v is replaced by another value. The same holds for $\Delta_v(z)$, since it is computed in the tree from which ℓ_v is deleted. Hence $F(\omega) = F(\omega^{\mathcal{A} \leftarrow 1})$, and similarly for $\Delta_v(z)$, which proves the claim.

Let $t = Y_{\ell(v)}/p$ and $p = z\Delta_v(z)$. On $\mathcal{T}$, (9.10) gives $G_{vv}(z) = z/(t - p)$, so that (R7) is the event $|z| \geq T_n|t - p|$. On the event that (R1)–(R6) hold, we have $E^2/2 \leq \operatorname{Re} p \leq 3E^2/2$ by Lemma 10.8, and $|\operatorname{Im} p| \leq |z|/(2T_n)$ by (R6). Further, p is not real. Indeed, $\operatorname{Im} \Delta_v(z) > 0$, since $\Delta_v(z) = -1/G_{vv}^{\{\ell_v\}}(z)$ and $\operatorname{Im} G_{vv}^{\{\ell_v\}}(z) > 0$, and $\operatorname{Re} \Delta_v(z) \geq 5E/8$ by Lemma 9.6; thus $\operatorname{Im} p = E \operatorname{Im} \Delta_v(z) + \eta \operatorname{Re} \Delta_v(z) > 0$. Let K be the set of $p \in \mathbb{C} \setminus \mathbb{R}$ with $E^2/2 \leq \operatorname{Re} p \leq 3E^2/2$ and $|\operatorname{Im} p| \leq |z|/(2T_n)$. We apply the substitution identity (4.14) (consequence (c) after Definition 4.16) with $m = 1$, the revealed variable $X = p$, and $\Phi(p, y) = \mathbf{1}_K(p) \mathbf{1}\{|z| \geq T_n|y/p - p|\}$. Since y/p has the density pe^{-pt} when y has law $\operatorname{Exp}(1)$, Lemma 9.7(ii) gives, almost surely on the event that (R1)–(R6) hold,

$$Q((R7) \mid \mathcal{H}_{\mathcal{A}}) \geq \int \Phi(p, y) e^{-y} dy \geq \frac{c_{\text{tune}}}{T_n};$$

here we used $T_n \geq 4(E+1)/E^2$, which holds since $n \geq n_1$. Multiplying by F and taking expectations, we conclude that

$$Q(v \text{ is resonant}) \geq \frac{c_{\text{tune}}}{T_n} Q((R1)–(R6)) \geq \frac{c_{\text{tune}}}{T_n} \left(Q((R1)–(R5)) - Q(\mathcal{T} \setminus (R6)) \right). \quad (10.15)$$

STEP 3. The single-child segment and the seed. We first integrate over the squared moduli $Y_{v_{n+1}}, \dots, Y_{v_N}$, so that $t_{n+i} = Y_{v_{n+i}}/p$ for $1 \leq i \leq c$. Let $\mathcal{G}_4$ be the σ -field of Lemma 5.18(ii) for the set $\{v_{n+1}, \dots, v_N\}$, which is a function of the endpoint. It is generated by the offspring numbers, the auxiliary marks, the endpoint v , the phases of all marks, and the marks w_u with $u \notin \{v_{n+1}, \dots, v_N\}$, and the c squared moduli $Y_{v_{n+1}}, \dots, Y_{v_N}$ are fresh given $\mathcal{G}_4$. By the discussion after Definition 10.7, the event $\mathcal{T} \cap (R3) \cap (R5)$ and the variable $X_n(z)$ do not depend on these squared moduli, and they are therefore $\mathcal{G}_4$ -measurable.

Condition (R4) is the event $\Phi(X_n(z), Y_{v_{n+1}}, \dots, Y_{v_N}) = 1$, where

$$\begin{aligned} \Phi(X, y_1, \dots, y_c) &= \mathbf{1}\left\{\frac{1}{p} < \frac{y_1}{p} < \frac{2}{p}, \left|z - \frac{y_1}{p}X\right| > \varepsilon_E\right\} \\ &\quad \times \mathbf{1}\left\{\frac{b_2}{2} < \frac{y_2}{p} < b_2\right\} \prod_{i=3}^c \mathbf{1}\left\{\frac{b_3}{2} < \frac{y_i}{p} < b_3\right\}. \end{aligned}$$

By the substitution identity (4.14), applied with the revealed variable $X = X_n(z)$ and $m = c$, $\mathbb{Q}((R4) \mid \mathcal{G}_4)$ is the integral of $\Phi(X_n(z), \cdot)$ against $\text{Exp}(1)^{\otimes c}$. This integral factorizes, and Lemma 9.5(i), which applies since $\eta \leq E/8$, gives

$$\mathbb{Q}((R4) \mid \mathcal{G}_4) \geq \frac{\rho_1}{2} \rho_2 \rho_3^{c-2} = c_{\text{seg}},$$

where $\rho_2 = \mathbb{P}(pb_2/2 < Y < pb_2)$ and $\rho_3 = \mathbb{P}(pb_3/2 < Y < pb_3)$ for $Y \sim \text{Exp}(1)$. Multiplying by the indicator of $\mathcal{T} \cap (R3) \cap (R5)$ and taking expectations, we obtain $\mathbb{Q}((R1)-(R5)) \geq c_{\text{seg}} \mathbb{Q}(\mathcal{T} \cap (R3) \cap (R5))$.

It remains to bound the probability of $\mathcal{T} \cap (R3) \cap (R5)$. By Lemma 4.7(iii), under $\mathbb{Q}$ the side sequences $S_0, \dots, S_{N-1}$, the spine edge marks, and the endpoint configuration $\theta_v \omega$ are independent, and $\theta_v \omega$ has law $\mathbb{P}^\infty$. Condition (R3) is a function of $S_0, \dots, S_{n-1}$ and $w_{v_1}, \dots, w_{v_n}$; condition (R1) is the event that $S_n, \dots, S_{N-1}$ are empty, which has probability e^{-pc} ; and the event that (R2) and (R5) hold is a function of $\theta_v \omega$. On (R2), we have $|h_{vy_v}|^2 = |w_{y_v}|^2/p$ and $\Gamma_{y_v}(z) = \Gamma_o(z) \circ \theta_{y_v}$. Applying (4.21) to the endpoint configuration with $F_2 = F_3 = 1$ and

$$F_1(w, \omega'') = \mathbf{1}\left\{\frac{b_{\text{aux}}}{2} < \frac{|w|^2}{p} < b_{\text{aux}}\right\} \mathbf{1}\left\{|\Gamma_o(z)(\omega'')| < R_{\text{aux}}, \text{Im } \Gamma_o(z)(\omega'') \geq \frac{\zeta_\alpha(\eta)}{2}\right\},$$

and using Lemma 9.9(ii), we find that the probability of (R2) and (R5) equals

$$p^2 e^{-2p} \int F_1 \, d\mathbf{m}^\infty = p^2 e^{-2p} \rho_{\text{aux}} \mathbb{P}^\infty\left(|\Gamma_o(z)| < R_{\text{aux}}, \text{Im } \Gamma_o(z) \geq \frac{\zeta_\alpha(\eta)}{2}\right) \geq p^2 e^{-2p} \rho_{\text{aux}} \frac{3\alpha}{4},$$

where $\rho_{\text{aux}} = \mathbb{P}(pb_{\text{aux}}/2 < Y < pb_{\text{aux}})$ for $Y \sim \text{Exp}(1)$. By the independence of the three groups of random elements, and since $Q_* = e^{-pc} p^2 e^{-2p}$, we conclude that

$$\mathbb{Q}((R1)-(R5)) \geq c_{\text{seg}} \mathbb{Q}(\mathcal{T} \cap (R3) \cap (R5)) \geq c_{\text{seg}} Q_* \frac{3\alpha \rho_{\text{aux}}}{4} \mathbb{Q}((R3)). \quad (10.16)$$

STEP 4. Regular growth at the real axis. For $k < n$, the number $d_k(z)$ is the side denominator $D_{v_k}^{\text{side}}(z)$ of Section 5.6. By Proposition 5.25(i) and Lemma 5.21(i), $\mathbb{Q}$ -almost surely $d_k(E + i\eta') \rightarrow d_k(E) = D_{v_k}^{\text{side}}(E)$ as $\eta' \downarrow 0$, and by Lemma 5.21(ii), the factors $(d_k(E), t_{k+1})$, $0 \leq k < n$, are independent with law ν_{fac} . Let $V_{\eta'} = (d_k(E + i\eta'), t_{k+1})_{0 \leq k < n} \in \mathbb{C}^{2n}$ for $0 < \eta' \leq 1$, and let V_0 be the vector of the factors at E . Let O be the set of vectors

$(d_k, t_{k+1})_{0 \leq k < n} \in \mathbb{C}^{2n}$ such that $R^{-1} < |d_k| < R$ and $R^{-1} < \operatorname{Re} t_{k+1} < R$ for every k , and such that the growths Z_l of the blocks of the factors $M(d_k, \operatorname{Re} t_{k+1})$ satisfy $|Z_1 + \dots + Z_i - im\hat{\gamma}| < b_n$ for $1 \leq i \leq L$. The entries of $M(d, t)$ are continuous on $\mathbb{C} \times (0, \infty)$, and the norm of a product of factors is at least one, so each Z_l is a continuous function on the set where all $\operatorname{Re} t_{k+1}$ are positive. Hence O is open. Since the squared weights t_{k+1} are real, (R3) at $E + i\eta'$ is the event $\{V_{\eta'} \in O\}$, and $\{V_0 \in O\}$ is the event of Lemma 9.12(ii) with $k = L$ and $b = b_n$ for the independent factors at E . By Lemma 9.15(i), Lemma 9.12(ii), and the choice of n_1 ,

$$\liminf_{\eta' \downarrow 0} Q((R3) \text{ at } E + i\eta') \geq e^{-m-sb_n} \left(1 - \frac{L(m \log C_R)^2}{b_n^2}\right) \geq \frac{1}{2} e^{-m-sb_n}.$$

STEP 5. *The pole width.* On $\mathcal{T}$, the number $G_{\mathbf{w}}^{\{\ell_v\}}(E + i\eta')$ is a coordinate of the depth- $(N+1)$ data, selected by a function of the genealogy and the ranks. By Proposition 5.25(i), Lemma 5.21(i), and Proposition 5.11(ii), it converges almost surely as $\eta' \downarrow 0$ to a real nonzero number. Then $\operatorname{Im}(z' \Delta_v(z')) \rightarrow 0$ almost surely on $\mathcal{T}$, where $z' = E + i\eta'$, while $|z'|/(2T_n) \geq E/(2T_n) > 0$. By dominated convergence, $Q(\mathcal{T} \setminus (R6) \text{ at } E + i\eta') \rightarrow 0$ as $\eta' \downarrow 0$.

STEP 6. *Conclusion.* By Steps 4 and 5, there exists $\eta_1(n) \in (0, \min\{1, E/8\}]$ such that for every $\eta \in (0, \eta_1(n)]$, we have $Q((R3)) \geq e^{-m-sb_n}/4$ and $Q(\mathcal{T} \setminus (R6)) \leq c_{\text{seg}} Q_* \alpha \rho_{\text{aux}} e^{-m-sb_n}/16$. Inserting these bounds into (10.16) and (10.15), we obtain

$$\mathbb{E}^\infty N_n(\eta) \geq p^N \frac{c_{\text{tune}}}{T_n} \cdot \frac{c_{\text{seg}} Q_* \alpha \rho_{\text{aux}}}{8} e^{-m-sb_n}.$$

Since $\log p - \iota - \hat{\gamma} - \delta = \hat{\varphi}(s) + s\hat{\gamma} - \hat{\gamma} - \delta = \hat{F} - \delta$, we have $p^N T_n^{-1} e^{-m} = p^c e^{(\hat{F}-\delta)n}$, and the claim follows with $c_4 = p^c c_{\text{tune}} c_{\text{seg}} Q_* \alpha \rho_{\text{aux}}/8$. $\square$

10.5 The second moment

For pairs of resonant vertices, we use the backbone two-spine laws $Q_{N,j}^\infty$ of Definition 4.6, with endpoints u and v and fork $v_j = u_j$ at generation j . We apply the notation (10.10) to $x = u$ and to $x = v$, and we indicate the endpoint by a superscript: t_k^u , $d_k^u(z)$, $Z_l^u(z)$, and so on. For $k < j$, the vertex $u_k = v_k$ has the spine child $u_{k+1} = v_{k+1}$, so the factors of u and v with index $k < j$ coincide; we call them the common factors. The factor with index j differs for u and v : the side denominator $d_j^u(z)$ contains the forward cavity of v_{j+1} , which depends on the whole subtree of v_{j+1} , including the pendant leaf of v . We therefore discard the block that contains the factor with index j . Let $\lambda_j = \lfloor j/m \rfloor$. The blocks $1, \dots, \lambda_j$ consist of common factors, the block $\lambda_j + 1$ contains the factor with index j , and the blocks $\lambda_j + 2, \dots, L$ of u consist of factors with index $k > j$, which we call the factors of the arm of u ; similarly for v .

Definition 10.10. Let $n \in m\mathbb{N}$ and $0 \leq j \leq n-1$. For $\eta \in (0, 1]$, set $z = E + i\eta$; for $\eta = 0$, all quantities below are evaluated at the boundary values $E + i0$. The *enlarged pair event* $\mathcal{E}_j(\eta)$ is the event that the following conditions hold under $\mathbb{Q}_{N,j}^\infty$.

- (E1) Conditions (R1) and (R2) hold for u and for v .
- (E2) If $\lambda_j \geq 1$, then the common blocks $1, \dots, \lambda_j$ are retained, and $Z_1(z) + \dots + Z_{\lambda_j}(z) \leq \lambda_j m\hat{\gamma} + b_n$.
- (E3) For $w \in \{u, v\}$, if $\lambda_j + 2 \leq L$, then the blocks $\lambda_j + 2, \dots, L$ of w are retained, and $\sum_{l=\lambda_j+2}^L Z_l^w(z) \leq (L - \lambda_j - 1)m\hat{\gamma} + 2b_n$.

The *interaction event* $\mathcal{E}'_j(\eta)$ is the event that (E1) and (E2) hold. Further, we set

$$\mathfrak{D}_\eta = |z| (t_N^u t_N^v)^{1/2} |G_{u_{N-1}v_{N-1}}^{V \setminus \{u, v\}}(z)|, \quad (10.17)$$

where $G^{V \setminus \{u, v\}}$ denotes the resolvent of the tree from which u and v are deleted.

We first estimate the enlarged pair event and the interaction event at the real axis.

Lemma 10.11. *There exists $C_5 < \infty$, depending on p , E , and the constants (a)–(d), such that for every $n \in m\mathbb{N}$ and every $0 \leq j \leq n-1$, with $\mathbb{Q} = \mathbb{Q}_{N,j}^\infty$,*

$$\mathbb{Q}(\mathcal{E}_j(0)) \leq Q_*^2 e^{-\iota(2n-j)+2m\iota+5sb_n}, \quad (10.18)$$

$$\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\mathcal{E}'_j(0)} \min\{1, \mathfrak{D}_0\}] \leq C_5 Q_*^2 e^{-\iota(j-m)+sb_n+2(n-j)(\varphi(s)-\log p)}. \quad (10.19)$$

Proof. **STEP 1. Independence.** By Lemma 4.7(iv), under $\mathbb{Q}$ the side sequences at the spine vertices other than the endpoints, the spine edge marks, and the two endpoint configurations are independent. The side sequences and edge marks have their unconditioned laws, and the endpoint configurations have law $\mathbb{P}^\infty$. Since $j < n$, the vertices u_k and v_k , $n \leq k \leq N-1$, of the two single-child segments are $2c$ distinct vertices, each with an empty side sequence with probability e^{-p} . Together with Lemma 4.21(ii) and (v), this gives $\mathbb{Q}(\text{E1}) = Q_*^2$. Condition (E1) depends only on the side sequences of the vertices of the single-child segments and on the endpoint configurations. Condition (E2) depends only on the common factors with index $k < m\lambda_j \leq j$, that is, on the side sequences at $v_0, \dots, v_{j-1}$ and on the marks $w_{v_1}, \dots, w_{v_j}$. Condition (E3) for w depends only on the factors of the arm of w with index $k \geq m(\lambda_j + 1) > j$ and $k < n$. These four groups of random elements are independent. By Lemma 5.21(ii), at $\eta = 0$ the common factors are independent with law ν_{fac} , and so are the factors of each arm with index $j < k < n$.

STEP 2. Proof of (10.18). By Step 1 and Lemma 9.12(i), applied to the λ_j common blocks with width b_n and to the $L - \lambda_j - 1$ blocks of each arm with width $2b_n$ (the probabilities are at most 1 if there are no such blocks), we obtain

$$\mathbb{Q}(\mathcal{E}_j(0)) \leq Q_*^2 e^{-\iota m\lambda_j + sb_n} (e^{-\iota m(L-\lambda_j-1)+2sb_n})^2 = Q_*^2 e^{-\iota(2n-m\lambda_j-2m)+5sb_n}.$$

Since $m\lambda_j \leq j$ and $\iota \geq 0$ (Lemma 9.11(iii)), this implies (10.18).

STEP 3. *The path between the parents of the endpoints.* On (E1), let C be the component of the tree with u and v deleted that contains u_{N-1} and v_{N-1} . The path from u_{N-1} to v_{N-1} in C is

$$u_{N-1}, u_{N-2}, \dots, u_{j+1}, v_j, v_{j+1}, \dots, v_{N-1},$$

of length $r = 2(N - 1 - j)$. For $\eta > 0$, we compute its side denominators and weights in the sense of Definition 9.1(ii). By Lemma 3.5(iii), the cavity variables of the branches hanging off the path are the same in C as in the whole tree, since these branches contain neither u nor v . The side denominator at u_{N-1} is z , because its only child u is deleted, and similarly at v_{N-1} . At u_k and v_k with $j < k < N - 1$, the side denominators are $d_k^u(z)$ and $d_k^v(z)$, which equal z for $k \geq n$. At the fork, the side denominator is

$$d_f(z) = z + \frac{1}{p} \sum_u Y_u \Gamma_u(z) + \mathbf{1}_{\{j \geq 1\}} t_j \Gamma_{v_{j-1} \rightarrow v_j}(z),$$

where the sum runs over the children of v_j other than u_{j+1} and v_{j+1} . The edge from u_k to u_{k-1} has squared weight t_k^u , and the edge from v_k to v_{k+1} has squared weight t_{k+1}^v . By (9.4), we have

$$|G_{u_{N-1}v_{N-1}}^{V \setminus \{u,v\}}(z)| = |e_2^T G_1 A_1 G_2 M_f G_3 A_2 G_4(1, z)^T|^{-1}, \quad (10.20)$$

where $M_f = M(d_f(z), t_{j+1}^v)$ is the *fork factor*, and, writing $d_k^u = d_k^u(z)$ and $d_k^v = d_k^v(z)$,

$$\begin{aligned} G_1 &= M(z, t_{N-1}^u) M(z, t_{N-2}^u) M(z, t_{N-3}^u), \\ A_1 &= M(d_{N-4}^u, t_{N-4}^u) \cdots M(d_{j+4}^u, t_{j+4}^u), \\ G_2 &= M(d_{j+3}^u, t_{j+3}^u) M(d_{j+2}^u, t_{j+2}^u) M(d_{j+1}^u, t_{j+1}^u), \\ G_3 &= M(d_{j+1}^v, t_{j+2}^v) M(d_{j+2}^v, t_{j+3}^v) M(d_{j+3}^v, t_{j+4}^v), \\ A_2 &= M(d_{j+4}^v, t_{j+5}^v) \cdots M(d_{N-5}^v, t_{N-4}^v), \\ G_4 &= M(z, t_{N-3}^v) M(z, t_{N-2}^v) M(z, t_{N-1}^v). \end{aligned}$$

Here we used $N - 4 \geq n$, which holds since $c = 10$, so that the side denominators in G_1 and G_4 equal z . Since $N - 1 - j \geq c = 10$, the products A_1 and A_2 have $N - j - 7 \geq 3$ and $N - j - 8 \geq 2$ factors. Each squared weight t_k^u , $j < k < N$, and t_k^v , $j < k < N$, appears in exactly one factor; in particular, the edge from the fork to u_{j+1} enters G_2 , and the edge from the fork to v_{j+1} enters the fork factor. The four products $G_1, \dots, G_4$ of three factors are the *averaging triples*.

As $\eta \downarrow 0$, each side denominator in (10.20) converges $\mathbb{Q}$ -almost surely to its boundary value, which is real. Indeed, the numbers $d_k^u(z)$ and $d_k^v(z)$ are side denominators of spine vertices which are not endpoints, and they converge by Lemma 5.21(i). The fork denominator $d_f(z)$ is the sum of z and of finitely many products of squared moduli

with cavity variables, which are coordinates of the depth- $(N+1)$ data by (5.6). These converge by Proposition 5.25(i), and their limits are real by Proposition 5.11(ii). The left side of (10.20) converges to $|G_{u_{N-1}v_{N-1}}^{V \setminus \{u,v\}}(E + i0)|$, which is finite by Definition 5.7. The bilinear form on the right side is a polynomial in z and in the side denominators, with coefficients depending on the squared moduli, and it converges to its value at $\eta = 0$. Since its modulus is the reciprocal of the left side, its limit is nonzero. It follows that, almost surely, (10.20) holds at $\eta = 0$, with z replaced by E and the side denominators replaced by their boundary values.

STEP 4. *Averaging over $G_1, \dots, G_4$.* We work at $\eta = 0$ and condition on (E1). Let $\mathcal{G}$ be the σ -field generated by the event (E1), the side sequences at $v_0, \dots, v_j$, the marks $w_{v_1}, \dots, w_{v_{j+1}}$, the side sequences and marks entering the factors of A_1 and A_2 , the marks w_u and w_v , and the two endpoint configurations. Then (E2), A_1 , M_f , A_2 , t_N^u , and t_N^v are $\mathcal{G}$ -measurable. The twelve factors of the averaging triples involve the marks w_{u_k} with $k \in \{j+1, j+2, j+3, N-3, N-2, N-1\}$, the marks w_{v_k} with $k \in \{j+2, j+3, j+4, N-3, N-2, N-1\}$, and the side sequences at u_k and v_k , $k \in \{j+1, j+2, j+3\}$. None of these is a generator of $\mathcal{G}$. By Lemma 4.7(iv), the side sequences at the spine vertices other than the endpoints, the spine edge marks, and the two endpoint configurations are independent under $\mathbb{Q}$. The event (E1) is the intersection of events each of which depends on a single side sequence at a vertex of a single-child segment or on a single endpoint configuration. Conditioning on (E1) therefore preserves this independence, and it changes only the laws of the side sequences at the vertices of the single-child segments, which become empty, and of the endpoint configurations. Since the averaging triples and the generators of $\mathcal{G}$ other than (E1) are functions of disjoint groups of these random elements, the averaging triples $G_1, \dots, G_4$ and $\mathcal{G}$ are independent conditionally on (E1). Each averaging triple is a product of three independent factors $M(b, \beta)$, where $\beta = Y/p$ has the exponential law of rate p , and b is either the constant E (at a vertex of a single-child segment) or a side denominator, which has law π by Lemma 5.21(ii). These factors satisfy the hypotheses of Proposition 6.18 with $K = p$, $a_+ = \max\{E^s, \mathbb{E}|D|^s\}$, and $a_- = \max\{E^{-s}, \mathbb{E}|D|^{-s}\}$, where $D \sim \pi$. These moments are finite by Proposition 5.5(iii). By Lemma 9.13, applied conditionally on (E1) with $w = (1, E)^T$, which satisfies $\|w\| \geq 1$, we obtain, almost surely on (E1),

$$\mathbb{E}_{\mathbb{Q}} \left[\left| G_{u_{N-1}v_{N-1}}^{V \setminus \{u,v\}}(E + i0) \right|^s \middle| \mathcal{G} \right] \leq C_*^4 \|A_1\|^{-s} \|A_2\|^{-s}.$$

STEP 5. *Proof of (10.19).* Since $\min\{1, \mathfrak{D}_0\} \leq \mathfrak{D}_0^s$, Step 4 gives

$$\mathbb{E}_{\mathbb{Q}} [\mathbf{1}_{\mathcal{E}'_j(0)} \min\{1, \mathfrak{D}_0\}] \leq E^s C_*^4 \mathbb{E}_{\mathbb{Q}} [\mathbf{1}_{\mathcal{E}'_j(0)} (t_N^u t_N^v)^{s/2} \|A_1\|^{-s} \|A_2\|^{-s}].$$

By Step 1 and the discussion in Step 4, conditionally on (E1), the event (E2), the variables t_N^u and t_N^v , and the matrices A_1 and A_2 are independent: they are functions of the

common factors with index $k < m\lambda_j$, of the marks w_u and w_v , and of the factors of A_1 and of A_2 , respectively. Since $\mathcal{E}'_j(0) = (E1) \cap (E2)$, the expectation on the right side equals

$$\mathbb{Q}(\mathcal{E}'_j(0)) \mathbb{E}[(t_N^u)^{s/2}] \mathbb{E}[(t_N^v)^{s/2}] \mathbb{E}_{\mathbb{Q}}[\|A_1\|^{-s} \mid (E1)] \mathbb{E}_{\mathbb{Q}}[\|A_2\|^{-s} \mid (E1)].$$

We have $\mathbb{E}(Y/p)^{s/2} = p^{-s/2} \Gamma(1 + s/2) \leq 1$ for $Y \sim \text{Exp}(1)$, since $p > 1$ and $\Gamma \leq 1$ on $[1, 2]$. Further, $A_1 = HV$, where H is the product of the factors of A_1 at the vertices u_k of the single-child segment, $\max\{n, j+4\} \leq k \leq N-4$, whose denominators equal E , and V is the product of the $(n-j-4)_+$ factors at u_k , $j+4 \leq k \leq n-1$, which are independent with law ν_{fac} . By Lemma 9.14 with $u = s$, with at most c constant factors, and with $C_{\text{arm}} \geq 1$, and since $\varphi(s) < \log p$ by Proposition 6.15(v),

$$\mathbb{E}_{\mathbb{Q}}[\|A_1\|^{-s} \mid (E1)] \leq C_{\text{arm}}^c e^{(n-j-4)_+(\varphi(s)-\log p)} \leq C_{\text{arm}}^c e^{-4(\varphi(s)-\log p)} e^{(n-j)(\varphi(s)-\log p)},$$

and the same bound holds for $A_2 = V'H'$, where V' is the product of the factors at v_k , $j+4 \leq k \leq n-1$, and H' is the product of the factors at the vertices v_k of the single-child segment, $\max\{n, j+4\} \leq k \leq N-5$. Finally, $\mathbb{Q}(\mathcal{E}'_j(0)) \leq Q_*^2 e^{-im\lambda_j + sb_n}$ by Steps 1 and 2, and $m\lambda_j \geq j-m$. Combining these bounds, we obtain (10.19) with $C_5 = \max\{1, E\} C_*^4 C_{\text{arm}}^{2c} e^{-8(\varphi(s)-\log p)}$. $\square$

We now bound the second factorial moment of the number of resonant vertices.

Lemma 10.12. *There exist $C_6 < \infty$ and $n_2 \geq n_1$, depending on p , E , and the constants (a)–(d), such that for every $n \in m\mathbb{N}$ with $n \geq n_2$ there exists $\eta_2(n) \in (0, \min\{1, E/8\}]$ with*

$$\mathbb{E}^\infty [N_n(\eta)(N_n(\eta) - 1)] \leq C_6 e^{2(\hat{F}-\delta)n + 5sb_n} \quad \text{for every } \eta \in (0, \eta_2(n)].$$

Proof. Let $n_2 = n_1$, fix $n \in m\mathbb{N}$ with $n \geq n_2$, let $\eta \in (0, \min\{1, E/8\}]$, and write $z = E + i\eta$ and $\mathbb{Q}_j = \mathbb{Q}_{N,j}^\infty$.

STEP 1. Many-to-two formula. Resonant vertices belong to $\mathcal{V}_N^\infty$. By (4.10), we have

$$\mathbb{E}^\infty [N_n(\eta)(N_n(\eta) - 1)] = \sum_{j=0}^{N-1} q_p p^{2N-j} \mathbb{Q}_j(u \text{ and } v \text{ are resonant at } E + i\eta).$$

If $j \geq n$, then the fork v_j is a vertex of the single-child segment of v with the two children $u_{j+1} \neq v_{j+1}$, so (R1) fails for v , and the corresponding term vanishes. Let $0 \leq j \leq n-1$.

STEP 2. Inclusion in the enlarged pair event. Suppose that u and v are resonant at $E + i\eta$. Then (E1) holds. The common blocks $1, \dots, \lambda_j$ are blocks of u , so (R3) for u with $i = \lambda_j$ implies (E2). For $w \in \{u, v\}$, (R3) for w with $i = L$ and $i = \lambda_j + 1$ gives

$$\sum_{l=\lambda_j+2}^L Z_l^w(z) < (Lm\hat{\gamma} + b_n) - ((\lambda_j + 1)m\hat{\gamma} - b_n),$$

and strictly retained blocks are retained; this is (E3). We conclude that the event in Step 1 is contained in $\mathcal{E}_j(\eta) \cap \{|G_{uu}(z)| \geq T_n, |G_{vv}(z)| \geq T_n\}$.

STEP 3. The two leaf weights. Let $\mathcal{A}$ be the set of selected leaf children $\ell(w)$ of the endpoints $w \in \{u, v\}$ which have a leaf child; on (E1), we have $\mathcal{A} = \{\ell_u, \ell_v\}$. By Lemma 4.21(iv) with $X = \{u, v\}$, applied under $\mathbf{Q}_j$, the squared moduli of the marks of the edges from u to $\ell(u)$ and from v to $\ell(v)$ are fresh given the σ -field $\mathcal{H}_{\mathcal{A}}$ of Lemma 4.17(iii). We write $t_u = |h_{u\ell_u}|^2$ and $t_v = |h_{v\ell_v}|^2$ for the corresponding squared weights. The event $\mathcal{E}_j(\eta)$ and the number $\mathfrak{D}_\eta$ do not depend on the marks of these two edges: the factors in the retained common and arm blocks, which exclude the block containing the branching point, involve only the side subtrees of u_k and v_k and the spine edges, and $\mathfrak{D}_\eta$ is computed in the tree from which u and v are deleted. As in Step 2 of the proof of Lemma 10.9, they are therefore $\mathcal{H}_{\mathcal{A}}$ -measurable.

On (E1), the vertices u and v have the same generation and are therefore not adjacent, and we apply Lemma 9.4 to the whole tree with $x = u$ and $y = v$. The neighbors of u are u_{N-1} , ℓ_u , and y_u . In the tree with u and v deleted, the leaf ℓ_u is an isolated vertex, with diagonal resolvent entry $-1/z$, the child y_u is the root of the component $V(\mathcal{T}_{y_u})$, with diagonal resolvent entry $\Gamma_{y_u}(z)$, and u_{N-1} lies in the component that contains the root. We obtain

$$S_{uu}(z) = -z + \frac{t_u}{z} - \sigma_u(z), \quad \sigma_u(z) = t_N^u G_{u_{N-1}u_{N-1}}^{V \setminus \{u, v\}}(z) + |h_{uy_u}|^2 \Gamma_{y_u}(z),$$

and similarly for v . Let B be the matrix with the diagonal entries $z^2 + z\sigma_u(z)$ and $z^2 + z\sigma_v(z)$ and the off-diagonal entries $B_{uv} = -zS_{uv}(z)$ and $B_{vu} = -zS_{vu}(z)$. Then $zS(z) = \text{diag}(t_u, t_v) - B$, and (9.6) shows that the 2×2 block of the resolvent at $\{u, v\}$ equals $R(t_u, t_v) = z(\text{diag}(t_u, t_v) - B)^{-1}$. The matrix B does not depend on the marks of the leaf edges, and it is $\mathcal{H}_{\mathcal{A}}$ -measurable. Further, $\text{diag}(t_1, t_2) - B$ is invertible for all $t_1, t_2 > 0$. Indeed, let $S'(z)$ be the matrix of Lemma 9.4 for the tree from which the two leaves are deleted. Its entries are those of $S(z)$ without the terms t_u/z and t_v/z , so that $\text{diag}(t_1, t_2) - B = z(S'(z) + \text{diag}(t_1, t_2)/z)$. By Lemma 9.4, $\text{Im } S'(z) \leq -\eta I$, and $\text{Im}(\text{diag}(t_1, t_2)/z) = -\eta|z|^{-2} \text{diag}(t_1, t_2) \leq 0$. Thus $\text{Im}\langle f, (S'(z) + \text{diag}(t_1, t_2)/z)f \rangle \leq -\eta\|f\|^2 < 0$ for $f \neq 0$, and the matrix is invertible. By (9.7), applied with $x' = u_{N-1}$ and $y' = v_{N-1}$, we have $|B_{uv}B_{vu}|^{1/2} = |z| |S_{uv}(z)S_{vu}(z)|^{1/2} = \mathfrak{D}_\eta$.

We now apply Lemma 9.8 conditionally on $\mathcal{H}_{\mathcal{A}}$, using the substitution identity (4.14) (consequence (c) after Definition 4.16) with the revealed variable $X = B \in \mathbb{C}^4$ and the squared moduli of the two leaf marks as fresh variables. With $|z| \leq E + 1$, we find that, almost surely on (E1),

$$\mathbf{Q}_j(|G_{uu}(z)| \geq T_n, |G_{vv}(z)| \geq T_n \mid \mathcal{H}_{\mathcal{A}}) \leq \frac{C_{\text{tt}}}{T_n^2} + \frac{C_{\text{tt}}}{T_n} \min\{1, \mathfrak{D}_\eta\}, \quad C_{\text{tt}} = 8p^2(E + 1)^2.$$

Combining this with Step 2, and using $\mathcal{E}_j(\eta) \subset \mathcal{E}'_j(\eta)$, we obtain

$$Q_j(\text{u and v are resonant}) \leq \frac{C_{\text{tt}}}{T_n^2} Q_j(\mathcal{E}_j(\eta)) + \frac{C_{\text{tt}}}{T_n} \mathbb{E}_{Q_j} [\mathbf{1}_{\mathcal{E}'_j(\eta)} \min\{1, \mathfrak{D}_\eta\}]. \quad (10.21)$$

STEP 4. *Passage to the real axis.* By Proposition 5.25(i), Q_j -almost surely, the factors of u and v at $E + i\eta'$ converge to their values at $\eta' = 0$ as $\eta' \downarrow 0$, and $\mathfrak{D}_{\eta'} \rightarrow \mathfrak{D}_0$, since $G_{u_{N-1}v_{N-1}}^{V \setminus \{u, v\}}(E + i\eta')$ is a coordinate of the depth- $(N+1)$ data. Let $V_{\eta'}$ be the vector of the factors of u and v with index $k < n$ at $E + i\eta'$. Conditions (E2) and (E3) are non-strict inequalities for partial sums of growths of retained blocks, so that $\mathcal{E}_j(\eta') = (\text{E1}) \cap \{V_{\eta'} \in F\}$ and $\mathcal{E}'_j(\eta') = (\text{E1}) \cap \{V_{\eta'} \in F'\}$ for closed sets F and F' , by the discussion after Lemma 9.15. The event (E1) does not depend on η' . By Lemma 9.15(ii), applied with $h_{\eta'} = \mathbf{1}_{(\text{E1})}$ and with $h_{\eta'} = \mathbf{1}_{(\text{E1})} \min\{1, \mathfrak{D}_{\eta'}\}$, which are bounded and converge almost surely, the lim sup as $\eta' \downarrow 0$ of the right side of (10.21) at $E + i\eta'$ is at most its value at $\eta' = 0$. By Lemma 10.11, this value is at most

$$B_j = \frac{C_{\text{tt}}}{T_n^2} Q_*^2 e^{-\iota(2n-j)+2m\iota+5sb_n} + \frac{C_{\text{tt}}C_5}{T_n} Q_*^2 e^{-\iota(j-m)+sb_n+2(n-j)(\varphi(s)-\log p)} > 0.$$

Since there are finitely many j , there exists $\eta_2(n) \in (0, \min\{1, E/8\}]$ such that the left side of (10.21) is at most $2B_j$ for every $\eta \in (0, \eta_2(n)]$ and every $0 \leq j \leq n-1$.

STEP 5. *Summation.* Let $\eta \in (0, \eta_2(n)]$, and set $M_n = p^N T_n^{-1} Q_* e^{-\iota m} = p^c Q_* e^{(\hat{F}-\delta)n}$, as in the proof of Lemma 10.9. Then $M_n^2 = p^{2N} T_n^{-2} Q_*^2 e^{-2\iota m}$, and for $0 \leq j \leq n-1$ we have

$$\begin{aligned} p^{2N-j} \frac{Q_*^2}{T_n^2} e^{-\iota(2n-j)+2m\iota+5sb_n} &= M_n^2 e^{2m\iota+5sb_n} e^{-(\log p - \iota)j}, \\ p^{2N-j} \frac{Q_*^2}{T_n^2} e^{-\iota(j-m)+sb_n+2(n-j)(\varphi(s)-\log p)} &= M_n^2 e^{m\iota+sb_n} e^{\Psi_n(j)}, \end{aligned}$$

where

$$\Psi_n(j) = -j \log p + (\hat{\gamma} + \delta)n + \iota(2n-j) + 2(n-j)(\varphi(s) - \log p).$$

Indeed, the first identity follows from $p^{2N-j} e^{\iota j} = p^{2N} e^{-(\log p - \iota)j}$, and the second from $p^{2N-j} = p^{2N} e^{-j \log p}$, $T_n^{-1} = T_n^{-2} e^{(\hat{\gamma} + \delta)n}$, and $e^{-\iota(j-m)} = e^{-2\iota m} e^{\iota(2n-j)} e^{m\iota}$. By Steps 1 and 4, we obtain

$$\begin{aligned} \mathbb{E}^\infty [N_n(\eta)(N_n(\eta) - 1)] &\leq \sum_{j=0}^{n-1} 2q_p p^{2N-j} B_j \\ &= 2q_p C_{\text{tt}} M_n^2 \sum_{j=0}^{n-1} \left(e^{2m\iota+5sb_n} e^{-(\log p - \iota)j} + C_5 e^{m\iota+sb_n} e^{\Psi_n(j)} \right). \end{aligned}$$

Using $\iota = \log p - \widehat{\varphi}(s) - s\widehat{\gamma}$, we find that Ψ_n is affine in j ,

$$\begin{aligned}\Psi_n(j) &= (\widehat{\gamma} + \delta + 2\log p - 2\widehat{\varphi}(s) - 2s\widehat{\gamma} + 2\varphi(s) - 2\log p)n \\ &\quad + (-\log p - \iota - 2\varphi(s) + 2\log p)j \\ &= -\kappa_0 n + (\widehat{\varphi}(s) + s\widehat{\gamma} - 2\varphi(s))j,\end{aligned}$$

with κ_0 as in (10.9). Its values at the endpoints are $\Psi_n(0) = -\kappa_0 n$ and $\Psi_n(n) = ((1-s)\widehat{\gamma} - \widehat{\varphi}(s) + \delta)n = -(\widehat{F} - \delta)n$. Since an affine function attains its maximum over an interval at an endpoint, we have $\Psi_n(j) \leq -\kappa_1 n$ for $0 \leq j \leq n$, where $\kappa_1 = \min\{\kappa_0, \widehat{F} - \delta\} > 0$. Further, $\log p - \iota > 0$ by Lemma 9.11(iii). Thus

$$\sum_{j=0}^{n-1} e^{-(\log p - \iota)j} \leq \frac{1}{1 - e^{-(\log p - \iota)}}, \quad \sum_{j=0}^{n-1} e^{\Psi_n(j)} \leq ne^{-\kappa_1 n} \leq \frac{1}{e\kappa_1}.$$

Inserting these bounds, and using $M_n^2 = p^{2c} Q_*^2 e^{2(\widehat{F} - \delta)n}$, we obtain the claim. $\square$

10.6 Proof of the delocalization criterion

Let $p \geq p_*$, let $E \in \mathcal{E}_p$ with $E > 0$, and suppose that $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Suppose, for a contradiction, that $\varphi_{\pi_E}(1) > 0$. We fix the constants (a)–(d) of Section 10.3, and the numbers c_4 , n_1 , C_6 , and n_2 of Lemmas 10.9 and 10.12. For $n \in m\mathbb{N}$ with $n \geq n_2$, we set

$$\eta_n = \min\{\eta_{\text{reg}}, \eta_1(n), \eta_2(n)\}.$$

Then $\eta_n \in (0, \min\{1, E/8\}]$, and Lemmas 10.9 and 10.12 apply at $\eta = \eta_n$. The proof has three steps: we bound the probability that a resonant vertex exists from below, we amplify this probability, and we compare the result with the definition of the quantile.

STEP 1. *A resonant vertex exists with subexponential probability.* Write $N_n = N_n(\eta_n)$, $\mu_n = c_4 e^{(\widehat{F} - \delta)n - sb_n}$, and $\chi_n = C_6 e^{2(\widehat{F} - \delta)n + 5sb_n}$. Since $N_n \leq |\mathcal{V}_N|$ and $\mathbb{E}^\infty |\mathcal{V}_N| \leq \varrho_p^{-1} p^N$ by (4.3), the expectation $\mathbb{E}^\infty N_n$ is finite. By the Cauchy–Schwarz inequality, we have

$$(\mathbb{E}^\infty N_n)^2 = (\mathbb{E}^\infty [N_n \mathbf{1}\{N_n \geq 1\}])^2 \leq \mathbb{E}^\infty [N_n^2] \mathbb{P}^\infty(N_n \geq 1);$$

this is the Paley–Zygmund inequality [108]. Since $\mathbb{E}^\infty [N_n^2] = \mathbb{E}^\infty N_n + \mathbb{E}^\infty [N_n(N_n - 1)] \leq \mathbb{E}^\infty N_n + \chi_n$ by Lemma 10.12, and since $t \mapsto t^2/(t + \chi_n)$ is increasing on $(0, \infty)$, Lemma 10.9 gives

$$\mathbb{P}^\infty(N_n \geq 1) \geq \frac{(\mathbb{E}^\infty N_n)^2}{\mathbb{E}^\infty N_n + \chi_n} \geq \frac{\mu_n^2}{\mu_n + \chi_n}.$$

Since $\widehat{F} - \delta > 0$ and $b_n = o(n)$, we have $\mu_n \geq 1$ for all large n . Then $\mu_n + \chi_n \leq \mu_n^2 + \chi_n \leq 2 \max\{\mu_n^2, \chi_n\}$, and

$$\frac{\mu_n^2}{\mu_n + \chi_n} \geq \min\left\{\frac{1}{2}, \frac{\mu_n^2}{2\chi_n}\right\} = \min\left\{\frac{1}{2}, \frac{c_4^2}{2C_6} e^{-7sb_n}\right\} \geq c_7 e^{-7sb_n}, \quad c_7 = \min\left\{\frac{1}{2}, \frac{c_4^2}{2C_6}\right\}.$$

STEP 2. *Amplification.* On the event $\{N_n \geq 1\}$, there exists a vertex which is resonant at $E + i\eta_n$, and Lemma 10.8 gives $\operatorname{Im} \Gamma_o(E + i\eta_n) \geq y_n$, where $y_n = c_{\text{gain}} e^{2\delta n - 2b_n} \zeta_\alpha(\eta_n)$. By Step 1, for all large n ,

$$\mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta_n) \geq y_n) \geq e^{-a_n}, \quad a_n = 7sb_n + \log(1/c_7) \geq 0.$$

Since $\eta_n \leq \eta_{\text{reg}}$, Corollary 10.6 implies that

$$\mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta_n) \geq c_3 e^{-C_3(1+a_n)} y_n) \geq q_* > \alpha.$$

STEP 3. *The quantile.* By Lemma 9.9(i), we have $\mathbb{P}^\infty(\operatorname{Im} \Gamma_o(E + i\eta_n) \geq t) < \alpha$ for every $t > \zeta_\alpha(\eta_n)$. By Step 2, this implies that $c_3 e^{-C_3(1+a_n)} y_n \leq \zeta_\alpha(\eta_n)$. Since $0 < \zeta_\alpha(\eta_n) < \infty$, dividing by $\zeta_\alpha(\eta_n)$, we obtain

$$c_3 c_{\text{gain}} \exp\left(2\delta n - 2b_n - C_3(1 + 7sb_n + \log(1/c_7))\right) \leq 1$$

for every large $n \in m\mathbb{N}$. The constants $c_3, C_3, c_{\text{gain}}, c_7, s$, and $\delta > 0$ do not depend on n , and $b_n = n^{3/4}$. Therefore the left side tends to infinity as $n \rightarrow \infty$, which is a contradiction. We conclude that $\varphi_{\pi_E}(1) \leq 0$. The threshold p_* was fixed in (10.4) and does not depend on E . This completes the proof of Theorem 10.1.

PART III

The free energy at large degree

The principal solution

In this section, we construct the principal solution $\nu_{p,E}$ of the cavity equation at large degree and establish the properties of it that are used in the rest of the paper. We work in the normalization of Definition 5.2. For $E > 2$ we write $\theta = \theta(E) = g(E)^2$, and we recall that $g(E)E = 1 + \theta$. If D satisfies the real cavity equation $D \stackrel{d}{=} E - p^{-1} \sum_{j \leq N} Y_j / D_j$, then multiplying by $g(E)$ shows that $B = g(E)D$ satisfies

$$B \stackrel{d}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j}, \quad (11.1)$$

where $N \sim \text{Pois}(p)$, $Y_j \sim \text{Exp}(1)$, and B_j are copies of B , all independent. Conversely, every law of B on $\mathbb{R} \setminus \{0\}$ satisfying (11.1) with $\theta = \theta(E)$ is the law of $g(E)D$ for a real solution D at E in the sense of Definition 5.1. Since $g(E) = 2/(E + \sqrt{E^2 - 4})$, we have $E^{-2} \leq \theta \leq 4E^{-2}$ for $E \geq 2$; for a scaled energy $E = \xi\sqrt{p}$ with ξ in a compact subset of $(0, \infty)$, the parameter θ is of order p^{-1} .

The main results of this section are Theorem 11.21, which constructs $\nu_{p,E}$ on compact scaled windows as the unique fixed point of (11.1) in an explicit class of laws; Proposition 11.22 and Corollary 11.23, which treat large energies and the whole half-line $E \geq \xi_0\sqrt{p}$; and Proposition 11.27, which shows that on compact scaled windows every real solution, and in particular the real boundary law of Proposition 5.11, coincides with $\nu_{p,E}$ once p is large.

We now describe the mechanism. When p is large and $\theta \leq C_0/p$, the sum in (11.1) has about p terms, and a term with $|B_j|$ of order one has size of order p^{-2} . Three effects then control the law of B . First, the children with $|B_j|$ bounded below contribute a sum that concentrates around θ on a scale much smaller than $r_p = p^{-1/4}$, so B is close to 1 with overwhelming probability. Second, for these children to bring B close to zero, their sum must be of order one. Since the squared moduli Y_j are exponential, an exponential tilt shows that this costs a factor e^{-cp^2} , at the level of densities. Third, a child with B_j close to zero produces a jump $\theta Y_j / (p B_j)$ that may be of order one. The jumps produced in this way have an intensity bounded by $\theta H / y^2$, where H bounds the density of B_j near zero, and a one-large-jump argument shows that their contribution to the density of B near zero is of order θH . Since $\theta \leq C_0/p$, the density near zero contracts. These three effects define a class of laws that is invariant under the map defined by (11.1) (Section 11.3). To prove uniqueness, we construct a metric for which this map is a contraction. Differences of laws

are measured in a Kantorovich norm near 1, in the supremum norm of densities near 0, and in a bounded-Lipschitz norm elsewhere, with exponentially large weights on the last two components (Sections 11.4 and 11.5). The contraction is proved in Section 11.6, the small-parameter regime, which contains the large-energy regime, is treated in Section 11.7, and real solutions are identified in Section 11.8.

Throughout this section, Y , Y_j , and T denote standard exponential random variables, $N \sim \text{Pois}(p)$, and these variables are independent of each other and of all other variables, unless stated otherwise. For a finite signed measure α on $\mathbb{R}$ and a bounded Borel function ψ , we write $\langle \alpha, \psi \rangle = \int \psi d\alpha$, $\|\alpha\|_{\text{TV}} = |\alpha|(\mathbb{R})$, and $\alpha^\vee$ for the image of α under $y \mapsto -y$. A constant is called numerical if it is absolute.

11.1 The law map

Definition 11.1. Let $p > 1$ and $\theta > 0$. For a probability measure ν on $\mathbb{R}$ with $\nu(\{0\}) = 0$, let $\mathcal{R}_{p,\theta}(\nu)$ be the law of

$$1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j},$$

where $N \sim \text{Pois}(p)$, $Y_j \sim \text{Exp}(1)$, and $B_j \sim \nu$, all independent. We call $\mathcal{R}_{p,\theta}$ the law map. A fixed point of $\mathcal{R}_{p,\theta}$ is a probability measure ν on $\mathbb{R}$ with $\nu(\{0\}) = 0$ and $\mathcal{R}_{p,\theta}(\nu) = \nu$.

By the discussion at the beginning of this section, for $E > 2$ a probability measure π on $\mathbb{R} \setminus \{0\}$ is a real solution at E if and only if the law of $g(E)D$, $D \sim \pi$, is a fixed point of $\mathcal{R}_{p,\theta(E)}$.

For $b \neq 0$ and $\theta > 0$, the variable $\theta Y / (pb)$ is exponential with rate $p|b|/\theta$ and has the sign of b . Its density is

$$j_\theta(b; y) = \frac{p|b|}{\theta} e^{-p|b||y|/\theta} \mathbf{1}_{\{by > 0\}}, \quad y \in \mathbb{R}. \quad (11.2)$$

For a finite signed measure σ on $\mathbb{R}$ with $|\sigma|(\{0\}) = 0$, define the finite signed measure

$$\mathsf{K}_\theta \sigma(dy) = \left(\int j_\theta(b; y) \sigma(db) \right) dy.$$

Since $\int j_\theta(b; y) dy = 1$, we have $\|\mathsf{K}_\theta \sigma\|_{\text{TV}} \leq \|\sigma\|_{\text{TV}}$ and $\mathsf{K}_\theta \sigma(\mathbb{R}) = \sigma(\mathbb{R})$. For a finite signed measure α , set

$$\mathsf{P}_p(\alpha) = e^{-p} \sum_{k=0}^{\infty} \frac{p^k}{k!} \alpha^{*k}, \quad \alpha^{*0} = \delta_0.$$

The series converges in total variation, since $\|\alpha^{*k}\|_{\text{TV}} \leq \|\alpha\|_{\text{TV}}^k$. Finally, let $s_\theta(y) = 1 + \theta - y$, and write $(s_\theta)_\# \alpha$ for the image of α under s_θ . Since $s_\theta(y_1 + y_2) = s_\theta(y_1) - y_2$, we have

$$(s_\theta)_\#(\alpha * \beta) = ((s_\theta)_\# \alpha) * \beta^\vee. \quad (11.3)$$

Lemma 11.2. Let $p > 1$, $\theta > 0$, and let ν be a probability measure on $\mathbb{R}$ with $\nu(\{0\}) = 0$.

- (i) We have $\mathcal{R}_{p,\theta}(\nu) = (s_\theta)_\# P_p(K_\theta \nu)$.
- (ii) Let $A_1, \dots, A_k$ be disjoint Borel sets with union $\mathbb{R} \setminus \{0\}$, let $B' = 1 + \theta - (\theta/p) \sum_{j \leq N} Y_j / B_j$ be as in Definition 11.1, and set

$$S_{A_i} = \frac{\theta}{p} \sum_{j \leq N: B_j \in A_i} \frac{Y_j}{B_j}.$$

Then $B' = 1 + \theta - \sum_i S_{A_i}$, and the variables $S_{A_1}, \dots, S_{A_k}$ are independent. For each i , the points $\theta Y_j / (p B_j)$ with $B_j \in A_i$ form a Poisson process on $\mathbb{R} \setminus \{0\}$ with intensity measure $\ell_{A_i}(y) dy$, where

$$\ell_A(y) = p \int_A j_\theta(b; y) \nu(db), \quad \int \ell_A(y) dy = p \nu(A). \quad (11.4)$$

In particular, the law of S_A is $e^{-p\nu(A)} \sum_{m \geq 0} \ell_A^{*m} / m!$, where $\ell_A^{*0} = \delta_0$.

- (iii) The measure $\mathcal{R}_{p,\theta}(\nu)$ is the sum of $e^{-p} \delta_{1+\theta}$ and an absolutely continuous measure.

Proof. The pairs (B_j, Y_j) , $j \leq N$, form a mixed binomial process with Poisson mixing distribution, which is a Poisson process on $(\mathbb{R} \setminus \{0\}) \times (0, \infty)$ with intensity measure $p \nu \otimes \text{Exp}(1)$. The restrictions of a Poisson process to disjoint sets are independent Poisson processes, and the image of a Poisson process under the map $(b, y) \mapsto \theta y / (pb)$ is a Poisson process whose intensity is the image of the original intensity [94, Chapters 3 and 5]. The image of $p \nu|_A \otimes \text{Exp}(1)$ under this map is $\ell_A(y) dy$ by (11.2). This proves (ii); the formula for the law of S_A follows by conditioning on the number of points.

For (i), conditionally on $N = n$, the jumps $\theta Y_j / (p B_j)$, $j \leq n$, are independent with law $K_\theta \nu$, and the law of their sum is $(K_\theta \nu)^{*n}$. Averaging over N shows that the sum $\sum_{j \leq N} \theta Y_j / (p B_j)$ has law $P_p(K_\theta \nu)$, and applying s_θ gives (i). For (iii), the event $N = 0$ contributes $e^{-p} \delta_{1+\theta}$. Conditionally on $N = n \geq 1$ and on $(B_j)_{j \leq n}$, the sum is absolutely continuous because the summand $\theta Y_1 / (p B_1)$ has a density, and the convolution of an absolutely continuous law with any law is absolutely continuous. $\square$

Lemma 11.3. Let $p > 1$ and $\theta > 0$. For a probability measure λ on $\mathbb{R}$ with $\lambda(\{0\}) = 0$ and a finite signed measure σ with $|\sigma|(\{0\}) = 0$, define

$$\mathcal{L}_{p,\theta;\lambda} \sigma = p \mathcal{R}_{p,\theta}(\lambda) * (K_\theta \sigma)^\vee. \quad (11.5)$$

Let $D_\lambda \sim \mathcal{R}_{p,\theta}(\lambda)$ and $T \sim \text{Exp}(1)$ be independent, and for $b \neq 0$ set $Z_b = D_\lambda - \theta T / (pb)$.

- (i) For every bounded Borel function ψ ,

$$\langle \mathcal{L}_{p,\theta;\lambda} \sigma, \psi \rangle = p \int \mathbb{E}[\psi(Z_b)] \sigma(db). \quad (11.6)$$

(ii) For $b \neq 0$, the law of Z_b has the density k_b^λ given, with $\mu_\lambda = \mathcal{R}_{p,\theta}(\lambda)$, by

$$k_b^\lambda(x) = \begin{cases} \frac{pb}{\theta} \int_{(x,\infty)} e^{-pb(d-x)/\theta} \mu_\lambda(\mathrm{d}d), & b > 0, \\ \frac{p|b|}{\theta} \int_{(-\infty,x)} e^{-p|b|(x-d)/\theta} \mu_\lambda(\mathrm{d}d), & b < 0. \end{cases} \quad (11.7)$$

The measure $\mathcal{L}_{p,\theta;\lambda}\sigma$ is absolutely continuous with density $x \mapsto p \int k_b^\lambda(x) \sigma(\mathrm{d}b)$. Moreover, $\|\mathcal{L}_{p,\theta;\lambda}\sigma\|_{\mathrm{TV}} \leq p\|\sigma\|_{\mathrm{TV}}$ and $\mathcal{L}_{p,\theta;\lambda}\sigma(\mathbb{R}) = p\sigma(\mathbb{R})$.

(iii) Let ν, μ be probability measures on $\mathbb{R}$ without atom at 0, set $\sigma = \nu - \mu$ and $\lambda_t = (1-t)\mu + t\nu$ for $t \in [0, 1]$. Then, for every bounded Borel function ψ ,

$$\langle \mathcal{R}_{p,\theta}(\nu) - \mathcal{R}_{p,\theta}(\mu), \psi \rangle = \int_0^1 \langle \mathcal{L}_{p,\theta;\lambda_t}\sigma, \psi \rangle \mathrm{d}t, \quad (11.8)$$

and

$$\|\mathcal{R}_{p,\theta}(\nu) - \mathcal{R}_{p,\theta}(\mu) - \mathcal{L}_{p,\theta;\mu}\sigma\|_{\mathrm{TV}} \leq \frac{p^2}{2} \|\sigma\|_{\mathrm{TV}}^2. \quad (11.9)$$

The operator $\mathcal{L}_{p,\theta;\lambda}$ is the linearization of the law map at λ ; by (11.6), it inserts one additional child with denominator distributed according to σ into an independent output of the law map applied to λ .

Proof. By (11.5) and Fubini's theorem, we have

$$\langle \mathcal{L}_{p,\theta;\lambda}\sigma, \psi \rangle = p \iint \psi(d-y) \mu_\lambda(\mathrm{d}d) \mathcal{K}_\theta\sigma(\mathrm{d}y) = p \int \sigma(\mathrm{d}b) \int j_\theta(b;y) \mathbb{E}[\psi(D_\lambda - y)] \mathrm{d}y,$$

which is (11.6), since $j_\theta(b;\cdot)$ is the density of $\theta T/(pb)$. For (ii), let $b > 0$. Then $\theta T/(pb)$ has density $j_\theta(b;\cdot)$ on $(0, \infty)$, and for a Borel set A ,

$$\mathbb{P}(Z_b \in A) = \int \mu_\lambda(\mathrm{d}d) \int_{(0,\infty)} \mathbf{1}_A(d-y) j_\theta(b;y) \mathrm{d}y = \int_A \left(\int_{(x,\infty)} j_\theta(b;d-x) \mu_\lambda(\mathrm{d}d) \right) \mathrm{d}x,$$

which is the first line of (11.7). The case $b < 0$ is the same, with $d-x < 0$. Applying (11.6) with $\psi = \mathbf{1}_A$ and using Fubini's theorem (which applies since $\int k_b^\lambda = 1$ and $|\sigma|$ is finite), we find that $\mathcal{L}_{p,\theta;\lambda}\sigma(A) = \int_A p \int k_b^\lambda(x) \sigma(\mathrm{d}b) \mathrm{d}x$. The total variation bound and the mass identity follow from (11.6) with $|\psi| \leq 1$ and $\psi = 1$.

For (iii), set $\alpha_t = \mathcal{K}_\theta\lambda_t = \mathcal{K}_\theta\mu + t\mathcal{K}_\theta\sigma$. Each α_t is a probability measure. Since convolution is bilinear and commutative, $t \mapsto \alpha_t^{*k}$ is a polynomial with values in the Banach space of finite signed measures with the total variation norm, with derivative $k\alpha_t^{*(k-1)} * \mathcal{K}_\theta\sigma$. The differentiated series $\sum_{k \geq 1} e^{-p} p^k k \alpha_t^{*(k-1)} * \mathcal{K}_\theta\sigma / k!$ converges uniformly in $t \in [0, 1]$, because its terms have norm at most $e^{-p} p^k \|\sigma\|_{\mathrm{TV}} / (k-1)!$. This implies that $t \mapsto \mathcal{P}_p(\alpha_t)$ is continuously differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{P}_p(\alpha_t) = p \mathcal{P}_p(\alpha_t) * \mathcal{K}_\theta\sigma,$$

and the fundamental theorem of calculus in this Banach space gives $P_p(\alpha_1) - P_p(\alpha_0) = \int_0^1 p P_p(\alpha_t) * K_\theta \sigma dt$. Applying $(s_\theta)_\sharp$, which is a linear isometry, and using (i) of Lemma 11.2 and (11.3), we obtain $\mathcal{R}_{p,\theta}(\nu) - \mathcal{R}_{p,\theta}(\mu) = \int_0^1 \mathcal{L}_{p,\theta;\lambda_t} \sigma dt$. Pairing with ψ , which is a bounded linear functional, gives (11.8). Further, since each $P_p(\alpha_u)$ is a probability measure, the same computation shows that $\|P_p(\alpha_t) - P_p(\alpha_0)\|_{TV} \leq tp\|\sigma\|_{TV}$, and therefore

$$\|\mathcal{L}_{p,\theta;\lambda_t} \sigma - \mathcal{L}_{p,\theta;\mu} \sigma\|_{TV} = p \|(P_p(\alpha_t) - P_p(\alpha_0)) * K_\theta \sigma\|_{TV} \leq tp^2 \|\sigma\|_{TV}^2.$$

Integrating over $t \in [0, 1]$ proves (11.9). $\square$

11.2 Estimates for compound Poisson sums

In this subsection, we fix $p > 1$ and $\theta > 0$, and we consider a finite measure Λ on $\mathbb{R} \setminus \{0\}$ with $\Lambda(\mathbb{R}) \leq p$, a Poisson process $\{(b_j, Y_j)\}$ on $(\mathbb{R} \setminus \{0\}) \times (0, \infty)$ with intensity measure $\Lambda \otimes \text{Exp}(1)$, and the sum

$$S = \sum_j \frac{\theta Y_j}{p b_j}. \quad (11.10)$$

By Lemma 11.2, each sum S_A in that lemma is of this form, with $\Lambda = p\nu|_A$. The first estimate is a Chernoff bound for the children whose denominators are bounded away from zero.

Lemma 11.4. *Let $\delta \in (0, 1]$ and suppose that Λ is carried by $\{b : |b| \geq \delta\}$. Then $\mathbb{E}S = (\theta/p) \int b^{-1} \Lambda(db)$, and for every $x > 0$,*

$$\mathbb{P}(|S - \mathbb{E}S| > x) \leq 2 \exp \left(- \min \left\{ \frac{p\delta^2 x^2}{8\theta^2}, \frac{p\delta x}{4\theta} \right\} \right), \quad (11.11)$$

$$\mathbb{P} \left(\sum_j \frac{\theta Y_j}{p|b_j|} > x \right) \leq \exp \left(p - \frac{p\delta x}{2\theta} \right). \quad (11.12)$$

Proof. Conditioning on the number of points, which is Poisson with mean $\Lambda(\mathbb{R})$, we have $\mathbb{E}e^{tS} = \exp \left(\int (\mathbb{E}e^{t\theta Y/(pb)} - 1) \Lambda(db) \right)$ whenever the integral converges. For $u < 1$, we have $\mathbb{E}e^{uY} = (1 - u)^{-1}$. Let $|t| \leq p\delta/(2\theta)$ and $u_b = t\theta/(pb)$, so that $|u_b| \leq 1/2$ on the support of Λ . Differentiating at $t = 0$ gives the formula for $\mathbb{E}S$, and we obtain

$$\log \mathbb{E}e^{t(S - \mathbb{E}S)} = \int \left(\frac{1}{1 - u_b} - 1 - u_b \right) \Lambda(db) = \int \frac{u_b^2}{1 - u_b} \Lambda(db) \leq 2\Lambda(\mathbb{R}) \frac{t^2 \theta^2}{p^2 \delta^2} \leq \frac{2t^2 \theta^2}{p\delta^2}.$$

By Markov's inequality, $\mathbb{P}(S - \mathbb{E}S > x) \leq \exp(-tx + 2t^2 \theta^2/(p\delta^2))$. If $2\theta \leq \delta x$, we choose $t = p\delta/(2\theta)$; then $2t^2 \theta^2/(p\delta^2) = t\theta/\delta \leq tx/2$, and the exponent is at most $-p\delta x/(4\theta)$. If $2\theta > \delta x$, we choose $t = p\delta^2 x/(4\theta^2) < p\delta/(2\theta)$, and the exponent equals $-p\delta^2 x^2/(8\theta^2)$.

The same argument applies to $-(S - \mathbb{E}S)$, which proves (11.11). For (11.12), let $t_* = p\delta/(2\theta)$. Since $(1 - u)^{-1} - 1 \leq 1$ for $0 \leq u \leq 1/2$, we have $\mathbb{E} \exp(t_* \sum_j \theta Y_j / (p|b_j|)) \leq \exp(\Lambda(\mathbb{R})) \leq e^p$, and Markov's inequality gives (11.12). $\square$

The second estimate expresses that a large sum of independent variables has a large summand.

Lemma 11.5. *Let $m \geq 1$, let $V_1, \dots, V_m$ be independent real random variables, and let $v > 0$.*

- (i) *We have $\mathbb{P}(|V_1 + \dots + V_m| \geq v) \leq \sum_{i=1}^m \mathbb{P}(|V_i| \geq v/m)$.*
- (ii) *Suppose that, for each i , the restriction of the law of V_i to $\{y : |y| \geq v/m\}$ is absolutely continuous with density at most κ_i . Then the restriction of the law of $V_1 + \dots + V_m$ to $\{u : |u| \geq v\}$ is absolutely continuous with density at most $\kappa_1 + \dots + \kappa_m$.*
- (iii) *If $V_1, \dots, V_m$ are exponential with arbitrary rates $q_1, \dots, q_m > 0$, then $V_1 + \dots + V_m$ has a density f with $f(u) \leq m^2/(eu)$ for almost every $u > 0$. The same holds if the rates are random and independent of the standard exponential variables $q_i V_i$.*

Proof. Let $\Sigma = V_1 + \dots + V_m$ and $\Sigma^{(i)} = \Sigma - V_i$. On every outcome, some i satisfies $|V_i| \geq |\Sigma|/m$. Part (i) follows. For (ii), let $A \subset \{u : |u| \geq v\}$ be a Borel set. Then

$$\begin{aligned} \mathbb{P}(\Sigma \in A) &\leq \sum_{i=1}^m \mathbb{P}(\Sigma \in A, |V_i| \geq v/m) \\ &= \sum_{i=1}^m \mathbb{E} \left[\mathbb{P}(V_i \in A - \Sigma^{(i)}, |V_i| \geq v/m \mid \Sigma^{(i)}) \right] \leq \sum_{i=1}^m \kappa_i |A|, \end{aligned}$$

where we used that V_i is independent of $\Sigma^{(i)}$. For (iii), since $\sup_{q>0} qe^{-qy} = 1/(ey)$ for $y > 0$, the density of V_i on $[v/m, \infty)$ is at most $m/(ev)$. By (ii), the density of Σ on $[v, \infty)$ is at most $m^2/(ev)$. Applying this for all rational v smaller than u gives $f(u) \leq m^2/(eu)$ for almost every $u > 0$. For random rates, apply this conditionally on the rates and average. $\square$

The third estimate is the fresh cancellation bound. It says that the children with denominators bounded away from zero can move the output by a distance of order one only with probability $e^{-cp/\theta}$, and that the same holds for densities.

Lemma 11.6. *Let $\delta \in (0, 1]$, suppose that Λ is carried by $\{b : |b| \geq \delta\}$, and let $c \in \mathbb{R}$. Then the restriction of the law of $c - S$ to $(-\infty, c - 1/2]$ is absolutely continuous with density at most*

$$F_{p,\theta}(\delta) = 6p^2 \exp\left(p - \frac{\delta p}{4\theta}\right). \quad (11.13)$$

Proof. We write $S = S_+ - S_-$, where S_+ is the sum over the points with $b_j > 0$ and $S_- \geq 0$ is minus the sum over the points with $b_j < 0$. By the restriction property of

Poisson processes, S_+ and S_- are independent. If $c - S = x$ with $x \leq c - 1/2$, then $S_+ = c - x + S_- \geq 1/2$. Conditioning on S_- , we see that it suffices to show that the restriction of the law of S_+ to $[1/2, \infty)$ has density at most $F_{p,\theta}(\delta)$.

Let $\Lambda_+ = \Lambda([\delta, \infty)) \leq p$; if $\Lambda_+ = 0$ then $S_+ = 0$ and there is nothing to prove. The variable S_+ is a sum of $N_+ \sim \text{Pois}(\Lambda_+)$ independent jumps with common law $\pi_J(dy) = \Lambda_+^{-1} \int_{[\delta, \infty)} j_\theta(b; y) \Lambda(db) dy$. Let f_n denote the density of the sum of n independent jumps with law π_J . We use an exponential tilt (see [61, Section 2.2]). Let $\alpha = \delta/2$ and $t = \alpha p / \theta$. For $b \geq \delta$, we have

$$e^{ty} j_\theta(b; y) = \frac{b}{b - \alpha} \cdot \frac{p(b - \alpha)}{\theta} e^{-p(b - \alpha)y/\theta}, \quad y > 0,$$

and $1 \leq b/(b - \alpha) \leq 2$. Set $M = \int e^{ty} \pi_J(dy) \in [1, 2]$ and $\tilde{\pi}_J(dy) = M^{-1} e^{ty} \pi_J(dy)$. Then $\tilde{\pi}_J$ is a mixture of exponential laws with positive rates $p(b - \alpha)/\theta$, and $e^{tu} f_n(u) = M^n \tilde{f}_n(u)$, where $\tilde{f}_n$ is the density of the sum of n independent jumps with law $\tilde{\pi}_J$. By part (iii) of Lemma 11.5, $\tilde{f}_n(u) \leq n^2/(eu)$. We obtain, for almost every $u \geq 1/2$,

$$\sum_{n \geq 1} e^{-\Lambda_+} \frac{\Lambda_+^n}{n!} f_n(u) = e^{-tu} e^{\Lambda_+(M-1)} \sum_{n \geq 1} e^{-\Lambda_+ M} \frac{(\Lambda_+ M)^n}{n!} \tilde{f}_n(u) \leq e^{-tu} e^p \frac{2}{e} \mathbb{E}[\tilde{N}^2],$$

where $\tilde{N} \sim \text{Pois}(\Lambda_+ M)$. In the inequality, we used $\Lambda_+(M - 1) \leq p$ and $1/(eu) \leq 2/e$. Since $\Lambda_+ M \leq 2p$, we have $\mathbb{E}\tilde{N}^2 \leq 2p + 4p^2 \leq 6p^2$, and $e^{-tu} \leq e^{-\delta p/(4\theta)}$ for $u \geq 1/2$. The left side is the density of the absolutely continuous part of the law of S_+ , and the only atom of this law is at 0. This proves the lemma. $\square$

The last lemma concerns children whose denominators are close to zero.

Lemma 11.7. (i) Let $\rho' > 0$, $H \geq 0$, and let ν be a finite measure on $\mathbb{R}$ whose restriction to $(-\rho', \rho')$ is absolutely continuous with density at most H . Then $\ell(y) = p \int_{(-\rho', \rho')} j_\theta(b; y) \nu(db)$ satisfies

$$\ell(y) \leq \min \left\{ \frac{\theta H}{y^2}, \frac{p^2 \rho'^2 H}{2\theta} \right\}, \quad \int \ell(y) dy = p \nu((-\rho', \rho')) \leq 2\rho' p H. \quad (11.14)$$

(ii) Let $\ell \geq 0$ be integrable with $\lambda = \int \ell \leq 1$ and $\ell(y) \leq \varkappa/y^2$ for $y \neq 0$, and let S have law $e^{-\lambda} \sum_{m \geq 0} \ell^{*m}/m!$. Then the law of S is $e^{-\lambda} \delta_0 + d_S(u) du$ with $d_S(u) \leq 5\varkappa/u^2$ for almost every u , and $\mathbb{P}(|S| \geq u) \leq 4\varkappa/u$ for every $u > 0$.

Proof. For (i), we have $\ell(y) \leq (p^2 H / \theta) \int_0^{\rho'} b e^{-pb|y|/\theta} db$. Bounding the integral by $\int_0^\infty b e^{-pb|y|/\theta} db = \theta^2/(p^2 y^2)$, and alternatively by $\rho'^2/2$, gives the two bounds. The identity for $\int \ell$ follows from $\int j_\theta(b; y) dy = 1$.

For (ii), we may assume $\lambda > 0$. Let $m \geq 1$. Then $\ell^{*m} = \lambda^m \text{Law}(V_1 + \cdots + V_m)$, where V_i are independent with density ℓ/λ . On $\{|y| \geq v/m\}$ this density is at most $\varkappa m^2/(\lambda v^2)$.

By part (ii) of Lemma 11.5, ℓ^{*m} has density at most $\varkappa m^3 \lambda^{m-1} / v^2$ on $\{|u| \geq v\}$. This implies that $\ell^{*m}(u) \leq \varkappa m^3 \lambda^{m-1} / u^2$ for almost every u . Summing, we obtain

$$d_S(u) \leq \frac{\varkappa}{u^2} e^{-\lambda} \sum_{m \geq 1} \frac{m^3 \lambda^{m-1}}{m!} = \frac{\varkappa}{u^2} e^{-\lambda} \sum_{k \geq 0} \frac{(k+1)^2 \lambda^k}{k!} = \frac{\varkappa}{u^2} (1 + 3\lambda + \lambda^2) \leq \frac{5\varkappa}{u^2}.$$

Similarly, by part (i) of Lemma 11.5, $\mathbb{P}(|V_1 + \cdots + V_m| \geq u) \leq m \mathbb{P}(|V_1| \geq u/m) \leq 2\varkappa m^2 / (\lambda u)$, and therefore

$$\mathbb{P}(|S| \geq u) \leq \frac{2\varkappa}{u} e^{-\lambda} \sum_{m \geq 1} \frac{m^2 \lambda^{m-1}}{m!} = \frac{2\varkappa}{u} (1 + \lambda) \leq \frac{4\varkappa}{u}. \quad \square$$

11.3 The class $\mathfrak{S}_{p,\theta}$ and its invariance

Definition 11.8. A tuple $\mathbf{q} = (a, A, K', \rho)$ is admissible if $a > 0$, $A \geq 0$, $K' > 0$, and $0 < \rho \leq 1/16$. For $p > 1$, set

$$r_p = p^{-1/4}, \quad \mathcal{C}_p = [1 - r_p, 1 + r_p], \quad \mathfrak{h}_p = p^A e^{-ap}.$$

The ambient class $\mathcal{A}_p(\mathbf{q})$ is the set of probability measures ν on $\mathbb{R}$ such that

- (C) $\nu(\mathbb{R} \setminus \mathcal{C}_p) \leq e^{-a\sqrt{p}}$;
- (B) the restriction of ν to $(-\rho, \rho)$ is absolutely continuous with density at most $\mathfrak{h}_p$ almost everywhere;
- (T) $\nu(\{b : |b| > L\}) \leq K' \mathfrak{h}_p / L + e^{-apL}$ for every $L \geq 3$.

For $\theta > 0$, the class $\mathfrak{S}_{p,\theta}(\mathbf{q})$ is the set of $\nu \in \mathcal{A}_p(\mathbf{q})$ of the form $\nu = e^{-p} \delta_{1+\theta} + f(b) db$ with $f \in L^1(\mathbb{R})$. We call (C), (B), and (T) the core, boundary, and tail bounds. The canonical tuple is $\mathbf{q}_{\text{can}} = (1, 0, 1, 1/16)$, and we write $\mathcal{A}_p = \mathcal{A}_p(\mathbf{q}_{\text{can}})$ and $\mathfrak{S}_{p,\theta} = \mathfrak{S}_{p,\theta}(\mathbf{q}_{\text{can}})$.

The ambient class is convex, it contains δ_1 , and it does not depend on θ . Its members have no atom in $(-\rho, \rho)$, and in particular the law map is defined on $\mathcal{A}_p(\mathbf{q})$. The members of $\mathfrak{S}_{p,\theta}(\mathbf{q})$ have a single atom, of mass e^{-p} at $1 + \theta$, which is the atom forced by the event $N = 0$ in Lemma 11.2. The ambient class is introduced because $\mathfrak{S}_{p,\theta}(\mathbf{q})$ is not complete for the metric used below; see Remark 11.14.

Constants and the threshold. We now fix the constants used in Sections 11.3–11.6, and we make the order of choices explicit. First, an admissible tuple $\mathbf{q} = (a, A, K', \rho)$ is fixed, and with it

$$\rho_0 = \rho/4, \quad \rho_1 = \rho/2.$$

Second, a number $C_0 > 0$ is fixed; the parameter θ will always satisfy $0 < \theta \leq C_0/p$. Third, with

$$c_1 = \frac{\rho^2}{128(1+C_0)^2}, \quad \omega_p = 2e^{-c_1 p^{7/4}} + 2\rho p \mathfrak{h}_p, \quad \delta_p = p^{A+3}e^{-ap},$$

$$t_p = \frac{\rho_0 p^{7/4}}{2C_0}, \quad x_p = \frac{\rho \rho_0 p^2}{C_0},$$

we consider the following conditions on p .

- (S1) $p \geq 256$, $p \geq (2/\rho)^4$, and $p \geq 12C_0$; in particular $r_p \leq \min\{1/4, \rho/2\}$ and $\theta \leq 1/12$.
- (S2) $(\rho^{-1} + 2)e^{-a\sqrt{p}} \leq r_p$.
- (S3) $2e^{-c_1 p^{7/4}} + (t_p + 2)e^{-t_p} \leq \rho p \mathfrak{h}_p$ and $4\rho p \mathfrak{h}_p \leq e^{-a\sqrt{p}}/2$; in particular $(t_p + 2)e^{-t_p} \leq \omega_p \leq 3\rho p \mathfrak{h}_p \leq e^{-a\sqrt{p}}/2$.
- (S4) $2\rho p \mathfrak{h}_p \leq 1$ and $20C_0 \leq \rho^2 p$.
- (S5) $48p^2 \exp(p - \rho p^2/(4C_0)) \leq \mathfrak{h}_p$.
- (S6) $16C_0 \leq K'p$ and $\rho p^2/(18C_0) \geq (a+1)p + 1$.
- (S7) $x_p \geq 2$ and $2(\rho \rho_0)^{-1} x_p^2 e^{-x_p} \leq \mathfrak{h}_p/2$.
- (S8) $p^{3/4} \geq 3100(1+C_0)\rho^{-2}$ and $160C_0 \leq \rho^2 p^2$.
- (S9) $20C_0/p + 7pe^{-a\sqrt{p}/2} \leq 1/2$ and $2p^{A+3}e^{-ap/2} \leq pe^{-a\sqrt{p}/2}$.

Each condition holds for all sufficiently large p , because in each of them a quantity decaying like e^{-cp} or faster is compared with a quantity decaying at most like $p^{-1}e^{-ap}$ or $e^{-a\sqrt{p}}$, or a polynomial is compared with a larger power of p . We let $p_*(\mathbf{q}, C_0)$ be the least integer such that (S1)–(S9) hold for every $p \geq p_*(\mathbf{q}, C_0)$. It depends only on $\mathbf{q}$ and C_0 . The exponents a and A and the constant K' of the class are never changed in the arguments below: the lemma that follows shows that the law map sends the ambient class into the class $\mathfrak{S}_{p,\theta}(\mathbf{q})$ with the same tuple $\mathbf{q}$, and with the right sides of the three bounds halved.

Lemma 11.9. *Let $\mathbf{q}$ be admissible, $C_0 > 0$, $p \geq p_*(\mathbf{q}, C_0)$, and $0 < \theta \leq C_0/p$. For every $\nu \in \mathcal{A}_p(\mathbf{q})$, the law $\mu = \mathcal{R}_{p,\theta}(\nu)$ satisfies the following.*

- (i) $\mu = e^{-p}\delta_{1+\theta} + h(b)db$, where $h \leq \mathfrak{h}_p/2$ almost everywhere on $[-2\rho, 2\rho]$.
- (ii) $\mu(\{b : |b-1| > r_p/2\}) \leq \omega_p \leq e^{-a\sqrt{p}}/2$.
- (iii) $\mu(\{b : |b| > L\}) \leq K'\mathfrak{h}_p/(2L) + e^{-apL}/2$ for every $L \geq 3$.

In particular, $\mathcal{R}_{p,\theta}$ maps $\mathcal{A}_p(\mathbf{q})$ into $\mathfrak{S}_{p,\theta}(\mathbf{q})$.

Proof. We split the children according to whether $|B_j| \geq \rho$. By Lemma 11.2, a variable B' with law μ can be written as

$$B' = D - S_{\text{bd}}, \quad D = 1 + \theta - S_{\text{nb}}, \quad S_{\text{nb}} = \frac{\theta}{p} \sum_{j \leq N: |B_j| \geq \rho} \frac{Y_j}{B_j}, \quad S_{\text{bd}} = \frac{\theta}{p} \sum_{j \leq N: |B_j| < \rho} \frac{Y_j}{B_j}, \quad (11.15)$$

with S_{nb} and S_{bd} independent. The sum S_{nb} is of the form (11.10) with $\Lambda = p\nu|_{\{|b| \geq \rho\}}$, and S_{bd} is of this form with $\Lambda = p\nu|_{(-\rho, \rho)}$.

STEP 1. Core concentration. We have $\mathbb{E}S_{\text{nb}} = \theta \int_{\{|b| \geq \rho\}} b^{-1} \nu(db)$ by Lemma 11.4. Since $\int (b^{-1} - 1) \mathbf{1}_{\{|b| \geq \rho\}} d\nu - \nu((-\rho, \rho)) = \int_{\{|b| \geq \rho\}} b^{-1} d\nu - 1$, and $|b^{-1} - 1| \leq r_p/(1 - r_p) \leq 2r_p$ on $\mathcal{C}_p$, we have

$$\begin{aligned} \left| \int_{\{|b| \geq \rho\}} \frac{\nu(db)}{b} - 1 \right| &\leq 2r_p + (\rho^{-1} + 1)\nu(\mathbb{R} \setminus \mathcal{C}_p) + \nu((-\rho, \rho)) \\ &\leq 2r_p + (\rho^{-1} + 2)e^{-a\sqrt{p}} \leq 3r_p. \end{aligned}$$

In the second inequality, we used (C) and $(-\rho, \rho) \subset \mathbb{R} \setminus \mathcal{C}_p$, and in the last one (S2). Since $\theta \leq 1/12$ by (S1), we obtain $|\mathbb{E}S_{\text{nb}} - \theta| \leq 3\theta r_p \leq r_p/4$. Next, (11.11) with $\delta = \rho$ and $x = r_p/4$, together with $\theta \leq C_0/p$ and $r_p = p^{-1/4}$, gives

$$\mathbb{P}(|S_{\text{nb}} - \mathbb{E}S_{\text{nb}}| > r_p/4) \leq 2 \exp \left(- \min \left\{ \frac{\rho^2 p^{5/2}}{128 C_0^2}, \frac{\rho p^{7/4}}{16 C_0} \right\} \right) \leq 2e^{-c_1 p^{7/4}}.$$

Finally, $\mathbb{P}(S_{\text{bd}} \neq 0) = 1 - e^{-p\nu((-\rho, \rho))} \leq 2\rho p \mathfrak{h}_p$ by (B). Since $|B' - 1| \leq |\mathbb{E}S_{\text{nb}} - \theta| + |S_{\text{nb}} - \mathbb{E}S_{\text{nb}}|$ on the event $S_{\text{bd}} = 0$, we conclude that $\mathbb{P}(|B' - 1| > r_p/2) \leq \omega_p$. The bound $\omega_p \leq e^{-a\sqrt{p}}/2$ is (S3). This proves (ii).

STEP 2. Fresh cancellation. By Lemma 11.6 with $\delta = \rho$ and $c = 1 + \theta$, the restriction of the law of D to $(-\infty, 1/2]$ is absolutely continuous with density at most $F = F_{p,\theta}(\rho)$. Since $\theta \leq C_0/p$, we have $F \leq 6p^2 \exp(p - \rho p^2/(4C_0)) \leq \mathfrak{h}_p/8$ by (S5).

STEP 3. Boundary renewal. By part (i) of Lemma 11.7 with $\rho' = \rho$ and $H = \mathfrak{h}_p$, the intensity ℓ_{bd} of the jumps of S_{bd} satisfies $\ell_{\text{bd}}(y) \leq \theta \mathfrak{h}_p/y^2$ and $\int \ell_{\text{bd}} \leq 2\rho p \mathfrak{h}_p \leq 1$, by (S4). By part (ii) of the same lemma, the law of S_{bd} is $e^{-\lambda} \delta_0 + d_{\text{bd}}(u) du$ with $\lambda = \int \ell_{\text{bd}}$ and $d_{\text{bd}}(u) \leq 5\theta \mathfrak{h}_p/u^2$. Let $A \subset [-2\rho, 2\rho]$ be a Borel set. Since D and S_{bd} are independent,

$$\mathbb{P}(B' \in A) = e^{-\lambda} \mathbb{P}(D \in A) + \int \left(\int \mathbf{1}_A(d - u) d_{\text{bd}}(u) du \right) \mathbb{P}(D \in dd).$$

The first term is at most $F|A|$, because $A \subset (-\infty, 1/2]$. We split the integral in the second term according to whether $|d| \leq 3\rho$. Since $3\rho \leq 1/2$, the part with $|d| \leq 3\rho$ equals $\int d_{\text{bd}}(u) \mathbb{P}(D \in (A + u) \cap [-3\rho, 3\rho]) du \leq F|A|$. If $|d| > 3\rho$ and $d - u \in A$, then $|u| \geq \rho$

and $d_{\text{bd}}(u) \leq 5\theta\mathfrak{h}_p/\rho^2$, so the part with $|d| > 3\rho$ is at most $5\theta\mathfrak{h}_p\rho^{-2}|A|$. We conclude that the restriction of μ to $[-2\rho, 2\rho]$ has density at most

$$2F + \frac{5\theta}{\rho^2}\mathfrak{h}_p \leq \frac{\mathfrak{h}_p}{4} + \frac{5C_0}{\rho^2 p}\mathfrak{h}_p \leq \frac{\mathfrak{h}_p}{2}, \quad (11.16)$$

where we used Step 2 and (S4). Together with part (iii) of Lemma 11.2, this proves (i).

STEP 4. Far tail. Let $L \geq 3$. If $|D| > L/2$, then $\sum_{|B_j| \geq \rho} \theta Y_j / (p|B_j|) \geq |S_{\text{nb}}| > L/2 - 1 - \theta \geq L/9$, where we used $\theta \leq 1/12$ and $L \geq 3$. By (11.12) with $\delta = \rho$ and (S6),

$$\mathbb{P}(|D| > L/2) \leq \exp\left(p - \frac{\rho p^2 L}{18C_0}\right) \leq \exp(p - ((a+1)p + 1)L) \leq \frac{1}{2}e^{-apL}.$$

Further, part (ii) of Lemma 11.7 gives $\mathbb{P}(|S_{\text{bd}}| \geq L/2) \leq 8\theta\mathfrak{h}_p/L \leq 8C_0\mathfrak{h}_p/(pL) \leq K'\mathfrak{h}_p/(2L)$, where we used (S6) in the last step. Since $|B'| > L$ implies $|D| > L/2$ or $|S_{\text{bd}}| \geq L/2$, this proves (iii).

Finally, (i)–(iii) show that μ satisfies (C), (B), and (T), since $\{b : |b-1| > r_p\} \subset \{b : |b-1| > r_p/2\}$ and $(-\rho, \rho) \subset [-2\rho, 2\rho]$. We conclude that $\mu \in \mathfrak{S}_{p,\theta}(\mathbf{q})$. $\square$

Remark 11.10. The proof gives more than the bounds (C), (B), and (T). The output of the law map has density at most $2F_{p,\theta}(\rho) + 5\theta\rho^{-2}\mathfrak{h}$ on $[-2\rho, 2\rho]$ whenever the input has density at most $\mathfrak{h}$ on $(-\rho, \rho)$ and $2\rho p\mathfrak{h} \leq 1$; the first term is of order e^{-cp^2} . In particular, the boundary bound is not saturated at a fixed point, and any admissible choice of a and A can be used. This is why the tuple $\mathbf{q}$ in Definition 11.8 can be fixed once and for all, independently of the energy window; see also part (iii) of Theorem 11.19.

11.4 The mixed metric

We fix once and for all a C^∞ function $\chi : \mathbb{R} \rightarrow [0, 1]$ which is even, equal to 1 on $[-1, 1]$, equal to 0 outside $(-2, 2)$, and satisfies $|\chi'| \leq 2$. Its higher derivatives are then bounded by numerical constants.

Definition 11.11. Let $\mathbf{q} = (a, A, K', \rho)$ be admissible, let $p \geq \max\{256, (2/\rho)^4\}$, and recall $\rho_0 = \rho/4$ and $\rho_1 = \rho/2$.

- (i) The cutoffs are $\chi_c(b) = \chi((b-1)/r_p)$, $\chi_b(b) = \chi(4b/\rho)$, and $\chi_i = 1 - \chi_c - \chi_b$.
- (ii) The core region is $[1 - 2r_p, 1 + 2r_p]$, the boundary region is $[-\rho_1, \rho_1]$, and the intermediate region is

$$I_p = (-\infty, -\rho_0] \cup [\rho_0, 1 - r_p] \cup \{1\} \cup [1 + r_p, \infty).$$

- (iii) For a finite signed measure σ on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$ and $\alpha \in \{c, b, i\}$, the compensated projections are

$$\Pi_\alpha \sigma = \chi_\alpha \sigma - m_\alpha(\sigma) \delta_1, \quad m_\alpha(\sigma) = \int \chi_\alpha d\sigma.$$

- (iv) For a finite signed measure τ with $\tau(\mathbb{R}) = 0$, the component norms, with values in $[0, \infty]$, are

$$\|\tau\|_c = \sup \left\{ \left| \int \varphi d\tau \right| : \varphi(1) = 0, \text{Lip}(\varphi) \leq 1 \right\},$$

$$\|\tau\|_i = \sup \left\{ \left| \int \varphi d\tau \right| : \|\varphi\|_\infty \leq 1, \text{Lip}(\varphi) \leq 1 \right\},$$

and $\|\tau\|_b = \|g\|_{L^\infty}$ if $\tau = g(b) db - (\int g)\delta_1$ for some $g \in L^\infty$ vanishing outside $[-\rho_1, \rho_1]$, and $\|\tau\|_b = \infty$ otherwise. We write $n_\alpha(\sigma) = \|\Pi_\alpha \sigma\|_\alpha$ for $\sigma(\mathbb{R}) = 0$.

- (v) The weights are $W_c = 1$, $W_b = e^{ap/2}$, and $W_i = e^{a\sqrt{p}/2}$. The mixed norm of σ with $\sigma(\mathbb{R}) = 0$ is

$$N_p(\sigma) = n_c(\sigma) + W_b n_b(\sigma) + W_i n_i(\sigma) \in [0, \infty],$$

and the mixed metric on probability measures is $d_p(\nu, \mu) = N_p(\nu - \mu)$.

The regions, the cutoffs, and the mixed metric depend on p and on $\mathbf{q}$ through ρ and a , but not on θ . The norm $\|\cdot\|_c$ is the Kantorovich–Rubinstein norm and $\|\cdot\|_i$ is the dual bounded–Lipschitz norm. In the boundary norm, the decomposition $\tau = g db - (\int g)\delta_1$ is unique when it exists, since the absolutely continuous part and the atom at 1 of a measure are uniquely determined.

Lemma 11.12. *Let $\mathbf{q}$ be admissible, $p \geq \max\{256, (2/\rho)^4\}$, and let σ be a finite signed measure with $\sigma(\mathbb{R}) = 0$.*

- (i) *The supports of χ_c and χ_b are contained in the core and boundary regions, which are disjoint; $0 \leq \chi_i \leq 1$; and χ_i vanishes on $\mathcal{C}_p \cup [-\rho_0, \rho_0]$, so that $\{\chi_i \neq 0\} \subset I_p \setminus \{1\}$. We have $\sigma = \Pi_c \sigma + \Pi_b \sigma + \Pi_i \sigma$, each $\Pi_\alpha \sigma$ has total mass zero, $\Pi_c \sigma$ is carried by the core region, and $\Pi_i \sigma$ is carried by I_p .*
- (ii) *For every Lipschitz function φ with $\varphi(1) = 0$, we have $\langle \Pi_c \sigma, \varphi \rangle = \langle \sigma, \chi_c \varphi \rangle$. For every bounded Lipschitz φ , we have $\langle \Pi_i \sigma, \varphi \rangle = \langle \sigma, \chi_i(\varphi - \varphi(1)) \rangle$.*
- (iii) *If $\varphi(1) = 0$ and $\text{Lip}(\varphi) \leq 1$, then $\psi_c = \chi_c \varphi$ satisfies $\|\psi_c\|_\infty \leq 2r_p$ and $\text{Lip}(\psi_c) \leq 5$. If $\|\varphi\|_\infty \leq 1$ and $\text{Lip}(\varphi) \leq 1$, then $\psi_i = \chi_i(\varphi - \varphi(1))$ satisfies $\|\psi_i\|_\infty \leq 2$ and $\text{Lip}(\psi_i) \leq 21/\rho$, and ψ_i vanishes on $\mathcal{C}_p \cup [-\rho_0, \rho_0]$.*
- (iv) *In the suprema defining $\|\cdot\|_c$ and $\|\cdot\|_i$, applied to a measure carried by a compact set in the first case, one may restrict to C^∞ functions φ .*
- (v) *Let τ be a finite signed measure with $\tau(\mathbb{R}) = 0$. If τ is carried by an interval $I \ni 1$ and $G : I \rightarrow \mathbb{R}$ is Lipschitz, then $|\int G d\tau| \leq \text{Lip}_I(G) \|\tau\|_c$. If τ is carried by a closed set I and $G : I \rightarrow \mathbb{R}$ is bounded and Lipschitz, then $|\int G d\tau| \leq (\sup_I |G| + \text{Lip}_I(G)) \|\tau\|_i$. If G is continuously differentiable on each of the four pieces of I_p , with $|G'| \leq M$ there and $|G| \leq S$ on I_p , then $\text{Lip}_{I_p}(G) \leq M + 2S/r_p$.*
- (vi) *N_p is a norm on the space of finite signed measures σ with $\sigma(\mathbb{R}) = 0$ and $N_p(\sigma) < \infty$. For every $\nu, \mu \in \mathcal{A}_p(\mathbf{q})$, $d_p(\nu, \mu) < \infty$, and d_p is a metric on $\mathcal{A}_p(\mathbf{q})$.*

Proof. For (i), the support of χ_c is in $[1 - 2r_p, 1 + 2r_p] \subset [1/2, 3/2]$ and that of χ_b is in $[-\rho/2, \rho/2]$, and these are disjoint. Since at each point at most one of χ_c, χ_b is nonzero, and both take values in $[0, 1]$, we have $0 \leq \chi_i \leq 1$. Since $\chi_c = 1$ on $\mathcal{C}_p$ and $\chi_b = 1$ on $[-\rho_0, \rho_0]$, the function χ_i vanishes there. The identity $\sum_\alpha \Pi_\alpha \sigma = \sigma$ follows from $\sum_\alpha \chi_\alpha = 1$ and $\sum_\alpha m_\alpha(\sigma) = \sigma(\mathbb{R}) = 0$, and each $\Pi_\alpha \sigma$ has mass $m_\alpha(\sigma) - m_\alpha(\sigma) = 0$. Part (ii) follows from the definition of Π_α , since $\langle \chi_i \sigma - m_i(\sigma) \delta_1, \varphi \rangle = \langle \sigma, \chi_i \varphi \rangle - \varphi(1) \langle \sigma, \chi_i \rangle$.

For (iii), on the support of χ_c we have $|b - 1| \leq 2r_p$ and $|\varphi(b)| \leq 2r_p$. Since $|\chi'_c| \leq 2/r_p$, we obtain $|(\chi_c \varphi)'| \leq (2/r_p)(2r_p) + 1 = 5$ almost everywhere. For ψ_i , we have $\chi'_i = -\chi'_c - \chi'_b$. On the support of χ'_c , $|\chi'_c(\varphi - \varphi(1))| \leq (2/r_p)(2r_p) = 4$; and $|\chi'_b(\varphi - \varphi(1))| \leq (8/\rho) \cdot 2$. Adding $\chi_i |\varphi'| \leq 1$, we find $\text{Lip}(\psi_i) \leq 5 + 16/\rho \leq 21/\rho$. The vanishing of ψ_i follows from (i).

For (iv), let φ be Lipschitz and let φ_ε be its convolution with a smooth probability density supported in $[-\varepsilon, \varepsilon]$. Then φ_ε is smooth, $\text{Lip}(\varphi_\varepsilon) \leq \text{Lip}(\varphi)$, $\|\varphi_\varepsilon\|_\infty \leq \|\varphi\|_\infty$, and $|\varphi_\varepsilon - \varphi| \leq \varepsilon \text{Lip}(\varphi)$. In the first case, $\varphi_\varepsilon - \varphi_\varepsilon(1)$ vanishes at 1. Since τ is finite, $\int \varphi_\varepsilon d\tau \rightarrow \int \varphi d\tau$, and (iv) follows.

For (v), if I is an interval, extend $G - G(1)$ to $\mathbb{R}$ by constants outside I . The extension φ satisfies $\varphi(1) = 0$ and $\text{Lip}(\varphi) = \text{Lip}_I(G)$, and $\int G d\tau = \int \varphi d\tau$ since τ has zero mass. If I is closed, let $\Lambda = \text{Lip}_I(G)$ and $S' = \sup_I |G|$. The McShane extension $\tilde{G}(x) = \inf_{y \in I} (G(y) + \Lambda|x - y|)$ agrees with G on I and has Lipschitz constant Λ , and truncating it at levels $\pm S'$ preserves both properties. The truncated extension divided by $\max\{S', \Lambda\}$ is one of the test functions in the definition of $\|\tau\|_i$. For the last claim, note that $p \geq (2/\rho)^4$ gives $r_p \leq \rho/2 = 2\rho_0$, so distinct pieces of I_p are at distance at least r_p . For b, b' in the same piece we have $|G(b) - G(b')| \leq M|b - b'|$, and otherwise $|G(b) - G(b')| \leq 2S \leq (2S/r_p)|b - b'|$.

For (vi), suppose $N_p(\sigma) = 0$. Then $\|\Pi_c \sigma\|_c = 0$, and we have $\int \varphi d\Pi_c \sigma = 0$ for every Lipschitz φ (subtract $\varphi(1)$ and use that $\Pi_c \sigma$ has zero mass). Since bounded Lipschitz functions determine finite measures, $\Pi_c \sigma = 0$. Similarly $\Pi_b \sigma = 0$ and $\Pi_i \sigma = 0$, and (i) gives $\sigma = 0$. Homogeneity and the triangle inequality are clear. If $\nu, \mu \in \mathcal{A}_p(\mathbf{q})$ and $\sigma = \nu - \mu$, then $\|\Pi_c \sigma\|_c \leq 2r_p \|\Pi_c \sigma\|_{\text{TV}} \leq 8r_p$, because $\Pi_c \sigma$ is carried by the core region, where the test functions in the definition of $\|\cdot\|_c$ are bounded by $2r_p$; $\Pi_b \sigma = \chi_b(f_\nu - f_\mu) db - m_b(\sigma) \delta_1$, where f_ν, f_μ are the densities on $(-\rho, \rho)$ given by (B), and $n_b(\sigma) \leq 2h_p$; and $n_i(\sigma) \leq \|\Pi_i \sigma\|_{\text{TV}} \leq 4$. In particular, $d_p(\nu, \mu) < \infty$. $\square$

Lemma 11.13. *Let $\mathbf{q}$ be admissible and $p \geq \max\{256, (2/\rho)^4\}$.*

- (i) *The class $\mathcal{A}_p(\mathbf{q})$ is tight and closed under weak convergence of probability measures.*
- (ii) *If $\nu_n, \nu \in \mathcal{A}_p(\mathbf{q})$ and $\nu_n \Rightarrow \nu$, then $d_p(\mu, \nu) \leq \liminf_n d_p(\mu, \nu_n)$ for every $\mu \in \mathcal{A}_p(\mathbf{q})$.*
- (iii) *The metric space $(\mathcal{A}_p(\mathbf{q}), d_p)$ is complete, and convergence in d_p implies weak convergence.*
- (iv) *Let $\theta_n \rightarrow \theta > 0$, and let ν_n, ν be probability measures on $\mathbb{R}$ without atom at 0 with $\nu_n \Rightarrow \nu$. Then $\mathcal{R}_{p, \theta_n}(\nu_n) \Rightarrow \mathcal{R}_{p, \theta}(\nu)$.*

Proof. For (i), tightness follows from the uniform tail bound (T). Let $\nu_n \in \mathcal{A}_p(\mathbf{q})$ converge weakly to ν . Since the sets $\mathbb{R} \setminus \mathcal{C}_p$ and $\{|b| > L\}$ are open, the portmanteau theorem [37, Theorem 2.1] shows that ν satisfies (C) and (T). For an open interval $J \subset (-\rho, \rho)$, we have $\nu(J) \leq \liminf_n \nu_n(J) \leq \mathfrak{h}_p |J|$ by the same theorem. Since every open subset of $(-\rho, \rho)$ is a countable disjoint union of open intervals, we obtain $\nu(U) \leq \mathfrak{h}_p |U|$ for open $U \subset (-\rho, \rho)$, and by outer regularity of Lebesgue measure, $\nu(A) \leq \mathfrak{h}_p |A|$ for every Borel set $A \subset (-\rho, \rho)$. By the Radon–Nikodym theorem, ν satisfies (B).

For (ii), let $\mu \in \mathcal{A}_p(\mathbf{q})$ and write $\sigma_n = \mu - \nu_n$ and $\sigma = \mu - \nu$. For each smooth φ with $\varphi(1) = 0$, the function $\chi_c \varphi$ is bounded and continuous, and we have $\langle \Pi_c \sigma_n, \varphi \rangle = \langle \sigma_n, \chi_c \varphi \rangle \rightarrow \langle \sigma, \chi_c \varphi \rangle = \langle \Pi_c \sigma, \varphi \rangle$. By part (iv) of Lemma 11.12, $n_c(\sigma)$ is a supremum of such limits, which implies that $n_c(\sigma) \leq \liminf_n n_c(\sigma_n)$. The same argument applies to n_i , using the bounded continuous functions $\chi_i(\varphi - \varphi(1))$. For n_b , let f_n, f be the densities of ν_n, ν on $(-\rho, \rho)$ and f_μ that of μ . Since L^∞ is the dual of L^1 and continuous functions with compact support are dense in L^1 , we have

$$n_b(\sigma) = \|\chi_b(f_\mu - f)\|_{L^\infty} = \sup \left\{ \left| \int \chi_b(f_\mu - f) u \right| : u \in C_c((-\rho, \rho)), \|u\|_{L^1} \leq 1 \right\}.$$

For each such u , the function $\chi_b u$ is bounded and continuous, and $\int \chi_b f_n u = \langle \nu_n, \chi_b u \rangle \rightarrow \langle \nu, \chi_b u \rangle = \int \chi_b f u$. We conclude that $n_b(\sigma) \leq \liminf_n n_b(\sigma_n)$. Combining the three components proves (ii).

For (iii), let (ν_n) be a Cauchy sequence for d_p in $\mathcal{A}_p(\mathbf{q})$. By (i) and Prohorov's theorem [37, Theorem 5.1], a subsequence (ν_{n_k}) converges weakly to some $\nu \in \mathcal{A}_p(\mathbf{q})$. By (ii), for each n ,

$$d_p(\nu_n, \nu) \leq \liminf_{k \rightarrow \infty} d_p(\nu_n, \nu_{n_k}) \leq \sup_{m \geq n} d_p(\nu_n, \nu_m),$$

which tends to zero as $n \rightarrow \infty$. Hence $\nu_n \rightarrow \nu$ in d_p . Now suppose $d_p(\nu_n, \nu) \rightarrow 0$ with $\nu_n, \nu \in \mathcal{A}_p(\mathbf{q})$. By (i), every subsequence has a further subsequence converging weakly to some $\nu' \in \mathcal{A}_p(\mathbf{q})$, and by (ii), $d_p(\nu', \nu) \leq \liminf d_p(\nu_{n_k}, \nu) = 0$. Then $\nu' = \nu$, and $\nu_n \Rightarrow \nu$.

For (iv), let G_n and G be the quantile functions of ν_n and ν , and let $U_1, U_2, \dots$ be independent uniform variables on $(0, 1)$, independent of N and (Y_j) . Then $B_{n,j} = G_n(U_j)$ has law ν_n , $B_j = G(U_j)$ has law ν , and $B_{n,j} \rightarrow B_j$ almost surely for each j , because $G_n \rightarrow G$ at every continuity point of G and G has countably many discontinuities (this is the quantile form of Skorokhod's representation theorem; see [37, Theorem 6.7]). Since $B_j \neq 0$ almost surely,

$$1 + \theta_n - \frac{\theta_n}{p} \sum_{j \leq N} \frac{Y_j}{B_{n,j}} \longrightarrow 1 + \theta - \frac{\theta}{p} \sum_{j \leq N} \frac{Y_j}{B_j}$$

almost surely, and almost sure convergence implies convergence in distribution. $\square$

Remark 11.14. The class $\mathfrak{S}_{p,\theta}(\mathbf{q})$ is not complete for d_p . For example, let $v_n = e^{-p}\delta_{1+\theta} + (1 - e^{-p})u_n$, where u_n is the uniform law on $[1 - 1/n, 1 + 1/n]$. If $\theta \leq r_p$ and $n \geq 1/r_p$, then $v_n \in \mathfrak{S}_{p,\theta}(\mathbf{q})$, and $d_p(v_n, v_m) = n_c(v_n - v_m) \leq 1/n + 1/m$, because $v_n - v_m$ is carried by $\mathcal{C}_p$, where $\chi_c = 1$. The d_p -limit $e^{-p}\delta_{1+\theta} + (1 - e^{-p})\delta_1$ has a second atom and is not in $\mathfrak{S}_{p,\theta}(\mathbf{q})$. The core and intermediate norms metrize weak convergence on their regions, and weak limits of absolutely continuous laws may have atoms. The boundary norm is a supremum norm of densities, and its limits keep a density; this is why atoms are allowed in the ambient class exactly outside the open interval $(-\rho, \rho)$, which contains the boundary region. The law map sends $\mathcal{A}_p(\mathbf{q})$ back into $\mathfrak{S}_{p,\theta}(\mathbf{q})$ by Lemma 11.9, and in particular fixed points in the ambient class belong to $\mathfrak{S}_{p,\theta}(\mathbf{q})$.

11.5 Column estimates for the linearized map

Throughout this subsection, $\mathbf{q} = (a, A, K', \rho)$ is admissible, $C_0 > 0$, $p \geq p_*(\mathbf{q}, C_0)$, $0 < \theta \leq C_0/p$, and $\lambda \in \mathcal{A}_p(\mathbf{q})$ is arbitrary. We use the notation of Lemma 11.3: $\mu_\lambda = \mathcal{R}_{p,\theta}(\lambda)$, $D_\lambda \sim \mu_\lambda$, $T \sim \text{Exp}(1)$ independent of D_λ , $Z_b = D_\lambda - \theta T/(pb)$, and k_b^λ is the density of Z_b . By Lemma 11.9, μ_λ has a density $h_\lambda \leq \mathfrak{h}_p/2$ on $[-2\rho, 2\rho]$ and satisfies $\mu_\lambda(\{d : |d - 1| > r_p/2\}) \leq \omega_p$. In applications, λ is a point of a segment joining two members of $\mathcal{A}_p(\mathbf{q})$, or a fixed point of the law map; all estimates below are uniform in $\lambda \in \mathcal{A}_p(\mathbf{q})$.

Lemma 11.15. *Under the assumptions above, the following hold.*

- (i) *For every b with $|b| \geq \rho_0$, we have $\mathbb{P}(|Z_b - 1| \geq r_p) + \mathbb{E}[T \mathbf{1}_{\{|Z_b - 1| \geq r_p\}}] \leq 3\omega_p$.*
- (ii) *For every $x \in [-\rho_1, \rho_1]$, the function $b \mapsto k_b^\lambda(x)$ is continuously differentiable on $(-\infty, -\rho_0]$ and on $[\rho_0, \infty)$, and there*

$$0 \leq k_b^\lambda(x) \leq \mathfrak{h}_p, \quad |\partial_b k_b^\lambda(x)| \leq \mathfrak{h}_p/\rho_0.$$

- (iii) *For every $x \in [-\rho_1, \rho_1]$ and every $g \in L^\infty$ vanishing outside $[-\rho_1, \rho_1]$,*

$$\left| p \int k_b^\lambda(x) g(b) db \right| \leq \left(2\theta + \rho p \mathfrak{h}_p + \frac{2\theta \omega_p}{\rho^2} \right) \|g\|_{L^\infty}.$$

Proof. For (i), let $|b| \geq \rho_0$. On the event $|D_\lambda - 1| \leq r_p/2$, the inequality $|Z_b - 1| \geq r_p$ forces $\theta T/(p|b|) \geq r_p/2$, that is, $T \geq p|b|r_p/(2\theta) \geq t_p$, where we used $|b| \geq \rho_0$ and $\theta \leq C_0/p$. Since T is independent of D_λ , we obtain

$$\begin{aligned} \mathbb{P}(|Z_b - 1| \geq r_p) + \mathbb{E}[T \mathbf{1}_{\{|Z_b - 1| \geq r_p\}}] &\leq \omega_p + e^{-t_p} + \mathbb{E}[T] \omega_p + \mathbb{E}[T \mathbf{1}_{\{T \geq t_p\}}] \\ &= 2\omega_p + (t_p + 2)e^{-t_p}, \end{aligned}$$

which is at most $3\omega_p$ by (S3).

For (ii), let $e_s(u) = se^{-su}$ for $s > 0$ and $u \geq 0$. Let $b \geq \rho_0$; the case $b \leq -\rho_0$ is identical after the reflection $d - x \mapsto x - d$ in (11.7). By (11.7), $k_b^\lambda(x) = \int_{(x,\infty)} e_{pb/\theta}(d-x) \mu_\lambda(dd)$, and

$$\partial_b e_{pb/\theta}(u) = \frac{p}{\theta} \left(1 - \frac{pbu}{\theta}\right) e^{-pbu/\theta}.$$

This derivative is bounded by p/θ and continuous in b . By dominated convergence, $b \mapsto k_b^\lambda(x)$ is continuously differentiable with derivative $\int_{(x,\infty)} \partial_b e_{pb/\theta}(d-x) \mu_\lambda(dd)$. We split the integrals into $d \in (x, 2\rho]$ and $d > 2\rho$. On $(x, 2\rho] \subset [-2\rho, 2\rho]$, the measure μ_λ has density at most $\mathfrak{h}_p/2$, and this part of $k_b^\lambda(x)$ is at most $(\mathfrak{h}_p/2) \int_0^\infty e_{pb/\theta} = \mathfrak{h}_p/2$, and this part of the derivative is at most

$$\frac{\mathfrak{h}_p}{2} \int_0^\infty \frac{p}{\theta} \left|1 - \frac{pbu}{\theta}\right| e^{-pbu/\theta} du = \frac{\mathfrak{h}_p}{2b} \int_0^\infty |1 - v| e^{-v} dv = \frac{\mathfrak{h}_p}{eb} \leq \frac{\mathfrak{h}_p}{2\rho_0}.$$

If $d > 2\rho$, then $u = d - x \geq 2\rho - \rho_1 \geq \rho$. Let $S = p/\theta$, $w = Sbu$, and $x_0 = \rho\rho_0 S$. Since $S \geq p^2/C_0$, we have $x_0 \geq x_p \geq 2$ by (S7), and since $b \geq \rho_0$ and $u \geq \rho$, we have $w \geq x_0$. The functions we^{-w} and $(1+w)e^{-w}$ are decreasing on $[1, \infty)$, and x^2e^{-x} is decreasing on $[2, \infty)$. Hence

$$\begin{aligned} e_{pb/\theta}(u) &= \frac{w}{u} e^{-w} \leq \frac{x_0 e^{-x_0}}{\rho} \leq \frac{2x_p^2 e^{-x_p}}{\rho\rho_0}, \\ |\partial_b e_{pb/\theta}(u)| &\leq S(1+w)e^{-w} \leq \frac{x_0(1+x_0)e^{-x_0}}{\rho\rho_0} \leq \frac{2x_p^2 e^{-x_p}}{\rho\rho_0}, \end{aligned}$$

and the right sides are at most $\mathfrak{h}_p/2$ by (S7). Since μ_λ is a probability measure, the parts of $k_b^\lambda(x)$ and of its derivative coming from $d > 2\rho$ are at most $\mathfrak{h}_p/2$. Adding the two parts proves (ii).

For (iii), consider first the contribution of $b \in (0, \rho_1]$. We split (x, ∞) into $P_1 = (x, 2\rho]$, $P_2 = (2\rho, \infty) \cap \{|d-1| > r_p/2\}$, and $P_3 = [1 - r_p/2, 1 + r_p/2]$; these sets cover (x, ∞) because $1 - r_p/2 > 2\rho$. As in the proof of (ii), the part of $p k_b^\lambda(x)$ coming from P_1 is at most $p\mathfrak{h}_p/2$, so the contribution of P_1 is at most $p\rho_1\mathfrak{h}_p\|g\|_{L^\infty}/2$. For $P_2 \cup P_3$, we use Tonelli's theorem and

$$\int_0^\infty p e_{pb/\theta}(u) db = \int_0^\infty \frac{p^2 b}{\theta} e^{-pbu/\theta} db = \frac{\theta}{u^2},$$

so that the contribution of $P_2 \cup P_3$ is at most $\|g\|_{L^\infty} \int_{P_2 \cup P_3} \theta(d-x)^{-2} \mu_\lambda(dd)$. On P_2 we have $d-x \geq \rho$ and $\mu_\lambda(P_2) \leq \omega_p$, and on P_3 we have $d-x \geq 1 - r_p/2 - \rho_1 \geq 27/32$, where we used $r_p \leq 1/4$ and $\rho \leq 1/16$. Since $(32/27)^2 \leq 2$, this contribution is at most $(\theta\omega_p/\rho^2 + 2\theta)\|g\|_{L^\infty}$. For $b \in [-\rho_1, 0)$, the kernel integrates over $d < x$, and we split into $[-2\rho, x)$ and $(-\infty, -2\rho) \subset \{|d-1| > r_p/2\}$. By the same arguments, the contribution of $b \in [-\rho_1, 0)$ is at most $(p\rho_1\mathfrak{h}_p/2 + \theta\omega_p/\rho^2)\|g\|_{L^\infty}$. Adding the contributions and using $\rho_1 = \rho/2$ proves (iii). $\square$

For $\alpha, \beta \in \{c, b, i\}$, the quantity $n_\alpha(\mathcal{L}\Pi_\beta\sigma)$ below measures the α component of the output produced by the β component of the input. Recall from Definition 11.11 that $n_\beta(\sigma) = \|\Pi_\beta\sigma\|_\beta$ is the norm of the measure $\Pi_\beta\sigma$ itself; the compensated projections are not idempotent, and we never project an input twice.

Lemma 11.16. *Let $\mathbf{q}$ be admissible, $C_0 > 0$, $p \geq p_*(\mathbf{q}, C_0)$, $0 < \theta \leq C_0/p$, $\lambda \in \mathcal{A}_p(\mathbf{q})$, and $\mathcal{L} = \mathcal{L}_{p,\theta,\lambda}$. Let $\mathcal{D}_p$ be the real vector space of finite signed measures σ on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$ whose restriction to $(-\rho_1, \rho_1)$ is absolutely continuous with bounded density. Then $\Pi_\alpha\sigma \in \mathcal{D}_p$ and $\mathcal{L}\sigma \in \mathcal{D}_p$ for $\sigma \in \mathcal{D}_p$, and with $\delta_p = p^{A+3}e^{-ap}$ and*

$$M_{p,\theta} = \begin{pmatrix} M_{cc} & M_{cb} & M_{ci} \\ M_{bc} & M_{bb} & M_{bi} \\ M_{ic} & M_{ib} & M_{ii} \end{pmatrix} = \begin{pmatrix} 20\theta & p & 6p \\ \delta_p & 2\theta + \delta_p & \delta_p \\ \delta_p & p & \delta_p \end{pmatrix}, \quad (11.17)$$

we have, for every $\sigma \in \mathcal{D}_p$ and all $\alpha, \beta \in \{c, b, i\}$,

$$n_\alpha(\mathcal{L}\Pi_\beta\sigma) \leq M_{\alpha\beta} n_\beta(\sigma), \quad \text{and hence} \quad n_\alpha(\mathcal{L}\sigma) \leq \sum_\beta M_{\alpha\beta} n_\beta(\sigma). \quad (11.18)$$

Proof. The proof is a direct computation with the dual formula (11.6) for the core and intermediate outputs and the density formula of Lemma 11.3 for the boundary output, combined with Lemma 11.15. We first record these formulas.

STEP 1. The three outputs. Let $\tau \in \mathcal{D}_p$. Then $|\tau|(\{0\}) = 0$, and by Lemma 11.3, $\mathcal{L}\tau$ has total mass zero and density $G_\tau(x) = p \int k_b^\lambda(x) \tau(db)$. Then $\Pi_b\mathcal{L}\tau = \chi_b G_\tau dx - m_b(\mathcal{L}\tau)\delta_1$ and

$$n_b(\mathcal{L}\tau) \leq \sup_{|x| \leq \rho_1} |G_\tau(x)|. \quad (11.19)$$

For a C^∞ function ψ with bounded derivative, set $F_\psi(b) = p \mathbb{E}\psi(Z_b)$ for $b \neq 0$. Since $\partial_b Z_b = \theta T/(pb^2)$, dominated convergence shows that F_ψ is continuously differentiable on $(-\infty, 0)$ and on $(0, \infty)$, with

$$F'_\psi(b) = \frac{\theta}{b^2} \mathbb{E}[T \psi'(Z_b)]. \quad (11.20)$$

By parts (ii) and (iv) of Lemma 11.12 and (11.6), we have

$$n_c(\mathcal{L}\tau) = \sup_\varphi \left| \int F_{\chi_c\varphi} d\tau \right|, \quad n_i(\mathcal{L}\tau) = \sup_{\varphi'} \left| \int F_{\chi_i(\varphi' - \varphi'(1))} d\tau \right|, \quad (11.21)$$

where φ ranges over C^∞ functions with $\varphi(1) = 0$ and $\text{Lip}(\varphi) \leq 1$, and φ' over C^∞ functions with $\|\varphi'\|_\infty \leq 1$ and $\text{Lip}(\varphi') \leq 1$. We write $\psi_c = \chi_c\varphi$ and $\psi_i = \chi_i(\varphi' - \varphi'(1))$, and we recall the bounds of part (iii) of Lemma 11.12. Since ψ_i is smooth and vanishes on $\mathcal{C}_p$, its derivative vanishes on $\mathcal{C}_p$, so that

$$|\psi_i(z)| \leq 2 \cdot \mathbf{1}_{\{|z-1| \geq r_p\}}, \quad |\psi'_i(z)| \leq \frac{21}{\rho} \mathbf{1}_{\{|z-1| \geq r_p\}}. \quad (11.22)$$

STEP 2. *Core inputs.* Let $\tau = \Pi_c \sigma$, which is carried by $[1 - 2r_p, 1 + 2r_p] \subset [1/2, 3/2]$ and satisfies $\|\tau\|_c = n_c(\sigma)$. By (11.20) and $\text{Lip}(\psi_c) \leq 5$, we have $|F'_{\psi_c}(b)| \leq 20\theta$ for $b \in [1/2, 3/2]$. Part (v) of Lemma 11.12 and (11.21) give $n_c(\mathcal{L}\tau) \leq 20\theta n_c(\sigma)$. For $|x| \leq \rho_1$, part (ii) of Lemma 11.15 shows that $b \mapsto p k_b^\lambda(x)$ is $p\hbar_p/\rho_0$ -Lipschitz on $[1/2, 3/2]$. By part (v) of Lemma 11.12, we have $|G_\tau(x)| \leq (p\hbar_p/\rho_0)n_c(\sigma)$, and (11.19) gives $n_b(\mathcal{L}\tau) \leq (p\hbar_p/\rho_0)n_c(\sigma)$. Finally, (11.20), (11.22), and part (i) of Lemma 11.15 give $|F'_{\psi_i}(b)| \leq 4\theta(21/\rho) \cdot 3\omega_p$ for $b \in [1/2, 3/2]$, which implies that $n_i(\mathcal{L}\tau) \leq (252\theta\omega_p/\rho) n_c(\sigma)$.

STEP 3. *Boundary inputs.* Let $\tau = \Pi_b \sigma = g db - (\int g)\delta_1$, where $g = \chi_b f_\sigma$ vanishes outside $[-\rho_1, \rho_1]$ and $\|g\|_{L^\infty} = n_b(\sigma)$. Then $\|\tau\|_{TV} \leq 2\|g\|_{L^1} \leq 2\rho n_b(\sigma)$. Since $|F_{\psi_c}| \leq 2pr_p$ and $|F_{\psi_i}| \leq 2p$, (11.21) gives $n_c(\mathcal{L}\tau) \leq 4\theta pr_p n_b(\sigma) \leq p n_b(\sigma)$ and $n_i(\mathcal{L}\tau) \leq 4\theta p n_b(\sigma) \leq p n_b(\sigma)$. For the boundary output, we have

$$G_\tau(x) = p \int k_b^\lambda(x) g(b) db - \left(\int g \right) p k_1^\lambda(x).$$

By part (iii) of Lemma 11.15, the first term is at most $(2\theta + \rho p\hbar_p + 2\theta\omega_p/\rho^2)\|g\|_{L^\infty}$ in absolute value for $|x| \leq \rho_1$. By part (ii) of the same lemma with $b = 1$, and $|\int g| \leq 2\rho_1\|g\|_{L^\infty} = \rho\|g\|_{L^\infty}$, the second term is at most $\rho p\hbar_p\|g\|_{L^\infty}$. Combining these bounds with (11.19), we obtain

$$n_b(\mathcal{L}\tau) \leq \left(2\theta + 2\rho p\hbar_p + \frac{2\theta\omega_p}{\rho^2} \right) n_b(\sigma).$$

STEP 4. *Intermediate inputs.* Let $\tau = \Pi_i \sigma$, which is carried by the closed set I_p and satisfies $\|\tau\|_i = n_i(\sigma)$. Every $b \in I_p$ satisfies $|b| \geq \rho_0$. On I_p , we have $|F_{\psi_c}| \leq 2pr_p$, and $|F'_{\psi_c}| \leq 5\theta/\rho_0^2$ on each piece of I_p . By part (v) of Lemma 11.12, $\text{Lip}_{I_p}(F_{\psi_c}) \leq 5\theta/\rho_0^2 + 4p$, and therefore

$$n_c(\mathcal{L}\tau) \leq (2pr_p + 5\theta\rho_0^{-2} + 4p)n_i(\sigma) \leq 6p n_i(\sigma),$$

where we used $r_p \leq 1/4$ and $5\theta\rho_0^{-2} \leq 80C_0/(\rho^2 p) \leq p/2$, which follows from (S8). For $|x| \leq \rho_1$, part (ii) of Lemma 11.15 and part (v) of Lemma 11.12 show that the bounded-Lipschitz norm of $b \mapsto p k_b^\lambda(x)$ on I_p is at most $p\hbar_p(1 + \rho_0^{-1} + 2r_p^{-1})$. Together with (11.19), this implies that $n_b(\mathcal{L}\tau) \leq p\hbar_p(1 + \rho_0^{-1} + 2r_p^{-1}) n_i(\sigma)$. Finally, (11.22) and part (i) of Lemma 11.15 give $|F_{\psi_i}| \leq 6p\omega_p$ on I_p and $|F'_{\psi_i}| \leq 63\theta\omega_p/(\rho\rho_0^2)$ on each piece, so that

$$n_i(\mathcal{L}\tau) \leq \left(6p\omega_p(1 + 2r_p^{-1}) + \frac{63\theta\omega_p}{\rho\rho_0^2} \right) n_i(\sigma).$$

STEP 5. *The small entries.* By (S3), $\omega_p \leq 3\rho p\hbar_p$, and $\theta \leq C_0/p$. Using these bounds, $\rho \leq 1$, $\rho_0 = \rho/4$, and $r_p^{-1} = p^{1/4}$, each of the five coefficients

$$\frac{p\hbar_p}{\rho_0}, \quad \frac{252\theta\omega_p}{\rho}, \quad 2\rho p\hbar_p + \frac{2\theta\omega_p}{\rho^2}, \quad p\hbar_p \left(1 + \frac{1}{\rho_0} + \frac{2}{r_p} \right), \quad 6p\omega_p \left(1 + \frac{2}{r_p} \right) + \frac{63\theta\omega_p}{\rho\rho_0^2}$$

obtained in Steps 2–4 is at most $3100(1 + C_0)\rho^{-2}p^{9/4}\mathfrak{h}_p$. For instance, the last one is at most $54\rho p^{9/4}\mathfrak{h}_p + 3024C_0\rho^{-2}\mathfrak{h}_p$. By (S8), each of them is at most $p^3\mathfrak{h}_p = \delta_p$. This proves the first inequality in (11.18).

STEP 6. Conclusion. Let $\sigma \in \mathcal{D}_p$. The projections $\Pi_\alpha\sigma$ have zero mass, and their restrictions to $(-\rho_1, \rho_1)$ are 0, $\chi_b f_\sigma db$, and $\chi_i f_\sigma db$, so they belong to $\mathcal{D}_p$. The measure $\mathcal{L}\sigma$ has zero mass and is absolutely continuous, and the bounds on $\sup_{|x| \leq \rho_1} |G_{\Pi_\beta\sigma}(x)|$ obtained in Steps 2–4 show that its density is bounded on $(-\rho_1, \rho_1)$, since $n_\beta(\sigma) < \infty$ for $\sigma \in \mathcal{D}_p$ (see the proof of part (vi) of Lemma 11.12). It follows that $\mathcal{L}\sigma \in \mathcal{D}_p$. Since $\sigma = \sum_\beta \Pi_\beta\sigma$ and $\Pi_\alpha\mathcal{L}$ is linear, we have $n_\alpha(\mathcal{L}\sigma) \leq \sum_\beta n_\alpha(\mathcal{L}\Pi_\beta\sigma)$, and the second inequality in (11.18) follows from the first. $\square$

Remark 11.17. The estimates of Lemma 11.16 are linear and apply to arbitrary inputs in $\mathcal{D}_p$, not only to differences of probability measures; the side law λ enters only through $\mu_\lambda = \mathcal{R}_{p,\theta}(\lambda)$. At a fixed point $\nu = \mathcal{R}_{p,\theta}(\nu)$ in $\mathcal{A}_p(\mathbf{q})$, we have $\mathcal{L}_{p,\theta;\nu}\sigma = p\nu * (\mathcal{K}_\theta\sigma)^\vee$. By part (vi) of Lemma 11.12, N_p is a norm on $\mathcal{D}_p$. Let $\mathcal{Y}_p$ be the completion of $(\mathcal{D}_p, N_p)$. Since $n_\alpha \leq W_\alpha^{-1}N_p$, each component seminorm n_α extends uniquely to a continuous seminorm on $\mathcal{Y}_p$, and Lemma 11.18 below shows that $\mathcal{L}$ is bounded on $(\mathcal{D}_p, N_p)$. Then $\mathcal{L}$ extends uniquely to a bounded operator on $\mathcal{Y}_p$. By density and continuity, the inequalities (11.18), in their second form, and the bound $N_p(\mathcal{L}\sigma) \leq \kappa_{p,\theta}N_p(\sigma)$ of Lemma 11.18, continue to hold on $\mathcal{Y}_p$.

11.6 The principal fixed point

Lemma 11.18. Let $\mathbf{q}$ be admissible, $C_0 > 0$, $p \geq p_*(\mathbf{q}, C_0)$, and $0 < \theta \leq C_0/p$. Set

$$\kappa_{p,\theta} = 20\theta + 7pe^{-a\sqrt{p}/2}.$$

Then $\kappa_{p,\theta} \leq 20C_0/p + 7pe^{-a\sqrt{p}/2} \leq 1/2$, and the following hold.

- (i) For every $\lambda \in \mathcal{A}_p(\mathbf{q})$ and $\sigma \in \mathcal{D}_p$, we have $N_p(\mathcal{L}_{p,\theta;\lambda}\sigma) \leq \kappa_{p,\theta}N_p(\sigma)$.
- (ii) For all $\nu, \mu \in \mathcal{A}_p(\mathbf{q})$, we have $d_p(\mathcal{R}_{p,\theta}(\nu), \mathcal{R}_{p,\theta}(\mu)) \leq \kappa_{p,\theta}d_p(\nu, \mu)$.

Proof. The bound on $\kappa_{p,\theta}$ is (S9). For (i), by Lemma 11.16, we have

$$N_p(\mathcal{L}\sigma) = \sum_\alpha W_\alpha n_\alpha(\mathcal{L}\sigma) \leq \sum_\alpha W_\alpha \sum_\beta M_{\alpha\beta} n_\beta(\sigma) = \sum_\beta c_\beta W_\beta n_\beta(\sigma), \quad c_\beta = \sum_\alpha \frac{W_\alpha M_{\alpha\beta}}{W_\beta}.$$

It suffices to show that $c_\beta \leq \kappa_{p,\theta}$ for each β . By (S9), $2p^{A+3}e^{-ap/2} \leq pe^{-a\sqrt{p}/2}$; since $\delta_p \leq p^{A+3}e^{-ap/2}$ and $\delta_p W_b = p^{A+3}e^{-ap/2}$, each of δ_p , $\delta_p W_b$, and $\delta_p W_i$ is at most $pe^{-a\sqrt{p}/2}/2$.

Using (11.17), we find

$$\begin{aligned} c_c &= 20\theta + (W_b + W_i)\delta_p \leq 20\theta + pe^{-a\sqrt{p}/2}, \\ c_b &= pe^{-ap/2} + 2\theta + \delta_p + pe^{-a(p-\sqrt{p})/2} \leq 2\theta + 3pe^{-a\sqrt{p}/2}, \\ c_i &= 6pe^{-a\sqrt{p}/2} + (W_b + W_i)W_i^{-1}\delta_p \leq 7pe^{-a\sqrt{p}/2}, \end{aligned}$$

where in the second line we used $p - \sqrt{p} \geq \sqrt{p}$. This proves (i).

For (ii), let $\sigma = \nu - \mu$ and $\lambda_t = (1-t)\mu + t\nu$. Then $\sigma \in \mathcal{D}_p$ by (B), and $\lambda_t \in \mathcal{A}_p(\mathbf{q})$ by convexity. We write $\mathcal{L}_t = \mathcal{L}_{p,\theta;\lambda_t}$ and $\zeta = \mathcal{R}_{p,\theta}(\nu) - \mathcal{R}_{p,\theta}(\mu)$. For a smooth φ with $\varphi(1) = 0$ and $\text{Lip}(\varphi) \leq 1$, (11.8) and part (ii) of Lemma 11.12 give

$$\langle \Pi_c \zeta, \varphi \rangle = \langle \zeta, \chi_c \varphi \rangle = \int_0^1 \langle \mathcal{L}_t \sigma, \chi_c \varphi \rangle dt = \int_0^1 \langle \Pi_c \mathcal{L}_t \sigma, \varphi \rangle dt,$$

so that $n_c(\zeta) \leq \sup_t n_c(\mathcal{L}_t \sigma)$. The same argument gives $n_i(\zeta) \leq \sup_t n_i(\mathcal{L}_t \sigma)$. By Lemma 11.9, ζ has a density on $(-\rho, \rho)$, and if G_t denotes the density of $\mathcal{L}_t \sigma$, then (11.8) with $\psi = \chi_b u$, for bounded Borel u supported in $(-\rho_1, \rho_1)$, gives $|\langle \zeta, \chi_b u \rangle| \leq \sup_t \|\chi_b G_t\|_{L^\infty} \|u\|_{L^1}$. This implies that $n_b(\zeta) \leq \sup_t n_b(\mathcal{L}_t \sigma)$. Since the bounds of Lemma 11.16 are uniform in t , the computation in (i) gives $N_p(\zeta) \leq \kappa_{p,\theta} N_p(\sigma)$. $\square$

Theorem 11.19. *Let $\mathbf{q} = (a, A, K', \rho)$ be admissible, $C_0 > 0$, and $p \geq p_*(\mathbf{q}, C_0)$. For every $\theta \in (0, C_0/p]$, the following hold.*

- (i) *The law map $\mathcal{R}_{p,\theta}$ has exactly one fixed point in $\mathcal{A}_p(\mathbf{q})$. This fixed point, denoted ν_p^θ , belongs to $\mathfrak{S}_{p,\theta}(\mathbf{q})$; in particular, it is the unique fixed point in $\mathfrak{S}_{p,\theta}(\mathbf{q})$. It has the properties (i)–(iii) of Lemma 11.9.*
- (ii) *For every $\mu \in \mathcal{A}_p(\mathbf{q})$, in particular for $\mu = \delta_1$, we have $d_p(\mathcal{R}_{p,\theta}^n(\mu), \nu_p^\theta) \leq \kappa_{p,\theta}^n d_p(\mu, \nu_p^\theta)$ for $n \geq 0$, and $\mathcal{R}_{p,\theta}^n(\mu) \Rightarrow \nu_p^\theta$.*
- (iii) *Let $\mathbf{q}' = (a', A', K'', \rho')$ be another admissible tuple, and let*

$$\mathbf{q} \vee \mathbf{q}' = (\max\{a, a'\}, \min\{A, A'\}, \min\{K', K''\}, \max\{\rho, \rho'\}).$$

If $p \geq \max\{p_*(\mathbf{q}, C_0), p_*(\mathbf{q}', C_0), p_*(\mathbf{q} \vee \mathbf{q}', C_0)\}$, then the fixed points given by (i) for $\mathbf{q}$ and for $\mathbf{q}'$ coincide.

- (iv) *The map $\theta \mapsto \nu_p^\theta$ is weakly continuous on $(0, C_0/p]$.*
- (v) *For every $s \in (0, 1)$, we have*

$$\int |b|^{-s} \nu_p^\theta(db) \leq \rho^{-s} + \frac{2\mathfrak{h}_p \rho^{1-s}}{1-s}, \quad \int |b|^s \nu_p^\theta(db) \leq 4 + \frac{K' \mathfrak{h}_p}{1-s},$$

and

$$\int |b-1|^s \nu_p^\theta(db) \leq \theta^s \left(1 + p^{1-s} \int |b|^{-s} \nu_p^\theta(db) \right).$$

In particular, for fixed p , $\nu_p^\theta \Rightarrow \delta_1$ as $\theta \downarrow 0$.

Proof. By Lemma 11.13, $(\mathcal{A}_p(\mathbf{q}), d_p)$ is a complete metric space. By Lemmas 11.9 and 11.18, $\mathcal{R}_{p,\theta}$ maps it into $\mathfrak{S}_{p,\theta}(\mathbf{q}) \subset \mathcal{A}_p(\mathbf{q})$ and is a contraction with factor $\kappa_{p,\theta} \leq 1/2$. The Banach fixed-point theorem gives a unique fixed point ν_p^θ in $\mathcal{A}_p(\mathbf{q})$, and $\nu_p^\theta = \mathcal{R}_{p,\theta}(\nu_p^\theta)$ belongs to $\mathfrak{S}_{p,\theta}(\mathbf{q})$ and has the properties of Lemma 11.9. A fixed point in $\mathfrak{S}_{p,\theta}(\mathbf{q})$ lies in $\mathcal{A}_p(\mathbf{q})$ and therefore equals ν_p^θ . For (ii), we have $d_p(\mathcal{R}_{p,\theta}^n(\mu), \nu_p^\theta) = d_p(\mathcal{R}_{p,\theta}^n(\mu), \mathcal{R}_{p,\theta}^n(\nu_p^\theta)) \leq \kappa_{p,\theta}^n d_p(\mu, \nu_p^\theta)$, which is finite by part (vi) of Lemma 11.12, and weak convergence follows from part (iii) of Lemma 11.13. Since $\delta_1 \in \mathcal{A}_p(\mathbf{q})$, this applies to $\mu = \delta_1$.

For (iii), let $\mathbf{q}'' = \mathbf{q} \vee \mathbf{q}'$. Comparing the bounds in Definition 11.8 term by term, we see that $\mathfrak{S}_{p,\theta}(\mathbf{q}'') \subset \mathfrak{S}_{p,\theta}(\mathbf{q}) \cap \mathfrak{S}_{p,\theta}(\mathbf{q}')$: a larger a and smaller A and K' make the right sides of (C), (B), and (T) smaller, and a larger ρ enlarges the interval in (B). Then the fixed point given by (i) for $\mathbf{q}''$ is a fixed point in $\mathfrak{S}_{p,\theta}(\mathbf{q})$ and in $\mathfrak{S}_{p,\theta}(\mathbf{q}')$, and by uniqueness it equals both of the other fixed points.

For (iv), let $\theta_n \rightarrow \theta$ in $(0, C_0/p]$ and $\nu_n = \nu_{p,\theta_n}^\theta$. By part (i) of Lemma 11.13, every subsequence of (ν_n) has a further subsequence converging weakly to some $\nu' \in \mathcal{A}_p(\mathbf{q})$. Along it, part (iv) of Lemma 11.13 gives $\nu_n = \mathcal{R}_{p,\theta_n}(\nu_n) \Rightarrow \mathcal{R}_{p,\theta}(\nu')$, so $\nu' = \mathcal{R}_{p,\theta}(\nu')$. By (i), $\nu' = \nu_p^\theta$. Since every subsequence has a further subsequence converging to ν_p^θ , we have $\nu_n \Rightarrow \nu_p^\theta$.

For (v), the first bound follows by splitting at $|b| = \rho$ and using (B). For the second, we have $\int |b|^s d\nu = \int_0^\infty st^{s-1} \nu(|b| > t) dt \leq 3^s + \int_3^\infty st^{s-1} (K' \mathfrak{h}_p/t + e^{-apt}) dt \leq 4 + K' \mathfrak{h}_p/(1-s)$ by (T), where we used that $ap \geq 1$, which follows from (S2). For the third, let B satisfy (11.1) with $B_j \sim \nu_p^\theta$. Since $x \mapsto x^s$ is subadditive on $[0, \infty)$, we have

$$\mathbb{E}|B-1|^s \leq \theta^s + \left(\frac{\theta}{p}\right)^s \mathbb{E} \sum_{j \leq N} Y_j^s |B_j|^{-s} = \theta^s + \left(\frac{\theta}{p}\right)^s p \Gamma(1+s) \int |b|^{-s} d\nu_p^\theta,$$

and $\Gamma(1+s) \leq 1$. The first bound in (v) does not depend on θ , so $\mathbb{E}|B-1|^s \rightarrow 0$ as $\theta \downarrow 0$, which gives the last claim. $\square$

The principal law is thus a fixed point which does not depend on the choice of the admissible tuple, and which is obtained by iterating the law map from δ_1 . We now show that functionals of finite-depth recursions built from it depend continuously on θ . Let $L \geq 0$. Let $\mathcal{T}$ be the marked PGW(p) tree of Definition 3.2, let $\mathcal{T}_{\leq L}$ be its restriction to the generations $0, \dots, L$, with vertex set $V_{\leq L}$, squared moduli Y_e on its edges, and numbers of children N_v . Given θ , let $(B_v)_{|v|=L}$ be, conditionally on $\mathcal{T}_{\leq L}$ and its marks, independent with law ν_p^θ , and define recursively, for $|v| < L$,

$$B_v = 1 + \theta - \frac{\theta}{p} \sum_{u \text{ child of } v} \frac{Y_{vu}}{B_u}. \quad (11.23)$$

The recursion is well defined almost surely: by induction from generation L , each B_v is almost surely nonzero, since conditionally on the children values the law of B_v is $\delta_{1+\theta}$ if

$N_v = 0$ and absolutely continuous otherwise. We call $(\mathcal{T}_{\leq L}, (Y_e), (B_v)_{v \in V_{\leq L}})$ the depth- L principal recursion at θ , and we write $\mathbb{P}_\theta$ and $\mathbb{E}_\theta$ for its law and expectation. Since ν_p^θ is a fixed point, every B_v has law ν_p^θ .

Lemma 11.20. *Let $\mathbf{q}$ be admissible, $C_0 > 0$, $p \geq p_*(\mathbf{q}, C_0)$, $J \subset (0, C_0/p]$ an interval, $L \geq 0$, $s_0 \in [0, 1)$, $k \geq 0$, and $C_\Psi \geq 1$. For each $\theta \in J$, let Ψ_θ be a real Borel function of the depth- L data $\omega = (t, y, b)$, where t is a rooted tree of depth at most L , $y = (y_e)$ are positive edge marks, and $b = (b_v) \in (\mathbb{R} \setminus \{0\})^{V(t)}$. Assume:*

- (a) *for every $\theta \in J$ and $\mathbb{P}_\theta$ -almost every $\omega = (t, y, b)$, we have $\Psi_{\theta'}(t, y, b') \rightarrow \Psi_\theta(t, y, b)$ as $(\theta', b') \rightarrow (\theta, b)$ with $\theta' \in J$;*
- (b) *for all $\theta \in J$ and ω ,*

$$|\Psi_\theta(\omega)| \leq C_\Psi^{|V(t)|} \prod_e (1 + y_e)^k \prod_{v \in V(t)} (1 + |b_v|^{-1})^{s_0} \prod_{|v|=L} (1 + |b_v|)^{s_0}.$$

Then the law of the depth- L principal recursion is weakly continuous in $\theta \in (0, C_0/p]$, we have $\sup_{\theta \in J} \mathbb{E}_\theta |\Psi_\theta|^r < \infty$ with $r = s_0^{-1/2}$ (and $r = 2$ if $s_0 = 0$), and $\theta \mapsto \mathbb{E}_\theta \Psi_\theta$ is finite and continuous on J .

Condition (b) allows, at every vertex, a total inverse power of B_v at most $s_0 < 1$; positive powers are allowed only at generation L . Positive powers of B_v at internal vertices can be reduced to this form using (11.23), at the cost of increasing the inverse powers of the children. Indicator functions of events whose boundaries are $\mathbb{P}_\theta$ -null sets satisfy (a).

Proof. **STEP 1. Coupling.** Let $\theta_n \rightarrow \theta$ in J . We realize the depth- L data at all parameters on one probability space. Let $\mathcal{T}$ and its marks be given, and let (U_v) be independent uniform variables on $(0, 1)$, indexed by the Ulam–Harris labels and independent of $\mathcal{T}$ and its marks. With G_θ the quantile function of ν_p^θ , set $B_v^{(n)} = G_{\theta_n}(U_v)$ and $B_v = G_\theta(U_v)$ for $|v| = L$, and define the internal values by (11.23) with θ_n and θ , respectively. By part (iv) of Theorem 11.19, $G_{\theta_n} \rightarrow G_\theta$ at the continuity points of G_θ , which are all but countably many points, so $B_v^{(n)} \rightarrow B_v$ almost surely for $|v| = L$. Since the limiting values are almost surely nonzero, induction over the generations shows that $B_v^{(n)} \rightarrow B_v$ almost surely for every $v \in V_{\leq L}$. This proves the weak continuity of the law, and by (a), $\Psi_{\theta_n}(\omega^{(n)}) \rightarrow \Psi_\theta(\omega)$ almost surely.

STEP 2. A uniform moment bound. First suppose that $s_0 = 0$, so $r = 2$. Let $M_k = \mathbb{E}(1 + Y)^{2k} < \infty$ for $Y \sim \text{Exp}(1)$. By condition (b) and independence of the edge marks, for every $\theta \in J$,

$$\mathbb{E}_\theta [|\Psi_\theta|^2 \mid \text{shape}] \leq C_\Psi^{2|V_{\leq L}|} M_k^{|V_{\leq L}|-1} \leq (C_\Psi^2 M_k)^{|V_{\leq L}|}.$$

The fixed-depth exponential moment calculation below therefore gives the required uniform L^2 bound. Now suppose that $0 < s_0 < 1$, set $s_1 = rs_0 = s_0^{1/2} \in (0, 1)$, and let $\vartheta \in J$. Since $(1+x)^{s_1} \leq 1+x^{s_1}$ for $x \geq 0$, condition (b) gives $|\Psi_\vartheta|^r \leq C_\Psi^{r|V_{\leq L}|} Q_0$, where, for $0 \leq n \leq L$,

$$Q_n = \prod_{|v| \geq n} (1 + |B_v|^{-s_1}) \prod_{e: |e^-| \geq n} (1 + Y_e)^{kr} \prod_{|v|=L} (1 + |B_v|^{s_1}),$$

and e^- denotes the endpoint of e closer to the root. Let $\mathcal{G}_n$ be the σ -field generated by the shape of $\mathcal{T}_{\leq L}$, the marks of the edges e with $|e^-| \geq n$, and the leaf values $(B_v)_{|v|=L}$. Then Q_n and all B_v with $|v| \geq n$ are $\mathcal{G}_n$ -measurable, and conditionally on $\mathcal{G}_{n+1}$, the marks of the edges e with $|e^-| = n$ are independent standard exponential variables, because the marks are independent of each other, of the shape, and of the leaf values. For $|v| = n < L$, conditionally on $\mathcal{G}_{n+1}$, the variable B_v is of the form $c + \sum_u a_u Y_{vu}$ with $c = 1 + \vartheta$ and $\mathcal{G}_{n+1}$ -measurable real coefficients $a_u = -\vartheta/(pB_u)$. Let $c_{k,r} = \sup_{y \geq 0} (1+y)^{kr} e^{-y/2} \geq 1$. Since $(1+y)^{kr} e^{-y} \leq c_{k,r} e^{-y/2}$, we have, for $m = N_v$ and independent standard exponential variables $Y'_1, \dots, Y'_m$,

$$\begin{aligned} \mathbb{E} \left[(1 + |B_v|^{-s_1}) \prod_u (1 + Y_{vu})^{kr} \mid \mathcal{G}_{n+1} \right] &\leq (2c_{k,r})^m \left(1 + \mathbb{E} \left| c + \sum_{j \leq m} 2a_j Y'_j \right|^{-s_1} \right) \\ &\leq (2c_{k,r})^m (1 + C_{s_1,1}(m+1)^{s_1/2}), \end{aligned}$$

where we used the conditional version of Lemma 4.18 with $d = 1$. The right side is at most $c_2^{N_v+1}$ with $c_2 = 4c_{k,r}(1 + C_{s_1,1})$. The factors for distinct vertices of generation n involve disjoint sets of marks. We obtain $\mathbb{E}[Q_n \mid \mathcal{G}_{n+1}] \leq \prod_{|v|=n} c_2^{N_v+1} Q_{n+1}$. Iterating from $n = 0$ to $n = L - 1$ and then integrating the leaf values, which are independent with law ν_p^ϑ , we find

$$\mathbb{E}_\vartheta[Q_0 \mid \text{shape}] \leq \prod_{|v| < L} c_2^{N_v+1} \cdot M^{Z_L}, \quad M = \sup_{\vartheta \in (0, C_0/p]} \int (1 + |b|^{-s_1}) (1 + |b|^{s_1}) \nu_p^\vartheta(db),$$

where Z_L is the number of vertices of generation L . By part (v) of Theorem 11.19, $M < \infty$. Since $\sum_{|v| < L} (N_v + 1) \leq 2|V_{\leq L}|$, we conclude that $\mathbb{E}_\vartheta|\Psi_\vartheta|^r \leq \mathbb{E}[x^{|V_{\leq L}|}]$ with $x = C_\Psi^r c_2^2 \max\{M, 1\}$. Finally, $\mathbb{E}[x^{|V_{\leq L}|}] < \infty$: if Z_n denotes the size of generation n , then $\mathbb{E}[x^{Z_0 + \dots + Z_{n+1}}] = \mathbb{E}[x^{Z_0 + \dots + Z_n} e^{pZ_n(x-1)}] \leq \mathbb{E}[(xe^{p(x-1)})^{Z_0 + \dots + Z_n}]$, and induction on n shows that these expectations are finite for every $x \geq 1$.

STEP 3. Conclusion. By Steps 1 and 2, $\Psi_{\theta_n}(\omega^{(n)}) \rightarrow \Psi_\theta(\omega)$ almost surely and the family is bounded in L^r with $r > 1$, which implies that it is uniformly integrable. By Vitali's convergence theorem, $\mathbb{E}_{\theta_n} \Psi_{\theta_n} \rightarrow \mathbb{E}_\theta \Psi_\theta$. $\square$

We now state the result on compact scaled windows. Recall that $\theta(E) = g(E)^2$ and that $E^{-1} \leq g(E) \leq 2E^{-1}$ for $E \geq 2$, since $g(E) = 2/(E + \sqrt{E^2 - 4})$.

Theorem 11.21. *Let $\Xi \subset (0, \infty)$ be compact and $\xi_- = \min \Xi$. Let $C_\Xi = 4/\xi_-^2$ and $p_{\text{st}}(\Xi) = \max\{p_*(\mathbf{q}_{\text{can}}, C_\Xi), \lceil 5/\xi_-^2 \rceil\}$. For every $p \geq p_{\text{st}}(\Xi)$, $\xi \in \Xi$, and $E = \xi\sqrt{p}$, with $\theta = \theta(E)$, the following hold.*

- (i) *We have $E > 2$ and $\theta \leq C_\Xi/p$. The law map $\mathcal{R}_{p,\theta}$ has exactly one fixed point in the class $\mathfrak{S}_{p,\theta}$, which is also its unique fixed point in $\mathcal{A}_p$. We denote it by $\nu_{p,E} = \nu_p^{\theta(E)}$, and we write $\nu_{p,E}^x$ for the law of $1/B$, $B \sim \nu_{p,E}$.*
- (ii) *For every $\mu \in \mathcal{A}_p$, in particular for $\mu = \delta_1$, the iterates $\mathcal{R}_{p,\theta}^n(\mu)$ converge to $\nu_{p,E}$ in d_p and weakly, with $d_p(\mathcal{R}_{p,\theta}^n(\mu), \nu_{p,E}) \leq \kappa^n d_p(\mu, \nu_{p,E})$ and*

$$\kappa = \kappa_{p,\theta} \leq \frac{80}{\xi_-^2 p} + 7pe^{-\sqrt{p}/2}.$$
- (iii) *The law $\nu_{p,E}$ satisfies, uniformly in $\xi \in \Xi$: $\nu_{p,E} = e^{-p}\delta_{1+\theta} + f(b) \, db$ with $f \leq e^{-p}/2$ almost everywhere on $[-1/8, 1/8]$; $\nu_{p,E}(\{|b-1| > r_p/2\}) \leq \omega_p \leq e^{-\sqrt{p}/2}$; $\nu_{p,E}(\{|b| > L\}) \leq e^{-p}/(2L) + e^{-pL}/2$ for $L \geq 3$; and, for $s \in (0, 1)$, $\int |b|^{-s} d\nu_{p,E} \leq 16 + (1-s)^{-1}$ and $\int |b|^s d\nu_{p,E} \leq 4 + (1-s)^{-1}$. In particular, if $B \sim \nu_{p,E}$, then $\sup_{\xi \in \Xi} \mathbb{P}(|B-1| > r_p) \rightarrow 0$ as $p \rightarrow \infty$.*
- (iv) *The map $E \mapsto \nu_{p,E}$ is weakly continuous on $\{\xi\sqrt{p} : \xi \in \Xi\}$. For every family (Ψ_θ) as in Lemma 11.20, with $\mathbf{q} = \mathbf{q}_{\text{can}}$ and $J \subset (0, C_\Xi/p]$ an interval containing $\{\theta(\xi\sqrt{p}) : \xi \in \Xi\}$, the map $E \mapsto \mathbb{E}_{\theta(E)} \Psi_{\theta(E)}$ is continuous on $\{\xi\sqrt{p} : \xi \in \Xi\}$.*
- (v) *If $B \sim \nu_{p,E}$, then the law of $D = B/g(E)$ is a real solution at E .*

Proof. Since $p \geq 5/\xi_-^2$, we have $E \geq \xi_- \sqrt{p} > 2$, and $\theta = g(E)^2 \leq 4/E^2 \leq C_\Xi/p$. Parts (i)–(iii) follow from Theorem 11.19 with $\mathbf{q} = \mathbf{q}_{\text{can}}$ and $C_0 = C_\Xi$, from Lemma 11.9, and from $\kappa_{p,\theta} \leq 20C_\Xi/p + 7pe^{-\sqrt{p}/2}$; for the moment bounds we used $\rho = 1/16$, $K' = 1$, and $\mathfrak{h}_p = e^{-p} \leq 1$ in part (v) of Theorem 11.19. For (iv), the function $E \mapsto \theta(E)$ is continuous on $(2, \infty)$, so (iv) follows from part (iv) of Theorem 11.19 and Lemma 11.20. Part (v) follows from the discussion at the beginning of this section, since $\nu_{p,E}(\{0\}) = 0$. $\square$

The canonical tuple plays no special role. If $\mathbf{q}$ is any admissible tuple and $p \geq \max\{p_{\text{st}}(\Xi), p_*(\mathbf{q}, C_\Xi), p_*(\mathbf{q} \vee \mathbf{q}_{\text{can}}, C_\Xi)\}$, then part (iii) of Theorem 11.19 shows that $\nu_{p,E}$ is also the unique fixed point of $\mathcal{R}_{p,\theta(E)}$ in $\mathfrak{S}_{p,\theta(E)}(\mathbf{q})$, and Lemma 11.9 gives the bounds of (iii) with e^{-p} , $e^{-\sqrt{p}}$, and $1/16$ replaced by $p^A e^{-ap}$, $e^{-a\sqrt{p}}$, and ρ . The same applies to the mixed metric of Definition 11.11 built with $\mathbf{q}$.

11.7 Small parameter and the half-line

The fixed point ν_p^θ of Theorem 11.19 is characterized as the unique fixed point of $\mathcal{R}_{p,\theta}$ in $\mathcal{A}_p(\mathbf{q})$, a class which does not involve C_0 ; the number C_0 only enters the threshold $p_*(\mathbf{q}, C_0)$. In particular, the construction covers all small values of θ , which correspond to large energies, with a single class.

Proposition 11.22. Let $c_{\text{ent}} = 1$, let $\mathfrak{S}_{p,\theta}^{\text{ent}} = \mathfrak{S}_{p,\theta}$ be the class of Definition 11.8 for the canonical tuple $\mathbf{q}_{\text{can}} = (1, 0, 1, 1/16)$, which we call the small-parameter class, and let $p_{\text{ent}} = p_*(\mathbf{q}_{\text{can}}, c_{\text{ent}})$. For every $p \geq p_{\text{ent}}$ and $0 < \theta \leq c_{\text{ent}}/p$, the following hold.

- (i) The law map $\mathcal{R}_{p,\theta}$ has exactly one fixed point in $\mathfrak{S}_{p,\theta}^{\text{ent}}$, namely ν_p^θ ; it is also the unique fixed point in $\mathcal{A}_p$. The iterates $\mathcal{R}_{p,\theta}^n(\delta_1)$ converge to ν_p^θ in d_p and weakly, with contraction factor at most $20/p + 7pe^{-\sqrt{p}/2}$.
- (ii) For every $s_0 \in [0, 1)$, $\sup_{0 < \theta \leq c_{\text{ent}}/p} \int |b|^{-s_0} \nu_p^\theta(db) \leq 16 + (1 - s_0)^{-1}$.
- (iii) The map $\theta \mapsto \nu_p^\theta$ is weakly continuous on $(0, c_{\text{ent}}/p]$, and for every family (Ψ_θ) as in Lemma 11.20 with $J \subset (0, c_{\text{ent}}/p]$, the map $\theta \mapsto \mathbb{E}_\theta \Psi_\theta$ is continuous on J .
- (iv) As $\theta \downarrow 0$, $\nu_p^\theta \Rightarrow \delta_1$, and the law of $1/B$, $B \sim \nu_p^\theta$, converges weakly to δ_1 .
- (v) Let $\Xi \subset (0, \infty)$ be compact, $p \geq \max\{p_{\text{ent}}, p_{\text{st}}(\Xi)\}$, $\xi \in \Xi$, and $E = \xi\sqrt{p}$. If $\theta(E) \leq c_{\text{ent}}/p$, then $\nu_p^{\theta(E)} = \nu_{p,E}$, the law of Theorem 11.21.

The same statements hold with c_{ent} replaced by any $c > 0$ and p_{ent} by $p_*(\mathbf{q}_{\text{can}}, c)$.

Proof. Parts (i), (iii), and (iv) follow from Theorem 11.19 and Lemma 11.20 with $\mathbf{q} = \mathbf{q}_{\text{can}}$ and $C_0 = c_{\text{ent}}$; for the reciprocal law in (iv), note that $B \rightarrow 1$ in probability implies $1/B \rightarrow 1$ in probability. For (ii), part (v) of Theorem 11.19 with $\rho = 1/16$ and $\mathfrak{h}_p = e^{-p} \leq 1$ gives the bound $16^{s_0} + 2e^{-p}16^{s_0-1}/(1 - s_0) \leq 16 + (1 - s_0)^{-1}$. For (v), both $\nu_p^{\theta(E)}$ and $\nu_{p,E}$ are fixed points of $\mathcal{R}_{p,\theta(E)}$ in $\mathfrak{S}_{p,\theta(E)}(\mathbf{q}_{\text{can}})$, which has at most one fixed point by part (i) of Theorem 11.19. The last sentence follows by the same arguments with $C_0 = c$. $\square$

Corollary 11.23. Let $M_0 = 3$. Then $g(E)^2 \leq c_{\text{ent}}/p$ for every $p > 1$ and $E \geq M_0\sqrt{p}$. Fix $\xi_0 > 0$, and let

$$p_0(\xi_0) = \max\{p_*(\mathbf{q}_{\text{can}}, \max\{4\xi_0^{-2}, 1\}), \lceil 5\xi_0^{-2} \rceil, p_{\text{ent}}\}.$$

For every $p \geq p_0(\xi_0)$ and $E \geq \xi_0\sqrt{p}$, the following hold.

- (i) We have $E > 2$, and the iterates $\mathcal{R}_{p,\theta(E)}^n(\delta_1)$ converge weakly, and in d_p , to the unique fixed point $\nu_{p,E} = \nu_p^{\theta(E)}$ of $\mathcal{R}_{p,\theta(E)}$ in $\mathfrak{S}_{p,\theta(E)}$. The law $\nu_{p,E}$ has the properties (i)–(iii) of Lemma 11.9 with $\mathbf{q} = \mathbf{q}_{\text{can}}$ and $C_0 = \max\{4\xi_0^{-2}, 1\}$. In particular, $\nu_{p,E}(\{b : |b| > L\}) \leq e^{-p}/(2L) + e^{-pL}/2$ for $L \geq 3$, and $\int |b|^{-s} d\nu_{p,E} \leq 16 + (1 - s)^{-1}$ and $\int |b|^s d\nu_{p,E} \leq 4 + (1 - s)^{-1}$ for $s \in (0, 1)$.
- (ii) Let $\Xi \subset [\xi_0, \infty)$ be compact with $E/\sqrt{p} \in \Xi$. Then $p \geq p_{\text{st}}(\Xi)$, and $\nu_{p,E}$ is the law given by Theorem 11.21 for the window Ξ ; for example, this applies with $\Xi = [\xi_0, M_0 + 1]$ if $E \leq (M_0 + 1)\sqrt{p}$. If $E \geq M_0\sqrt{p}$, then $\nu_{p,E}$ is the law given by Proposition 11.22 with $\theta = \theta(E)$.
- (iii) The map $E \mapsto \nu_{p,E}$ is weakly continuous on $[\xi_0\sqrt{p}, \infty)$.
- (iv) If $B \sim \nu_{p,E}$, then the law of $D = B/g(E)$ is a real solution at E .

Proof. For $E \geq 3\sqrt{p}$ we have $E > 2$ and $g(E)^2 \leq 4/E^2 \leq 4/(9p) \leq c_{\text{ent}}/p$. Let $C_0 = \max\{4\xi_0^{-2}, 1\}$. For $p \geq p_0(\xi_0)$ and $E \geq \xi_0\sqrt{p}$, we have $E \geq \sqrt{5} > 2$ and $\theta(E) \leq 4/E^2 \leq C_0/p$. Part (i) then follows from parts (i), (ii), and (v) of Theorem 11.19 with $\mathbf{q} = \mathbf{q}_{\text{can}}$, as in the proof of Theorem 11.21. For (ii), let $\xi_- = \min \Xi \geq \xi_0$. Then $p_{\text{st}}(\Xi) = \max\{p_*(\mathbf{q}_{\text{can}}, 4\xi_-^{-2}), \lceil 5\xi_-^{-2} \rceil\} \leq p_0(\xi_0)$, because $4\xi_-^{-2} \leq \max\{4\xi_0^{-2}, 1\}$ and $p_*(\mathbf{q}, C_0)$ is nondecreasing in C_0 (the conditions (S1)–(S9) become stronger as C_0 grows). The laws in Theorem 11.21 and Proposition 11.22 are fixed points of $\mathcal{R}_{p, \theta(E)}$ in $\mathfrak{S}_{p, \theta(E)}$, so they coincide with $\nu_{p, E}$ by uniqueness. Part (iii) follows from part (iv) of Theorem 11.19 and the continuity of $E \mapsto \theta(E)$, and part (iv) from the discussion at the beginning of this section. $\square$

Remark 11.24. If a later argument requires a smaller value of c_{ent} or a larger value of M_0 , nothing changes: for $M \geq 3$, the energies in the large-energy regime $E \geq M\sqrt{p}$ satisfy $\theta(E) \leq 4/(M^2p)$, and Proposition 11.22 holds with c_{ent} replaced by $4/M^2$ and p_{ent} by $p_*(\mathbf{q}_{\text{can}}, 4/M^2) \leq p_{\text{ent}}$. The principal law $\nu_{p, E}$ itself does not depend on these choices.

11.8 Real solutions and the real boundary law

In this subsection, we show that on compact scaled windows and for large p , every real solution of the cavity equation is the principal solution. We begin with an a priori bound for the physical cavity variables and with the observation that real limit points of their laws are real solutions.

Lemma 11.25. *Let $p > 1$, $E > 2$, $\theta = \theta(E)$, and for $\eta > 0$ set $X_\eta = -\Gamma_o(E + i\eta)/g(E)$.*

- (i) *For every $r \in (0, 1)$ and $\eta > 0$, we have $\mathbb{E}|X_\eta|^r \leq C_{r,1}(p+1)^{r/2} \leq 2C_{r,1}p^{r/2}$ and $\mathbb{E}^\infty|X_\eta|^r \leq 2\varrho_p^{-1}C_{r,1}p^{r/2}$, where $C_{r,1}$ is the constant of Lemma 4.18 with $d = 1$.*
- (ii) *Let $\eta_k \downarrow 0$, and suppose that the laws of X_{η_k} under $\mathbb{P}$ converge weakly to a probability measure carried by $\mathbb{R}$. Let X have this law. Then $\mathbb{P}(X = 0) = 0$, and the law of $B = 1/X$ is a fixed point of $\mathcal{R}_{p, \theta}$; equivalently, the law of $B/g(E)$ is a real solution at E .*

Proof. By Proposition 3.16, $\Gamma_o \stackrel{d}{=} -(z + p^{-1} \sum_{j \leq N} Y_j \Gamma_j)^{-1}$ with $z = E + i\eta$, where $N \sim \text{Pois}(p)$, $Y_j \sim \text{Exp}(1)$, and the Γ_j are copies of $\Gamma_o(z)$, all independent. Dividing by $-g(E)$ and using $g(E)E = 1 + \theta$, we obtain

$$X_\eta \stackrel{d}{=} \left(1 + \theta + ig(E)\eta - \frac{\theta}{p} \sum_{j \leq N} Y_j X_{\eta, j}\right)^{-1}, \quad X_{\eta, j} = -\Gamma_j/g(E). \quad (11.24)$$

Conditionally on $N = m$ and on $(X_{\eta, j})_{j \leq m}$, the denominator is $c + \sum_{j \leq m} a_j Y_j$ with $c = 1 + \theta + ig(E)\eta$, $|c| \geq 1$, and complex coefficients $a_j = -(\theta/p)X_{\eta, j}$ that are measurable with respect to the conditioning, while $Y_1, \dots, Y_m$ are standard exponential variables which are independent of each other and of the conditioning. The conditional version of Lemma 4.18

gives $\mathbb{E}[|X_\eta|^r \mid N, (X_{\eta,j})] \leq C_{r,1}(N+1)^{r/2}$. Averaging and using Jensen's inequality, we find $\mathbb{E}|X_\eta|^r \leq C_{r,1}(p+1)^{r/2}$. The bound under $\mathbb{P}^\infty$ follows from $\mathbb{E}^\infty Z \leq \varrho_p^{-1} \mathbb{E} Z$ for $Z \geq 0$ (Lemma 4.5).

For (ii), write $X_k = X_{\eta_k}$, let $(X_j)_{j \geq 1}$ be independent copies of X , independent of N and (Y_j) , and let $X_{k,j}$ be independent copies of X_k , independent of N and (Y_j) . Set

$$D_k = 1 + \theta + \mathrm{i}g(E)\eta_k - \frac{\theta}{p} \sum_{j \leq N} Y_j X_{k,j}, \quad D = 1 + \theta - \frac{\theta}{p} \sum_{j \leq N} Y_j X_j.$$

Fix $m \geq 0$. Since the components are independent and converge weakly, the vector $(Y_1, \dots, Y_m, X_{k,1}, \dots, X_{k,m})$ converges weakly to $(Y_1, \dots, Y_m, X_1, \dots, X_m)$ [37, Theorem 2.8]. Since $\eta_k \rightarrow 0$, the continuous mapping theorem [37, Theorem 2.7] shows that the conditional law of D_k given $N = m$ converges weakly to that of D given $N = m$. Averaging over m with dominated convergence, we obtain $D_k \Rightarrow D$. By (11.24), $1/D_k$ has the law of X_k . The laws of X_k converge, and in particular they are tight. For every $R > 0$, the portmanteau theorem applied to the open set $\{|w| < R^{-1}\}$ gives

$$\mathbb{P}(|D| < R^{-1}) \leq \liminf_k \mathbb{P}(|D_k| < R^{-1}) = \liminf_k \mathbb{P}(|X_k| > R) \leq \sup_k \mathbb{P}(|X_k| > R),$$

which tends to zero as $R \rightarrow \infty$. Hence $\mathbb{P}(D = 0) = 0$, and the continuous mapping theorem gives $1/D_k \Rightarrow 1/D$. We obtain $X \stackrel{\mathrm{d}}{=} 1/D$. Since D is real and finite, X is real and almost surely nonzero, and $1/X \stackrel{\mathrm{d}}{=} D$. By the definition of D , this says that the law of $1/X$ is a fixed point of $\mathcal{R}_{p,\theta}$. The equivalence with real solutions was noted after Definition 11.1. $\square$

The next lemma is the main step. It shows that every real fixed point on a scaled window satisfies the bounds defining the class $\mathfrak{S}_{p,\theta}$. The only input is the fixed-point equation: we start from the bound $\sqrt{p}$ on the density near zero given by Lemma 4.18, and we improve it by a one-big-jump argument.

Lemma 11.26. *Let $0 < c_0 \leq C_0$. There exists $p_{\mathrm{boot}}(c_0, C_0) \geq p_*(\mathbf{q}_{\mathrm{can}}, C_0)$ such that, for every $p \geq p_{\mathrm{boot}}(c_0, C_0)$ and $c_0/p \leq \theta \leq C_0/p$, every fixed point of $\mathcal{R}_{p,\theta}$ belongs to $\mathfrak{S}_{p,\theta}$ and therefore equals v_p^θ . Moreover, its density on $[-2\rho, 2\rho]$ is at most e^{-2p} , where $\rho = 1/16$.*

Proof. We work with the canonical tuple: $\rho = 1/16$, $a = 1$, $A = 0$, $K' = 1$, and $\mathfrak{h}_p = e^{-p}$. Set

$$\begin{aligned} h_0 &= 2\sqrt{p}, & c^{(0)} &= \frac{C_0 h_0}{p}, & F_p &= 6p^2 \exp\left(p - \frac{\rho p^2}{2C_0}\right), \\ M_p &= \frac{2\rho^2 p^3 h_0}{c_0}, & h_1 &= 4F_p + 4e^2 M_p e^{-\rho p^{1/12}}. \end{aligned}$$

We choose $p_{\mathrm{boot}}(c_0, C_0) \geq \max\{p_*(\mathbf{q}_{\mathrm{can}}, C_0), 16\}$ so large that, for $p \geq p_{\mathrm{boot}}$:

- (B1) $c^{(0)} \leq p^{-1/3} \leq 1/4$, $(c^{(0)})^{3/4}(7 \log p + 2) \leq 1$, and $\sqrt{2}\rho p^2 \leq c_0 p^3$;
 (B2) $4C_0 \leq \rho^2 p$ and $h_0 \leq 8^{-4}(p/C_0)^5$;
 (B3) $C_0 h_1 \leq \rho^4 p^{-7}$ and $4F_p + 4e^2 M_p e^{-p^2} \leq e^{-2p}$;
 (B4) $36C_0 \leq p^{3/4}$, $2e^{-c_1 p^{7/4}} + 4\rho p e^{-2p} \leq e^{-\sqrt{p}}/2$, $\rho p^2/(9C_0) \geq 2p + 1$, and $8C_0 \leq p e^p$.

Each condition holds for large p . Let ν_B be a fixed point of $\mathcal{R}_{p,\theta}$ with $c_0/p \leq \theta \leq C_0/p$, and realize $B \sim \nu_B$ as $B = 1 + \theta - (\theta/p) \sum_{j \leq N} Y_j/B_j$ with $B_j \sim \nu_B$, as in Definition 11.1.

STEP 1. An initial density bound. Conditionally on $N = m \geq 1$ and on $(B_j)_{j \leq m}$, the variable B equals $c + \sum_{j \leq m} a_j Y_j$ with $c = 1 + \theta$ and $a_j = -\theta/(pB_j)$, which are fixed by the conditioning, and $Y_1, \dots, Y_m$ are standard exponential variables which are independent of each other and of the conditioning. Since $a_j \neq 0$, its conditional law is absolutely continuous. Since $|x - c| \geq 1 - 2\rho \geq 1/2$ for $x \in [-2\rho, 2\rho]$, part (iii) of Lemma 4.18 with $d = 1/2$ bounds its density on $[-2\rho, 2\rho]$ by $\sqrt{m+1}$. On $\{N = 0\}$, $B = 1 + \theta \notin [-2\rho, 2\rho]$. Averaging, we find that the restriction of ν_B to $[-2\rho, 2\rho]$ is absolutely continuous with density at most $\mathbb{E}\sqrt{N+1} \leq \sqrt{p+1} \leq h_0$. Let H be the essential supremum of this density, so that $H \leq h_0$. If $H = 0$, the density bound of the lemma holds, and we go directly to Step 6. In Steps 2–5, we assume that $H > 0$.

STEP 2. Decomposition. We split the children according to whether $|B_j| \geq 2\rho$, and write $B = D - S$ with $D = 1 + \theta - (\theta/p) \sum_{|B_j| \geq 2\rho} Y_j/B_j$ and $S = (\theta/p) \sum_{|B_j| < 2\rho} Y_j/B_j$, which are independent by Lemma 11.2. By Lemma 11.6 with $\delta = 2\rho$ and $c = 1 + \theta$, the law of D restricted to $(-\infty, 1/2]$ has density at most $F_{p,\theta}(2\rho) \leq F_p$. By part (i) of Lemma 11.7 with $\rho' = 2\rho$, the jumps of S form a Poisson process with intensity $\ell(y) dy$, where

$$\ell(y) \leq \min \left\{ \frac{c_*}{y^2}, M \right\}, \quad c_* = \theta H \leq c^{(0)}, \quad M = \frac{2p^2 \rho^2 H}{\theta} \leq M_p. \quad (11.25)$$

Let $a_* = c_*^{1/4} \leq 1$, and write $S = S_{<} + S_{>}$, where $S_{<}$ and $S_{>}$ are the sums of the jumps with $|y| \leq a_*$ and $|y| > a_*$. These are independent compound Poisson variables with intensities $\ell_{<} = \ell \mathbf{1}_{\{|y| \leq a_*\}}$ and $\ell_{>} = \ell \mathbf{1}_{\{|y| > a_*\}}$. By (11.25),

$$\lambda_{>} = \int \ell_{>} \leq 2 \int_{a_*}^{\infty} \frac{c_*}{y^2} dy = 2c_*^{3/4} \leq 1, \quad \|\ell_{>}\|_{\infty} \leq \frac{c_*}{a_*^2} = c_*^{1/2}. \quad (11.26)$$

STEP 3. The small jumps. We claim that the law of $S_{<}$ restricted to $\{|s| \geq \rho\}$ has density at most $e^2 M e^{-\rho/a_*}$. Let $m_{<} = \int y \ell_{<}(y) dy$, $v_{<} = \int y^2 \ell_{<}(y) dy$, and $y_0 = (c_*/M)^{1/2} = \theta/(\sqrt{2}p\rho)$. By (11.25), if $y_0 < a_*$, then

$$|m_{<}| \leq 2 \int_0^{y_0} M y dy + 2 \int_{y_0}^{a_*} \frac{c_*}{y} dy = c_* + 2c_* \log \frac{a_*}{y_0},$$

and if $y_0 \geq a_*$, then $|m_{<}| \leq Ma_*^2 \leq c_*$. Since $a_*/y_0 \leq \sqrt{2}\rho p/\theta \leq \sqrt{2}\rho p^2/c_0 \leq p^3$ by (B1), we have $|m_{<}| \leq 7c_* \log p$. Further, $v_{<} \leq 2 \int_0^{a_*} c_* dy = 2c_* a_*$. Let $t = \pm 1/a_*$. Since $|ty| \leq 1$ on the support of $\ell_{<}$ and $e^u - 1 - u \leq u^2$ for $|u| \leq 1$, we obtain

$$\int (e^{ty} - 1) \ell_{<}(y) dy \leq \frac{|m_{<}|}{a_*} + \frac{v_{<}}{a_*^2} \leq c_*^{3/4} (7 \log p + 2) \leq 1,$$

where we used (B1) and $c_* \leq c^{(0)}$. Let $\tilde{\ell}(y) = e^{ty} \ell_{<}(y)$, so that $\|\tilde{\ell}\|_\infty \leq eM$ and $\|\tilde{\ell}\|_{L^1} - \int \ell_{<} \leq 1$. Since $e^{t(y_1 + \dots + y_m)} = \prod_i e^{ty_i}$, we have $\ell_{<}^{*m}(s) = e^{-ts} \tilde{\ell}^{*m}(s)$, and since $\|q^{*m}\|_\infty \leq \|q\|_\infty \|q\|_{L^1}^{m-1}$, the density $d_{<}$ of the law of $S_{<}$ away from 0 satisfies

$$d_{<}(s) = e^{-ts} e^{-\int \ell_{<}} \sum_{m \geq 1} \frac{\tilde{\ell}^{*m}(s)}{m!} \leq e^{-ts} \|\tilde{\ell}\|_\infty \exp \left(\|\tilde{\ell}\|_{L^1} - \int \ell_{<} \right) \leq e^2 M e^{-ts}.$$

Choosing the sign of t equal to that of s proves the claim.

Let $W = D - S_{<}$. We claim that the law of W restricted to $[-3\rho, 3\rho]$ has density at most $G = F_p + e^2 M e^{-\rho/a_*}$. Indeed, for a Borel set $A \subset [-3\rho, 3\rho]$, the event $\{W \in A, |S_{<}| < \rho\}$ requires $D \in (A + S_{<}) \cap [-4\rho, 4\rho]$, and $4\rho \leq 1/2$, so conditioning on $S_{<}$ bounds its probability by $F_p |A|$; and conditioning on D , the probability of $\{W \in A, |S_{<}| \geq \rho\}$ is at most $e^2 M e^{-\rho/a_*} |A|$ by the previous claim.

STEP 4. *The large jumps.* We have $B = W - S_{>}$, and the law of $S_{>}$ is $e^{-\lambda_{>}} \sum_{m \geq 0} \ell_{>}^{*m}/m!$. Let $A \subset [-2\rho, 2\rho]$ be a Borel set. The term $m = 0$ contributes at most $G|A|$ to $\mathbb{P}(B \in A)$. The term $m = 1$ contributes at most

$$\int_A \left(\int \ell_{>}(w - x) \mathbb{P}(W \in dw) \right) dx \leq \left(G \lambda_{>} + \frac{c_*}{\rho^2} \right) |A|,$$

where we split the inner integral according to whether $|w - x| < \rho$, in which case $w \in [-3\rho, 3\rho]$ and the density of W is at most G , or $|w - x| \geq \rho$, in which case $\ell_{>}(w - x) \leq c_*/\rho^2$ by (11.25). The terms $m \geq 2$ contribute at most $\sum_{m \geq 2} \|\ell_{>}^{*m}\|_\infty |A|/m! \leq \|\ell_{>}\|_\infty \lambda_{>} |A| \leq 2c_*^{5/4} |A|$, by (11.26). Since $\lambda_{>} \leq 1$, we conclude that

$$H \leq 2G + \frac{\theta H}{\rho^2} + 2(\theta H)^{5/4}. \quad (11.27)$$

STEP 5. *The bootstrap.* By (B2), $\theta/\rho^2 \leq C_0/(p\rho^2) \leq 1/4$. Further, $2(\theta H)^{5/4} \leq H/4$ is equivalent to $H \leq 8^{-4}\theta^{-5}$, which holds since $H \leq h_0 \leq 8^{-4}(p/C_0)^5$ by (B2). Inserting these bounds into (11.27), we obtain $H \leq 4G$. Since $M \leq M_p$ and $a_* = (\theta H)^{1/4}$, this reads $H \leq \phi(H)$, where

$$\phi(h) = 4F_p + 4e^2 M_p \exp(-\rho(\theta h)^{-1/4}), \quad h > 0,$$

is nondecreasing. We now iterate this inequality. Since $H \leq h_0$ and $\theta h_0 \leq c^{(0)} \leq p^{-1/3}$ by (B1), we have $H \leq \phi(h_0) \leq h_1$. Next, $\theta h_1 \leq C_0 h_1/p \leq \rho^4 p^{-8}$ by (B3), so that

$\rho(\theta h_1)^{-1/4} \geq p^2$ and $H \leq \phi(h_1) \leq 4F_p + 4e^2 M_p e^{-p^2} \leq e^{-2p}$, again by (B3). This proves the density bound in the lemma, and in particular ν_B satisfies the boundary bound (B) with $\mathfrak{h}_p = e^{-p}$.

STEP 6. Core and tail bounds. By Lemma 11.4 with $\delta = 2\rho$, $\mathbb{E}(1 + \theta - D) = \theta \int_{|b| \geq 2\rho} b^{-1} \nu_B(db)$, which has absolute value at most $\theta/(2\rho) = 8\theta$. This gives $|\mathbb{E}D - 1| \leq 9\theta \leq 9C_0/p \leq r_p/4$, by (B4). Applying (11.11) with $\delta = 2\rho$ and $x = r_p/4$ as in Step 1 of the proof of Lemma 11.9, we obtain $\mathbb{P}(|D - \mathbb{E}D| > r_p/4) \leq 2e^{-c_1 p^{7/4}}$. Further, $\mathbb{P}(S \neq 0) \leq p \nu_B((-2\rho, 2\rho)) \leq 4\rho p H \leq 4\rho p e^{-2p}$. By (B4), we obtain

$$\mathbb{P}(|B - 1| > r_p) \leq \mathbb{P}(|B - 1| > r_p/2) \leq 2e^{-c_1 p^{7/4}} + 4\rho p e^{-2p} \leq \frac{1}{2}e^{-\sqrt{p}},$$

which is the core bound (C). For $L \geq 3$, as in Step 4 of the proof of Lemma 11.9, (11.12) with $\delta = 2\rho$ gives $\mathbb{P}(|D| > L/2) \leq \exp(p - \rho p^2 L / (9C_0)) \leq e^{-pL}$ by (B4). By part (ii) of Lemma 11.7, which applies since $\int \ell \leq 4\rho p H \leq 1$ and $\ell(y) \leq \theta H / y^2$, we have $\mathbb{P}(|S| \geq L/2) \leq 8\theta H / L \leq 8C_0 e^{-2p} / (pL) \leq e^{-p} / L$, by (B4). We obtain $\nu_B(\{|b| > L\}) \leq e^{-p} / L + e^{-pL}$, which is the tail bound (T) for the canonical tuple. Finally, $\nu_B = \mathcal{R}_{p,\theta}(\nu_B)$ is of the form $e^{-p} \delta_{1+\theta} + f db$ by part (iii) of Lemma 11.2. Therefore $\nu_B \in \mathfrak{S}_{p,\theta}$, and $\nu_B = \nu_p^\theta$ by part (i) of Theorem 11.19. $\square$

Proposition 11.27. *Let $\Xi \subset (0, \infty)$ be compact, $\xi_- = \min \Xi$, $\xi_+ = \max \Xi$, and*

$$p_{\text{phys}}(\Xi) = \max \{p_{\text{boot}}(\xi_+^{-2}, 4\xi_-^{-2}), p_{\text{st}}(\Xi)\}.$$

For every $p \geq p_{\text{phys}}(\Xi)$, $\xi \in \Xi$, and $E = \xi\sqrt{p}$, the following hold.

- (i) *For every $r \in (0, 1)$, we have $\sup_{\eta > 0} \mathbb{E}|\Gamma_o(E + i\eta)/g(E)|^r \leq C_r p^{r/2}$ and $\sup_{\eta > 0} \mathbb{E}^\infty|\Gamma_o(E + i\eta)/g(E)|^r \leq q_p^{-1} C_r p^{r/2}$, where C_r depends only on r .*
- (ii) *For every real solution π at E , the law of $g(E)D$, $D \sim \pi$, is $\nu_{p,E}$.*
- (iii) *If $\eta_k \downarrow 0$ and the laws under $\mathbb{P}$ of $-\Gamma_o(E + i\eta_k)/g(E)$ converge weakly to a probability measure carried by $\mathbb{R}$, then this limit is the law of $1/B$ with $B \sim \nu_{p,E}$. In particular, if $\text{Im } \Gamma_o(E + i\eta) \rightarrow 0$ in $\mathbb{P}$ -probability as $\eta \downarrow 0$, then the law of $-\Gamma_o(E + i\eta)/g(E)$ under $\mathbb{P}$ converges weakly to the law of $1/B$.*
- (iv) *Suppose that $E \in \mathcal{E}_p$ and $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Then the real boundary law at E of Proposition 5.11, in the B normalization, equals $\nu_{p,E}$. Moreover, $-\Gamma_o(E + i\eta)/g(E)$ converges $\mathbb{P}$ -almost surely as $\eta \downarrow 0$ to a random variable with the law of $1/B$, $B \sim \nu_{p,E}$.*

Proof. Part (i) is part (i) of Lemma 11.25 with $C_r = 2C_{r,1}$. For (ii), since $p \geq p_{\text{st}}(\Xi)$ we have $E > 2$, and $E^{-1} \leq g(E) \leq 2E^{-1}$ gives $\xi_+^{-2}/p \leq \theta(E) \leq 4\xi_-^{-2}/p$. By the discussion after Definition 11.1, the law of $g(E)D$ is a fixed point of $\mathcal{R}_{p,\theta(E)}$, and Lemma 11.26 with $c_0 = \xi_+^{-2}$ and $C_0 = 4\xi_-^{-2}$ shows that it equals $\nu_p^{\theta(E)} = \nu_{p,E}$. The first claim in (iii) follows

from part (ii) of Lemma 11.25 and (ii). For the second claim, let $X_\eta = -\Gamma_o(E + i\eta)/g(E)$ and let $\eta_k \downarrow 0$. By (i) and Markov's inequality, the laws of X_{η_k} are tight, so every subsequence has a further subsequence along which X_{η_k} converges in law to some X . Since $\text{Im } X_{\eta_k} \rightarrow 0$ in probability and Im is continuous, $\text{Im } X = 0$ almost surely, and the first claim identifies the law of X . This shows that the laws of X_η converge to the law of $1/B$ along every sequence $\eta_k \downarrow 0$. For (iv), since $E \in \mathcal{E}_p$, the boundary value $\Gamma_o(E + i0) = \lim_{\eta \downarrow 0} \Gamma_o(E + i\eta)$ exists almost surely (Definition 5.7). By part (ii) of Proposition 5.11, $\Gamma_o(E + i0)$ is $\mathbb{P}$ -almost surely finite, real, and nonzero, and the law of $D = -1/\Gamma_o(E + i0)$ under $\mathbb{P}$ is a real solution at E . By (ii), the law of $B = g(E)D$ is $\nu_{p,E}$, and $-\Gamma_o(E + i\eta)/g(E) \rightarrow -\Gamma_o(E + i0)/g(E) = 1/B$ almost surely. In other words, under the no-ac hypothesis the laws of $-\Gamma_o(E + i\eta)/g(E)$ have a unique limit point as $\eta \downarrow 0$, this limit point is carried by $\mathbb{R}$, and (iii) identifies it with $\nu_{p,E}^x$. $\square$

Remark 11.28. Part (ii) of Proposition 11.27 does not use the a priori bound (i): the one-big-jump argument of Lemma 11.26 starts from the density bound $\sqrt{p+1}$ near zero, which holds for every real fixed point by Lemma 4.18. The bound (i) is used only for tightness, in the second claim of (iii). Under survival conditioning, the corresponding boundary law is not $\nu_{p,E}$, since a surviving vertex has at least one child; it is dominated by q_p^{-1} times the unconditioned law, as in Proposition 4.14, and therefore inherits the upper bounds of Theorem 11.21 with an additional factor q_p^{-1} .

Analyticity of the principal solution and bounds on its derivatives

In this section, we study how the principal solution $\nu_{p,E}$ of Theorem 11.21 depends on the scaled energy $\xi = E/\sqrt{p}$. We prove three results. The first, Theorem 12.19, states that for each fixed large p , the law of the principal denominator, written in the scaling $g = 1/E$, is a real-analytic function of the energy with values in a Banach space X_q of signed measures. The second, Proposition 12.21, deduces from this that the critical growth rate $\lambda_p(\xi)$ is real-analytic in ξ . The third, Proposition 12.24 and its consequence Corollary 12.25, bounds the first four ξ -derivatives of the law with constants that are explicit in p : polynomial for the weighted total variation, exponentially small for the parts of the derivatives near zero and away from one, and of order p^{-1} for the parts near one that are seen by smooth test functions. These bounds are used in Section 14.

The analyticity statements in Sections 12.4 and 12.5 are qualitative: they hold at every fixed p above a threshold, and the constants in their proofs depend on p . The quantitative bounds in Section 12.6 are proved separately. There we differentiate the fixed point equation and estimate the derivatives with the component norms of Definition 11.11; analyticity is used only to know that the derivatives exist and satisfy the differentiated equations.

Section 12.1 introduces the scaling $g = 1/E$ and collects the properties of the principal law that we use. Section 12.2 defines the space X_q and proves that the law map is holomorphic on it. Section 12.3 shows that the linearization of the law map at the principal law is a contraction for the mixed norm of Section 11 and deduces that I minus the linearization is invertible on X_q . Section 12.4 proves Theorem 12.19, and Section 12.5 proves Proposition 12.21. Section 12.6 proves Proposition 12.24 and Corollary 12.25.

12.1 The scaling $g = 1/E$

Conventions

Throughout this section, $\Xi = [\xi_-, \xi_+] \subset (0, \infty)$ is a compact interval. Every compact set $\Xi \subset (0, \infty)$ is contained in its convex hull, which is a compact interval in $(0, \infty)$, and the

results below for intervals imply those for compact sets. We set

$$\Xi' = [\tilde{\zeta}_-/2, 2\tilde{\zeta}_+], \quad (12.1)$$

so that Ξ is contained in the interior of Ξ' , and we apply the results of Section 11 on the window Ξ' . Let $p_{\text{st}}(\Xi')$ be the threshold of Theorem 11.21 for this window. By Theorem 11.21, for $p \geq p_{\text{st}}(\Xi')$ and $E = \tilde{\zeta}\sqrt{p}$ with $\tilde{\zeta} \in \Xi'$, the law $\nu_{p,E}$ is the weak limit of the iteration of the cavity map in the B -normalization started from δ_1 . In particular, $\nu_{p,E}$ does not depend on the window used to construct it.

For $\tilde{\zeta} \in \Xi'$ and $p \geq p_{\text{st}}(\Xi')$, we write $E = \tilde{\zeta}\sqrt{p}$, $g = g(E)$, $\theta = \theta(\tilde{\zeta}) = g(E)^2$, and

$$\tau = \tau(\tilde{\zeta}) = \frac{1}{pE^2} = \frac{1}{p^2\tilde{\zeta}^2}. \quad (12.2)$$

Recall that $Eg(E) = 1 + \theta$, so that $\theta/(1 + \theta)^2 = g^2/(Eg)^2 = E^{-2}$ and

$$\tau = \frac{\theta}{p(1 + \theta)^2}. \quad (12.3)$$

Constants $c, C > 0$ in this section may depend on Ξ , on the exponent $q \in (0, 1)$ when one is fixed, and on the constants of Section 11 for the window Ξ' (which depend only on Ξ); they never depend on p or $\tilde{\zeta}$ unless this is indicated, and they may change from line to line. We write $Y, T, Y_1, Y_2, \dots$ for independent $\text{Exp}(1)$ variables and $N \sim \text{Pois}(p)$, all independent of each other and of the other variables in the same formula.

We first record the size of the parameters and of their derivatives.

Lemma 12.1. *There exist $p_0(\Xi) \geq p_{\text{st}}(\Xi')$ and $C > 0$ such that for every $p \geq p_0(\Xi)$, the functions $\tilde{\zeta} \mapsto \theta(\tilde{\zeta})$ and $\tilde{\zeta} \mapsto \tau(\tilde{\zeta})$ are real-analytic on a neighborhood of Ξ' , and for all $\tilde{\zeta} \in \Xi'$ and $k \in \{0, 1, \dots, 4\}$,*

$$\frac{1}{Cp} \leq \theta(\tilde{\zeta}) \leq \frac{C}{p}, \quad |\partial_{\tilde{\zeta}}^k \theta(\tilde{\zeta})| \leq \frac{C}{p}, \quad |\partial_{\tilde{\zeta}}^k \tau(\tilde{\zeta})| \leq \frac{C}{p^2}. \quad (12.4)$$

In particular, $\theta(\tilde{\zeta}) \leq 1/24$ and $1 \leq 1 + \theta(\tilde{\zeta}) \leq 9/8$ for $\tilde{\zeta} \in \Xi'$.

Proof. The bounds for τ follow from (12.2), since $\partial_{\tilde{\zeta}}^k \tilde{\zeta}^{-2}$ is bounded on Ξ' . For θ , recall from Definition 5.2 that $\sqrt{p}g(\tilde{\zeta}\sqrt{p}) = a_p(\tilde{\zeta})^{-1}$, where $a_p(\tilde{\zeta}) = (\tilde{\zeta} + \sqrt{\tilde{\zeta}^2 - 4/p})/2$. Then

$$\theta(\tilde{\zeta}) = \frac{1}{p a_p(\tilde{\zeta})^2}.$$

Let $p \geq 128/\tilde{\zeta}_-^2$. For $\tilde{\zeta} \geq \tilde{\zeta}_-/4$ we have $8/p \leq \tilde{\zeta}_-^2/16 \leq \tilde{\zeta}^2$, that is, $\tilde{\zeta}^2 - 4/p \geq \tilde{\zeta}^2/2$. On the half-line $(\tilde{\zeta}_-/4, \infty)$, which contains a neighborhood of Ξ' , the function a_p is real-analytic and satisfies $\tilde{\zeta}/2 \leq a_p(\tilde{\zeta}) \leq \tilde{\zeta}$, and on Ξ' its derivatives of order at most four are bounded by a constant depending only on $\tilde{\zeta}_-$ and $\tilde{\zeta}_+$, uniformly in $p \geq 128/\tilde{\zeta}_-^2$. This gives the bounds for θ . The last assertion holds once $C/p \leq 1/24$. $\square$

The principal law in the scaling $g = 1/E$

Let $p \geq p_0(\Xi)$, $\xi \in \Xi'$, and let $B \sim \nu_{p,E}$. Recall from Theorem 11.21 that $\nu_{p,E}$ solves

$$B \stackrel{d}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{\tilde{B}_j}, \quad (12.5)$$

where $B_1, B_2, \dots$ are independent with law $\nu_{p,E}$. Set $\tilde{B} = B/(1 + \theta) = D/E$, where $D = B/g$ is the corresponding real solution (Corollary 11.23). Dividing (12.5) by $1 + \theta$ and using (12.3), we obtain

$$\tilde{B} \stackrel{d}{=} 1 - \tau \sum_{j=1}^N \frac{Y_j}{\tilde{B}_j}, \quad (12.6)$$

where $\tilde{B}_1, \tilde{B}_2, \dots$ are independent copies of $\tilde{B}$. The event $N = 0$ gives $\tilde{B}$ an atom of mass at least e^{-p} at 1, whose location does not depend on ξ .

Definition 12.2. For $p \geq p_0(\Xi)$ and $E = \xi\sqrt{p}$ with $\xi \in \Xi'$, the *principal law in the scaling $g = 1/E$* is the law $\tilde{\nu}_{p,E}$ of $\tilde{B} = B/(1 + \theta)$, where $B \sim \nu_{p,E}$. When p is fixed, we abbreviate $\tilde{\nu}_\xi = \tilde{\nu}_{p,\xi\sqrt{p}}$.

The reason for this scaling is the following. The law $\nu_{p,E}$ of B has the atom $e^{-p}\delta_{1+\theta(\xi)}$, and $\xi \mapsto \theta(\xi)$ is strictly monotone. For $\xi \neq \xi'$, the measure $\nu_{p,\xi\sqrt{p}} - \nu_{p,\xi'\sqrt{p}}$ gives mass e^{-p} to the point $1 + \theta(\xi)$, since the second law has no atom there. Its total variation is at least e^{-p} , and $\xi \mapsto \nu_{p,\xi\sqrt{p}}$ is not continuous in total variation, let alone differentiable. After the dilation by $(1 + \theta)^{-1}$, the compulsory atom sits at 1 for every ξ , and we show that $\xi \mapsto \tilde{\nu}_\xi$ is real-analytic in a norm that dominates the weighted total variation. Smooth test functions of B are recovered from $\tilde{\nu}_\xi$ by the chain rule (see Corollary 12.25).

Inputs from the construction of the principal law

We use the following facts from Section 11, applied on the window Ξ' . We state them in the B -coordinate b , as in Section 11, and write $r_p = p^{-1/4}$ and $\mathcal{C}_p = [1 - r_p, 1 + r_p]$. Let $C_{\Xi'} = 16/\xi_-^2$, which is the constant C_Ξ of Theorem 11.21 for the window Ξ' , so that $\theta \leq C_{\Xi'}/p$ for $\xi \in \Xi'$ and $p \geq p_{\text{st}}(\Xi')$.

(S1) (Definition 11.8 and Theorem 11.21.) There exist constants $a_{\text{st}}, K_{\text{st}} > 0$, $A_{\text{st}} \geq 0$, and $\rho \in (0, 1/16]$ such that for all $p \geq p_{\text{st}}(\Xi')$ and $\xi \in \Xi'$, we have $\nu_{p,E} = e^{-p}\delta_{1+\theta} + f(b) db$ with

$$\begin{aligned} \nu_{p,E}(\mathcal{C}_p^c) &\leq e^{-a_{\text{st}}\sqrt{p}}, & \sup_{|b| < \rho} f(b) &\leq p^{A_{\text{st}}} e^{-a_{\text{st}}p}, \\ \nu_{p,E}(|b| > L) &\leq \frac{K_{\text{st}} p^{A_{\text{st}}} e^{-a_{\text{st}}p}}{L} + e^{-a_{\text{st}}pL} \end{aligned} \quad (12.7)$$

for $L \geq 3$. Indeed, $\nu_{p,E}$ belongs to the class $\mathfrak{S}_{p,\theta}$ of Definition 11.8 with the canonical tuple, and we may take $(a_{\text{st}}, A_{\text{st}}, K_{\text{st}}, \rho) = (1, 0, 1, 1/16)$; we keep general constants to indicate how they enter. Further, $\xi \mapsto \nu_{p,\xi\sqrt{p}}$ is weakly continuous on Ξ' , and $\nu_{p,E}$ is a fixed point of the law map $\mathcal{R}_{p,\theta}$ of Definition 11.1, that is, it satisfies (12.5).

- (S2) (Definition 11.11 and Lemma 11.12.) The cutoffs $\chi_c, \chi_b, \chi_i: \mathbb{R} \rightarrow [0, 1]$ satisfy $\chi_c + \chi_b + \chi_i = 1$, $\chi_c = 1$ on $\mathcal{C}_p$, $\text{supp } \chi_c \subset \{|b - 1| \leq 2r_p\}$, $|\chi'_c| \leq 2r_p^{-1}$, $\chi_b = 1$ on $[-\rho/4, \rho/4]$, and $\text{supp } \chi_b \subset [-\rho/2, \rho/2]$. For a finite real measure σ on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$, we have $m_\alpha(\sigma) = \int \chi_\alpha d\sigma$ and $\Pi_\alpha \sigma = \chi_\alpha \sigma - m_\alpha(\sigma)\delta_1$ for $\alpha \in \{c, b, i\}$, and the component norms are

$$\begin{aligned} \|\Pi_c \sigma\|_c &= \sup \left\{ \left| \int \varphi d\Pi_c \sigma \right| : \varphi(1) = 0, \text{Lip}(\varphi) \leq 1 \right\}, \\ \|\Pi_b \sigma\|_b &= \|h\|_{L^\infty([-\rho/2, \rho/2])} \quad \text{when } \chi_b \sigma = h(b) db, \\ \|\Pi_i \sigma\|_i &= \sup \left\{ \left| \int \varphi d\Pi_i \sigma \right| : \|\varphi\|_\infty \leq 1, \text{Lip}(\varphi) \leq 1 \right\}. \end{aligned}$$

The second norm is finite when $\chi_b \sigma$ is absolutely continuous with bounded density. In the notation of Definition 11.11, $\|\Pi_\alpha \sigma\|_\alpha = n_\alpha(\sigma)$.

- (S3) (Lemma 11.3.) The linearization of the law map at a law λ is the operator $\mathcal{L}_{p,\theta;\lambda}$, given on finite real measures σ with $|\sigma|(\{0\}) = 0$ by

$$\int \varphi d(\mathcal{L}_{p,\theta;\lambda} \sigma) = p \int \mathbb{E} \left[\varphi \left(D_\lambda - \frac{\theta T}{pb} \right) \right] \sigma(db) \quad (12.8)$$

for bounded Borel φ , where $D_\lambda \sim \mathcal{R}_{p,\theta}(\lambda)$ and $T \sim \text{Exp}(1)$ are independent. We use only the case $\lambda = \nu_{p,E}$, for which $D_\lambda \sim \nu_{p,E}$.

- (S4) (Lemma 11.16.) Let $\mathcal{L} = \mathcal{L}_{p,\theta;\nu_{p,E}}$, and let σ be a finite real measure with $\sigma(\mathbb{R}) = 0$ whose restriction to $(-\rho/2, \rho/2)$ is absolutely continuous with bounded density. Then the vectors $v(\sigma) = (\|\Pi_c \sigma\|_c, \|\Pi_b \sigma\|_b, \|\Pi_i \sigma\|_i)$ satisfy $v(\mathcal{L}\sigma) \leq M_{p,\theta} v(\sigma)$ entry-wise, with the matrix $M_{p,\theta}$ of (11.17); this applies since $\nu_{p,E}$ belongs to the ambient class of Definition 11.8. We use this inequality only through a cruder form. Let

$$P_p = p^{A_{\text{col}}}, \quad \delta_{b,p} = p^{A_{\text{col}}} e^{-c_b p}, \quad \delta_{i,p} = p^{A_{\text{col}}} e^{-c_i \sqrt{p}}, \quad \varepsilon_p = \frac{C_{\text{col}}}{p} + \delta_{i,p}, \quad (12.9)$$

with $A_{\text{col}} = A_{\text{st}} + 3$, $c_b = c_i = a_{\text{st}}$, and $C_{\text{col}} = 20C_{\Xi'}$. Since $\theta \leq C_{\Xi'}/p$ and $e^{-a_{\text{st}} p} \leq e^{-a_{\text{st}} \sqrt{p}}$, every entry of $M_{p,\theta}$ is at most the corresponding entry of

$$M_p = \begin{pmatrix} \varepsilon_p & P_p & P_p \\ \delta_{b,p} & \varepsilon_p & \delta_{b,p} \\ \delta_{i,p} & P_p & \delta_{i,p} + \delta_{b,p} \end{pmatrix} \quad (12.10)$$

once $p \geq 6$, and therefore $v(\mathcal{L}\sigma) \leq M_p v(\sigma)$. The proofs below use only this form, with the structure of polynomial and exponentially small entries; any constants $A_{\text{col}}, C_{\text{col}}, c_b, c_i > 0$ depending only on Ξ would do.

We write $D_t\mu$ for the pushforward of a measure μ on $\mathbb{R}$ under $x \mapsto tx$ ($t > 0$), and $R\mu$ for its pushforward under $x \mapsto -x$ (denoted $\mu^\vee$ in Section 11). In the scaling $g = 1/E$, we denote the variable by x , so that $b = (1 + \theta)x$. The following sets are used throughout:

$$\begin{aligned} \mathcal{Z} &= (-\rho/5, \rho/5), & J_0 &= [1/4, 7/4], \\ \mathcal{S}_\xi &= \left\{x : |(1 + \theta)x - 1| \leq \frac{r_p}{4}\right\}, & \mathcal{O}_\xi &= \left\{x : |(1 + \theta)x - 1| > \frac{r_p}{2}\right\}. \end{aligned} \quad (12.11)$$

The set $\mathcal{S}_\xi$ is a strict core, the set $\mathcal{O}_\xi$ is the complement of a core of half radius, and $\mathcal{Z}$ is an interval around zero on which the boundary cutoff equals one. We also set $x_\xi = (1 + \theta)^{-1}$, the point corresponding to the anchor $b = 1$.

Lemma 12.3. *There exists $p_1(\Xi) \geq p_0(\Xi)$ such that the following holds for all $p \geq p_1(\Xi)$ and $\xi \in \Xi'$. Set $H_p = 2p^{A_{\text{st}}}e^{-a_{\text{st}}p}$.*

- (i) *The law $\tilde{v}_\xi$ satisfies (12.6) and has the form $\tilde{v}_\xi = e^{-p}\delta_1 + \tilde{f}_\xi(x) dx$ with $\tilde{f}_\xi \leq H_p$ on $(-3\rho/4, 3\rho/4)$.*
- (ii) *For every $q' \in (0, 1)$, there exists $C_{q'}$ depending only on q' and Ξ such that $\int (1 + |x|^{q'}) \tilde{v}_\xi(dx) \leq C_{q'}$.*
- (iii) *We have $\tilde{v}_\xi(\mathcal{S}_\xi^c) \leq e^{-a_{\text{st}}\sqrt{p}/2}$.*
- (iv) *The map $\xi \mapsto \tilde{v}_\xi$ is weakly continuous on Ξ' .*

Proof. We derived (12.6) above. The law $\tilde{v}_\xi = D_{1/(1+\theta)}\nu_{p,E}$ has the atom $e^{-p}\delta_1$ and the density $\tilde{f}_\xi(x) = (1 + \theta)f((1 + \theta)x)$. For $|x| < 3\rho/4$, we have $|(1 + \theta)x| < (9/8)(3\rho/4) < \rho$ by Lemma 12.1, and (12.7) implies that $\tilde{f}_\xi(x) \leq (9/8)p^{A_{\text{st}}}e^{-a_{\text{st}}p} \leq H_p$. This proves (i).

For (ii), we have $\tilde{v}_\xi(|x| > L) \leq \nu_{p,E}(|b| > L) \leq C/L$ for $L \geq 3$ by (12.7), since $1 + \theta \geq 1$. Then

$$\int |x|^{q'} \tilde{v}_\xi(dx) \leq 3^{q'} + \int_3^\infty q' L^{q'-1} \tilde{v}_\xi(|x| > L) dL \leq 3^{q'} + Cq' \int_3^\infty L^{q'-2} dL,$$

which is finite and depends only on q' and Ξ .

For (iii), we work in the B -coordinate, where $\mathcal{S}_\xi^c$ corresponds to $\{|b - 1| > r_p/4\}$. Let B' denote the right side of (12.5), with $B_j \sim \nu_{p,E}$, so that $B' \sim \nu_{p,E}$. We split the children $j \leq N$ according to whether $B_j \in \mathcal{C}_p$. By the thinning property of Poisson processes [94, Chapter 5], the number N_2 of children with $B_j \notin \mathcal{C}_p$ is $\text{Pois}(p\nu_{p,E}(\mathcal{C}_p^c))$, so (12.7) gives

$$\mathbb{P}(N_2 \geq 1) \leq p\nu_{p,E}(\mathcal{C}_p^c) \leq pe^{-a_{\text{st}}\sqrt{p}}.$$

The sum $S = (\theta/p) \sum_{j \leq N: B_j \in \mathcal{C}_p} Y_j/B_j$ is compound Poisson with intensity $p_1 = p\nu_{p,E}(\mathcal{C}_p) \leq p$ and jumps distributed as $Z = \theta Y/(p\hat{B})$, where $\hat{B}$ has law $\nu_{p,E}(\cdot | \mathcal{C}_p)$ and is independent of Y . On the event $\{N_2 = 0\}$ we have $B' = 1 + \theta - S$. Since $r_p \leq 1/2$, we have $1/\hat{B} \in [1 - 2r_p, 1 + 2r_p]$, so

$$|\mathbb{E}S - \theta| = |\theta\nu_{p,E}(\mathcal{C}_p)\mathbb{E}[\hat{B}^{-1}] - \theta| \leq \theta(\nu_{p,E}(\mathcal{C}_p^c) + 2r_p) \leq 3\theta r_p$$

once $e^{-a_{\text{st}}\sqrt{p}} \leq r_p$. Let $\lambda \in \mathbb{R}$ with $|\lambda| \leq p/(4\theta)$. Then $|\lambda Z| \leq Y/2$, and the inequality $e^w - 1 - w \leq w^2 e^{|w|}$ gives

$$\log \mathbb{E} e^{\lambda(S - \mathbb{E}S)} = p_1 \mathbb{E} [e^{\lambda Z} - 1 - \lambda Z] \leq p \lambda^2 \left(\frac{2\theta}{p} \right)^2 \mathbb{E} [Y^2 e^{Y/2}] = \frac{64\lambda^2 \theta^2}{p}.$$

In the last equality we used that $\mathbb{E}[Y^2 e^{Y/2}] = 16$. Taking $\lambda = \pm p/(4\theta)$ in the exponential Chebyshev inequality, we obtain

$$\mathbb{P}(|S - \mathbb{E}S| \geq r_p/8) \leq 2 \exp\left(-\frac{pr_p}{32\theta} + 4p\right) \leq 2 \exp(-c p^{7/4} + 4p),$$

where we used $\theta \leq C/p$. On the event $\{N_2 = 0\} \cap \{|S - \mathbb{E}S| < r_p/8\}$, we have $|B' - 1| \leq 3\theta r_p + r_p/8 \leq r_p/4$, because $\theta \leq 1/24$. We conclude that

$$\nu_{p,E}(|b - 1| > r_p/4) \leq p e^{-a_{\text{st}}\sqrt{p}} + 2e^{-cp^{7/4}+4p} \leq e^{-a_{\text{st}}\sqrt{p}/2}$$

for p large, which proves (iii).

For (iv), let $\xi_n \rightarrow \xi$ in Ξ' . By (S1), $B_n \sim \nu_{p,\xi_n\sqrt{p}}$ converges in law to $B \sim \nu_{p,\xi\sqrt{p}}$, and $1 + \theta(\xi_n) \rightarrow 1 + \theta(\xi)$. By Slutsky's lemma, $B_n/(1 + \theta(\xi_n))$ converges in law to $B/(1 + \theta(\xi))$. $\square$

Component norms in the scaling $g = 1/E$

We transport the component norms of (S2) to the scaling $g = 1/E$ by the dilation $b = (1 + \theta)x$.

Definition 12.4. Let $p \geq p_1(\Xi)$ and $\xi \in \Xi'$. Let σ be a finite real measure on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$ whose restriction to $[-\rho/2, \rho/2]$ is absolutely continuous with bounded density. Set $\sigma^B = D_{1+\theta}\sigma$, and for $\alpha \in \{c, b, i\}$ set $\Pi_\alpha^\xi \sigma = D_{1/(1+\theta)} \Pi_\alpha \sigma^B$ and

$$\|\sigma\|_{c,\xi} = \|\Pi_c \sigma^B\|_c, \quad \|\sigma\|_{b,\xi} = \|\Pi_b \sigma^B\|_b, \quad \|\sigma\|_{i,\xi} = \|\Pi_i \sigma^B\|_i.$$

For $\gamma > 0$ we define the mixed seminorm

$$N_{\xi,\gamma}(\sigma) = \|\sigma\|_{c,\xi} + e^{\gamma p} \|\sigma\|_{b,\xi} + e^{\gamma\sqrt{p}} \|\sigma\|_{i,\xi}.$$

For a complex measure σ whose real and imaginary parts satisfy these assumptions, each of the four quantities is defined as its value at $\text{Re } \sigma$ plus its value at $\text{Im } \sigma$.

The measure σ^B is absolutely continuous on $[-\rho/2, \rho/2]$, because σ is absolutely continuous on $[-\rho/(2(1+\theta)), \rho/(2(1+\theta))]$. In particular, all four quantities are finite. We record the properties of these norms that we use.

Lemma 12.5. Let $p \geq p_1(\Xi)$, $\xi \in \Xi'$, and let σ be as in Definition 12.4. Set $S_c = \{x : |(1+\theta)x - 1| \leq 2r_p\}$ and $S_i = \{x : |(1+\theta)x| \geq \rho/4, |(1+\theta)x - 1| \geq r_p\} \cup \{x_\xi\}$.

- (i) We have $\sigma = \Pi_c^\xi \sigma + \Pi_b^\xi \sigma + \Pi_i^\xi \sigma$, and each summand has total mass zero. The measure $\Pi_c^\xi \sigma$ is carried by $S_c \subset J_0$, the measure $\Pi_i^\xi \sigma$ is carried by $S_i \subset \{|x| \geq 2\rho/9\}$, and $\Pi_b^\xi \sigma = g(x) dx - (\int g) \delta_{x_\xi}$ with g supported in $[-\rho/2, \rho/2]$ and $|g| \leq 2\|\sigma\|_{b,\xi}$.
- (ii) The restriction of σ to $\mathcal{Z}$ is absolutely continuous with density bounded by $2\|\sigma\|_{b,\xi}$.
- (iii) We have

$$\|\sigma\|_{c,\xi} \leq 2r_p \|\sigma\|_{TV}, \quad \|\sigma\|_{b,\xi} \leq \sup_{|x| \leq \rho/2} \left| \frac{d\sigma}{dx}(x) \right|, \quad \|\sigma\|_{i,\xi} \leq 2|\sigma|(\mathcal{O}_\xi).$$

- (iv) Let V be a real Banach space. If $f: S_c \rightarrow V$ is Lipschitz, then

$$\left\| \int f d\Pi_c^\xi \sigma \right\|_V \leq \text{Lip}_{S_c}(f) \|\sigma\|_{c,\xi}.$$

If $f: S_i \rightarrow V$ is bounded and Lipschitz, then

$$\left\| \int f d\Pi_i^\xi \sigma \right\|_V \leq \left(\sup_{S_i} \|f\|_V + \text{Lip}_{S_i}(f) \right) \|\sigma\|_{i,\xi}.$$

Proof. Part (i) follows from (S2) after the change of variables $b = (1 + \theta)x$: the measure $\Pi_c \sigma^B$ is carried by $\text{supp } \chi_c \cup \{1\} \subset \{|b - 1| \leq 2r_p\}$, and the measure $\Pi_i \sigma^B$ is carried by $\text{supp } \chi_i \cup \{1\}$, where $\text{supp } \chi_i \subset \{|b| \geq \rho/4, |b - 1| \geq r_p\}$ because χ_i vanishes on $\mathcal{C}_p \cup [-\rho/4, \rho/4]$. If $\Pi_b \sigma^B = h(b) db - (\int h) \delta_1$, then $g(x) = (1 + \theta)h((1 + \theta)x)$, which gives $|g| \leq (9/8)\|\sigma\|_{b,\xi}$ by Lemma 12.1. The inclusions $S_c \subset J_0$ and $S_i \subset \{|x| \geq 2\rho/9\}$ follow from $r_p \leq 1/4$ and $1 \leq 1 + \theta \leq 9/8$.

For (ii), if $x \in \mathcal{Z}$ then $|(1 + \theta)x| < \rho/4$, so $\chi_b((1 + \theta)x) = 1$ and $\chi_c((1 + \theta)x) = \chi_i((1 + \theta)x) = 0$; further, $x_\xi \notin \mathcal{Z}$. The restriction of σ to $\mathcal{Z}$ is therefore the restriction of $g(x) dx$, and (i) gives the bound.

For (iii), let φ be one of the test functions in the definition of $\|\cdot\|_c$. Since $\varphi(1) = 0$ and $\text{Lip}(\varphi) \leq 1$, we have $|\chi_c \varphi| \leq 2r_p$, and $\int \varphi d\Pi_c \sigma^B = \int \chi_c \varphi d\sigma^B$ is bounded by $2r_p \|\sigma\|_{TV}$. The second bound holds because the density of σ^B at b is $(1 + \theta)^{-1}(d\sigma/dx)(b/(1 + \theta))$ and $\chi_b \leq 1$. For the third, if $\|\varphi\|_\infty \leq 1$, then

$$\left| \int \varphi d\Pi_i \sigma^B \right| \leq \int \chi_i d|\sigma^B| + |m_i(\sigma^B)| \leq 2 \int \chi_i d|\sigma^B| \leq 2|\sigma^B|(\mathcal{C}_p^c),$$

and $|\sigma^B|(\mathcal{C}_p^c) = |\sigma|(\{|(1 + \theta)x - 1| > r_p\}) \leq |\sigma|(\mathcal{O}_\xi)$.

For (iv), let $\lambda \in V^*$ with $\|\lambda\| \leq 1$, and set $F = \lambda \circ f$ and $F^B(b) = F(b/(1 + \theta))$ on $(1 + \theta)S_c$. Then $\text{Lip}(F^B) \leq \text{Lip}_{S_c}(f)$. Let $\tilde{F}$ be the McShane extension of F^B to $\mathbb{R}$, which has the same Lipschitz constant, and set $\varphi = \tilde{F} - \tilde{F}(1)$. Since $\Pi_c \sigma^B$ has mass zero and is carried by $(1 + \theta)S_c$, we obtain

$$\lambda \left(\int f d\Pi_c^\xi \sigma \right) = \int F^B d\Pi_c \sigma^B = \int \varphi d\Pi_c \sigma^B,$$

and the right side is bounded by $\text{Lip}_{S_c}(f) \|\sigma\|_{c,\xi}$. The first identity holds because Bochner integrals commute with bounded functionals; the integral exists because f is continuous and bounded on the compact set S_c . Taking the supremum over λ and using the Hahn–Banach theorem, we obtain the first inequality. For the second, we argue in the same way, with $\tilde{F}$ replaced by the McShane extension truncated at $\pm s$, where $s = \sup_{S_i} |F|$. Since the truncation does not increase the Lipschitz constant, $\tilde{F} / \max\{s, \text{Lip}(\tilde{F})\}$ is one of the test functions in the definition of $\|\cdot\|_i$, and the bound follows since $\max\{s, \text{Lip}\} \leq s + \text{Lip}$. $\square$

12.2 An analytic Banach space for the law map

The spaces

Fix $q \in (0, 1)$, and set

$$R_0 = \frac{5\rho}{8}. \quad (12.12)$$

Then $\rho/2 < R_0 < 3\rho/4$ and $R_0 < 1/2$. We use the convention that a map between open subsets of complex Banach spaces is *holomorphic* if it is complex Fréchet differentiable at every point. Compositions of holomorphic maps are holomorphic by the chain rule. A holomorphic map f of one complex variable with values in a complex Banach space V is given near each point by a norm-convergent power series; in particular, its restriction to a real interval is real-analytic. Indeed, if the closed disc of radius r around z_0 lies in the domain of f , then the Cauchy formula $f(z) = (2\pi i)^{-1} \oint_{|w-z_0|=r} f(w)(w-z)^{-1} dw$ holds for $|z-z_0| < r$, since both sides are continuous in z and agree after applying any $\lambda \in V^*$ (Hahn–Banach), and expanding $(w-z)^{-1}$ in powers of $(z-z_0)/(w-z_0)$ gives a power series that converges in norm for $|z-z_0| < r$ (see also [82, Chapter III]).

Definition 12.6. Let $\mathcal{M}_q$ be the space of complex Borel measures μ on $\mathbb{R}$ with

$$\|\mu\|_{q,\text{TV}} = \int (1 + |x|^q) |\mu|(dx) < \infty.$$

Let X_q be the space of $\sigma \in \mathcal{M}_q$ whose restriction to $[-R_0, R_0]$ is absolutely continuous with a bounded density h_σ , with the norm

$$\|\sigma\|_{X_q} = \|\sigma\|_{q,\text{TV}} + \|h_\sigma\|_{L^\infty([-R_0, R_0])}.$$

Let $X_q^0 = \{\sigma \in X_q : \sigma(\mathbb{R}) = 0\}$ and $\mathcal{A}_q = \{\sigma \in X_q : \sigma(\mathbb{R}) = 1\}$. Let W_q be the space of $\mu \in \mathcal{M}_q$ whose restriction to $[1-R_0, 1+R_0]$ is absolutely continuous with a bounded density, normed in the same way with $[1-R_0, 1+R_0]$ in place of $[-R_0, R_0]$. Finally, let Z_q be the space of complex measures $\alpha = \ell(y) dy$ with

$$\|\alpha\|_{Z_q} = \int (1 + |y|^q) |\ell(y)| dy + \text{ess sup}_{y \in \mathbb{R}} |y| |\ell(y)| < \infty.$$

The density of an element of X_q is defined up to a null set. No element of X_q charges zero, and atoms outside $[-R_0, R_0]$ are allowed.

Lemma 12.7. *The spaces $\mathcal{M}_q$, X_q , W_q , and Z_q are complex Banach spaces, and X_q^0 is a closed subspace of X_q . With convolution as product, $\mathcal{M}_q$ is a commutative Banach algebra with unit δ_0 , that is, $\|\mu * \mu'\|_{q,\text{TV}} \leq \|\mu\|_{q,\text{TV}} \|\mu'\|_{q,\text{TV}}$. Further, $\|\mathbf{R}\mu\|_{q,\text{TV}} = \|\mu\|_{q,\text{TV}}$ and $\|\delta_1 * \mu\|_{q,\text{TV}} \leq 2\|\mu\|_{q,\text{TV}}$. Finally, for $\sigma \in X_q$,*

$$\int (1 + |x|^{-q}) |\sigma|(\mathrm{d}x) \leq C_{q,\rho} \|\sigma\|_{X_q}, \quad C_{q,\rho} = 1 + R_0^{-q} + \frac{2R_0^{1-q}}{1-q}. \quad (12.13)$$

Proof. The map $\mu \mapsto (1 + |x|^q)\mu$ is an isometry from $\mathcal{M}_q$ onto the space of finite complex measures with the total variation norm, which is complete. Let (σ_n) be a Cauchy sequence in X_q . Then $\sigma_n \rightarrow \sigma$ in $\mathcal{M}_q$ for some σ , and $h_{\sigma_n} \rightarrow h$ in $L^\infty([-R_0, R_0])$ for some h . The restrictions of σ_n to $[-R_0, R_0]$ converge in total variation both to the restriction of σ and to $h(x) \mathrm{d}x$, since $\|h_{\sigma_n} - h\|_{L^1([-R_0, R_0])} \leq 2R_0 \|h_{\sigma_n} - h\|_{L^\infty}$. We conclude that $\sigma|_{[-R_0, R_0]} = h \mathrm{d}x$ and $\sigma_n \rightarrow \sigma$ in X_q . The same argument applies to W_q . The space Z_q is the intersection of two weighted L^1 and L^∞ spaces with the sum norm; a Cauchy sequence converges in both, and the two limits agree almost everywhere along a subsequence. This proves that Z_q is complete. The functional $\sigma \mapsto \sigma(\mathbb{R})$ is bounded by $\|\sigma\|_{q,\text{TV}}$, so X_q^0 is closed.

Since $q \leq 1$, we have $|x + y|^q \leq |x|^q + |y|^q$, so $1 + |x + y|^q \leq (1 + |x|^q)(1 + |y|^q)$; integrating this against $|\mu| \otimes |\mu'|$, we obtain the algebra property. The reflection bound is immediate, and the translation bound follows from $1 + |x + 1|^q \leq 2(1 + |x|^q)$. For (12.13), we split the integral at $|x| = R_0$:

$$\int |x|^{-q} |\sigma|(\mathrm{d}x) \leq \|h_\sigma\|_{L^\infty([-R_0, R_0])} \int_{-R_0}^{R_0} |x|^{-q} \mathrm{d}x + R_0^{-q} \|\sigma\|_{\text{TV}},$$

$$\text{and } \int_{-R_0}^{R_0} |x|^{-q} \mathrm{d}x = 2R_0^{1-q}/(1-q). \quad \square$$

The jump kernel

Let $\Omega = \{\tau \in \mathbb{C} : \operatorname{Re} \tau > 0\}$. For $\tau \in \Omega$, $x \in \mathbb{R} \setminus \{0\}$, and $y \in \mathbb{R}$, define

$$k_\tau(x; y) = \frac{|x|}{\tau} \exp\left(-\frac{|x||y|}{\tau}\right) \mathbf{1}_{\{xy > 0\}}, \quad (12.14)$$

and for $\mu \in X_q$ let $\mathcal{K}_\tau \mu$ be the measure with density $y \mapsto \int k_\tau(x; y) \mu(\mathrm{d}x)$. This is the analogue in the scaling $g = 1/E$ of the kernel j_θ and the operator $\mathbf{K}_\theta$ of Section 11.1. For real $\tau > 0$, $k_\tau(x; \cdot)$ is the density of $\tau Y/x$. If μ is a probability measure with $\mu(\{0\}) = 0$, then $\mathcal{K}_\tau \mu$ is the law of the jump $\tau Y/X$, where $X \sim \mu$ and $Y \sim \operatorname{Exp}(1)$ are independent. The parameter τ changes the density of the jump but not its support.

- Lemma 12.8.** (i) For all $\tau \in \Omega$ and $x \neq 0$, we have $\int k_\tau(x; y) dy = 1$. In particular $(\mathcal{K}_\tau \mu)(\mathbb{R}) = \mu(\mathbb{R})$ for $\mu \in X_q$.
- (ii) The map $\tau \mapsto \mathcal{K}_\tau$ is holomorphic from Ω to the space of bounded operators from X_q to Z_q , with the operator norm. For $j \geq 1$, its j -th derivative is the integral operator with kernel $\partial_\tau^j k_\tau(x; y)$. For every compact $\Omega' \subset \Omega$ and $j \geq 0$, $\sup_{\tau \in \Omega'} \|\partial_\tau^j \mathcal{K}_\tau\|_{X_q \rightarrow Z_q}$ is bounded by a constant depending only on Ω' , j , q , and ρ .

Proof. We write $\kappa = 1/\tau$, which maps Ω biholomorphically onto itself, so that $k_\tau(x; y) = |x| \kappa e^{-\kappa s} \mathbf{1}_{\{xy > 0\}}$ with $s = |x||y|$. Part (i) follows from $\int_0^\infty |x| \kappa e^{-\kappa |x|y} dy = 1$ for $\operatorname{Re} \kappa > 0$.

For (ii), fix $\kappa_0 \in \Omega$ and let $\mathbb{D}$ be the closed disc of radius $\alpha = \operatorname{Re} \kappa_0/2$ around κ_0 , so that $\operatorname{Re} \kappa \geq \alpha$ and $|\kappa| \leq 2|\kappa_0|$ for $\kappa \in \mathbb{D}$. By the Leibniz rule,

$$\partial_\kappa^j (|x| \kappa e^{-\kappa s}) = |x| ((-s)^j \kappa + j(-s)^{j-1}) e^{-\kappa s},$$

so for $\kappa \in \mathbb{D}$, with the substitution $u = |x||y|$,

$$\begin{aligned} \int (1 + |y|^q) |\partial_\kappa^j k| dy &\leq \int_0^\infty (2|\kappa_0|u^j + ju^{j-1}) e^{-\alpha u} \left(1 + \frac{u^q}{|x|^q}\right) du \leq C_j (1 + |x|^{-q}), \\ |y| |\partial_\kappa^j k| &\leq u (2|\kappa_0|u^j + ju^{j-1}) e^{-\alpha u} \leq C_j, \end{aligned}$$

where C_j depends only on κ_0 , j , and q . Integrating against $|\mu|$ and using (12.13), we find that the integral operator with kernel $\partial_\kappa^j k$ is bounded from X_q to Z_q , with norm at most $2C_j C_{q,\rho}$, uniformly for $\kappa \in \mathbb{D}$. For $\kappa, \kappa + h \in \mathbb{D}$ we have pointwise

$$k_{\kappa+h} - k_\kappa - h \partial_\kappa k_\kappa = h^2 \int_0^1 (1-t) \partial_\kappa^2 k_{\kappa+th} dt,$$

and the previous bound with $j = 2$ gives that the corresponding operator has norm at most $2C_2 C_{q,\rho} |h|^2$. Then $\kappa \mapsto \mathcal{K}$ is complex differentiable in operator norm, with derivative the integral operator with kernel $\partial_\kappa k$. Applying the same argument to $\partial_\kappa^j k$ in place of k and inducting on j , we obtain the statement about higher derivatives. Composing with $\tau \mapsto 1/\tau$ and covering a compact Ω' by finitely many discs proves (ii). $\square$

Convolution powers

The local density of the law map is controlled by the following estimate: if a sum of k jumps lands at distance $|z|$ from zero, then one of the jumps has modulus at least $|z|/k$.

Lemma 12.9. Let $k \geq 1$ and let $\ell_1, \dots, \ell_k \in L^1(\mathbb{R})$ be complex functions with $|\ell_i(y)| \leq M_i/|y|$ for almost every y . Then for almost every $z \neq 0$,

$$(|\ell_1| * \dots * |\ell_k|)(z) \leq \frac{k}{|z|} \sum_{i=1}^k M_i \prod_{j \neq i} \|\ell_j\|_{L^1}. \quad (12.15)$$

In particular, $|(\ell_1 * \dots * \ell_k)(z)| \leq k^2 |z|^{-1} \prod_{i=1}^k \max\{M_i, \|\ell_i\|_{L^1}\}$ for almost every $z \neq 0$.

Proof. For $k = 1$, the conclusion is the assumed almost-everywhere bound. Suppose $k \geq 2$, and let $H_z = \{(y_1, \dots, y_k) \in \mathbb{R}^k : y_1 + \dots + y_k = z\}$. The left side of (12.15) is the integral of $\prod_j |\ell_j(y_j)|$ over H_z with respect to the Lebesgue measure in any $k - 1$ of the coordinates; all these parametrizations give the same measure on H_z , since the changes of variables between them have Jacobian ± 1 . On H_z we have $\max_i |y_i| \geq |z|/k$. Then H_z is covered by the sets $A_i = H_z \cap \{|y_i| \geq |z|/k\}$. On A_i we bound $|\ell_i(y_i)| \leq kM_i/|z|$; this holds outside a set of y_i of measure zero, whose preimage in H_z is null in the parametrization by $(y_j)_{j \neq i}$. Integrating the other $k - 1$ factors over $(y_j)_{j \neq i} \in \mathbb{R}^{k-1}$, we obtain

$$\int_{A_i} \prod_{j=1}^k |\ell_j(y_j)| \leq \frac{kM_i}{|z|} \prod_{j \neq i} \|\ell_j\|_{L^1}.$$

Summing over i proves (12.15), and the second bound follows. $\square$

For $\alpha \in Z_q$, define the compound Poisson measure (as in Section 11.1)

$$P_p(\alpha) = e^{-p} \sum_{k=0}^{\infty} \frac{p^k}{k!} \alpha^{*k}, \quad \alpha^{*0} = \delta_0. \quad (12.16)$$

For a probability density α , this is the law of a sum of $N \sim \text{Pois}(p)$ independent variables with law α .

Lemma 12.10. *For $k \geq 1$ and $\alpha_1, \dots, \alpha_k \in Z_q$, the measure $\alpha_1 * \dots * \alpha_k$ is absolutely continuous and*

$$\|\alpha_1 * \dots * \alpha_k\|_{W_q} \leq 3k^2 \prod_{i=1}^k \|\alpha_i\|_{Z_q}. \quad (12.17)$$

The series (12.16) converges in W_q , and $P_p: Z_q \rightarrow W_q$ is holomorphic. Its mass is $P_p(\alpha)(\mathbb{R}) = \exp(p(\alpha(\mathbb{R}) - 1))$, and for $m \geq 1$ and $\beta_1, \dots, \beta_m \in Z_q$,

$$D^m P_p(\alpha)[\beta_1, \dots, \beta_m] = p^m P_p(\alpha) * \beta_1 * \dots * \beta_m, \quad (12.18)$$

where the right side is the series $e^{-p} \sum_{k \geq 0} p^{k+m} (k!)^{-1} \alpha^{*k} * \beta_1 * \dots * \beta_m$, which converges in W_q .

Proof. Let $\alpha_i = \ell_i(y) dy$. The weighted total variation of $\alpha_1 * \dots * \alpha_k$ is at most $\prod_i \|\alpha_i\|_{q, \text{TV}}$ by Lemma 12.7. The convolution is absolutely continuous with density $\ell_1 * \dots * \ell_k$. For $z \in [1 - R_0, 1 + R_0]$ we have $|z| \geq 1 - R_0 \geq 1/2$, and Lemma 12.9 with $M_i = \|\alpha_i\|_{Z_q} \geq \|\ell_i\|_{L^1}$ implies that $|(\ell_1 * \dots * \ell_k)(z)| \leq 2k^2 \prod_i \|\alpha_i\|_{Z_q}$ for almost every such z . This proves (12.17). The term $k = 0$ is δ_0 , which does not charge $[1 - R_0, 1 + R_0]$ and has norm one in W_q . Since $\sum_k p^k 3k^2 r^k / k! < \infty$ for every $r > 0$, the series (12.16) and the series in (12.18) converge in W_q , uniformly for α and β_i in bounded sets.

Let $A_k(\alpha_1, \dots, \alpha_k) = \alpha_1 * \dots * \alpha_k$, a bounded symmetric k -linear map by (12.17) and commutativity. For $\alpha, \beta \in Z_q$ and $k \geq 2$, we have

$$\alpha_+^{*k} - \alpha^{*k} - k A_k(\alpha, \dots, \alpha, \beta) = \sum_{j=2}^k \binom{k}{j} A_k(\underbrace{\alpha, \dots, \alpha}_{k-j}, \underbrace{\beta, \dots, \beta}_j), \quad \alpha_+ = \alpha + \beta.$$

Since $\binom{k}{j} \leq k^2 \binom{k-2}{j-2}$ for $j \geq 2$, the norm of the right side is at most $3k^4 \|\beta\|_{Z_q}^2 (\|\alpha\|_{Z_q} + \|\beta\|_{Z_q})^{k-2}$. Multiplying by $e^{-p} p^k / k!$ and summing over k , we find that P_p is complex differentiable at α with derivative $\beta \mapsto e^{-p} \sum_{k \geq 1} p^k k (k!)^{-1} \alpha^{*(k-1)} * \beta = p P_p(\alpha) * \beta$. The m -th derivative is computed in the same way, by induction on m : at each stage the derivative is a series of the same form in α , multilinear in $\beta_1, \dots, \beta_m$, and it converges in W_q uniformly on bounded sets by (12.17). This gives (12.18). The mass formula holds because the mass of a convolution is the product of the masses. $\square$

The law map

For $\tau \in \Omega$ and $\nu \in X_q$, define

$$\tilde{\Phi}_\tau(\nu) = \delta_1 * R P_p(\mathcal{K}_\tau \nu). \quad (12.19)$$

For real $\tau > 0$ and a probability measure $\nu \in X_q$, the measure $\mathcal{K}_\tau \nu$ is the law of $\tau Y / X$ with $X \sim \nu$, and $\tilde{\Phi}_\tau(\nu)$ is the law of $1 - \tau \sum_{j \leq N} Y_j / X_j$ with X_j independent with law ν . This is the law map of the rescaled equation (12.6).

Lemma 12.11. (i) The map $(\tau, \nu) \mapsto \tilde{\Phi}_\tau(\nu)$ is holomorphic from $\Omega \times X_q$ to X_q .

(ii) For $\tau \in \Omega$ and $\nu \in \mathcal{A}_q$, we have $\tilde{\Phi}_\tau(\nu) \in \mathcal{A}_q$, and $\tilde{\Phi}_\tau(\nu) - e^{-p} \delta_1$ is absolutely continuous.

(iii) For $\tau \in \Omega$, $\nu \in X_q$, $m \geq 1$, and $\sigma_1, \dots, \sigma_m \in X_q$,

$$D_\nu^m \tilde{\Phi}_\tau(\nu)[\sigma_1, \dots, \sigma_m] = p^m \tilde{\Phi}_\tau(\nu) * R \mathcal{K}_\tau \sigma_1 * \dots * R \mathcal{K}_\tau \sigma_m. \quad (12.20)$$

(iv) Let $p \geq p_1(\Xi)$ and $\xi \in \Xi'$. Then $\tilde{\nu}_\xi \in \mathcal{A}_q$ and $\tilde{\Phi}_{\tau(\xi)}(\tilde{\nu}_\xi) = \tilde{\nu}_\xi$.

Proof. The map $\mu \mapsto \delta_1 * R \mu$ is a bounded linear map from W_q to X_q : it is the pushforward under $z \mapsto 1 - z$, which maps $[1 - R_0, 1 + R_0]$ onto $[-R_0, R_0]$ and preserves the size of densities, and it at most doubles the weighted total variation by Lemma 12.7. The map $(\tau, \nu) \mapsto \mathcal{K}_\tau \nu$ is holomorphic from $\Omega \times X_q$ to Z_q : by Lemma 12.8, for $h \in \mathbb{C}$ and $\mu \in X_q$,

$$\begin{aligned} & \|\mathcal{K}_{\tau+h}(\nu + \mu) - \mathcal{K}_\tau \nu - h(\partial_\tau \mathcal{K}_\tau) \nu - \mathcal{K}_\tau \mu\|_{Z_q} \\ & \leq \|\mathcal{K}_{\tau+h} - \mathcal{K}_\tau - h \partial_\tau \mathcal{K}_\tau\| \|\nu\|_{X_q} + \|\mathcal{K}_{\tau+h} - \mathcal{K}_\tau\| \|\mu\|_{X_q}, \end{aligned}$$

which is $o(|h| + \|\mu\|_{X_q})$. Composing with the holomorphic map P_p of Lemma 12.10 and with $\delta_1 * R$ proves (i).

For (ii), we have $(\mathcal{K}_\tau \nu)(\mathbb{R}) = 1$ by Lemma 12.8(i), and Lemma 12.10 implies that $P_p(\mathcal{K}_\tau \nu)$ has mass one. The term $k = 0$ of (12.16) gives the atom $e^{-p} \delta_1$ of $\tilde{\Phi}_\tau(\nu)$, and the terms $k \geq 1$ are absolutely continuous.

Part (iii) follows from (12.18), the linearity of $\nu \mapsto \mathcal{K}_\tau \nu$, and the identity $\delta_1 * R(\mu * \beta) = (\delta_1 * R\mu) * R\beta$.

For (iv), Lemma 12.3(i) shows that $\tilde{\nu}_\xi$ is a probability measure with a density bounded by H_p on $[-R_0, R_0] \subset (-3\rho/4, 3\rho/4)$, and Lemma 12.3(ii) shows that its weighted total variation is finite. Then $\tilde{\nu}_\xi \in \mathcal{A}_q$. The fixed point identity is (12.6), by the description of $\tilde{\Phi}_\tau$ for real τ given before the lemma. $\square$

12.3 Invertibility of the linearization

Score polynomials

For real parameters, derivatives of the jump kernel in the input x and in the energy ξ are polynomials in the normalized jump $u = |x||y|/\tau$ times the kernel itself. We use this in the rest of the section.

Lemma 12.12. *For all integers $m, r \geq 0$ there exists a polynomial $P_{m,r}$ of degree at most $m + r$ with integer coefficients such that the following holds. Let $p > 1$, $\xi > 0$, $\tau = \tau(\xi) = (p\xi)^{-2}$, and let $x, y \in \mathbb{R}$ with $xy > 0$. Then, with $u = |x||y|/\tau$,*

$$\partial_x^m \partial_\xi^r k_{\tau(\xi)}(x; y) = x^{-m} \xi^{-r} P_{m,r}(u) k_{\tau(\xi)}(x; y). \quad (12.21)$$

Proof. For fixed $y \neq 0$ and x of the sign of y , we have $k_{\tau(\xi)}(x; y) = G(u)/|y|$ with $G(u) = ue^{-u}$ and $u = |x||y|p^2\xi^2$. Then $\partial_\xi u = 2u/\xi$, $\partial_x u = u/x$, and $G'(u) = (1 - u)G(u)/u$. Suppose that $\partial_x^m \partial_\xi^r k = x^{-m} \xi^{-r} P(u)G(u)/|y|$ for a polynomial P . Then

$$\begin{aligned} \partial_\xi (x^{-m} \xi^{-r} P(u)G(u)) &= x^{-m} \xi^{-r-1} G(u) (-rP(u) + 2uP'(u) + 2(1-u)P(u)), \\ \partial_x (x^{-m} \xi^{-r} P(u)G(u)) &= x^{-m-1} \xi^{-r} G(u) (-mP(u) + uP'(u) + (1-u)P(u)). \end{aligned}$$

Both brackets are polynomials with integer coefficients whose degree exceeds that of P by at most one. Starting from $P_{0,0} = 1$ and using that ∂_x and ∂_ξ commute, we obtain (12.21) by induction on $m + r$. $\square$

Lemma 12.13. *Let $q \in (0, 1)$ and let $m, r \geq 0$ be integers with $m + r \leq 5$. There exists $C > 0$, depending only on m, r, q , and Ξ , such that for all $p > 1$, $\xi \in \Xi'$, $\tau = \tau(\xi)$, and $x \neq 0$:*

- (a) $\int (1 + |y|^q) |\partial_x^m \partial_\xi^r k_\tau(x; y)| dy \leq C|x|^{-m} (1 + (\tau/|x|)^q)$;
- (b) for $t > 0$, $\int_{|y| \geq t} (1 + |y|^q) |\partial_x^m \partial_\xi^r k_\tau(x; y)| dy \leq C|x|^{-m} (1 + (\tau/|x|)^q) e^{-|x|t/(2\tau)}$;
- (c) for $y \neq 0$, $|\partial_x^m \partial_\xi^r k_\tau(x; y)| \leq C|x|^{-m} |y|^{-1} e^{-|x||y|/(2\tau)}$;
- (d) if $m = 0$, then $\int_{\mathbb{R}} |\partial_\xi^r k_\tau(x; y)| dx \leq C\tau y^{-2}$ for $y \neq 0$.

Proof. On the half-line $\{xy > 0\}$ we substitute $u = |x||y|/\tau$, so that $k_\tau(x; y) dy = e^{-u} du$, $|y|^q = (\tau u/|x|)^q$, and $|y| k_\tau(x; y) = ue^{-u}$. By Lemma 12.12 and $\xi^{-r} \leq (2/\xi_-)^r$, parts (a) and (b) follow from

$$\int_{u_0}^{\infty} |P_{m,r}(u)| \left(1 + \left(\frac{\tau u}{|x|}\right)^q\right) e^{-u} du \leq \left(1 + \left(\frac{\tau}{|x|}\right)^q\right) e^{-u_0/2} \int_0^{\infty} |P_{m,r}(u)| (1 + u^q) e^{-u/2} du,$$

with $u_0 = 0$ and $u_0 = |x|t/\tau$, respectively. Part (c) follows from $|P_{m,r}(u)| k_\tau(x; y) = |y|^{-1} |P_{m,r}(u)| ue^{-u} \leq C|y|^{-1} e^{-u/2}$. For (d), we substitute $x = \tau u/y$ for x of the sign of y and obtain $\int |P_{0,r}(u)| k_\tau(x; y) dx = \tau y^{-2} \int_0^{\infty} |P_{0,r}(u)| ue^{-u} du$. $\square$

The linearization at the principal law

For $p \geq p_1(\Xi)$ and $\xi \in \Xi'$, let

$$\tilde{\mathcal{L}}_\xi = D_\nu \tilde{\Phi}_{\tau(\xi)}(\tilde{v}_\xi), \quad \text{so that} \quad \tilde{\mathcal{L}}_\xi \sigma = p \tilde{v}_\xi * \text{RK}_{\tau(\xi)} \sigma \quad (12.22)$$

by Lemma 12.11(iii),(iv). By Lemma 12.8(i), $\tilde{\mathcal{L}}_\xi$ maps X_q^0 into itself. Let $\tilde{D} \sim \tilde{v}_\xi$ and $T \sim \text{Exp}(1)$ be independent. For $\sigma \in X_q$ and bounded Borel φ , by Fubini's theorem we have

$$\int \varphi d\tilde{\mathcal{L}}_\xi \sigma = p \int \mathbb{E} \left[\varphi \left(\tilde{D} - \frac{\tau T}{x} \right) \right] \sigma(dx). \quad (12.23)$$

Further, $\tilde{\mathcal{L}}_\xi \sigma$ is absolutely continuous, with density

$$\frac{d\tilde{\mathcal{L}}_\xi \sigma}{dz}(z) = p \int \bar{k}_{\xi,x}(z) \sigma(dx), \quad \bar{k}_{\xi,x}(z) = \int k_{\tau(\xi)}(x; d - z) \tilde{v}_\xi(dd). \quad (12.24)$$

Here $\bar{k}_{\xi,x}$ is the density of $\tilde{D} - \tau T/x$.

Lemma 12.14. *Let $p \geq p_1(\Xi)$ and $\xi \in \Xi'$, and let $\mathcal{L} = \mathcal{L}_{p,\theta;\nu_{p,E}}$. For every real $\sigma \in X_q^0$, we have $D_{1+\theta} \tilde{\mathcal{L}}_\xi \sigma = \mathcal{L} D_{1+\theta} \sigma$. Consequently, for every $\sigma \in X_q^0$, real or complex, the vector $v_\xi(\sigma) = (\|\sigma\|_{c,\xi}, \|\sigma\|_{b,\xi}, \|\sigma\|_{i,\xi})$ satisfies $v_\xi(\tilde{\mathcal{L}}_\xi \sigma) \leq M_p v_\xi(\sigma)$ entrywise, with M_p as in (12.10).*

Proof. Let φ be bounded and Borel, and set $\sigma^B = D_{1+\theta} \sigma$. Let $D = (1 + \theta)\tilde{D}$, which has law $\nu_{p,E}$. Using (12.23) with the test $x \mapsto \varphi((1 + \theta)x)$ and substituting $b = (1 + \theta)x$, we find

$$\int \varphi dD_{1+\theta} \tilde{\mathcal{L}}_\xi \sigma = p \int \mathbb{E} \left[\varphi \left(D - \frac{(1 + \theta)^2 \tau T}{b} \right) \right] \sigma^B(db) = p \int \mathbb{E} \left[\varphi \left(D - \frac{\theta T}{pb} \right) \right] \sigma^B(db),$$

where we used (12.3) in the second equality. By (12.8) the right side is $\int \varphi d\mathcal{L} \sigma^B$; the measure σ^B is real, has mass zero, does not charge zero, and $\chi_b \sigma^B$ has a bounded density, as noted after Definition 12.4. This proves the identity. Since the components of $\tilde{\mathcal{L}}_\xi \sigma$ in the sense of Definition 12.4 are those of $\mathcal{L} \sigma^B$, the matrix inequality for real σ follows from (12.10). Since $\tilde{\mathcal{L}}_\xi$ maps real measures to real measures, the complex case follows by applying the real case to $\text{Re } \sigma$ and $\text{Im } \sigma$ and adding. $\square$

In the next lemma and below we write $L_q^1(G)$, for a Borel set $G \subset \mathbb{R}$, for the space of real functions on G with norm $\int_G (1 + |z|^q) |F(z)| dz$. We also set

$$q' = \frac{1+q}{2}, \quad a_1 = \frac{(1-q) a_{\text{st}}}{2(1+q)}. \quad (12.25)$$

By Hölder's inequality with exponents q'/q and $q'/(q' - q)$, Lemma 12.3(ii),(iii), and $(1 + |d|^q)^{q'/q} \leq 2^{q'/q} (1 + |d|^{q'})$, we have

$$\int_{\mathcal{S}_\xi^\xi} (1 + |d|^q) \tilde{v}_\xi(d) \leq C \tilde{v}_\xi(\mathcal{S}_\xi^\xi)^{1-q/q'} \leq C e^{-a_1 \sqrt{p}}, \quad (12.26)$$

since $1 - q/q' = (1 - q)/(1 + q)$.

Lemma 12.15. *There exists $C > 0$, depending only on q and Ξ , such that for all $p \geq p_1(\Xi)$, $\xi \in \Xi'$, and $\sigma \in X_q^0$, writing $\|\sigma\|_{\text{cbi}} = \|\sigma\|_{\text{c},\xi} + \|\sigma\|_{\text{b},\xi} + \|\sigma\|_{\text{i},\xi}$:*

- (i) $\|\tilde{\mathcal{L}}_\xi \sigma\|_{q,\text{TV}} \leq Cp \|\sigma\|_{\text{cbi}}$;
- (ii) *the density of $\tilde{\mathcal{L}}_\xi \sigma$ is bounded by $Cp \|\sigma\|_{\text{cbi}}$ on $[-R_0, R_0]$;*
- (iii) $\int_{\mathcal{O}_\xi} (1 + |z|^q) |\tilde{\mathcal{L}}_\xi \sigma|(dz) \leq C p e^{-a_1 \sqrt{p}} (\|\sigma\|_{\text{c},\xi} + \|\sigma\|_{\text{i},\xi}) + Cp \|\sigma\|_{\text{b},\xi}$;
- (iv) *for every $\gamma > 0$, $N_{\xi,\gamma}(\tilde{\mathcal{L}}_\xi \sigma) \leq C p e^{\gamma p} \|\sigma\|_{X_q}$.*

Proof. It suffices to consider real σ . We decompose $\sigma = \Pi_{\text{c}}^\xi \sigma + \Pi_{\text{b}}^\xi \sigma + \Pi_{\text{i}}^\xi \sigma$ as in Lemma 12.5, write $\Pi_{\text{b}}^\xi \sigma = g dx - (\int g) \delta_{x_\xi}$, and treat the three components separately. Throughout, $\tau = \tau(\xi) \leq 4/(p\xi_-)^2$. We use the following consequence of Lemma 12.13(a) for $x \neq 0$ and $s \in \mathbb{R}$:

$$\int (1 + |y|^q) |k_\tau(x; y) - k_\tau(s; y)| dy \leq |x - s| \sup_{w \in [x, s]} \int (1 + |y|^q) |\partial_x k_\tau(w; y)| dy, \quad (12.27)$$

valid when the segment $[x, s]$ does not contain zero, which follows from the fundamental theorem of calculus in x and Tonelli's theorem. The same inequality holds with the weight and the domain of integration restricted to any Borel set of y , and after integrating in d against $\tilde{v}_\xi$ with the shift $y = d - z$.

STEP 1. Weighted total variation. Let $f(x) = k_\tau(x; \cdot) \in L_q^1(\mathbb{R})$. By Lemma 12.13(a), $\|f(x)\| \leq C$ and $\|\partial_x f(x)\| \leq C$ for $|x| \geq 2\rho/9$. By (12.27), f is Lipschitz with constant at most C on $S_{\text{c}} \subset J_0$ and on each of $S_{\text{i}} \cap (0, \infty)$ and $S_{\text{i}} \cap (-\infty, 0)$. If $x, s \in S_{\text{i}}$ have opposite signs, then $|x - s| \geq 4\rho/9$, and therefore $\|f(x) - f(s)\| \leq 2C \leq (9C/(2\rho))|x - s|$. We conclude that $\text{Lip}_{S_{\text{i}}}(f) \leq C$. The density of $\mathcal{K}_\tau \Pi_{\alpha}^\xi \sigma$ is the Bochner integral $\int f d\Pi_{\alpha}^\xi \sigma$, and Lemma 12.5(iv) implies that $\|\mathcal{K}_\tau \Pi_{\text{c}}^\xi \sigma\|_{q,\text{TV}} \leq C \|\sigma\|_{\text{c},\xi}$ and $\|\mathcal{K}_\tau \Pi_{\text{i}}^\xi \sigma\|_{q,\text{TV}} \leq C \|\sigma\|_{\text{i},\xi}$. For the boundary component, Lemma 12.13(a) gives

$$\|\mathcal{K}_\tau \Pi_{\text{b}}^\xi \sigma\|_{q,\text{TV}} \leq \int_{-\rho/2}^{\rho/2} |g(x)| C(1 + |x|^{-q}) dx + C \left| \int g \right| \leq C \|\sigma\|_{\text{b},\xi},$$

since $q < 1$ and $|g| \leq 2\|\sigma\|_{b,\xi}$. By Lemma 12.7 and Lemma 12.3(ii), $\|\tilde{\mathcal{L}}_\xi \sigma\|_{q,\text{TV}} \leq p \|\tilde{v}_\xi\|_{q,\text{TV}} \|\mathcal{K}_\tau \sigma\|_{q,\text{TV}}$, which proves (i).

STEP 2. *Local density.* Fix $|z| \leq R_0$ and split the integral defining $\bar{k}_{\xi,x}(z)$ into $d \in (-3\rho/4, 3\rho/4)$, where $\tilde{v}_\xi$ has density at most H_p , and the complement, where $|d - z| \geq 3\rho/4 - R_0 = \rho/8$. For $|x| \geq 2\rho/9$, Lemma 12.8(i) and Lemma 12.13(a),(c) give

$$\begin{aligned} |\bar{k}_{\xi,x}(z)| + |\partial_x \bar{k}_{\xi,x}(z)| &\leq H_p \int (k_\tau(x;y) + |\partial_x k_\tau(x;y)|) dy + \sup_{|y| \geq \rho/8} (k_\tau(x;y) + |\partial_x k_\tau(x;y)|) \\ &\leq C(H_p + e^{-cp^2}). \end{aligned}$$

As in Step 1, the scalar function $x \mapsto \bar{k}_{\xi,x}(z)$ is then Lipschitz on S_c and bounded and Lipschitz on S_i , with constants at most C . By (12.24) and Lemma 12.5(iv), the core and intermediate components contribute at most $Cp(\|\sigma\|_{c,\xi} + \|\sigma\|_{i,\xi})$ to the density at z . For the boundary component, the same splitting and Lemma 12.13(d) give

$$\int_{-\rho/2}^{\rho/2} |g(x)| \bar{k}_{\xi,x}(z) dx \leq \|g\|_\infty \left(\rho H_p + \sup_{|y| \geq \rho/8} \int_{\mathbb{R}} k_\tau(x;y) dx \right) \leq \|g\|_\infty \left(\rho H_p + \frac{64\tau}{\rho^2} \right),$$

and the compensating atom contributes $|\int g| \bar{k}_{\xi,x_\xi}(z) \leq C\|\sigma\|_{b,\xi}$. This proves (ii).

STEP 3. *Variation away from the core.* Let $f(x) = \bar{k}_{\xi,x} \mathbf{1}_{\mathcal{O}_\xi} \in L_q^1(\mathcal{O}_\xi)$. For $|x| \geq 2\rho/9$, we have

$$\|f(x)\| + \|\partial_x f(x)\| \leq \int \tilde{v}_\xi(dd) \int (k_\tau(x;y) + |\partial_x k_\tau(x;y)|) \mathbf{1}_{\{d-y \in \mathcal{O}_\xi\}} (1 + |d-y|^q) dy.$$

If $d \in \mathcal{S}_\xi$ and $d-y \in \mathcal{O}_\xi$, then $(1+\theta)|y| \geq r_p/2 - r_p/4$, which gives $|y| \geq r_p/5$; further, $|d| \leq 2$ implies that $1 + |d-y|^q \leq 3(1 + |y|^q)$. By Lemma 12.13(b), the part of the integral with $d \in \mathcal{S}_\xi$ is at most $Ce^{-(2\rho/9)r_p/(10\tau)} \leq Ce^{-cp^{7/4}}$. For the part with $d \notin \mathcal{S}_\xi$, we bound $1 + |d-y|^q \leq (1 + |d|^q)(1 + |y|^q)$ and use Lemma 12.13(a) and (12.26), which give at most $Ce^{-a_1\sqrt{p}}$. Arguing as in Step 1, f is Lipschitz on S_c and bounded and Lipschitz on S_i with constants at most $Ce^{-a_1\sqrt{p}}$. By Lemma 12.5(iv) with $V = L_q^1(\mathcal{O}_\xi)$, the core and intermediate components contribute at most $Cpe^{-a_1\sqrt{p}}(\|\sigma\|_{c,\xi} + \|\sigma\|_{i,\xi})$ to the left side of (iii). The boundary component contributes at most $\|\tilde{\mathcal{L}}_\xi \Pi_b^\xi \sigma\|_{q,\text{TV}} \leq Cp\|\sigma\|_{b,\xi}$ by Step 1. This proves (iii).

STEP 4. *Comparison with the norm of X_q .* For real τ we have $\|\mathcal{K}_\tau \sigma\|_{\text{TV}} \leq \|\sigma\|_{\text{TV}}$, and therefore $\|\tilde{\mathcal{L}}_\xi \sigma\|_{\text{TV}} \leq p\|\sigma\|_{\text{TV}}$. Since $\sup_x k_\tau(x;y) = (e|y|)^{-1}$, we have $|\bar{k}_{\xi,x}(z)| \leq H_p + 8/(e\rho)$ for all $x \neq 0$ and $|z| \leq R_0$. Then the density of $\tilde{\mathcal{L}}_\xi \sigma$ on $[-R_0, R_0]$ is at most $Cp\|\sigma\|_{\text{TV}}$. Part (iv) now follows from Lemma 12.5(iii). $\square$

Lemma 12.16. *Let $\gamma = \min\{c_b, c_i\}/4$. There exists $p_2(\Xi) \geq p_1(\Xi)$ such that for all $p \geq p_2(\Xi)$, $\xi \in \Xi'$, and $\sigma \in X_q^0$,*

$$N_{\xi, \gamma}(\tilde{\mathcal{L}}_\xi \sigma) \leq \frac{1}{2} N_{\xi, \gamma}(\sigma). \quad (12.28)$$

Consequently, there exists $C_p < \infty$ such that $\|\tilde{\mathcal{L}}_\xi^n\|_{X_q^0 \rightarrow X_q^0} \leq C_p 2^{-n}$ for all $n \geq 0$, and $I - \tilde{\mathcal{L}}_\xi$ has a bounded inverse on X_q^0 , given by the norm-convergent series $\sum_{n \geq 0} \tilde{\mathcal{L}}_\xi^n$.

Proof. We set $v_\xi(\sigma) = (c, b, i)$ and $v_\xi(\tilde{\mathcal{L}}_\xi \sigma) = (c', b', i')$. By Lemma 12.14,

$$\begin{aligned} N_{\xi, \gamma}(\tilde{\mathcal{L}}_\xi \sigma) &= c' + e^{\gamma p} b' + e^{\gamma \sqrt{p}} i' \\ &\leq (\varepsilon_p c + P_p b + P_p i) + e^{\gamma p} (\delta_{b,p} c + \varepsilon_p b + \delta_{b,p} i) \\ &\quad + e^{\gamma \sqrt{p}} (\delta_{i,p} c + P_p b + (\delta_{i,p} + \delta_{b,p}) i) \\ &= k_c c + k_b e^{\gamma p} b + k_i e^{\gamma \sqrt{p}} i, \end{aligned}$$

where

$$\begin{aligned} k_c &= \varepsilon_p + e^{\gamma p} \delta_{b,p} + e^{\gamma \sqrt{p}} \delta_{i,p}, & k_b &= P_p e^{-\gamma p} + \varepsilon_p + P_p e^{-\gamma(p - \sqrt{p})}, \\ k_i &= P_p e^{-\gamma \sqrt{p}} + \delta_{b,p} e^{\gamma(p - \sqrt{p})} + \delta_{i,p} + \delta_{b,p}. \end{aligned}$$

Since $\gamma \leq c_b/4$ and $\gamma \leq c_i/4$, every term is at most C/p or $p^{A_{\text{col}}} e^{-\gamma \sqrt{p}}$, and $\max\{k_c, k_b, k_i\} \leq 1/2$ for p larger than a threshold depending only on Ξ . This proves (12.28).

By Lemma 12.15(i),(ii) and $N_{\xi, \gamma} \geq \|\cdot\|_{\text{cbi}}$, we have $\|\tilde{\mathcal{L}}_\xi \mu\|_{X_q} \leq C_p N_{\xi, \gamma}(\mu)$ for $\mu \in X_q^0$. For $n \geq 2$, this, (12.28), and Lemma 12.15(iv) give

$$\|\tilde{\mathcal{L}}_\xi^n \sigma\|_{X_q} \leq C_p N_{\xi, \gamma}(\tilde{\mathcal{L}}_\xi^{n-1} \sigma) \leq C_p 2^{-(n-2)} N_{\xi, \gamma}(\tilde{\mathcal{L}}_\xi \sigma) \leq C^2 p^2 e^{\gamma p} 2^{-(n-2)} \|\sigma\|_{X_q}.$$

The cases $n = 0, 1$ follow from the boundedness of $\tilde{\mathcal{L}}_\xi$ on X_q , which holds by Lemma 12.11(i). The Neumann series therefore converges in operator norm on the Banach space X_q^0 , and its sum inverts $I - \tilde{\mathcal{L}}_\xi$. $\square$

Remark 12.17. The mixed seminorm $N_{\xi, \gamma}$ is weaker than $\|\cdot\|_{X_q}$: it measures the core and intermediate parts of a measure only through Lipschitz test functions. The operator $\tilde{\mathcal{L}}_\xi$ is a contraction for the weak seminorm and maps it back into the strong norm, because its kernel is smooth in the input variable away from zero. The proof above uses only these two properties.

12.4 Analyticity of the principal law

We first show that the principal law depends continuously on ξ in the norm of X_q . This is used to identify the analytic branch produced by the implicit function theorem with the principal law.

Lemma 12.18. *Let $p \geq p_1(\Xi)$. Then $\xi \mapsto \tilde{v}_\xi$ is continuous from Ξ' to X_q .*

Proof. Let $\xi_n \rightarrow \xi$ in Ξ' , and write $v_n = \tilde{v}_{\xi_n}$, $v = \tilde{v}_\xi$, $\tau_n = \tau(\xi_n)$, and $\tau_0 = \tau(\xi)$. By Lemma 12.11(iv), $v_n = \delta_1 * \mathbf{R}P_p(\mathcal{K}_{\tau_n}v_n)$ and $v = \delta_1 * \mathbf{R}P_p(\mathcal{K}_{\tau_0}v)$. Since $P_p: Z_q \rightarrow W_q$ is continuous and $\mu \mapsto \delta_1 * \mathbf{R}\mu$ is bounded from W_q to X_q , it suffices to show that $\mathcal{K}_{\tau_n}v_n \rightarrow \mathcal{K}_{\tau_0}v$ in Z_q . We write

$$\mathcal{K}_{\tau_n}v_n - \mathcal{K}_{\tau_0}v = (\mathcal{K}_{\tau_n} - \mathcal{K}_{\tau_0})v_n + \mathcal{K}_{\tau_0}(v_n - v).$$

By Lemma 12.3(i),(ii), $\sup_n \|v_n\|_{X_q} < \infty$, and the first term tends to zero by the continuity of $\tau \mapsto \mathcal{K}_\tau$ in operator norm (Lemma 12.8(ii)).

For the second term, we note that Lemma 12.13(a),(c) with $m \in \{0, 1\}$ and $r = 0$ give

$$\begin{aligned} \|k_\tau(x; \cdot)\|_{Z_q} &\leq C(1 + |x|^{-q}), \\ \|k_\tau(x; \cdot) - k_\tau(s; \cdot)\|_{Z_q} &\leq C \frac{|x - s|}{\min\{|x|, |s|\}} \left(1 + \frac{1}{\min\{|x|, |s|\}^q}\right) \end{aligned} \quad (12.29)$$

for x, s of the same sign, where $\tau = \tau_0$ and we identify $k_\tau(x; \cdot)$ with the measure it is the density of. The second bound follows from the bounds for $\partial_x k_\tau$ and the fundamental theorem of calculus in x . Let $\varepsilon > 0$. Let $0 < \delta < R_0$ and $M \geq 3$, and let $\psi: \mathbb{R} \rightarrow [0, 1]$ be continuous with $\psi = 1$ on $\{\delta \leq |x| \leq M\}$ and $\psi = 0$ on $\{|x| \leq \delta/2\} \cup \{|x| \geq 2M\}$. For $\mu \in \{\nu, \nu_1, \nu_2, \dots\}$, the first bound in (12.29), the density bound of Lemma 12.3(i), and the tail bound (12.7) give

$$\|\mathcal{K}_{\tau_0}((1 - \psi)\mu)\|_{Z_q} \leq C \int_{|x| < \delta} (1 + |x|^{-q}) H_p dx + 2C\mu(|x| > M) \leq CH_p \delta^{1-q} + \frac{C}{M}.$$

We fix δ and M so that the right side is at most ε . The Z_q -valued function $F(x) = \psi(x)k_{\tau_0}(x; \cdot)$ vanishes for $|x| \leq \delta/2$ and $|x| \geq 2M$ and is uniformly continuous by (12.29). Let $r > 0$ be such that $\|F(x) - F(s)\|_{Z_q} \leq \varepsilon$ for $|x - s| < r$, and let $(\phi_i)_{i \leq m}$ be a finite family of continuous functions $\mathbb{R} \rightarrow [0, 1]$, each supported in an interval of length less than r containing a point x_i , with $\sum_i \phi_i = 1$ on the support of F and $\sum_i \phi_i \leq 1$ everywhere. Then $\|F(x) - \sum_i \phi_i(x)F(x_i)\|_{Z_q} \leq \varepsilon$ for every x , and therefore

$$\|\mathcal{K}_{\tau_0}(\psi(v_n - v))\|_{Z_q} = \left\| \int F d(v_n - v) \right\|_{Z_q} \leq 2\varepsilon + \sum_{i=1}^m \|F(x_i)\|_{Z_q} \left| \int \phi_i d(v_n - v) \right|.$$

The last sum tends to zero by the weak convergence $v_n \rightarrow v$ (Lemma 12.3(iv)). We conclude that $\limsup_n \|\mathcal{K}_{\tau_0}(v_n - v)\|_{Z_q} \leq 4\varepsilon$, and ε was arbitrary. $\square$

Theorem 12.19. *Let $\Xi \subset (0, \infty)$ be a compact interval. There exists $p_{\text{an}}(\Xi)$ such that for every $q \in (0, 1)$ and $p \geq p_{\text{an}}(\Xi)$ the following holds, with $R_0 = R_0(\Xi) \in (0, 1/2)$ and X_q as in Definition 12.6.*

- (i) For every $\xi \in \Xi'$, the principal law $\tilde{v}_{p,\xi\sqrt{p}}$ in the scaling $g = 1/E$ belongs to X_q .
- (ii) There exists an open interval $I \supset \Xi$ such that $\xi \mapsto \tilde{v}_{p,\xi\sqrt{p}}$ is real-analytic from I to X_q . Equivalently, $E \mapsto \tilde{v}_{p,E}$ is real-analytic on $\sqrt{p}I$, and $\tau \mapsto \tilde{v}_{p,(p\tau)^{-1/2}}$ is real-analytic on $\tau(I)$.
- (iii) For every $\xi_0 \in \Xi$, there exist an open disc $U \subset \Omega$ centered at $\tau_0 = \tau(\xi_0)$ and a holomorphic map $v(\cdot): U \rightarrow \mathcal{A}_q$ such that $\tilde{\Phi}_\tau(v(\tau)) = v(\tau)$ for $\tau \in U$, and $v(\tau(\xi)) = \tilde{v}_{p,\xi\sqrt{p}}$ whenever $\xi > 0$ and $\tau(\xi) \in U$.

Proof. Let $p_{\text{an}}(\Xi) = p_2(\Xi)$, which does not depend on q . Part (i) is Lemma 12.11(iv). Fix $\xi_0 \in \Xi$, set $\tau_0 = \tau(\xi_0)$ and $v_0 = \tilde{v}_{\xi_0}$, and define

$$F: \Omega \times X_q^0 \rightarrow X_q^0, \quad F(\tau, \sigma) = \sigma - \tilde{\Phi}_\tau(v_0 + \sigma) + v_0.$$

Since $\tilde{\Phi}_\tau(v_0 + \sigma)$ has mass one by Lemma 12.11(ii), F takes values in X_q^0 . By Lemma 12.11(i), F is holomorphic. Its partial derivatives are

$$D_\sigma F(\tau, \sigma) = I - D_v \tilde{\Phi}_\tau(v_0 + \sigma), \quad D_\tau F(\tau, \sigma)h = -h p \tilde{\Phi}_\tau(v_0 + \sigma) * R(\partial_\tau \mathcal{K}_\tau)(v_0 + \sigma),$$

where the second formula follows from (12.18) with $m = 1$ and the chain rule. These are continuous functions of (τ, σ) in operator norm: by Lemma 12.10, the map $\alpha \mapsto DP_p(\alpha)$ has a derivative that is bounded on bounded sets, which implies that it is locally Lipschitz in operator norm, and $\tau \mapsto \mathcal{K}_\tau$ and $\tau \mapsto \partial_\tau \mathcal{K}_\tau$ are continuous in operator norm by Lemma 12.8(ii). We conclude that F is continuously Fréchet differentiable when $\Omega \subset \mathbb{C}$ is regarded as an open subset of $\mathbb{R}^2$. Further, $F(\tau_0, 0) = 0$ by Lemma 12.11(iv), and $D_\sigma F(\tau_0, 0) = I - \tilde{\mathcal{L}}_{\xi_0}$ is invertible on X_q^0 by Lemma 12.16.

By the implicit function theorem in Banach spaces [60, Section 15], there exist an open disc $U \subset \Omega$ centered at τ_0 , an open neighborhood V of 0 in X_q^0 , and a continuously differentiable map $\sigma: U \rightarrow V$ such that, for $(\tau, \sigma') \in U \times V$, we have $F(\tau, \sigma') = 0$ if and only if $\sigma' = \sigma(\tau)$. The uniqueness in this pointwise form is part of the standard proof, and it can be checked directly. Let $A = D_\sigma F(\tau_0, 0)$. By the continuity of $D_\sigma F$, after replacing V by a small ball around 0 and then shrinking U so that $\sigma(U) \subset V$, we have $\|I - A^{-1}D_\sigma F(\tau, \sigma')\| \leq 1/2$ on $U \times V$. By the mean value inequality, $\sigma' \mapsto \sigma' - A^{-1}F(\tau, \sigma')$ then has Lipschitz constant at most 1/2 on V for each $\tau \in U$, so it has at most one fixed point in V , and the zeros of $F(\tau, \cdot)$ in V are exactly its fixed points. Shrinking U , we may assume that $D_\sigma F(\tau, \sigma(\tau))$ is invertible for $\tau \in U$, since the invertible operators form an open set. Differentiating the identity $F(\tau, \sigma(\tau)) = 0$, we obtain

$$D\sigma(\tau)h = -(D_\sigma F(\tau, \sigma(\tau)))^{-1} D_\tau F(\tau, \sigma(\tau))h, \quad h \in \mathbb{C}.$$

The right side is complex-linear in h , so σ is complex differentiable on U , that is, holomorphic. We set $v(\tau) = v_0 + \sigma(\tau)$, which lies in $\mathcal{A}_q$ and satisfies $\tilde{\Phi}_\tau(v(\tau)) = v(\tau)$.

Since $\Xi \subset \text{int } \Xi'$, Lemma 12.18 and the continuity of $\xi \mapsto \tau(\xi)$ give $\eta > 0$ such that, for all real ξ with $|\xi - \xi_0| < \eta$, we have $\xi \in \Xi'$, $\tilde{\nu}_\xi - \nu_0 \in V$, and $\tau(\xi) \in U$. For such ξ , we have $F(\tau(\xi), \tilde{\nu}_\xi - \nu_0) = 0$ by Lemma 12.11(iv), and the uniqueness in the implicit function theorem gives $\tilde{\nu}_\xi = \nu(\tau(\xi))$. This proves (iii), after shrinking U so that its real points are of the form $\tau(\xi)$ with $|\xi - \xi_0| < \eta$.

By the power series expansion recalled at the beginning of Section 12.2, $\tau \mapsto \nu(\tau)$ is real-analytic on $U \cap \mathbb{R}$. Since $\xi \mapsto \tau(\xi) = (p\xi)^{-2}$ extends holomorphically to a complex neighborhood of $(0, \infty)$, the map $\xi \mapsto \nu(\tau(\xi)) = \tilde{\nu}_\xi$ extends holomorphically to a complex neighborhood of $(\xi_0 - \eta, \xi_0 + \eta)$, and it is real-analytic there. The union of these intervals over $\xi_0 \in \Xi$ is an open set containing the compact interval Ξ , and it contains an open interval $I \supset \Xi$. This proves (ii); the reformulations in $E = \xi\sqrt{p}$ and in τ hold because $\xi \mapsto \xi\sqrt{p}$ and $\xi \mapsto \tau(\xi)$ are real-analytic with real-analytic inverses on $(0, \infty)$. $\square$

Remark 12.20. The constants in the proof of Theorem 12.19 depend on p , for instance through the factor $e^{\gamma p}$ in Lemma 12.16, and the theorem gives no bound on the derivatives that is uniform in p . Such bounds are proved in Section 12.6. By Theorem 12.19(ii), the derivatives $\partial_\xi^j \tilde{\nu}_{p, \xi\sqrt{p}}$ exist in X_q for all $j \geq 1$; they are real measures of total mass zero, since $\tilde{\nu}_{p, \xi\sqrt{p}}$ is a probability measure for every real ξ , and they have no atoms, since the atom $e^{-p}\delta_1$ does not depend on ξ .

12.5 Analyticity of the free energy at $s = 1$

The transfer operator in the scaling $g = 1/E$

We use the transfer matrices $M(b, \beta)$, the vectors v_x , the projective maps $f_{b, \beta}$, and the transfer operators $\mathcal{Q}_s$ on $C(\mathbb{RP}^1)$ of Definition 6.1. Let $p \geq p_1(\Xi)$, $\xi \in \Xi'$, and $E = \xi\sqrt{p}$. By Corollary 11.23, the law π of $D = B/g(E)$, where $B \sim \nu_{p, E}$, is a real solution at E . With the scaling $g = g(E)$, the law of $gD = B$ is $\nu_{p, E}$ and the rate is $K_g = p/\theta$; this is the B -normalization, and we write $\mathcal{Q}_{E, 1}$ for the corresponding critical transfer operator. With the scaling $g' = 1/E$, the law of $g'D = \tilde{B}$ is $\tilde{\nu}_\xi$ and the rate is $K_{g'} = pE^2 = 1/\tau(\xi)$. We denote the corresponding critical transfer operator by $\tilde{\mathcal{Q}}_\xi$:

$$(\tilde{\mathcal{Q}}_\xi F)(x) = p \int \tilde{\nu}_\xi(d\mathbf{d}) \int_0^\infty \frac{1}{\tau} e^{-\beta/\tau} \|M(d, \beta)v_x\|^{-1} F(f_{d, \beta}(x)) d\beta, \quad \tau = \tau(\xi). \quad (12.30)$$

All dependence on ξ is in the law $\tilde{\nu}_\xi$ and in the exponential density; the function F is evaluated at real projective points. Since $g'/g = (Eg)^{-1} = (1 + \theta)^{-1}$, we obtain from Lemma 6.13(ii) that

$$\begin{aligned} \tilde{\mathcal{Q}}_\xi &= \Theta_\xi^{-1} \mathcal{Q}_{E, 1} \Theta_\xi, & (\Theta_\xi F)(y) &= \|S_\xi v_y\|^{-1} F(S_\xi \cdot y), \\ S_\xi &= \text{diag}((1 + \theta)^{1/2}, (1 + \theta)^{-1/2}), \end{aligned} \quad (12.31)$$

where Θ_{ξ} is a positive bounded operator on $C(\mathbb{RP}^1)$ with positive bounded inverse. At the level of matrices, this is the constant diagonal conjugacy $M(B, \theta Y/p) = S_{\xi}^{-1} M(\tilde{B}, \tau Y) S_{\xi}$, which follows from (6.17) with $\gamma = 1 + \theta$ and (12.3). In particular, the two operators have the same spectrum and the same Perron–Frobenius eigenvalue, and by Definition 6.14 this eigenvalue is the exponential of the free energy at $E = \xi\sqrt{p}$ and $s = 1$, which we denote by $\lambda_p(\xi)$.

The two scalings differ in one respect that matters here. In the B -normalization the law $\nu_{p,E}$ has its atom at $1 + \theta(\xi)$, which moves with ξ , and this atom produces an atom in the row of $\mathcal{Q}_{E,1}$ at $x = 0$: since $\|M(b, \beta)v_0\|^{-1} = \beta^{1/2}(1 + b^2)^{-1/2}$ and $f_{b,\beta}(0) = 1/b$, the measure $F \mapsto (\mathcal{Q}_{E,1}F)(0)$ has an atom at $1/(1 + \theta(\xi))$ of mass $pe^{-p}\mathbb{E}[\beta^{1/2}](1 + (1 + \theta)^2)^{-1/2}$, and it has no atom at $1/(1 + \theta(\xi'))$ for $\xi' \neq \xi$. Since $\xi \mapsto \theta(\xi)$ is strictly monotone, $\|\mathcal{Q}_{\xi\sqrt{p},1} - \mathcal{Q}_{\xi'\sqrt{p},1}\|$ is bounded below by the mass of this atom, and $\xi \mapsto \mathcal{Q}_{\xi\sqrt{p},1}$ is not norm-continuous; by Proposition 6.12, its third power is norm-continuous (see also Remark 6.8). In the scaling $g = 1/E$, the atom of $\tilde{\nu}_{\xi}$ is at 1 for every ξ , and we show that $\xi \mapsto \tilde{\mathcal{Q}}_{\xi}$ itself is real-analytic in operator norm.

Analyticity of the critical growth rate

Proposition 12.21. *Let $\Xi \subset (0, \infty)$ be compact, and let $\hat{\Xi}$ denote its convex hull. There exists $p_{\text{cr}}(\Xi)$ such that for every $p \geq p_{\text{cr}}(\Xi)$ the following holds.*

- (i) *For every $\xi \in \hat{\Xi}$, $\lambda_p(\xi)$ is the Perron–Frobenius eigenvalue of the critical transfer operator $\tilde{\mathcal{Q}}_{\xi}$ in the scaling $g = 1/E$, and of the critical transfer operator $\mathcal{Q}_{\xi\sqrt{p},1}$ in the B -normalization.*
- (ii) *The maps $\xi \mapsto \tilde{\mathcal{Q}}_{\xi}$ and $\xi \mapsto \mathcal{Q}_{\xi\sqrt{p},1}^3$ are continuous on $\hat{\Xi}$ in the operator norm of $C(\mathbb{RP}^1)$. Moreover, there exists an open interval $I \supset \hat{\Xi}$ such that $\xi \mapsto \tilde{\mathcal{Q}}_{\xi}$ is real-analytic from I to the bounded operators on $C(\mathbb{RP}^1)$ with the operator norm.*
- (iii) *The function $\xi \mapsto \lambda_p(\xi)$ is real-analytic on I . In particular, it is continuous on Ξ and real-analytic on the interior of Ξ .*

Proof. We may assume that Ξ is an interval, and we set $p_{\text{cr}}(\Xi) = p_{\text{an}}(\Xi)$. We fix $q = 3/4$ and apply Theorem 12.19 with this q . Let $p \geq p_{\text{cr}}(\Xi)$.

STEP 1. Identification of λ_p and continuity. Part (i) for $\xi \in \Xi'$ is the discussion after (12.30); the hypotheses (6.1) hold for $\nu_{p,E}$ and $\tilde{\nu}_{\xi}$ by Lemma 6.13(i), since both are rescalings of the real solution π . Next we apply Proposition 6.12 on the interval of energies $\sqrt{p}\Xi'$, with $\nu_E = \nu_{p,E}$ and $K(E) = p/\theta$. The map $E \mapsto K(E)$ is continuous by Lemma 12.1, $E \mapsto \nu_{p,E}$ is weakly continuous by (S1), and by the tail bound in (12.7), we have $\sup_{\xi \in \Xi'} \int |b|^{3/4} \nu_{p,\xi\sqrt{p}}(db) < \infty$, as in the proof of Lemma 12.3(ii). Since $q_0 = 3/4 > 1/2$, Proposition 6.12 shows that $\xi \mapsto \mathcal{Q}_{\xi\sqrt{p},1}^3$ is norm-continuous and $\xi \mapsto \lambda_p(\xi)$ is continuous on Ξ' .

STEP 2. *Complex extension of the operator.* Fix $\xi_0 \in \Xi$, let $\tau_0 = \tau(\xi_0)$, and let U and $\tau \mapsto \nu(\tau)$ be as in Theorem 12.19(iii). Shrinking U , we may assume that its radius is at most $\tau_0/2$. Then for $\tau \in U$, the number $a(\tau) = \operatorname{Re}(1/\tau) = \operatorname{Re} \tau/|\tau|^2$ lies in $[2/(9\tau_0), 2/\tau_0]$, and $|\tau^{-1}e^{-\beta/\tau}| = (|\tau|a(\tau))^{-1}a(\tau)e^{-a(\tau)\beta}$. For a complex measure $\mu \in \mathcal{M}_q$, $\tau \in U$, and a bounded Borel function F on $\mathbb{RP}^1$, define

$$(\mathcal{Q}(\mu, \tau)F)(x) = p \int \mu(d\beta) \int_0^\infty \frac{1}{\tau} e^{-\beta/\tau} \|M(d, \beta)v_x\|^{-1} F(f_{d,\beta}(x)) d\beta.$$

By Lemma 6.5 with $s = 1$ and $K = a(\tau)$, whose constant $C_{1,K}$ is bounded for K in a compact subset of $(0, \infty)$, and by $|d|^{1/2} \leq 1 + |d|^q$, this is a bounded operator on the space $\mathcal{B}(\mathbb{RP}^1)$ of bounded Borel functions with the supremum norm, and $\|\mathcal{Q}(\mu, \tau)\| \leq C\|\mu\|_{q,\text{TV}}$ for $\tau \in U$, where C depends on p and τ_0 .

We claim that $\tau \mapsto \mathcal{Q}(\mu, \tau)$ is holomorphic on U in the operator norm of $\mathcal{B}(\mathbb{RP}^1)$, with derivative given by the same formula with $\partial_\tau(\tau^{-1}e^{-\beta/\tau})$ in place of $\tau^{-1}e^{-\beta/\tau}$, and that the first-order Taylor remainder at τ in the direction h has norm at most $C|h|^2\|\mu\|_{q,\text{TV}}$. To see this, write $\kappa = 1/\tau$ and $e_\kappa(\beta) = \kappa e^{-\kappa\beta} = \tau^{-1}e^{-\beta/\tau}$. The image of U under $\tau \mapsto 1/\tau$ is a disc, on which $\operatorname{Re} \kappa \geq 2/(9\tau_0)$ and $|\kappa| \leq 2/\tau_0$. Then $|\partial_\kappa^2 e_\kappa(\beta)| = |(-2\beta + \kappa\beta^2)e^{-\kappa\beta}| \leq C a_* e^{-a_*\beta}$ with $a_* = 1/(9\tau_0)$. Applying Lemma 6.5 with $K = a_*$ to the remainder $e_{\kappa+h} - e_\kappa - h\partial_\kappa e_\kappa = h^2 \int_0^1 (1-t) \partial_\kappa^2 e_{\kappa+th} dt$ proves the claim, as in the proof of Lemma 12.8. Since $\mathcal{Q}(\mu, \tau)$ is linear in μ with $\|\mathcal{Q}(\mu, \tau)\| \leq C\|\mu\|_{q,\text{TV}}$, the map $(\mu, \tau) \mapsto \mathcal{Q}(\mu, \tau)$ is holomorphic on $\mathcal{M}_q \times U$: the first-order remainder at (μ, τ) in the direction (μ', h) is $\mathcal{Q}(\mu', \tau + h) - \mathcal{Q}(\mu', \tau)$ plus the Taylor remainder in τ at μ , whose norms are $O(|h|\|\mu'\|_{q,\text{TV}})$ and $O(|h|^2)$. By Theorem 12.19(iii) and the inequality $\|\cdot\|_{q,\text{TV}} \leq \|\cdot\|_{X_q}$, the map $\tau \mapsto \mathcal{Q}(\tau) = \mathcal{Q}(\nu(\tau), \tau)$ is holomorphic on U .

Next, $\mathcal{Q}(\tau)$ maps $C(\mathbb{RP}^1)$ into itself. Let $F \in C(\mathbb{RP}^1)$ and $x_n \rightarrow x$ in $\mathbb{RP}^1$. Consider the finite positive measure $m_\tau(d\beta) = |\nu(\tau)|(\ddot{d}\beta) |\tau^{-1}e^{-\beta/\tau}| d\beta$ and the functions $g_n(d, \beta) = \|M(d, \beta)v_{x_n}\|^{-1} F(f_{d,\beta}(x_n))$. For fixed (d, β) with $\beta > 0$, these converge to the same expression $g(d, \beta)$ at x , by the continuity noted after (6.3). By Lemma 6.5 with $s = 5/4$, we have $\int |g_n|^{5/4} dm_\tau \leq C\|F\|_\infty^{5/4} \int (1 + |d|^{5/8}) |\nu(\tau)|(\ddot{d}\beta)$, which is finite and independent of n because $5/8 < q$. Then (g_n) is uniformly integrable with respect to m_τ . By Vitali's convergence theorem, $\int |g_n - g| dm_\tau \rightarrow 0$, which implies that $(\mathcal{Q}(\tau)F)(x_n) \rightarrow (\mathcal{Q}(\tau)F)(x)$. Since $C(\mathbb{RP}^1)$ is a closed subspace of $\mathcal{B}(\mathbb{RP}^1)$ and the difference quotients of $\tau \mapsto \mathcal{Q}(\tau)$ map $C(\mathbb{RP}^1)$ into itself, the restrictions to $C(\mathbb{RP}^1)$ form a holomorphic family of bounded operators on $C(\mathbb{RP}^1)$. For real $\tau = \tau(\xi) \in U$, the identity $\nu(\tau) = \tilde{\nu}_\xi$ gives $\mathcal{Q}(\tau(\xi)) = \tilde{\mathcal{Q}}_\xi$. As in the proof of Theorem 12.19, $\xi \mapsto \tilde{\mathcal{Q}}_\xi$ is real-analytic in operator norm near ξ_0 , and taking the union over $\xi_0 \in \Xi$ proves (ii).

STEP 3. *Perturbation of the Perron–Frobenius eigenvalue.* Let $T_0 = \mathcal{Q}(\tau_0) = \tilde{\mathcal{Q}}_{\xi_0}$ and $\lambda_0 = \lambda_p(\xi_0)$. By Theorem 6.11(ii), applied to $\tilde{\nu}_{\xi_0}$ and the rate $1/\tau_0$, the point $\lambda_0 > 0$ is an eigenvalue of T_0 whose Riesz projection Π_0 has rank one, it has an eigenfunction ψ_1 with

$\min \psi_1 > 0$, and there exists $r_0 < \lambda_0$ with $|z| \leq r_0$ for every $z \in \sigma(T_0) \setminus \{\lambda_0\}$. Since ψ_1 is an eigenfunction, $\Pi_0 \psi_1 = \psi_1$. Let ℓ be the functional $F \mapsto \int F d\ell_1$, where ℓ_1 is the measure of Theorem 6.11(iii), so that $\ell(\Pi_0 \psi_1) = 1$. Fix $\delta \in (0, (\lambda_0 - r_0)/3)$, and let $\varepsilon > 0$ be given by Lemma 6.10 for T_0 , $\mu_0 = \lambda_0$, r_0 , δ , $\psi = \psi_1$, and ℓ . By Step 2, there exists an open disc $U_1 \subset U$ centered at τ_0 with $\|\mathcal{Q}(\tau) - T_0\| < \varepsilon$ for $\tau \in U_1$. By Lemma 6.10(iii), the function $\tau \mapsto \mu(\mathcal{Q}(\tau))$ is holomorphic on U_1 .

Let $\xi > 0$ with $\tau(\xi) \in U_1$. By Step 1 and Theorem 6.11(ii), $\lambda_p(\xi)$ is the spectral radius of $\mathcal{Q}(\tau(\xi)) = \tilde{\mathcal{Q}}_\xi$ and belongs to its spectrum. By Lemma 6.10(i), $\mu(\mathcal{Q}(\tau(\xi)))$ is the unique point of maximal modulus in this spectrum, and we conclude that $\lambda_p(\xi) = \mu(\mathcal{Q}(\tau(\xi)))$. Then λ_p coincides near ξ_0 with the composition of the holomorphic function $\tau \mapsto \mu(\mathcal{Q}(\tau))$ and the real-analytic function $\xi \mapsto \tau(\xi)$, and it is real-analytic near ξ_0 . The union over $\xi_0 \in \Xi$ of these neighborhoods is an open set containing the compact interval Ξ , and it contains an open interval $I' \supset \Xi$. Replacing I by $I \cap I'$, which does not affect (ii), we obtain (iii); continuity on Ξ also follows from Step 1. $\square$

Remark 12.22. By Proposition 12.21(iii), if $\lambda_p - 1$ does not vanish identically on the interval I , then the equation $\lambda_p(\xi) = 1$ has only finitely many solutions in Ξ . Section 14 proves the stronger statement that it has exactly one solution in a suitable window when p is large.

12.6 Quantitative derivative bounds

Statements

In this section, we bound the derivatives $\sigma_j = \partial_\xi^j \tilde{v}_\xi$ for $1 \leq j \leq 4$, which exist in X_q by Theorem 12.19. The constants in these bounds are explicit in p . We proceed by induction on j , as follows. Differentiating the fixed point equation $\tilde{v}_\xi = \tilde{\Phi}_{\tau(\xi)}(\tilde{v}_\xi)$ j times, we obtain $(I - \tilde{\mathcal{L}}_\xi)\sigma_j = R_j$ (Lemma 12.26). The source R_j is a sum of convolutions of the principal law with at most j jump measures. Each jump measure is a derivative of the jump kernel, integrated against the principal law or against a lower-order derivative. The derivatives fall only on the exponential jump kernels, never on the principal law, and they produce the polynomial factors of Lemma 12.12. We bound R_j in three strong norms: the weighted total variation, the density near zero, and the weighted variation away from the core (Lemma 12.27). The last two bounds are exponentially small. Indeed, near zero the principal law has an exponentially small density, and an output that leaves the core from the strict core requires a large jump. Inverting the component matrix of (S4), we obtain bounds on the components of σ_j (Lemma 12.28), and the identity $\sigma_j = R_j + \tilde{\mathcal{L}}_\xi \sigma_j$ and Lemma 12.15 then give the strong norms of σ_j . Finally, we show that the core parts of σ_1 and σ_2 are of order p^{-1} when tested against Lipschitz and C^2 functions, respectively.

The reason is that each derivative of the kernel in ξ or in the input variable costs a factor $|\partial_{\xi}\tau| \leq Cp^{-2}$ or $\tau \leq Cp^{-2}$, while the prefactor is p .

We fix a function $\eta \in C^2(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $[-1/2, 1/2]$, and $\text{supp } \eta \subset [-1, 1]$, and set

$$\hat{\chi}_{\xi}(x) = \eta\left(\frac{(1+\theta)x-1}{r_p}\right). \quad (12.32)$$

Then $\hat{\chi}_{\xi} = 1$ on $\mathbb{R} \setminus \mathcal{O}_{\xi}$, the support of $\hat{\chi}_{\xi}$ is contained in $\{|(1+\theta)x-1| \leq r_p\}$, which lies in the interior of J_0 , and $|\hat{\chi}_{\xi}^{(k)}| \leq Cr_p^{-k}$ for $k = 1, 2$.

Definition 12.23. For a finite real measure σ on $\mathbb{R}$, the *smooth core norm* is

$$n_{\xi}(\sigma) = \sup\left\{\left|\int \hat{\chi}_{\xi}\varphi \, d\sigma\right| : \varphi \in C^2(J_0), \varphi(x_{\xi}) = 0, \|\varphi'\|_{L^{\infty}(J_0)} + \|\varphi''\|_{L^{\infty}(J_0)} \leq 1\right\},$$

where $\hat{\chi}_{\xi}\varphi$ is extended by zero outside J_0 .

Proposition 12.24. Let $\Xi \subset (0, \infty)$ be a compact interval and $q \in (0, 1)$. There exist constants $A, a, C > 0$ and $p_{\text{der}}(\Xi, q) \geq p_{\text{an}}(\Xi)$, depending only on Ξ and q , such that for all $p \geq p_{\text{der}}(\Xi, q)$ and $\xi \in \Xi$ the following holds. For $j \in \{1, 2, 3, 4\}$, the derivative $\sigma_j = \partial_{\xi}^j \tilde{\nu}_{p, \xi, \sqrt{p}} \in X_q^0$ satisfies:

- (i) $\|\sigma_j\|_{q, \text{TV}} \leq p^A$;
- (ii) $\|\sigma_j\|_{b, \xi} \leq p^A e^{-ap}$, and the density of σ_j on $\mathcal{Z}$ is bounded by $p^A e^{-ap}$;
- (iii) $\|\sigma_j\|_{i, \xi} \leq p^A e^{-a\sqrt{p}}$;
- (iv) $\int_{\mathcal{O}_{\xi}} (1 + |x|^q) |\sigma_j|(\,dx) \leq p^A e^{-a\sqrt{p}}$.

Further,

- (v) $\|\sigma_1\|_{c, \xi} \leq C/p$;
- (vi) $n_{\xi}(\sigma_2) \leq C/p$;
- (vii) $\int_{\mathcal{O}_{\xi}} (1 + |x|^q) \tilde{\nu}_{\xi}(\,dx) \leq C e^{-a\sqrt{p}}$.

We choose the constants in the following order. We first fix Ξ and q . The constants of (S1)–(S4) for the window Ξ' are then given, and they depend only on Ξ . Next, we set $a = \min\{a_1, c_b, c_i\}$, with a_1 from (12.25), and $A = A_4$, where

$$A_0 = 0, \quad A_j = 4A_{j-1} + A_{\text{st}} + 3A_{\text{col}} + 13 \quad (1 \leq j \leq 4); \quad (12.33)$$

Finally, we choose p_{der} large depending on all of the above. In particular, A depends only on A_{st} and A_{col} , and a depends only on q , a_{st} , c_b , and c_i .

Corollary 12.25. Let $\Xi \subset (0, \infty)$ be a compact interval and let $J \subset \mathbb{R}$ be compact. There exist $A, a, C > 0$ and $p_{\text{test}}(\Xi, J)$ such that for all $p \geq p_{\text{test}}(\Xi, J)$, $\xi \in \Xi$, $j \in \{1, 2\}$, and $\varphi \in C^2(\mathbb{R})$ supported in J ,

$$|\partial_{\xi}^j \mathbb{E}\varphi(B)| \leq \left(\frac{C}{p} + p^A e^{-a\sqrt{p}}\right) \|\varphi\|_{C^2}, \quad \|\varphi\|_{C^2} = \|\varphi\|_{\infty} + \|\varphi'\|_{\infty} + \|\varphi''\|_{\infty}, \quad (12.34)$$

where $B \sim \nu_{p,\xi}\sqrt{p}$, so that $B = (1 + \theta)\tilde{B}$ with $\tilde{B} \sim \tilde{\nu}_{p,\xi}\sqrt{p}$. More generally, let $I_0 \supset \Xi$ be an open interval and let $\varphi: I_0 \times \mathbb{R} \rightarrow \mathbb{R}$ be such that $\xi \mapsto \varphi(\xi, \cdot)$ is twice continuously differentiable from I_0 to the space of C^2 functions supported in J with the norm $\|\cdot\|_{C^2}$. Then, for $j \in \{1, 2\}$,

$$\left| \partial_\xi^j \mathbb{E}\varphi(\xi, B) - \mathbb{E}[(\partial_\xi^j \varphi)(\xi, B)] \right| \leq \left(\frac{C}{p} + p^A e^{-a\sqrt{p}} \right) \sum_{i=0}^{j-1} \binom{j}{i} \|\partial_\xi^i \varphi(\xi, \cdot)\|_{C^2}. \quad (12.35)$$

We note that the support of the test function in Corollary 12.25 may be much larger than the core region $\{|b - 1| \leq 2r_p\}$. For instance, it may be a neighborhood of 1 of radius $1/\log p$, or any fixed compact set. We control the part of the test function outside the core with the weighted variation bound in Proposition 12.24(iv).

The differentiated fixed point equation

For $r \geq 0$ we write $\mathcal{K}_\xi^{(r)}$ for the r -th derivative of $\xi \mapsto \mathcal{K}_{\tau(\xi)}$ as a map into the bounded operators from X_q to Z_q ; by Lemma 12.8(ii) and the chain rule, it is the integral operator with kernel $\partial_\xi^r k_{\tau(\xi)}(x; y)$, and $\mathcal{K}_\xi^{(0)} = \mathcal{K}_{\tau(\xi)}$.

Lemma 12.26. *Let $p \geq p_{\text{an}}(\Xi)$, $q \in (0, 1)$, and $\xi \in \Xi$, and write $\nu = \tilde{\nu}_\xi$ and $\sigma_j = \partial_\xi^j \tilde{\nu}_\xi$. For every $j \geq 1$,*

$$(I - \tilde{\mathcal{L}}_\xi) \sigma_j = R_j, \quad (12.36)$$

where R_j is a finite sum, over an index set depending only on j , of terms of the form

$$c p^n \nu * \text{RK}_\xi^{(r_1)} \mu_1 * \cdots * \text{RK}_\xi^{(r_n)} \mu_n, \quad (12.37)$$

in which c is a positive integer depending only on j and on the term, $1 \leq n \leq j$, $r_1 + \cdots + r_n \leq j$, and each μ_i belongs to $\{\nu, \sigma_1, \dots, \sigma_{j-1}\}$. In particular,

$$\begin{aligned} R_1 &= p \nu * \text{RK}_\xi^{(1)} \nu, \\ R_2 &= p \nu * \text{RK}_\xi^{(2)} \nu + p^2 \nu * \text{RK}_\xi^{(1)} \nu * \text{RK}_\xi^{(1)} \nu + 2p \nu * \text{RK}_\xi^{(1)} \sigma_1 \\ &\quad + 2p^2 \nu * \text{RK}_\xi^{(1)} \nu * \text{RK}_\xi^{(0)} \sigma_1 + p^2 \nu * \text{RK}_\xi^{(0)} \sigma_1 * \text{RK}_\xi^{(0)} \sigma_1. \end{aligned}$$

Proof. Let I be the interval of Theorem 12.19(ii) and set $\mathcal{F}(\xi, \mu) = \tilde{\Phi}_{\tau(\xi)}(\mu)$ for $\xi \in I$ and $\mu \in X_q$. The map $\mathcal{F}$ is infinitely differentiable, as the composition of the infinitely differentiable maps $\xi \mapsto \mathcal{K}_{\tau(\xi)}$ (Lemma 12.8(ii)), $(\mathcal{K}, \mu) \mapsto \mathcal{K}\mu$ (bounded and bilinear), P_p (Lemma 12.10), and the bounded linear map $\mu \mapsto \delta_1 * R\mu$. The map $\nu(\xi) = \tilde{\nu}_\xi$ is infinitely differentiable from I to X_q by Theorem 12.19(ii), and $\nu(\xi) = \mathcal{F}(\xi, \nu(\xi))$ by Lemma 12.11(iv). We claim that for $j \geq 1$,

$$\partial_\xi^j [\mathcal{F}(\xi, \nu(\xi))] = \sum c_{r,m,j_1,\dots,j_m} \partial_\xi^r D_\mu^m \mathcal{F}(\xi, \nu(\xi)) [\nu^{(j_1)}, \dots, \nu^{(j_m)}], \quad (12.38)$$

where the sum is over $r, m \geq 0$ and $j_1, \dots, j_m \geq 1$ with $r + j_1 + \dots + j_m = j$, the coefficients are nonnegative integers, and the only term containing $\nu^{(j)}$ is the term $(r, m, j_1) = (0, 1, j)$, with coefficient one. For $j = 1$, this is the chain rule. If it holds for j , we differentiate each term once more. By the chain rule, the derivative of the term indexed by $(r, m, j_1, \dots, j_m)$ is the sum of the terms indexed by $(r + 1, m, j_1, \dots, j_m)$, by $(r, m + 1, j_1, \dots, j_m, 1)$, and by $(r, m, j_1, \dots, j_\ell + 1, \dots, j_m)$ for $\ell \leq m$. All of them satisfy $r + \sum j_\ell = j + 1$. A term containing $\nu^{(j+1)}$ arises only from a term containing $\nu^{(j)}$ by raising that index, which proves the claim. Since $\partial_{\tilde{\zeta}}^j[\mathcal{F}(\tilde{\zeta}, \nu(\tilde{\zeta}))] = \sigma_j$ and $D_\mu \mathcal{F}(\tilde{\zeta}, \nu) = \tilde{\mathcal{L}}_{\tilde{\zeta}}$, (12.38) implies (12.36) with R_j equal to the sum of all other terms.

It remains to compute $\partial_{\tilde{\zeta}}^r D_\mu^m \mathcal{F}(\tilde{\zeta}, \mu)[\mu_1, \dots, \mu_m]$ for fixed $\mu, \mu_1, \dots, \mu_m$. By (12.20),

$$D_\mu^m \mathcal{F}(\tilde{\zeta}, \mu)[\mu_1, \dots, \mu_m] = p^m \mathcal{F}(\tilde{\zeta}, \mu) * \text{RK}_{\tilde{\zeta}}^{(0)} \mu_1 * \dots * \text{RK}_{\tilde{\zeta}}^{(0)} \mu_m.$$

By (12.18) with $m = 1$ and the chain rule, $\partial_{\tilde{\zeta}} \mathcal{F}(\tilde{\zeta}, \mu) = p \mathcal{F}(\tilde{\zeta}, \mu) * \text{RK}_{\tilde{\zeta}}^{(1)} \mu$, and $\partial_{\tilde{\zeta}} \mathcal{K}_{\tilde{\zeta}}^{(r)} = \mathcal{K}_{\tilde{\zeta}}^{(r+1)}$. Since convolution is a bounded multilinear operation on $\mathcal{M}_q$, the Leibniz rule applies, and by induction on r we find that $\partial_{\tilde{\zeta}}^r D_\mu^m \mathcal{F}(\tilde{\zeta}, \mu)[\mu_1, \dots, \mu_m]$ is a sum, with positive integer coefficients depending only on r and m , of the terms

$$p^{m+n'} \mathcal{F}(\tilde{\zeta}, \mu) * \text{RK}_{\tilde{\zeta}}^{(r_1)} \mu_1 * \dots * \text{RK}_{\tilde{\zeta}}^{(r_m)} \mu_m * \text{RK}_{\tilde{\zeta}}^{(s_1)} \mu * \dots * \text{RK}_{\tilde{\zeta}}^{(s_{n'})} \mu,$$

with $n' \geq 0$, $s_1, \dots, s_{n'} \geq 1$, and $r_1 + \dots + r_m + s_1 + \dots + s_{n'} = r$. At $\mu = \nu$ we have $\mathcal{F}(\tilde{\zeta}, \nu) = \nu$. In each term of R_j , the number of jump factors is $n = m + n' \leq m + r \leq j$, and $n \geq 1$ because $r + \sum j_\ell = j \geq 1$ forces $m \geq 1$ or $r \geq 1$. The inputs are ν or $\nu^{(j_\ell)} = \sigma_{j_\ell}$ with $j_\ell \leq j - 1$, since the only term with $j_\ell = j$ has been removed. The formulas for R_1 and R_2 follow from the same computation. $\square$

Estimates for selected jumps

For a finite real measure μ on $\mathbb{R}$ whose restriction to $\mathcal{Z}$ is absolutely continuous with a bounded density, we set

$$h(\mu) = \text{ess sup}_{x \in \mathcal{Z}} \left| \frac{d\mu}{dx}(x) \right|, \quad \|\mu\|_{q, \mathfrak{h}} = \|\mu\|_{q, \text{TV}} + h(\mu).$$

By Lemma 12.3(i),(ii), we have $h(\tilde{\nu}_{\tilde{\zeta}}) \leq H_p$ and $\|\tilde{\nu}_{\tilde{\zeta}}\|_{q, \mathfrak{h}} \leq C$.

In this subsection, the notation $\mathcal{K}_{\tilde{\zeta}}^{(r)}$ also denotes its integral-kernel extension to finite real measures μ with $\|\mu\|_{q, \mathfrak{h}} < \infty$.

Lemma 12.27. *There exists $C > 0$, depending only on q and Ξ , such that the following holds for $p \geq p_1(\Xi)$ and $\tilde{\zeta} \in \Xi'$. Let $1 \leq n \leq 4$, let $r_1, \dots, r_n \geq 0$ with $r_1 + \dots + r_n \leq 4$, and let $\mu_1, \dots, \mu_n$ be finite real measures with $\|\mu_i\|_{q, \mathfrak{h}} < \infty$. Let*

$$T = \tilde{\nu}_{\tilde{\zeta}} * \text{RK}_{\tilde{\zeta}}^{(r_1)} \mu_1 * \dots * \text{RK}_{\tilde{\zeta}}^{(r_n)} \mu_n, \quad \Pi = \prod_{i=1}^n \|\mu_i\|_{q, \mathfrak{h}}, \quad \Sigma_h = \sum_{i=1}^n h(\mu_i) \prod_{k \neq i} \|\mu_k\|_{q, \mathfrak{h}}.$$

Then T is absolutely continuous, and

$$\|T\|_{q,\text{TV}} \leq C\Pi, \quad (12.39)$$

$$\text{ess sup}_{|z| \leq R_0} \left| \frac{dT}{dz}(z) \right| \leq C(H_p + e^{-p^2/C})\Pi + C\Sigma_h, \quad (12.40)$$

$$\int_{\mathcal{O}_\xi} (1 + |z|^q) |T|(dz) \leq C(e^{-a_1\sqrt{p}} + e^{-p^{7/4}/C})\Pi + C\Sigma_h. \quad (12.41)$$

Proof. We abbreviate $\nu = \tilde{\nu}_\xi$ and $\tau = \tau(\xi) \leq 4(p\xi_-)^{-2}$, and we let $\ell_i(y) = \int \partial_\xi^{r_i} k_\tau(x; y) \mu_i(dx)$ be the density of $\mathcal{K}_\xi^{(r_i)} \mu_i$. We first record three bounds on ℓ_i , which follow from Lemma 12.13 with $m = 0$ after splitting the integral over x into $x \in \mathcal{Z}$, where μ_i has density at most $h(\mu_i)$, and $|x| \geq \rho/5$:

$$\int (1 + |y|^q) |\ell_i(y)| dy \leq C \int (1 + (\tau/|x|)^q) |\mu_i|(dx) \leq C\|\mu_i\|_{q,\mathfrak{h}}, \quad (12.42)$$

$$\text{ess sup}_{|y| \geq \rho/32} |\ell_i(y)| \leq Ce^{-p^2/C} \|\mu_i\|_{\text{TV}} + Ch(\mu_i), \quad (12.43)$$

$$\int_{|y| \geq r_p/20} (1 + |y|^q) |\ell_i(y)| dy \leq Ce^{-p^{7/4}/C} \|\mu_i\|_{\text{TV}} + Ch(\mu_i). \quad (12.44)$$

For (12.42) we used Lemma 12.13(a) and $\int_{\mathcal{Z}} |x|^{-q} dx < \infty$. For (12.43), the part with $|x| \geq \rho/5$ is bounded by Lemma 12.13(c), which gives $C(32/\rho)e^{-(\rho/5)(\rho/32)/(2\tau)}$, and the part with $x \in \mathcal{Z}$ by Lemma 12.13(d), which gives $Ch(\mu_i)\tau(32/\rho)^2$. For (12.44), the part with $|x| \geq \rho/5$ is bounded by Lemma 12.13(b), with exponent $(\rho/5)(r_p/20)/(2\tau) \geq p^{7/4}/C$, and the part with $x \in \mathcal{Z}$ by Lemma 12.13(a).

The measure T is absolutely continuous since $n \geq 1$, with density $z \mapsto \int \nu(dd) G(d - z)$, where $G = \ell_1 * \cdots * \ell_n$. The bound (12.39) follows from Lemma 12.7 and (12.42), since $\|\nu\|_{q,\text{TV}} \leq C$.

For (12.40), consider $|z| \leq R_0$. The part of the integral with $d \in (-3\rho/4, 3\rho/4)$ is at most $H_p \|G\|_{L^1} \leq H_p \prod_i \|\ell_i\|_{L^1} \leq CH_p \Pi$. For the remaining d , we have $|d - z| \geq \rho/8$. The argument in the proof of Lemma 12.9 shows that for almost every w with $|w| \geq \rho/8$,

$$|G(w)| \leq \sum_{i=1}^n \text{ess sup}_{|y| \geq |w|/n} |\ell_i(y)| \prod_{k \neq i} \|\ell_k\|_{L^1},$$

and $|w|/n \geq \rho/32$. By Fubini's theorem, this bound holds at $w = d - z$ for ν -almost every d and Lebesgue-almost every z . By (12.42) and (12.43), the remaining part is therefore at most $Ce^{-p^2/C}\Pi + C\Sigma_h$ for almost every z .

For (12.41), we have

$$\begin{aligned} & \int_{\mathcal{O}_{\xi}} (1 + |z|^q) |T|(dz) \\ & \leq \int \nu(dd) \int \prod_{i=1}^n |\ell_i(y_i)| \mathbf{1}\left\{d - \sum_i y_i \in \mathcal{O}_{\xi}\right\} \left(1 + \left|d - \sum_i y_i\right|^q\right) dy_1 \cdots dy_n. \end{aligned}$$

We use $1 + |d - \sum_i y_i|^q \leq (1 + |d|^q) \prod_i (1 + |y_i|^q)$. The part with $d \notin \mathcal{S}_{\xi}$ is at most $C \int_{\mathcal{S}_{\xi}^c} (1 + |d|^q) \nu(dd) \Pi \leq C e^{-a_1 \sqrt{p}} \Pi$ by (12.42) and (12.26). If $d \in \mathcal{S}_{\xi}$ and $d - \sum_i y_i \in \mathcal{O}_{\xi}$, then $(1 + \theta) |\sum_i y_i| > r_p/4$, which gives $|\sum_i y_i| \geq r_p/5$, and some $|y_i| \geq r_p/(5n) \geq r_p/20$; further, $|d| \leq 2$. The part with $d \in \mathcal{S}_{\xi}$ is therefore at most

$$3 \sum_{i=1}^n \int_{|y| \geq r_p/20} (1 + |y|^q) |\ell_i(y)| dy \prod_{k \neq i} \int (1 + |y|^q) |\ell_k(y)| dy,$$

which is at most $C e^{-p^{7/4}/C} \Pi + C \Sigma_h$ by (12.42) and (12.44). $\square$

Inversion of the component matrix

Lemma 12.28. *Let $P \geq 1$, $\varepsilon \in [0, 1/4]$, and $\delta_b, \delta_i \geq 0$ with $P^3(\delta_b + \delta_i) \leq 1/16$. Let $v = (v_c, v_b, v_i)$ and $s = (s_c, s_b, s_i)$ belong to $[0, \infty)^3$ and satisfy, entrywise,*

$$v \leq Mv + s, \quad M = \begin{pmatrix} \varepsilon & P & P \\ \delta_b & \varepsilon & \delta_b \\ \delta_i & P & \delta_i + \delta_b \end{pmatrix}.$$

Then

$$\begin{aligned} v_c & \leq 3s_c + 11P^2(s_b + s_i), \\ v_b & \leq 6\delta_b s_c + 4s_b + 25P^2\delta_b s_i, \\ v_i & \leq 6(\delta_i + P\delta_b)s_c + 4Ps_b + 4s_i. \end{aligned}$$

Proof. The hypotheses give $\delta_b, \delta_i, P\delta_b, P\delta_i, P^2\delta_b \leq 1/16$. Since $\varepsilon \leq 1/4$, the second inequality gives

$$v_b \leq \frac{4}{3}(\delta_b v_c + \delta_b v_i + s_b). \quad (12.45)$$

Inserting this into the third inequality, we find that the coefficient of v_i on the right side is $4P\delta_b/3 + \delta_i + \delta_b \leq 1/4$, and therefore

$$v_i \leq \frac{4}{3} \left((\delta_i + \frac{4}{3}P\delta_b) v_c + \frac{4}{3}Ps_b + s_i \right) \leq 2(\delta_i + P\delta_b)v_c + 2Ps_b + 2s_i. \quad (12.46)$$

Inserting (12.46) into (12.45) and using $2\delta_i + 2P\delta_b \leq 1/4$, we obtain

$$v_b \leq 2\delta_b v_c + 2s_b + 3\delta_b s_i. \quad (12.47)$$

Inserting (12.46) and (12.47) into the first inequality, the coefficient of v_c on the right side is $\varepsilon + 2P\delta_b + 2P\delta_i + 2P^2\delta_b \leq 5/8$, and we find

$$\frac{3}{8}v_c \leq s_c + (2P + 2P^2)s_b + (3P\delta_b + 2P)s_i \leq s_c + 4P^2s_b + 3Ps_i,$$

which gives the bound for v_c . Inserting it into (12.47) and (12.46), and using $22P^2\delta_b \leq 2$ and $22P^2(\delta_i + P\delta_b) \leq 22P^3(\delta_i + \delta_b) \leq 2$, we obtain the bounds for v_b and v_i . $\square$

Proof of the quantitative bounds

We use the following representation of the pairing of a test function with a selected tensor. Let ψ be bounded and Borel, let T be as in Lemma 12.27, and let $\tilde{D} \sim \tilde{v}_\xi$ and $T_1, \dots, T_n \sim \text{Exp}(1)$ be independent. By Fubini's theorem,

$$\begin{aligned} \int \psi dT &= \int \cdots \int G_\psi(x_1, \dots, x_n) \mu_1(dx_1) \cdots \mu_n(dx_n), \\ G_\psi(x_1, \dots, x_n) &= \partial_{\eta_1}^{r_1} \cdots \partial_{\eta_n}^{r_n} \mathbb{E} \psi \left(\tilde{D} - \sum_{i=1}^n \frac{\tau(\eta_i) T_i}{x_i} \right) \Big|_{\eta_1 = \dots = \eta_n = \xi'} \end{aligned} \quad (12.48)$$

where the law of $\tilde{D}$ is held fixed when differentiating in η_i . To justify (12.48), write the expectation as $\int \tilde{v}_\xi(dd) \int \psi(d - \sum_i y_i) \prod_i k_{\tau(\eta_i)}(x_i; y_i) dy$ and differentiate under the integral sign; this is allowed because, by Lemma 12.12, for η_i in a compact neighborhood of ξ in $(0, \infty)$ the functions $|\partial_{\eta_i}^{r_i} k_{\tau(\eta_i)}(x_i; \cdot)|$ are dominated by a fixed integrable function of y_i . In particular, $|G_\psi| \leq \|\psi\|_\infty \prod_i \int |\partial_{\xi}^{r_i} k_\tau(x_i; y)| dy \leq C\|\psi\|_\infty$ for all $x_1, \dots, x_n \neq 0$, since the substitution $u = |x||y|/\tau$ and Lemma 12.12 give $\int |\partial_{\xi}^r k_\tau(x; y)| dy = \xi^{-r} \int_0^\infty |P_{0,r}(u)| e^{-u} du$, which does not depend on x . If $\psi \in C^2$ with bounded derivatives, we may also differentiate under the expectation in (12.48) when the total differentiation order is at most two.

Proof of Proposition 12.24. Fix Ξ and q , and let a and $A_0, \dots, A_4$ be as in (12.33). Throughout the proof, $p \geq p_{\text{der}}$, where $p_{\text{der}} \geq p_{\text{an}}(\Xi)$ is large enough for the finitely many conditions stated below, each of which depends only on Ξ , q , and the constants of (S1)–(S4). Fix $\xi \in \Xi$ and write $v = \tilde{v}_\xi$, $\tau = \tau(\xi)$, $\mathcal{O} = \mathcal{O}_\xi$, and $\|\cdot\|_c, \|\cdot\|_b, \|\cdot\|_i$ for the norms of Definition 12.4. Recall that $|\partial_\xi^k \tau| \leq Cp^{-2}$ for $k \leq 4$ by Lemma 12.1.

STEP 1. Small quantities. Since $a \leq a_1 < a_{\text{st}}$, $a \leq c_b$, and $a \leq c_i$, we have

$$H_p \leq p^{A_{\text{st}}+1} e^{-ap}, \quad \delta_{b,p} \leq p^{A_{\text{col}}} e^{-ap}, \quad \delta_{i,p} \leq p^{A_{\text{col}}} e^{-a\sqrt{p}}, \quad e^{-a_1\sqrt{p}} \leq e^{-a\sqrt{p}},$$

and for p large, $e^{-p^2/C} + e^{-p^{7/4}/C} \leq e^{-ap}$, $\varepsilon_p \leq 1/4$, and $P_p^3(\delta_{b,p} + \delta_{i,p}) \leq 1/16$, so that Lemma 12.28 applies to M_p . Part (vii) follows from (12.26), since $\mathcal{O} \subset \mathcal{S}_\xi^c$.

STEP 2. *Induction hypothesis.* For $k \in \{0, \dots, 4\}$, let $\mathbf{H}(k)$ denote the statement that (i)–(iv) hold for $1 \leq j \leq k$ with A replaced by A_j . The statement $\mathbf{H}(0)$ is empty. Let $j \in \{1, \dots, 4\}$ and assume $\mathbf{H}(j-1)$. We claim that every $\mu \in \{\nu, \sigma_1, \dots, \sigma_{j-1}\}$ satisfies

$$\|\mu\|_{q, \mathfrak{h}} \leq p^{A_{j-1}+1}, \quad h(\mu) \leq p^{A_{j-1}+A_{\text{st}}+1} e^{-ap}. \quad (12.49)$$

For $\mu = \nu$, this follows from $\|\nu\|_{q, \mathfrak{h}} \leq C \leq p$ and $h(\nu) \leq H_p$. For $\mu = \sigma_k$ with $k < j$, Lemma 12.5(ii) and $\mathbf{H}(j-1)$ give $h(\sigma_k) \leq 2\|\sigma_k\|_{\mathfrak{b}} \leq 2p^{A_k} e^{-ap}$ and $\|\sigma_k\|_{q, \mathfrak{h}} \leq 3p^{A_k}$, and $A_k \leq A_{j-1}$.

STEP 3. *Bounds on the source.* By Lemma 12.26, R_j is a sum of a number of terms depending only on j , each of the form $c p^n T$ with c depending only on j , $n \leq 4$, and T as in Lemma 12.27 with inputs among $\nu, \sigma_1, \dots, \sigma_{j-1}$. By (12.49), the quantities of Lemma 12.27 satisfy $\Pi \leq p^{4A_{j-1}+4}$ and $\Sigma_h \leq 4p^{4A_{j-1}+A_{\text{st}}+4} e^{-ap}$. Using Lemma 12.27, Step 1, and $p^n \leq p^4$, and absorbing the constants into one power of p , we find, with $B_j = 4A_{j-1} + A_{\text{st}} + 10$,

$$\|R_j\|_{q, \text{TV}} \leq p^{B_j}, \quad \text{ess sup}_{|z| \leq R_0} \left| \frac{dR_j}{dz}(z) \right| \leq p^{B_j} e^{-ap}, \quad \int_{\mathcal{O}} (1 + |z|^q) |R_j|(\mathrm{d}z) \leq p^{B_j} e^{-a\sqrt{p}}. \quad (12.50)$$

The measure $R_j = (I - \tilde{\mathcal{L}}_{\xi})\sigma_j$ belongs to X_q^0 and is real, so Lemma 12.5(iii) applies and gives

$$\|R_j\|_{\mathfrak{c}} \leq p^{B_j}, \quad \|R_j\|_{\mathfrak{b}} \leq p^{B_j} e^{-ap}, \quad \|R_j\|_{\mathfrak{i}} \leq 2p^{B_j} e^{-a\sqrt{p}}. \quad (12.51)$$

STEP 4. *Components of σ_j .* By (12.36), $\sigma_j = \tilde{\mathcal{L}}_{\xi}\sigma_j + R_j$. The triangle inequality for each component norm and Lemma 12.14 show that the finite vector $v = (\|\sigma_j\|_{\mathfrak{c}}, \|\sigma_j\|_{\mathfrak{b}}, \|\sigma_j\|_{\mathfrak{i}})$ satisfies $v \leq M_p v + s$, with s the vector of components of R_j . By Lemma 12.28, (12.51), and Step 1, we obtain for p large

$$\|\sigma_j\|_{\mathfrak{c}} \leq p^{B_j+1}, \quad \|\sigma_j\|_{\mathfrak{b}} \leq p^{B_j+3A_{\text{col}}+1} e^{-ap}, \quad \|\sigma_j\|_{\mathfrak{i}} \leq p^{B_j+2A_{\text{col}}+1} e^{-a\sqrt{p}}. \quad (12.52)$$

For instance, the second bound follows from $6\delta_{\mathfrak{b}, p} p^{B_j} + 4p^{B_j} e^{-ap} + 50P_p^2 \delta_{\mathfrak{b}, p} p^{B_j} e^{-a\sqrt{p}} \leq p^{B_j+3A_{\text{col}}+1} e^{-ap}$.

STEP 5. *Strong norms of σ_j .* By $\sigma_j = \tilde{\mathcal{L}}_{\xi}\sigma_j + R_j$, Lemma 12.15(i), (12.50), and (12.52),

$$\|\sigma_j\|_{q, \text{TV}} \leq p^{B_j} + Cp(\|\sigma_j\|_{\mathfrak{c}} + \|\sigma_j\|_{\mathfrak{b}} + \|\sigma_j\|_{\mathfrak{i}}) \leq p^{B_j+3A_{\text{col}}+3} = p^{A_j}.$$

By Lemma 12.15(iii), we have similarly

$$\int_{\mathcal{O}} (1 + |z|^q) |\sigma_j|(\mathrm{d}z) \leq p^{B_j} e^{-a\sqrt{p}} + Cp e^{-a_1\sqrt{p}} (\|\sigma_j\|_{\mathfrak{c}} + \|\sigma_j\|_{\mathfrak{i}}) + Cp \|\sigma_j\|_{\mathfrak{b}} \leq p^{A_j} e^{-a\sqrt{p}}.$$

By (12.52) and $B_j + 3A_{\text{col}} + 1 = A_j - 2$, we have $\|\sigma_j\|_{\mathfrak{b}} \leq p^{A_j} e^{-ap}$ and $\|\sigma_j\|_{\mathfrak{i}} \leq p^{A_j} e^{-a\sqrt{p}}$, and by Lemma 12.5(ii) the density of σ_j on $\mathcal{Z}$ is at most $2\|\sigma_j\|_{\mathfrak{b}} \leq p^{A_j} e^{-ap}$. This proves $\mathbf{H}(j)$. By induction, $\mathbf{H}(4)$ holds, and (i)–(iv) follow with $A = A_4$, since $A_j \leq A_4$.

STEP 6. *The core norm of the first derivative.* Let φ be a test function in the definition of $\|\cdot\|_c$ in (S2), that is, $\varphi(1) = 0$ and $\text{Lip}(\varphi) \leq 1$, and set $\psi(x) = \chi_c((1+\theta)x)\varphi((1+\theta)x)$. Then $\int \varphi d\Pi_c R_1^B = \int \psi dR_1$. Since $|\varphi| \leq 2r_p$ on $\text{supp } \chi_c$ and $|\chi'_c| \leq Cr_p^{-1}$, we have $\|\psi\|_\infty \leq 2r_p$ and $\text{Lip}(\psi) \leq C$. By Lemma 12.26 and (12.48),

$$\int \psi dR_1 = p \int G(x) \nu(dx), \quad G(x) = \partial_\eta \mathbb{E} \psi \left(\tilde{D} - \frac{\tau(\eta)T}{x} \right) \Big|_{\eta=\xi}.$$

The function $\eta \mapsto \mathbb{E} \psi(\tilde{D} - \tau(\eta)T/x)$ is Lipschitz with constant at most $\text{Lip}(\psi) \sup |\tau'| \mathbb{E}[T]/|x| \leq C/(p^2|x|)$, and its derivative satisfies $|G(x)| \leq C/(p^2|x|)$. For $x \in \mathcal{Z}$ we use instead $|G(x)| \leq C\|\psi\|_\infty \leq Cr_p$. Hence

$$\left| \int \psi dR_1 \right| \leq p \left(\frac{5C}{\rho p^2} + Cr_p \nu(\mathcal{Z}) \right) \leq \frac{C}{p} + Cpr_p H_p,$$

and taking the supremum over φ gives $\|R_1\|_c \leq C/p$ for p large. By Lemma 12.28, (12.51), and Step 1, we conclude that $\|\sigma_1\|_c \leq 3\|R_1\|_c + 11P_p^2(\|R_1\|_b + \|R_1\|_i) \leq C/p$, which proves (v).

STEP 7. *Smooth contraction.* Let φ be a test function in Definition 12.23 and let $\psi = \hat{\chi}_\xi \varphi$. On the support of $\hat{\chi}_\xi$ we have $|x - x_\xi| \leq r_p$, and since $\varphi(x_\xi) = 0$ and $|\varphi'| \leq 1$, this implies that $|\varphi| \leq r_p$ there. Together with $|\hat{\chi}_\xi^{(k)}| \leq Cr_p^{-k}$, this gives

$$\|\psi\|_\infty \leq r_p, \quad \|\psi'\|_\infty \leq C, \quad \|\psi''\|_\infty \leq Cr_p^{-1}. \quad (12.53)$$

Let $\sigma \in X_q^0$ be real. By (12.23), $\int \psi d\tilde{\mathcal{L}}_\xi \sigma = p \int G d\sigma$, where $G(x) = \mathbb{E} \psi(\tilde{D} - \tau T/x)$. Since $\sigma(\mathbb{R}) = 0$, we have

$$\int \psi d\tilde{\mathcal{L}}_\xi \sigma = p \int \hat{\chi}_\xi (G - G(x_\xi)) d\sigma + p \int (1 - \hat{\chi}_\xi) (G - G(x_\xi)) d\sigma.$$

The function $1 - \hat{\chi}_\xi$ vanishes outside $\mathcal{O}$ and $|G| \leq r_p$. Then the second term is at most $2pr_p|\sigma|(\mathcal{O})$. On J_0 , where $x \geq 1/4$, we may differentiate under the expectation, and

$$G'(x) = \mathbb{E} \left[\psi' \left(\tilde{D} - \frac{\tau T}{x} \right) \frac{\tau T}{x^2} \right], \quad G''(x) = \mathbb{E} \left[\psi'' \left(\tilde{D} - \frac{\tau T}{x} \right) \frac{\tau^2 T^2}{x^4} - 2\psi' \left(\tilde{D} - \frac{\tau T}{x} \right) \frac{\tau T}{x^3} \right].$$

By (12.53) and $\tau \leq Cp^{-2} \leq r_p$, we have $\|G'\|_{L^\infty(J_0)} + \|G''\|_{L^\infty(J_0)} \leq C\tau \leq Cp^{-2}$. Since the function $G - G(x_\xi)$ vanishes at x_ξ , it is Cp^{-2} times a test function in Definition 12.23, and the first term is at most $Cp^{-1}n_\xi(\sigma)$. Taking the supremum over φ , we obtain

$$n_\xi(\tilde{\mathcal{L}}_\xi \sigma) \leq \frac{C}{p} n_\xi(\sigma) + 2pr_p|\sigma|(\mathcal{O}). \quad (12.54)$$

STEP 8. *The smooth core norm of the second source.* Let ψ be as in Step 7. We bound the pairing of ψ with each of the five terms of R_2 in Lemma 12.26, using (12.48) and (12.53).

We split $\nu = \nu|_{\mathcal{Z}} + \nu|_{\mathcal{Z}^c}$ where needed; the part $\nu|_{\mathcal{Z}}$ has mass at most ρH_p , and since $|G_\psi| \leq Cr_p$ in all cases, every term in which some input equal to ν is replaced by $\nu|_{\mathcal{Z}}$ contributes at most $Cp^2 r_p H_p (1 + \|\sigma_1\|_{\text{TV}})^2$, which is at most Cp^{-1} . For the derivative σ_1 , we use the decomposition $\sigma_1 = \Pi_c^\xi \sigma_1 + \vartheta$, where $\vartheta = (1 - \chi_c((1 + \theta)\cdot))\sigma_1 + m_c \delta_{x_\xi}$ and $m_c = \int \chi_c((1 + \theta)x) \sigma_1(dx) = -\int (1 - \chi_c((1 + \theta)x)) \sigma_1(dx)$. Since $1 - \chi_c((1 + \theta)\cdot)$ vanishes outside $\mathcal{O}$, by (iv) we have $\|\vartheta\|_{\text{TV}} \leq 2|\sigma_1|(\mathcal{O}) \leq 2p^A e^{-a\sqrt{p}}$. By Step 6 and Lemma 12.5(iv), integration against $\Pi_c^\xi \sigma_1$ is bounded by C/p times the Lipschitz constant of the test function on S_c ; this measure is carried by $S_c \subset J_0$.

The term $p \nu * \text{RK}_\xi^{(2)} \nu$. For $|x| \geq \rho/5$, we have

$$G_\psi(x) = \mathbb{E} \left[\psi''(\cdot) \left(\frac{\tau' T}{x} \right)^2 - \psi'(\cdot) \frac{\tau'' T}{x} \right], \quad |G_\psi(x)| \leq C \left(\frac{\tau'^2}{r_p} + |\tau''| \right) \leq \frac{C}{p^2},$$

where the argument of ψ' and ψ'' is $\tilde{D} - \tau T/x$. The pairing is therefore at most Cp^{-1} .

The term $p^2 \nu * \text{RK}_\xi^{(1)} \nu * \text{RK}_\xi^{(1)} \nu$. For $|x_1|, |x_2| \geq \rho/5$, we have $G_\psi(x_1, x_2) = \tau'^2 \mathbb{E}[\psi''(\cdot) T_1 T_2 / (x_1 x_2)]$, which gives $|G_\psi| \leq C \tau'^2 r_p^{-1} \leq Cp^{-15/4}$. The pairing is at most $Cp^{-7/4}$.

The term $2p \nu * \text{RK}_\xi^{(1)} \sigma_1$. Here $G_\psi(x) = -\tau' \mathbb{E}[\psi'(\tilde{D} - \tau T/x) T/x]$, and for $x \in J_0$,

$$|G'_\psi(x)| = \left| \tau' \mathbb{E} \left[\psi''(\cdot) \frac{\tau T^2}{x^3} - \psi'(\cdot) \frac{T}{x^2} \right] \right| \leq C |\tau'| \left(\frac{\tau}{r_p} + 1 \right) \leq \frac{C}{p^2}.$$

By Lemma 12.5(iv), $|\int G_\psi d\Pi_c^\xi \sigma_1| \leq Cp^{-2} \|\sigma_1\|_c \leq Cp^{-3}$, and $|\int G_\psi d\vartheta| \leq Cr_p \|\vartheta\|_{\text{TV}}$. The pairing is at most $Cp^{-2} + Cpr_p p^A e^{-a\sqrt{p}}$.

The term $2p^2 \nu * \text{RK}_\xi^{(1)} \nu * \text{RK}_\xi^{(0)} \sigma_1$. After removing $\nu|_{\mathcal{Z}}$, the pairing is $2p^2 \int \tilde{G}(x_2) \sigma_1(dx_2)$, where

$$\tilde{G}(x_2) = -\tau' \int_{|x_1| \geq \rho/5} \mathbb{E} \left[\psi' \left(\tilde{D} - \frac{\tau T_1}{x_1} - \frac{\tau T_2}{x_2} \right) \frac{T_1}{x_1} \right] \nu(dx_1).$$

For $x_2 \in J_0$, we have $|\tilde{G}'(x_2)| \leq C |\tau'| \tau r_p^{-1} \leq Cp^{-15/4}$, and $|\tilde{G}| \leq Cr_p$ everywhere. As in the previous term, the pairing is at most $2p^2 (Cp^{-15/4} \|\sigma_1\|_c + Cr_p \|\vartheta\|_{\text{TV}}) \leq Cp^{-11/4} + Cp^{2+A} r_p e^{-a\sqrt{p}}$.

The term $p^2 \nu * \text{RK}_\xi^{(0)} \sigma_1 * \text{RK}_\xi^{(0)} \sigma_1$. Here $G_\psi(x_1, x_2) = \mathbb{E} \psi(\tilde{D} - \tau T_1/x_1 - \tau T_2/x_2)$, and for $x_1, x_2 \in J_0$,

$$|\partial_{x_1} \partial_{x_2} G_\psi(x_1, x_2)| = \left| \mathbb{E} \left[\psi''(\cdot) \frac{\tau T_1}{x_1^2} \frac{\tau T_2}{x_2^2} \right] \right| \leq C \tau^2 r_p^{-1} \leq Cp^{-15/4}.$$

We expand $\sigma_1 \otimes \sigma_1$ using $\sigma_1 = \Pi_c^\xi \sigma_1 + \vartheta$. For the term $\Pi_c^\xi \sigma_1 \otimes \Pi_c^\xi \sigma_1$, let $F(x_2) = \int G_\psi(x_1, x_2) \Pi_c^\xi \sigma_1(dx_1)$ for $x_2 \in S_c$. Applying Lemma 12.5(iv) to $x_1 \mapsto G_\psi(x_1, x_2) -$

$G_\psi(x_1, x'_2)$, whose Lipschitz constant on the interval S_c is at most $Cp^{-15/4}|x_2 - x'_2|$, we find $\text{Lip}_{S_c}(F) \leq Cp^{-15/4}\|\sigma_1\|_c$. Applying Lemma 12.5(iv) again, the double integral is at most $Cp^{-15/4}\|\sigma_1\|_c^2 \leq Cp^{-23/4}$. The remaining three terms each contain a factor ϑ , and since $|G_\psi| \leq r_p$ and $\|\Pi_c^\xi \sigma_1\|_{\text{TV}} \leq 2\|\sigma_1\|_{\text{TV}} \leq 2p^A$, they are at most $Cr_p p^A \|\vartheta\|_{\text{TV}}$. The pairing is at most $Cp^{-15/4} + Cp^{2+2A}r_p e^{-a\sqrt{p}}$.

Adding the five bounds and taking the supremum over φ , we obtain $n_\xi(R_2) \leq C/p$ for p large.

STEP 9. Conclusion. The quantity $n_\xi(\sigma_2)$ is finite, since $|\hat{\chi}_\xi \varphi| \leq r_p$. By (12.36), Step 8, (12.54), and (iv),

$$n_\xi(\sigma_2) \leq n_\xi(R_2) + n_\xi(\tilde{\mathcal{L}}_\xi \sigma_2) \leq \frac{C}{p} + \frac{C}{p} n_\xi(\sigma_2) + 2pr_p p^A e^{-a\sqrt{p}}.$$

For p large we have $C/p \leq 1/2$, and (vi) follows. This completes the proof. $\square$

Proof of Corollary 12.25. We apply Proposition 12.24 with $q = 1/2$, and we let A, a, C be its constants and $p \geq p_{\text{der}}(\Xi, 1/2)$. Let $M_J = \max_{b \in J} |b|$. We first record two bounds for a finite real measure $\sigma \in X_q^0$ and $\xi \in \Xi$. If f is bounded and Lipschitz, then

$$\left| \int f d\sigma \right| \leq \text{Lip}(f) \|\sigma\|_{c, \xi} + 2\|f\|_\infty |\sigma|(\mathcal{O}_\xi), \quad (12.55)$$

and if $f \in C^2$ is bounded, then

$$\left| \int f d\sigma \right| \leq (\|f'\|_{L^\infty(J_0)} + \|f''\|_{L^\infty(J_0)}) n_\xi(\sigma) + 2\|f\|_\infty |\sigma|(\mathcal{O}_\xi). \quad (12.56)$$

For (12.55), we write $\int f d\sigma = \int (f - f(x_\xi)) d\sigma$, which holds because $\sigma(\mathbb{R}) = 0$, and split it with the cutoff $\chi_c((1 + \theta) \cdot)$. The first part equals $\int (f - f(x_\xi)) d\Pi_c^\xi \sigma$, because $f - f(x_\xi)$ vanishes at x_ξ , and it is bounded by Lemma 12.5(iv). The second part is supported in $\mathcal{O}_\xi$. The proof of (12.56) is the same, with $\hat{\chi}_\xi$ in place of $\chi_c((1 + \theta) \cdot)$ and Definition 12.23 in place of Lemma 12.5(iv).

Let $\varphi \in C^2(\mathbb{R})$ be supported in J . By Definition 12.2, $\mathbb{E}\varphi(B) = \int \varphi((1 + \theta(\xi))x) \tilde{v}_\xi(dx)$. The map $\xi \mapsto \tilde{v}_\xi$ is infinitely differentiable into X_q by Theorem 12.19, and hence in total variation. The map $\eta \mapsto \varphi((1 + \theta(\eta)) \cdot)$ is twice continuously differentiable into the bounded functions with the supremum norm, with derivatives $\theta'(\eta)x\varphi'((1 + \theta)x)$ and $\theta''(\eta)x\varphi'((1 + \theta)x) + \theta'(\eta)^2 x^2 \varphi''((1 + \theta)x)$; this uses that φ' and φ'' are uniformly continuous and that $|x| \leq M_J$ on the support of these functions. By the product rule for the bounded bilinear pairing between bounded functions and finite measures, we obtain

$$\begin{aligned} \partial_\xi \mathbb{E}\varphi(B) &= \int \varphi_\theta d\sigma_1 + \theta' \int x\varphi'((1 + \theta)x) \tilde{v}_\xi(dx), \\ \partial_\xi^2 \mathbb{E}\varphi(B) &= \int \varphi_\theta d\sigma_2 + 2\theta' \int x\varphi'((1 + \theta)x) \sigma_1(dx) \\ &\quad + \int (\theta'' x\varphi'((1 + \theta)x) + \theta'^2 x^2 \varphi''((1 + \theta)x)) \tilde{v}_\xi(dx), \end{aligned}$$

where $\varphi_\theta(x) = \varphi((1 + \theta)x)$. We have $\|\varphi_\theta^{(k)}\|_\infty \leq (9/8)^k \|\varphi^{(k)}\|_\infty$ for $k \leq 2$, and the function $x \mapsto x\varphi'((1 + \theta)x)$ is bounded by $M_J\|\varphi'\|_\infty$ and Lipschitz with constant at most $\|\varphi'\|_\infty + 2M_J\|\varphi''\|_\infty$. By (12.55) and Proposition 12.24(iv),(v), we have $|\int \varphi_\theta d\sigma_1| \leq (C/p + 2p^A e^{-a\sqrt{p}})\|\varphi\|_{C^2}$; by (12.56) and Proposition 12.24(iv),(vi), the same bound holds for $|\int \varphi_\theta d\sigma_2|$; by (12.55), $|\int x\varphi'((1 + \theta)x) \sigma_1(dx)| \leq C_J\|\varphi\|_{C^2}$. The remaining terms are bounded by $C_J(|\theta'| + |\theta''| + \theta'^2)\|\varphi\|_{C^2} \leq C_J p^{-1}\|\varphi\|_{C^2}$ by Lemma 12.1. This proves (12.34).

For (12.35), let $\Psi(\eta, \zeta) = \int \varphi(\eta, (1 + \theta(\zeta))x) \tilde{v}_\zeta(dx)$. By the assumption on φ and the argument above, Ψ is twice continuously differentiable near (ζ, ζ) , and

$$\partial_\zeta^j \mathbb{E}\varphi(\zeta, B) = \sum_{i=0}^j \binom{j}{i} \partial_\eta^i \partial_\zeta^{j-i} \Psi(\zeta, \zeta).$$

The term $i = j$ is $\mathbb{E}[(\partial_\zeta^j \varphi)(\zeta, B)]$. For $i < j$, the term $\partial_\zeta^{j-i} \Psi(\zeta, \zeta)$ at $\zeta = \zeta$ is the $(j - i)$ -th derivative of $\mathbb{E}g(B)$ for the fixed test $g = \partial_\zeta^i \varphi(\zeta, \cdot)$, which is supported in J , and (12.34) bounds it by $(C/p + p^A e^{-a\sqrt{p}})\|g\|_{C^2}$. This proves (12.35). $\square$

Asymptotics of the free energy

Let $\nu_{p,E}$ be the principal solution constructed in Section 11, and recall that for $\xi > 0$ we write $\lambda_p(\xi) = \exp \varphi_{\xi\sqrt{p}}(1)$, where φ_E is the free energy of $\nu_{p,E}$ (Definition 6.14). The goal of this section is to prove the following two results. The first identifies the leading behavior of λ_p on compact scaled windows. The second shows that the free energy at $s = 1$ is negative at all sufficiently large energies.

Theorem 13.1. *Fix a compact set $\Xi \subset (0, \infty)$. Then*

$$\lim_{p \rightarrow \infty} \sup_{\xi \in \Xi} \left| \lambda_p(\xi) - \frac{\mu_*}{\xi} \right| = 0.$$

Here $\lambda_p(\xi)$ is defined for $p \geq p_0(\min \Xi)$, the threshold of Corollary 11.23 (on the window, $\nu_{p,\xi\sqrt{p}}$ is the law of Theorem 11.21), and $\mu_* = \sqrt{\pi}/2$.

Proposition 13.2. *Let M_0 be the numerical constant of Corollary 11.23. There exist a numerical constant $M_{\text{ext}} \geq M_0$ and a numerical threshold $p_{\text{ext}} > 1$ such that, for every $p \geq p_{\text{ext}}$ and every $E \geq M_{\text{ext}}\sqrt{p}$, the principal solution $\nu_{p,E}$ satisfies*

$$\varphi_E(1) \leq \varphi_E(1 - 1/\log p) \leq -\log 2. \quad (13.1)$$

Theorem 13.1 is used in Section 14 to locate the critical energy, and Proposition 13.2 is used in Sections 14 and 16 to show that the free energy at $s = 1$ has no zero at energies above the scaled windows.

We now explain how the proofs go. Both rest on one expansion of the free energy. For $s < 1$, Proposition 6.27 expresses $\varphi_E(s)$ as the exponential growth rate of the path moments $\Phi_L(s)$. In Section 13.1 we write $\Phi_L(s)$ through the iterates of a normalized operator $T_{p,s,\xi}^*$, which differs from the operator $\mathcal{A}_s$ of Definition 6.1 by an explicit scalar $\alpha_{p,s}(\xi)$. We find that $\varphi_E(s) = \log \alpha_{p,s}(\xi) + \log r_{p,s}(\xi)$, where $r_{p,s}(\xi)$ is the exponential growth rate of the normalized iterates. As $s \uparrow 1$, the scalar $\alpha_{p,s}(\xi)$ converges to $\mu_*/a_p(\xi)$, and $a_p(\xi) \rightarrow \xi$ as $p \rightarrow \infty$. It therefore suffices to show that $r_{p,s}(\xi) = 1 + o(1)$ for s close to one, uniformly in $\xi \in \Xi$.

The lower bound $r_{p,s}(\xi) \geq 1 - o(1)$ is elementary. Since the principal law concentrates near one, a test function supported near one is a subsolution for $T_{p,s,\xi}^*$ (Section 13.4). The upper bound is the main work. There are two parts of the state space on which the normalized operator has growth close to one. The first is the core, near $x = 1$. The

second is the tail, where x is of order $K = p^2 a_p(\xi)^2$. There a state x produces a state of the same order when the side variable B nearly cancels the term $T_s x / K$, and at $s = 1$ the expected weight of such near-cancellations is also close to one. We cover the real line by six regions (Definition 13.14), estimate the norm of every block of the operator between two regions (Proposition 13.17), and show that every closed chain of blocks other than the two diagonal blocks of the core and the tail has a small product. A diagonal balancing of the resulting six-by-six matrix (Lemma 13.23) then produces a supersolution with eigenvalue $1 + o(1)$.

The same construction applies, with uniform constants, at energies $E \geq M\sqrt{p}$ for large M , if the exponent is $s_p = 1 - 1/\log p$. This choice keeps $p^{1-s_p} = e$ bounded, and it replaces the losses of order $\log K$ in the capped integrals by losses of order $\log p$; this matters because K is unbounded at such energies. Since the scalar α_{p,s_p} is small there, the free energy at s_p is negative, and Proposition 13.2 follows from the monotonicity of the free energy on $(0, 1]$.

13.1 The normalized transfer operator

Throughout this section, $p > 1$ and $E > 2$. We use the normalizations of Definition 5.2:

$$g(E) = \frac{E - \sqrt{E^2 - 4}}{2}, \quad \theta = g(E)^2, \quad K = \frac{p}{\theta}. \quad (13.2)$$

When $E = \xi\sqrt{p}$, we also write

$$a_p(\xi) = \frac{1}{\sqrt{p}g(\xi\sqrt{p})} = \frac{\xi + \sqrt{\xi^2 - 4/p}}{2}, \quad \text{so that} \quad K = p^2 a_p(\xi)^2. \quad (13.3)$$

Let π be a real solution at energy E (Definition 5.1), let D have law π , and let ν be the law of $B = g(E)D$. Since $g(E)E = 1 + \theta$ and $g(E)^2 = \theta$, the law ν solves

$$B \stackrel{d}{=} 1 + \theta - \frac{\theta}{p} \sum_{j=1}^N \frac{Y_j}{B_j}, \quad N \sim \text{Pois}(p), \quad Y_j \sim \text{Exp}(1), \quad (13.4)$$

with all variables on the right side independent. Conversely, if a probability measure ν on $\mathbb{R} \setminus \{0\}$ solves (13.4), then the law of $B/g(E)$ is a real solution at E . We denote by ν^x the law of $X = B^{-1}$. In the notation of Lemma 6.13 and Lemma 6.23(ii), ν is the law ν_g with $g = g(E)$, the rate $K_g = p/g^2$ equals K , and ν^x is the law π_g^x ; the variable $\beta = g^2 Y/p$ with $Y \sim \text{Exp}(1)$ equals Y/K . By Lemma 6.13(i), ν satisfies (6.1).

Definition 13.3. For $s \in (0, 2)$, set $\mu_s = \Gamma(1 + s/2)$ and let T_s be a random variable with density

$$\omega_s(t) = \frac{t^{s/2} e^{-t}}{\mu_s} \mathbf{1}_{\{t>0\}}. \quad (13.5)$$

Let $B \sim \nu$ and T_s be independent. For $x \in \mathbb{R}$, set

$$D_x = B - \frac{T_s x}{K}, \quad Y_x = D_x^{-1}. \quad (13.6)$$

For a measurable function $F: \mathbb{R} \rightarrow [0, \infty]$, define

$$(\mathsf{T}^* F)(x) = |x|^s \mathbb{E}[F(Y_x)], \quad x \in \mathbb{R}, \quad (13.7)$$

and, for $E = \xi \sqrt{p}$, set

$$\alpha_{p,s}(\xi) = p^{1-s} \mu_s a_p(\xi)^{-s}. \quad (13.8)$$

When $\nu = \nu_{p,E}$ is the principal solution, we write $\mathsf{T}_{p,s,\xi}^*$ for T^* .

The variable Y_x is defined almost surely. Indeed, for $x \neq 0$ the variable D_x has a density because T_s has one and is independent of B , and for $x = 0$ we have $D_0 = B$, which has no atom at zero because π is a law on $\mathbb{R} \setminus \{0\}$. We also write $e_s(x) = |x|^s$.

Lemma 13.4. *Let π be a real solution at $E = \xi \sqrt{p} > 2$, and let ν, ν^x be as above. Let $\mathcal{A}_s$ be the transfer operator in the x -coordinate of Definition 6.1(vi), defined with the law ν and the rate K . For every $s \in (0, 2)$ and every Borel $F: \mathbb{R} \rightarrow [0, \infty]$, we have*

$$\mathcal{A}_s F = \alpha_{p,s}(\xi) \mathsf{T}^* F. \quad (13.9)$$

Moreover, for $s \in (0, 1)$ and $L \geq 1$, the path moment of Definition 6.24 satisfies

$$\Phi_L(s; E) = g(E)^s \alpha_{p,s}(\xi)^L \int (\mathsf{T}^*)^L e_s \, d\nu^x, \quad (13.10)$$

the limit

$$r_{\pi,s} = \lim_{L \rightarrow \infty} \left(\int (\mathsf{T}^*)^L e_s \, d\nu^x \right)^{1/L} \quad (13.11)$$

exists in $(0, \infty)$, and

$$\varphi_\pi(s) = \log \alpha_{p,s}(\xi) + \log r_{\pi,s}. \quad (13.12)$$

Proof. Let $Y \sim \text{Exp}(1)$. For every measurable $G \geq 0$, the definition of ω_s gives

$$\mathbb{E}[Y^{s/2} G(Y)] = \int_0^\infty y^{s/2} e^{-y} G(y) \, dy = \mu_s \int_0^\infty \omega_s(y) G(y) \, dy = \mu_s \mathbb{E}[G(T_s)]. \quad (13.13)$$

By Definition 6.1(iii) and (vi) with $\beta = Y/K$, we have

$$(\mathcal{A}_s F)(x) = p \mathbb{E} \left[\left(\frac{Y}{K} \right)^{s/2} |x|^s F \left(\left(B - \frac{Yx}{K} \right)^{-1} \right) \right] = p K^{-s/2} \mu_s |x|^s \mathbb{E}[F(Y_x)],$$

where in the second equality we conditioned on B and applied (13.13). By (13.3), we have $p K^{-s/2} \mu_s = p (p a_p(\xi))^{-s} \mu_s = \alpha_{p,s}(\xi)$, which proves (13.9). Iterating (13.9) and inserting

it into the definition $\Phi_L(s; E) = g^s \int \mathcal{A}_s^{(g)L} e_s d\pi_g^x$ of Definition 6.24 with $g = g(E)$, for which $\mathcal{A}_s^{(g)} = \mathcal{A}_s$ and $\pi_g^x = \nu^x$, we obtain (13.10).

Let $s \in (0, 1)$. By Proposition 6.27(iii), the law ν and the rate K satisfy the hypotheses of Proposition 6.27, namely (6.1) and the fractional cancellation estimate (6.25) for some $q \in (s, 1)$. Part (i) of that proposition gives $C \geq 1$ with

$$C^{-1} \lambda(s)^L \leq \int \mathcal{A}_s^L e_s d\nu^x \leq C \lambda(s)^L, \quad L \geq 1,$$

where $\lambda(s)$ is the Perron–Frobenius eigenvalue of Theorem 6.11 for ν and K . By Definition 6.14(iii), $\lambda(s) = e^{\varphi_\pi(s)}$. Dividing by $\alpha_{p,s}(\xi)^L$ and using (13.9), we find

$$C^{-1} \left(\frac{e^{\varphi_\pi(s)}}{\alpha_{p,s}(\xi)} \right)^L \leq \int (\mathbb{T}^*)^L e_s d\nu^x \leq C \left(\frac{e^{\varphi_\pi(s)}}{\alpha_{p,s}(\xi)} \right)^L.$$

Taking L -th roots, we conclude that the limit in (13.11) exists, equals $e^{\varphi_\pi(s)} / \alpha_{p,s}(\xi) \in (0, \infty)$, and satisfies (13.12). $\square$

When π is the principal solution at $E = \xi \sqrt{p}$, that is, the law of $B/g(E)$ under $\nu_{p,E}$, we write $r_{p,s}(\xi) = r_{\pi,s}$. Then (13.12) reads $\varphi_E(s) = \log \alpha_{p,s}(\xi) + \log r_{p,s}(\xi)$.

Remark 13.5. By Remark 6.25 and Proposition 8.3, $\Phi_L(s; E)$ is the expected generation- L sum of the s -th powers of the formal inverse column. In this interpretation each generation contributes the factor p from the Mecke formula, the factor $p^{-s/2} Y^{s/2}$ from $|h_e|^s$, and the factor $g(E)^s$ from the normalization $B = g(E)D$. The tilt $Y^{s/2}$ turns the exponential mark into T_s , and the remaining scalar is $\alpha_{p,s}(\xi) = p^{1-s/2} \mu_s g(E)^s$, which agrees with (13.8) because $g(E) = (\sqrt{p} a_p(\xi))^{-1}$. We do not use this interpretation below.

Lemma 13.6. *Let π be a real solution at $E = \xi \sqrt{p} > 2$. Then $\mu_1 = \Gamma(3/2) = \mu_*$, the limit $\lim_{s \uparrow 1} r_{\pi,s}$ exists, and*

$$\exp \varphi_\pi(1) = \lim_{s \uparrow 1} \alpha_{p,s}(\xi) r_{\pi,s} = \frac{\mu_*}{a_p(\xi)} \lim_{s \uparrow 1} r_{\pi,s}. \quad (13.14)$$

Proof. For a standard complex Gaussian w , the variable $|w|^2$ is exponential with mean one. Then

$$\mu_* = \mathbb{E}|w| = \int_0^\infty y^{1/2} e^{-y} dy = \Gamma(3/2) = \mu_1,$$

and $\Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2$. By Theorem 6.11(ii) and (iv), $s \mapsto \lambda(s)$ is positive and real-analytic on $(0, 2)$. In particular, $\varphi_\pi(s) = \log \lambda(s)$ is continuous at $s = 1$ (see also Proposition 6.15(i)). Using (13.12), we have $\alpha_{p,s}(\xi) r_{\pi,s} = \exp \varphi_\pi(s) \rightarrow \exp \varphi_\pi(1)$ as $s \uparrow 1$. Since Γ is continuous, $\alpha_{p,s}(\xi) \rightarrow \mu_1 a_p(\xi)^{-1} = \mu_* / a_p(\xi) > 0$. Dividing, we conclude that $r_{\pi,s}$ converges as $s \uparrow 1$ and that (13.14) holds. $\square$

By Lemma 13.6, the proof of Theorem 13.1 reduces to showing that $\lim_{s \uparrow 1} r_{\pi,s} = 1 + o(1)$ for the principal solution, uniformly on the window. The rest of this section is devoted to two-sided bounds on $r_{\pi,s}$.

13.2 Inputs from the principal solution

We use the following properties of the principal solution from Section 11. Let $\mathbf{q}_{\text{can}} = (1, 0, 1, 1/16)$ be the canonical tuple of Definition 11.8, and for $\theta > 0$ let $\mathfrak{S}_{p,\theta} = \mathfrak{S}_{p,\theta}(\mathbf{q}_{\text{can}})$ be the corresponding class. Every $\nu \in \mathfrak{S}_{p,\theta}$ has the form $e^{-p}\delta_{1+\theta} + f(b)db$ and satisfies the core, boundary and tail bounds of Definition 11.8 with $(a, A, K', \rho) = (1, 0, 1, 1/16)$:

$$\nu(\{b : |b - 1| > p^{-1/4}\}) \leq e^{-\sqrt{p}}, \quad (13.15)$$

$$f(b) \leq e^{-p} \quad \text{for almost every } b \in (-1/16, 1/16), \quad (13.16)$$

$$\nu(\{b : |b| > L\}) \leq \frac{e^{-p}}{L} + e^{-pL}, \quad L \geq 3. \quad (13.17)$$

By Corollary 11.23, with the numerical constants $M_0 = 3$ and $c_{\text{ent}} = 1$, we have $g(E)^2 \leq c_{\text{ent}}/p$ for $E \geq M_0\sqrt{p}$, and for every $\xi_0 > 0$ there exists $p_0(\xi_0)$ such that for every $p \geq p_0(\xi_0)$ and $E \geq \xi_0\sqrt{p}$ the following hold: the principal solution $\nu_{p,E}$ is the unique fixed point of the law map in $\mathfrak{S}_{p,\theta(E)}$, where $\theta(E) = g(E)^2$; it is the law of Theorem 11.21 if $E \leq (M_0 + 1)\sqrt{p}$, and the fixed point of Proposition 11.22 with $\theta = \theta(E)$ if $E \geq M_0\sqrt{p}$; and the law of $B/g(E)$ under $\nu_{p,E}$ is a real solution at E . In particular, for $p \geq p_0(\xi_0)$, the bounds (13.15)–(13.17) hold for $\nu_{p,E}$ with numerical constants, uniformly in $E \geq \xi_0\sqrt{p}$. On compact scaled windows, part (iii) of Theorem 11.21 gives the same bounds with halved right sides, and part (v) gives the real-solution property.

We do not use the other properties of Theorem 11.21, Proposition 11.22, and Corollary 11.23. To treat both regimes at once, we isolate the inequalities used below.

Definition 13.7. Let $\mathfrak{c} = (c_0, A_0, K_0, \rho_0, c_*, C_*)$ be constants with $A_0 \geq 0$, $\rho_0 \in (0, 1/2]$, and $c_0, K_0, c_*, C_* > 0$, and let $p \geq 3$. For $K \geq 1$ and $s \in (0, 1)$, set

$$\ell_{K,s} = 1 + \min\left\{\log K, \frac{1}{1-s}\right\}. \quad (13.18)$$

A triple (ν, K, s) , consisting of a Borel probability measure ν on $\mathbb{R}$, a number $K \geq 1$ and an exponent s , is $\mathfrak{c}$ -admissible at p if the following hold.

- (A1) $\nu(\{b : |b - 1| > p^{-1/4}\}) \leq e^{-c_0\sqrt{p}}$.
- (A2) The restriction of ν to $[-\rho_0, \rho_0]$ has a density bounded by $p^{A_0}e^{-c_0p}$.
- (A3) $\nu(\{b : |b| > L\}) \leq K_0p^{A_0}e^{-c_0p}L^{-1} + e^{-c_0pL}$ for every $L \geq 3$.
- (A4) $K \geq c_*p^2$.
- (A5) $3/4 \leq s < 1$ and $\ell_{K,s} \leq C_* \log p$.

Conditions (A1)–(A3) are the core bound, the boundary density bound and the tail bound. In the scaled windows, (A5) holds for all s because $\log K \leq 3 \log p$. At large energies K is unbounded, and (A5) is obtained from the term $(1-s)^{-1}$ at $s = 1 - 1/\log p$.

Lemma 13.8. *The following statements hold.*

- (i) *Fix a compact set $\Xi \subset (0, \infty)$ and write $\xi_- = \min \Xi$, $\xi_+ = \max \Xi$. There exist constants c_Ξ and a threshold $p_\Xi \geq p_0(\xi_-)$ such that for every $p \geq p_\Xi$, $\xi \in \Xi$ and $s \in [3/4, 1)$, the triple $(\nu_{p,\xi\sqrt{p}}, p^2 a_p(\xi)^2, s)$ is c_Ξ -admissible at p .*
- (ii) *Let $M \geq M_0$ and $s_p = 1 - 1/\log p$. There exist numerical constants c_{ext} such that for every $p \geq \max\{p_0(M), e^4\}$ and every $E \geq M\sqrt{p}$, the triple $(\nu_{p,E}, p/g(E)^2, s_p)$ is c_{ext} -admissible at p .*

Proof. For (i), we take

$$p_\Xi = \max\{3, p_0(\xi_-), 16\xi_-^{-2}, e\xi_+^2\}, \quad c_\Xi = (1, 0, 1, 1/32, \xi_-^2/4, 3).$$

Conditions (A1)–(A3) follow from (13.15)–(13.17), since $\xi\sqrt{p} \geq \xi_- \sqrt{p}$ and $p \geq p_0(\xi_-)$; for (A2) we used that the density bound on $(-1/16, 1/16)$ implies the same bound on $[-1/32, 1/32]$, and that $\nu_{p,E}$ has no atom there. Since $p \geq 16\xi_-^{-2}$, we have $\xi^2 - 4/p \geq 0$, and (13.3) gives $\xi/2 \leq a_p(\xi) \leq \xi$. Then $K = p^2 a_p(\xi)^2$ satisfies $\xi_-^2 p^2/4 \leq K \leq \xi_+^2 p^2$, which gives (A4) and $K \geq 1$, and

$$\ell_{K,s} \leq 1 + \log K \leq 1 + 2 \log p + 2 \log \xi_+ \leq 3 \log p,$$

where the last inequality holds because $p \geq e\xi_+^2$. This proves (A5).

For (ii), we take $c_{\text{ext}} = (1, 0, 1, 1/32, 1, 2)$. Conditions (A1)–(A3) hold as in (i), since $E \geq M\sqrt{p}$ and $p \geq p_0(M)$. Since $M \geq 3$, we have $\theta = g(E)^2 \leq 4/E^2 \leq 4/(9p) \leq 1/p$, and $K = p/\theta \geq p^2$. Since $p \geq e^4$, we have $s_p \geq 3/4$, and $\ell_{K,s_p} \leq 1 + (1 - s_p)^{-1} = 1 + \log p \leq 2 \log p$. $\square$

Conventions. In Sections 13.3–13.6 and 13.8, the constants c are fixed, and (ν, K, s) is c -admissible at p . Unless stated otherwise, C and c denote positive constants depending only on c , and p_1 denotes a threshold depending only on c ; we increase p_1 finitely many times, and each time we say which condition it must satisfy. We write $\ell = \ell_{K,s}$. We set

$$I_* = \left[\frac{99}{100}, \frac{101}{100} \right], \quad \chi = \mathbf{1}_{I_*}, \quad (13.19)$$

and we always take p_1 so large that $p_1^{-1/4} \leq 1/100$. Then $\{b : \chi(b) = 0\} \subset \{b : |b - 1| > p^{-1/4}\}$ for $p \geq p_1$, and (A1) gives

$$\nu(\{b : \chi(b) = 0\}) \leq e^{-c_0 \sqrt{p}}. \quad (13.20)$$

For $x \in \mathbb{R}$ we write $m = |x|/K$ and $\sigma = \text{sgn } x$ (with $\sigma = 1$ if $x = 0$), so that $D_x = B - \sigma m T_s$.

13.3 Elementary estimates

We begin with properties of the tilted variable T_s .

Lemma 13.9. *Let $s \in [3/4, 1]$. Then the following statements hold.*

(i) $0.88 \leq \mu_s \leq 1$, and for every $t > 0$,

$$\omega_s(t) \leq 2t^{s/2}e^{-t} \leq 2t^{s/2}, \quad \omega_s(t) \leq 2e^{-t/2}. \quad (13.21)$$

(ii) For every $\tau > 0$, we have $\mathbb{P}(T_s \leq \tau) \leq 2\tau^{1+s/2}$ and $\mathbb{P}(T_s \geq \tau) \leq 4e^{-\tau/2}$.

(iii) $\mathbb{E}[T_s^{-s}] = \Gamma(1 - s/2)/\Gamma(1 + s/2) \leq 3$.

(iv) For $b > 0$, $u > 0$ and $d < b$, we have

$$\omega_s(u(b-d)) = \omega_s(ub) \left(1 - \frac{d}{b}\right)^{s/2} e^{ud}. \quad (13.22)$$

(v) For $b > 0$, we have

$$\int_0^\infty u^{1-s} \omega_s(ub) du = \frac{\Gamma(2 - s/2)}{\Gamma(1 + s/2)} b^{s-2}, \quad (13.23)$$

and there exists a numerical constant C_Γ such that $\Gamma(2 - s/2)/\Gamma(1 + s/2) \leq 1 + C_\Gamma(1 - s)$.

Proof. On $(0, \infty)$ the function Γ is positive with minimum larger than 0.88, and it is log-convex with $\Gamma(1) = \Gamma(2) = 1$; log-convexity implies that $\Gamma \leq 1$ on $[1, 2]$. Since $1 + s/2 \in [1, 2]$, this implies $0.88 \leq \mu_s \leq 1$. The first bound in (13.21) follows from $1/\mu_s \leq 2$. For the second, if $t \leq 1$ then $t^{s/2}e^{-t} \leq e^{-t/2}$, and if $t \geq 1$ then $t^{s/2}e^{-t} \leq t^{1/2}e^{-t/2}e^{-t/2} \leq e^{-1/2}e^{-t/2}$, because $t \mapsto t^{1/2}e^{-t/2}$ is decreasing on $[1, \infty)$. In both cases $\omega_s(t) \leq 2e^{-t/2}$.

For (ii), we integrate the two bounds in (13.21) and find $\mathbb{P}(T_s \leq \tau) \leq \int_0^\tau 2t^{s/2} dt \leq 2\tau^{1+s/2}$ and $\mathbb{P}(T_s \geq \tau) \leq \int_\tau^\infty 2e^{-t/2} dt = 4e^{-\tau/2}$.

For (iii), we have $\mathbb{E}[T_s^{-s}] = \mu_s^{-1} \int_0^\infty t^{-s/2} e^{-t} dt = \Gamma(1 - s/2)/\mu_s$. Since $1 - s/2 \in [1/2, 5/8]$ and Γ is decreasing on $(0, 1]$, we have $\Gamma(1 - s/2) \leq \Gamma(1/2) = \sqrt{\pi}$, and $\sqrt{\pi}/0.88 \leq 3$.

Identity (13.22) follows from the definition (13.5), since $(u(b-d))^{s/2} = (ub)^{s/2}(1 - d/b)^{s/2}$ and $e^{-u(b-d)} = e^{-ub}e^{ud}$.

For (v), the substitution $w = ub$ gives

$$\int_0^\infty u^{1-s} \omega_s(ub) du = \frac{b^{s-2}}{\mu_s} \int_0^\infty w^{1-s/2} e^{-w} dw = \frac{\Gamma(2 - s/2)}{\Gamma(1 + s/2)} b^{s-2}.$$

The function $s \mapsto \Gamma(2 - s/2)/\Gamma(1 + s/2)$ is smooth on $[3/4, 1]$ and equals one at $s = 1$. The mean value theorem then implies the last claim, with C_Γ the maximum of the absolute value of its derivative on $[3/4, 1]$. $\square$

We next introduce the capped weight. For $K \geq 1$ and $s \in (0, 1)$, set

$$v(y) = \frac{K^s |y|^s}{K^s + |y|^s}, \quad M_K(d) = \frac{K^s}{1 + K^s |d|^s}, \quad y, d \in \mathbb{R}. \quad (13.24)$$

Then $v(d^{-1}) = M_K(d)$ for $d \neq 0$; we extend v by $v(\pm\infty) = K^s = M_K(0)$. The weight v agrees with $|y|^s$ up to a factor two when $|y| \leq K$, and it is capped at K^s for larger $|y|$. The following estimate controls the integral of M_K near its peak; the term $\min\{\log K, (1-s)^{-1}\}$ in $\ell_{K,s}$ comes from it.

Lemma 13.10. *Let $K \geq 1$ and $s \in (0, 1)$. Then*

$$\frac{1}{2} \min\{|y|^s, K^s\} \leq v(y) \leq \min\{|y|^s, K^s\}, \quad M_K(d) \leq \min\{K^s, |d|^{-s}\}. \quad (13.25)$$

Moreover, for every interval $I \subset \mathbb{R}$ of length at most two,

$$\int_I M_K(d) \, dd \leq 2\ell_{K,s}. \quad (13.26)$$

Proof. The bounds (13.25) follow from $\max\{K^s, |y|^s\} \leq K^s + |y|^s \leq 2\max\{K^s, |y|^s\}$ and from $1 + K^s |d|^s \geq \max\{1, K^s |d|^s\}$.

Since M_K is even and nonincreasing in $|d|$, its integral over an interval of length at most two is at most its integral over $[-1, 1]$. Substituting $w = K|d|$, we have

$$\int_{-1}^1 M_K(d) \, dd = 2K^{s-1} \int_0^K \frac{dw}{1+w^s} \leq 2K^{s-1} \left(1 + \int_1^K w^{-s} \, dw\right) = 2K^{s-1} + 2 \frac{1 - K^{s-1}}{1-s}.$$

We have $K^{s-1} \leq 1$. Writing $K^{s-1} = e^{-y}$ with $y = (1-s) \log K \geq 0$ and using $1 - e^{-y} \leq \min\{1, y\}$, we find $(1 - K^{s-1})/(1-s) \leq \min\{(1-s)^{-1}, \log K\}$. This proves (13.26). $\square$

The next lemma bounds the contribution of side variables outside I_* . It is the only place where the boundary density bound and the tail bound are used. Since such contributions occur in every block below, we state it for all $x \in \mathbb{R}$.

Lemma 13.11. *There exist C_N and p_1 , depending only on $\mathfrak{c}$, such that for $p \geq p_1$ and every $x \in \mathbb{R}$,*

$$\left(1 + \left|\frac{x}{K}\right|^s\right) \mathbb{E}[(1 - \chi(B)) M_K(D_x)] \leq C_N \ell e^{-c_0 \sqrt{p}}. \quad (13.27)$$

This uses (A1), (A2) and (A3).

Proof. Recall that $D_x = B - \sigma m T_s$ with $m = |x|/K$. We consider the cases $m < 1$ and $m \geq 1$ separately. In both cases we use (13.20).

STEP 1. *The case $m < 1$. We claim that*

$$\mathbb{E}[(1 - \chi(B)) M_K(D_x)] \leq 2\ell p^{A_0} e^{-c_0 p} + 8\ell \rho_0^{-1} e^{-c_0 \sqrt{p}}. \quad (13.28)$$

Since $1 + m^s \leq 2$, this implies (13.27) for $m < 1$ after enlarging p_1 so that $p^{A_0} e^{-c_0 p} \leq e^{-c_0 \sqrt{p}}$.

We first bound the contribution of $|B| \leq \rho_0$, dropping the factor $1 - \chi(B)$. Fix $t \geq 0$. By (A2) and (13.26) applied to the interval $[-\rho_0, \rho_0] - \sigma mt$ of length $2\rho_0 \leq 1$,

$$\int_{[-\rho_0, \rho_0]} M_K(b - \sigma mt) \nu(\mathrm{d}b) \leq p^{A_0} e^{-c_0 p} \int_{-\rho_0}^{\rho_0} M_K(b - \sigma mt) \mathrm{d}b \leq 2\ell p^{A_0} e^{-c_0 p}.$$

Since T_s is independent of B , we may integrate this bound over the law of T_s (Tonelli's theorem), and we find $\mathbb{E}[M_K(D_x); |B| \leq \rho_0] \leq 2\ell p^{A_0} e^{-c_0 p}$.

Next fix b with $|b| > \rho_0$. We show that $\mathbb{E}[M_K(b - \sigma m T_s)] \leq 8\ell/\rho_0$. If $m = 0$ or $\sigma b < 0$, then $|b - \sigma m T_s| \geq |b| > \rho_0$, so $M_K(b - \sigma m T_s) \leq \rho_0^{-s} \leq \rho_0^{-1}$ by (13.25). Suppose $m > 0$ and $\sigma b > 0$, so that $|b - \sigma m T_s| = ||b| - m T_s|$. On the event $||b| - m T_s| > \rho_0/2$, we have $M_K \leq (\rho_0/2)^{-s} \leq 2\rho_0^{-1}$. On its complement $T_s \geq (|b| - \rho_0/2)/m \geq \rho_0/(2m)$. Changing variables from t to $d = |b| - m t$, and using (13.21) and (13.26), we obtain

$$\begin{aligned} \mathbb{E}[M_K(b - \sigma m T_s); ||b| - m T_s| \leq \rho_0/2] &= \frac{1}{m} \int_{|d| \leq \rho_0/2} \omega_s\left(\frac{|b| - d}{m}\right) M_K(d) \mathrm{d}d \\ &\leq \frac{1}{m} 2e^{-\rho_0/(4m)} \cdot 2\ell \leq \frac{16\ell}{e\rho_0}. \end{aligned}$$

In the last inequality we used that $m^{-1}e^{-\rho_0/(4m)} \leq 4/(e\rho_0)$ for all $m > 0$. Adding the two parts, we have $\mathbb{E}[M_K(b - \sigma m T_s)] \leq (2 + 16/e)\ell\rho_0^{-1} \leq 8\ell\rho_0^{-1}$, since $\ell \geq 1$. Integrating this bound against $(1 - \chi(b))\mathbf{1}_{\{|b| > \rho_0\}}\nu(\mathrm{d}b)$, whose total mass is at most $e^{-c_0 \sqrt{p}}$ by (13.20), and adding the contribution of $|B| \leq \rho_0$, we obtain (13.28).

STEP 2. *A conditional bound for $m \geq 1$.* Let $m \geq 1$, fix $b \in \mathbb{R}$ and write $\tau = |b|$. We claim that

$$(1 + m^s) \mathbb{E}[M_K(b - \sigma m T_s)] \leq 8\ell(1 + \tau)^{1/2} + 12 + 12m^s \min\{1, (\tau/m)^{1+s/2}\}. \quad (13.29)$$

We write $d = b - \sigma m T_s$ and split according to the events $\{|d| \leq 1\}$, $\{|d| > 1, T_s \geq 2\tau/m\}$ and $\{|d| > 1, T_s < 2\tau/m\}$.

On $\{|d| \leq 1\}$, we change variables from t to $d = b - \sigma m t$, which has Jacobian $1/m$; the corresponding value of t is $\sigma(b - d)/m \leq (\tau + 1)/m$. Using the first bound in (13.21) and (13.26), we find

$$\begin{aligned} \mathbb{E}[M_K(d); |d| \leq 1] &= \frac{1}{m} \int_{|d| \leq 1} \omega_s\left(\frac{\sigma(b - d)}{m}\right) M_K(d) \mathrm{d}d \\ &\leq \frac{2}{m} \left(\frac{\tau + 1}{m}\right)^{s/2} \cdot 2\ell \leq 4\ell m^{-1-s/2} (1 + \tau)^{1/2}. \end{aligned}$$

On $\{T_s \geq 2\tau/m\}$, we have $|d| \geq m T_s - \tau \geq m T_s/2$, and (13.25) implies that $M_K(d) \leq (m T_s/2)^{-s}$. By Lemma 13.9(iii), the contribution of this event is at most $2m^{-s} \mathbb{E}[T_s^{-s}] \leq$

$6m^{-s}$. On $\{|d| > 1, T_s < 2\tau/m\}$, we have $M_K(d) \leq 1$, and by Lemma 13.9(ii) the contribution is at most $\min\{1, 2(2\tau/m)^{1+s/2}\} \leq \min\{1, 6(\tau/m)^{1+s/2}\}$. Multiplying the sum of the three contributions by $1 + m^s \leq 2m^s$ and using $m^{s/2-1} \leq 1$, we obtain (13.29).

STEP 3. *Two tail integrals.* We record two consequences of (A3). For $q \in (0, 2]$ and $M \geq 3$, we have by the layer cake formula that

$$\int_{\{3 < |b| \leq M\}} |b|^q \nu(db) \leq 3^q \nu(\{|b| > 3\}) + \int_3^M qL^{q-1} \nu(\{|b| > L\}) dL. \quad (13.30)$$

We enlarge p_1 so that $c_0 p \geq 1$, which implies $e^{-c_0 p L} \leq e^{-c_0 p} e^{1-L}$ for $L \geq 1$. Inserting (A3) and this bound into (13.30), and using $e^{-c_0 p} \leq p^{A_0} e^{-c_0 p}$, we obtain, for $q = 1/2$ and $M = \infty$, and for $q = 1 + s/2$ and $M = m \geq 3$,

$$\int_{\{|b| > 3\}} |b|^{1/2} \nu(db) \leq C p^{A_0} e^{-c_0 p}, \quad \int_{\{3 < |b| \leq m\}} |b|^{1+s/2} \nu(db) \leq C p^{A_0} e^{-c_0 p} (1 + m^{s/2}). \quad (13.31)$$

For the second bound we used $\int_3^m L^{s/2-1} dL \leq (2/s)m^{s/2}$.

STEP 4. *The case $m \geq 1$.* We integrate (13.29) against $\nu_{\text{irr}}(db) = (1 - \chi(b))\nu(db)$. By (13.20) and the first bound in (13.31),

$$\begin{aligned} \int (8\ell(1 + |b|)^{1/2} + 12)\nu_{\text{irr}}(db) &\leq (16\ell + 12)e^{-c_0\sqrt{p}} + 16\ell \int_{\{|b| > 3\}} |b|^{1/2} \nu(db) \\ &\leq C\ell(e^{-c_0\sqrt{p}} + p^{A_0} e^{-c_0 p}). \end{aligned}$$

For the last term in (13.29), we split the range of b into $\{|b| \leq 3\}$, $\{3 < |b| \leq m\}$ and $\{|b| > \max\{3, m\}\}$. On the first set, $\min\{1, (|b|/m)^{1+s/2}\} \leq (3/m)^{1+s/2}$, and (13.20) bounds its contribution by $12 \cdot 3^{3/2} m^{s/2-1} e^{-c_0\sqrt{p}} \leq 63e^{-c_0\sqrt{p}}$. On the second set, which is empty unless $m > 3$, the second bound in (13.31) bounds the contribution by

$$12 m^s m^{-1-s/2} C p^{A_0} e^{-c_0 p} (1 + m^{s/2}) \leq 24C p^{A_0} e^{-c_0 p},$$

since $m^{s-1} \leq 1$. On the third set, (A3) bounds the contribution by

$$12 m^s \left(\frac{K_0 p^{A_0} e^{-c_0 p}}{\max\{3, m\}} + e^{-c_0 p \max\{3, m\}} \right) \leq 12K_0 p^{A_0} e^{-c_0 p} + 36e^{-3c_0 p} \leq C p^{A_0} e^{-c_0 p}.$$

Here we used $m^s \leq \max\{3, m\}$, and that $y \mapsto ye^{-c_0 p y}$ is decreasing on $[3, \infty)$ because $c_0 p \geq 1$. Adding these bounds, and enlarging p_1 so that $p^{A_0} e^{-c_0 p} \leq e^{-c_0\sqrt{p}}$, we conclude that (13.27) holds for $m \geq 1$. This completes the proof. $\square$

We now introduce the profile that defines the similarity on the tail region. For $u > 0$, set

$$\varrho_s(u) = \mathbb{E}[\chi(B) \omega_s(uB)], \quad \phi_s(u) = u^{1-s} \varrho_s(u). \quad (13.32)$$

In the tail region, we will see that the function $x \mapsto K^s \phi_s(K/x)$ is an approximate eigenfunction of T^* with eigenvalue $K^{s-1} \int_0^\infty \phi_s$.

Lemma 13.12. *There exists p_1 , depending only on $\mathfrak{c}$, such that for $p \geq p_1$ the following hold.*

(i) *For every $u > 0$,*

$$\frac{1}{3} \min\{1, u\}^{3/4} e^{-3u/2} \leq \phi_s(u) \leq (1 + u^s) \phi_s(u) \leq 5. \quad (13.33)$$

(ii) *We have*

$$\int_0^\infty \phi_s(u) du \leq \min\left\{(1 + C_\Gamma(1 - s))(1 + 4p^{-1/4}), 2\right\}. \quad (13.34)$$

This uses (A1).

Proof. For the lower bound in (i), we restrict the expectation defining q_s to the event $\{|B - 1| \leq p^{-1/4}\}$, which is contained in $\{\chi(B) = 1\}$ and has probability at least $1/2$ by (A1), after enlarging p_1 so that $e^{-c_0\sqrt{p}} \leq 1/2$. For $b \in [1/2, 3/2]$ we have $\omega_s(ub) \geq (ub)^{s/2} e^{-ub} \geq (u/2)^{s/2} e^{-3u/2}$, because $\mu_s \leq 1$. If $u \leq 2$ then $(u/2)^{s/2} \geq (u/2)^{1/2}$, and if $u \geq 2$ then $(u/2)^{s/2} \geq 1$. This gives $q_s(u) \geq 2^{-3/2} \min\{1, u\}^{1/2} e^{-3u/2}$. Since $u^{1-s} \geq \min\{1, u\}^{1/4}$ for $s \geq 3/4$, the lower bound in (13.33) follows.

For the upper bound, (13.21) and $\chi = \mathbf{1}_{I_*}$ give $q_s(u) \leq \sup_{b \in I_*} 2(ub)^{s/2} e^{-ub} \leq 2(1.01u)^{s/2} e^{-0.99u}$. If $u \leq 1$, then $(1 + u^s)u^{1-s}q_s(u) \leq 2 \cdot 2 \cdot 1.01 \leq 5$. If $u \geq 1$, then $(1 + u^s)u^{1-s}q_s(u) \leq 4.04 u^{3/2} e^{-0.99u} \leq 5$, since $u^{3/2} e^{-0.99u} \leq 1$ for all $u \geq 0$.

For (ii), Tonelli's theorem and (13.23) give

$$\int_0^\infty \phi_s(u) du = \frac{\Gamma(2 - s/2)}{\Gamma(1 + s/2)} \mathbb{E}[\chi(B) B^{s-2}].$$

On $\{|B - 1| \leq p^{-1/4}\}$, we have $B^{s-2} \leq (1 - p^{-1/4})^{-2} \leq 1 + 3p^{-1/4}$, because $p^{-1/4} \leq 1/100$. On $\{\chi(B) = 1, |B - 1| > p^{-1/4}\}$, we have $B^{s-2} \leq 0.99^{-2} \leq 1.03$, and this event has probability at most $e^{-c_0\sqrt{p}}$ by (A1). Then $\mathbb{E}[\chi(B) B^{s-2}] \leq 1 + 3p^{-1/4} + 1.03e^{-c_0\sqrt{p}} \leq 1 + 4p^{-1/4}$ after enlarging p_1 . Together with Lemma 13.9(v), this proves the first bound in (13.34). For the second, we have $\Gamma(2 - s/2) \leq 1$ and $\Gamma(1 + s/2) \geq 0.88$ by Lemma 13.9(i), and $\mathbb{E}[\chi(B) B^{s-2}] \leq 0.99^{s-2} \leq 1.03$ because $B \geq 0.99$ on $\{\chi(B) = 1\}$. Then the left side of (13.34) is at most $1.03/0.88 \leq 2$. $\square$

13.4 The lower bound

The lower bound uses only the concentration of B near one: a function supported near one is a subsolution for T^* .

Lemma 13.13. *There exists p_1 , depending only on $\mathfrak{c}$, such that the following holds for $p \geq p_1$. Let π be a real solution at $E = \xi\sqrt{p} > 2$, let ν be the law of $g(E)D$, and suppose that (ν, K, s) is $\mathfrak{c}$ -admissible at p , where $K = p/g(E)^2$. Then*

$$r_{\pi, s} \geq 1 - 2p^{-1/4}. \quad (13.35)$$

This uses (A1) and (A4).

Proof. Let $S = [1/2, 2]$ and $f(x) = |x|^s \mathbf{1}_S(x)$. Fix $x \in S$, and consider the event

$$G = \{|B - 1| \leq p^{-1/4}\} \cap \{T_s \leq K^{1/2}\}.$$

On G , we have $0 \leq T_s x / K \leq 2K^{-1/2}$ and $D_x \in [1 - p^{-1/4} - 2K^{-1/2}, 1 + p^{-1/4}]$. By (A4), we have $2K^{-1/2} \leq 2c_*^{-1/2} p^{-1} \leq p^{-1/4}$ after enlarging p_1 . On G , this gives $D_x \in [1 - 2p^{-1/4}, 1 + p^{-1/4}] \subset [1/2, 2]$. In particular, $Y_x \in S$ and $|Y_x|^s \geq (1 + p^{-1/4})^{-s} \geq 1 - p^{-1/4}$ on G . By (A1) and Lemma 13.9(ii),

$$\mathbb{P}(G^c) \leq e^{-c_0 \sqrt{p}} + 4e^{-\sqrt{K}/2} \leq p^{-1/4},$$

after enlarging p_1 and using (A4). We conclude that for $x \in S$,

$$(\mathsf{T}^* f)(x) = |x|^s \mathbb{E}[|Y_x|^s \mathbf{1}_S(Y_x)] \geq |x|^s (1 - p^{-1/4}) \mathbb{P}(G) \geq (1 - 2p^{-1/4}) f(x).$$

For $x \notin S$, we have $f(x) = 0 \leq (\mathsf{T}^* f)(x)$. Therefore $\mathsf{T}^* f \geq (1 - 2p^{-1/4}) f$ on $\mathbb{R}$. Since T^* is positive and $e_s \geq f$, we have by induction that $(\mathsf{T}^*)^L e_s \geq (\mathsf{T}^*)^L f \geq (1 - 2p^{-1/4})^L f$ for every $L \geq 1$. Integrating against ν^x , we obtain

$$\int (\mathsf{T}^*)^L e_s d\nu^x \geq (1 - 2p^{-1/4})^L \int f d\nu^x, \quad \int f d\nu^x = \mathbb{E}[|B|^{-s}; B \in S] \geq \frac{1}{2} \nu(S) \geq \frac{1}{4},$$

where we used that $X = B^{-1} \in S$ if and only if $B \in S$, and that $\nu(S) \geq 1 - e^{-c_0 \sqrt{p}} \geq 1/2$ by (A1). Taking L -th roots, letting $L \rightarrow \infty$ and using (13.11), we conclude that (13.35) holds. $\square$

13.5 Six regions and the block matrix

For the upper bound, we conjugate T^* by the capped weight v of (13.24). For a measurable function $F: \mathbb{R} \rightarrow [0, \infty]$, or a bounded measurable F , define

$$(\mathsf{L}F)(x) = \left(1 + \left|\frac{x}{K}\right|^s\right) \mathbb{E}[M_K(D_x) F(Y_x)], \quad x \in \mathbb{R}. \quad (13.36)$$

Since $v(Y_x) = M_K(D_x)$ and $v(x)(1 + |x/K|^s) = |x|^s$, we have

$$\mathsf{T}^*(vF) = v \mathsf{L}F \quad (13.37)$$

for every measurable $F \geq 0$; at $x = 0$ both sides vanish. The point of the cap is that $M_K \leq K^s$: a single step of L is finite even when D_x is close to zero. The price is the logarithmic factor ℓ in (13.26).

We now cover the real line by six regions. For a state $x > 0$ we use the reciprocal coordinate $u = K/x$, and for $x < 0$ we write $x = -K/u$. The two scales of the construction are

$$R = \log p, \quad u_* = p^{-1}. \quad (13.38)$$

We always take p_1 so large that $R \geq 2$ and $R^2 \leq K$ for $p \geq p_1$; this is possible by (A4).

Definition 13.14. Let $\mathcal{C} = \{c, t, 0, +, +\infty, -\}$. The *regions* are the sets

$$\begin{aligned} S_c &= [\tfrac{1}{2}, 2], & S_0 &= [-R, R] \setminus S_c, & S_- &= (-\infty, -R), \\ S_+ &= (R, K/R), & S_t &= [K/R, K/u_*], & S_{+\infty} &= (K/u_*, \infty). \end{aligned}$$

We call S_c the core region, S_t the tail region, S_0 the bounded region, S_+ the intermediate region, $S_{+\infty}$ the supertail region, and S_- the negative region. The labels c and t are *recurrent*, and the other four labels are *transient*. The similarity functions are $r_i \equiv 1$ on S_i for $i \neq t$, and

$$r_t(x) = (1 + u^s)\phi_s(u), \quad u = K/x, \quad x \in S_t. \quad (13.39)$$

Since $2 \leq R$ and $R < K/R$, the regions are pairwise disjoint and their union is $\mathbb{R}$. In the reciprocal coordinate, $x \in S_+$ if and only if $u \in (R, K/R)$, $x \in S_t$ if and only if $u \in [u_*, R]$, and $x \in S_{+\infty}$ if and only if $u \in (0, u_*)$. For $x \in S_-$ we have $u \in (0, K/R)$. By Lemma 13.12(i), for $p \geq p_1$,

$$\tfrac{1}{3} p^{-9/4} \leq r_t(x) \leq 5, \quad x \in S_t; \quad (13.40)$$

for the lower bound we used $\min\{1, u\}^{3/4} e^{-3u/2} \geq u_*^{3/4} e^{-3R/2} = p^{-9/4}$ for $u \in [u_*, R]$. In particular, $1/\inf_{S_i} r_i \leq 3p^{9/4}$ and $\sup_{S_i} r_i \leq 5$ for every $i \in \mathcal{C}$.

The tail similarity is chosen so that, for $y = K/w > 0$ with $w > 0$,

$$v(y) (1 + w^s)\phi_s(w) = \frac{K^s}{1 + w^s} (1 + w^s)\phi_s(w) = K^s \phi_s(w). \quad (13.41)$$

In the tail region the weight $v r_t$ is therefore the profile $x \mapsto K^s \phi_s(K/x)$.

Definition 13.15. For $i, j \in \mathcal{C}$ and $x \in S_i$, set

$$a_{ij}(x) = r_i(x)^{-1} (L(r_j \mathbf{1}_{S_j}))(x), \quad a_{ij} = \sup_{x \in S_i} a_{ij}(x) \in [0, \infty]. \quad (13.42)$$

We call a_{ij} the *block norm* from i to j , and $A = (a_{ij})_{i,j \in \mathcal{C}}$ the block matrix; the current region indexes the row and the future region indexes the column. Since L is positive, a_{ij} is the norm, on bounded Borel functions with the supremum norm, of the operator $q \mapsto r_i^{-1} L(r_j \mathbf{1}_{S_j} q)|_{S_i}$. We split $a_{ij}(x) = a_{ij}^{\text{reg}}(x) + a_{ij}^{\text{irr}}(x)$, where the regular part $a_{ij}^{\text{reg}}(x)$ is defined as $a_{ij}(x)$ with the factor $\chi(B)$ inserted in the expectation in (13.36), and the irregular part $a_{ij}^{\text{irr}}(x)$ with the factor $1 - \chi(B)$ inserted.

A vector d with positive entries and $Ad \leq \lambda d$ entrywise gives a supersolution for T^* , and a supersolution bounds the growth rate.

Lemma 13.16. Suppose that $a_{ij} < \infty$ for all $i, j \in \mathcal{C}$, and that there exist $\lambda > 0$ and $d_i > 0$, $i \in \mathcal{C}$, with

$$\sum_{j \in \mathcal{C}} a_{ij} d_j \leq \lambda d_i, \quad i \in \mathcal{C}. \quad (13.43)$$

Then the function $h = v \sum_{i \in \mathcal{C}} d_i r_i \mathbf{1}_{S_i}$ satisfies $\mathsf{T}^* h \leq \lambda h$ on $\mathbb{R}$. If moreover v is the law of $g(E)D$ for a real solution π at $E = \xi \sqrt{p} > 2$ and $K = p/g(E)^2$, then $r_{\pi,s} \leq \lambda$.

Proof. Let $w = \sum_j d_j r_j \mathbf{1}_{S_j}$, so that $h = vw$. For $x \in S_i$, by the definition of a_{ij} we have

$$(\mathsf{L}w)(x) = \sum_j d_j (\mathsf{L}(r_j \mathbf{1}_{S_j}))(x) \leq r_i(x) \sum_j a_{ij} d_j \leq \lambda d_i r_i(x) = \lambda w(x).$$

By (13.37), $\mathsf{T}^* h = v \mathsf{L}w \leq \lambda vw = \lambda h$.

For the second claim, we use the fractional cancellation estimate (6.25) with $q = s$, which holds for the law v of $g(E)D$ by Proposition 5.5; the variable T_q in (6.25) with $q = s$ is our T_s . Then there exists a constant $C_\pi < \infty$ with $\mathbb{E}|B - \sigma m T_s|^{-s} \leq C_\pi \min\{1, m^{-s}\}$ for all $m \geq 0$ and $\sigma \in \{-1, 1\}$. With $m = |x|/K$ and (13.25), this gives

$$(\mathsf{T}^* e_s)(x) = |x|^s \mathbb{E}|D_x|^{-s} \leq C_\pi \min\{|x|^s, K^s\} \leq 2C_\pi v(x) \leq \frac{2C_\pi}{w_{\min}} h(x),$$

where $w_{\min} = \min_i d_i \inf_{S_i} r_i > 0$ by (13.40). By positivity and induction, $(\mathsf{T}^*)^L e_s \leq 2C_\pi w_{\min}^{-1} \lambda^{L-1} h$ for $L \geq 1$. Further, $h \leq w_{\max} v \leq w_{\max} e_s$ with $w_{\max} = \max_i d_i \sup_{S_i} r_i$, and $\int e_s d\nu^x = \mathbb{E}|B|^{-s} < \infty$ by Lemma 6.13(i). Then

$$\int (\mathsf{T}^*)^L e_s d\nu^x \leq \frac{2C_\pi w_{\max}}{w_{\min}} \mathbb{E}|B|^{-s} \lambda^{L-1},$$

and taking L -th roots in (13.11), we conclude that $r_{\pi,s} \leq \lambda$. $\square$

13.6 Block estimates

In this section we bound every block norm. We first collect the result; the proofs, one row of the block matrix at a time, follow. Recall that $\ell = \ell_{K,s}$ and $R = \log p$, $u_* = p^{-1}$. Set

$$\varepsilon_K = 40K e^{-\sqrt{K}/2}, \quad E_{\text{irr}} = 15 C_N \ell p^{9/4} e^{-c_0 \sqrt{p}}, \quad (13.44)$$

where C_N is the constant of Lemma 13.11.

Proposition 13.17. *There exist a numerical constant C_+ and a threshold p_1 depending only on $\mathbf{c}$ such that, for $p \geq p_1$, the block norms satisfy $a_{ij} \leq \bar{a}_{ij}$ for all $i, j \in \mathcal{C}$, where the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})$ is given in Table 13.1.*

Before the proof, we explain the structure of Table 13.1. The quantities ε_K and E_{irr} decay faster than every power of p by (A4) and (A5), while the factors $\ell \leq C_* \log p$ and the large entries $3p^{1.01}$ and $4p$ are at most polynomial. From a bounded state, the output is in the core unless T_s is larger than $\sqrt{K}$, which explains the entries ε_K . From an intermediate state, an output outside the core requires T_s of order $u \geq R$, which costs

Current region i	Future region j	Bound $\bar{a}_{ij}$
c	c	$1 + 4p^{-1/4}$
c or 0	$j \neq c$	$\varepsilon_K + E_{\text{irr}}$
0 or +	c	4
+	$j \neq c$	$C_+ \ell p^{-1/5} + E_{\text{irr}}$
$+\infty$	c	$8p^{-1/2} + E_{\text{irr}}$
$+\infty$	0	$14 + E_{\text{irr}}$
$+\infty$	t, +, $+\infty$, -	$80 \ell p^{-1/2} + E_{\text{irr}}$
-	c or 0	$6 + E_{\text{irr}}$
-	t, +, $+\infty$, -	E_{irr}
t	t	$K^{s-1} e^{R^2/K} (1 + C_\Gamma(1-s)) (1 + 4p^{-1/4}) + E_{\text{irr}}$
t	c	$3p^{1.01} + E_{\text{irr}}$
t	0	$2p^{1/2} \log p + 4p + E_{\text{irr}}$
t	+, $+\infty$, -	$12 \ell + E_{\text{irr}}$

Table 13.1. Bounds on the block norms a_{ij} for c-admissible data and $p \geq p_1$. The quantities ε_K and E_{irr} are defined in (13.44). The two recurrent diagonal entries are close to one (the tail entry for s close to one); the only transient blocks of order one are $+\infty \rightarrow 0$ and $- \rightarrow 0$, and both end in the bounded region, whose exits other than to the core are bounded by $\varepsilon_K + E_{\text{irr}}$.

$e^{-cR} = p^{-c}$. From a supertail state $x = K/u$ with $u < u_*$, the term T_s/u is typically very large and the output is a small negative number in the bounded region; other outputs require $T_s \lesssim u$ and cost a power of u_* . From a negative state, the output $1/(B + T_s|x|/K)$ is bounded and positive. From a tail state, the output is again in the tail with weight close to the profile mass $K^{s-1} \int \phi_s$, and the exits from the tail can be large but are at most polynomial in p .

Proof of Proposition 13.17. By Lemma 13.11, (13.25) and (13.40), for all $i, j \in \mathcal{C}$ and $x \in S_i$,

$$a_{ij}^{\text{irr}}(x) \leq \frac{\sup_{S_j} r_j}{\inf_{S_i} r_i} \left(1 + \left|\frac{x}{K}\right|^s\right) \mathbb{E}[(1 - \chi(B))M_K(D_x)] \leq 5 \cdot 3p^{9/4} \cdot C_N \ell e^{-c_0 \sqrt{p}} = E_{\text{irr}}. \quad (13.45)$$

It therefore suffices to bound the regular parts, except in the entries without E_{irr} in Table 13.1, which we bound directly. The bounds for the rows c and 0 are Lemma 13.18, for the row + Lemma 13.19, for the row $+\infty$ Lemma 13.20, for the row - Lemma 13.21, and for the row t Lemma 13.22. In the row t we used $R = \log p$ and $u_* = p^{-1}$ to

write $3e^{1.01R} = 3p^{1.01}$ and $2Re^{R/2} + 4u_*^{-1} = 2p^{1/2} \log p + 4p$, and in the row + we wrote $e^{-R/5} = p^{-1/5}$. $\square$

Lemma 13.18. *For $p \geq p_1$ the following hold.*

- (i) $a_{cc} \leq 1 + 4p^{-1/4}$.
- (ii) $a_{0c} \leq 4$.
- (iii) For $i \in \{c, 0\}$, $j \neq c$ and $x \in S_i$, we have $a_{ij}^{\text{reg}}(x) \leq \varepsilon_K$.

This uses (A1) and (A4).

Proof. Let $x \in S_c \cup S_0 = [-R, R]$, so that $m = |x|/K \leq R/K \leq 1$ and $1 + m^s \leq 2$. We have $r_i(x) = 1$. For $Y_x \in S_c$, we have $M_K(D_x) = v(Y_x) \leq |Y_x|^s \leq 2$ by (13.25), and since $1 + m^s \leq 2$,

$$a_{ic}(x) = (1 + m^s) \mathbb{E}[M_K(D_x) \mathbf{1}_{S_c}(Y_x)] \leq 4.$$

This proves (ii).

For (i), let $x \in S_c$ and let G be the event of the proof of Lemma 13.13. On G we showed that $D_x \in [1 - 2p^{-1/4}, 2]$, which implies $M_K(D_x) \leq |D_x|^{-s} \leq (1 - 2p^{-1/4})^{-1} \leq 1 + 3p^{-1/4}$. On G^c we use $M_K(D_x) \mathbf{1}_{S_c}(Y_x) \leq 2$ and $\mathbb{P}(G^c) \leq e^{-c_0\sqrt{p}} + 4e^{-\sqrt{K}/2}$. Since $m \leq 2/K$ and $s \geq 3/4$, we have $1 + m^s \leq 1 + (2/K)^{3/4}$. Then

$$a_{cc}(x) \leq (1 + (2/K)^{3/4}) (1 + 3p^{-1/4} + 2e^{-c_0\sqrt{p}} + 8e^{-\sqrt{K}/2}) \leq 1 + 4p^{-1/4},$$

after enlarging p_1 and using (A4).

For (iii), suppose that $\chi(B) = 1$ and $T_s \leq \sqrt{K}$. Then $|T_s x/K| \leq R/\sqrt{K} \leq 1/100$ after enlarging p_1 . It follows that $D_x \in [98/100, 102/100]$ and $Y_x \in S_c$. We conclude that $\{\chi(B) = 1, Y_x \notin S_c\} \subset \{T_s > \sqrt{K}\}$. Using $M_K \leq K^s \leq K$, $r_j \leq 5$ and Lemma 13.9(ii), for $j \neq c$ we obtain

$$a_{ij}^{\text{reg}}(x) \leq (1 + m^s) \cdot 5K \cdot \mathbb{P}(T_s > \sqrt{K}) \leq 2 \cdot 5K \cdot 4e^{-\sqrt{K}/2} = \varepsilon_K. \quad \square$$

In the remaining rows we use the following change of variables. Let $u > 0$, $b > 0$, and let $F \geq 0$ be measurable. Substituting $d = b - t/u$, so that $t = u(b - d)$ and $dt = u dd$, we have

$$\mathbb{E}\left[F\left(b - \frac{T_s}{u}\right)\right] = \int_0^\infty \omega_s(t) F\left(b - \frac{t}{u}\right) dt = u \int_{-\infty}^b \omega_s(u(b - d)) F(d) dd. \quad (13.46)$$

For $x = K/u > 0$ and $B = b$, the variable $b - T_s/u$ is D_x .

Lemma 13.19. *There exist a numerical constant C_+ and p_1 such that for $p \geq p_1$ the following hold.*

- (i) $a_{+c} \leq 4$.

(ii) For $j \neq c$ and $x \in S_+$, we have $a_{+j}^{\text{reg}}(x) \leq C_+ \ell e^{-R/5}$.

Proof. Let $x \in S_+$ and $u = K/x \in (R, K/R)$, so that $m = u^{-1} \leq 1$, $1 + m^s \leq 2$ and $r_+(x) = 1$. Part (i) follows as in the proof of Lemma 13.18(ii).

For (ii), fix $b \in I_*$. Since $D_x = b - T_s/u \leq b \leq 101/100 < 2$, the event $Y_x \notin S_c$ forces $D_x < 1/2$, hence $T_s > u(b - 1/2) \geq 0.49u$. Using $r_j \leq 5$, we have

$$a_{+j}^{\text{reg}}(x) \leq 10 \sup_{b \in I_*} \mathbb{E}[M_K(b - T_s/u); T_s > 0.49u].$$

On the event $\{|b - T_s/u| \leq 1\}$, we use (13.46), the bound $\omega_s(t) \leq 2e^{-t/2}$ for $t > 0.49u$, and (13.26):

$$\mathbb{E}[M_K(b - T_s/u); T_s > 0.49u, |b - T_s/u| \leq 1] \leq u \cdot 2e^{-0.245u} \int_{-1}^1 M_K(d) dd \leq 4\ell u e^{-0.245u}.$$

On the complement, $M_K \leq 1$, and Lemma 13.9(ii) implies that this part is at most $\mathbb{P}(T_s > 0.49u) \leq 4e^{-0.245u}$. Adding, we find $a_{+j}^{\text{reg}}(x) \leq 40\ell(1+u)e^{-0.245u} \leq C_+ \ell e^{-u/5} \leq C_+ \ell e^{-R/5}$, with $C_+ = 40 \sup_{u \geq 0} (1+u)e^{-0.045u}$. $\square$

Lemma 13.20. For $p \geq p_1$ and $x \in S_{+\infty}$, the following hold.

- (i) $a_{+\infty, c}^{\text{reg}}(x) \leq 8u_*^{1/2}$.
- (ii) $a_{+\infty, 0}^{\text{reg}}(x) \leq 14$.
- (iii) For $j \in \{t, +, +\infty, -\}$, we have $a_{+\infty, j}^{\text{reg}}(x) \leq 80\ell u_*^{1/2}$.

Proof. Let $x \in S_{+\infty}$ and $u = K/x \in (0, u_*)$, so that $m = u^{-1}$, $1 + m^s \leq 2u^{-s}$ and $r_{+\infty}(x) = 1$. Fix $b \in I_*$; we bound the conditional expectation given $B = b$, uniformly in b .

For (i), $Y_x \in S_c$ requires $D_x \geq 1/2$, hence $T_s \leq u(b - 1/2) \leq u$. Since $M_K(D_x) \leq 2$ on $\{Y_x \in S_c\}$, Lemma 13.9(ii) gives $a_{+\infty, c}^{\text{reg}}(x) \leq 2u^{-s} \cdot 2 \cdot 2u^{1+s/2} = 8u^{1-s/2} \leq 8u_*^{1/2}$, since $u < 1$ and $1 - s/2 \geq 1/2$.

For (ii), $Y_x \in S_0$ implies $|Y_x| \leq R$, that is $|D_x| \geq 1/R$, equivalently $|ub - T_s| \geq u/R$. By (13.25),

$$(1 + m^s)M_K(D_x) \leq (1 + u^{-s})|b - T_s/u|^{-s} = (1 + u^s)|ub - T_s|^{-s} \leq 2|ub - T_s|^{-s}.$$

On $\{T_s \geq 2ub\}$, we have $|ub - T_s| \geq T_s/2$, and Lemma 13.9(iii) bounds the contribution by $2 \cdot 2 \cdot 3 = 12$. On $\{T_s < 2ub\}$, we have $T_s < 3u$ and $\omega_s(T_s) \leq 2(3u)^{s/2}$, and the variable $z = |ub - T_s|$ ranges over $[u/R, ub]$, each value being attained at most twice. Then

$$\begin{aligned} \mathbb{E}[2|ub - T_s|^{-s}; T_s < 2ub, |ub - T_s| \geq u/R] &\leq 2 \cdot 2(3u)^{s/2} \cdot 2 \int_{u/R}^{ub} z^{-s} dz \\ &\leq 24 u^{1-s/2} \log(2R), \end{aligned}$$

where we used $\int_{u/R}^{ub} z^{-s} dz \leq (ub)^{1-s} \log(bR)$ and $b \leq 2$. Since $u < u_* = p^{-1}$, we have $24u^{1-s/2} \log(2R) \leq 24p^{-1/2} \log(2 \log p) \leq 2$ after enlarging p_1 . This proves (ii).

For (iii), $Y_x \in S_j$ with $j \in \{t, +, +\infty, -\}$ implies $|Y_x| > R$, that is $|D_x| < 1/R$. Using $r_j \leq 5$, (13.46), the bound $\omega_s(u(b-d)) \leq 2(2u)^{s/2}$ for $|d| < 1/R$ (since $u(b-d) \leq 2u$), and (13.26), we obtain

$$\begin{aligned} a_{+\infty, j}^{\text{reg}}(x) &\leq 2u^{-s} \cdot 5 \cdot u \int_{|d| < 1/R} 2(2u)^{s/2} M_K(d) dd \\ &\leq 2u^{-s} \cdot 5 \cdot u \cdot 4u^{s/2} \cdot 2\ell = 80\ell u^{1-s/2} \leq 80\ell u_*^{1/2}. \end{aligned} \quad \square$$

Lemma 13.21. *For $p \geq p_1$ and $x \in S_-$, we have $a_{-,c}^{\text{reg}}(x) + a_{-,0}^{\text{reg}}(x) \leq 6$, and $a_{-,j}^{\text{reg}}(x) = 0$ for $j \in \{t, +, +\infty, -\}$.*

Proof. Let $x = -K/u \in S_-$, with $u \in (0, K/R)$, so that $m = u^{-1}$ and $r_-(x) = 1$. If $\chi(B) = 1$, then $D_x = B + T_s/u \geq 0.99$ and $Y_x \in (0, 1/0.99] \subset S_0 \cup S_c$. This proves the second claim. For the first, (13.25) gives

$$(1 + m^s) M_K(D_x) \leq (1 + u^{-s})(B + T_s/u)^{-s} = (1 + u^s)(uB + T_s)^{-s}.$$

If $u \leq 1$, this is at most $2T_s^{-s}$, whose expectation is at most 6 by Lemma 13.9(iii). If $u > 1$, it is at most $2u^s(0.99u)^{-s} \leq 3$. $\square$

For the tail row, we write $x = K/u$ with $u \in [u_*, R]$. Since $(1 + u^{-s})/r_t(x) = (1 + u^{-s})/((1 + u^s)u^{1-s}\varrho_s(u)) = 1/(u\varrho_s(u))$, the change of variables (13.46) gives, for every measurable $F \geq 0$,

$$\frac{1 + m^s}{r_t(x)} \mathbb{E}[\chi(B) M_K(D_x) F(Y_x)] = \frac{1}{\varrho_s(u)} \mathbb{E}\left[\chi(B) \int_{-\infty}^B \omega_s(u(B-d)) M_K(d) F(d^{-1}) dd\right]. \quad (13.47)$$

By (13.22), for $b \in I_*$ and $d < b$ we have $\omega_s(u(b-d)) \leq \omega_s(ub) W_u(d)$, where

$$W_u(d) = \begin{cases} e^{ud} \mathbf{1}_{\{d < 1.01\}}, & d \geq 0, \\ (1 + 2|d|)^{s/2} e^{-u|d|}, & d < 0; \end{cases} \quad (13.48)$$

here we used $(1 - d/b)^{s/2} \leq 1$ for $d \geq 0$ and $(1 + |d|/b)^{s/2} \leq (1 + 2|d|)^{s/2}$ for $d < 0$. Since $\mathbb{E}[\chi(B)\omega_s(uB)] = \varrho_s(u)$, Tonelli's theorem and (13.47) give

$$\frac{1 + m^s}{r_t(x)} \mathbb{E}[\chi(B) M_K(D_x) F(Y_x)] \leq \int_{\mathbb{R}} W_u(d) M_K(d) F(d^{-1}) dd. \quad (13.49)$$

Lemma 13.22. *For $p \geq p_1$ and $x \in S_t$, the following hold.*

- (i) $a_{tt}^{\text{reg}}(x) \leq K^{s-1} e^{R^2/K} \int_0^\infty \phi_s(w) dw$.
- (ii) $a_{tc}^{\text{reg}}(x) \leq 3e^{1.01R}$.

- (iii) $a_{t0}^{\text{reg}}(x) \leq 2Re^{R/2} + 4u_*^{-1}$.
- (iv) $\sum_{j \in \{+, +\infty, -\}} a_{tj}^{\text{reg}}(x) \leq 12\ell$.

This uses (A1), through Lemma 13.12.

Proof. We apply (13.49) with $F = r_j \mathbf{1}_{S_j}$, and we determine the set of reciprocals $d = 1/Y_x$ for each region. Recall that $u \leq R$.

For (i), $Y_x \in S_t$ if and only if $d \in [u_*/K, R/K]$. For such d , we have $W_u(d) \leq e^{uR/K} \leq e^{R^2/K}$, and (13.41) with $w = Kd$ implies that $M_K(d)r_t(d^{-1}) = K^s \phi_s(Kd)$. The substitution $w = Kd$ gives

$$a_{tt}^{\text{reg}}(x) \leq e^{R^2/K} \int_{u_*/K}^{R/K} K^s \phi_s(Kd) dd \leq K^{s-1} e^{R^2/K} \int_0^\infty \phi_s(w) dw.$$

For (ii), $Y_x \in S_c$ if and only if $d \in [1/2, 2]$. There $W_u(d) \leq e^{1.01u} \leq e^{1.01R}$ and $M_K(d) \leq d^{-s} \leq 2$, and the interval has length $3/2$. This gives (ii).

For (iii), $Y_x \in S_0$ if and only if $d \in (-\infty, -1/R] \cup [1/R, 1/2) \cup (2, \infty)$. On $(2, \infty)$ we have $W_u = 0$. On $[1/R, 1/2)$, we have $W_u(d) \leq e^{u/2} \leq e^{R/2}$ and $M_K(d) \leq d^{-s} \leq R$, which contributes at most $Re^{R/2}/2$. On $(-\infty, -1/R]$, write $t = |d|$ and use $M_K(d) \leq t^{-s}$. For $t \in [1/R, 1]$, we have $W_u \leq \sqrt{3}$ and $\int_{1/R}^1 t^{-s} dt \leq \log R$. For $t \geq 1$, we have $(1+2t)^{s/2} t^{-s} \leq (3t)^{s/2} t^{-s} \leq \sqrt{3} t^{-s/2}$, and

$$\int_1^\infty \sqrt{3} t^{-s/2} e^{-ut} dt \leq \sqrt{3} \Gamma(1-s/2) u^{s/2-1} \leq \sqrt{3\pi} \max\{1, u_*^{-1}\} \leq 4u_*^{-1},$$

since $u \geq u_*$ and $\Gamma(1-s/2) \leq \sqrt{\pi}$. Adding, we find $a_{t0}^{\text{reg}}(x) \leq Re^{R/2}/2 + \sqrt{3} \log R + 4u_*^{-1} \leq 2Re^{R/2} + 4u_*^{-1}$.

For (iv), $Y_x \in S_+ \cup S_{+\infty} \cup S_-$ implies $|Y_x| > R$, that is $|d| < 1/R$, and $r_j = 1$ for these regions. For $|d| < 1/R$ we have $W_u(d) \leq (1+2/R)^{s/2} e^{u/R} \leq 2e \leq 6$, since $u \leq R$ and $R \geq 2$. By (13.26), the sum in (iv) is at most $6 \int_{|d| < 1/R} M_K(d) dd \leq 12\ell$. $\square$

Combining Lemma 13.22(i) with Lemma 13.12(ii), we find

$$a_{tt}^{\text{reg}} \leq K^{s-1} e^{R^2/K} (1 + C_\Gamma(1-s)) (1 + 4p^{-1/4}). \quad (13.50)$$

This is the diagonal entry of the tail region in Table 13.1. The factor K^{s-1} is at most one, and it tends to one as $s \uparrow 1$ for fixed K ; the profile mass tends to a number close to one. This is the second source of growth of T^* at $s = 1$.

13.7 Balancing a nonnegative matrix

The following lemma bounds the growth of a nonnegative matrix whose large entries are confined to a set of diagonal positions and whose other closed chains have small

products. It is the standard feasibility criterion for a system of difference constraints: such a system is solvable if and only if no cycle has negative total weight, and a solution is given by shortest-path potentials.

Lemma 13.23. *Let $n \geq 1$, let $\bar{A} = (\bar{a}_{ij})_{i,j=1}^n$ have nonnegative entries, and let $\Delta \subset \{1, \dots, n\}$. Call (i, j) an allowed edge if $\bar{a}_{ij} > 0$ and $(i, j) \neq (k, k)$ for every $k \in \Delta$. A simple cycle of length $k \geq 1$ is a sequence $(i_1, \dots, i_k)$ of distinct indices such that $(i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i_1)$ are allowed edges, and its weight is the product of $\bar{a}$ over these k edges. Let q be the maximum of $(\text{weight})^{1/k}$ over all simple cycles, with $q = 0$ if there are none. Then for every $\varepsilon > q$ there exist $d_1, \dots, d_n \in (0, 1]$ such that $\bar{a}_{ij}d_j \leq \varepsilon d_i$ whenever $(i, j) \neq (k, k)$ for all $k \in \Delta$. In particular,*

$$\sum_{j=1}^n \bar{a}_{ij}d_j \leq \left(\max_{k \in \Delta} \bar{a}_{kk} + n\varepsilon \right) d_i, \quad i = 1, \dots, n, \quad (13.51)$$

where the maximum over the empty set is zero.

Proof. Since $\varepsilon > q \geq 0$, the number $\log \varepsilon$ is finite. Give each allowed edge (i, j) the weight $w_{ij} = \log \varepsilon - \log \bar{a}_{ij}$. A walk of length $k \geq 0$ is a sequence $i_0, i_1, \dots, i_k$ such that each (i_{l-1}, i_l) is an allowed edge, and its weight is $\sum_{l=1}^k w_{i_{l-1}i_l}$; a walk of length zero has weight zero. A walk is closed if $k \geq 1$ and $i_k = i_0$.

STEP 1. Closed walks have positive weight. We argue by induction on the length k . If $i_0, \dots, i_{k-1}$ are distinct, the closed walk is a simple cycle of some weight $\Pi > 0$, and its weight as a walk is $k \log \varepsilon - \log \Pi = k(\log \varepsilon - \log \Pi^{1/k}) > 0$, since $\Pi^{1/k} \leq q < \varepsilon$. Otherwise there exist $0 \leq a < b \leq k-1$ with $i_a = i_b$. Then $(i_a, \dots, i_b)$ and $(i_0, \dots, i_a, i_{b+1}, \dots, i_k)$ are closed walks of lengths $b-a$ and $k-(b-a)$, both between 1 and $k-1$, and their weights add up to the weight of the original walk. The induction hypothesis completes the proof of the claim.

STEP 2. Potentials. For each j , let u_j be the infimum of the weights of all walks ending at j . The walk of length zero gives $u_j \leq 0$. If a walk ending at j visits some index twice, it contains a closed subwalk, and removing that subwalk produces a shorter walk ending at j whose weight is not larger, by Step 1. Repeating this, we see that the infimum equals the minimum over walks with distinct indices, of which there are finitely many. Therefore $u_j \in (-\infty, 0]$.

STEP 3. Conclusion. If (i, j) is an allowed edge, appending it to a walk ending at i of weight u_i gives $u_j \leq u_i + w_{ij}$. Set $d_j = e^{u_j} \in (0, 1]$. Then $d_j \leq d_i e^{w_{ij}} = \varepsilon d_i / \bar{a}_{ij}$, that is $\bar{a}_{ij}d_j \leq \varepsilon d_i$. If $\bar{a}_{ij} = 0$, this inequality is trivial. This proves the first claim. For (13.51), if $i \in \Delta$ we bound the diagonal term by $\bar{a}_{ii}d_i$ and the other $n-1$ terms by εd_i ; if $i \notin \Delta$, we bound all n terms by εd_i . $\square$

Let $D = \text{diag}(d_1, \dots, d_n)$. The matrix $D^{-1}\bar{A}D$ has the same spectral radius as $\bar{A}$, and by (13.51) its row sums are at most $\max_{k \in \Delta} \bar{a}_{kk} + n\varepsilon$. Since the spectral radius is at most

the maximal row sum, the spectral radius of $\bar{A}$ is at most $\max_{k \in \Delta} \bar{a}_{kk} + n\varepsilon$. We use (13.51) directly, through Lemma 13.16.

13.8 The upper bound

We now apply Lemma 13.23 to the matrix $\bar{A}$ of Table 13.1, with $\Delta = \{c, t\}$. The first step is to show that every simple cycle other than the two recurrent loops has small weight. Define

$$\begin{aligned} \bar{E}_{\text{irr}} &= 15C_N C_* p^{9/4} (\log p) e^{-c_0 \sqrt{p}}, & \theta_1 &= 40c_* p^2 e^{-\sqrt{c_*} p/2} + \bar{E}_{\text{irr}}, \\ \theta_2 &= C_+ C_* (\log p) p^{-1/5} + \bar{E}_{\text{irr}}, & \theta_3 &= 88C_* (\log p) p^{-1/2} + \bar{E}_{\text{irr}}, \\ \Theta &= \max\{\theta_1, \theta_2, \theta_3\}, \end{aligned} \quad (13.52)$$

and

$$Q_p = \max\{\theta_1 p^{10}, 13C_* (\log p) \Theta\}. \quad (13.53)$$

These quantities depend only on c and p , and $Q_p \rightarrow 0$ as $p \rightarrow \infty$, since θ_1 decays faster than every power of p and $\Theta \leq C(\log p) p^{-1/5}$ for large p .

Proposition 13.24. *There exists p_1 , depending only on c , such that for $p \geq p_1$ every simple cycle of $\bar{A}$ with respect to $\Delta = \{c, t\}$ has weight at most $Q_p \leq 1$. In particular, the number q of Lemma 13.23 for $\bar{A}$ and Δ satisfies $q \leq Q_p^{1/6}$.*

Proof. We first compare the entries of Table 13.1 with the quantities (13.52). By (A5), $E_{\text{irr}} \leq \bar{E}_{\text{irr}}$ and $1 \leq \ell \leq C_* \log p$. The function $K \mapsto 40K e^{-\sqrt{K}/2}$ is decreasing on $[16, \infty)$. By (A4), we have $\varepsilon_K \leq 40c_* p^2 e^{-\sqrt{c_*} p/2}$ once $c_* p^2 \geq 16$. Then, after enlarging p_1 so that $\Theta \leq 1$, $Q_p \leq 1$ and $\bar{E}_{\text{irr}} \leq 1$:

- (a) every edge leaving c or 0 , except the edges into c , is at most θ_1 ;
- (b) every edge leaving $+$, except the edge into c , is at most θ_2 ;
- (c) every edge from $+\infty$ to t , $+$, $+\infty$ or $-$ is at most θ_3 , since $80\ell p^{-1/2} \leq 88C_* (\log p) p^{-1/2}$;
- (d) every edge from $-$ to t , $+$, $+\infty$ or $-$ is at most $\bar{E}_{\text{irr}} \leq \theta_1$;
- (e) every edge from t to $+$, $+\infty$ or $-$ is at most $12\ell + 1 \leq 13C_* \log p$;
- (f) every entry of $\bar{A}$ is at most p^2 , after enlarging p_1 .

Let γ be a simple cycle. We distinguish four cases.

If γ passes through c , then, since the loop at c is not allowed, γ contains an edge from c to some $j \neq c$. By (a) this edge is at most θ_1 , and by (f) each of the at most five other edges is at most p^2 . The weight is at most $\theta_1 p^{10}$.

If γ passes through 0 but not through c , it contains an edge from 0 to some $j \neq c$ (possibly $j = 0$). As in the previous case, the weight is at most $\theta_1 p^{10}$.

If γ passes through t but through neither c nor 0 , then its indices lie in $\{t, +, +\infty, -\}$, and since the loop at t is not allowed, it has length at least two. Then γ contains exactly one edge into t , which starts at $+$, $+\infty$ or $-$ and is at most Θ by (b)–(d); exactly one edge out of t , which is at most $13C_* \log p$ by (e); and other edges between two of the labels $+$, $+\infty$, $-$, each at most $\Theta \leq 1$ by (b)–(d). The weight is at most $13C_*(\log p)\Theta$.

If γ passes only through $+$, $+\infty$ and $-$, each of its edges is at most $\Theta \leq 1$ by (b)–(d), and its weight is at most $\Theta \leq Q_p$.

This proves that every simple cycle has weight at most Q_p . Since simple cycles have length at most six and $Q_p \leq 1$, we conclude that $(\text{weight})^{1/k} \leq Q_p^{1/k} \leq Q_p^{1/6}$. $\square$

The proof shows how the two recurrent regions are separated. Every cycle through the core leaves it through an edge of size θ_1 , which decays faster than every power of p , while the return edges $t \rightarrow c$ and $t \rightarrow 0$ are at most polynomially large. Every cycle through the tail that avoids the core and the bounded region returns to the tail through an edge of size Θ , which is at most $C(\log p)p^{-1/5}$, while the exits from the tail to the transient regions $+$, $+\infty$, $-$ are at most logarithmic. The only transient blocks of order one, $+\infty \rightarrow 0$ and $- \rightarrow 0$, end in the bounded region, and every exit from the bounded region other than to the core costs θ_1 .

Remark 13.25. In the notation $R = L_0 \log p$ and $u_* = p^{-L_1}$, the argument works for any fixed $L_0, L_1 > 0$, and we have taken $L_0 = L_1 = 1$. The choice of the two scales enters only through the factors $p^{-L_0/5}$ and $p^{-L_1/2}$ in θ_2 and θ_3 , which must beat the logarithmic exit bound $13C_* \log p$ from the tail, and through polynomial factors $p^{O(L_0+L_1)}$ in $\bar{E}_{\text{irr}}$ and in the large entries, which are absorbed by θ_1 . No order of choice between L_0 and L_1 is needed, because the exits from the tail into the transient regions other than the bounded region are bounded by 12ℓ through the capped estimate (13.26).

Proposition 13.26. *There exist p_1 , depending only on c , and $\delta_p = 6(Q_p^{1/6} + p^{-1})$ such that the following holds for $p \geq p_1$. Let π be a real solution at $E = \xi\sqrt{p} > 2$, let ν be the law of $g(E)D$, let $K = p/g(E)^2$, and suppose that (ν, K, s) is c -admissible at p . Then*

$$r_{\pi,s} \leq \max\left\{1 + 4p^{-1/4}, K^{s-1}e^{R^2/K}(1 + C_\Gamma(1-s))(1 + 4p^{-1/4}) + \bar{E}_{\text{irr}}\right\} + \delta_p. \quad (13.54)$$

Proof. By Proposition 13.17, $a_{ij} \leq \bar{a}_{ij} < \infty$ for all i, j , and the two diagonal entries $\bar{a}_{cc}$ and $\bar{a}_{tt}$ are at most the two terms in the maximum in (13.54), since $E_{\text{irr}} \leq \bar{E}_{\text{irr}}$. Let $\varepsilon = Q_p^{1/6} + p^{-1}$, which is larger than q by Proposition 13.24. Lemma 13.23 with $n = 6$ and $\Delta = \{c, t\}$ gives $d_i \in (0, 1]$ with

$$\sum_j a_{ij}d_j \leq \sum_j \bar{a}_{ij}d_j \leq (\max\{\bar{a}_{cc}, \bar{a}_{tt}\} + 6\varepsilon)d_i, \quad i \in \mathcal{C}.$$

Lemma 13.16 with $\lambda = \max\{\bar{a}_{cc}, \bar{a}_{tt}\} + 6\varepsilon$ now implies that $r_{\pi,s} \leq \lambda$, which is (13.54). $\square$

13.9 Proof of the leading asymptotics

Proof of Theorem 13.1. Fix a compact set $\Xi \subset (0, \infty)$, and let $\xi_- = \min \Xi$. Let c_Ξ and p_Ξ be given by Lemma 13.8(i), and let $p_1 = p_1(c_\Xi)$ be so large that Lemma 13.13 and Proposition 13.26 hold. Let $p \geq \max\{p_\Xi, p_1\}$ and $\xi \in \Xi$, and set $E = \xi\sqrt{p}$. Let π be the law of $B/g(E)$ under $\nu_{p,E}$; it is a real solution at E because $p \geq p_0(\xi_-)$ (Section 13.2), and $\lambda_p(\xi) = \exp \varphi_\pi(1)$. For every $s \in [3/4, 1)$, the triple $(\nu_{p,E}, p^2 a_p(\xi)^2, s)$ is c_Ξ -admissible at p . Using $K^{s-1} \leq 1$ and $R^2/K \leq (\log p)^2/(c_* p^2)$ with $c_* = \xi_-^2/4$, Lemma 13.13 and Proposition 13.26 give

$$1 - 2p^{-1/4} \leq r_{\pi,s} \leq \max\left\{1 + 4p^{-1/4}, e^{(\log p)^2/(c_* p^2)}(1 + C_\Gamma(1-s))(1 + 4p^{-1/4}) + \bar{E}_{\text{irr}}\right\} + \delta_p.$$

Letting $s \uparrow 1$ and using Lemma 13.6, we obtain

$$\frac{\mu_*}{a_p(\xi)}(1 - 2p^{-1/4}) \leq \lambda_p(\xi) \leq \frac{\mu_*}{a_p(\xi)}(1 + \delta'_p), \quad (13.55)$$

where

$$\delta'_p = \max\left\{4p^{-1/4}, e^{(\log p)^2/(c_* p^2)}(1 + 4p^{-1/4}) - 1 + \bar{E}_{\text{irr}}\right\} + \delta_p$$

depends only on Ξ and p and tends to zero as $p \rightarrow \infty$.

It remains to compare $a_p(\xi)$ with ξ . By (13.3),

$$0 \leq \xi - a_p(\xi) = \frac{\xi - \sqrt{\xi^2 - 4/p}}{2} = \frac{2}{p(\xi + \sqrt{\xi^2 - 4/p})} \leq \frac{2}{p\xi},$$

and $a_p(\xi) \geq \xi/2$. Together these give $|a_p(\xi)^{-1} - \xi^{-1}| \leq 4/(p\xi^3)$ and $a_p(\xi)^{-1} \leq 2/\xi_-$. Combining these bounds with (13.55), we find

$$\sup_{\xi \in \Xi} \left| \lambda_p(\xi) - \frac{\mu_*}{\xi} \right| \leq \frac{4\mu_*}{p\xi_-^3} + \frac{2\mu_*}{\xi_-} \max\{2p^{-1/4}, \delta'_p\},$$

which tends to zero as $p \rightarrow \infty$. □

13.10 The free energy at large energies

At energies $E \geq M\sqrt{p}$ with M large, the parameter $\theta = g(E)^2$ is at most $4/(M^2 p)$ and $K = p/\theta$ is unbounded as $E \rightarrow \infty$. We work at the exponent

$$s_p = 1 - \frac{\kappa}{\log p}, \quad \kappa = 1. \quad (13.56)$$

At this exponent $p^{1-s_p} = e^\kappa$, and the capped integrals of Lemma 13.10 are bounded by $2\ell_{K,s_p} \leq 2(1 + \kappa^{-1} \log p)$ uniformly in K . We fix the constants

$$c_1 = (4e)^{-8/3}, \quad M_{\text{ext}} = \max\{M_0, 2c_1^{-1/2}\} = 2(4e)^{4/3}, \quad (13.57)$$

where $M_0 = 3$ is the constant of Corollary 11.23. These constants satisfy $c_1 \leq c_{\text{ent}} = 1$, $e^\kappa c_1^{3/8} = 1/4$, and $4/M_{\text{ext}}^2 = c_1$. By Remark 11.24, the principal law at energies $E \geq M_{\text{ext}}\sqrt{p}$ is not affected by the choice of a constant larger than M_0 .

Remark 13.27. In the large-energy regime, the estimates of Sections 13.3–13.6 and 13.8 hold uniformly in $E \geq M_{\text{ext}}\sqrt{p}$, that is, in $\theta \in (0, c_1/p]$, because they depend on K only through the following quantities. The term $\varepsilon_K = 40Ke^{-\sqrt{K}/2}$ is decreasing for $K \geq 16$, and the terms R^2/K , $K^{-1/2}$ and K^{s-1} are nonincreasing in K ; we bound each by its value at $K = c_*p^2$. The capped integrals are bounded by $2\ell_{K,s}$, and at $s = s_p$ we have $\ell_{K,s_p} \leq 1 + \kappa^{-1} \log p$ for every $K \geq 1$. At a fixed $s < 1$ the second point would also hold, but then p^{1-s} would be unbounded; at $s = 1$ the capped integrals grow like $\log K$. The exponent s_p is chosen between these two cases. In addition, since $K \geq p^2$, the tail diagonal entry satisfies

$$\bar{a}_{\text{tt}} \leq K^{s_p-1} e^{R^2/K} (1 + C_\Gamma(1 - s_p)) (1 + 4p^{-1/4}) + \bar{E}_{\text{irr}} \leq e^{-2\kappa+o(1)},$$

because $K^{s_p-1} = \exp(-\kappa \log K / \log p) \leq e^{-2\kappa}$. In this regime the tail region is strictly subcritical, and the growth of $\mathbb{T}^*$ comes from the core alone.

Proof of Proposition 13.2. Let c_{ext} be the constants of Lemma 13.8(ii), and let $p_1 = p_1(c_{\text{ext}})$ be so large that Proposition 13.26 holds. Let $p_0(M_{\text{ext}})$ be the threshold of Corollary 11.23 for $\xi_0 = M_{\text{ext}}$. Let $p_{\text{ext}} \geq \max\{e^4, p_0(M_{\text{ext}}), p_1\}$ be so large that for $p \geq p_{\text{ext}}$,

$$\max\left\{1 + 4p^{-1/4}, e^{-2} e^{(\log p)^2/p^2} \left(1 + \frac{C_\Gamma}{\log p}\right) (1 + 4p^{-1/4}) + \bar{E}_{\text{irr}}\right\} + \delta_p \leq 2. \quad (13.58)$$

All these thresholds are numerical.

Let $p \geq p_{\text{ext}}$ and $E \geq M_{\text{ext}}\sqrt{p}$, and set $\theta = g(E)^2$ and $K = p/\theta$. Since $g(E) = 2/(E + \sqrt{E^2 - 4}) \leq 2/E$, we have

$$\theta \leq \frac{4}{E^2} \leq \frac{4}{M_{\text{ext}}^2 p} = \frac{c_1}{p}.$$

Since $M_{\text{ext}} \geq M_0$ and $p \geq p_0(M_{\text{ext}})$, Corollary 11.23 shows that $\nu_{p,E}$ is the fixed point of Proposition 11.22 with parameter θ , and that the law π of $B/g(E)$ under $\nu_{p,E}$ is a real solution at E . By Lemma 13.8(ii) with $M = M_{\text{ext}}$, the triple $(\nu_{p,E}, K, s_p)$ is c_{ext} -admissible at p , with $c_* = 1$.

Since $K \geq p^2$, we have $K^{s_p-1} \leq e^{-2}$ and $R^2/K \leq (\log p)^2/p^2$. Then Proposition 13.26 and (13.58) imply that $r_{\pi, s_p} \leq 2$. Next we bound the scalar. With $\xi = E/\sqrt{p}$, we have $a_p(\xi)^{-1} = \sqrt{p}g(E) = (p\theta)^{1/2}$. Then

$$\alpha_{p, s_p}(\xi) = p^{1-s_p} \mu_{s_p}(p\theta)^{s_p/2} \leq e \cdot 1 \cdot c_1^{s_p/2} \leq e c_1^{3/8} \leq \frac{1}{4},$$

where we used $p^{1-s_p} = e$, $\mu_{s_p} \leq 1$ (Lemma 13.9(i)), $p\theta \leq c_1 \leq 1$ and $s_p \geq 3/4$. By Lemma 13.4, we conclude that

$$\varphi_E(s_p) = \log \alpha_{p, s_p}(\xi) + \log r_{\pi, s_p} \leq \log \frac{1}{4} + \log 2 = -\log 2.$$

Finally, since φ_E is nonincreasing on $(0, 1]$ by Proposition 6.15(ii), we have $\varphi_E(1) \leq \varphi_E(s_p)$. This proves (13.1). $\square$

Uniqueness of the critical energy

In this section we prove that, for every fixed $\xi_0 \in (0, \mu_*)$ and all large p , the free energy at $s = 1$ of the principal solution vanishes at exactly one energy in $[\xi_0\sqrt{p}, \infty)$, and that this zero is transverse. Recall from Section 11 that for every compact window $\Xi \subset (0, \infty)$ and $p \geq p_{\text{st}}(\Xi)$, the principal solution $v_{p,E}$ is defined at every energy $E = \xi\sqrt{p}$ with $\xi \in \Xi$ (Theorem 11.21), and that we write

$$\lambda_p(\xi) = \exp \varphi_{\xi\sqrt{p}}(1)$$

for the exponential of its free energy at $s = 1$. By Theorem 13.1, $\lambda_p(\xi)$ converges to μ_*/ξ uniformly on compact windows. This places every solution of $\lambda_p(\xi) = 1$ near μ_* , but it does not exclude several crossings of the level one in a neighborhood of μ_* whose size tends to zero as $p \rightarrow \infty$. The analyticity of λ_p (Proposition 12.21) gives no bound that is uniform in p either. The main task of this section is to show that λ_p is strictly decreasing near μ_* .

14.1 Main results and strategy

Theorem 14.1. *Let $\Xi_0 \subset (0, \infty)$ be a compact interval with μ_* in its interior. There exists $p(\Xi_0)$ such that the following holds for every $p \geq p(\Xi_0)$. The equation $\lambda_p(\xi) = 1$ has exactly one solution $\xi_c(p) \in \Xi_0$. For $\xi \in \Xi_0$, we have $\lambda_p(\xi) > 1$ if $\xi < \xi_c(p)$ and $\lambda_p(\xi) < 1$ if $\xi > \xi_c(p)$. The function λ_p is continuously differentiable in a neighborhood of $\xi_c(p)$, and $\partial_{\xi}\lambda_p(\xi_c(p)) < 0$. Finally, $\xi_c(p) \rightarrow \mu_*$ as $p \rightarrow \infty$.*

Corollary 14.2. *Fix $\xi_0 \in (0, \mu_*)$. There exists $p(\xi_0)$ such that for every $p \geq p(\xi_0)$ the following holds, where $\varphi_E(1)$ denotes the free energy at $s = 1$ of the principal solution $v_{p,E}$ of Corollary 11.23. There exists a unique $E_c(p) \in [\xi_0\sqrt{p}, \infty)$ with $\varphi_{E_c(p)}(1) = 0$. Further, $\varphi_E(1) > 0$ for $E \in [\xi_0\sqrt{p}, E_c(p))$ and $\varphi_E(1) < 0$ for $E \in (E_c(p), \infty)$. The function $E \mapsto \varphi_E(1)$ is continuously differentiable in a neighborhood of $E_c(p)$, with negative derivative at $E_c(p)$. Finally, $E_c(p)/\sqrt{p} \rightarrow \mu_*$ as $p \rightarrow \infty$.*

Strategy

We work on a fixed compact window Ξ around μ_* and in the scaling $g = 1/E$, that is, with $\tilde{B} = D/E$, in which the squared modulus Y enters through $\beta = \tau Y$ with $\tau = 1/(pE^2)$. The proof has four steps.

First, we construct an L^2 version of the critical transfer operator, on a Hilbert space $\mathcal{L}$ that does not depend on ξ . Its states are pairs (x, β) , where x is the reciprocal of the projective coordinate and β is the rescaled squared modulus that the next step will use. The operator A_ξ sends a function F to an average of $F(y - \beta/x, \beta')$ over the side variable y and a fresh rescaled squared modulus β' . The maps $(x, \beta) \mapsto y - \beta/x$ do not depend on ξ ; the parameter enters only through the law ν_ξ of $\tilde{B}$ and through a density factor. Since ν_ξ has a fixed atom, the derivative bounds of Section 12 show that $\xi \mapsto A_\xi$ is four times differentiable in operator norm, with derivatives bounded by a power of p . There exists an involution J of $\mathcal{L}$, which does not depend on ξ , with $JA_\xi J = A_\xi^*$. We prove in Lemma 14.11 that the spectral radius of A_ξ is $\lambda_p(\xi)$, and that it is an eigenvalue with a positive eigenvector.

Second, we decompose the square as

$$A_\xi^2 = G_\xi + h_c \otimes \ell_c + h_t \otimes \ell_t,$$

where all three terms are positive operators. The first rank-one term (the core component) is supported on an ordinary set of states with x close to one and small β ; there the kernel of A_ξ^2 is close to a product kernel, and we subtract a rank-one minorant of it. The second rank-one term (the tail component) is the image of the first under J . The remainder G_ξ has spectral radius $o(1)$. To see this, we split the state space into four blocks and observe that the transitions of G_ξ with norm larger than a negative power of p form an acyclic graph, apart from two diagonal blocks of norm $O(p^{-1} \log^D p)$. In particular, $(\lambda - G_\xi)^{-1}$ is a positive operator with norm bounded by a power of p , for $\lambda \in [1/2, 2]$.

Third, for $\lambda > r(G_\xi)$, a positive eigenvector of A_ξ^2 with eigenvalue λ corresponds to a positive eigenvector of the 2×2 renewal matrix $T(\xi, \lambda)$ with entries $\lambda \langle \ell_a, (\lambda - G_\xi)^{-1} h_b \rangle$. The square of $\lambda_p(\xi)$ is therefore the solution of $\lambda = r(T(\xi, \lambda))$, where $r(T)$ is the Perron–Frobenius eigenvalue. The symmetry under J makes the two diagonal entries of T equal, so that $r(T) = T_{cc} + (T_{ct}T_{tc})^{1/2}$. The diagonal entry is close to $(\mu_*/\xi)^2$ and its derivative is $-2\mu_*^2/\xi^3 + o(1)$. The entry T_{ct} , which measures transitions from the core component to the tail component, is exponentially small, while T_{tc} is bounded by a power of p .

Fourth, we control $\partial_\xi(T_{ct}T_{tc})^{1/2}$ without dividing by any small quantity: neither by T_{ct} , nor by the gap between the two eigenvalues of T , nor by a spectral gap of A_ξ . For a nonnegative function f , Glaeser’s inequality bounds $|f'|$ by $(2f\|f''\|_\infty)^{1/2}$ plus a boundary term, and the second derivatives of T_{ct} and T_{tc} are controlled by interpolating between the bounds on their values and polynomial bounds on their fourth derivatives. Since the entries of T are nonincreasing in λ , implicit differentiation of $\lambda = r(T(\xi, \lambda))$ has a denominator at least one. This gives $\partial_\xi \lambda_p < 0$ near μ_* . The abstract form of this last step is Proposition 14.8.

The section is organized as follows. Section 14.2 collects elementary facts on positive operators and two inequalities for real functions. Section 14.3 states and proves the

abstract renewal criterion. Section 14.4 fixes the parameters and lists the inputs from earlier sections. Section 14.5 constructs the L^2 version of the transfer operator. Section 14.6 constructs the decomposition of the square, Section 14.7 proves the bounds on the remainder, and Section 14.8 analyzes the renewal matrix. Section 14.9 contains the proofs of Theorem 14.1 and Corollary 14.2.

14.2 Elementary lemmas

Positive operators

Throughout this section, $\mathcal{H}$ denotes a real Hilbert space of the form $L^2(\Omega, \mathfrak{m})$ for a σ -finite measure space $(\Omega, \mathfrak{m})$, with inner product $\langle \cdot, \cdot \rangle$. All operators are bounded. Spectra and spectral radii $r(\cdot)$ refer to the complexifications of the operators; by the spectral radius formula [113, Theorem VI.6], $r(X) = \lim_{n \rightarrow \infty} \|X^n\|^{1/n}$. For $f, g \in \mathcal{H}$, we write $f \otimes g$ for the rank-one operator $F \mapsto \langle g, F \rangle f$.

Definition 14.3. An operator X on $\mathcal{H}$ is *positive*, written $X \succeq 0$, if $XF \geq 0$ almost everywhere whenever $F \geq 0$ almost everywhere. We write $X \preceq Y$ if $Y - X \succeq 0$. A function $F \in \mathcal{H}$ is positive if $F > 0$ almost everywhere. For a measurable set $\Omega' \subset \Omega$, we let $P_{\Omega'}$ denote multiplication by $\mathbf{1}_{\Omega'}$.

The order $\succeq$ is not the order of quadratic forms; it is the order of positivity-preserving operators.

Lemma 14.4. Let X, Y, X_1, X_2, Y_1, Y_2 be operators on $\mathcal{H}$.

- (i) If $0 \preceq X \preceq Y$, then $\|X\| \leq \|Y\|$ and $r(X) \leq r(Y)$.
- (ii) If $X \succeq 0$, then $X^* \succeq 0$.
- (iii) If $0 \preceq X_i \preceq Y_i$ for $i = 1, 2$, then $0 \preceq X_1 X_2 \preceq Y_1 Y_2$.
- (iv) If $X \succeq 0$ and $\lambda > r(X)$, then $(\lambda - X)^{-1} = \sum_{n \geq 0} \lambda^{-n-1} X^n \succeq 0$, where the series converges in norm. If moreover $0 \preceq X \preceq Y$ and $\lambda > r(Y)$, then $(\lambda - X)^{-1} \preceq (\lambda - Y)^{-1}$.
- (v) Let $\Omega = \Omega_1 \cup \dots \cup \Omega_m$ be a partition into measurable sets and write $P_i = P_{\Omega_i}$. If $X \succeq 0$, then $P_i X P_j \succeq 0$ for all i, j , and $\sum_{(i,j) \in \mathcal{S}} P_i X P_j \preceq X$ for every set $\mathcal{S}$ of pairs of indices.

Proof. For (i), let $F \in \mathcal{H}$. Since $X(|F| \pm F) \geq 0$, we have $|XF| \leq X|F| \leq Y|F|$ almost everywhere, which gives $\|XF\| \leq \|Y|F|\| \leq \|Y\| \|F\|$. By (iii) below, $0 \preceq X^n \preceq Y^n$, and the first claim gives $\|X^n\| \leq \|Y^n\|$ for every n ; the spectral radius formula implies that $r(X) \leq r(Y)$. For (ii), if $F, H \geq 0$, then $\langle X^* F, H \rangle = \langle F, XH \rangle \geq 0$; taking $H = \mathbf{1}_{\{X^* F < 0\}} \mathbf{1}_{\Omega''}$ for sets Ω'' of finite measure shows that $X^* F \geq 0$. For (iii), we have $Y_1 Y_2 - X_1 X_2 = (Y_1 - X_1) Y_2 + X_1 (Y_2 - X_2)$, and both terms are compositions of positive operators. For (iv), by the spectral radius formula, we have $\sum_n \lambda^{-n-1} \|X^n\| < \infty$, which implies that the Neumann series converges in norm to $(\lambda - X)^{-1}$; each term is positive. If $0 \preceq X \preceq Y$,

then $X^n \preceq Y^n$ by (iii), and summing the series gives the second claim. For (v), we have $P_i X P_j \succeq 0$ by (iii), since P_i and P_j are positive. Since $\sum_i P_i$ is the identity, $X = \sum_{i,j} P_i X P_j$, and $X - \sum_{(i,j) \in \mathcal{S}} P_i X P_j$ is a sum of positive operators. $\square$

Lemma 14.5. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $X: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $Y: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ be bounded. Then $\sigma(XY) \setminus \{0\} = \sigma(YX) \setminus \{0\}$. If $z \neq 0$ is an isolated point of this set, then the Riesz projections of XY and YX at z have the same rank. If $YXv = zv$ with $z \neq 0$ and $v \neq 0$, then $Xv \neq 0$ and $XY(Xv) = zXv$.*

Proof. Let $w \neq 0$ with $w \notin \sigma(YX)$. A direct multiplication shows that

$$(w - XY)^{-1} = w^{-1}(1 + X(w - YX)^{-1}Y), \quad (14.1)$$

so $w \notin \sigma(XY)$. By symmetry, the two sets agree away from zero. Let $z \neq 0$ be isolated, and let γ be a small circle around z that encloses no other point of the spectrum and does not enclose zero. Integrating (14.1) over γ , and using that $\oint_{\gamma} w^{-1} dw = 0$, we find that the Riesz projection Π_{XY} of XY at z equals $X\mathcal{R}Y$, where $\mathcal{R} = (2\pi i)^{-1} \oint_{\gamma} w^{-1}(w - YX)^{-1} dw$. The operator $\mathcal{R}$ vanishes on the kernel of the Riesz projection Π_{YX} of YX at z and maps into its range [82, Section III.6.4]. In particular, the rank of Π_{XY} is at most the rank of Π_{YX} . By symmetry, the ranks agree. For the last claim, $Xv = 0$ would give $zv = YXv = 0$, and $XY(Xv) = X(YXv) = zXv$. $\square$

Two inequalities for real functions

The next lemma bounds the first two derivatives of a function with small values and polynomially bounded fourth derivative.

Lemma 14.6. *Let $[s_-, s_+]$ be a compact interval, let $f \in C^4([s_-, s_+])$ be real valued, and set*

$$\epsilon = \sup_{[s_-, s_+]} |f|, \quad m_4 = \sup_{[s_-, s_+]} |f^{(4)}|.$$

Let $r_0 > 0$ and let $s \in [s_- + r_0, s_+ - r_0]$. Then

$$|f''(s)| \leq 2\sqrt{\epsilon m_4} + \frac{8\epsilon}{r_0^2}, \quad |f'(s)| \leq 2\epsilon^{3/4} m_4^{1/4} + \frac{4\epsilon}{r_0}. \quad (14.2)$$

Proof. If $\epsilon = 0$, the result is immediate. We henceforth assume $\epsilon > 0$ and interpret $h_* = \infty$ whenever $m_4 = 0$ in both optimizations below.

For $|t| \leq r_0$, Taylor's formula with Lagrange remainder gives

$$f(s+t) = f(s) + tf'(s) + \frac{t^2}{2}f''(s) + \frac{t^3}{6}f'''(s) + \frac{t^4}{24}f^{(4)}(s_t) \quad (14.3)$$

for some s_t between s and $s+t$.

We begin with the second derivative. Let $0 < h \leq r_0$. Adding (14.3) at $t = h$ and $t = -h$, we obtain

$$\left| \frac{f(s+h) - 2f(s) + f(s-h)}{h^2} - f''(s) \right| \leq \frac{h^2 m_4}{12}.$$

The difference quotient is at most $4\epsilon/h^2$ in absolute value, so

$$|f''(s)| \leq \frac{4\epsilon}{h^2} + \frac{m_4 h^2}{12}. \quad (14.4)$$

Set $h_* = (48\epsilon/m_4)^{1/4}$ (with $h_* = \infty$ if $m_4 = 0$). If $h_* \leq r_0$, we choose $h = h_*$ in (14.4), which gives $|f''(s)| \leq 2(\epsilon m_4/3)^{1/2} \leq 2\sqrt{\epsilon m_4}$. If $h_* > r_0$, we choose $h = r_0$; then $m_4 r_0^4 < 48\epsilon$, so $m_4 r_0^2/12 < 4\epsilon/r_0^2$ and $|f''(s)| \leq 8\epsilon/r_0^2$.

For the first derivative, let $0 < h \leq r_0/2$ and consider the difference quotient

$$D_h f(s) = \frac{8(f(s+h) - f(s-h)) - (f(s+2h) - f(s-2h))}{12h},$$

which is exact on cubic polynomials. Subtracting (14.3) at $t = -u$ from (14.3) at $t = u$, we find that

$$f(s+u) - f(s-u) = 2uf'(s) + \frac{u^3}{3}f'''(s) + R(u), \quad |R(u)| \leq \frac{u^4 m_4}{12},$$

for $0 < u \leq r_0$. Using this with $u = h$ and $u = 2h$, the terms with $f'''(s)$ cancel and we obtain

$$|D_h f(s) - f'(s)| = \frac{|8R(h) - R(2h)|}{12h} \leq \frac{8h^4 m_4 + 16h^4 m_4}{144h} = \frac{h^3 m_4}{6}.$$

Since $|D_h f(s)| \leq 18\epsilon/(12h)$, this gives

$$|f'(s)| \leq \frac{3\epsilon}{2h} + \frac{m_4 h^3}{6}. \quad (14.5)$$

Set $h_* = (3\epsilon/m_4)^{1/4}$. If $h_* \leq r_0/2$, the choice $h = h_*$ in (14.5) gives $|f'(s)| \leq 2\epsilon/h_* = 2 \cdot 3^{-1/4} \epsilon^{3/4} m_4^{1/4}$. If $h_* > r_0/2$, we choose $h = r_0/2$; then $m_4 h^4 < 3\epsilon$, which gives $m_4 h^3/6 < \epsilon/(2h)$ and $|f'(s)| \leq 2\epsilon/h = 4\epsilon/r_0$. $\square$

The second inequality is a version of Glaeser's inequality [75], which bounds the derivative of a nonnegative function by the square root of its value. We include its short proof, since we use a form with a boundary term.

Lemma 14.7. *Let I_1 be an interval, let $f \in C^2(I_1)$ be nonnegative with $m_2 = \sup_{I_1} |f''| < \infty$, and let $s \in I_1$ be a point whose distance to the complement of I_1 is at least $r_0 > 0$. Then*

$$|f'(s)| \leq \sqrt{2f(s)m_2} + \frac{2f(s)}{r_0}. \quad (14.6)$$

Proof. We may suppose that $f'(s) \neq 0$; let $\sigma = -\operatorname{sgn} f'(s)$. For $0 < h < r_0$, the segment between s and $s + \sigma h$ lies in I_1 , and Taylor's formula and $f \geq 0$ give

$$0 \leq f(s + \sigma h) \leq f(s) - h|f'(s)| + \frac{h^2 m_2}{2}.$$

Then $|f'(s)| \leq f(s)/h + hm_2/2$ for $0 < h < r_0$, and letting $h \uparrow r_0$, we obtain

$$|f'(s)| \leq \frac{f(s)}{h} + \frac{hm_2}{2} \quad \text{for all } h \in (0, r_0]. \quad (14.7)$$

If $f(s) = 0$, letting $h \downarrow 0$ in (14.7) gives $f'(s) = 0$, a contradiction; so $f(s) > 0$. Set $h_* = (2f(s)/m_2)^{1/2}$ (with $h_* = \infty$ if $m_2 = 0$). If $h_* \leq r_0$, the choice $h = h_*$ gives $|f'(s)| \leq \sqrt{2f(s)m_2}$. If $h_* > r_0$, the choice $h = r_0$ gives $|f'(s)| \leq f(s)/r_0 + r_0 m_2/2 \leq 2f(s)/r_0$, where we used that $r_0^2 m_2 < 2f(s)$. $\square$

14.3 A positive rank-two renewal criterion

In this subsection we state the abstract form of the derivative argument. The setting is a family of positive operators that differ from a positive operator of small spectral radius by two positive rank-one operators. For a 2×2 matrix T with nonnegative entries, we write $r(T)$ for its spectral radius (the Perron–Frobenius eigenvalue).

Proposition 14.8. *Let $\mathcal{H} = L^2(\Omega, \mathfrak{m})$ be a real Hilbert space, let $I \subset I_1$ be open intervals such that every point of I has distance at least $r_0 > 0$ to the complement of I_1 , and let $0 < \lambda_- < \lambda_+$. For $\xi \in I_1$, let $G_\xi \succeq 0$ be an operator on $\mathcal{H}$, let $h_{1,\xi}, h_{2,\xi}, \ell_{1,\xi}, \ell_{2,\xi} \in \mathcal{H}$ be nonnegative, and set*

$$U_\xi = G_\xi + h_{1,\xi} \otimes \ell_{1,\xi} + h_{2,\xi} \otimes \ell_{2,\xi}.$$

Suppose that $\xi \mapsto G_\xi$ and $\xi \mapsto h_{i,\xi}, \ell_{i,\xi}$ are twice continuously differentiable on I_1 in the operator norm and in the norm of $\mathcal{H}$, respectively, and that $r(G_\xi) < \lambda_-$ for every $\xi \in I_1$. For $\xi \in I_1$ and $\lambda > r(G_\xi)$, define the renewal matrix

$$T_{ij}(\xi, \lambda) = \lambda \langle \ell_{i,\xi}, (\lambda - G_\xi)^{-1} h_{j,\xi} \rangle, \quad i, j \in \{1, 2\}. \quad (14.8)$$

Assume the following.

- (i) *For all $\xi \in I_1$ and $\lambda \in [\lambda_-, \lambda_+]$, we have $T_{11}(\xi, \lambda) = T_{22}(\xi, \lambda)$, $T_{12}(\xi, \lambda) > 0$ and $T_{21}(\xi, \lambda) > 0$.*
- (ii) *There exists $\kappa > 0$ such that $\partial_\xi T_{11}(\xi, \lambda) \leq -\kappa$ for all $\xi \in I$ and $\lambda \in [\lambda_-, \lambda_+]$, and for every $\lambda \in [\lambda_-, \lambda_+]$, writing $b = T_{12}(\cdot, \lambda)$ and $d = T_{21}(\cdot, \lambda)$ and taking suprema over I_1 ,*

$$\sqrt{\|d\|_\infty \|\partial_\xi^2 b\|_\infty} + \sqrt{\|b\|_\infty \|\partial_\xi^2 d\|_\infty} + \frac{2\sqrt{\|b\|_\infty \|d\|_\infty}}{r_0} \leq \frac{\kappa}{2}. \quad (14.9)$$

(iii) For every $\xi \in I$, there exist $\lambda(\xi) \in (\lambda_-, \lambda_+)$ and a nonzero $F_\xi \in \mathcal{H}$ with $F_\xi \geq 0$ such that $U_\xi F_\xi = \lambda(\xi) F_\xi$.

Then the following hold for every $\xi \in I$.

- (a) The entries of $T(\xi, \lambda)$ are nonincreasing in $\lambda \in [\lambda_-, \lambda_+]$, we have $r(T(\xi, \lambda)) = T_{11} + \sqrt{T_{12}T_{21}}$ there, and $\partial_\lambda r(T(\xi, \lambda)) \leq 0$ for $\lambda \in (\lambda_-, \lambda_+)$.
- (b) The number $\lambda(\xi)$ is the unique solution $\lambda \in [\lambda_-, \lambda_+]$ of $\lambda = r(T(\xi, \lambda))$. The numbers $\langle \ell_{1,\xi}, F_\xi \rangle$ and $\langle \ell_{2,\xi}, F_\xi \rangle$ are positive.
- (c) The function $\xi \mapsto \lambda(\xi)$ is continuously differentiable on I , and

$$\lambda'(\xi) = \frac{\partial_\xi r(T(\xi, \lambda))}{1 - \partial_\lambda r(T(\xi, \lambda))} \Big|_{\lambda=\lambda(\xi)} \leq -\frac{\kappa}{2(1 - \partial_\lambda r(T(\xi, \lambda(\xi))))} < 0. \quad (14.10)$$

The hypotheses involve only the sizes of the off-diagonal entries and of their second derivatives. No lower bound on T_{12} or T_{21} is required, and the proof does not divide by the gap $2\sqrt{T_{12}T_{21}}$ between the two eigenvalues of T , nor by a spectral gap of U_ξ .

Proof. We abbreviate $G = G_\xi$, $h_i = h_{i,\xi}$ and $\ell_i = \ell_{i,\xi}$ when ξ is fixed.

STEP 1. *The renewal matrix.* We first claim that, for $\xi \in I_1$ and $\lambda > r(G_\xi)$,

$$T_{ij}(\xi, \lambda) = \sum_{n \geq 0} \lambda^{-n} \langle \ell_i, G^n h_j \rangle, \quad (14.11)$$

that T is continuously differentiable in (ξ, λ) , and that the entries of T are nonincreasing in λ . By Lemma 14.4(iv), we have $\lambda(\lambda - G)^{-1} = \sum_{n \geq 0} \lambda^{-n} G^n$ in norm, which gives (14.11). Each coefficient $\langle \ell_i, G^n h_j \rangle$ is nonnegative, since $G^n \succeq 0$ and $h_j, \ell_i \geq 0$. The series (14.11) is a power series in λ^{-1} whose radius of convergence is at least $1/r(G)$. It may therefore be differentiated termwise, and

$$\partial_\lambda T_{ij}(\xi, \lambda) = - \sum_{n \geq 1} n \lambda^{-n-1} \langle \ell_i, G^n h_j \rangle \leq 0.$$

For the joint differentiability, note that $(\xi, \lambda) \mapsto (\lambda - G_\xi)^{-1}$ is continuous in norm on $\{(\xi, \lambda) : \lambda > r(G_\xi)\}$, and that its partial derivatives are $(\lambda - G_\xi)^{-1}(\partial_\xi G_\xi)(\lambda - G_\xi)^{-1}$ and $-(\lambda - G_\xi)^{-2}$, which are continuous as well. The set $\{(\xi, \lambda) : \lambda > r(G_\xi)\}$ is open, since the spectral radius is upper semicontinuous in norm [82, Section IV.3.1]. Combined with the differentiability of h_j and ℓ_i , this shows that T is continuously differentiable, and twice continuously differentiable in ξ for fixed λ .

STEP 2. *The Perron–Frobenius eigenvalue.* Let $\xi \in I_1$ and $\lambda \in [\lambda_-, \lambda_+]$, and write $a = T_{11} = T_{22}$, $b = T_{12}$ and $d = T_{21}$. The eigenvalues of T are $a \pm \sqrt{bd}$. Since $a \geq 0$, we have $r(T) = a + \sqrt{bd}$. Since $b, d > 0$ on $I_1 \times [\lambda_-, \lambda_+]$, Step 1 shows that $r(T)$ is continuously differentiable on $I_1 \times (\lambda_-, \lambda_+)$. By Step 1, b and d are positive and nonincreasing in λ .

The functions $\sqrt{bd}$ and a are therefore nonincreasing in λ , and $\partial_\lambda r(T) \leq 0$. This proves (a).

STEP 3. *The eigenvector relation.* Let $\xi \in I$, write $\lambda = \lambda(\xi)$ and $F = F_\xi$, and set $u_i = \langle \ell_i, F \rangle \geq 0$. The eigenvalue equation $U_\xi F = \lambda F$ reads $(\lambda - G)F = u_1 h_1 + u_2 h_2$. Since $\lambda > r(G)$, we obtain

$$F = (\lambda - G)^{-1}(u_1 h_1 + u_2 h_2). \quad (14.12)$$

If $u_1 = u_2 = 0$, then (14.12) implies that $F = 0$, which is excluded. Pairing (14.12) with ℓ_i and multiplying by λ , we find that

$$\lambda u_i = \sum_{j=1}^2 T_{ij}(\xi, \lambda) u_j, \quad i = 1, 2,$$

that is, $u = (u_1, u_2)$ is a nonzero nonnegative eigenvector of $T(\xi, \lambda)$ with eigenvalue λ . If $u_1 = 0$, the first equation reads $bu_2 = 0$, and $b > 0$ forces $u_2 = 0$, which is impossible. Then $u_1 > 0$, and similarly $u_2 > 0$. The two equations now read $(\lambda - a)u_1 = bu_2$ and $(\lambda - a)u_2 = du_1$. Multiplying them, we obtain $(\lambda - a)^2 = bd$, and the first one shows that $\lambda - a > 0$. We conclude that $\lambda = a + \sqrt{bd} = r(T(\xi, \lambda))$.

Let $\Psi(\xi, \lambda) = \lambda - r(T(\xi, \lambda))$. By (a), $\lambda \mapsto r(T(\xi, \lambda))$ is nonincreasing on $[\lambda_-, \lambda_+]$, which implies that $\Psi(\xi, \cdot)$ is strictly increasing there and has at most one zero. This proves (b).

STEP 4. *The slope in ξ .* We claim that $\partial_\xi r(T(\xi, \lambda)) \leq -\kappa/2$ for $\xi \in I$ and $\lambda \in [\lambda_-, \lambda_+]$. Fix λ , and let b, d be as in (ii). We have

$$\partial_\xi \sqrt{bd} = \frac{\sqrt{d}}{2\sqrt{b}} \partial_\xi b + \frac{\sqrt{b}}{2\sqrt{d}} \partial_\xi d.$$

Since $b > 0$ is twice continuously differentiable on I_1 , we may apply Lemma 14.7, and we obtain, at every $\xi \in I$,

$$\frac{|\partial_\xi b|}{2\sqrt{b}} \leq \frac{\sqrt{2b\|\partial_\xi^2 b\|_\infty} + 2b/r_0}{2\sqrt{b}} \leq \sqrt{\|\partial_\xi^2 b\|_\infty} + \frac{\sqrt{b}}{r_0},$$

and similarly for d . Inserting these bounds, we obtain

$$|\partial_\xi \sqrt{bd}| \leq \sqrt{d\|\partial_\xi^2 b\|_\infty} + \sqrt{b\|\partial_\xi^2 d\|_\infty} + \frac{2\sqrt{bd}}{r_0} \leq \frac{\kappa}{2},$$

where in the last inequality we used (14.9). Combined with $\partial_\xi T_{11} \leq -\kappa$, this proves the claim.

STEP 5. *Implicit differentiation.* Fix $\xi_1 \in I$. The function Ψ of Step 3 is continuously differentiable on an open neighborhood of $(\xi_1, \lambda(\xi_1))$, it vanishes at this point, and

$\partial_\lambda \Psi \geq 1$ there. By the implicit function theorem, there exist a neighborhood $I' \subset I$ of ξ_1 and a continuously differentiable function $\tilde{\lambda}$ on I' with values in (λ_-, λ_+) , such that $\tilde{\lambda}(\xi_1) = \lambda(\xi_1)$ and $\Psi(\xi, \tilde{\lambda}(\xi)) = 0$ for $\xi \in I'$. By the uniqueness in (b), $\tilde{\lambda} = \lambda$ on I' . Then λ is continuously differentiable on I , and differentiating $\Psi(\xi, \lambda(\xi)) = 0$ gives the identity in (14.10). The numerator is at most $-\kappa/2$ by Step 4, and the denominator is at least one by (a). This proves (c). $\square$

14.4 Parameters and inputs

Normalization

Throughout Sections 14.4–14.8, we fix a compact interval $\Xi = [\xi_-, \xi_+] \subset (0, \infty)$ and write $E = \xi \sqrt{p}$ for $\xi \in \Xi$. We always assume that p is so large that $\xi_- \sqrt{p} \geq 3$. In the notation of Definition 5.2, we set

$$g = g(E), \quad \theta = g^2, \quad c = 1 + \theta = Eg, \quad \tau = \frac{1}{pE^2} = \frac{1}{p^2 \xi^2} = \frac{\theta}{pc^2}, \quad \alpha = \sqrt{p} g. \quad (14.13)$$

All of these are functions of ξ (and p). From the formula for g , we have $\alpha = 1/a_p(\xi) = 2/(\xi + (\xi^2 - 4/p)^{1/2})$ and $g \leq 2/E$, and we note the identities

$$\tau c^2 = \frac{\theta}{p}, \quad pc\sqrt{\tau} = \alpha. \quad (14.14)$$

Elementary calculus shows that there exists $C(\Xi)$ such that for $0 \leq j \leq 5$ and $\xi \in \Xi$,

$$|\partial_\xi^j \theta| + |\partial_\xi^j c - \mathbf{1}_{j=0}| + \left| \partial_\xi^j \left(\alpha - \frac{1}{\xi} \right) \right| \leq \frac{C}{p}, \quad \frac{|\partial_\xi^j \tau|}{\tau} + \frac{|\partial_\xi^j (\tau c)|}{\tau c} \leq C. \quad (14.15)$$

In particular, $\alpha = \xi^{-1} + O(p^{-1})$ and $\partial_\xi \alpha = -\xi^{-2} + O(p^{-1})$, uniformly on Ξ .

We write $\omega_\xi(d\beta) = \tau^{-1} e^{-\beta/\tau} d\beta$ for the law of $\beta = \tau Y$ with $Y \sim \text{Exp}(1)$, and $\gamma(du) = 2u e^{-u^2} du$ on $[0, \infty)$ for the law of $\sqrt{Y}$. We record that

$$\int_0^\infty u \gamma(du) = \int_0^\infty 2u^2 e^{-u^2} du = \frac{\sqrt{\pi}}{2} = \mu_*. \quad (14.16)$$

Inputs from earlier sections

We use the following results, for $\xi \in \Xi$ and $p \geq p_I(\Xi)$, where $p_I(\Xi)$ is at least the thresholds of the cited results for the window Ξ ; in Theorem 11.21, we use the canonical tuple of Definition 11.8.

(I1) *The principal law.* By Theorem 11.21(i),(v), the principal solution $\nu_{p,E}$ (the law of B) is defined, and the law of $D = B/g$ is a real solution at E . We let ν_ξ denote the law of

$\tilde{B} = B/c$ under $\nu_{p,E}$; this is the principal law in the scaling $g = 1/E$ of Definition 12.2, denoted $\tilde{\nu}_\xi$ there. By Proposition 5.5 applied to $D = E\tilde{B}$, the law ν_ξ has an atom of mass e^{-p} at 1, no other atom (in particular no atom at 0), full support $\mathbb{R}$, and $\mathbb{E}|\tilde{B}|^q < \infty$ for $q \in (0, 1)$.

(I2) *Bounds defining the class $\mathfrak{S}_{p,\theta}$.* By Theorem 11.21(iii) and $r_p = p^{-1/4}$, the core bound

$$\nu_{p,E}(\{b : |b - 1| > p^{-1/4}\}) \leq e^{-a\sqrt{p}} \quad (14.17)$$

holds with $a = 1$, and it continues to hold if a is decreased. The tail bound of Theorem 11.21(iii) gives $\nu_{p,E}(\{b : |b| > L\}) \leq e^{-p}/(2L) + e^{-pL}/2 \leq C_p/L$ for all $L \geq 3$, with C_p depending only on p . (We use the tail bound only at fixed p . In (14.17), any core radius r_p with $r_p \log p \rightarrow 0$ would do.)

(I3) *The critical transfer operator.* Let $\mathcal{Q}_1 = \mathcal{Q}_{1,\xi}$ be the transfer operator of Definition 6.1 at $s = 1$, defined with the law ν_ξ and an independent $\beta \sim \omega_\xi$, which has the exponential law of rate $1/\tau = pE^2$. This is the scaling $g = 1/E$ in Definition 6.14(iii): the law of gD is ν_ξ , and $p/g^2 = pE^2$. The Perron–Frobenius eigenvalue of $\mathcal{Q}_1$ is therefore $\exp \varphi_E(1) = \lambda_p(\xi)$; see also Proposition 12.21(i). By Theorem 6.11(i),(ii), $\mathcal{Q}_1$ is bounded and positive on $C(\mathbb{RP}^1)$, and $\lambda_p(\xi)$ is an eigenvalue of $\mathcal{Q}_1$ on $C(\mathbb{RP}^1)$ whose Riesz projection has rank one, with an eigenfunction $\psi_\xi \in C(\mathbb{RP}^1)$ satisfying $\min \psi_\xi > 0$. Further, there exists $r_\xi < \lambda_p(\xi)$ such that $|z| \leq r_\xi$ for every $z \in \sigma(\mathcal{Q}_1) \setminus \{\lambda_p(\xi)\}$.

(I4) *Analyticity of the law.* We fix $q = 1/2$. By Theorem 12.19(ii), $\xi \mapsto \nu_\xi$ is real-analytic on a neighborhood of Ξ as a map into the space X_q of Definition 12.6, whose norm dominates the weighted total variation norm $\|\sigma\|_{q,\text{TV}}$, which in turn dominates the total variation norm $|\sigma|(\mathbb{R})$. In particular, $\xi \mapsto \nu_\xi$ is four times continuously differentiable in total variation, and $\partial_\xi^j \nu_\xi$ is a signed measure of total mass zero for $j \geq 1$.

(I5) *Quantitative derivative bounds.* By Proposition 12.24(i), there exists $A \geq 1$, depending only on Ξ , such that $|\partial_\xi^j \nu_\xi|(\mathbb{R}) \leq \|\partial_\xi^j \nu_\xi\|_{q,\text{TV}} \leq p^A$ for $1 \leq j \leq 4$.

(I6) *Smooth tests.* We apply Corollary 12.25 with $J = [1/2, 3/2]$, after enlarging A and decreasing a if necessary, and we write $\|\zeta\|_{C^2} = \|\zeta\|_\infty + \|\zeta'\|_\infty + \|\zeta''\|_\infty$ and

$$\epsilon_p = \frac{C}{p} + p^A e^{-a\sqrt{p}}.$$

Let $B \sim \nu_{p,\xi\sqrt{p}}$ and $j \in \{1, 2\}$. For every $\zeta \in C^2(\mathbb{R})$ supported in $[1/2, 3/2]$,

$$|\partial_\xi^j \mathbb{E}\zeta(B)| \leq \epsilon_p \|\zeta\|_{C^2}. \quad (14.18)$$

More generally, if $I_0 \supset \Xi$ is an open interval and $\xi \mapsto \zeta_\xi$ is twice continuously differentiable from I_0 into the space of C^2 functions supported in $[1/2, 3/2]$ with the norm $\|\cdot\|_{C^2}$, then

$$|\partial_\xi^j \mathbb{E} \zeta_\xi(B)| \leq \epsilon_p \sum_{i=0}^{j-1} \binom{j}{i} \|\partial_\xi^i \zeta_\xi\|_{C^2} + \sup |\partial_\xi^j \zeta_\xi|. \quad (14.19)$$

The constants a and A are fixed from now on; they depend only on Ξ . The leading asymptotics (Theorem 13.1) and the large-energy bound (Proposition 13.2) are used only in Section 14.9.

Order of the parameters

The construction below involves a fixed power n_0 , a target exponent N_* and a cutoff level M_* . They are chosen in the following order.

- (1) The window Ξ is fixed; then a and A are fixed by (I2), (I5) and (I6).
- (2) We fix $n_0 = 12$.
- (3) We define the exponents

$$\begin{aligned} A_1 &= 2A + 2, & A_2 &= 10n_0 + 8A_1 + 5, \\ A_j &= 2A_1 + A_2 + 1, & A_v &= 2A_1 + 4n_0 + 2. \end{aligned} \quad (14.20)$$

They depend only on A , so only on Ξ .

- (4) We fix an integer $N_* > \max\{2n_0 + 10, A_v + A_j + 2\}$.
- (5) We fix $M_* = 4N_* + 20$.
- (6) Finally, we take $p \geq p_1$, where $p_1 = p_1(\Xi, N_*, M_*) \geq p_I(\Xi)$ is a threshold that may be enlarged finitely many times below.

In Sections 14.5–14.8, the letters C and D denote positive constants that may depend on Ξ , N_* and M_* , and may change from line to line, but not on p , ξ or λ . The point of the order above is that every bound of the form $Cp^e \log^D p$ below has an exponent e that does not depend on M_* ; quantities such as e^{M_*+1} enter only through C . In particular, $Cp^e \log^D p \leq p^{e+1/2}$ for $p \geq p_1$, which is how such bounds are converted into the pure powers in (14.20).

14.5 The L^2 version of the transfer operator

The operator

Let $\kappa_0 = p^2 \xi_-^2 / 2$, so that $\kappa_0 < 1/\tau$ for every $\xi \in \Xi$, and define

$$\mu_0(d\beta) = \kappa_0 e^{-\kappa_0 \beta} d\beta, \quad \omega_\xi(\beta) = \left(\frac{1}{\tau \kappa_0}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{1}{\tau} - \kappa_0\right)\beta\right), \quad \beta > 0.$$

Then $\omega_\xi^2 \mu_0 = \omega_\xi$, and in particular $\int \omega_\xi^2 d\mu_0 = 1$.

Definition 14.9. Let $\mathcal{L} = L^2(\mathbb{R} \times (0, \infty), dx \mu_0(d\beta))$ (real valued), with inner product $\langle \cdot, \cdot \rangle$, and let $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx)$. Define $I_\xi: L^2(\mathbb{R}) \rightarrow \mathcal{L}$ by $(I_\xi f)(x, \beta) = \omega_\xi(\beta) f(x)$. For a finite signed measure σ on $\mathbb{R}$, define $\mathcal{C}_\sigma$ on $L^2(\mathbb{R})$ by $(\mathcal{C}_\sigma f)(x) = \int f(y - x) \sigma(dy)$. Define J on $\mathcal{L}$ by

$$(JF)(x, \beta) = \frac{\sqrt{\beta}}{|x|} F\left(\frac{\beta}{x}, \beta\right),$$

and the L^2 version of the critical transfer operator

$$A_\xi = p J I_\xi \mathcal{C}_{\nu_\xi} I_\xi^*. \quad (14.21)$$

Explicitly, since $(I_\xi^* F)(x) = \int \omega_\xi(\beta') F(x, \beta') \mu_0(d\beta')$, we have

$$(A_\xi F)(x, \beta) = p \omega_\xi(\beta) \frac{\sqrt{\beta}}{|x|} \int \nu_\xi(dy) \int \mu_0(d\beta') \omega_\xi(\beta') F\left(y - \frac{\beta}{x}, \beta'\right). \quad (14.22)$$

The operator A_ξ is not the transfer operator $\mathcal{A}_s$ in the x -coordinate of Definition 6.1; it is related to the transfer operator $\mathcal{Q}_1$ in Lemma 14.11. The state (x, β) consists of the reciprocal x of the projective coordinate and the rescaled squared modulus β used in the next step; the new state is $(y - \beta/x, \beta')$ with a side variable $y \sim \nu_\xi$ and a fresh rescaled squared modulus β' . The maps $(x, \beta, y) \mapsto y - \beta/x$ do not depend on ξ , and ξ enters (14.22) only through ν_ξ and ω_ξ .

Lemma 14.10. Let $p \geq p_1$.

- (i) The operator J is a positive unitary involution: $J \succeq 0$ and $J = J^* = J^{-1}$. For every measurable $\Omega' \subset \mathbb{R} \times (0, \infty)$, we have $J P_{\Omega'} J = P_{J\Omega'}$, where $J\Omega' = \{(\beta/x, \beta) : (x, \beta) \in \Omega'\}$.
- (ii) The operator I_ξ is an isometry, and for every finite signed measure σ , $\mathcal{C}_\sigma$ is self-adjoint with $\|\mathcal{C}_\sigma\| \leq |\sigma|(\mathbb{R})$.
- (iii) We have $A_\xi \succeq 0$, $\|A_\xi\| \leq p$, and $A_\xi^* = J A_\xi J$.
- (iv) The map $\xi \mapsto A_\xi$ is four times continuously differentiable on Ξ in operator norm, and $\|\partial_\xi^j A_\xi\| \leq p^{A_1}$ for $0 \leq j \leq 4$.

Proof. For fixed $\beta > 0$, the map $x \mapsto \beta/x$ is an involution of $\mathbb{R} \setminus \{0\}$ with Jacobian β/x^2 . This substitution gives $\int |(JF)(x, \beta)|^2 dx = \int |F(x, \beta)|^2 dx$ for every β , and $J^2 F = F$, since $(J^2 F)(x, \beta) = (\sqrt{\beta}/|x|)(\sqrt{\beta}/|\beta/x|) F(x, \beta)$. A unitary involution is self-adjoint. Positivity is evident, and the formula for $J P_{\Omega'} J$ follows from $\mathbf{1}_{\Omega'}(\beta/x, \beta) = \mathbf{1}_{J\Omega'}(x, \beta)$. This proves (i).

For (ii), we have $\|I_\xi f\|^2 = \|f\|^2 \int \omega_\xi^2 d\mu_0 = \|f\|^2$. By Minkowski's integral inequality, we have $\|\mathcal{C}_\sigma f\| \leq \int \|f(y - \cdot)\| |\sigma|(dy) = |\sigma|(\mathbb{R}) \|f\|$, and the substitution $x \mapsto y - x$ gives $\langle \mathcal{C}_\sigma f_1, f_2 \rangle = \langle f_1, \mathcal{C}_\sigma f_2 \rangle$.

For (iii), positivity is evident from (14.22), and (i), (ii) give $\|A_{\xi}\| \leq p$. Since J and $\mathcal{C}_{\nu_{\xi}}$ are self-adjoint, we have $A_{\xi}^* = pI_{\xi}\mathcal{C}_{\nu_{\xi}}I_{\xi}^*J = J(pJI_{\xi}\mathcal{C}_{\nu_{\xi}}I_{\xi}^*)J = JA_{\xi}J$.

For (iv), we find by direct computation that $\partial_{\xi}^j \omega_{\xi}(\beta) = \omega_{\xi}(\beta)P_{j,\xi}(\beta/\tau)$ for $j \leq 5$, where $P_{j,\xi}$ is a polynomial of degree j whose coefficients are bounded on Ξ by (14.15). Since β/τ has law $\text{Exp}(1)$ under $\omega_{\xi}^2\mu_0$, we have $\|\partial_{\xi}^j \omega_{\xi}\|_{L^2(\mu_0)} \leq C$. Moreover, $\sup_{\xi \in \Xi} |\partial_{\xi}^j \omega_{\xi}(\beta)| \leq C(1 + p^2\beta)^{2j}\kappa_0^{-1}p^2\xi_+^2 e^{-(p^2\xi_-^2 - \kappa_0)\beta}$, which is integrable against μ_0 ; by dominated convergence, $\xi \mapsto \omega_{\xi}$ is five times continuously differentiable as a map into $L^2(\mu_0)$. Since $\|\partial_{\xi}^j I_{\xi}\| = \|\partial_{\xi}^j \omega_{\xi}\|_{L^2(\mu_0)}$, the map $\xi \mapsto I_{\xi}$ is four times continuously differentiable with $\|\partial_{\xi}^j I_{\xi}\| \leq C$. By (I4) and (ii), $\xi \mapsto \mathcal{C}_{\nu_{\xi}}$ is four times continuously differentiable with $\partial_{\xi}^j \mathcal{C}_{\nu_{\xi}} = \mathcal{C}_{\partial_{\xi}^j \nu_{\xi}}$, and (I5) gives $\|\partial_{\xi}^j \mathcal{C}_{\nu_{\xi}}\| \leq p^A$. The Leibniz rule applied to (14.21) yields $\|\partial_{\xi}^j A_{\xi}\| \leq Cp^{A+1} \leq p^{A_1}$. $\square$

The spectral radius of the L^2 version

We now relate A_{ξ} to the transfer operator $\mathcal{Q}_1$ of (I3). We identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$ through the reciprocal $x = 1/q$ of the projective coordinate q of Definition 6.1. For $x \in \mathbb{R} \setminus \{0\}$, $\tilde{B} \in \mathbb{R}$ and $\beta > 0$, set

$$x' = \tilde{B} - \frac{\beta}{x}, \quad w = \frac{\sqrt{\beta}(1+x^2)^{1/2}}{|x|(1+x'^2)^{1/2}}, \quad (14.23)$$

and extend these by $x' = \infty$, $w = \beta^{-1/2}$ at $x = 0$ and by $x' = \tilde{B}$, $w = \beta^{1/2}(1+\tilde{B}^2)^{-1/2}$ at $x = \infty$. For fixed $\tilde{B}$ and β , the maps $x \mapsto x'$ and $x \mapsto w$ are continuous on $\mathbb{RP}^1$. Writing $M(\tilde{B}, \beta)v_q$ in the coordinates of Definition 6.1, we have $M(\tilde{B}, \beta)v_q \propto (1, \tilde{B} - \beta q)$. The image of $q = 1/x$ has reciprocal coordinate x' , and $\|M(\tilde{B}, \beta)v_q\|^{-1} = w$. We obtain

$$(\mathcal{Q}_1 u)(x) = p\mathbb{E}[w u(x')], \quad x \in \mathbb{RP}^1, \quad (14.24)$$

with $\tilde{B} \sim \nu_{\xi}$ and $\beta \sim \omega_{\xi}$ independent; at $x \in \{0, \infty\}$, this agrees with Definition 6.1 by (6.3). As in Definition 6.1, (14.24) applies to every bounded Borel function u on $\mathbb{RP}^1$, and we write $\mathcal{Q}_1(x, \cdot)$ for the finite measure $u \mapsto (\mathcal{Q}_1 u)(x)$, and similarly for powers of $\mathcal{Q}_1$.

Lemma 14.11. *Let $p \geq p_1$ and $\xi \in \Xi$, and write $\rho_{\xi} = \lambda_p(\xi)$.*

- (i) *The spectral radius of A_{ξ} on $\mathcal{L}$ equals ρ_{ξ} . Further, ρ_{ξ} is an isolated point of the spectrum of A_{ξ} , and its Riesz projection has rank one.*
- (ii) *There exists $F_{\xi} \in \mathcal{L}$ with $F_{\xi} > 0$ almost everywhere and $A_{\xi}F_{\xi} = \rho_{\xi}F_{\xi}$. In particular, $A_{\xi}^2 F_{\xi} = \rho_{\xi}^2 F_{\xi}$.*

Only part (ii) is used in the proof of Theorem 14.1; part (i) shows that the eigenvalue in (ii) is the spectral radius.

Proof. We fix ξ and drop it from the notation; we write $\nu = \nu_\xi$, $\omega = \omega_\xi$, $\psi = \psi_\xi$ and $\rho = \rho_\xi$. Let $\varsigma(x) = (1 + x^2)^{-1/2}$ and $\tilde{m}(dx) = \varsigma(x)^2 dx$, a finite measure on $\mathbb{R}$ (equivalently, on $\mathbb{RP}^1$, with $\{\infty\}$ a null set). Let $0 < \psi_- \leq \psi \leq \psi_+$ be the bounds of the positive continuous function ψ on the compact space $\mathbb{RP}^1$. The proof proceeds by transporting the problem to the transfer operator $\mathcal{Q}_1$, where the Perron–Frobenius theory of (I3) applies, and then to A_ξ .

STEP 1. The scalar operator and the weight. Define $\mathcal{W}$ on $L^2(\mathbb{R})$ by $(\mathcal{W}f)(x) = \int \omega(d\beta) \sqrt{\beta}|x|^{-1} f(\beta/x)$, and set $A^s = p\mathcal{W}C_\nu$. For each β , the operator $f \mapsto \sqrt{\beta}|x|^{-1} f(\beta/x)$ is a unitary involution of $L^2(\mathbb{R})$, as in the proof of Lemma 14.10(i). The operator $\mathcal{W}$ is an average of these involutions, and it is self-adjoint with $\|\mathcal{W}\| \leq 1$. We have

$$(A^s f)(x) = p \int \nu(dy) \int \omega(d\beta) \frac{\sqrt{\beta}}{|x|} f\left(y - \frac{\beta}{x}\right).$$

By (14.23), $w = \sqrt{\beta}|x|^{-1} \varsigma(x')/\varsigma(x)$ for $x \neq 0$, and (14.24) gives

$$(\mathcal{Q}_1 u)(x) = \varsigma(x)^{-1} (A^s(\varsigma u))(x) \quad \text{for almost every } x, \quad (14.25)$$

for every bounded Borel u . The map $\Theta f = f/\varsigma$ is unitary from $L^2(\mathbb{R})$ onto $L^2(\tilde{m})$, and $\tilde{Q} = \Theta A^s \Theta^{-1}$ is a bounded operator on $L^2(\tilde{m})$ that acts on bounded functions by the kernel of $\mathcal{Q}_1$. Set $f = \varsigma\psi$. Then $f \in L^2(\mathbb{R})$, $f > 0$ everywhere, and (14.25) with $\mathcal{Q}_1\psi = \rho\psi$ gives $A^s f = \rho f$.

STEP 2. A left eigenvector. Set $f^* = C_\nu f$. Since $\mathcal{W}$ and C_ν are self-adjoint, $(A^s)^* = pC_\nu\mathcal{W}$, and

$$(A^s)^* f^* = pC_\nu\mathcal{W}C_\nu f = C_\nu A^s f = \rho f^*.$$

We claim that $\chi = f^*/\varsigma$ satisfies $c_\chi \leq \chi \leq C_\chi$ for constants $0 < c_\chi \leq C_\chi < \infty$ (which may depend on p and ξ). Since $\psi_- C_\nu \varsigma \leq f^* \leq \psi_+ C_\nu \varsigma$, it suffices to compare $C_\nu \varsigma(x) = \int \varsigma(y-x) \nu(dy)$ with $\varsigma(x)$. For the lower bound, if $|y| \leq 1$ then $1 + (y-x)^2 \leq 1 + (1+|x|)^2 \leq 3(1+x^2)$, which gives $C_\nu \varsigma(x) \geq \nu([-1,1])\varsigma(x)/\sqrt{3}$, and $\nu([-1,1]) > 0$ by the full support in (I1). For the upper bound, if $|x| \leq 6$ then $C_\nu \varsigma(x) \leq 1 \leq \sqrt{37}\varsigma(x)$. If $|x| > 6$, we split the integral according to whether $|y| \leq |x|/2$. On the first part, $\varsigma(y-x) \leq |y-x|^{-1} \leq 2/|x|$. The second part is at most $\nu(\{|y| > |x|/2\}) = \nu_{p,E}(\{|b| > c|x|/2\}) \leq 2C_p/|x|$, by the tail bound in (I2). Then $C_\nu \varsigma(x) \leq (2 + 2C_p)/|x| \leq (4 + 4C_p)\varsigma(x)$.

STEP 3. The Doob transform. For bounded Borel u on $\mathbb{RP}^1$, set $\mathcal{P}u = (\rho\psi)^{-1}\mathcal{Q}_1(\psi u)$. Since $\mathcal{Q}_1\psi = \rho\psi$ holds at every point of $\mathbb{RP}^1$, $\mathcal{P}(x, \cdot)$ is a probability measure for every $x \in \mathbb{RP}^1$. On $L^2(\tilde{m})$, the operator $\mathcal{P} = \rho^{-1}\psi^{-1}\tilde{Q}\psi$ is bounded. Let $\pi(dx) = Z^{-1}\psi\chi d\tilde{m}$, with Z chosen so that π is a probability measure. By Step 2, the density of π with respect to $\tilde{m}$ lies in $[Z^{-1}\psi_- c_\chi, Z^{-1}\psi_+ C_\chi]$, and the norms of $L^2(\pi)$ and $L^2(\tilde{m})$ are therefore equivalent. We claim that π is invariant: $\int \mathcal{P}u d\pi = \int u d\pi$ for $u \in L^2(\tilde{m})$, and then for

every measurable $u \geq 0$ by monotone convergence. Indeed, the adjoint of $\tilde{Q}$ in $L^2(\tilde{m})$ is $\Theta(A^s)^* \Theta^{-1}$. By Step 2, we have $\tilde{Q}^* \chi = \Theta(A^s)^* f^* = \rho \chi$, and

$$Z \int \mathcal{P}u \, d\pi = \rho^{-1} \int \tilde{Q}(\psi u) \chi \, d\tilde{m} = \rho^{-1} \int \psi u \tilde{Q}^* \chi \, d\tilde{m} = Z \int u \, d\pi.$$

STEP 4. The spectral radius. Since the rows of $\mathcal{P}$ are probability measures, we have $|\mathcal{P}u|^2 \leq \mathcal{P}(|u|^2)$ pointwise by the Cauchy–Schwarz inequality, and Step 3 gives $\int |\mathcal{P}u|^2 \, d\pi \leq \int |u|^2 \, d\pi$. It follows that $\|\mathcal{P}^n\|_{L^2(\pi)} \leq 1$ for all n . Since $\tilde{Q}^n = \rho^n \psi \mathcal{P}^n \psi^{-1}$ and the norms are equivalent, we find that $\|\tilde{Q}^n\|_{L^2(\tilde{m})} \leq C' \rho^n$ for a constant C' independent of n . By the spectral radius formula, we conclude that $r(A^s) = r(\tilde{Q}) \leq \rho$, and Step 1 gives equality.

STEP 5. Three-step smoothing. We claim that for every bounded Borel u on $\mathbb{RP}^1$, the function $\mathcal{P}^3 u$ is continuous on $\mathbb{RP}^1$ and depends only on the $\tilde{m}$ -equivalence class of u . Since $\mathcal{P}^3 u = \rho^{-3} \psi^{-1} \mathcal{Q}_1^3(\psi u)$, it suffices to show that $x \mapsto \mathcal{Q}_1^3(x, \cdot)$ is continuous on $\mathbb{RP}^1$ in total variation and that each $\mathcal{Q}_1^3(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.

Let $x \in \mathbb{R} \setminus \{0\}$. Substituting $z = y - \beta/x$ in (14.24), so that $\beta = x(y - z)$ and $d\beta = |x| \, dz$, we find that $\mathcal{Q}_1(x, dz) = p\zeta(x)^{-1} \zeta(z)(\nu \star k_x)(z) \, dz$, where

$$k_x(t) = \tau^{-1} \sqrt{xt} e^{-xt/\tau} \mathbf{1}_{\{xt > 0\}}, \quad (\nu \star k_x)(z) = \int k_x(y - z) \nu(dy).$$

For $t \neq 0$, $k_x(t)$ is continuous in $x \in \mathbb{R} \setminus \{0\}$, and $\|k_x\|_{L^1} = \Gamma(3/2) \tau^{1/2} |x|^{-1}$ is continuous as well. By Scheffé's lemma [64], the map $x \mapsto k_x$ is continuous into $L^1(\mathbb{R})$. Since $\|\nu \star (k_x - k_{x_0})\|_{L^1} \leq \|k_x - k_{x_0}\|_{L^1}$ and $\zeta \leq 1$, the map $x \mapsto \mathcal{Q}_1(x, \cdot)$ is continuous in total variation on $\mathbb{R} \setminus \{0\}$, and each $\mathcal{Q}_1(x, \cdot)$ with $x \in \mathbb{R} \setminus \{0\}$ is absolutely continuous.

Next, $x \mapsto w$ is continuous on $\mathbb{RP}^1$ for fixed $(\tilde{B}, \beta)$, and $\mathbb{E}w = p^{-1}(\mathcal{Q}_1 \mathbf{1})(x)$ is continuous on $\mathbb{RP}^1$ since $\mathcal{Q}_1$ maps $C(\mathbb{RP}^1)$ into itself. By Scheffé's lemma, $\mathbb{E}|w(x) - w(x_0)| \rightarrow 0$ as $x \rightarrow x_0$ in $\mathbb{RP}^1$, where we indicate the dependence of w and x' on x . Let $\mathcal{K}(y, \cdot)$ be either $\mathcal{Q}_1(y, \cdot)$ or $\mathcal{Q}_1^2(y, \cdot)$, and suppose that $y \mapsto \mathcal{K}(y, \cdot)$ is continuous in total variation at almost every value of $x'(x_0)$. For a Borel function $|u| \leq 1$, we have $(\mathcal{Q}_1 \mathcal{K}u)(x) = p\mathbb{E}[w(x)(\mathcal{K}u)(x'(x))]$, so

$$\begin{aligned} |(\mathcal{Q}_1 \mathcal{K}u)(x) - (\mathcal{Q}_1 \mathcal{K}u)(x_0)| &\leq pm_{\mathcal{K}} \mathbb{E}|w(x) - w(x_0)| \\ &\quad + p\mathbb{E}[w(x_0) \|\mathcal{K}(x'(x), \cdot) - \mathcal{K}(x'(x_0), \cdot)\|_{TV}], \end{aligned}$$

where $m_{\mathcal{K}} = \sup_y \mathcal{K}(y, \mathbb{RP}^1) \leq \max\{\|\mathcal{Q}_1\|, \|\mathcal{Q}_1\|^2\}$ and $\|\mathcal{Q}_1\|$ is the norm on $C(\mathbb{RP}^1)$, which bounds the total mass of every row of $\mathcal{Q}_1$. The first term tends to zero. In the second term, the integrand tends to zero almost surely, since $x'(x) \rightarrow x'(x_0)$, and it is bounded by $2m_{\mathcal{K}} w(x_0)$, which is integrable; by dominated convergence, it tends to zero. This shows that $\mathcal{Q}_1 \mathcal{K}$ is continuous in total variation at x_0 .

We apply this twice. If $x_0 \in \mathbb{R} \setminus \{0\}$, then $x'(x_0) = \tilde{B} - \beta/x_0$ has a density, and if $x_0 = \infty$, then $x'(x_0) = \tilde{B}$; in both cases $x'(x_0) \in \mathbb{R} \setminus \{0\}$ almost surely, by (I1). With $\mathcal{K} = \mathcal{Q}_1$, we conclude that $\mathcal{Q}_1^2$ is continuous in total variation on $\mathbb{RP}^1 \setminus \{0\}$. For every $x_0 \in \mathbb{RP}^1$, we have $x'(x_0) \neq 0$ almost surely (at $x_0 = 0$, $x' = \infty$). With $\mathcal{K} = \mathcal{Q}_1^2$, we conclude that $\mathcal{Q}_1^3$ is continuous in total variation on $\mathbb{RP}^1$. From the same representation, we see that $\mathcal{Q}_1^2(x, \cdot) = p\mathbb{E}[w \mathcal{Q}_1(x', \cdot)]$ is absolutely continuous for $x \neq 0$, since $x' \in \mathbb{R} \setminus \{0\}$ almost surely, and that $\mathcal{Q}_1^3(x, \cdot) = p\mathbb{E}[w \mathcal{Q}_1^2(x', \cdot)]$ is absolutely continuous for every $x \in \mathbb{RP}^1$, since $x' \neq 0$ almost surely. This proves the claim. Since the rows of $\mathcal{P}^3$ are probability measures, we also have $\sup |\mathcal{P}^3 u| \leq \|u\|_{L^\infty(\pi)}$.

STEP 6. *Decay in $C(\mathbb{RP}^1)$ and in $L^\infty(\pi)$.* On $C(\mathbb{RP}^1)$, the operator $\mathcal{P}$ is similar to $\rho^{-1}\mathcal{Q}_1$ through multiplication by ψ . By (I3), 1 is an algebraically simple eigenvalue of $\mathcal{P}$ on $C(\mathbb{RP}^1)$, with eigenfunction $\mathbf{1}$, and the rest of the spectrum lies in a disc of radius $\theta_1 < 1$. Let Π_C be the Riesz projection at 1. It has rank one and range spanned by $\mathbf{1}$; we write $\Pi_C v = \phi(v)\mathbf{1}$ with a bounded linear functional ϕ . Further, for $\theta_2 \in (\theta_1, 1)$ there exists C_0 with $\sup |\mathcal{P}^n v - \phi(v)| \leq C_0 \theta_2^n \sup |v|$ for $v \in C(\mathbb{RP}^1)$. Integrating against π and using the invariance of π , we find that $\int v d\pi = \int \mathcal{P}^n v d\pi \rightarrow \phi(v)$, which shows that $\phi(v) = \int v d\pi$. Now let $u \in L^\infty(\pi)$ and $n \geq 3$. The function $v = \mathcal{P}^3 u$ lies in $C(\mathbb{RP}^1)$ by Step 5, and $\int v d\pi = \int u d\pi$ by invariance. Applying the bound in $C(\mathbb{RP}^1)$ to v , we find that

$$\left\| \mathcal{P}^n u - \int u d\pi \right\|_{L^\infty(\pi)} = \left\| \mathcal{P}^{n-3} v - \int v d\pi \right\|_\infty \leq C_0 \theta_2^{n-3} \|u\|_{L^\infty(\pi)}. \quad (14.26)$$

STEP 7. *The gap in $L^2(\pi)$.* Let $\Pi u = (\int u d\pi)\mathbf{1}$ and $\mathcal{P}_0 = \mathcal{P} - \Pi$. Since $\mathcal{P}\mathbf{1} = \mathbf{1}$ and π is invariant, we have $\Pi\mathcal{P} = \mathcal{P}\Pi = \Pi^2 = \Pi$. Then $\Pi\mathcal{P}_0 = \mathcal{P}_0\Pi = 0$ and $\mathcal{P}_0^n = \mathcal{P}^n - \Pi$. On $L^1(\pi)$, we have $|\mathcal{P}u| \leq \mathcal{P}|u|$ and $\int \mathcal{P}|u| d\pi = \int |u| d\pi$, which give $\|\mathcal{P}_0^n\|_{L^1(\pi)} \leq 2$. Applying (14.26) to real and imaginary parts, $\|\mathcal{P}_0^n\|_{L^\infty(\pi)} \leq 2C_0\theta_2^{n-3}$ on complex functions. The Riesz–Thorin interpolation theorem [112, Section IX.4] gives $\|\mathcal{P}_0^n\|_{L^2(\pi)} \leq 2(C_0\theta_2^{n-3})^{1/2}$, and the spectral radius formula yields $r(\mathcal{P}_0) \leq \theta_2^{1/2} < 1$ on $L^2(\pi)$. For $z \neq 1$ with $|z| > \theta_2^{1/2}$, a direct computation using $\Pi\mathcal{P}_0 = \mathcal{P}_0\Pi = 0$ shows that

$$(z - \mathcal{P})^{-1} = (z - 1)^{-1}\Pi + (z - \mathcal{P}_0)^{-1}(1 - \Pi).$$

Therefore, 1 is an isolated point of the spectrum of $\mathcal{P}$ on $L^2(\pi)$, with Riesz projection Π of rank one. Since $\tilde{Q} = \rho\psi\mathcal{P}\psi^{-1}$ on the space $L^2(\tilde{\mathfrak{m}}) = L^2(\pi)$ (with equivalent norms), and $A^s = \Theta^{-1}\tilde{Q}\Theta$, the same holds for ρ and A^s .

STEP 8. *The augmented operator.* Let $\mathcal{L}' = L^2(\mathbb{R} \times (0, \infty), dx \omega(d\beta))$ and define $V: L^2(\mathbb{R}) \rightarrow \mathcal{L}'$ and $\Pi': \mathcal{L}' \rightarrow L^2(\mathbb{R})$ by

$$(Vf)(x, \beta) = p \frac{\sqrt{\beta}}{|x|} (C_\nu f)\left(\frac{\beta}{x}\right), \quad (\Pi'F)(x) = \int \omega(d\beta) F(x, \beta).$$

For fixed β , the substitution $x \mapsto \beta/x$ shows that $\|Vf\|^2 = p^2\|C_\nu f\|^2 \leq p^2\|f\|^2$, and by Jensen's inequality, we have $\|\Pi'\| \leq 1$. Further, $\Pi'V = p\mathcal{W}C_\nu = A^s$. The map $\mathcal{U}F = \omega_\xi F$ is unitary from $\mathcal{L}'$ onto $\mathcal{L}$, since $\omega_\xi^2 \mu_0 = \omega$, and comparing with (14.22) shows that $A_\xi = \mathcal{U}V\Pi'\mathcal{U}^{-1}$. By Lemma 14.5 with $X = V$ and $Y = \Pi'$, the operators A_ξ and A^s have the same nonzero spectrum, and their Riesz projections at ρ have the same rank. With Steps 4 and 7, this proves (i). For (ii), set $F_\xi = \mathcal{U}Vf$. Then $A_\xi F_\xi = \mathcal{U}V\Pi'Vf = \mathcal{U}VA^s f = \rho F_\xi$. Since $f > 0$ everywhere and ν is a probability measure, $C_\nu f > 0$ everywhere, so $F_\xi(x, \beta) > 0$ for all $x \neq 0$ and $\beta > 0$. $\square$

14.6 A positive decomposition of the square

In this subsection we construct the decomposition $A_\xi^2 = G_\xi + h_c \otimes \ell_c + h_t \otimes \ell_t$. We assume $p \geq p_1$ and $\xi \in \Xi$ throughout, and use the conventions of Section 14.4.

Regions and cutoffs

Definition 14.12. Set

$$r = \frac{1}{\log p}, \quad W = ((M_* + 1) \log p)^{1/2}, \quad \beta_{\max} = \frac{4W^2}{p^2 \xi_-^2}.$$

The *core region* is $\Omega_c = \{(x, \beta) \in \mathbb{R} \times (0, \infty) : |x| > \sqrt{\beta}\}$, and the *tail region* is its complement Ω_t . The *ordinary set* is $O = [1 - r, 1 + r] \times (0, \beta_{\max}]$. We also set $U_c = \Omega_c \setminus O$ and $U_t = J U_c$, with the notation $J\Omega'$ of Lemma 14.10(i).

The map $(x, \beta) \mapsto (\beta/x, \beta)$ sends Ω_c onto $\{0 < |x| < \sqrt{\beta}\}$. Then $J\Omega_c$ and Ω_t differ by a null set, and $JP_{\Omega_c}J = P_{\Omega_t}$ as operators. For $p \geq p_1$, we have $\sqrt{\beta_{\max}} < 1 - r$, which implies that $O \subset \Omega_c$ and $JO \subset \Omega_t$ up to a null set. Up to null sets, $\mathbb{R} \times (0, \infty)$ is the disjoint union of the four sets

$$X_1 = JO, \quad X_2 = U_t, \quad X_3 = U_c, \quad X_4 = O, \quad (14.27)$$

with $\Omega_t = X_1 \cup X_2$ and $\Omega_c = X_3 \cup X_4$.

We fix, once and for all, smooth functions $\chi_0, \psi_0: \mathbb{R} \rightarrow [0, 1]$ with $\chi_0 = 1$ on $[-1/8, 1/8]$, $\text{supp } \chi_0 \subset (-1/4, 1/4)$, $\psi_0 = 1$ on $(-\infty, 0]$ and $\psi_0 = 0$ on $[1, \infty)$. Define

$$\chi(b) = \chi_0\left(\frac{b-1}{r}\right), \quad \psi(u) = \psi_0\left(\frac{u^2}{\log p} - M_*\right), \quad u \geq 0.$$

Then $\chi(b) = 1$ for $|b-1| \leq r/8$, $\chi(b) = 0$ for $|b-1| \geq r/4$, $\psi(u) = 1$ for $u^2 \leq M_* \log p$, and $\psi(u) = 0$ for $u \geq W$. For $l \leq 5$, we have

$$|\chi^{(l)}| \leq C \log^l p, \quad |(u\partial_u)^l \psi(u)| \leq C, \quad (14.28)$$

where the second bound holds because $(u\partial_u)^l \psi(u)$ is a combination of $(u^2/\log p)^i \psi_0^{(i)}(u^2/\log p - M_*)$, $i \leq l$, and $u^2/\log p \leq M_* + 1$ on the support of ψ'_0 . For $x > 0$ and $\beta > 0$, define

$$w_\xi(x, \beta) = \left(\frac{\beta}{\tau c x} \right)^{1/2},$$

and define the multipliers

$$\Psi_\xi^{\text{out}}(x, \beta) = \mathbf{1}_O(x, \beta) \psi(w_\xi(x, \beta)), \quad \Psi_\xi^{\text{in}}(x, \beta) = \mathbf{1}_O(x, \beta) \chi(cx) \psi(w_\xi(x, \beta)).$$

The retained operator is $B_\xi = \Psi_\xi^{\text{out}} A_\xi \Psi_\xi^{\text{in}}$, where the multipliers act as multiplication operators.

- Lemma 14.13.** (i) If $x > 0$, $\beta > 0$ and $\chi(cx)\psi(w_\xi(x, \beta)) \neq 0$, then $(x, \beta) \in O$ and $|cx - 1| < r/4$.
(ii) The functions $\Psi_\xi^{\text{out}}, \Psi_\xi^{\text{in}}$ take values in $[0, 1]$, and $\xi \mapsto \Psi_\xi^{\text{out}}, \Psi_\xi^{\text{in}}$ are four times continuously differentiable into L^∞ , with $\sup |\partial_\xi^j \Psi_\xi^{\text{out}}| + \sup |\partial_\xi^j \Psi_\xi^{\text{in}}| \leq C \log^j p$ for $j \leq 4$.
(iii) We have $0 \leq B_\xi \leq P_O A_\xi P_O$, the map $\xi \mapsto B_\xi$ is four times continuously differentiable in norm, and $\|\partial_\xi^j B_\xi\| \leq C p^{A_1} \log^8 p$ for $j \leq 4$.

Proof. For (i), $|cx - 1| < r/4$ and $c - 1 = \theta \leq 4/(p\xi_-^2)$ give $|x - 1| \leq r$ for $p \geq p_1$. Further, $\psi(w_\xi) \neq 0$ implies that $w_\xi < W$, and then $\beta < \tau c x W^2 \leq 2\tau(1+r)W^2 < \beta_{\max}$, since $c \leq 2$ and $\tau \leq 1/(p^2 \xi_-^2)$. For (ii), we have $\partial_\xi w_\xi = -w_\xi \partial_\xi(\tau c)/(2\tau c)$. By induction, $\partial_\xi^j[\psi(w_\xi)]$ is a combination of the functions $((u\partial_u)^l \psi)(w_\xi)$, $l \leq j$, with coefficients that are polynomials in $\partial_\xi^i(\tau c)/(\tau c)$, $i \leq j$. These coefficients are bounded by (14.15), and (14.28) bounds the functions. Similarly, $\partial_\xi^j[\chi(cx)]$ is a combination of $x^l \chi^{(l)}(cx)$ with coefficients that are polynomials in $\partial_\xi^i c$; since $x \leq 2$ on O , these are bounded by $C \log^j p$. The same bounds hold for $j = 5$, and Taylor's formula implies that the L^∞ -valued maps are four times continuously differentiable. For (iii), $0 \leq \Psi_\xi^{\text{out}}, \Psi_\xi^{\text{in}} \leq \mathbf{1}_O$ gives

$$P_O A_\xi P_O - B_\xi = (P_O - \Psi_\xi^{\text{out}}) A_\xi P_O + \Psi_\xi^{\text{out}} A_\xi (P_O - \Psi_\xi^{\text{in}}) \geq 0.$$

The derivative bounds follow from (ii), Lemma 14.10(iv) and the Leibniz rule. $\square$

Factorization of the retained operator

Let $\mathcal{V} = L^2([0, W], \gamma)$, a real Hilbert space of the form considered in Section 14.2; we write $\mathbf{1}$ for the constant function one in $\mathcal{V}$. Define

$$h_\xi(x, \beta) = \mathbf{1}_O(x, \beta) p \omega_\xi(\beta) \frac{\sqrt{\beta}}{x} \psi(w_\xi(x, \beta)), \quad (14.29)$$

and define $R_{\xi}: \mathcal{V} \rightarrow \mathcal{L}$ and $S_{\xi}: \mathcal{L} \rightarrow \mathcal{V}$ by

$$(R_{\xi}\phi)(x, \beta) = h_{\xi}(x, \beta) \phi(w_{\xi}(x, \beta)), \quad (14.30)$$

$$(S_{\xi}F)(w) = \int \nu_{\xi}(dy) \int \mu_0(d\beta') \omega_{\xi}(\beta') \Psi_{\xi}^{\text{in}}(x', \beta') F(x', \beta'), \quad x' = y - \tau c w^2, \quad (14.31)$$

where in (14.30) we set $\phi(w) = 0$ for $w > W$ (where h_{ξ} vanishes anyway). Let $Y(b, u) = \sqrt{b} e^{-(b-1)u^2} \chi(b)$ for $b > 0$ and $Y(b, u) = 0$ for $b \leq 0$; this is a smooth function, since χ vanishes for $b \leq 1/2$. For $w, u \in [0, W]$, define

$$Y_{\xi}(w, u) = \mathbb{E} Y\left(B - \frac{\theta w^2}{p}, u\right), \quad Y_{\xi}^0(u) = \mathbb{E} Y(B, u), \quad (14.32)$$

where $B \sim \nu_{p, \xi} \sqrt{p}$, that is, $B = c\tilde{B}$ with $\tilde{B} \sim \nu_{\xi}$.

Lemma 14.14. *The operators R_{ξ} and S_{ξ} are positive, $\|R_{\xi}\| \leq C$, $\|S_{\xi}\| \leq Cp$, and $B_{\xi} = R_{\xi}S_{\xi}$. The operator $N_{\xi} = S_{\xi}R_{\xi}$ on $\mathcal{V}$ is the integral operator*

$$(N_{\xi}\phi)(w) = \alpha \int_0^W \gamma(du) u \psi(u)^2 Y_{\xi}(w, u) \phi(u). \quad (14.33)$$

Moreover, $R_{\xi}\mathbf{1} = h_{\xi}$.

Proof. Positivity is evident. To bound R_{ξ} , we substitute $\beta = \tau c x u^2$ for fixed $x \in [1-r, 1+r]$; then $\omega_{\xi}(\beta)^2 \mu_0(d\beta) = \omega_{\xi}(d\beta) = e^{-cxu^2} 2cxu du$ and $p^2 \beta x^{-2} = p^2 \tau c u^2 / x$, and we have

$$\|R_{\xi}\phi\|^2 \leq \int_{1-r}^{1+r} dx \int_0^W \gamma(du) p^2 \tau c^2 u^2 e^{-(cx-1)u^2} |\phi(u)|^2 \leq \frac{c^2}{\xi^2} \cdot 2r \cdot W^2 e^{2rW^2} \|\phi\|^2.$$

Here we used $p^2 \tau = \xi^{-2}$, $|cx-1| \leq 2r$ and $u \leq W$. Since $rW^2 = M_* + 1$, this gives $\|R_{\xi}\| \leq C$. To bound S_{ξ} , we use the Cauchy–Schwarz inequality in β' with $\int \omega_{\xi}^2 d\mu_0 = 1$, Jensen's inequality in y , and $\Psi_{\xi}^{\text{in}} \leq 1$, which give $|(S_{\xi}F)(w)|^2 \leq \int \nu_{\xi}(dy) \int \mu_0(d\beta') |F(y - \tau c w^2, \beta')|^2$. For fixed y , the substitution $x' = y - \tau c w^2$ has $dx' = 2\tau c w dw$, which gives

$$\|S_{\xi}F\|^2 \leq \int \nu_{\xi}(dy) \int \mu_0(d\beta') \int_0^{\infty} 2w e^{-w^2} |F(y - \tau c w^2, \beta')|^2 dw \leq \frac{1}{\tau c} \|F\|^2 \leq Cp^2 \|F\|^2.$$

For the factorization, let (x, β) with $\Psi_{\xi}^{\text{out}}(x, \beta) \neq 0$. Then $(x, \beta) \in \mathcal{O}$ and $x > 0$, and $\beta/x = \tau c w^2$ with $w = w_{\xi}(x, \beta) < W$. By (14.22),

$$\begin{aligned} (B_{\xi}F)(x, \beta) &= \psi(w) \mathbf{1}_{\mathcal{O}}(x, \beta) p \omega_{\xi}(\beta) \frac{\sqrt{\beta}}{x} \\ &\quad \times \int \nu_{\xi}(dy) \int \mu_0(d\beta') \omega_{\xi}(\beta') \Psi_{\xi}^{\text{in}}(y - \tau c w^2, \beta') F(y - \tau c w^2, \beta'), \end{aligned}$$

which is $h_\xi(x, \beta)(S_\xi F)(w) = (R_\xi S_\xi F)(x, \beta)$. Where $\Psi_\xi^{\text{out}}(x, \beta) = 0$, both sides vanish.

For (14.33), let $\phi \in \mathcal{V}$ and $w \in [0, W]$, and write $x' = y - \tau c w^2$ and $w' = w_\xi(x', \beta')$ for $x' > 0$. By Lemma 14.13(i), $\Psi_\xi^{\text{in}}(x', \beta') h_\xi(x', \beta') = \mathbf{1}_{\{x' > 0\}} \chi(cx') \psi(w')^2 p \omega_\xi(\beta') \sqrt{\beta'}/x'$, and therefore

$$(S_\xi R_\xi \phi)(w) = \int \nu_\xi(dy) \mathbf{1}_{\{x' > 0\}} \chi(cx') \int \omega_\xi(d\beta') \psi(w')^2 \frac{p \sqrt{\beta'}}{x'} \phi(w').$$

We substitute $\beta' = \tau c x' u^2$ in the inner integral. Then $w' = u$, $\omega_\xi(d\beta') = 2c x' u e^{-c x' u^2} du$ and $p \sqrt{\beta'}/x' = p(\tau c)^{1/2} u x'^{-1/2}$, so the inner integral equals

$$\begin{aligned} & \int_0^\infty \gamma(du) u \psi(u)^2 p \tau^{1/2} c^{3/2} x'^{1/2} e^{-(cx'-1)u^2} \phi(u) \\ &= \alpha \int_0^W \gamma(du) u \psi(u)^2 (cx')^{1/2} e^{-(cx'-1)u^2} \phi(u), \end{aligned}$$

where we used $p \tau^{1/2} c = \alpha$ from (14.14). Since $c x' = c y - \tau c^2 w^2 = B - \theta w^2/p$ with $B = c y$, again by (14.14), we obtain (14.33) from Fubini's theorem. Finally, $R_\xi \mathbf{1} = h_\xi$ since h_ξ vanishes where $w_\xi > W$. $\square$

The rank-one minorant

Let $\eta_p = C_\eta \log^3 p / p^2$, where C_η is the constant of Lemma 14.15(ii) below, and define

$$m_\xi(u) = Y_\xi^0(u) - \eta_p, \quad \hat{k}_\xi(u) = \alpha u \psi(u)^2 m_\xi(u), \quad k_\xi(\phi) = \int_0^W \hat{k}_\xi(u) \phi(u) \gamma(du),$$

and the operators $N_\xi^0 = \mathbf{1} \otimes \hat{k}_\xi$ (so that $N_\xi^0 \phi = k_\xi(\phi) \mathbf{1}$) and $\Delta_\xi = N_\xi - N_\xi^0$ on $\mathcal{V}$. We also set $\sigma_\xi = k_\xi(\mathbf{1})$.

Lemma 14.15. *For $u, w \in [0, W]$ and $j \in \{1, 2\}$, the following hold.*

- (i) $|Y_\xi^0(u) - 1| \leq C p^{-1/4} \log p$; in particular, $m_\xi(u) \geq 1/2$.
- (ii) *There exists C_η such that $|Y_\xi(w, u) - Y_\xi^0(u)| \leq \eta_p$, so that $0 \leq Y_\xi(w, u) - m_\xi(u) \leq 2\eta_p$. Further, $|\partial_\xi^j(Y_\xi(w, u) - Y_\xi^0(u))| \leq C p^{-2} \log^D p$.*
- (iii) $|\partial_\xi^j Y_\xi^0(u)| \leq C \epsilon_p \log^4 p$, with ϵ_p as in (14.18).
- (iv) $\Delta_\xi \succeq 0$, $N_\xi^0 \succeq 0$, and for $j \leq 2$, $\|\partial_\xi^j \Delta_\xi\| \leq C p^{-2} \log^D p$, $\|\partial_\xi^j N_\xi^0\| \leq C$ and $\|\partial_\xi^j \hat{k}_\xi\|_{\mathcal{V}} \leq C$.

Proof. We first record properties of Y . It vanishes unless $|b - 1| < r/4$, and for such b and $u \leq W$ we have $|(b - 1)u^2| \leq rW^2/4 = (M_* + 1)/4$. Differentiating, we find that for $l, l' \leq 5$,

$$\sup_{b \in \mathbb{R}, u \in [0, W]} |\partial_b^l (u \partial_u)^{l'} Y(b, u)| \leq C \log^{2l} p, \quad (14.34)$$

since each b -derivative falls on χ , costing $r^{-1} = \log p$, or on the exponential, costing $u^2 \leq W^2 \leq C \log p$, or on $\sqrt{b}$, while $(u\partial_u)e^{-(b-1)u^2} = -2(b-1)u^2e^{-(b-1)u^2}$ is bounded on the support.

For (i), let $\mathcal{C}_B = \{|B-1| \leq p^{-1/4}\}$. On $\mathcal{C}_B$, we have $\chi(B) = 1$ for $p \geq p_1$, and $|\sqrt{B}e^{-(B-1)u^2} - 1| \leq |\sqrt{B} - 1|e^{p^{-1/4}W^2} + |e^{-(B-1)u^2} - 1| \leq Cp^{-1/4}\log p$. Off $\mathcal{C}_B$, the integrand is bounded by C , and (14.17) bounds the probability by $e^{-a\sqrt{p}}$. Combining the two cases, we obtain $|Y_\xi^0(u) - 1| \leq Cp^{-1/4}\log p$, and $m_\xi \geq 1 - Cp^{-1/4}\log p - \eta_p \geq 1/2$.

For (ii), let $\delta_\xi = \theta w^2/p$, so that $0 \leq \delta_\xi \leq 4W^2/(p^2\xi_-^2)$ and $|\partial_\xi^i \delta_\xi| \leq CW^2/p^2$ for $i \leq 2$, by (14.15). Set $\zeta_\xi(b) = Y(b - \delta_\xi, u) - Y(b, u)$. We have $Y_\xi(w, u) - Y_\xi^0(u) = \mathbb{E}\zeta_\xi(B)$, and ζ_ξ is supported in $(1 - r/4, 1 + r/4 + \delta_\xi) \subset [1/2, 3/2]$. By (14.34) and the mean value theorem, $\sup |\zeta_\xi| \leq \delta_\xi \sup |\partial_b Y| \leq C \log^3 p / p^2$, which gives the first claim with a suitable C_η . Further, $\|\zeta_\xi\|_{C^2} \leq \delta_\xi \sup_{l \leq 3} |\partial_b^l Y| \leq Cp^{-2} \log^D p$, and $\partial_\xi \zeta_\xi(b) = -(\partial_\xi \delta_\xi) \partial_b Y(b - \delta_\xi, u)$ and $\partial_\xi^2 \zeta_\xi$ have C^2 norms bounded by $Cp^{-2} \log^D p$ as well. Then (14.19) implies the claimed derivative bounds.

For (iii), we apply (14.18) to the test $Y(\cdot, u)$, which is supported in $[1/2, 3/2]$ and has C^2 norm at most $C \log^4 p$ by (14.34).

For (iv), the kernel of Δ_ξ with respect to $\gamma \otimes \gamma$ is $\alpha u \psi(u)^2 (Y_\xi(w, u) - m_\xi(u))$, which is nonnegative by (ii), and $N_\xi^0 \geq 0$ since $\hat{k}_\xi \geq 0$ by (i). By the Hilbert–Schmidt bound and $\int u^2 \gamma(du) = 1$,

$$\|\partial_\xi^j \Delta_\xi\| \leq \sup_{u, w \in [0, W]} |\partial_\xi^j [\alpha (Y_\xi(w, u) - Y_\xi^0(u) + \eta_p)]| \leq Cp^{-2} \log^D p,$$

by (ii) and (14.15). (The kernels are twice continuously differentiable in ξ , uniformly in $(w, u) \in [0, W]^2$, because $\xi \mapsto \nu_\xi$ is twice continuously differentiable in total variation; this justifies differentiating under the Hilbert–Schmidt norm.) In the same way, $|Y_\xi| \leq C$ and the bounds in (ii) and (iii) give $\|\partial_\xi^j N_\xi\| \leq C$ and $\|\partial_\xi^j \hat{k}_\xi\|_{\mathcal{V}} \leq C$. $\square$

We now define the rank-one operator of the core component and its left vector. Set

$$h_c = h_\xi = R_\xi \mathbf{1}, \quad \ell_c = S_\xi^* \hat{k}_\xi, \quad L_c = h_c \otimes \ell_c = R_\xi N_\xi^0 S_\xi,$$

where the last identity holds because $R_\xi N_\xi^0 S_\xi F = k_\xi(S_\xi F) R_\xi \mathbf{1} = \langle \ell_c, F \rangle h_c$. We suppress the dependence of h_c , ℓ_c and L_c on ξ in the notation. For $t \in \mathbb{R}$, let

$$q_\xi(t) = \frac{\alpha}{\tau c} \left(\frac{t}{\tau c}\right)^{1/2} e^{-t/(\tau c)} \psi\left(\left(\frac{t}{\tau c}\right)^{1/2}\right)^2 m_\xi\left(\left(\frac{t}{\tau c}\right)^{1/2}\right) \mathbf{1}_{\{t > 0\}}. \quad (14.35)$$

This is a smoothly truncated density of Gamma type whose threshold $t = 0$ does not depend on ξ .

Lemma 14.16. (i) For almost every (x, β) ,

$$\ell_c(x, \beta) = \Psi_\xi^{\text{in}}(x, \beta) \omega_\xi(\beta) \int \nu_\xi(dy) q_\xi(y - x). \quad (14.36)$$

In particular, h_c and ℓ_c are nonnegative and vanish outside O .

(ii) The maps $\xi \mapsto h_c, \ell_c$ are four times continuously differentiable into $\mathcal{L}$, and for $j \leq 4$,

$$\|\partial_\xi^j h_c\| \leq C, \quad \|\partial_\xi^j \ell_c\| \leq Cp^{2A+1} \log^D p \leq p^{A_1}, \quad \|\partial_\xi^j L_c\| \leq p^{A_1+1}.$$

(iii) There exist a nonempty open set $V_0 \subset O$ and $c_0 > 0$, both possibly depending on p and ξ , such that $h_c \geq c_0$ and $\ell_c \geq c_0$ on V_0 .

Proof. For (i), let $F \in \mathcal{L}$. By the definition of ℓ_c and (14.31),

$$\langle \ell_c, F \rangle = \int \nu_\xi(dy) \int \mu_0(d\beta') \omega_\xi(\beta') \int_0^W \gamma(dw) \hat{k}_\xi(w) (\Psi_\xi^{\text{in}} F)(y - \tau c w^2, \beta').$$

For fixed y , we substitute $x = y - \tau c w^2$ in the inner integral. Then $w = ((y - x)/(\tau c))^{1/2}$ and $\gamma(dw) = (\tau c)^{-1} e^{-(y-x)/(\tau c)} dx$ on $\{x < y\}$, and comparing with (14.35) shows that $\gamma(dw) \hat{k}_\xi(w) = q_\xi(y - x) dx$. By Fubini's theorem, we obtain (14.36). Since h_c contains the factor $\mathbf{1}_O$ and $\Psi_\xi^{\text{in}} \leq \mathbf{1}_O$, both vanish outside O ; they are nonnegative since $m_\xi \geq 0$.

For (ii), we begin with h_c . By Lemma 14.10(iv) and the proof of Lemma 14.13(ii), we have $|\partial_\xi^j [\omega_\xi(\beta) \psi(w_\xi(x, \beta))]| \leq C \omega_\xi(\beta) (1 + \beta/\tau)^j$, which gives

$$\|\partial_\xi^j h_c\|^2 \leq C \int_{1-r}^{1+r} \frac{dx}{x^2} \int \omega_\xi(d\beta) p^2 \beta (1 + \beta/\tau)^{2j} \leq Cp^2 \tau r \leq C,$$

where we used that $\beta/\tau \sim \text{Exp}(1)$ under ω_ξ and $p^2 \tau = \xi^{-2}$. Differentiability follows by dominated convergence, as in the proof of Lemma 14.10(iv).

We now turn to ℓ_c . We write $s_\xi = \tau c$ and $Q_\xi(s) = s^{1/2} e^{-s} \psi(s^{1/2})^2 m_\xi(s^{1/2})$ for $s > 0$, so that $q_\xi(t) = \alpha s_\xi^{-1} Q_\xi(t/s_\xi) \mathbf{1}_{\{t>0\}}$. We claim that for $i \leq 4, l \leq 4$ and $u \in [0, W]$,

$$|(u \partial_u)^l \partial_\xi^i m_\xi(u)| \leq Cp^A \log^D p. \quad (14.37)$$

For $i = l = 0$ this follows from (i) of Lemma 14.15. Otherwise, the constant η_p does not contribute, and $(u \partial_u)^l \partial_\xi^i m_\xi(u) = \partial_\xi^i \int f(c(\xi)y) \nu_\xi(dy)$ with $f = (u \partial_u)^l Y(\cdot, u)$. By the Leibniz rule, this is a sum of terms $\int \partial_\xi^{i_2} [f(c(\xi)y)] (\partial_\xi^{i_1} \nu_\xi)(dy)$ with $i_1 + i_2 = i$. The function $f(c(\xi)y)$ vanishes unless $|y| \leq 2$, and on this set its ξ -derivatives of order i_2 are bounded by $C \log^D p$, by (14.34) and (14.15). Using $|\partial_\xi^{i_1} \nu_\xi|(\mathbb{R}) \leq p^A$ from (I5) (and $\nu_\xi(\mathbb{R}) = 1$ for $i_1 = 0$), we find (14.37).

By induction on i , $\partial_\xi^i q_\xi(t)$ is αs_ξ^{-1} times a combination of the functions $((s \partial_s)^l \partial_\xi^{i'} Q_\xi)(t/s_\xi)$ with $l + i' \leq i$, with coefficients that are polynomials in $\partial_\xi^{i''} \alpha / \alpha$ and

$\partial_{\xi}'' s_{\xi}/s_{\xi}$, which are bounded by (14.15). Since $(s\partial_s)$ acts on $s^{1/2}e^{-s}$ by multiplication with $1/2 - s$, and on a function of $s^{1/2}$ as $(u\partial_u)/2$ with $u = s^{1/2}$, the bounds (14.28) and (14.37) give

$$|\partial_{\xi}^i q_{\xi}(t)| \leq Cp^A \log^D p \frac{\alpha}{s_{\xi}} \left(\frac{t}{s_{\xi}}\right)^{1/2} e^{-t/s_{\xi}} \left(1 + \frac{t}{s_{\xi}}\right)^4 \mathbf{1}_{\{t>0\}}, \quad i \leq 4.$$

Integrating the square, we obtain $\|\partial_{\xi}^i q_{\xi}\|_{L^2(\mathbb{R})}^2 \leq Cp^{2A} \log^{2D} p \alpha^2 s_{\xi}^{-1} \leq Cp^{2A+2} \log^{2D} p$, since $s_{\xi}^{-1} \leq p^2 \xi_+^2$. The supremum over $\xi \in \Xi$ of the right-hand side is square integrable, and dominated convergence shows that $\xi \mapsto q_{\xi}$ is four times continuously differentiable into $L^2(\mathbb{R})$.

Now let $\bar{q}_{\xi} = \nu_{\xi} \star q_{\xi}$, that is, $\bar{q}_{\xi}(x) = \int \nu_{\xi}(dy) q_{\xi}(y - x)$. As in Lemma 14.10(ii), $\|\sigma \star f\|_{L^2} \leq \|\sigma\|(\mathbb{R}) \|f\|_{L^2}$ for a finite signed measure σ . By (I4) and the Leibniz rule, $\xi \mapsto \bar{q}_{\xi}$ is four times continuously differentiable into $L^2(\mathbb{R})$, and by (I5),

$$\|\partial_{\xi}^i \bar{q}_{\xi}\|_{L^2} \leq \sum_{i_1+i_2=i} \binom{i}{i_1} |\partial_{\xi}^{i_1} \nu_{\xi}|(\mathbb{R}) \|\partial_{\xi}^{i_2} q_{\xi}\|_{L^2} \leq Cp^{2A+1} \log^D p, \quad i \leq 4.$$

By (14.36), $\ell_c(x, \beta) = \Psi_{\xi}^{\text{in}}(x, \beta) \omega_{\xi}(\beta) \bar{q}_{\xi}(x)$. By Lemma 14.13(ii) and the proof of Lemma 14.10(iv), $|\partial_{\xi}^i [\Psi_{\xi}^{\text{in}}(x, \beta) \omega_{\xi}(\beta)]| \leq C \log^4 p \omega_{\xi}(\beta) (1 + \beta/\tau)^i$ for $i \leq 4$. It follows that, for every $f \in L^2(\mathbb{R})$, the function $\partial_{\xi}^i [\Psi_{\xi}^{\text{in}} \omega_{\xi}] f$ has norm at most $C \log^4 p \|f\|_{L^2}$ in $\mathcal{L}$. The Leibniz rule now gives the differentiability of $\xi \mapsto \ell_c$ and the bound $\|\partial_{\xi}^j \ell_c\| \leq Cp^{2A+1} \log^D p \leq p^{A_1}$. Finally, $\partial_{\xi}^j L_c = \sum_i \binom{j}{i} \partial_{\xi}^i h_c \otimes \partial_{\xi}^{j-i} \ell_c$ and $\|f \otimes g\| = \|f\| \|g\|$ give $\|\partial_{\xi}^j L_c\| \leq 16Cp^{A_1} \leq p^{A_1+1}$.

For (iii), let $x_1 = 1 - 3\tau c/2$, and let $V_0 = (x_1 - \tau c/4, x_1 + \tau c/4) \times (\beta_1/2, \beta_1)$ with $\beta_1 = \tau$. For $(x, \beta) \in V_0$ and $p \geq p_1$, we have $|cx - 1| \leq \theta + 2\tau c^2 \leq r/8$ and $w_{\xi}(x, \beta)^2 \leq 2 \leq M_* \log p$. Then $\Psi_{\xi}^{\text{in}}(x, \beta) = 1$ and $\psi(w_{\xi}(x, \beta)) = 1$; also $(x, \beta) \in \mathcal{O}$. On V_0 , the function $h_c = p\omega_{\xi}(\beta) \sqrt{\beta}/x$ is bounded below by a positive constant. Further, the atom of ν_{ξ} at 1 gives $\bar{q}_{\xi}(x) \geq e^{-p} q_{\xi}(1 - x)$, and for $(x, \beta) \in V_0$ we have $(1 - x)/(\tau c) \in [5/4, 7/4]$, where $q_{\xi}(1 - x) \geq \alpha(\tau c)^{-1} e^{-2}/2$ by Lemma 14.15(i). We conclude that ℓ_c is bounded below by a positive constant on V_0 as well. $\square$

Lemma 14.17. Define $G_{\xi}^{\text{ord}} = B_{\xi}^2 - L_c$, the operator $L_t = JL_c^*J$, the vectors $h_t = J\ell_c$ and $\ell_t = Jh_c$, and

$$G_{\xi} = A_{\xi}^2 - L_c - L_t. \quad (14.38)$$

(i) We have $0 \leq L_c \preceq B_{\xi}^2 \preceq (P_0 A_{\xi} P_0)^2 \preceq P_0 A_{\xi}^2 P_0$.

(ii) We have $G_{\xi}^{\text{ord}} = R_{\xi} \Delta_{\xi} S_{\xi} \succeq 0$, $G_{\xi}^{\text{ord}} = P_0 G_{\xi}^{\text{ord}} P_0$, and $\|G_{\xi}^{\text{ord}}\| \leq Cp^{-1} \log^D p$. The map $\xi \mapsto G_{\xi}^{\text{ord}}$ is four times continuously differentiable, with $\|\partial_{\xi}^j G_{\xi}^{\text{ord}}\| \leq p^{2A_1+1}$ for $j \leq 4$.

- (iii) We have $L_t = h_t \otimes \ell_t$, the vectors h_t, ℓ_t are nonnegative and vanish outside JO , and $0 \preceq L_t \preceq P_{JO} A_\xi^2 P_{JO}$.
- (iv) We have $G_\xi \succeq 0$, $G_\xi^* = J G_\xi J$ and $\|G_\xi\| \leq p^2$. The map $\xi \mapsto G_\xi$ is four times continuously differentiable, with $\|\partial_\xi^j G_\xi\| \leq p^{2A_1+1}$ for $j \leq 4$. Finally,

$$A_\xi^2 = G_\xi + h_c \otimes \ell_c + h_t \otimes \ell_t. \quad (14.39)$$

Proof. For (i) and (ii), by Lemma 14.14, we have $B_\xi^2 = R_\xi N_\xi S_\xi$ and $L_c = R_\xi N_\xi^0 S_\xi$. Then $G_\xi^{\text{ord}} = R_\xi \Delta_\xi S_\xi$. Since $R_\xi, S_\xi, \Delta_\xi, N_\xi^0$ are positive (Lemmas 14.14 and 14.15), we have $L_c \succeq 0$ and $G_\xi^{\text{ord}} \succeq 0$. By Lemma 14.13(iii) and Lemma 14.4(iii), $B_\xi^2 \preceq (P_O A_\xi P_O)^2$, and $(P_O A_\xi P_O)^2 = P_O A_\xi P_O A_\xi P_O \preceq P_O A_\xi^2 P_O$ since $P_O \preceq 1$. The range of R_ξ consists of functions vanishing outside O , and $S_\xi F$ depends only on $F \mathbf{1}_O$ since $\Psi_\xi^{\text{in}} \leq \mathbf{1}_O$; this gives $G_\xi^{\text{ord}} = P_O G_\xi^{\text{ord}} P_O$. The norm bound follows from Lemma 14.14 and Lemma 14.15(iv): $\|R_\xi \Delta_\xi S_\xi\| \leq C \cdot C p^{-2} \log^D p \cdot C p$. For the derivatives, we do not differentiate R_ξ and S_ξ separately; instead, $G_\xi^{\text{ord}} = B_\xi^2 - L_c$, and Lemma 14.13(iii) and Lemma 14.16(ii) give $\|\partial_\xi^j G_\xi^{\text{ord}}\| \leq C p^{2A_1} \log^{16} p + p^{A_1+1} \leq p^{2A_1+1}$.

For (iii), we have $L_c^* = \ell_c \otimes h_c$ and $L_t F = \langle h_c, JF \rangle J \ell_c = \langle J h_c, F \rangle J \ell_c$, which gives $L_t = h_t \otimes \ell_t$. The support claim follows from Lemma 14.16(i) and $J P_O J = P_{JO}$. Since $J \succeq 0$, conjugation by J and taking adjoints preserve the order $\preceq$ (Lemma 14.4(ii),(iii)). Applying them to (i) and using $J P_O J = P_{JO}$ and $J(A_\xi^*)^2 J = (J A_\xi^* J)^2 = A_\xi^2$ (Lemma 14.10(iii)), we find that $L_t \preceq J P_O (A_\xi^*)^2 P_O J = P_{JO} A_\xi^2 P_{JO}$.

For (iv), (14.39) is the definition (14.38). Since the sets O and JO are disjoint up to a null set, Lemma 14.4(v), applied to the partition (14.27), implies that $P_O A_\xi^2 P_O + P_{JO} A_\xi^2 P_{JO} \preceq A_\xi^2$. With (i) and (iii), we find that

$$G_\xi = (P_O A_\xi^2 P_O - L_c) + (P_{JO} A_\xi^2 P_{JO} - L_t) + (A_\xi^2 - P_O A_\xi^2 P_O - P_{JO} A_\xi^2 P_{JO}) \succeq 0.$$

Since $J A_\xi^2 J = (A_\xi^*)^2$, $J L_c J = L_t^*$ and $J L_t J = L_c^*$, we have $J G_\xi J = G_\xi^*$. Since $0 \preceq G_\xi \preceq A_\xi^2$, Lemma 14.4(i) gives $\|G_\xi\| \leq \|A_\xi\|^2 \leq p^2$. The derivative bound follows from Lemma 14.10(iv), Lemma 14.16(ii) and $\|\partial_\xi^j L_t\| = \|\partial_\xi^j L_c\|$: we have $\|\partial_\xi^j G_\xi\| \leq 16 p^{2A_1} + 2 p^{A_1+1} \leq p^{2A_1+1}$. $\square$

14.7 Cutoff bounds and a polynomial inverse

We continue to assume $p \geq p_1$ and $\xi \in \Xi$. The next lemma shows that, apart from transitions of size p^{-N_*} or $e^{-a\sqrt{p}/2}$, the core region Ω_c feeds only into the ordinary set O , and the retained operator captures $P_O A_\xi P_O$. In the notation $P_X A_\xi P_Y$, the set X constrains the output state (x, β) and Y constrains the input state $(y - \beta/x, \beta')$ in (14.22).

Lemma 14.18. For $p \geq p_1$ and $\xi \in \Xi$,

$$\|P_{\Omega_c} A_\xi P_{U_c}\| \leq p^{-N_*-2}, \quad (14.40)$$

$$\|P_{\Omega_c} A_\xi P_{\Omega_t}\| \leq e^{-a\sqrt{p}}/2, \quad (14.41)$$

$$\|P_O A_\xi P_O - B_\xi\| \leq p^{-N_*-3}, \quad \|(P_O A_\xi P_O)^2 - B_\xi^2\| \leq p^{-N_*-2}. \quad (14.42)$$

Proof. We begin with a general bound. Let $\mathcal{C} = \{y : |cy - 1| \leq p^{-1/4}\}$; by (14.17), we have $\nu_\xi(\mathbb{R} \setminus \mathcal{C}) \leq e^{-a\sqrt{p}}$. For thresholds $b_1, b_2 > 0$, let $P_{\{\beta > b\}}$ denote multiplication by $\mathbf{1}_{\{\beta > b\}}$, and let $A_\xi^{\text{exc}} = pJ I_\xi \mathcal{C}_\sigma I_\xi^*$ with σ the restriction of ν_ξ to $\mathbb{R} \setminus \mathcal{C}$. Since J commutes with $P_{\{\beta > b\}}$ and $\|P_{\{\beta > b\}} I_\xi\|^2 = \int_{\{\beta > b\}} \omega_\xi^2 d\mu_0 = e^{-b/\tau}$, we have

$$\|P_{\{\beta > b_1\}} A_\xi\| \leq p e^{-b_1/(2\tau)}, \quad \|A_\xi P_{\{\beta > b_2\}}\| \leq p e^{-b_2/(2\tau)}, \quad \|A_\xi^{\text{exc}}\| \leq p e^{-a\sqrt{p}}. \quad (14.43)$$

Let X, Y be measurable sets, and suppose that $(y - \beta/x, \beta') \notin Y$ whenever $(x, \beta) \in X$, $\beta \leq b_1$, $y \in \mathcal{C}$ and $\beta' \leq b_2$. Then

$$\mathbf{1}_X(x, \beta) \mathbf{1}_Y(y - \beta/x, \beta') \leq \mathbf{1}_{\{\beta > b_1\}} + \mathbf{1}_{\{\beta' > b_2\}} + \mathbf{1}_{\{y \notin \mathcal{C}\}},$$

and (14.22) gives $P_X A_\xi P_Y \preceq P_{\{\beta > b_1\}} A_\xi + A_\xi P_{\{\beta > b_2\}} + A_\xi^{\text{exc}}$. By Lemma 14.4(i) and (14.43),

$$\|P_X A_\xi P_Y\| \leq p(e^{-b_1/(2\tau)} + e^{-b_2/(2\tau)} + e^{-a\sqrt{p}}). \quad (14.44)$$

For (14.40), take $X = \Omega_c$, $Y = U_c$, $b_1 = r^2/16$ and $b_2 = \beta_{\max}$. If $(x, \beta) \in \Omega_c$ and $\beta \leq b_1$, then $|\beta/x| < \sqrt{\beta} \leq r/4$. If $y \in \mathcal{C}$, then $|y - 1| \leq p^{-1/4} + \theta \leq r/4$. Then $|y - \beta/x - 1| \leq r/2$, and if $\beta' \leq \beta_{\max}$, the point $(y - \beta/x, \beta')$ lies in O , which is disjoint from U_c . Since $1/\tau = p^2 \xi^2$, we have $b_1/(2\tau) \geq p^2 \xi^2 / (32 \log^2 p)$ and $b_2/(2\tau) \geq 2W^2 = 2(M_* + 1) \log p$. Inserting these bounds into (14.44), we obtain $\|P_{\Omega_c} A_\xi P_{U_c}\| \leq 2p^{-2M_*-1} \leq p^{-N_*-2}$, for $p \geq p_1$.

For (14.41), take $X = \Omega_c$, $Y = \Omega_t$ and $b_1 = b_2 = 1/64$. If $(x, \beta) \in \Omega_c$ and $\beta \leq 1/64$, then $|\beta/x| < 1/8$; if $y \in \mathcal{C}$, then $y \geq 3/4$; so $y - \beta/x \geq 5/8 > \sqrt{\beta'}$ for $\beta' \leq 1/64$, and $(y - \beta/x, \beta') \in \Omega_c$. Since $1/(128\tau) = p^2 \xi^2 / 128$, the bound (14.41) follows from (14.44) for $p \geq p_1$.

For (14.42), write $D_1 = P_O A_\xi P_O - B_\xi = (P_O - \Psi_\xi^{\text{out}}) A_\xi P_O + \Psi_\xi^{\text{out}} A_\xi (P_O - \Psi_\xi^{\text{in}})$, a sum of positive operators. Since $x \geq 1/2$ and $c \geq 1$ on O , we have $1 - \psi(w_\xi(x, \beta)) \leq \mathbf{1}_{\{\beta > \tau M_* \log p/2\}}$. Hence $(P_O - \Psi_\xi^{\text{out}}) A_\xi P_O \preceq P_{\{\beta > \tau M_* \log p/2\}} A_\xi$, which has norm at most $p^{1-M_*/4}$ by (14.43). For the second term, $\mathbf{1}_O(1 - \chi(cx')\psi(w')) \leq \mathbf{1}_O(1 - \chi(cx')) + (1 - \psi(w'))\mathbf{1}_O$ at the input state (x', β') , and the second summand is at most $\mathbf{1}_{\{\beta' > \tau M_* \log p/2\}}$. If the output state (x, β) lies in O and $y \in \mathcal{C}$, then $c\beta/x \leq 4\beta_{\max}$ and $|cx' - 1| \leq |cy - 1| + c\beta/x \leq p^{-1/4} + 4\beta_{\max} \leq r/8$, and therefore $\chi(cx') = 1$. This shows that $\Psi_\xi^{\text{out}} A_\xi (P_O - \Psi_\xi^{\text{in}}) \preceq A_\xi P_{\{\beta > \tau M_* \log p/2\}} + A_\xi^{\text{exc}}$. Combining these bounds with (14.43) and Lemma 14.4(i), we obtain $\|D_1\| \leq 2p^{1-M_*/4} + p e^{-a\sqrt{p}} = 2p^{-N_*-4} + p e^{-a\sqrt{p}} \leq p^{-N_*-3}/2$

for $p \geq p_1$, since $M_*/4 = N_* + 5$. Finally, $(P_0 A_\xi P_0)^2 - B_\xi^2 = D_1 P_0 A_\xi P_0 + B_\xi D_1$, and $\|B_\xi\| \leq \|A_\xi\| \leq p$, so its norm is at most $2p\|D_1\| \leq p^{-N_*-2}$. $\square$

We now bound the blocks of G_ξ with respect to the partition (14.27). We write $G_{ij} = P_{X_i} G_\xi P_{X_j}$ for $i, j \in \{1, 2, 3, 4\}$, and set

$$E_0 = G_{44} - G_\xi^{\text{ord}}, \quad d_p = \|G_{44}\|.$$

Lemma 14.19. *For $p \geq p_1$ and $\xi \in \Xi$, the following hold.*

- (i) $\|G_{ij}\| \leq p^2$ for all i, j .
- (ii) $\|G_{43}\| + \|G_{33}\| + \|G_{22}\| + \|G_{21}\| \leq 8p^{-N_*-1}$.
- (iii) $\|G_{ij}\| \leq 2pe^{-a\sqrt{p}/2}$ for $i \in \{3, 4\}$ and $j \in \{1, 2\}$.
- (iv) $E_0 \succeq 0$, $\|E_0\| \leq 3p^{-N_*-1}$, $d_p \leq Cp^{-1} \log^D p$, and $G_{11} = JG_{44}^*J$, so that $\|G_{11}\| = d_p$.

Proof. Part (i) follows from $\|G_\xi\| \leq p^2$. The rank-one operators L_c and L_t have nonzero blocks only at $(4, 4)$ and $(1, 1)$, respectively. Then $G_{ij} = P_{X_i} A_\xi^2 P_{X_j}$ for $(i, j) \notin \{(1, 1), (4, 4)\}$. For any sets X, Y , we have $P_X A_\xi^2 P_Y = P_X A_\xi P_{\Omega_c} A_\xi P_Y + P_X A_\xi P_{\Omega_t} A_\xi P_Y$.

For (ii), let $X \subset \Omega_c$. By (14.40) and (14.41), we have $\|P_X A_\xi^2 P_{U_c}\| \leq \|A_\xi\| \|P_{\Omega_c} A_\xi P_{U_c}\| + \|P_{\Omega_c} A_\xi P_{\Omega_t}\| \|A_\xi\| \leq 2p^{-N_*-1}$. This bounds G_{43} and G_{33} . For the other two blocks, we use the symmetry $G_\xi^* = JG_\xi J$: for any sets X, Y ,

$$P_{JX} G_\xi P_{JY} = JP_X JG_\xi JP_Y J = JP_X G_\xi^* P_Y J = J(P_Y G_\xi P_X)^* J. \quad (14.45)$$

Since $X_2 = JX_3$ and $X_1 = JX_4$, this gives $\|G_{22}\| = \|G_{33}\|$ and $\|G_{21}\| = \|G_{43}\|$.

For (iii), let $X \subset \Omega_c$ and $Y \subset \Omega_t$. In the decomposition of $P_X A_\xi^2 P_Y$ above, the first term contains $P_{\Omega_c} A_\xi P_Y$ and the second contains $P_X A_\xi P_{\Omega_t}$. With (14.41) and $\|A_\xi\| \leq p$, we obtain $\|P_X A_\xi^2 P_Y\| \leq 2pe^{-a\sqrt{p}/2}$.

For (iv), we have

$$G_{44} = P_0 A_\xi^2 P_0 - L_c = (P_0 A_\xi P_0)^2 + P_0 A_\xi P_{U_c} A_\xi P_0 + P_0 A_\xi P_{\Omega_t} A_\xi P_0 - L_c,$$

so $E_0 = [(P_0 A_\xi P_0)^2 - B_\xi^2] + P_0 A_\xi P_{U_c} A_\xi P_0 + P_0 A_\xi P_{\Omega_t} A_\xi P_0$. Each term is positive (the first by Lemma 14.17(i)), and Lemma 14.18 bounds the norms by p^{-N_*-2} , $p \cdot p^{-N_*-2}$ and $pe^{-a\sqrt{p}/2}$, respectively. With Lemma 14.17(ii), this gives $d_p \leq \|G_\xi^{\text{ord}}\| + \|E_0\| \leq Cp^{-1} \log^D p$. The identity $G_{11} = JG_{44}^*J$ is (14.45) with $X = Y = O$. $\square$

Lemma 14.20. *For $p \geq p_1$ and $\xi \in \Xi$, we have $\|G_\xi^{n_0}\| \leq 2^{-2n_0}$; in particular, $r(G_\xi) \leq 1/4$. For $\lambda \in [1/2, 2]$, the operator $(\lambda - G_\xi)^{-1}$ is positive,*

$$\|(\lambda - G_\xi)^{-1}\| \leq p^{2n_0}, \quad \|\partial_\xi^j (\lambda - G_\xi)^{-1}\| \leq p^{A_2} \quad (j \leq 4), \quad (14.46)$$

and $\|(\lambda - G')^{-1}\| \leq p^{2n_0}$ for every operator G' with $0 \preceq G' \preceq G_\xi$.

Proof. Order the blocks as X_1, X_2, X_3, X_4 (that is, JO, U_t, U_c, O), and decompose $G_{\xi} = G^{\text{fw}} + G^{\text{dg}} + G^{\text{sm}}$, where

$$G^{\text{fw}} = \sum_{i < j} G_{ij}, \quad G^{\text{dg}} = G_{11} + G_{44}, \quad G^{\text{sm}} = \sum_{i > j} G_{ij} + G_{22} + G_{33}.$$

The blocks with $i > j$ are $(2, 1)$, $(4, 3)$ and the four blocks of Lemma 14.19(iii). By Lemma 14.19, we have

$$\|G^{\text{fw}}\| \leq 6p^2, \quad \|G^{\text{dg}}\| \leq d_p, \quad \|G^{\text{sm}}\| \leq 8p^{-N_*-1} + 8pe^{-a\sqrt{p}/2} \leq p^{-N_*}.$$

We expand $G_{\xi}^{n_0}$ as a sum of 3^{n_0} products of n_0 factors, each factor being G^{fw} , G^{dg} or G^{sm} . A product that contains at least one factor G^{sm} has norm at most $(7p^2)^{n_0-1}p^{-N_*}$. Consider a product of factors G^{fw} and G^{dg} only, and expand each factor into its blocks. A product of blocks $G_{i_1j_1}G_{i_2j_2}\cdots G_{i_{n_0}j_{n_0}}$ vanishes unless $j_k = i_{k+1}$ for each k . For the blocks of G^{fw} and G^{dg} , we have $i_k \leq j_k$, with strict inequality for the blocks of G^{fw} . Then the indices $i_1 \leq j_1 = i_2 \leq \dots$ are nondecreasing in $\{1, 2, 3, 4\}$ and increase strictly at each factor from G^{fw} . A nonvanishing product therefore contains at most three such factors. In other words, a product of factors G^{fw} and G^{dg} with at least four factors G^{fw} vanishes, and every other such product has norm at most $(6p^2)^3d_p^{n_0-3}$, using $d_p \leq 1$. Summing over the 3^{n_0} products, we obtain

$$\|G_{\xi}^{n_0}\| \leq 3^{n_0}(216p^6d_p^{n_0-3} + (7p^2)^{n_0-1}p^{-N_*}) \leq C(p^{6-9}\log^{9D}p + p^{2n_0-N_*}) \leq 2^{-2n_0},$$

for $p \geq p_1$, where we used $n_0 = 12$, Lemma 14.19(iv) and $N_* > 2n_0 + 10$. By the spectral radius formula, we have $r(G_{\xi}) \leq \|G_{\xi}^{n_0}\|^{1/n_0} \leq 1/4$.

Let $\lambda \in [1/2, 2]$. By Lemma 14.4(iv), we have $(\lambda - G_{\xi})^{-1} \succeq 0$. Since $\|G_{\xi}^{n_0}\| \leq \lambda^{n_0}/2$, the operator $\lambda^{n_0} - G_{\xi}^{n_0}$ is invertible with $\|(\lambda^{n_0} - G_{\xi}^{n_0})^{-1}\| \leq 2\lambda^{-n_0} \leq 2^{n_0+1}$, and the identity

$$(\lambda - G_{\xi})^{-1} = (\lambda^{n_0-1} + \lambda^{n_0-2}G_{\xi} + \cdots + G_{\xi}^{n_0-1})(\lambda^{n_0} - G_{\xi}^{n_0})^{-1}$$

and $\|G_{\xi}\| \leq p^2$ give $\|(\lambda - G_{\xi})^{-1}\| \leq n_0 2^{2n_0+1} p^{2n_0-2} \leq p^{2n_0}$. The map $\xi \mapsto (\lambda - G_{\xi})^{-1}$ is four times continuously differentiable, since $\xi \mapsto G_{\xi}$ is (Lemma 14.17(iv)). Its j th derivative is a sum of at most C products of at most $j+1$ factors $(\lambda - G_{\xi})^{-1}$ and at most j factors $\partial_{\xi}^i G_{\xi}$ with total order j ; for instance, $\partial_{\xi}(\lambda - G_{\xi})^{-1} = (\lambda - G_{\xi})^{-1}(\partial_{\xi} G_{\xi})(\lambda - G_{\xi})^{-1}$. With Lemma 14.17(iv), we find that $\|\partial_{\xi}^j(\lambda - G_{\xi})^{-1}\| \leq Cp^{2n_0(j+1)+j(2A_1+1)} \leq p^{A_2}$ for $j \leq 4$. Finally, if $0 \preceq G' \preceq G_{\xi}$, then $r(G') \leq r(G_{\xi})$ and $0 \preceq (\lambda - G')^{-1} \preceq (\lambda - G_{\xi})^{-1}$ by Lemma 14.4(i),(iv), which implies that $\|(\lambda - G')^{-1}\| \leq p^{2n_0}$. $\square$

14.8 The renewal matrix

For $\xi \in \Xi$ and $\lambda \in [1/2, 2]$, we define the renewal matrix

$$T_{ab}(\xi, \lambda) = \lambda \langle \ell_a, (\lambda - G_{\xi})^{-1} h_b \rangle, \quad a, b \in \{c, t\}, \quad (14.47)$$

with h_c, ℓ_c from Lemma 14.16 and h_t, ℓ_t from Lemma 14.17. This is the matrix (14.8) for the decomposition (14.39), with the index 1 for the core component c and 2 for the tail component t .

Lemma 14.21. *Let $p \geq p_1$, $\xi \in \Xi$ and $\lambda \in [1/2, 2]$.*

- (i) $T_{cc}(\xi, \lambda) = T_{tt}(\xi, \lambda)$.
- (ii) $T_{ct}(\xi, \lambda) > 0$ and $T_{tc}(\xi, \lambda) > 0$.
- (iii) *The entries of T are four times continuously differentiable in ξ , and $|\partial_\xi^j T_{ab}| \leq p^{A_j}$ for $j \leq 4$ and all a, b .*
- (iv) $T_{ct}(\xi, \lambda) \leq e^{-a\sqrt{p}/4}$.

Proof. For (i), we use $J\ell_c = h_t$, $Jh_c = \ell_t$ and $J(\lambda - G_\xi)^{-1}J = (\lambda - JG_\xi J)^{-1} = (\lambda - G_\xi^*)^{-1}$, which give

$$T_{tt} = \lambda \langle Jh_c, (\lambda - G_\xi)^{-1}J\ell_c \rangle = \lambda \langle h_c, (\lambda - G_\xi^*)^{-1}\ell_c \rangle = \lambda \langle (\lambda - G_\xi)^{-1}h_c, \ell_c \rangle = T_{cc}.$$

For (ii), we first claim that if $V \subset (\mathbb{R} \setminus \{0\}) \times (0, \infty)$ is a nonempty bounded open set, then $A_\xi \mathbf{1}_V$ has a representative that is lower semicontinuous and positive at every point (x, β) with $x \neq 0$; in particular, $A_\xi \mathbf{1}_V \geq c_1 \mathbf{1}_{V'}$ for some nonempty bounded open set V' and some $c_1 > 0$. Indeed, by (14.22),

$$(A_\xi \mathbf{1}_V)(x, \beta) = p \omega_\xi(\beta) \frac{\sqrt{\beta}}{|x|} \int v_\xi(dy) \int \mu_0(d\beta') \omega_\xi(\beta') \mathbf{1}_V\left(y - \frac{\beta}{x}, \beta'\right).$$

For fixed (x, β) , the set of (y, β') with $(y - \beta/x, \beta') \in V$ is open and nonempty. It has positive $v_\xi \otimes \mu_0$ measure, since v_ξ has full support by (I1) and μ_0 has a positive density. For fixed (y, β') , the function $(x, \beta) \mapsto \mathbf{1}_V(y - \beta/x, \beta')$ is lower semicontinuous on $\{x \neq 0\}$, and Fatou's lemma implies that the integral is lower semicontinuous, and the prefactor is continuous and positive. A positive lower semicontinuous function is bounded below by a positive constant on a neighborhood of each point, which proves the claim.

Now let V_0 and c_0 be as in Lemma 14.16(iii). Since $(x, \beta) \mapsto (\beta/x, \beta)$ is a homeomorphism of $(\mathbb{R} \setminus \{0\}) \times (0, \infty)$, the set JV_0 is open, and $J\ell_c \geq c_0 J\mathbf{1}_{V_0} \geq c_2 \mathbf{1}_{V_1}$ for a nonempty open set $V_1 \subset JV_0$ and some $c_2 > 0$. By the claim applied to V_1 , there exist a nonempty open set V' and $c_1 > 0$ with $A_\xi^2 \mathbf{1}_{V_1} \geq c_1 \mathbf{1}_{V'}$. Then $A_\xi^2 J\ell_c \geq c_2 c_1 A_\xi \mathbf{1}_{V'}$, and $A_\xi \mathbf{1}_{V'} > 0$ almost everywhere by the claim applied to V' . Since $\ell_c \geq c_0 \mathbf{1}_{V_0}$ and V_0 has positive measure, we obtain $\langle \ell_c, A_\xi^2 J\ell_c \rangle > 0$. Since ℓ_c vanishes outside O and $J\ell_c$ vanishes outside JO , we have $\langle \ell_c, J\ell_c \rangle = 0$, and so

$$\langle \ell_c, L_c J\ell_c \rangle = \langle \ell_c, J\ell_c \rangle \langle \ell_c, h_c \rangle = 0, \quad \langle \ell_c, L_t J\ell_c \rangle = \langle \ell_c, h_t \rangle \langle Jh_c, J\ell_c \rangle = \langle \ell_c, J\ell_c \rangle \langle h_c, \ell_c \rangle = 0.$$

Then $\langle \ell_c, G_\xi h_t \rangle = \langle \ell_c, A_\xi^2 J \ell_c \rangle > 0$. By (14.11), every term of the series for T_{ct} is nonnegative, and its term with $n = 1$ is $\lambda^{-1} \langle \ell_c, G_\xi h_t \rangle > 0$. The argument for T_{tc} is the same, with $\langle \ell_t, G_\xi h_c \rangle = \langle J h_c, A_\xi^2 h_c \rangle$ and the lower bound $h_c \geq c_0 \mathbf{1}_{V_0}$.

For (iii), the entries are compositions of the four times continuously differentiable maps $\xi \mapsto \ell_a, h_b, (\lambda - G_\xi)^{-1}$. By the Leibniz rule, Lemma 14.16(ii), $\|\partial_\xi^j h_t\| = \|\partial_\xi^j \ell_c\|$, $\|\partial_\xi^j \ell_t\| = \|\partial_\xi^j h_c\|$ and (14.46), we have $|\partial_\xi^j T_{ab}| \leq C p^{2A_1 + A_2} \leq p^{A_j}$.

For (iv), since ℓ_c vanishes outside Ω_c and h_t vanishes outside Ω_t , we have $T_{ct} = \lambda \langle \ell_c, P_{\Omega_c} (\lambda - G_\xi)^{-1} P_{\Omega_t} h_t \rangle$. Let $G' = G_\xi - P_{\Omega_c} G_\xi P_{\Omega_t}$. By Lemma 14.4(v), $0 \preceq G' \preceq G_\xi$. Further, $P_{\Omega_c} G' P_{\Omega_t} = 0$, which gives $P_{\Omega_c} G' = P_{\Omega_c} G' P_{\Omega_c}$; by induction, $P_{\Omega_c} (G')^n P_{\Omega_t} = 0$ for all $n \geq 0$; the Neumann series gives $P_{\Omega_c} (\lambda - G')^{-1} P_{\Omega_t} = 0$. By the resolvent identity, we then have

$$P_{\Omega_c} (\lambda - G_\xi)^{-1} P_{\Omega_t} = P_{\Omega_c} (\lambda - G')^{-1} (P_{\Omega_c} G_\xi P_{\Omega_t}) (\lambda - G_\xi)^{-1} P_{\Omega_t}.$$

By Lemma 14.19(iii), $\|P_{\Omega_c} G_\xi P_{\Omega_t}\| \leq 8pe^{-a\sqrt{p}/2}$, and Lemma 14.20 bounds both resolvents by p^{2n_0} . With $\|\ell_c\| = \|h_t\| \leq p^{A_1}$, we obtain $T_{ct} \leq 16p^{2A_1 + 4n_0 + 1} e^{-a\sqrt{p}/2} \leq e^{-a\sqrt{p}/4}$ for $p \geq p_1$. $\square$

The slope of the diagonal

Let $\Xi_1 \subset \Xi$ be a compact interval whose distance to $\mathbb{R} \setminus \Xi$ is at least $\delta_0 > 0$. In the following two lemmas, the threshold p_1 may also depend on δ_0 .

Lemma 14.22. *We have*

$$\lim_{p \rightarrow \infty} \sup_{\xi \in \Xi_1, \lambda \in [1/2, 2]} \left| \partial_\xi T_{cc}(\xi, \lambda) + \frac{2\mu_*^2}{\xi^3} \right| = 0.$$

Proof. We compare T_{cc} with the entry obtained by keeping only the ordinary part G_ξ^{ord} of the remainder,

$$a^{\text{ord}}(\xi, \lambda) = \lambda \langle \ell_c, (\lambda - G_\xi^{\text{ord}})^{-1} h_c \rangle,$$

which is well defined since $\|G_\xi^{\text{ord}}\| \leq 1/4$ for $p \geq p_1$. The slope of a^{ord} is computed from the factorization of Lemma 14.14, and the difference $T_{cc} - a^{\text{ord}}$ is small together with its first derivative.

STEP 1. *A series for a^{ord} .* We claim that

$$a^{\text{ord}}(\xi, \lambda) = \sum_{n \geq 0} \lambda^{-n} k_\xi (N_\xi (\Delta_\xi N_\xi)^n \mathbf{1}). \quad (14.48)$$

Indeed, $\lambda(\lambda - G_\xi^{\text{ord}})^{-1} = \sum_{n \geq 0} \lambda^{-n} (R_\xi \Delta_\xi S_\xi)^n$ in norm, and since $\langle \ell_c, F \rangle = k_\xi (S_\xi F)$ and $h_c = R_\xi \mathbf{1}$, we have

$$\langle \ell_c, (R_\xi \Delta_\xi S_\xi)^n h_c \rangle = k_\xi (S_\xi R_\xi (\Delta_\xi S_\xi R_\xi)^n \mathbf{1}) = k_\xi (N_\xi (\Delta_\xi N_\xi)^n \mathbf{1}).$$

The series (14.48) converges together with its first ξ -derivative, since $\|\Delta_\xi N_\xi\| \leq Cp^{-2} \log^D p$ by Lemma 14.15(iv). This representation involves only the operators N_ξ , Δ_ξ and the functional k_ξ on $\mathcal{V}$, whose first two derivatives are controlled by Lemma 14.15; it does not require differentiating R_ξ or S_ξ .

STEP 2. *The slope of a^{ord} .* We claim that $\partial_\xi a^{\text{ord}}(\xi, \lambda) = -2\mu_*^2 \xi^{-3} + o(1)$ uniformly in $\xi \in \Xi$ and $\lambda \in [1/2, 2]$. Since $N_\xi = N_\xi^0 + \Delta_\xi$ and $N_\xi^0 \mathbf{1} = \sigma_\xi \mathbf{1}$, the term with $n = 0$ in (14.48) is $k_\xi(N_\xi \mathbf{1}) = \sigma_\xi^2 + k_\xi(\Delta_\xi \mathbf{1})$. Every other term, and $k_\xi(\Delta_\xi \mathbf{1})$, contains at least one factor Δ_ξ . Differentiating such a product of $2n + 2$ factors by the Leibniz rule, and using Lemma 14.15(iv) and $\lambda^{-n} \leq 2^n$, we find that

$$|\partial_\xi(a^{\text{ord}} - \sigma_\xi^2)| \leq \sum_{n \geq 0} (2n + 2) 2^n C^{2n+2} (Cp^{-2} \log^D p)^{\max\{n, 1\}} \leq Cp^{-2} \log^D p.$$

It remains to differentiate σ_ξ^2 . By definition, $\sigma_\xi = \alpha \int \gamma(du) u \psi(u)^2 m_\xi(u)$. By (14.16), Lemma 14.15(i), and $1 - \psi^2 \leq \mathbf{1}_{\{u^2 > M_* \log p\}}$,

$$|\sigma_\xi - \alpha \mu_*| \leq \alpha \left(\mu_* \sup_{u \leq W} |m_\xi(u) - 1| + \int_{u^2 > M_* \log p} u \gamma(du) \right) \leq Cp^{-1/4} \log p.$$

Similarly, $\partial_\xi \sigma_\xi = (\partial_\xi \alpha) \int \gamma(du) u \psi^2 m_\xi + \alpha \int \gamma(du) u \psi^2 \partial_\xi Y_\xi^0$, and Lemma 14.15(iii) implies that $|\partial_\xi \sigma_\xi - \mu_* \partial_\xi \alpha| \leq C(p^{-1/4} \log p + \epsilon_p \log^4 p)$. By (14.15), $\alpha = \xi^{-1} + O(p^{-1})$ and $\partial_\xi \alpha = -\xi^{-2} + O(p^{-1})$. Then

$$\partial_\xi(\sigma_\xi^2) = 2\sigma_\xi \partial_\xi \sigma_\xi = 2\mu_*^2 \alpha \partial_\xi \alpha + o(1) = -\frac{2\mu_*^2}{\xi^3} + o(1),$$

since $\epsilon_p \log^4 p \rightarrow 0$. This proves the claim.

STEP 3. *The size of $T_{\text{cc}} - a^{\text{ord}}$.* We claim that $|T_{\text{cc}} - a^{\text{ord}}| \leq p^{A_v - N_*}$ on $\Xi \times [1/2, 2]$. Since ℓ_c and h_c vanish outside O , we have $T_{\text{cc}} = \lambda \langle \ell_c, P_O(\lambda - G_\xi)^{-1} P_O h_c \rangle$. Let $G_1 = G_\xi - P_O G_\xi P_{O^c}$, where O^c is the complement of O . As in the proof of Lemma 14.21(iv), $0 \preceq G_1 \preceq G_\xi$ and $P_O G_1 = G_{44} P_O$. By the Neumann series, $P_O(\lambda - G_1)^{-1} P_O = (\lambda - G_{44})^{-1} P_O$, where $(\lambda - G_{44})^{-1}$ exists since $\|G_{44}\| = d_p \leq 1/4$. By the resolvent identity, we have

$$P_O[(\lambda - G_\xi)^{-1} - (\lambda - G_1)^{-1}] P_O = P_O(\lambda - G_1)^{-1} (P_O G_\xi P_{O^c}) (\lambda - G_\xi)^{-1} P_O.$$

By Lemma 14.19(ii),(iii), $\|P_O G_\xi P_{O^c}\| \leq \|G_{43}\| + \|G_{42}\| + \|G_{41}\| \leq 9p^{-N_*-1}$, and Lemma 14.20 bounds the two resolvents by p^{2n_0} . Further, since $G_{44} = G_\xi^{\text{ord}} + E_O$, the difference

$$(\lambda - G_{44})^{-1} - (\lambda - G_\xi^{\text{ord}})^{-1} = (\lambda - G_{44})^{-1} E_O (\lambda - G_\xi^{\text{ord}})^{-1}$$

has norm at most $16\|E_O\| \leq 48p^{-N_*-1}$, by Lemma 14.19(iv) and since both resolvents have norm at most 4. Combining these bounds with $\|\ell_c\| \|h_c\| \leq Cp^{A_1}$, we conclude that $|T_{\text{cc}} - a^{\text{ord}}| \leq Cp^{A_1} (9p^{4n_0 - N_* - 1} + 48p^{-N_* - 1}) \leq p^{A_v - N_*}$.

STEP 4. Interpolation. Fix λ and let $f(\xi) = T_{cc}(\xi, \lambda) - a^{\text{ord}}(\xi, \lambda)$. By Step 3, $\sup_{\Xi} |f| \leq p^{A_v - N_*}$. By Lemma 14.21(iii), $|\partial_{\xi}^4 T_{cc}| \leq p^{A_j}$. The same bound holds for a^{ord} : by Lemma 14.17(ii), the derivatives of $(\lambda - G_{\xi}^{\text{ord}})^{-1}$ of order $j \leq 4$ are bounded by $Cp^{4(2A_1+1)}$, and with Lemma 14.16(ii), we have $|\partial_{\xi}^4 a^{\text{ord}}| \leq Cp^{10A_1+4} \leq p^{A_j}$. Applying Lemma 14.6 with $r_0 = \delta_0$, $\epsilon = p^{A_v - N_*}$ and $m_4 = 2p^{A_j}$, we find that for $\xi \in \Xi_1$,

$$|f'(\xi)| \leq 4p^{(3A_v + A_j - 3N_*)/4} + \frac{4p^{A_v - N_*}}{\delta_0} \rightarrow 0,$$

since $N_* > A_v + A_j$. Combined with Step 2, this proves the lemma. $\square$

Lemma 14.23. For $p \geq p_1$ and $\lambda \in [1/2, 2]$, write $b = T_{ct}(\cdot, \lambda)$ and $d = T_{tc}(\cdot, \lambda)$, and take suprema over Ξ_1 . Then

$$\|b\|_{\infty} \leq e^{-a\sqrt{p}/4}, \quad \|\partial_{\xi}^2 b\|_{\infty} \leq e^{-a\sqrt{p}/16}, \quad \|d\|_{\infty} + \|\partial_{\xi}^2 d\|_{\infty} \leq 2p^{A_j}.$$

In particular,

$$\sup_{\lambda \in [1/2, 2]} \left(\sqrt{\|d\|_{\infty} \|\partial_{\xi}^2 b\|_{\infty}} + \sqrt{\|b\|_{\infty} \|\partial_{\xi}^2 d\|_{\infty}} + \frac{2\sqrt{\|b\|_{\infty} \|d\|_{\infty}}}{\delta_0} \right) \rightarrow 0 \quad \text{as } p \rightarrow \infty.$$

Proof. The bounds on b , d and $\partial_{\xi}^2 d$ follow from Lemma 14.21(iii),(iv). For $\partial_{\xi}^2 b$, we apply Lemma 14.6 on Ξ with $r_0 = \delta_0$, $\epsilon = e^{-a\sqrt{p}/4}$ and $m_4 = p^{A_j}$, which gives $|\partial_{\xi}^2 b| \leq 2p^{A_j/2} e^{-a\sqrt{p}/8} + 8\delta_0^{-2} e^{-a\sqrt{p}/4} \leq e^{-a\sqrt{p}/16}$ on Ξ_1 for $p \geq p_1$. The last claim follows by inserting these bounds. $\square$

The bound on $\partial_{\xi}^2 b$ is an absolute bound; it is not a bound relative to the value of b , which may be much smaller than $e^{-a\sqrt{p}}$.

14.9 Proofs of the main results

Proof of Theorem 14.1. Let Ξ_0 be given, and fix $\delta_0 \in (0, \mu_*/8]$ such that $[\mu_* - 4\delta_0, \mu_* + 4\delta_0] \subset \Xi_0$. Set

$$\Xi = [\mu_* - 3\delta_0, \mu_* + 3\delta_0], \quad \Xi_1 = [\mu_* - 2\delta_0, \mu_* + 2\delta_0], \quad \Xi_2 = [\mu_* - \delta_0, \mu_* + \delta_0].$$

We apply the constructions of Sections 14.4–14.8 on the window Ξ , with the parameters chosen in the order of Section 14.4, and we take $p \geq p_1$, with p_1 also depending on δ_0 and enlarged finitely many times below. We also take p_1 at least the threshold $p_0(\min \Xi_0)$ of Corollary 11.23, so that λ_p is defined on Ξ_0 ; by the uniqueness in Theorem 11.21(i), it agrees on Ξ with the function used in the constructions.

STEP 1. Monotonicity on Ξ_2 . We apply Proposition 14.8 with $\mathcal{H} = \mathcal{L}$, $I = \text{int } \Xi_2$, $I_1 = \text{int } \Xi_1$, $r_0 = \delta_0$, $\lambda_- = 1/2$, $\lambda_+ = 2$, the operators G_{ξ} of (14.38), and $h_{1,\xi} = h_c$, $h_{2,\xi} = h_t$, $\ell_{1,\xi} = \ell_c$, $\ell_{2,\xi} = \ell_t$. By (14.39), $U_{\xi} = A_{\xi}^2$. We verify the hypotheses.

- The maps $\xi \mapsto G_\xi, h_c, \ell_c$ are four times continuously differentiable by Lemmas 14.16 and 14.17, and so are $h_t = J\ell_c$ and $\ell_t = Jh_c$ since J does not depend on ξ . By Lemma 14.20, $r(G_\xi) \leq 1/4 < \lambda_-$.
- Hypothesis (i) holds by Lemma 14.21(i),(ii).
- Let $\kappa = \mu_*^2(\mu_* + 2\delta_0)^{-3}$. By Lemma 14.22, $\partial_\xi T_{cc} \leq -2\mu_*^2\xi^{-3} + \kappa \leq -\kappa$ on $\Xi_1 \times [1/2, 2]$ for $p \geq p_1$, and by Lemma 14.23, (14.9) holds for $p \geq p_1$. So hypothesis (ii) holds.
- By Theorem 13.1, for $p \geq p_1$ we have $\sup_{\Xi_0} |\lambda_p(\xi) - \mu_*/\xi| \leq \delta_0/(4\mu_*)$. For $\xi \in \Xi_2$, we have $\mu_*/\xi \in [8/9, 8/7]$ since $\delta_0 \leq \mu_*/8$, and so $\lambda_p(\xi)^2 \in (1/2, 2)$. By Lemma 14.11(ii), $F_\xi > 0$ satisfies $A_\xi^2 F_\xi = \lambda_p(\xi)^2 F_\xi$. So hypothesis (iii) holds with $\lambda(\xi) = \lambda_p(\xi)^2$.

Proposition 14.8(c) shows that λ_p^2 is continuously differentiable on $\text{int } \Xi_2$ with negative derivative. Since $\lambda_p > 0$, the same holds for λ_p , with $\partial_\xi \lambda_p = \partial_\xi(\lambda_p^2)/(2\lambda_p) < 0$.

STEP 2. Signs on Ξ_0 . For $\xi \leq \mu_* - \delta_0/2$, we have $\mu_*/\xi \geq 1 + \delta_0/(2\mu_*)$, and for $\xi \geq \mu_* + \delta_0/2$, we have $\mu_*/\xi \leq 1 - \delta_0/(2\mu_* + \delta_0) \leq 1 - \delta_0/(3\mu_*)$, since $\delta_0 \leq \mu_*$. With the bound from Theorem 13.1 in Step 1, we obtain $\lambda_p > 1$ on $\Xi_0 \cap (-\infty, \mu_* - \delta_0/2]$ and $\lambda_p < 1$ on $\Xi_0 \cap [\mu_* + \delta_0/2, \infty)$. Since λ_p is strictly decreasing on $\text{int } \Xi_2 \supset [\mu_* - \delta_0/2, \mu_* + \delta_0/2]$ by Step 1, and $\lambda_p(\mu_* - \delta_0/2) > 1 > \lambda_p(\mu_* + \delta_0/2)$, it takes the value one at exactly one point $\xi_c(p) \in (\mu_* - \delta_0/2, \mu_* + \delta_0/2)$, and $\lambda_p - 1$ is positive to its left and negative to its right within $\text{int } \Xi_2$. Combined with the signs outside $[\mu_* - \delta_0/2, \mu_* + \delta_0/2]$, this shows that $\xi_c(p)$ is the only solution of $\lambda_p(\xi) = 1$ in Ξ_0 , with $\lambda_p > 1$ to its left and $\lambda_p < 1$ to its right in Ξ_0 . By Step 1, λ_p is continuously differentiable near $\xi_c(p)$ and $\partial_\xi \lambda_p(\xi_c(p)) < 0$.

STEP 3. Convergence. Let $\delta \in (0, \delta_0/2]$. By Theorem 13.1, there exists $p(\delta)$ such that $\sup_{\Xi_0} |\lambda_p(\xi) - \mu_*/\xi| \leq \delta/(4\mu_*)$ for $p \geq p(\delta)$; then $\lambda_p > 1$ on $\Xi_0 \cap (-\infty, \mu_* - \delta]$ and $\lambda_p < 1$ on $\Xi_0 \cap [\mu_* + \delta, \infty)$, by the argument of Step 2 with $\delta_0/2$ replaced by δ . Then $|\xi_c(p) - \mu_*| < \delta$ for $p \geq \max\{p_1, p(\delta)\}$, which proves $\xi_c(p) \rightarrow \mu_*$. $\square$

Proof of Corollary 14.2. Let $M_{\text{ext}} \geq 3$ and p_{ext} be as in Proposition 13.2, and let $\Xi_0 = [\xi_0, M_{\text{ext}} + 1]$; since $\mu_* < 1 < M_{\text{ext}}$, the point μ_* lies in the interior of Ξ_0 . Let $p \geq \max\{p(\Xi_0), p_{\text{ext}}, p_0(\xi_0)\}$, where $p(\Xi_0)$ is the threshold of Theorem 14.1 and $p_0(\xi_0)$ that of Corollary 11.23. By Corollary 11.23(i), for every $E \geq \xi_0\sqrt{p}$ the principal solution $v_{p,E}$ is the unique fixed point of the law map in the class $\mathfrak{S}_{p,\theta(E)}$. For $E \in [\xi_0\sqrt{p}, (M_{\text{ext}} + 1)\sqrt{p}]$, it therefore coincides with the fixed point of Theorem 11.21 for the window Ξ_0 , and $\varphi_E(1) = \log \lambda_p(E/\sqrt{p})$. By Theorem 14.1, the function $E \mapsto \varphi_E(1)$ has exactly one zero $E_c(p) = \xi_c(p)\sqrt{p}$ in $[\xi_0\sqrt{p}, (M_{\text{ext}} + 1)\sqrt{p}]$; it is positive on $[\xi_0\sqrt{p}, E_c(p))$ and negative on $(E_c(p), (M_{\text{ext}} + 1)\sqrt{p}]$. By Proposition 13.2, we have $\varphi_E(1) \leq -\log 2 < 0$ for every $E \geq M_{\text{ext}}\sqrt{p}$. Hence $E_c(p)$ is the only zero in $[\xi_0\sqrt{p}, \infty)$, with the stated signs. Near $E_c(p)$, we have $\partial_E \varphi_E(1) = p^{-1/2} \partial_\xi \lambda_p(\xi)/\lambda_p(\xi)$ at $\xi = E/\sqrt{p}$, which is continuous and negative at $E_c(p)$ by Theorem 14.1. Finally, $E_c(p)/\sqrt{p} = \xi_c(p) \rightarrow \mu_*$. $\square$

The free energy at small energies

In this section, we show that at large p , the free energy $\varphi_\pi(1)$ of every real solution π is bounded below by a positive numerical constant at all energies $0 < E \leq \xi_{\text{in}}\sqrt{p}$, where $\xi_{\text{in}} > 0$ is numerical. The bound holds for all real solutions, not only for the principal solution of Section 11. In Section 16, it is combined with Theorem 10.1 to show that the boundary values at these energies are not real.

The proof uses the elementary lower bound $\lambda(1) \geq p \mathbb{E}\|M_g\|^{-1}$ on the critical eigenvalue, valid for every scaling g , and two choices of scaling. With the scaling $g = 1$, the mean $p \mathbb{E}\|M_1\|^{-1}$ is large if $F = \mathbb{E}(1 + |D|)^{-1}$ is at least a large multiple of $p^{-1/2}$. With the scaling $g = \sqrt{p}$, the mean $p \mathbb{E}\|M_{\sqrt{p}}\|^{-1}$ is large if $q = \mathbb{P}(|D| \leq p^{-1/2})$ is at least a large multiple of p^{-1} . If both tests fail, then with probability bounded below no child denominator is small, and the cavity equation shows that $|D|$ is at most E plus a bounded quantity; this gives a lower bound on F of order $(1 + E)^{-1}$, which contradicts the failure of the first test when $E \leq \xi_{\text{in}}\sqrt{p}$ and ξ_{in} is small.

15.1 The critical transfer operator with two scalings

Throughout this section, π is a real solution at energy $E > 0$ (Definition 5.1), $D \sim \pi$, and $Y \sim \text{Exp}(1)$ is independent of D . For a scaling $g > 0$, we set

$$M_g = M\left(gD, \frac{g^2Y}{p}\right) = \left(\frac{g^2Y}{p}\right)^{-1/2} \begin{pmatrix} 0 & 1 \\ -g^2Y/p & gD \end{pmatrix},$$

as in Definition 5.2 and (6.2). The variable $\beta = g^2Y/p$ has the exponential law of rate $K_g = p/g^2$, and by Proposition 5.5 the law ν_g of gD satisfies the hypotheses (6.1) of Section 6. We write $\mathcal{Q}_1^{(g)}$ for the transfer operator of Definition 6.1 at $s = 1$, defined with ν_g and K_g :

$$(\mathcal{Q}_1^{(g)}\chi)(x) = p \mathbb{E}[\|M_g v_x\|^{-1} \chi(M_g \cdot x)], \quad \chi \in C(\mathbb{RP}^1), x \in \mathbb{RP}^1.$$

By Theorem 6.11(i), this operator is bounded and positive on $C(\mathbb{RP}^1)$, and we write $\lambda^{(g)}(1)$ for its spectral radius. By Definition 6.14(iii), the free energy of π at $s = 1$ is $\varphi_\pi(1) = \log \lambda^{(g)}(1)$.

We use the two scalings $g = 1$ and $g = \sqrt{p}$, and we recall why they give the same critical eigenvalue. Let $S = \text{diag}(p^{-1/4}, p^{1/4})$. Conjugation by S multiplies the (1,2)

entry of a matrix by $p^{-1/2}$ and the $(2, 1)$ entry by $p^{1/2}$, and leaves the diagonal entries unchanged. Since $M_1 = (Y/p)^{-1/2} \begin{pmatrix} 0 & 1 \\ -Y/p & D \end{pmatrix}$, we have

$$SM_1S^{-1} = \left(\frac{Y}{p}\right)^{-1/2} \begin{pmatrix} 0 & p^{-1/2} \\ -Y/\sqrt{p} & D \end{pmatrix} = Y^{-1/2} \begin{pmatrix} 0 & 1 \\ -Y & \sqrt{p}D \end{pmatrix} = M_{\sqrt{p}},$$

which is (6.17) with $\gamma = \sqrt{p}$. This identity between the random matrices of the two scalings, which holds for every realization of (D, Y) , induces the similarity $\mathcal{Q}_1^{(\sqrt{p})} = \Theta^{-1}\mathcal{Q}_1^{(1)}\Theta$ of Lemma 6.13(ii), where $(\Theta\chi)(x) = \|Sv_x\|^{-1}\chi(S \cdot x)$ is a bounded operator on $C(\mathbb{RP}^1)$ with bounded inverse. Similar operators have the same spectrum, which gives

$$\lambda^{(1)}(1) = \lambda^{(\sqrt{p})}(1) = e^{\varphi_\pi(1)}. \quad (15.1)$$

Lemma 15.1. *For every $g > 0$, we have $\lambda^{(g)}(1) \geq p \mathbb{E}\|M_g\|^{-1}$.*

Proof. Set $m = p \mathbb{E}\|M_g\|^{-1}$ and let $\mathbf{1}$ be the constant function one on $\mathbb{RP}^1$. For every $x \in \mathbb{RP}^1$, the vector v_x is a unit vector. Then $\|M_g v_x\| \leq \|M_g\|$, and

$$(\mathcal{Q}_1^{(g)}\mathbf{1})(x) = p \mathbb{E}\|M_g v_x\|^{-1} \geq m.$$

The operator $\mathcal{Q}_1^{(g)}$ is positive: if $\chi \geq \psi$ pointwise, then $\mathcal{Q}_1^{(g)}\chi \geq \mathcal{Q}_1^{(g)}\psi$ pointwise. By induction on n , we have $(\mathcal{Q}_1^{(g)})^n \mathbf{1} \geq m^n \mathbf{1}$ for every $n \geq 1$; indeed, if this holds for $n-1$, then

$$(\mathcal{Q}_1^{(g)})^n \mathbf{1} = \mathcal{Q}_1^{(g)}((\mathcal{Q}_1^{(g)})^{n-1} \mathbf{1}) \geq m^{n-1} \mathcal{Q}_1^{(g)} \mathbf{1} \geq m^n \mathbf{1}.$$

Since $\|\mathbf{1}\|_\infty = 1$, the operator norm on $C(\mathbb{RP}^1)$ satisfies $\|(\mathcal{Q}_1^{(g)})^n\| \geq \|(\mathcal{Q}_1^{(g)})^n \mathbf{1}\|_\infty \geq m^n$. By Gelfand's formula for the spectral radius [113, Theorem VI.6], this implies that $\lambda^{(g)}(1) = \lim_{n \rightarrow \infty} \|(\mathcal{Q}_1^{(g)})^n\|^{1/n} \geq m$. $\square$

Since M_g has determinant one, we have $\|M_g\| \geq 1$, and the bound of Lemma 15.1 is at most p . This is consistent with the bound $\varphi_\pi(1) < \log p$ of Proposition 6.15.

15.2 Proof of the lower bound

Let $r_0 = \mathbb{P}(1 \leq Y \leq 2) = e^{-1} - e^{-2}$ for $Y \sim \text{Exp}(1)$. We choose the constants in the following order:

$$A = \frac{6}{r_0}, \quad H = \frac{4}{r_0}, \quad \xi_{\text{in}} = \frac{e^{-H}}{8A}, \quad p_{\text{in}} = \left(\frac{1+4A}{\xi_{\text{in}}}\right)^2, \quad \delta = \log 2. \quad (15.2)$$

These are numerical constants. Numerically, $r_0 \approx 0.2325$, $A \approx 25.8$, $H \approx 17.2$ and $\xi_{\text{in}} \approx 1.6 \cdot 10^{-10}$; we make no attempt to optimize them.

Proposition 15.2. *Let ξ_{in} , p_{in} and δ be the numerical constants in (15.2). For every $p \geq p_{\text{in}}$, every $E \in (0, \xi_{\text{in}}\sqrt{p}]$, and every real solution π at energy E , we have $\varphi_{\pi}(1) \geq \delta$.*

Proof. Fix $p \geq p_{\text{in}}$, $E \in (0, \xi_{\text{in}}\sqrt{p}]$ and a real solution π at energy E . Let $D \sim \pi$ and $Y \sim \text{Exp}(1)$ be independent, and set

$$F = \mathbb{E} \frac{1}{1 + |D|}, \quad q = \mathbb{P}(|D| \leq p^{-1/2}).$$

Note that $p \geq p_{\text{in}} > 1$. We show that $F \geq A/\sqrt{p}$ or $pq \geq H$, and that each of these alternatives gives $\lambda(1) \geq 2$, where $\lambda(1) = e^{\varphi_{\pi}(1)}$ equals $\lambda^{(1)}(1)$ and $\lambda^{(\sqrt{p})}(1)$ by (15.1).

STEP 1. *The scaling $g = 1$.* We claim that if $F \geq A/\sqrt{p}$, then $\lambda(1) \geq 2$. On the event $\{1 \leq Y \leq 2\}$, we bound the operator norm of $M_1 = (Y/p)^{-1/2} \begin{pmatrix} 0 & 1 \\ -Y/p & D \end{pmatrix}$ by its Frobenius norm and obtain

$$\|M_1\| \leq \left(\frac{p}{Y}\right)^{1/2} \left(1 + \frac{Y^2}{p^2} + D^2\right)^{1/2} \leq \sqrt{p} \left(1 + \frac{Y}{p} + |D|\right) \leq 3\sqrt{p} (1 + |D|).$$

In the second inequality, we used $Y \geq 1$ and $(a^2 + b^2 + c^2)^{1/2} \leq a + b + c$ for $a, b, c \geq 0$, and in the third, $Y/p \leq 2$. Since Y is independent of D , we find

$$p \mathbb{E} \|M_1\|^{-1} \geq p \mathbb{E} \left[\mathbf{1}_{\{1 \leq Y \leq 2\}} \frac{1}{3\sqrt{p}(1 + |D|)} \right] = \frac{r_0 \sqrt{p}}{3} F.$$

If $F \geq A/\sqrt{p}$, the right side is at least $r_0 A/3 = 2$, and Lemma 15.1 gives $\lambda(1) \geq 2$.

STEP 2. *The scaling $g = \sqrt{p}$.* We claim that if $pq \geq H$, then $\lambda(1) \geq 2$. For $g = \sqrt{p}$, we have $gD = \sqrt{p}D$ and $g^2 Y/p = Y$. Then

$$M_{\sqrt{p}} = Y^{-1/2} \begin{pmatrix} 0 & 1 \\ -Y & \sqrt{p}D \end{pmatrix}, \quad \|M_{\sqrt{p}}\|^2 \leq \frac{1}{Y} + Y + \frac{pD^2}{Y}.$$

On the event $\{1 \leq Y \leq 2\} \cap \{|D| \leq p^{-1/2}\}$, the right side is at most $1 + 2 + 1 = 4$, and $\|M_{\sqrt{p}}\| \leq 2$. Since Y is independent of D , we find

$$p \mathbb{E} \|M_{\sqrt{p}}\|^{-1} \geq p \mathbb{P}(1 \leq Y \leq 2, |D| \leq p^{-1/2}) \cdot \frac{1}{2} = \frac{r_0}{2} pq.$$

If $pq \geq H$, the right side is at least $r_0 H/2 = 2$, and Lemma 15.1 gives $\lambda(1) \geq 2$.

STEP 3. *Both tests cannot fail.* Suppose, for contradiction, that $F < A/\sqrt{p}$ and $pq < H$. Let N , $(Y_j)_{j \geq 1}$ and $(D_j)_{j \geq 1}$ be as in Definition 5.1; then D has the law of $E - p^{-1} \sum_{j \leq N} Y_j / D_j$.

Call an index $j \leq N$ small if $|D_j| \leq p^{-1/2}$ and large otherwise, let Z be the number of small indices, and set

$$\Sigma = \frac{1}{p} \sum_{j \leq N: |D_j| > p^{-1/2}} \frac{Y_j}{|D_j|}.$$

The point process $\sum_{j \leq N} \delta_{(Y_j, D_j)}$ is a Poisson process on $(0, \infty) \times \mathbb{R}$ with intensity measure p times the law of (Y_1, D_1) , since $N \sim \text{Pois}(p)$ is independent of the independent and identically distributed pairs (Y_j, D_j) . Its restrictions to the disjoint sets $(0, \infty) \times [-p^{-1/2}, p^{-1/2}]$ and $(0, \infty) \times (\mathbb{R} \setminus [-p^{-1/2}, p^{-1/2}])$ are therefore independent Poisson processes [94, Chapter 5]. The variable Z is the total mass of the first restriction, which has law $\text{Pois}(pq)$, and Σ is a function of the second. In particular, Z and Σ are independent, and

$$\mathbb{P}(Z = 0) = e^{-pq} > e^{-H}.$$

We next bound the mean of Σ . On the event $\{|D| > p^{-1/2}\}$, we have $1 + |D| \leq (1 + \sqrt{p})|D|$, that is, $|D|^{-1} \leq (1 + \sqrt{p})(1 + |D|)^{-1}$. Using the independence of N , (Y_j) and (D_j) , we find

$$\mathbb{E}\Sigma = \frac{1}{p} \mathbb{E}N \mathbb{E}Y_1 \mathbb{E}[|D|^{-1} \mathbf{1}_{\{|D| > p^{-1/2}\}}] \leq (1 + \sqrt{p})F < \frac{(1 + \sqrt{p})A}{\sqrt{p}} \leq 2A,$$

where in the last inequality we used $p \geq 1$. By Markov's inequality, $\mathbb{P}(\Sigma \geq 4A) < 1/2$. On the event $\{Z = 0\} \cap \{\Sigma < 4A\}$, all indices are large, and $|E - p^{-1} \sum_{j \leq N} Y_j / D_j| \leq E + \Sigma < E + 4A$. By independence, this event has probability $\mathbb{P}(Z = 0) \mathbb{P}(\Sigma < 4A) > e^{-H}/2$, and we conclude that

$$F \geq \frac{e^{-H}}{2} \cdot \frac{1}{1 + E + 4A}.$$

Since $p \geq p_{\text{in}}$, we have $1 + 4A \leq \xi_{\text{in}} \sqrt{p}$, and since $E \leq \xi_{\text{in}} \sqrt{p}$, we have $1 + E + 4A \leq 2\xi_{\text{in}} \sqrt{p}$. By the choice of ξ_{in} in (15.2), we have

$$F \geq \frac{e^{-H}}{4\xi_{\text{in}} \sqrt{p}} = \frac{2A}{\sqrt{p}},$$

which contradicts $F < A/\sqrt{p}$.

STEP 4. Conclusion. By Step 3, we have $F \geq A/\sqrt{p}$ or $pq \geq H$. By Steps 1 and 2, in either case $\lambda^{(g)}(1) \geq 2$ for $g = 1$ or $g = \sqrt{p}$, and by (15.1) both spectral radii equal $\exp \varphi_\pi(1)$. We conclude that $\varphi_\pi(1) \geq \log 2 = \delta$. $\square$

PART IV

Proofs of the main results

Proofs of the main theorems

In this section we prove Theorems 1.1 and 1.2. All ingredients have been established in the previous sections, and we only combine them. Throughout, for $p > 1$ and $E \geq \xi_0\sqrt{p}$ in the range of Corollary 11.23, we write $\varphi_E(1)$ for the free energy at $s = 1$ of the principal solution: by Corollary 11.23, if B has law $\nu_{p,E}$, then the law of $B/g(E)$ is a real solution at energy E , and $\varphi_E(1)$ is the value at $s = 1$ of its free energy (Definition 6.14). This is the quantity $\varphi_E(1)$ of Theorem 1.2, and in the notation of Sections 12–14, we have $\lambda_p(\xi) = \exp \varphi_{\xi\sqrt{p}}(1)$.

16.1 Choice of constants

Let $\xi_{\text{in}} > 0$, $\delta > 0$, and p_{in} be the numerical constants of Proposition 15.2. Since the conclusion of Proposition 15.2 is preserved if ξ_{in} is decreased, we may and do assume that $\xi_{\text{in}} < \mu_*/2$. Set

$$\xi_0 = \frac{\xi_{\text{in}}}{2}.$$

Let p_* be the numerical constant of Theorem 10.1, and let M_{ext} and p_{ext} be the numerical constants of the large-energy estimate in Proposition 13.2. Let $\Xi = [\xi_0, M_{\text{ext}} + 1]$, and let $p_{\text{phys}}(\Xi)$ be the threshold of Proposition 11.27 for this window. By Corollary 14.2, there exists $p(\xi_0)$ such that for every $p \geq p(\xi_0)$ there is a unique $E_c(p) \geq \xi_0\sqrt{p}$ with

$$\varphi_E(1) > 0 \quad \text{for } \xi_0\sqrt{p} \leq E < E_c(p), \quad \varphi_E(1) < 0 \quad \text{for } E > E_c(p), \quad (16.1)$$

and such that $E_c(p)/\sqrt{p} \rightarrow \mu_*$ as $p \rightarrow \infty$. Since $\xi_{\text{in}} < \mu_*/2$, there exists p' such that $E_c(p) > \xi_{\text{in}}\sqrt{p}$ for every $p \geq p'$. We define p_0 to be the maximum of p_* , p_{in} , p_{ext} , $p_{\text{phys}}(\Xi)$, the threshold $p_0(\xi_0)$ of Corollary 11.23, $p(\xi_0)$, and p' . This is the maximum of finitely many finite numbers.

Fix $p \geq p_0$ for the remainder of this section. We record three consequences of the choice of p_0 . First, $\xi_{\text{in}}\sqrt{p} < E_c(p)$. Second, $E_c(p) < M_{\text{ext}}\sqrt{p}$, since the large-energy estimate of Proposition 13.2 gives $\varphi_E(1) < 0$ for every $E \geq M_{\text{ext}}\sqrt{p}$, while $\varphi_E(1) > 0$ for $E < E_c(p)$. Third, the principal solution $\nu_{p,E}$ is defined for every $E \geq \xi_0\sqrt{p}$.

16.2 Proof of Theorem 1.1

By Proposition 3.21, the operator $\mathcal{H}$ has pure point spectrum (respectively purely absolutely continuous spectrum) in a Borel set I if and only if it has the same property in $-I$. It therefore suffices to prove the statements about positive energies. Let $\mathcal{E}_p$ be the set of regular energies of Definition 5.7. By Lemma 5.8, its complement has Lebesgue measure zero.

STEP 1. Absolute continuity at small energies. Let $E \in \mathcal{E}_p \cap (0, \xi_{\text{in}}\sqrt{p}]$, and suppose that $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. By part (ii) of Proposition 5.11, the law π of the boundary denominator $-1/\Gamma_o(E + i0)$ under $\mathbb{P}$ is a real solution at energy E . By Proposition 15.2, its free energy satisfies $\varphi_\pi(1) \geq \delta > 0$. Since $p \geq p_*$, this contradicts Theorem 10.1. Therefore $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) < 1$ for every $E \in \mathcal{E}_p \cap (0, \xi_{\text{in}}\sqrt{p}]$. Since $\mathcal{E}_p$ has full measure, Corollary 10.2 implies that $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has purely absolutely continuous spectrum in $(0, \xi_{\text{in}}\sqrt{p})$.

STEP 2. Absolute continuity at energies of order $\sqrt{p}$ below $E_c(p)$. Let $E \in \mathcal{E}_p \cap (\xi_0\sqrt{p}, E_c(p))$, and suppose that $\mathbb{P}^\infty(\text{Im } \Gamma_o(E + i0) = 0) = 1$. Since $E_c(p) < M_{\text{ext}}\sqrt{p}$, we have $E/\sqrt{p} \in \Xi$. By part (iv) of Proposition 11.27, the law π of the boundary denominator $-1/\Gamma_o(E + i0)$ under $\mathbb{P}$ is a real solution at energy E , and the law of $g(E)D$, $D \sim \pi$, is $\nu_{p,E}$. Since the free energy does not depend on the scaling (Definition 6.14), the free energy of π at $s = 1$ is $\varphi_E(1)$, which is positive by (16.1). This again contradicts Theorem 10.1. As in the first step, Corollary 10.2 shows that $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has purely absolutely continuous spectrum in $(\xi_0\sqrt{p}, E_c(p))$.

Since $\xi_0\sqrt{p} < \xi_{\text{in}}\sqrt{p} < E_c(p)$, the open intervals in the first two steps cover $(0, E_c(p))$. If the restrictions of a finite measure to two open sets are absolutely continuous, then so is its restriction to their union. Hence $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has purely absolutely continuous spectrum in $(0, E_c(p))$.

STEP 3. Pure point spectrum above $E_c(p)$. Let $U = (E_c(p), \infty)$. For every $E \in U$, the law of $B/g(E)$, $B \sim \nu_{p,E}$, is a real solution at energy E by part (iv) of Corollary 11.23, and its free energy at $s = 1$ is $\varphi_E(1)$, which is negative by (16.1). By Theorem 8.4, $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has pure point spectrum in U .

STEP 4. Finitely supported eigenvectors. By part (ii) of Proposition 3.24, $\mathbb{P}^\infty$ -almost surely $\mathcal{H}$ has no finitely supported eigenvector with a nonzero eigenvalue. By part (ii) of Proposition 3.25, $\mathbb{P}^\infty$ -almost surely $\mathcal{H}$ has infinitely many pairwise orthogonal, finitely supported eigenvectors with eigenvalue zero.

Finally, $E_c(p)/\sqrt{p} \rightarrow \mu_*$ by Corollary 14.2. This completes the proof of Theorem 1.1.

Remark 16.1. The proof of pure point spectrum in the third step uses the principal solution only through the square-summable inverse column of Proposition 8.3, and it gives no

quantitative information about the Green function at spectral parameters with positive imaginary part.

16.3 Proof of Theorem 1.2

Fix $\xi'_0 \in (0, \mu_*)$, and let $p_1(\xi'_0)$ be the maximum of p_0 and of the thresholds of Corollaries 14.2 and 11.23 for the value ξ'_0 . Fix $p \geq p_1(\xi'_0)$.

Part (i) is part (i) of Corollary 11.23, applied with ξ'_0 in place of ξ_0 .

For part (ii), we first prove that $E \mapsto \varphi_E(1)$ is continuous on $[\xi'_0\sqrt{p}, \infty)$. We apply Proposition 6.12 with the scaling $g(E)$, that is, to the laws $\nu_E = \nu_{p,E}$ with the rates $K(E) = p/\theta(E)$, where $\theta(E) = g(E)^2$. The map $E \mapsto K(E)$ is continuous, and $E \mapsto \nu_{p,E}$ is weakly continuous by part (iii) of Corollary 11.23. Each $\nu_{p,E}$ satisfies (6.1), since it is the law of $g(E)D$ for a real solution D (part (iv) of Corollary 11.23 and Proposition 5.5). By part (i) of Corollary 11.23, we have $\int |b|^q \nu_{p,E}(db) \leq 4 + (1-q)^{-1}$ for every $q \in (0, 1)$ and every $E \geq \xi'_0\sqrt{p}$. Thus the hypotheses of Proposition 6.12 hold with every $q_0 \in (0, 1)$, and $E \mapsto \lambda_E(1) = \exp \varphi_E(1)$ is continuous on $[\xi'_0\sqrt{p}, \infty)$.

By Corollary 14.2, applied with ξ'_0 , there is a unique zero $E'_c(p)$ of $E \mapsto \varphi_E(1)$ on $[\xi'_0\sqrt{p}, \infty)$, with $\varphi_E(1) > 0$ to its left and $\varphi_E(1) < 0$ to its right. This zero coincides with the energy $E_c(p)$ of Section 16.1. Indeed, the half-lines $[\xi_0\sqrt{p}, \infty)$ and $[\xi'_0\sqrt{p}, \infty)$ are nested, and $E \mapsto \varphi_E(1)$ has exactly one zero on each of them. The zero on the smaller half-line is also a zero on the larger one, so the two zeros coincide. Finally, let $\Xi_c \subset (0, \infty)$ be a compact interval whose interior contains $E_c(p)/\sqrt{p}$ for all $p \geq p_1(\xi'_0)$; for instance $\Xi_c = [\mu_*/2, 2\mu_*]$, after increasing $p_1(\xi'_0)$. Increasing $p_1(\xi'_0)$ further, we may assume that $p \geq p_{\text{cr}}(\Xi_c)$, the threshold of Proposition 12.21. By that proposition, $\xi \mapsto \lambda_p(\xi)$ is real-analytic on an open interval containing Ξ_c , so $E \mapsto \varphi_E(1) = \log \lambda_p(E/\sqrt{p})$ is real-analytic in a neighborhood of $E_c(p)$, and its derivative at $E_c(p)$ is negative by Corollary 14.2. This proves part (ii).

Part (iii) is Theorem 13.1, applied to a compact set $\Xi \subset [\xi'_0, \infty)$.

The giant component

In this section, we transfer the spectral dichotomy of Theorem 1.1 to the giant component of the finite graph and prove Theorem 1.4. The section has three parts.

The first part, Sections 17.2 and 17.3, proves Proposition 17.13: the giant component, seen from finitely many uniformly chosen vertices, converges locally to independent copies of the marked PGW(p) tree conditioned on survival, and this convergence holds in probability for averages over the giant component. Membership in the giant component is not determined by a bounded neighborhood of a vertex. We therefore first prove the statement with the giant component replaced by the set of vertices whose component has at least K vertices, which is a local condition, and then let $K \rightarrow \infty$. Since the tree operator is unbounded, local convergence alone does not give convergence of resolvents; this step uses the essential self-adjointness of the tree operator from Theorem 3.10, as in [50].

The second part, Section 17.4, proves Proposition 17.14: on every compact interval separated from zero, the averaged root spectral measure of the survival-conditioned tree has a density bounded above and below. The upper bound comes from a Wegner-type estimate for an auxiliary bipartite random graph, whose squared operator is a sum of rank-one projections with independent Gamma-distributed weights. The lower bound comes from a configuration at the root (one surviving child and one leaf child) in which the weight of the leaf edge enters the root resolvent through a Poisson kernel.

The third part, Section 17.5, combines the first two to prove Propositions 17.20 and 17.22, and then Theorem 1.4.

17.1 Setting and notation

Throughout this section, $p > 1$ is fixed, and constants may depend on p without further mention. We use the finite model of Definition 3.1. The graph G_n is an Erdős–Rényi graph $G(n, p/n)$ on $[n] = \{1, \dots, n\}$. Conditionally on G_n , the variables w_{ij} , $1 \leq i < j \leq n$, are independent standard complex Gaussians. For an edge ij of G_n with $i < j$ we set $h_{ij} = w_{ij}/\sqrt{p}$ and $h_{ji} = \overline{h_{ij}}$, and $H_n(i, j) = \mathbf{1}_{\{ij \in E(G_n)\}} h_{ij}$. We write gc for the giant component of G_n , namely its largest connected component (if there are several, we take the one containing the smallest vertex), $N_n = |\text{gc}|$, and H_n^{gc} for the restriction of H_n to $\ell^2(\text{gc})$.

Since gc is a connected component of G_n , the matrix H_n is the direct sum of H_n^{gc} and

its restriction to $\ell^2([n] \setminus \text{gc})$. In particular, for all $u, v \in \text{gc}$, every $z \in \mathbb{C} \setminus \mathbb{R}$, and every bounded Borel function f , we have

$$\langle \delta_u, (H_n - z)^{-1} \delta_v \rangle = \langle \delta_u, (H_n^{\text{gc}} - z)^{-1} \delta_v \rangle, \quad \langle \delta_v, f(H_n) \delta_v \rangle = \langle \delta_v, f(H_n^{\text{gc}}) \delta_v \rangle. \quad (17.1)$$

For $v \in \text{gc}$, we let $\mu_{n,v}$ denote the spectral measure of δ_v for H_n^{gc} , so that $\mu_{n,v}(I) = \langle \delta_v, \mathbf{1}_I(H_n^{\text{gc}}) \delta_v \rangle$ as in Definition 1.3. By (17.1), it is also the spectral measure of δ_v for H_n . The empirical spectral measure of H_n^{gc} is

$$\nu_n^{\text{gc}} = \frac{1}{N_n} \sum_{v \in \text{gc}} \mu_{n,v}, \quad \text{so that} \quad \nu_n^{\text{gc}}(I) = \frac{1}{N_n} \text{Tr} \mathbf{1}_I(H_n^{\text{gc}}) = \bar{\mu}_n(I). \quad (17.2)$$

On the tree side, we use the marked PGW(p) tree $\mathcal{T}$ of Definition 3.2, with root o , edge marks $h_e = w_e / \sqrt{p}$, survival event $\mathcal{S}$, survival probability $\varrho_p = \mathbb{P}(\mathcal{S}) \in (0, 1)$, and conditioned law $\mathbb{P}^\infty = \mathbb{P}(\cdot \mid \mathcal{S})$. We write $|\mathcal{T}| \in \mathbb{N} \cup \{\infty\}$ for the number of vertices of $\mathcal{T}$, so that $\mathcal{S} = \{|\mathcal{T}| = \infty\}$. As in Definition 3.15, μ_o is the spectral measure of δ_o for $\mathcal{H}$. We define the averaged root spectral measures

$$\nu_p = \mathbb{E} \mu_o, \quad \nu_p^\infty = \mathbb{E}^\infty \mu_o, \quad (17.3)$$

that is, $\nu_p(B) = \mathbb{E}[\mu_o(B)]$ and $\nu_p^\infty(B) = \mathbb{E}^\infty[\mu_o(B)]$ for Borel sets $B \subset \mathbb{R}$. Both are probability measures. Their Stieltjes transforms are $\mathbb{E} G_{oo}(z)$ and $\mathbb{E}^\infty G_{oo}(z)$, by Fubini's theorem.

We use the following classical fact about the giant component [68], [133, Theorem 4.8]: as $n \rightarrow \infty$,

$$\frac{N_n}{n} \longrightarrow \varrho_p \quad \text{in probability.} \quad (17.4)$$

In these references, the limit is described as the unique positive solution ϱ of $1 - \varrho = e^{-p\varrho}$; this is the survival probability of a PGW(p) tree, since the extinction probability is the smallest fixed point in $[0, 1]$ of the generating function $s \mapsto e^{p(s-1)}$ of $\text{Pois}(p)$.

Several random quantities below, such as averages over k -tuples of distinct vertices of gc , are only defined on the event $\{N_n \geq k\}$. We set them equal to zero on the complement. Since $\mathbb{P}(N_n \geq k) \rightarrow 1$ by (17.4), this convention does not affect any limit in probability.

We say that random probability measures m_n on $\mathbb{R}$ converge *weakly in probability* to a deterministic probability measure m if $\int f dm_n \rightarrow \int f dm$ in probability for every bounded continuous $f: \mathbb{R} \rightarrow \mathbb{R}$.

17.2 Rooted marked graphs and resolvents

The local topology

Definition 17.1. Fix $k \geq 1$. A k -rooted marked graph is a triple $x = (G, h, o)$ with the following properties. The graph $G = (V, E)$ is simple and locally finite, with countable

vertex set. The marks h assign to each ordered pair (u, v) with $uv \in E$ a number $h_{uv} \in \mathbb{C}$, with $h_{vu} = \overline{h_{uv}}$. The roots $o = (o_1, \dots, o_k) \in V^k$ are such that every vertex of G is connected to at least one root. Two k -rooted marked graphs are isomorphic if there exists a graph isomorphism between them that maps the a -th root to the a -th root for every a and preserves the marks. We write $\mathcal{G}_k$ for the set of isomorphism classes, and we do not distinguish in notation between a k -rooted marked graph and its class. We call x *finite* if V is finite.

For $x \in \mathcal{G}_k$ and $R \in \mathbb{N} \cup \{0\}$, let $B_R(x)$ denote the k -rooted marked graph induced by x on the set of vertices at graph distance at most R from $\{o_1, \dots, o_k\}$. Let $R_*(x, x')$ be the supremum of the real numbers $r > 0$ for which there exists an isomorphism φ from $B_{\lfloor r \rfloor}(x)$ to $B_{\lfloor r \rfloor}(x')$ as k -rooted graphs (ignoring marks) such that $|h'_{\varphi(u)\varphi(v)} - h_{uv}| \leq r^{-1}$ for every edge uv of $B_{\lfloor r \rfloor}(x)$, with $R_* = 0$ if there is no such r . We set

$$d_{\text{loc}}(x, x') = \frac{1}{1 + R_*(x, x')}.$$

Local weak convergence was introduced by Benjamini and Schramm [35] and by Aldous and Steele [16]. The distance d_{loc} is the metric of [15, Section 2]; see also [134, Chapter 2 and Appendix A]. By these references, $(\mathcal{G}_k, d_{\text{loc}})$ is a complete separable metric space. (These references treat $k = 1$ and marks in a general complete separable metric space; the case of k roots is identical.) A sequence x_n converges to x if and only if, for every $R \geq 1$, there exists n_R such that for all $n \geq n_R$ there exists an isomorphism of $B_R(x)$ onto $B_R(x_n)$ as k -rooted graphs with all mark differences at most R^{-1} . Since a locally finite graph has finite balls, every $B_R(x)$ is finite, and $d_{\text{loc}}(x, B_R(x)) \leq (1 + R)^{-1}$. A function on $\mathcal{G}_k$ that depends only on $B_R(x)$ as a k -rooted graph (ignoring marks) is locally constant, hence continuous. Examples are $x \mapsto \mathbf{1}\{\deg(o_1) \leq L\}$, $x \mapsto \mathbf{1}\{\text{dist}(o_a, o_b) \leq R\}$ for $R \geq 0$, and

$$\chi_{K,a}(x) = \mathbf{1}\{\text{the component of } o_a \text{ has at least } K \text{ vertices}\}, \quad K \geq 1. \quad (17.5)$$

For $\chi_{K,a}$, this holds with $R = K - 1$, since a component with at least K vertices contains a connected set of K vertices containing o_a , and such a set lies within distance $K - 1$ of o_a . The set of finite elements of $\mathcal{G}_k$ is Borel, since it is the union over R of the sets $\{|B_R(x)| = |B_{R+1}(x)|\}$, which are open and closed.

For $1 \leq k' \leq k$ and an injective map $\sigma: [k'] \rightarrow [k]$, the *restriction* $r_\sigma: \mathcal{G}_k \rightarrow \mathcal{G}_{k'}$ keeps the roots $o_{\sigma(1)}, \dots, o_{\sigma(k')}$ and the union of their components. It satisfies $d_{\text{loc}}(r_\sigma x, r_\sigma x') \leq d_{\text{loc}}(x, x')$, because an isomorphism of $B_R(x)$ onto $B_R(x')$ restricts to an isomorphism of $B_R(r_\sigma x)$ onto $B_R(r_\sigma x')$. Here we used that rooted isomorphisms of R -balls preserve the distance to the roots, since for a vertex of $B_R(x)$ this distance is attained by a path inside $B_R(x)$.

We use two kinds of elements of $\mathcal{G}_k$. For distinct vertices $v_1, \dots, v_k$ of G_n , we write $[G_n, v]$ for the k -rooted marked graph formed by the union of the components

of $v_1, \dots, v_k$ in G_n , with the marks h_{ij} and roots $v = (v_1, \dots, v_k)$. For k rooted marked trees $(T_1, o_1), \dots, (T_k, o_k)$, we consider their disjoint union, rooted at $(o_1, \dots, o_k)$, as an element of $\mathcal{G}_k$. We write $\mathfrak{F}_k$ for the disjoint union of k independent marked PGW(p) trees with law $\mathbb{P}$, and $\mathfrak{F}_k^\infty$ for the disjoint union of k independent trees with law $\mathbb{P}^\infty$. We write $\mathbb{E}^{\infty, \otimes k}$ for the expectation with respect to the law of $\mathfrak{F}_k^\infty$.

Local convergence of the uniformly rooted graph

We need the local limit for two graph models. The second one is used only in Section 17.4.

- (a) The model of Section 17.1: $G_n = G(n, p/n)$ with vertex set $V_n = [n]$ and marks h_{ij} .
- (b) The bipartite model: $V_n = [n] \times \{0, 1\}$, and each pair $\{(i, 0), (j, 1)\}$ is an edge independently with probability p/n ; there are no other edges. Conditionally on the graph, the edges carry independent standard complex Gaussians w , and the edge $\{(i, 0), (j, 1)\}$ has marks $h_{(j,1),(i,0)} = w/\sqrt{p}$ and $h_{(i,0),(j,1)} = \bar{w}/\sqrt{p}$. In this model, we also write G_n for the marked graph.

In both models, the marks are independent of the graph. In the marked tree, the mark h_{uv} on the edge from a parent u to its child v is $w_{uv}/\sqrt{p}$. Since the standard complex Gaussian law is invariant under complex conjugation, the choice of orientation used to attach the Gaussian variable to an edge does not affect the law of a marked graph: in both models, conditionally on the graph, the marks h_{uv} , for any choice of one orientation (u, v) per edge, are independent with the law of $w/\sqrt{p}$.

Lemma 17.2. *Consider either model (a) or model (b). Fix $k \geq 1$, and let $\mathbf{V} = (V_1, \dots, V_k)$ be uniformly distributed among the k -tuples of distinct elements of V_n , independent of the marked graph. Then for every $R \geq 1$, the law of $B_R([G_n, \mathbf{V}])$ converges in total variation to the law of $B_R(\mathfrak{F}_k)$ as $n \rightarrow \infty$. In particular, $[G_n, \mathbf{V}] \rightarrow \mathfrak{F}_k$ in distribution in $\mathcal{G}_k$. Further, in model (a), $\mathbb{E}[\deg(V_1)^2] \leq p + p^2$ for all n .*

This is the standard exploration argument (see [134, Chapter 2] for local convergence of $G(n, p/n)$); we include the proof because we need k roots and marks.

Proof. We treat model (a) and indicate the changes for model (b) at the end. Fix $R \geq 1$ and $\varepsilon > 0$. Since the balls of radius R in PGW(p) trees are almost surely finite, there exists $M \geq k$ such that $\mathbb{P}(\sum_{a=1}^k |B_R(T_a)| > M) \leq \varepsilon$, where $T_1, \dots, T_k$ are independent PGW(p) trees.

We explore the balls of radius R around $V_1, \dots, V_k$ in G_n , one root at a time. Each vertex is first *discovered* and later possibly *processed*. When the exploration of the root V_a starts, V_a is discovered (if it was discovered earlier, we say that a *collision* occurs). Then, as long as there is an unprocessed vertex discovered during the exploration of V_a at exploration depth at most $R - 1$, we process the earliest discovered such vertex x . Processing x reveals, for every vertex y not yet processed, whether xy is an edge. Every

such y that was not yet discovered becomes discovered, as a child of x , at depth one more than x . When all roots have been explored, we reveal all pairs of discovered vertices that have not been revealed. Each pair of vertices is revealed at most once. Since the roots are independent of the graph, conditionally on everything revealed before a pair is revealed, its indicator is a Bernoulli variable with parameter p/n .

Let $\mathcal{C}$ be the event that no collision occurs and that every revealed pair $\{x, y\}$ of vertices that were both discovered before it was revealed is not an edge. On $\mathcal{C}$, the induced subgraph on the discovered vertices is the union of k disjoint trees, the exploration trees of $V_1, \dots, V_k$. Further, the vertices of the a -th tree are exactly the vertices at distance at most R from V_a . Then, on $\mathcal{C}$, the k -rooted graph $B_R([G_n, \mathbf{V}])$ (ignoring marks) is the disjoint union of the exploration trees. Its isomorphism class is determined by the sequence of numbers of children of the processed vertices, in the order of processing.

We stop the exploration as soon as more than M vertices are discovered. Before stopping, there are at most M processing steps and at most $(M+1)^2$ revealed pairs of the type excluded in $\mathcal{C}$, each of which is an edge with conditional probability p/n . Further, conditionally on the exploration of $V_1, \dots, V_{a-1}$, the root V_a is uniform on $[n] \setminus \{V_1, \dots, V_{a-1}\}$, and the probability of a collision at V_a is at most $(M+1)/(n-k)$. When a vertex x is processed and $m \leq M$ vertices are discovered, the number of its children is, conditionally on the past, binomial with parameters $n-m$ and p/n . By Le Cam's inequality and the bound $d_{\text{TV}}(\text{Pois}(\lambda), \text{Pois}(\lambda')) \leq |\lambda - \lambda'|$, we have

$$d_{\text{TV}}(\text{Bin}(n-m, p/n), \text{Pois}(p)) \leq (n-m) \frac{p^2}{n^2} + \frac{mp}{n} \leq \frac{p^2 + Mp}{n}.$$

Explore $T_1, \dots, T_k$ up to depth R in the same breadth-first order. Using at each processing step a maximal coupling of the two conditional offspring laws, we obtain a coupling under which, except on an event of probability at most

$$\varepsilon + M \frac{p^2 + Mp}{n} + (M+1)^2 \frac{p}{n} + \frac{k(M+1)}{n-k},$$

the event $\mathcal{C}$ holds and all offspring numbers agree. On this event, $B_R([G_n, \mathbf{V}])$ and $B_R(T_1) \sqcup \dots \sqcup B_R(T_k)$ are isomorphic as k -rooted graphs through the exploration. We transport the marks of G_n on the edges of the ball to the corresponding tree edges, and give the remaining tree edges independent marks. The marks are independent of the graph and of $\mathbf{V}$, and the exploration only reveals the graph. Hence, by the conjugation invariance noted above, the transported marks are independent with the law of $w/\sqrt{p}$, and independent of the trees. This produces marked trees with the law of $\mathfrak{F}_k$, and the marked balls agree on the event above. Since $\varepsilon > 0$ was arbitrary, the total variation distance tends to zero.

For the convergence in distribution, let $F: \mathcal{G}_k \rightarrow \mathbb{R}$ be bounded and uniformly continuous, with modulus of continuity ω_F . Since $d_{\text{loc}}(x, B_R(x)) \leq (1+R)^{-1}$, we have

$$|\mathbb{E}F([G_n, \mathbf{V}]) - \mathbb{E}F(\mathfrak{F}_k)| \leq 2\omega_F((1+R)^{-1}) + 2\|F\|_\infty d_{\text{TV}}(\text{Law } B_R([G_n, \mathbf{V}]), \text{Law } B_R(\mathfrak{F}_k)).$$

Letting $n \rightarrow \infty$ and then $R \rightarrow \infty$, we find $\mathbb{E}F([G_n, \mathbf{V}]) \rightarrow \mathbb{E}F(\mathfrak{F}_k)$. By the portmanteau theorem [37, Theorem 2.1], this proves the convergence in distribution. Finally, since the degree of V_1 is binomial with parameters $n-1$ and p/n , we have $\mathbb{E}[\deg(V_1)^2] \leq p + p^2$.

In model (b), the vertex x processed in the class $V_n \cap ([n] \times \{c\})$ only has potential neighbors in the other class, and the number of its children is binomial with parameters $n - m'$ and p/n , where $m' \leq M$ is the number of discovered vertices in the other class. The rest of the argument is unchanged. $\square$

Stieltjes transforms

For a finite Borel measure m on $\mathbb{R}$, we write $s_m(z) = \int m(d\lambda)/(\lambda - z)$, $z \in \mathbb{H}$, for its Stieltjes transform.

Lemma 17.3. *Let $D \subset \mathbb{H}$ be a set with an accumulation point in $\mathbb{H}$.*

- (i) *Let m_n and m be probability measures on $\mathbb{R}$ such that $s_{m_n}(z) \rightarrow s_m(z)$ for every $z \in D$. Then $m_n \rightarrow m$ weakly.*
- (ii) *Suppose that D is countable. Let m_n be random probability measures on $\mathbb{R}$ and m a deterministic probability measure such that $s_{m_n}(z) \rightarrow s_m(z)$ in probability for every $z \in D$. Then $m_n \rightarrow m$ weakly in probability.*

Proof. For (i), it suffices to show that every subsequence of (m_n) has a further subsequence converging weakly to m . By Helly's selection theorem, every subsequence has a further subsequence that converges vaguely to a measure m' with $m'(\mathbb{R}) \leq 1$. Since, for $z \in \mathbb{H}$, the function $\lambda \mapsto (\lambda - z)^{-1}$ is continuous and vanishes at infinity, we have $s_{m_n}(z) \rightarrow s_{m'}(z)$ along this subsequence. Then $s_{m'} = s_m$ on D . Both functions are analytic on $\mathbb{H}$, and the identity theorem implies that they agree on $\mathbb{H}$. By the Stieltjes inversion formula, $\pi^{-1} \text{Im } s_{m'}(E + i\varepsilon) dE$ converges vaguely to m' as $\varepsilon \downarrow 0$; the same holds for m , and we conclude that $m' = m$. Since $m(\mathbb{R}) = 1$, the vague convergence to m is weak convergence [64, Section 3.2].

For (ii), fix a bounded continuous f . It suffices to show that every subsequence has a further subsequence along which $\int f dm_n \rightarrow \int f dm$ almost surely. Since D is countable, by a diagonal argument there exists a further subsequence along which, almost surely, $s_{m_n}(z) \rightarrow s_m(z)$ for all $z \in D$ simultaneously. Along it, part (i) applies pathwise. $\square$

Resolvents on rooted marked graphs

For $x \in \mathcal{G}_k$ with vertex set V , let $H_{x,0}$ be the weighted adjacency operator on finitely supported functions $f: V \rightarrow \mathbb{C}$, given by $(H_{x,0}f)(u) = \sum_{v \sim u} h_{uv}f(v)$. It is a symmetric

operator on $\ell^2(V)$. For $R \geq 0$, let $H_x^{(R)}$ be the Hermitian matrix of the finite marked graph $B_R(x)$, acting on functions on the vertex set of $B_R(x)$. For $z \in \mathbb{C} \setminus \mathbb{R}$ and $a, b \in [k]$, set

$$G_{ab}^{(R)}(x, z) = \langle \delta_{o_a}, (H_x^{(R)} - z)^{-1} \delta_{o_b} \rangle,$$

and let $\mu_{x,a}^{(R)}$ be the spectral measure of δ_{o_a} for $H_x^{(R)}$.

Definition 17.4. For $x \in \mathcal{G}_k$, $z \in \mathbb{C} \setminus \mathbb{R}$ and $a, b \in [k]$, let $G_{ab}(x, z) = \lim_{R \rightarrow \infty} G_{ab}^{(R)}(x, z)$ if this limit exists, and $G_{ab}(x, z) = 0$ otherwise. Let $\mu_{x,a}$ be the weak limit of $\mu_{x,a}^{(R)}$ as $R \rightarrow \infty$ if it exists, and $\mu_{x,a} = \delta_0$ otherwise. Let $\mathcal{G}_k^{\text{esa}}$ be the set of $x \in \mathcal{G}_k$ such that $H_{x,0}$ is essentially self-adjoint; for such x , let H_x denote the closure of $H_{x,0}$.

If x is finite, then $B_R(x) = x$ for large R ; in this case, $G_{ab}(x, z)$ and $\mu_{x,a}$ are the resolvent entries and the root spectral measures of the matrix $H_{x,0}$. In particular, if $v_1, \dots, v_k$ are distinct vertices of gc , then (17.1) implies that $G_{ab}([G_n, v], z) = \langle \delta_{v_a}, (H_n^{\text{gc}} - z)^{-1} \delta_{v_b} \rangle$ and $\mu_{[G_n, v], a} = \mu_{n, v_a}$.

Lemma 17.5. For every $z \in \mathbb{C} \setminus \mathbb{R}$ and $a, b \in [k]$, the maps $x \mapsto G_{ab}(x, z)$ and $x \mapsto \mu_{x,a}$ are Borel, and $|G_{ab}(x, z)| \leq |\text{Im } z|^{-1}$. The set $\mathcal{G}_k^{\text{esa}}$ is a Borel subset of $\mathcal{G}_k$.

Proof. For fixed R , the quantities $G_{ab}^{(R)}(x, z)$ and $\mu_{x,a}^{(R)}$ depend only on the marked graph $B_R(x)$ and are invariant under its rooted isomorphisms. If $d_{\text{loc}}(x, x') < (1 + R')^{-1}$ with $R' \geq R$, then there exists an isomorphism of $B_{R'}(x)$ onto $B_{R'}(x')$ with mark differences at most $1/R'$, which restricts to $B_R(x)$; after relabeling, $H_{x'}^{(R)}$ is a Hermitian matrix with the same pattern as $H_x^{(R)}$ and entries within $1/R'$. The resolvent at a fixed nonreal z and the spectral measure of a fixed basis vector (in the weak topology) depend continuously on the entries of a Hermitian matrix. Then $x \mapsto G_{ab}^{(R)}(x, z)$ and $x \mapsto \mu_{x,a}^{(R)}$ are continuous, the latter as a map into the space $\mathcal{P}(\mathbb{R})$ of probability measures with the weak topology. Since $\mathcal{P}(\mathbb{R})$ with the Lévy–Prokhorov metric is a complete separable metric space [37, Theorem 6.8], the set where a sequence of continuous maps into $\mathbb{C}$ or into $\mathcal{P}(\mathbb{R})$ is Cauchy is Borel, and the limit is a Borel map on it. This proves the first claim. The bound on G_{ab} holds for each $G_{ab}^{(R)}$, hence for the limit.

For the second claim, recall that a symmetric operator A is essentially self-adjoint if and only if $\text{Ran}(A + i)$ and $\text{Ran}(A - i)$ are dense [113, Section VIII.2]. For $R' \geq R$ and a vertex v of $B_R(x)$, let $d_{R'}^{\pm}(x, v)$ be the distance from δ_v to the subspace of vectors $(H_{x,0} \pm i)f$ with f supported in the vertex set of $B_{R'}(x)$. These subspaces increase with R' , and their union is $\text{Ran}(H_{x,0} \pm i)$. Since the vectors δ_v span a dense subspace, we have $x \in \mathcal{G}_k^{\text{esa}}$ if and only if, for every R , we have $\max_{v \in B_R(x)} d_{R'}^{\pm}(x, v) \rightarrow 0$ as $R' \rightarrow \infty$ for both signs. For fixed $R \leq R'$, the function $x \mapsto \max_{v \in B_R(x)} d_{R'}^{\pm}(x, v)$ depends only on $B_{R'+1}(x)$, since $(H_{x,0} \pm i)f$ is supported in $B_{R'+1}(x)$ when f is supported in $B_{R'}(x)$. Let M_x denote

the matrix of $f \mapsto (H_{x,0} \pm i)f$ from functions on $B_{R'}(x)$ to functions on $B_{R'+1}(x)$. Since $H_{x,0}$ is symmetric, we have $\|M_x f\|^2 = \|H_{x,0} f\|^2 + \|f\|^2 \geq \|f\|^2$. Then $M_x^* M_x \geq 1$, and the orthogonal projection $M_x(M_x^* M_x)^{-1} M_x^*$ onto $\text{Ran } M_x$ depends continuously on the entries of M_x . As in the first part, it follows that $x \mapsto \max_{v \in B_R(x)} d_{R'}^\pm(x, v)$ is continuous. Then

$$\mathcal{G}_k^{\text{esa}} = \bigcap_{\pm} \bigcap_{R \geq 1} \bigcap_{j \geq 1} \bigcup_{R' \geq R} \left\{ x : \max_{v \in B_R(x)} d_{R'}^\pm(x, v) < j^{-1} \right\}$$

is Borel. □

The following lemma is the deterministic continuity statement behind resolvent convergence for locally convergent graphs; compare [50] and [47].

Lemma 17.6. *Let $x \in \mathcal{G}_k^{\text{esa}}$.*

(i) *Let $x_n \in \mathcal{G}_k$ be finite with $x_n \rightarrow x$. Then for all $a, b \in [k]$ and $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$G_{ab}(x_n, z) \longrightarrow \langle \delta_{o_a}, (H_x - z)^{-1} \delta_{o_b} \rangle.$$

(ii) *For all $a, b \in [k]$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we have $G_{ab}(x, z) = \langle \delta_{o_a}, (H_x - z)^{-1} \delta_{o_b} \rangle$, and $\mu_{x,a}$ is the spectral measure of δ_{o_a} for H_x .*

(iii) *In the setting of (i), $\mu_{x_n,a} \rightarrow \mu_{x,a}$ weakly for every $a \in [k]$.*

Proof. We begin with (i). Let V be the vertex set of x and V_n that of x_n . Since $x_n \rightarrow x$, there exist integers $R_n \rightarrow \infty$ and numbers $\varepsilon_n \rightarrow 0$ such that, for all sufficiently large n , there exists an isomorphism φ_n of $B_{R_n}(x)$ onto $B_{R_n}(x_n)$ as k -rooted graphs with mark differences at most ε_n ; we only consider such n . Let J_n be the partial isometry from $\ell^2(V)$ to $\ell^2(V_n)$ with $J_n \delta_v = \delta_{\varphi_n(v)}$ for $v \in B_{R_n}(x)$ and $J_n \delta_v = 0$ otherwise.

Fix $b \in [k]$ and $z \in \mathbb{C} \setminus \mathbb{R}$, and set $u_n = (H_{x_n,0} - z)^{-1} \delta_{o_b^n} \in \ell^2(V_n)$, where o^n denotes the roots of x_n . Then $\|u_n\| \leq |\text{Im } z|^{-1}$, and $J_n^* u_n$ is the pullback of u_n to $B_{R_n}(x)$, extended by zero. The set V is countable and every vertex of V lies in $B_{R_n}(x)$ for large n . By a diagonal argument, every subsequence has a further subsequence along which $(J_n^* u_n)(v)$ converges for every $v \in V$; let $u(v)$ denote the limit. By Fatou's lemma, $\|u\| \leq |\text{Im } z|^{-1}$.

Let f be finitely supported, with support in $B_r(x)$. The function $(H_{x,0} - \bar{z})f$ is supported in $B_{r+1}(x)$. For n with $R_n > r + 1$, the vertices of x_n adjacent to $\varphi_n(\text{supp } f)$ lie in $B_{r+1}(x_n) = \varphi_n(B_{r+1}(x))$, and the adjacency relations between them correspond under φ_n . Then $H_{x_n,0} J_n f - J_n H_{x,0} f$ is supported in $\varphi_n(B_{r+1}(x))$, its entries are bounded by $\varepsilon_n \sum_v |f(v)|$, and we obtain

$$\|H_{x_n,0} J_n f - J_n H_{x,0} f\| \leq \varepsilon_n |B_{r+1}(x)|^{1/2} \sum_v |f(v)| \longrightarrow 0.$$

Since $(H_{x,0} - \bar{z})f$ is finitely supported and $J_n^* u_n \rightarrow u$ pointwise along the subsequence, we obtain

$$\begin{aligned} \langle (H_{x,0} - \bar{z})f, u \rangle &= \lim_n \langle J_n(H_{x,0} - \bar{z})f, u_n \rangle = \lim_n \langle (H_{x_n,0} - \bar{z})J_n f, u_n \rangle \\ &= \lim_n \langle J_n f, (H_{x_n,0} - z)u_n \rangle = \lim_n \langle J_n f, \delta_{o_b^n} \rangle = \langle f, \delta_{o_b} \rangle. \end{aligned}$$

In the second equality, we used the previous display and $\|u_n\| \leq |\operatorname{Im} z|^{-1}$; in the third, that $H_{x_n,0}$ is a Hermitian matrix; in the last, that $\varphi_n(o_b) = o_b^n$. Since this holds for all finitely supported f , the vector u lies in the domain of $H_{x,0}^*$, and $H_{x,0}^* u = \delta_{o_b} + zu$. Since $H_{x,0}$ is essentially self-adjoint, $H_{x,0}^* = H_x$, and we conclude that $u = (H_x - z)^{-1} \delta_{o_b}$. Then $G_{ab}(x_n, z) = u_n(o_a^n) = (J_n^* u_n)(o_a) \rightarrow u(o_a)$ along the subsequence. The limit does not depend on the subsequence, which proves (i).

For (ii), we apply (i) to the finite graphs $x_R = B_R(x)$, which converge to x as $R \rightarrow \infty$. Since $B_{R'}(x_R) = B_R(x)$ for $R' \geq R$, we have $G_{ab}(x_R, z) = G_{ab}^{(R)}(x, z)$. By (i), $G_{ab}^{(R)}(x, z)$ converges to $\langle \delta_{o_a}, (H_x - z)^{-1} \delta_{o_b} \rangle$, which is therefore equal to $G_{ab}(x, z)$. In particular, the Stieltjes transforms of $\mu_{x,a}^{(R)}$ converge on $\mathbb{H}$ to that of the spectral measure of δ_{o_a} for H_x , and Lemma 17.3(i) shows that $\mu_{x,a}^{(R)}$ converges weakly to this measure. This proves (ii). Part (iii) follows from (i), (ii) and Lemma 17.3(i), since $s_{\mu_{x_n,a}}(z) = G_{aa}(x_n, z)$. $\square$

By Theorem 3.10, the operator of each tree of $\mathfrak{F}_k$ is almost surely essentially self-adjoint, and a direct sum of finitely many essentially self-adjoint operators is essentially self-adjoint. We conclude that $\mathbb{P}(\mathfrak{F}_k \in \mathcal{G}_k^{\text{esa}}) = 1$; this is an event by Lemma 17.5. On this event, Lemma 17.6(ii) shows that $G_{ab}(\mathfrak{F}_k, z)$ is the resolvent entry of the closed tree operator, which vanishes for $a \neq b$, and that $\mu_{\mathfrak{F}_k,a}$ is the root spectral measure of the a -th tree. In particular, $\mu_{\mathfrak{F}_1,1}$ has the law of μ_o under $\mathbb{P}$, and $\mu_{\mathfrak{F}_1^\infty,1}$ has the law of μ_o under $\mathbb{P}^\infty$.

Admissible functions

Definition 17.7. A bounded Borel function $F: \mathcal{G}_k \rightarrow \mathbb{C}$ is *admissible* if there exists a Borel set $\mathcal{A}_F \subset \mathcal{G}_k$ with $\mathbb{P}(\mathfrak{F}_k \in \mathcal{A}_F) = 1$ such that $F(x_n) \rightarrow F(x)$ for every $x \in \mathcal{A}_F$ and every sequence of finite $x_n \in \mathcal{G}_k$ with $x_n \rightarrow x$.

Lemma 17.8. *The following functions on $\mathcal{G}_k$ are admissible.*

- (i) *Every bounded continuous function.*
- (ii) *For $z_1, \dots, z_m \in \mathbb{C} \setminus \mathbb{R}$ and a bounded continuous $\phi: \mathbb{C}^{k \times k \times m} \rightarrow \mathbb{C}$, the function $x \mapsto \phi((G_{ab}(x, z_l))_{a,b \leq k, l \leq m})$.*
- (iii) *For intervals $I_1, \dots, I_m \subset \mathbb{R}$ whose endpoints c satisfy $\nu_p(\{c\}) = 0$, and a bounded continuous $\phi: [0, 1]^{k \times m} \rightarrow \mathbb{C}$, the function $x \mapsto \phi((\mu_{x,a}(I_l))_{a \leq k, l \leq m})$.*
- (iv) *Linear combinations, products and complex conjugates of admissible functions; and $F \circ r_\sigma$ for every admissible F on $\mathcal{G}_{k'}$ and every injective $\sigma: [k'] \rightarrow [k]$.*

Proof. For (i), take $\mathcal{A}_F = \mathcal{G}_k$. For (ii), the function is Borel by Lemma 17.5, and we take $\mathcal{A}_F = \mathcal{G}_k^{\text{esa}}$; the convergence follows from Lemma 17.6(i),(ii). For (iii), the function is Borel since $x \mapsto \mu_{x,a}$ is Borel and $\mu \mapsto \mu(I)$ is Borel on $\mathcal{P}(\mathbb{R})$. Let $\mathcal{A}_F$ be the set of $x \in \mathcal{G}_k^{\text{esa}}$ with $\mu_{x,a}(\{c\}) = 0$ for all a and all endpoints c of the intervals I_l ; it is Borel since $\mu \mapsto \mu(\{c\})$ is upper semicontinuous. By the discussion after Lemma 17.6, $\mu_{\mathfrak{F}_k,a}$ has the law of μ_o under $\mathbb{P}$ for each a , and $\mathbb{E}\mu_o(\{c\}) = \nu_p(\{c\}) = 0$; hence $\mathbb{P}(\mathfrak{F}_k \in \mathcal{A}_F) = 1$. For $x \in \mathcal{A}_F$ and finite $x_n \rightarrow x$, Lemma 17.6(iii) and the portmanteau theorem give $\mu_{x_n,a}(I_l) \rightarrow \mu_{x,a}(I_l)$, since the boundary of I_l is $\mu_{x,a}$ -null.

For (iv), the claim about linear combinations, products and conjugates follows by intersecting the sets $\mathcal{A}_F$. For $F \circ r_\sigma$, take $\mathcal{A} = r_\sigma^{-1}(\mathcal{A}_F)$, which is Borel since r_σ is continuous. Since the image $r_\sigma(\mathfrak{F}_k)$ is the disjoint union of the trees with indices $\sigma(1), \dots, \sigma(k')$, it has the law of $\mathfrak{F}_{k'}$, and $\mathbb{P}(\mathfrak{F}_k \in \mathcal{A}) = \mathbb{P}(\mathfrak{F}_{k'} \in \mathcal{A}_F) = 1$. If $x \in \mathcal{A}$ and $x_n \rightarrow x$ are finite, then the finite graphs $r_\sigma x_n$ converge to $r_\sigma x \in \mathcal{A}_F$, and $F(r_\sigma x_n) \rightarrow F(r_\sigma x)$. $\square$

Lemma 17.9. *Consider either model (a) or (b), and let $\mathbf{V}$ be as in Lemma 17.2. For every admissible $F: \mathcal{G}_k \rightarrow \mathbb{C}$, we have $\mathbb{E}F([G_n, \mathbf{V}]) \rightarrow \mathbb{E}F(\mathfrak{F}_k)$.*

Proof. By Lemma 17.2, $[G_n, \mathbf{V}] \rightarrow \mathfrak{F}_k$ in distribution in the complete separable metric space $\mathcal{G}_k$. By Skorokhod's representation theorem [37, Theorem 6.7], there exist random elements X_n and X of $\mathcal{G}_k$ on a common probability space such that X_n has the law of $[G_n, \mathbf{V}]$, X has the law of $\mathfrak{F}_k$, and $X_n \rightarrow X$ almost surely. Since the set of finite elements and $\mathcal{A}_F$ are Borel, almost surely every X_n is finite and $X \in \mathcal{A}_F$. Then $F(X_n) \rightarrow F(X)$ almost surely, and we conclude by the bounded convergence theorem that $\mathbb{E}F(X_n) \rightarrow \mathbb{E}F(X)$. $\square$

17.3 Local convergence on the giant component

For the rest of this section, except in Section 17.4, G_n is the graph of model (a). For a bounded function F on $\mathcal{G}_k$, define the averages over ordered k -tuples of distinct vertices

$$S_n(F) = \frac{1}{(n)_k} \sum_{\substack{v \in [n]^k \\ \text{distinct}}} F([G_n, v]), \quad S_n^{\text{gc}}(F) = \frac{1}{(N_n)_k} \sum_{\substack{v \in \text{gc}^k \\ \text{distinct}}} F([G_n, v]), \quad (17.6)$$

where $(m)_k = m(m-1) \cdots (m-k+1)$. Thus $S_n(F)$ and $S_n^{\text{gc}}(F)$ are the conditional expectations of $F([G_n, \mathbf{V}])$ given the marked graph, when $\mathbf{V}$ is a uniform k -tuple of distinct vertices of $[n]$ or of gc, respectively.

Lemma 17.10. *For every admissible $F: \mathcal{G}_k \rightarrow \mathbb{C}$, we have $S_n(F) \rightarrow \mathbb{E}F(\mathfrak{F}_k)$ in L^2 .*

Proof. By Lemma 17.9, we have $\mathbb{E}S_n(F) = \mathbb{E}F([G_n, \mathbf{V}]) \rightarrow \mathbb{E}F(\mathfrak{F}_k)$. It remains to show that $\mathbb{E}|S_n(F)|^2 \rightarrow |\mathbb{E}F(\mathfrak{F}_k)|^2$. We have

$$\mathbb{E}|S_n(F)|^2 = \frac{1}{(n)_k^2} \sum_{v, v'} \mathbb{E} \left[F([G_n, v]) \overline{F([G_n, v'])} \right],$$

where the sum is over pairs of k -tuples of distinct vertices. The number of pairs (v, v') such that some entry of v equals some entry of v' is at most $k^2(n)_k n^{k-1}$, and their contribution is at most $k^2 \|F\|_\infty^2 n^{k-1} / (n)_k = O(n^{-1})$. The remaining $(n)_{2k}$ pairs correspond to $2k$ -tuples (v, v') of distinct vertices. Let $F^\otimes: \mathcal{G}_{2k} \rightarrow \mathbb{C}$ be given by $F^\otimes(y) = F(r_1 y) \overline{F(r_2 y)}$, where r_1 and r_2 are the restrictions to the first k and the last k roots. Then $F([G_n, v]) \overline{F([G_n, v'])} = F^\otimes([G_n, (v, v')])$, and $F^\otimes$ is admissible by Lemma 17.8(iv). Applying Lemma 17.9 with $2k$ roots, we obtain

$$\mathbb{E}|S_n(F)|^2 = \frac{(n)_{2k}}{(n)_k^2} \mathbb{E}F^\otimes([G_n, \mathbf{V}']) + O(n^{-1}) \longrightarrow \mathbb{E}F^\otimes(\mathfrak{F}_{2k}) = |\mathbb{E}F(\mathfrak{F}_k)|^2,$$

where $\mathbf{V}'$ is a uniform $2k$ -tuple of distinct vertices. In the last equality, we used that $r_1 \mathfrak{F}_{2k}$ and $r_2 \mathfrak{F}_{2k}$ are independent copies of $\mathfrak{F}_k$. Expanding the square, we conclude that $\mathbb{E}|S_n(F) - \mathbb{E}F(\mathfrak{F}_k)|^2 \rightarrow 0$. $\square$

Next, we compare membership in gc with the local event that the component of a vertex has at least K vertices. For $K \geq 1$, set

$$a_K = \mathbb{P}(|\mathcal{T}| \geq K),$$

$$Z_{n,K} = \#\{v \in [n] \setminus \text{gc} : \text{the component of } v \text{ has at least } K \text{ vertices}\}.$$

Since $\{|\mathcal{T}| \geq K\}$ decreases to $\{|\mathcal{T}| = \infty\} = \mathcal{S}$ as $K \rightarrow \infty$, we have $a_K \downarrow \varrho_p$.

Lemma 17.11. *For every $K \geq 1$, we have $Z_{n,K}/n \rightarrow a_K - \varrho_p$ in probability.*

Proof. Let $\chi_K = \chi_{K,1}$ be the function from (17.5) with $k = 1$. It is continuous, and $\mathbb{E}\chi_K(\mathfrak{F}_1) = a_K$. By Lemma 17.10 with $k = 1$,

$$\frac{1}{n} \#\{v \in [n] : \text{the component of } v \text{ has at least } K \text{ vertices}\} = S_n(\chi_K) \longrightarrow a_K$$

in probability. On the event $\{N_n \geq K\}$, every vertex of gc belongs to a component with at least K vertices, which gives $Z_{n,K} = nS_n(\chi_K) - N_n$. Since $\mathbb{P}(N_n \geq K) \rightarrow 1$ and $N_n/n \rightarrow \varrho_p$ by (17.4), the claim follows. $\square$

Lemma 17.12. *For every admissible $F: \mathcal{G}_k \rightarrow \mathbb{C}$, we have $S_n^{\text{gc}}(F) \rightarrow \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty)$ in probability.*

Proof. We first replace membership in gc by the local event $\{\chi_{K,a} = 1\}$ for each root, then compute the limit of the resulting average for fixed K , and finally let $K \rightarrow \infty$.

STEP 1. The average with the local event. Fix $K \geq 1$ and set $\Phi_K(x) = F(x) \prod_{a=1}^k \chi_{K,a}(x)$. By Lemma 17.8(i),(iv), Φ_K is admissible. Since, in $\mathfrak{F}_k$, the component of the a -th root is the a -th tree T_a and the trees are independent, Lemma 17.10 implies that

$$S_n(\Phi_K) \longrightarrow \mathbb{E} \left[F(\mathfrak{F}_k) \prod_{a=1}^k \mathbf{1}\{|T_a| \geq K\} \right] = a_K^k \mathbb{E} \left[F(\mathfrak{F}_k) \mid |T_a| \geq K \text{ for all } a \right] \quad (17.7)$$

in L^2 . The factor a_K^k is the probability that each of the k roots lies in a component with at least K vertices.

STEP 2. *Comparison with the giant component.* On the event $\{N_n \geq \max(k, K)\}$, every k -tuple of distinct vertices of gc satisfies $\prod_a \chi_{K,a}([G_n, v]) = 1$. A k -tuple of distinct vertices with $\prod_a \chi_{K,a}([G_n, v]) = 1$ that is not contained in gc has an entry outside gc whose component has at least K vertices. There are at most $kZ_{n,K}n^{k-1}$ such k -tuples, and on this event we have

$$\left| (N_n)_k S_n^{\text{gc}}(F) - (n)_k S_n(\Phi_K) \right| \leq k\|F\|_\infty Z_{n,K} n^{k-1}.$$

Dividing by $(N_n)_k$, we find

$$\left| S_n^{\text{gc}}(F) - \frac{(n)_k}{(N_n)_k} S_n(\Phi_K) \right| \leq k\|F\|_\infty \frac{Z_{n,K}}{n} \cdot \frac{n^k}{(N_n)_k}. \quad (17.8)$$

STEP 3. *The limit $n \rightarrow \infty$ for fixed K .* By (17.4), we have $(N_n)_k/n^k \rightarrow \varrho_p^k$ and $(N_n)_k/(n)_k \rightarrow \varrho_p^k$ in probability. Combining this with (17.7), (17.8) and Lemma 17.11, we find that for every $\eta > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(|S_n^{\text{gc}}(F) - L_K| > \varepsilon_K + \eta\right) = 0, \quad (17.9)$$

where

$$L_K = \varrho_p^{-k} \mathbb{E}\left[F(\mathfrak{F}_k) \prod_{a=1}^k \mathbf{1}\{|T_a| \geq K\}\right], \quad \varepsilon_K = k\|F\|_\infty \varrho_p^{-k} (a_K - \varrho_p).$$

In other words, after normalization by the size of the giant component, the average over gc is close to $(a_K/\varrho_p)^k \mathbb{E}[F(\mathfrak{F}_k) \mid |T_a| \geq K \forall a]$.

STEP 4. *The limit $K \rightarrow \infty$.* As $K \rightarrow \infty$, the indicator $\prod_a \mathbf{1}\{|T_a| \geq K\}$ decreases to $\prod_a \mathbf{1}_{S_a}$, where S_a is the survival event of T_a . Since F is bounded, by dominated convergence we have

$$L_K \longrightarrow \varrho_p^{-k} \mathbb{E}\left[F(\mathfrak{F}_k) \prod_{a=1}^k \mathbf{1}_{S_a}\right] = \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty).$$

In the last equality, we used that the trees are independent, so that $\mathbb{P}(\cap_a S_a) = \varrho_p^k$ and, conditionally on $\cap_a S_a$, the trees are independent with law $\mathbb{P}^\infty$. Further, $\varepsilon_K \rightarrow 0$ since $a_K \rightarrow \varrho_p$. Given $\eta > 0$, we choose K such that $|L_K - \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty)| + \varepsilon_K < \eta$. Then (17.9) implies that $\mathbb{P}(|S_n^{\text{gc}}(F) - \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty)| > 2\eta) \rightarrow 0$. Since $S_n^{\text{gc}}(F)$ does not depend on K , this completes the proof. $\square$

We now state the main result of this part. For an oriented edge (u, w) of G_n , let $[G_n, u \rightarrow w] \in \mathcal{G}_2$ denote the component of u , rooted at (u, w) . Let $\mathfrak{E}$ denote the following random element of $\mathcal{G}_2$: take two independent marked PGW(p) trees T^- and T^+ with roots o^- and o^+ , join o^- and o^+ by an edge with mark $h_{o^- o^+} = w/\sqrt{p}$, where w is an independent standard complex Gaussian, and root the resulting tree at (o^-, o^+) . Let $\vec{E}(\text{gc})$ be the set of oriented edges of gc.

Proposition 17.13. Fix $p > 1$ and $k \geq 1$. On the event $\{N_n \geq k\}$, conditionally on the marked graph, let $\mathbf{V} = (V_1, \dots, V_k)$ be a uniformly chosen k -tuple of distinct vertices of gc.

- (i) (Local convergence.) For every admissible $F: \mathcal{G}_k \rightarrow \mathbb{C}$, in particular for every bounded continuous F ,

$$\mathbb{E}[F([G_n, \mathbf{V}]) \mid G_n, (w_{ij})] = S_n^{\text{gc}}(F) \longrightarrow \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty)$$

in probability. In particular, $[G_n, \mathbf{V}] \rightarrow \mathfrak{F}_k^\infty$ in distribution in $\mathcal{G}_k$, and $\text{dist}(V_a, V_b) \rightarrow \infty$ in probability for $a \neq b$. The same holds if $V_1, \dots, V_k$ are independent and uniform on gc.

- (ii) (Oriented edges.) We have

$$\frac{|\vec{E}(\text{gc})|}{n} \longrightarrow p(1 - (1 - q_p)^2) = pq_p(2 - q_p)$$

in probability. Further, for every bounded continuous $\Psi: \mathcal{G}_2 \rightarrow \mathbb{C}$,

$$\frac{1}{|\vec{E}(\text{gc})|} \sum_{(u,w) \in \vec{E}(\text{gc})} \Psi([G_n, u \rightarrow w]) \longrightarrow \mathbb{E}[\Psi(\mathfrak{E}) \mid T^- \text{ or } T^+ \text{ is infinite}]$$

in probability. In particular, a uniformly chosen oriented edge of gc converges in distribution to $\mathfrak{E}$ conditioned on the event that at least one of T^- and T^+ is infinite.

- (iii) (Resolvents.) For all $z_1, \dots, z_m \in \mathbb{C} \setminus \mathbb{R}$, the array $(\langle \delta_{V_a}, (H_n^{\text{gc}} - z_l)^{-1} \delta_{V_b} \rangle)_{a,b \leq k, l \leq m}$ converges in distribution to $(\delta_{ab} G_{00}^{(a)}(z_l))_{a,b \leq k, l \leq m}$, where $G_{00}^{(1)}, \dots, G_{00}^{(k)}$ are the root resolvents of k independent trees with law $\mathbb{P}^\infty$. In particular, the entries with $a \neq b$ converge to zero in probability. Further, for every bounded continuous $\phi: \mathbb{C}^m \rightarrow \mathbb{C}$,

$$\frac{1}{N_n} \sum_{v \in \text{gc}} \phi(\langle \delta_v, (H_n^{\text{gc}} - z_l)^{-1} \delta_v \rangle_{l \leq m}) \longrightarrow \mathbb{E}^\infty \phi((G_{00}(z_l))_{l \leq m})$$

in probability.

- (iv) (Empirical spectral measure.) The empirical spectral measure ν_n^{gc} converges weakly in probability to ν_p^∞ .
- (v) (Local spectral masses.) Let $I_1, \dots, I_m$ be intervals whose endpoints c satisfy $\nu_p(\{c\}) = 0$ (by Proposition 17.14, this holds for every $c \neq 0$). Then the array $(\mu_{n, V_a}(I_l))_{a \leq k, l \leq m}$ converges in distribution to $(\mu^{(a)}(I_l))_{a \leq k, l \leq m}$, where $\mu^{(1)}, \dots, \mu^{(k)}$ are the root spectral measures of k independent trees with law $\mathbb{P}^\infty$. Further, for every bounded continuous $\phi: [0, 1]^m \rightarrow \mathbb{C}$,

$$\frac{1}{N_n} \sum_{v \in \text{gc}} \phi((\mu_{n, v}(I_l))_{l \leq m}) \longrightarrow \mathbb{E}^\infty \phi((\mu_o(I_l))_{l \leq m})$$

in probability, and the analogous statement holds for averages over ordered pairs of distinct vertices of gc, with limit $\mathbb{E}^{\infty, \otimes 2}$.

Proof of Proposition 17.13. Proof of (i). The identity $\mathbb{E}[F([G_n, \mathbf{V}]) \mid G_n, (w_{ij})] = S_n^{\text{gc}}(F)$ holds by the definition of $\mathbf{V}$, and the convergence is Lemma 17.12. For bounded continuous F , taking expectations and using the bounded convergence theorem gives $\mathbb{E}F([G_n, \mathbf{V}]) \rightarrow \mathbb{E}^{\infty, \otimes k} F(\mathfrak{F}_k^\infty)$, which is the convergence in distribution. For $a \neq b$ and $R \geq 0$, the function $x \mapsto \mathbf{1}\{\text{dist}(o_a, o_b) \leq R\}$ is continuous and vanishes at $\mathfrak{F}_k^\infty$, whose roots lie in distinct components; hence $\mathbb{P}(\text{dist}(V_a, V_b) \leq R) \rightarrow 0$. If $V_1, \dots, V_k$ are independent and uniform on gc, then conditionally on the marked graph they are distinct with probability at least $1 - k^2/N_n$, and conditionally on being distinct they form a uniform k -tuple of distinct vertices of gc. Then their conditional expectation of $F([G_n, \mathbf{V}])$ differs from $S_n^{\text{gc}}(F)$ by at most $2k^2\|F\|_\infty/N_n$, which tends to zero in probability.

Proof of (ii). For $x \in \mathcal{G}_1$ with root o and a neighbor w of o , let $x^{o \rightarrow w} \in \mathcal{G}_2$ denote x rooted at (o, w) . Fix a bounded continuous $\Psi: \mathcal{G}_2 \rightarrow \mathbb{C}$, and define

$$F(x) = \sum_{w \sim o} \Psi(x^{o \rightarrow w}), \quad F_L(x) = \mathbf{1}\{\deg(o) \leq L\} F(x), \quad L \geq 1.$$

Since every neighbor of a vertex of gc lies in gc, we have

$$\sum_{(u,w) \in \tilde{E}(\text{gc})} \Psi([G_n, u \rightarrow w]) = \sum_{u \in \text{gc}} F([G_n, u]). \quad (17.10)$$

We first show that F_L is continuous. Let $x_m \rightarrow x$ in $\mathcal{G}_1$, and let $R \geq 1$. For m large, there exists an isomorphism φ_m of $B_{R+1}(x)$ onto $B_{R+1}(x_m)$ with mark differences at most $(R+1)^{-1}$. In particular, the roots of x and x_m have the same degree, and φ_m maps the neighbors of o bijectively onto those of the root o_m of x_m . For a neighbor w of o , the vertices within distance R of $\{o, w\}$ lie within distance $R+1$ of o . Then φ_m restricts to an isomorphism of $B_R(x^{o \rightarrow w})$ onto $B_R(x_m^{o_m \rightarrow \varphi_m(w)})$ with mark differences at most R^{-1} . This gives $d_{\text{loc}}(x_m^{o_m \rightarrow \varphi_m(w)}, x^{o \rightarrow w}) \leq (1+R)^{-1}$ and

$$|F_L(x_m) - F_L(x)| \leq \sum_{w \sim o} \sup \left\{ |\Psi(y) - \Psi(x^{o \rightarrow w})| : d_{\text{loc}}(y, x^{o \rightarrow w}) \leq (1+R)^{-1} \right\}$$

for m large. Since Ψ is continuous at the finitely many points $x^{o \rightarrow w}$, the right side tends to zero as $R \rightarrow \infty$. We conclude that F_L is continuous; it is bounded by $L\|\Psi\|_\infty$.

By part (i) with $k = 1$ and (17.4), we have

$$\frac{1}{n} \sum_{u \in \text{gc}} F_L([G_n, u]) = \frac{N_n}{n} S_n^{\text{gc}}(F_L) \longrightarrow q_p \mathbb{E}^\infty F_L(\mathcal{T}) = \mathbb{E}[F_L(\mathcal{T}) \mathbf{1}_S]$$

in probability, where we regard the rooted tree $\mathcal{T}$ as an element of $\mathcal{G}_1$. Since $|F - F_L|(x) \leq \|\Psi\|_\infty \deg(o) \mathbf{1}\{\deg(o) > L\} \leq \|\Psi\|_\infty \deg(o)^2/L$, by Lemma 17.2 we have

$$\mathbb{E} \left[\frac{1}{n} \sum_{u \in [n]} |F - F_L|([G_n, u]) \right] \leq \frac{\|\Psi\|_\infty}{L} \mathbb{E}[\deg(V_1)^2] \leq \frac{(p + p^2)\|\Psi\|_\infty}{L},$$

where V_1 is uniform on $[n]$. In the same way, since the number N of children of o satisfies $\mathbb{E}N^2 = p + p^2$, we have $\mathbb{E}|F - F_L|(\mathcal{T}) \leq (p + p^2)\|\Psi\|_\infty/L$. Letting $L \rightarrow \infty$ and using Markov's inequality for the first error term, we conclude from (17.10) that

$$\frac{1}{n} \sum_{(u,w) \in \vec{E}(\text{gc})} \Psi([G_n, u \rightarrow w]) \longrightarrow \mathbb{E}[F(\mathcal{T})\mathbf{1}_S] \quad (17.11)$$

in probability.

We now compute the right side of (17.11) using the Mecke formula. The children j of o , together with the marks h_{oj} and the marked forward subtrees $\mathcal{T}_j$, form a Poisson process $\Pi = \sum_j \delta_{M_j}$, $M_j = (h_{oj}, \mathcal{T}_j)$, whose intensity is p times the product of the law of $w/\sqrt{p}$ and the law of the marked PGW(p) tree. For each j , the rooted tree $\mathcal{T}^{o \rightarrow j}$ and the indicator $\mathbf{1}_S$ are measurable functions of $(M_j, \Pi - \delta_{M_j})$: the tree $\mathcal{T}^{o \rightarrow j}$ is obtained by joining the root of the tree with root o and child process $\Pi - \delta_{M_j}$ to the root of $\mathcal{T}_j$ by an edge with mark h_{oj} , and $\mathcal{T}$ is infinite if and only if one of these two trees is infinite. By the Mecke formula of Lemma 4.4, applied to the positive and negative parts of the real and imaginary parts of Ψ (each side is finite, since $|F(\mathcal{T})| \leq \|\Psi\|_\infty N$ and $\mathbb{E}N = p$), we have

$$\mathbb{E}[F(\mathcal{T})\mathbf{1}_S] = \mathbb{E} \sum_j \Psi(\mathcal{T}^{o \rightarrow j})\mathbf{1}_S = p \mathbb{E} \left[\Psi(\mathfrak{C})\mathbf{1}_{\{T^- \text{ or } T^+ \text{ is infinite}\}} \right].$$

Indeed, on the right side of the Mecke formula the residual process is an independent copy of Π , which means that the tree with root o^- built from it is a marked PGW(p) tree T^- , and the added point (h, T^+) is independent of it with the law of $(w/\sqrt{p}, \text{marked PGW}(p) \text{ tree})$. Taking $\Psi \equiv 1$ in (17.11), we obtain $|\vec{E}(\text{gc})|/n \rightarrow p \mathbb{P}(T^- \text{ or } T^+ \text{ is infinite}) = p(1 - (1 - q_p)^2)$, which is positive. Dividing (17.11) by this limit proves the second claim of (ii). The last claim follows by taking expectations, since the left side of the second display in (ii) is the conditional expectation of Ψ at a uniformly chosen oriented edge of gc, given the marked graph.

Proof of (iii). By Lemma 17.8(ii), the function $x \mapsto \phi((G_{ab}(x, z_l))_{a,b,l})$ is admissible for every bounded continuous ϕ . By the paragraph after Definition 17.4, its value at $[G_n, \mathbf{V}]$ is ϕ applied to the array in (iii). Since $\mathbb{P}^{\infty, \otimes k}$ is absolutely continuous with respect to the law of k independent unconditioned trees, $\mathfrak{F}_k^\infty \in \mathcal{G}_k^{\text{esa}}$ almost surely, and Lemma 17.6(ii) identifies $G_{ab}(\mathfrak{F}_k^\infty, z_l)$ with $\delta_{ab} G_{oo}^{(a)}(z_l)$. The convergence in distribution then follows from part (i) and the bounded convergence theorem, and the convergence of the average over gc follows from part (i) with $k = 1$.

Proof of (iv). Fix $z \in \mathbb{H}$. By (17.2), the Stieltjes transform of ν_n^{gc} is $N_n^{-1} \sum_{v \in \text{gc}} \langle \delta_v, (H_n^{\text{gc}} - z)^{-1} \delta_v \rangle$. We apply part (iii) with $m = 1$ and $\phi(\zeta) = \zeta \min\{1, (|\text{Im } z| |\zeta|)^{-1}\}$, which is bounded, continuous, and equal to ζ for $|\zeta| \leq |\text{Im } z|^{-1}$. Since all resolvent entries involved are bounded by $|\text{Im } z|^{-1}$, we find that the Stieltjes transform of ν_n^{gc} at z converges

in probability to $\mathbb{E}^\infty G_{00}(z)$, the Stieltjes transform of ν_p^∞ . Applying Lemma 17.3(ii) with $D = \{z \in \mathbb{H} : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$ completes the proof.

Proof of (v). By Lemma 17.8(iii), the function $x \mapsto \phi((\mu_{x,a}(I_l))_{a,l})$ is admissible for every bounded continuous ϕ , and by the paragraph after Definition 17.4 its value at $[G_n, \mathbf{V}]$ is $\phi((\mu_{n,V_a}(I_l))_{a,l})$. As in the proof of (iii), Lemma 17.6(ii) identifies $\mu_{\mathfrak{F}_k^\infty, a}$ with the root spectral measure of the a -th tree of $\mathfrak{F}_k^\infty$, almost surely. The claims follow from part (i) with $k, 1$ and 2 roots. $\square$

17.4 The density of states

In this section, we prove the following two-sided bound for the averaged root spectral measure ν_p^∞ of (17.3).

Proposition 17.14. *Fix $p > 1$ and a compact set $J \subset \mathbb{R} \setminus \{0\}$. There exist constants $0 < c_{p,J} \leq C_{p,J} < \infty$, depending only on p and J , such that*

$$c_{p,J}|I| \leq \nu_p^\infty(I) \leq C_{p,J}|I| \quad \text{and} \quad \nu_p(I) \leq C_{p,J}|I| \quad (17.12)$$

for every interval $I \subset J$. The restriction of ν_p^∞ to J is absolutely continuous, with a density ρ_p^∞ satisfying $c_{p,J} \leq \rho_p^\infty \leq C_{p,J}$ Lebesgue-almost everywhere on J . Further, for every $E \in \mathbb{R} \setminus \{0\}$, we have $\nu_p(\{E\}) = \nu_p^\infty(\{E\}) = 0$; in particular, $\mu_o(\{E\}) = 0$ almost surely under both $\mathbb{P}$ and $\mathbb{P}^\infty$.

We write ρ_p^∞ for a density of ν_p^∞ on $\mathbb{R} \setminus \{0\}$; by Proposition 17.14, it is unique up to null sets and bounded above and below by positive constants on every compact subset of $\mathbb{R} \setminus \{0\}$.

The upper bound

The upper bound uses an auxiliary bipartite random graph, model (b) of Section 17.2. Its local limit is again the marked PGW(p) tree, and its squared operator is a sum of rank-one projections with independent Gamma-distributed weights. We begin with a spectral averaging estimate for rank-one perturbations of matrices; compare [123] and [13].

Lemma 17.15. *Let L be a Hermitian $m \times m$ matrix and $v \in \mathbb{C}^m$ a unit vector. For $u \in \mathbb{R}$, let μ_u be the spectral measure of v for $L + u|v\rangle\langle v|$. Then for every bounded Borel function $g: \mathbb{R} \rightarrow [0, \infty)$ and every interval $B \subset \mathbb{R}$,*

$$\int_{\mathbb{R}} g(u) \mu_u(B) du \leq \|g\|_\infty |B|.$$

Proof. For $\zeta \in \mathbb{H}$, let $F(\zeta) = \langle v, (L - \zeta)^{-1}v \rangle$ and $F_u(\zeta) = \langle v, (L + u|v\rangle\langle v| - \zeta)^{-1}v \rangle$. Then $\operatorname{Im} F(\zeta) = \operatorname{Im} \zeta \|(L - \zeta)^{-1}v\|^2 > 0$, and the resolvent identity gives the Aronszajn–Krein formula $F_u = F/(1 + uF)$ [123]. Writing $F = \alpha + i\beta$ with $\beta > 0$, we have $\operatorname{Im} F_u = \beta/|1 + uF|^2$ and

$$\int_{\mathbb{R}} \operatorname{Im} F_u(\zeta) \, du = \int_{\mathbb{R}} \frac{\beta \, du}{(\alpha^2 + \beta^2)u^2 + 2\alpha u + 1} = \frac{2\pi\beta}{\sqrt{4(\alpha^2 + \beta^2) - 4\alpha^2}} = \pi.$$

Let B be an open interval and let f be continuous with compact support in B and $0 \leq f \leq 1$. Since $\pi^{-1} \operatorname{Im} F_u(E + i\varepsilon)$ is the convolution of μ_u with the Poisson kernel $P_\varepsilon(x) = \varepsilon/(\pi(x^2 + \varepsilon^2))$, and $P_\varepsilon * f \rightarrow f$ uniformly, we have $\int f \, d\mu_u = \lim_{\varepsilon \downarrow 0} \int f(E) \pi^{-1} \operatorname{Im} F_u(E + i\varepsilon) \, dE$ for every u . By Fatou's lemma and Tonelli's theorem,

$$\begin{aligned} \int g(u) \int f \, d\mu_u \, du &\leq \liminf_{\varepsilon \downarrow 0} \int f(E) \frac{1}{\pi} \int g(u) \operatorname{Im} F_u(E + i\varepsilon) \, du \, dE \\ &\leq \|g\|_\infty \int f(E) \, dE \leq \|g\|_\infty |B|. \end{aligned}$$

Letting f increase to $\mathbf{1}_B$ and using monotone convergence proves the claim for open intervals. A general interval B is contained in open intervals of length $|B| + \eta$ for every $\eta > 0$. $\square$

Let H_n^{bip} be the $2n \times 2n$ matrix of model (b), with the vertices $[n] \times \{0\}$ listed first. Then

$$H_n^{\text{bip}} = \begin{pmatrix} 0 & A_n^* \\ A_n & 0 \end{pmatrix},$$

where A_n is the $n \times n$ matrix with $A_n(j, i) = h_{(j,1),(i,0)}$ if $\{(i,0), (j,1)\}$ is an edge and $A_n(j, i) = 0$ otherwise. For $y \in [n]$, let $a_y = A_n^* \delta_y \in \mathbb{C}^n$ be the conjugate of the y -th row of A_n , let d_y be the degree of $(y, 1)$, and let $U_y = \|a_y\|^2$. If $d_y \geq 1$, set $v_y = a_y / \sqrt{U_y}$. Then

$$K_n = A_n^* A_n = \sum_{y \in [n]: d_y \geq 1} U_y |v_y\rangle \langle v_y|. \quad (17.13)$$

Lemma 17.16. *Let $0 < \alpha_- < \alpha_+$. For every integer $n > p$ and every interval $I \subset [\alpha_-, \alpha_+]$,*

$$\mathbb{E} \operatorname{Tr} \mathbf{1}_I(H_n^{\text{bip}}) \leq \frac{2\sqrt{p}\alpha_+}{\sqrt{2\pi}\alpha_-^2} n|I|.$$

Proof. We first record the conditional law of the rows. Conditionally on the graph, the rows of A_n are independent, since they involve disjoint sets of marks. If $d_y = d \geq 1$, the nonzero entries of a_y are d independent copies of $w/\sqrt{p}$ (up to conjugation, which does not change their law). The law of this vector in $\mathbb{C}^d$ has density $(p/\pi)^d e^{-p\|\zeta\|^2}$, which is invariant under unitary maps. Then its direction v_y is uniform on the unit sphere

of the coordinate subspace, independent of U_y , and $U_y = p^{-1} \sum_{j=1}^d |w_j|^2$ is a sum of d independent exponential variables with rate p , with density

$$f_{d,p}(u) = \frac{p^d}{\Gamma(d)} u^{d-1} e^{-pu} \mathbf{1}_{\{u>0\}}.$$

We conclude that, conditionally on the graph and on all directions $(v_y)_{y:d_y \geq 1}$, the squared row norms $(U_y)_{y:d_y \geq 1}$ are independent, and U_y has density $f_{d_y,p}$. This is the finite-volume analogue of Proposition 3.23.

Next, we reduce to the square block. Let $E > 0$. If $H_n^{\text{bip}}(\varphi, \chi) = E(\varphi, \chi)$, then $A_n^* \chi = E\varphi$ and $A_n \varphi = E\chi$, which gives $K_n \varphi = E^2 \varphi$; further, $\varphi \neq 0$ unless $\chi = 0$. Conversely, if $K_n \varphi = E^2 \varphi$, then $(\varphi, E^{-1} A_n \varphi)$ is an eigenvector of H_n^{bip} with eigenvalue E . This shows that $(\varphi, \chi) \mapsto \varphi$ is a linear bijection from $\ker(H_n^{\text{bip}} - E)$ onto $\ker(K_n - E^2)$. Summing over eigenvalues, we obtain $\text{Tr} \mathbf{1}_I(H_n^{\text{bip}}) = \text{Tr} \mathbf{1}_{I^2}(K_n)$ for every interval $I \subset (0, \infty)$, where $I^2 = \{\lambda^2 : \lambda \in I\}$.

Set $B = I^2 \subset [\alpha_-^2, \alpha_+^2]$. Since $K_n \mathbf{1}_B(K_n) \geq \alpha_-^2 \mathbf{1}_B(K_n)$, using (17.13) we obtain

$$\text{Tr} \mathbf{1}_B(K_n) \leq \alpha_-^{-2} \text{Tr}(K_n \mathbf{1}_B(K_n)) = \alpha_-^{-2} \sum_{y:d_y \geq 1} U_y \langle v_y, \mathbf{1}_B(K_n) v_y \rangle.$$

Fix y with $d_y = d \geq 1$, and let $L_y = K_n - U_y |v_y\rangle \langle v_y|$. Let $\mathcal{F}_y$ be the σ -algebra generated by the graph, all directions, and $(U_{y'})_{y' \neq y}$. Conditionally on $\mathcal{F}_y$, the matrix L_y and the vector v_y are fixed, and U_y has density $f_{d,p}$. By Lemma 17.15 with $g(u) = u f_{d,p}(u)$, we have

$$\mathbb{E}[U_y \langle v_y, \mathbf{1}_B(K_n) v_y \rangle \mid \mathcal{F}_y] = \int_0^\infty u f_{d,p}(u) \langle v_y, \mathbf{1}_B(L_y + u |v_y\rangle \langle v_y|) v_y \rangle du \leq c_d |B|,$$

where

$$c_d = \sup_{u>0} u f_{d,p}(u) = \frac{d^d e^{-d}}{\Gamma(d)} \leq \sqrt{\frac{d}{2\pi}}, \quad d \geq 1.$$

The supremum is attained at $u = d/p$, and the inequality follows from Stirling's lower bound $\Gamma(d+1) \geq \sqrt{2\pi d} (d/e)^d$. We set $c_0 = 0$; with this convention, the rows with $d_y = 0$, which have $U_y = 0$ and do not appear in (17.13), contribute $c_{d_y} = 0$. Taking expectations and using that d_y is binomial with parameters n and p/n , we obtain

$$\mathbb{E} \text{Tr} \mathbf{1}_B(K_n) \leq \alpha_-^{-2} |B| \sum_{y=1}^n \mathbb{E} c_{d_y} \leq \frac{|B|}{\sqrt{2\pi} \alpha_-^2} \sum_{y=1}^n \mathbb{E} \sqrt{d_y} \leq \frac{\sqrt{p} |B|}{\sqrt{2\pi} \alpha_-^2} n,$$

where in the last inequality we used Jensen's inequality and $\mathbb{E} d_y = p$. Finally, if I has endpoints $\alpha \leq \beta$, then $|B| = (\beta + \alpha)(\beta - \alpha) \leq 2\alpha_+ |I|$. $\square$

Lemma 17.17. *Let $J = [E_-, E_+] \subset (0, \infty)$ with $E_- > 0$, and set $C_J = 4(2\pi)^{-1/2} \sqrt{p} (E_+ + 1) E_-^{-2}$. For every interval $I \subset J$, we have $\nu_p(I) \leq C_J |I|$ and $\nu_p^\infty(I) \leq \varrho_p^{-1} C_J |I|$.*

Proof. Let V be uniform on the $2n$ vertices of model (b), independent of the marked graph, and let $m_n = \mathbb{E}[(2n)^{-1} \sum_v \mu_{n,v}^{\text{bip}}]$, where $\mu_{n,v}^{\text{bip}}$ is the spectral measure of δ_v for H_n^{bip} . Then m_n is a deterministic probability measure with $m_n(I) = (2n)^{-1} \mathbb{E} \text{Tr} \mathbf{1}_I(H_n^{\text{bip}})$, and its Stieltjes transform at $z \in \mathbb{H}$ is $\mathbb{E} G_{11}([G_n, V], z)$, in the notation of Definition 17.4. By Lemma 17.8(ii) (applied with a bounded continuous ϕ that equals the identity on the disc of radius $(\text{Im } z)^{-1}$) and Lemma 17.9 for model (b), this converges to $\mathbb{E} G_{11}(\mathfrak{F}_1, z) = \mathbb{E} G_{00}(z)$, which is the Stieltjes transform of ν_p . By Lemma 17.3(i), $m_n \rightarrow \nu_p$ weakly.

Let $J' = [E_-/2, E_+ + 1]$. For an open interval $O \subset J'$, Lemma 17.16 with $\alpha_- = E_-/2$ and $\alpha_+ = E_+ + 1$ implies that $m_n(O) \leq C_J |O|$, and by the portmanteau theorem, $\nu_p(O) \leq \liminf_n m_n(O) \leq C_J |O|$. If $I \subset J$ has endpoints $\alpha \leq \beta$ and $0 < \eta \leq \min\{1, E_-/2\}$, then $I \subset (\alpha - \eta, \beta + \eta) \subset J'$ and $\nu_p(I) \leq C_J(|I| + 2\eta)$. Letting $\eta \downarrow 0$ proves the first bound. By Lemma 4.5, $\mathbb{E}^\infty Z = \varrho_p^{-1} \mathbb{E}[Z \mathbf{1}_S] \leq \varrho_p^{-1} \mathbb{E} Z$ for every nonnegative random variable Z ; with $Z = \mu_o(I)$, this gives the second bound. $\square$

The lower bound

For the lower bound, we work directly on the tree. We use the following consequence of the parity symmetry. By Proposition 3.21(i), almost surely, the spectral measure μ_o of δ_o for $\mathcal{H}$ is symmetric under $\lambda \mapsto -\lambda$. The same holds for the spectral measure of δ_v for the forward subtree operator $\mathcal{H}_v$, for every vertex v , since $V(\mathcal{T}_v)$ is a component of the finite removal that deletes the edge $\{v^-, v\}$. Let κ_o be the image of μ_o under $\lambda \mapsto \lambda^2$, and let $F_o = s_{\kappa_o}$ be its Stieltjes transform. Then for every interval $I \subset (0, \infty)$, since $I \cup (-I)$ is the preimage of $I^2 = \{\lambda^2 : \lambda \in I\}$ under $\lambda \mapsto \lambda^2$,

$$\mu_o(I) = \frac{1}{2}(\mu_o(I) + \mu_o(-I)) = \frac{1}{2} \kappa_o(I^2). \quad (17.14)$$

Let $Q = \{z \in \mathbb{C} : \text{Re } z > 0, \text{Im } z > 0\}$ be the open first quadrant, so that $z^2 \in \mathbb{H}$ for $z \in Q$. For $z \in Q$, we have

$$G_{00}(z) = \int \frac{(\lambda + z) \mu_o(d\lambda)}{\lambda^2 - z^2} = z \int \frac{\mu_o(d\lambda)}{\lambda^2 - z^2} = z F_o(z^2), \quad (17.15)$$

where the term $\int \lambda \mu_o(d\lambda) / (\lambda^2 - z^2)$ vanishes by symmetry (the integrand is bounded, since $|\lambda^2 - z^2| \geq \text{Im } z^2 > 0$ and $|\lambda| / |\lambda^2 - z^2| \rightarrow 0$ as $|\lambda| \rightarrow \infty$). By Proposition 3.23(v), κ_o is the spectral measure of δ_o for A^*A , and the formula (17.16) below describes the effect of a pendant leaf as a rank-one perturbation of A^*A . We do not need this interpretation: the proof below only uses the Schur formula of Proposition 3.16 and (17.15).

Lemma 17.18. *Let $J = [E_-, E_+] \subset (0, \infty)$ with $E_- > 0$. There exists $c_J > 0$, depending only on p and J , such that $\nu_p^\infty(I) \geq c_J |I|$ for every interval $I \subset J$.*

Proof. We find a configuration at the root, of positive probability, on which $F_o(\lambda + i\varepsilon)$ is an explicit function of the weight t of a leaf edge. Averaging over t produces a Poisson

kernel, which gives a lower bound on $\mathbb{E}^\infty \operatorname{Im} F_o(\lambda + i\varepsilon)$ uniformly in $\lambda \in [E_-^2, E_+^2]$ and small $\varepsilon > 0$. We then let $\varepsilon \downarrow 0$ and use (17.14).

STEP 1. *The configuration.* Let $\mathcal{E}$ be the event that the root o has exactly two children, one of which has no children while the forward tree of the other is infinite. On $\mathcal{E}$, we call these children ℓ and y , respectively, and set $t = |h_{o\ell}|^2$ and $t_0 = |h_{oy}|^2$. Conditionally on $\{N = 2\}$, the pairs $(h_{oj}, \mathcal{T}_j)$, $j = 1, 2$, consisting of the edge mark and the marked forward tree of the j -th child, are independent, with h_{oj} distributed as $w/\sqrt{p}$ and $\mathcal{T}_j$ a marked $\operatorname{PGW}(p)$ tree. The event $\mathcal{E} \cap \{N = 2\} = \mathcal{E}$ is the disjoint union of $\{|\mathcal{T}_1| = 1, |\mathcal{T}_2| = \infty\}$ and $\{|\mathcal{T}_2| = 1, |\mathcal{T}_1| = \infty\}$, so

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(N = 2) \cdot 2e^{-p}q_p = p^2e^{-2p}q_p.$$

Further, conditionally on $\mathcal{E}$, the variables t , t_0 and the marked forward tree $\mathcal{T}_y$ are independent; t and t_0 have density $pe^{-ps}\mathbf{1}_{\{s>0\}}$, since $|w|^2$ is exponential with mean one; and $\mathcal{T}_y$ has law $\mathbb{P}^\infty$. Since $\mathcal{E} \subset \mathcal{S}$, by Lemma 4.5 we have $\mathbb{E}^\infty[Z\mathbf{1}_\mathcal{E}] = q_p^{-1}\mathbb{E}[Z\mathbf{1}_\mathcal{E}]$ for nonnegative Z .

STEP 2. *The Schur formula on $\mathcal{E}$.* On $\mathcal{E}$, the forward tree of ℓ is a single vertex, whose operator is zero, and we have $\Gamma_\ell(z) = -1/z$. By Proposition 3.16, for $z \in Q$,

$$G_{oo}(z) = -\left(z + t_0\Gamma_y(z) - \frac{t}{z}\right)^{-1} = \frac{z}{t - q(z)}, \quad q(z) = z^2 + zt_0\Gamma_y(z).$$

Combining this with (17.15), we obtain

$$F_o(z^2) = \frac{1}{t - q(z)} \quad \text{on } \mathcal{E}, \text{ for } z \in Q. \quad (17.16)$$

We claim that $\operatorname{Im} q(z) \geq \operatorname{Im} z^2 > 0$ for $z \in Q$. Let m_y be the spectral measure of δ_y for $\mathcal{H}_y$, which is symmetric. As in (17.15), we have

$$z\Gamma_y(z) = \int \frac{z^2 m_y(d\lambda)}{\lambda^2 - z^2} = -1 + \int \frac{\lambda^2 m_y(d\lambda)}{\lambda^2 - z^2},$$

and $\operatorname{Im}[\lambda^2/(\lambda^2 - z^2)] = \lambda^2 \operatorname{Im} z^2 / |\lambda^2 - z^2|^2 \geq 0$. This implies that $\operatorname{Im}(zt_0\Gamma_y(z)) \geq 0$, which proves the claim. Note that $q(z)$ is a function of t_0 and of the marked tree $\mathcal{T}_y$ only.

STEP 3. *A Poisson kernel bound.* Let $\Lambda = [E_-^2, E_+^2]$ and $\delta = E_-^2/4$. For $q \in \mathbb{H}$, let $P_q(s) = \pi^{-1} \operatorname{Im} q / ((s - \operatorname{Re} q)^2 + (\operatorname{Im} q)^2)$ be the Poisson kernel. We claim that if $\operatorname{dist}(\operatorname{Re} q, \Lambda) \leq \delta$ and $0 < \operatorname{Im} q \leq \delta$, then

$$\int_0^\infty pe^{-ps}P_q(s) ds \geq c_*, \quad c_* = \frac{p}{2}e^{-p(E_+^2+2\delta)}. \quad (17.17)$$

Indeed, the interval $[\operatorname{Re} q - \operatorname{Im} q, \operatorname{Re} q + \operatorname{Im} q]$ is contained in $[E_-^2 - 2\delta, E_+^2 + 2\delta] \subset (0, E_+^2 + 2\delta]$, where $pe^{-ps} \geq pe^{-p(E_+^2 + 2\delta)}$, and $\int_{\operatorname{Re} q - \operatorname{Im} q}^{\operatorname{Re} q + \operatorname{Im} q} P_q(s) ds = \pi^{-1}(\arctan 1 - \arctan(-1)) = 1/2$.

STEP 4. *Choice of parameters.* We choose, in this order, ε_0 , M and τ . For $\lambda \in \Lambda$ and $\varepsilon > 0$, let $z(\lambda, \varepsilon) \in Q$ be the square root of $\lambda + i\varepsilon$ in the first quadrant. Then

$$E_- \leq \sqrt{\lambda} \leq \operatorname{Re} z \leq |z| = (\lambda^2 + \varepsilon^2)^{1/4} \leq \sqrt{\lambda + \varepsilon} \leq \sqrt{\lambda} + \frac{\varepsilon}{2E_-}, \quad \operatorname{Im} z = \frac{\varepsilon}{2\operatorname{Re} z} \leq \frac{\varepsilon}{2E_-},$$

where the first inequality for $\operatorname{Re} z$ uses $\operatorname{Re} z = ((|\lambda + i\varepsilon| + \lambda)/2)^{1/2}$. Set $\varepsilon_0 = \min\{1, 2E_-, \delta/2\}$ and $J_+ = [E_-, E_+ + 1]$. Then for all $\lambda \in \Lambda$ and $0 < \varepsilon \leq \varepsilon_0$, the point $z = z(\lambda, \varepsilon)$ satisfies

$$\operatorname{Re} z \in J_+, \quad 0 < \operatorname{Im} z \leq 1, \quad |z| \leq E_+ + 1. \quad (17.18)$$

By Lemma 5.17 with $r = 1/2$ and the compact interval J_+ , there exists $C_1 = C(p, J_+, 1/2)$ with $\mathbb{E}^\infty |\Gamma_o(z)|^{1/2} \leq C_1$ for all z with $\operatorname{Re} z \in J_+$ and $0 < \operatorname{Im} z \leq 1$. (This bound follows from the cavity equation of Proposition 3.16 and Lemma 4.18, since the constant term z has modulus at least E_- .) Set $M = 16C_1^2$. By Markov's inequality, $\mathbb{P}^\infty(|\Gamma_o(z)| > M) \leq C_1 M^{-1/2} = 1/4$ for all such z . Finally, set $\tau = \delta/(2(E_+ + 1)M)$.

STEP 5. *Lower bound for the averaged transform.* Let $\lambda \in \Lambda$, $0 < \varepsilon \leq \varepsilon_0$, $\zeta = \lambda + i\varepsilon$ and $z = z(\lambda, \varepsilon)$, so that $z^2 = \zeta$. On the event $\mathcal{E} \cap \{t_0 \leq \tau\} \cap \{|\Gamma_y(z)| \leq M\}$, (17.18) gives

$$|q(z) - \lambda| \leq |z^2 - \lambda| + |z| t_0 |\Gamma_y(z)| \leq \varepsilon + (E_+ + 1)\tau M \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

so that $\operatorname{dist}(\operatorname{Re} q(z), \Lambda) \leq \delta$ and, by Step 2, $0 < \operatorname{Im} q(z) \leq \delta$. By (17.16), on $\mathcal{E}$ we have $\pi^{-1} \operatorname{Im} F_o(\zeta) = \pi^{-1} \operatorname{Im} q / |t - q|^2 = P_{q(z)}(t)$. Since $\operatorname{Im} F_o \geq 0$ everywhere, Step 1 gives

$$\frac{1}{\pi} \mathbb{E}^\infty \operatorname{Im} F_o(\zeta) \geq \varrho_p^{-1} \mathbb{E} \left[\mathbf{1}_{\mathcal{E}} \mathbf{1}_{\{t_0 \leq \tau\}} \mathbf{1}_{\{|\Gamma_y(z)| \leq M\}} P_{q(z)}(t) \right].$$

Conditionally on $\mathcal{E}$, the variable t is independent of $(t_0, \mathcal{T}_y)$ and has density pe^{-ps} , while $q(z)$ is a function of $(t_0, \mathcal{T}_y)$. Integrating first over t and using (17.17), then using the independence of t_0 and $\mathcal{T}_y$ given $\mathcal{E}$, we obtain

$$\begin{aligned} \frac{1}{\pi} \mathbb{E}^\infty \operatorname{Im} F_o(\lambda + i\varepsilon) &\geq \varrho_p^{-1} c_* \mathbb{P}(\mathcal{E}) \mathbb{P}(t_0 \leq \tau \mid \mathcal{E}) \mathbb{P}^\infty(|\Gamma_o(z)| \leq M) \\ &\geq \frac{3}{4} c_* p^2 e^{-2p} (1 - e^{-p\tau}) = c_1. \end{aligned} \quad (17.19)$$

In the second inequality, we used that $\Gamma_y(z)$ is the root resolvent of $\mathcal{T}_y$, which has law $\mathbb{P}^\infty$ given $\mathcal{E}$, and Step 4. The constant $c_1 > 0$ depends only on p and J , and (17.19) holds for all $\lambda \in \Lambda$ and $0 < \varepsilon \leq \varepsilon_0$.

STEP 6. *From the transform to the measure.* Let $\bar{\kappa} = \mathbb{E}^\infty \kappa_o$, and let $f \geq 0$ be continuous with compact support in (E_-^2, E_+^2) . For every sample, $\int f(\lambda) \pi^{-1} \operatorname{Im} F_o(\lambda + i\varepsilon) d\lambda = \int (P_{i\varepsilon} * f) d\kappa_o$, which converges to $\int f d\kappa_o$ as $\varepsilon \downarrow 0$ and is bounded by $\|f\|_\infty$. By dominated convergence, Tonelli's theorem and (17.19), we have

$$\int f d\bar{\kappa} = \lim_{\varepsilon \downarrow 0} \int f(\lambda) \frac{1}{\pi} \mathbb{E}^\infty \operatorname{Im} F_o(\lambda + i\varepsilon) d\lambda \geq c_1 \int f(\lambda) d\lambda.$$

Approximating the indicator of an open interval $O \subset (E_-^2, E_+^2)$ from below by such functions, we obtain $\bar{\kappa}(O) \geq c_1 |O|$.

STEP 7. *Conclusion.* Let $I \subset J$ be an interval with endpoints $\alpha < \beta$. Then $I^2 \supset (\alpha^2, \beta^2)$, and (17.14) and Step 6 give

$$\nu_p^\infty(I) = \frac{1}{2} \mathbb{E}^\infty \kappa_o(I^2) \geq \frac{1}{2} \bar{\kappa}((\alpha^2, \beta^2)) \geq \frac{c_1}{2} (\beta + \alpha)(\beta - \alpha) \geq c_1 E_- |I|.$$

This proves the lemma with $c_J = c_1 E_-$; the case of a degenerate interval is trivial. $\square$

Proof of Proposition 17.14. Since J is compact and does not contain 0, there exist $0 < E_- \leq E_+$ such that $J \subset [-E_+, -E_-] \cup [E_-, E_+]$, and every interval $I \subset J$ lies in one of these two intervals. For $I \subset [E_-, E_+]$, the bounds (17.12) follow from Lemmas 17.17 and 17.18, with $C_{p,J} = c_p^{-1} C_{[E_-, E_+]}$ and $c_{p,J} = \min\{c_{[E_-, E_+]}, C_{p,J}\}$. Since μ_o is almost surely symmetric (see the beginning of this subsection), ν_p and ν_p^∞ are symmetric, which gives the same bounds for $I \subset [-E_+, -E_-]$.

The upper bound implies that $\nu_p^\infty(B) \leq C_{p,J} |B|$ for every Borel set $B \subset [E_-, E_+]$: for $\eta > 0$, the set B is covered by countably many intervals of total length at most $|B| + \eta$, and their intersections with $[E_-, E_+]$ are intervals. It follows that the restriction of ν_p^∞ to $[E_-, E_+]$ is absolutely continuous with a density at most $C_{p,J}$ almost everywhere. By the Lebesgue differentiation theorem, for almost every $E \in (E_-, E_+)$ the density at E is the limit of $\nu_p^\infty((E - h, E + h)) / (2h)$ as $h \downarrow 0$, which is at least $c_{p,J}$ by the lower bound. The same holds on $[-E_+, -E_-]$.

Finally, let $E \neq 0$, and apply (17.12) to a compact interval around E not containing 0 and to $I = \{E\}$. This gives $\nu_p(\{E\}) = \nu_p^\infty(\{E\}) = 0$, that is, $\mathbb{E} \mu_o(\{E\}) = \mathbb{E}^\infty \mu_o(\{E\}) = 0$. $\square$

17.5 Eigenvalue counts and the transfer of phases

Eigenvalue counts

Lemma 17.19. *Let m_n be random probability measures on $\mathbb{R}$ that converge weakly in probability to a deterministic probability measure m .*

- (i) *For every interval I whose endpoints are not atoms of m , we have $m_n(I) \rightarrow m(I)$ in probability.*

- (ii) If m has no atoms in a compact interval $[a, b]$, then $\sup_I |m_n(I) - m(I)| \rightarrow 0$ in probability, where the supremum is over all intervals $I \subset [a, b]$.

Proof. For (i), first suppose that I is bounded, with endpoints $\alpha \leq \beta$, and let $r > 0$. Let f_+ be continuous with $\mathbf{1}_{[\alpha, \beta]} \leq f_+ \leq \mathbf{1}_{(\alpha-r, \beta+r)}$, and let f_- be continuous with $\mathbf{1}_{[\alpha+r, \beta-r]} \leq f_- \leq \mathbf{1}_{(\alpha, \beta)}$ (with $f_- = 0$ if $\beta - \alpha \leq 2r$). Then $f_- \leq \mathbf{1}_I \leq f_+$, and $\int (f_+ - f_-) dm \leq m((\alpha - r, \alpha + r)) + m((\beta - r, \beta + r))$, which tends to $m(\{\alpha\}) + m(\{\beta\}) = 0$ as $r \downarrow 0$. Given $\eta > 0$, we choose r such that $\int (f_+ - f_-) dm \leq \eta$. Since $\int f_- dm_n \leq m_n(I) \leq \int f_+ dm_n$ and $\int f_- dm \leq m(I) \leq \int f_+ dm$, we obtain

$$\mathbb{P}(|m_n(I) - m(I)| > 2\eta) \leq \sum_{\pm} \mathbb{P}\left(\left|\int f_{\pm} dm_n - \int f_{\pm} dm\right| > \eta\right) \rightarrow 0.$$

For a half-line, choose continuous upper and lower cutoffs only at its finite endpoint; their difference is supported in its r -neighborhood, so the same argument applies. The cases $I = \mathbb{R}$ and $I = \emptyset$ are immediate.

For (ii), if $a = b$, every nonempty interval $I \subset [a, b]$ is $\{a\}$, and part (i) gives $m_n(\{a\}) \rightarrow m(\{a\}) = 0$ in probability. Hence assume $a < b$, and fix $\eta > 0$. Since m has no atoms in $[a, b]$ and has countably many atoms, there exist points $x_0 < a < x_1 < \dots < x_L = b$ that are not atoms of m such that $m((x_{j-1}, x_j]) \leq \eta$ for all j . By part (i), with probability tending to one we have $|m_n((x_0, x_j]) - m((x_0, x_j])| \leq \eta$ for all $j \leq L$. On this event, let $I \subset [a, b]$ be an interval, and let I^- and I^+ be the unions of the intervals $(x_{j-1}, x_j]$ that are contained in I and that meet I , respectively. Both are of the form $(x_i, x_j]$ (or empty), which gives $|m_n(I^{\pm}) - m(I^{\pm})| \leq 2\eta$, and $m(I^+ \setminus I^-) \leq 2\eta$ since $I^+ \setminus I^-$ consists of at most two of the intervals $(x_{j-1}, x_j]$. Using $I^- \subset I \subset I^+$, we obtain

$$m(I) - 4\eta \leq m(I^-) - 2\eta \leq m_n(I^-) \leq m_n(I) \leq m_n(I^+) \leq m(I^+) + 2\eta \leq m(I) + 4\eta.$$

This proves (ii). $\square$

Proposition 17.20. Fix $p > 1$ and a compact set $J \subset \mathbb{R} \setminus \{0\}$, and let $c_{p,J}, C_{p,J}$ and ρ_p^{∞} be as in Proposition 17.14.

- (i) For every interval $I \subset J$, we have $N_n^{-1} \text{Tr } \mathbf{1}_I(H_n^{\text{gc}}) \rightarrow \nu_p^{\infty}(I)$ in probability. If J is an interval, the convergence is uniform over intervals $I \subset J$, in probability.
- (ii) Let $E \neq 0$ be a Lebesgue point of ρ_p^{∞} . Then for every $\delta > 0$,

$$\lim_{h \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\left|\frac{1}{2hN_n} \text{Tr } \mathbf{1}_{I_h(E)}(H_n^{\text{gc}}) - \rho_p^{\infty}(E)\right| > \delta\right) = 0.$$

- (iii) For every nondegenerate interval $I \subset J$,

$$\mathbb{P}\left(\frac{1}{2}c_{p,J}N_n|I| \leq \text{Tr } \mathbf{1}_I(H_n^{\text{gc}}) \leq 2C_{p,J}N_n|I|\right) \rightarrow 1.$$

Proof. By (17.2), $N_n^{-1} \text{Tr } \mathbf{1}_I(H_n^{\text{gc}}) = \nu_n^{\text{gc}}(I)$. By Proposition 17.13(iv), $\nu_n^{\text{gc}} \rightarrow \nu_p^\infty$ weakly in probability, and by Proposition 17.14, ν_p^∞ has no atoms in $\mathbb{R} \setminus \{0\}$. Part (i) now follows from Lemma 17.19. For (ii), let $h_0 > 0$ be such that $[E - h_0, E + h_0] \subset \mathbb{R} \setminus \{0\}$, and let $0 < h \leq h_0$. By (i), $(2h)^{-1} \nu_n^{\text{gc}}(I_h(E)) \rightarrow (2h)^{-1} \nu_p^\infty(I_h(E))$ in probability as $n \rightarrow \infty$. Since E is a Lebesgue point of ρ_p^∞ , we have $(2h)^{-1} \nu_p^\infty(I_h(E)) \rightarrow \rho_p^\infty(E)$ as $h \downarrow 0$. For h small enough that $|(2h)^{-1} \nu_p^\infty(I_h(E)) - \rho_p^\infty(E)| < \delta/2$, we conclude that the probability in (ii) tends to zero as $n \rightarrow \infty$. For (iii), Proposition 17.14 gives $c_{p,I}|I| \leq \nu_p^\infty(I) \leq C_{p,I}|I|$ with $|I| > 0$, and (i) implies that $\nu_n^{\text{gc}}(I)$ lies in $[\nu_p^\infty(I)/2, 2\nu_p^\infty(I)]$ with probability tending to one. $\square$

Deficits

Recall the normalized local masses $P_{I,n}(v)$ and the deficit $\mathcal{D}_{I,n}(R)$ of Definition 1.3, and that $I_h(E) = (E - h, E + h]$. We define their limiting counterparts as follows.

Definition 17.21. For an interval I with $\nu_p^\infty(I) > 0$ and $R \in [0, \infty)$, the *tree deficit* is

$$\mathcal{D}_I^\infty(R) = 1 - \mathbb{E}^\infty \min \left\{ \frac{\mu_o(I)}{\nu_p^\infty(I)}, R \right\}.$$

By Proposition 17.14, $\nu_p^\infty(I) > 0$ for every nondegenerate interval I whose closure does not contain 0.

Proposition 17.22. Fix $p > 1$.

- (i) For every nondegenerate interval I whose closure is contained in $\mathbb{R} \setminus \{0\}$ and every $R \in [0, \infty)$, we have $\mathcal{D}_{I,n}(R) \rightarrow \mathcal{D}_I^\infty(R)$ in probability.
- (ii) Let $U \subset \mathbb{R} \setminus \{0\}$ be open, and suppose that, $\mathbb{P}^\infty$ -almost surely, the restriction of μ_o to U is absolutely continuous. Then for Lebesgue-almost every $E \in U$, the limit $d_E(R) = \lim_{h \downarrow 0} \mathcal{D}_{I_h(E)}^\infty(R)$ exists for every R , $\lim_{R \rightarrow \infty} d_E(R) = 0$, and the giant component is uniformly integrable delocalized at E .
- (iii) Let $U \subset \mathbb{R} \setminus \{0\}$ be open, and suppose that, $\mathbb{P}^\infty$ -almost surely, the restriction of μ_o to U is singular with respect to Lebesgue measure (for instance, pure point). Then for Lebesgue-almost every $E \in U$ and every $R \in [0, \infty)$, we have $\lim_{h \downarrow 0} \mathcal{D}_{I_h(E)}^\infty(R) = 1$, and the giant component is spectrally condensed at E .

Proof. *Proof of (i).* Set $\bar{m} = \nu_p^\infty(I)$, which is positive by Proposition 17.14. For $v \in \text{gc}$, write $X_{n,v} = \mu_{n,v}(I) \in [0, 1]$, and let $\bar{X}_n = N_n^{-1} \sum_{v \in \text{gc}} X_{n,v} = \bar{\mu}_n(I)$. Since the endpoints of I are nonzero, Proposition 17.14 implies that they are not atoms of ν_p , and Proposition 17.13(v) applies to I : for every bounded continuous $\psi: [0, 1] \rightarrow \mathbb{R}$,

$$\frac{1}{N_n} \sum_{v \in \text{gc}} \psi(X_{n,v}) \longrightarrow \mathbb{E}^\infty \psi(\mu_o(I)) \quad (17.20)$$

in probability. With $\psi(x) = x$, we find $\bar{X}_n \rightarrow \bar{m}$ in probability. On the event $\{|\bar{X}_n - \bar{m}| \leq \bar{m}/2\}$, whose probability tends to one, we have $\bar{X}_n > 0$ and $\mathcal{D}_{I,n}(R) = 1 - N_n^{-1} \sum_{v \in \text{gc}} \min\{X_{n,v}/\bar{X}_n, R\}$, and for $x \in [0, 1]$,

$$\left| \min\left\{\frac{x}{\bar{X}_n}, R\right\} - \min\left\{\frac{x}{\bar{m}}, R\right\} \right| \leq x \left| \frac{1}{\bar{X}_n} - \frac{1}{\bar{m}} \right| \leq \frac{2|\bar{X}_n - \bar{m}|}{\bar{m}^2}.$$

Averaging over $v \in \text{gc}$, we find that $|\mathcal{D}_{I,n}(R) - 1 + N_n^{-1} \sum_{v \in \text{gc}} \min\{X_{n,v}/\bar{m}, R\}| \leq 2|\bar{X}_n - \bar{m}|/\bar{m}^2$ on this event. Applying (17.20) with $\psi(x) = \min\{x/\bar{m}, R\}$, we conclude that $\mathcal{D}_{I,n}(R) \rightarrow 1 - \mathbb{E}^\infty \min\{\mu_o(I)/\bar{m}, R\} = \mathcal{D}_I^\infty(R)$ in probability.

Differentiation of the root measure. For the remaining parts, we use the Lebesgue differentiation theorem in the following form [118, Chapter 7]: if m is a locally finite Borel measure on an open set $U \subset \mathbb{R}$, with Lebesgue decomposition $m = f d\lambda + m_s$, then $m((E - h, E + h))/(2h) \rightarrow f(E)$ as $h \downarrow 0$ for Lebesgue-almost every $E \in U$. The same holds with $I_h(E)$ in place of $(E - h, E + h)$, since $(E - h, E + h) \subset I_h(E) \subset (E - h', E + h')$ for $h' > h$; taking $h' = (1 + \theta)h$ and then $\theta \downarrow 0$ gives the claim.

For $E \in \mathbb{R}$ and $h > 0$, set

$$A_h(E) = \frac{\mu_o(I_h(E))}{2h}, \quad b_h(E) = \mathbb{E}^\infty A_h(E) = \frac{\nu_p^\infty(I_h(E))}{2h}.$$

For fixed h , the map $(\omega, E) \mapsto A_h(E)$ is jointly measurable, since $c \mapsto \mu_o((-\infty, c])$ is right-continuous and measurable in ω for each c . Since $h \mapsto \mu_o(I_h(E))$ is nondecreasing for fixed E , we have $(h_1/h)A_{h_1}(E) \leq A_h(E) \leq (h_2/h)A_{h_2}(E)$ for $h_1 < h < h_2$. Then the upper and lower limits of $A_h(E)$ as $h \downarrow 0$ coincide with those along positive rationals, and they are jointly measurable functions of (ω, E) ; we denote them by $A^+(E)$ and $A^-(E)$.

By Proposition 17.14, ν_p^∞ is absolutely continuous on U with density ρ_p^∞ , and ρ_p^∞ is bounded below by a positive constant on every compact subset of U . Then there exists a Lebesgue-null set $\mathcal{N}_0 \subset U$ such that for $E \in U \setminus \mathcal{N}_0$,

$$b_h(E) \longrightarrow \rho_p^\infty(E) \in (0, \infty) \quad (h \downarrow 0). \quad (17.21)$$

Proof of (ii). Since, for $\mathbb{P}^\infty$ -almost every sample, μ_o restricted to U is absolutely continuous, the differentiation theorem implies that $A^-(E) = A^+(E) < \infty$ for Lebesgue-almost every $E \in U$. Then $\rho_o = A^+ \mathbf{1}_{\{A^+ < \infty\}}$, which is jointly measurable, is a density of μ_o on U . The set of (ω, E) with $E \in U$ and $A^-(E) \neq A^+(E)$ or $A^+(E) = \infty$ is jointly measurable, and its sections at fixed ω are Lebesgue-null for $\mathbb{P}^\infty$ -almost every ω . By Tonelli's theorem, its sections at fixed E are $\mathbb{P}^\infty$ -null for E outside a Lebesgue-null set $\mathcal{N}_1$. Further, by Tonelli's theorem we have $\nu_p^\infty(B) = \mathbb{E}^\infty \int_B \rho_o = \int_B \mathbb{E}^\infty \rho_o$ for Borel $B \subset U$, and we obtain $\mathbb{E}^\infty \rho_o(E) = \rho_p^\infty(E)$ for E outside a Lebesgue-null set $\mathcal{N}_2$.

Fix $E \in U \setminus (\mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2)$ and a sequence $h_j \downarrow 0$. Then $A_{h_j}(E) \geq 0$, $A_{h_j}(E) \rightarrow \rho_o(E)$ $\mathbb{P}^\infty$ -almost surely, and $\mathbb{E}^\infty A_{h_j}(E) = b_{h_j}(E) \rightarrow \rho_p^\infty(E) = \mathbb{E}^\infty \rho_o(E) < \infty$. By Scheffé's

lemma, $A_{h_j}(E) \rightarrow \rho_o(E)$ in $L^1(\mathbb{P}^\infty)$. Indeed, $\mathbb{E}^\infty |A_{h_j}(E) - \rho_o(E)| = \mathbb{E}^\infty [A_{h_j}(E) - \rho_o(E)] + 2\mathbb{E}^\infty (\rho_o(E) - A_{h_j}(E))^+$; the first term tends to zero, and so does the second, by dominated convergence with the integrable dominating function $\rho_o(E)$. Since the sequence was arbitrary, this holds as $h \downarrow 0$. Set

$$Z_h = \frac{\mu_o(I_h(E))}{v_p^\infty(I_h(E))} = \frac{A_h(E)}{b_h(E)}, \quad Z_E = \frac{\rho_o(E)}{\rho_p^\infty(E)}.$$

Then

$$\mathbb{E}^\infty |Z_h - Z_E| \leq \frac{\mathbb{E}^\infty |A_h(E) - \rho_o(E)|}{b_h(E)} + \mathbb{E}^\infty \rho_o(E) \left| \frac{1}{b_h(E)} - \frac{1}{\rho_p^\infty(E)} \right| \rightarrow 0,$$

by (17.21). Since $x \mapsto \min\{x, R\}$ is 1-Lipschitz, we obtain

$$\mathcal{D}_{I_h(E)}^\infty(R) = 1 - \mathbb{E}^\infty \min\{Z_h, R\} \rightarrow 1 - \mathbb{E}^\infty \min\{Z_E, R\} = d_E(R) \quad (h \downarrow 0).$$

Since $\mathbb{E}^\infty Z_E = 1$, monotone convergence gives $\mathbb{E}^\infty \min\{Z_E, R\} \uparrow 1$, that is, $d_E(R) \downarrow 0$ as $R \rightarrow \infty$.

It remains to deduce uniformly integrable delocalization at E . Fix $\delta > 0$, and choose R_0 with $d_E(R_0) < \delta$. Let $R \geq R_0$, so that $d_E(R) < \delta$. For all sufficiently small $h > 0$, we have $\mathcal{D}_{I_h(E)}^\infty(R) < \delta$ and the closure of $I_h(E)$ is contained in $\mathbb{R} \setminus \{0\}$. For such h , part (i) implies that $\mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > \delta) \rightarrow 0$ as $n \rightarrow \infty$. We conclude that $\limsup_{h \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > \delta) = 0$ for every $R \geq R_0$, and in particular the limit as $R \rightarrow \infty$ is zero. This is uniformly integrable delocalization at E in the sense of Definition 1.3.

Proof of (iii). Since, for $\mathbb{P}^\infty$ -almost every sample, μ_o restricted to U is singular, the differentiation theorem implies that $A^+(E) = 0$ for Lebesgue-almost every $E \in U$. As in the proof of (ii), Tonelli's theorem gives a Lebesgue-null set $\mathcal{N}'_1$ such that for $E \in U \setminus \mathcal{N}'_1$, we have $A_h(E) \rightarrow 0$ as $h \downarrow 0$, $\mathbb{P}^\infty$ -almost surely. Fix $E \in U \setminus (\mathcal{N}_0 \cup \mathcal{N}'_1)$ and $R \in [0, \infty)$. By (17.21), $Z_h = A_h(E)/b_h(E) \rightarrow 0$ almost surely, and $0 \leq \min\{Z_h, R\} \leq R$. By dominated convergence, $\mathbb{E}^\infty \min\{Z_h, R\} \rightarrow 0$, that is, $\mathcal{D}_{I_h(E)}^\infty(R) \rightarrow 1$.

Now fix $\delta > 0$. For all sufficiently small $h > 0$, we have $\mathcal{D}_{I_h(E)}^\infty(R) > 1 - \delta$ and the closure of $I_h(E)$ is contained in $\mathbb{R} \setminus \{0\}$, and part (i) implies that $\mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > 1 - \delta) \rightarrow 1$ as $n \rightarrow \infty$. This shows that $\lim_{h \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{D}_{I_h(E),n}(R) > 1 - \delta) = 1$. Since R and δ were arbitrary, the giant component is spectrally condensed at E . $\square$

In both (ii) and (iii), the limit $n \rightarrow \infty$ is taken first, at fixed h and R , then $h \downarrow 0$, and in (ii) finally $R \rightarrow \infty$; this is the order in Definition 1.3.

Proof of Theorem 1.4

Proof of Theorem 1.4. For part (i), let $p > 1$ and let $J \subset \mathbb{R} \setminus \{0\}$ be compact. By Proposition 17.20(iii), the conclusion holds with the constants $c_{p,J}/2$ and $2C_{p,J}$ from Proposition 17.14.

For part (ii), let $p \geq p_0$, and set $U_{ac} = (-E_c(p), 0) \cup (0, E_c(p))$ and $U_{pp} = (-\infty, -E_c(p)) \cup (E_c(p), \infty)$, which are open subsets of $\mathbb{R} \setminus \{0\}$. By Theorem 1.1, $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ is purely absolutely continuous on U_{ac} . This means that the spectral measure of every vector, restricted to U_{ac} , is absolutely continuous; in particular, so is the restriction of μ_o . By Proposition 17.22(ii), the giant component is uniformly integrable delocalized at Lebesgue-almost every $E \in U_{ac}$. Again by Theorem 1.1, $\mathbb{P}^\infty$ -almost surely, $\mathcal{H}$ has pure point spectrum on U_{pp} . In particular, the restriction of μ_o to U_{pp} is a countable sum of point masses, which is singular with respect to Lebesgue measure. By Proposition 17.22(iii), the giant component is spectrally condensed at Lebesgue-almost every $E \in U_{pp}$. $\square$

Remark 17.23. The restriction to the giant component in Theorem 1.4 cannot be removed. For every $p > 1$, the graph G_n has linearly many finite components, and these support localized eigenvectors in every interval of nonzero energies. For instance, fix a compact interval $I \subset (0, \infty)$ with nonempty interior, and let $X_n(I)$ be the number of components of G_n that consist of a single edge e with $|h_e| \in I$. The function on $\mathcal{G}_1$ equal to 1 if the component of the root is a single edge e with $|h_e| \in I$, and 0 otherwise, is continuous at every point except those whose root component is a single edge e with $|h_e|$ an endpoint of I . Since the latter have probability zero under the law of $\mathfrak{F}_1$, this function is admissible, and Lemma 17.10 implies that

$$\frac{2X_n(I)}{n} \longrightarrow \mathbb{P}(\mathcal{T} \text{ is a single edge } e \text{ with } |h_e| \in I) = p e^{-2p} \mathbb{P}(|w| \in \sqrt{p} I) > 0$$

in probability. Each such component carries an eigenvector of H_n with eigenvalue $|h_e| \in I$, supported on the two vertices of e , and these eigenvectors are orthogonal. Since $\text{Tr } \mathbf{1}_I(H_n) \leq n$, a fraction at least $X_n(I)/n$ of the eigenvalues of H_n in I belongs to eigenvectors supported on two vertices, and this fraction is bounded below in probability by a positive constant. This holds for every $p > 1$, independently of the transition at $E_c(p)$; by (17.1), these eigenvectors do not affect H_n^{gc} .

References

- [1] Ragi Abou-Chacra and David J. Thouless. Self-consistent theory of localization. II. Localization near the band edges. *J. Phys. C*, 7(1):65, 1974.
- [2] Ragi Abou-Chacra, David J. Thouless, and Philip W. Anderson. A selfconsistent theory of localization. *J. Phys. C: Solid State Phys.*, 6(10):1734–1752, 1973.
- [3] Ievgenii Afanasiev. On the correlation functions of the characteristic polynomials of the sparse Hermitian random matrices. *J. Stat. Phys.*, 163(2):324–356, 2016. doi: 10.1007/s10955-016-1486-z. arXiv:1508.06623.
- [4] Amol Aggarwal and Patrick Lopatto. Mobility edge for the Anderson model on the Bethe lattice, 2025. Preprint, arXiv:2503.08949.
- [5] Amol Aggarwal, Patrick Lopatto, and Jake Marcinek. Eigenvector statistics of Lévy matrices. *Ann. Probab.*, 49(4):1778–1846, 2021. doi: 10.1214/20-AOP1493. arXiv:2002.09355.
- [6] Amol Aggarwal, Patrick Lopatto, and Horng-Tzer Yau. GOE statistics for Lévy matrices. *J. Eur. Math. Soc. (JEMS)*, 23(11):3707–3800, 2021. doi: 10.4171/JEMS/1089. arXiv:1806.07363.
- [7] Amol Aggarwal, Charles Bordenave, and Patrick Lopatto. Mobility edge for Lévy matrices, 2022. Preprint, arXiv:2210.09458.
- [8] Michael Aizenman. Localization at weak disorder: some elementary bounds. *Rev. Math. Phys.*, 6(5A):1163–1182, 1994. doi: 10.1142/S0129055X94000419.
- [9] Michael Aizenman and Stanislav Molchanov. Localization at large disorder and at extreme energies: an elementary derivation. *Comm. Math. Phys.*, 157(2):245–278, 1993.
- [10] Michael Aizenman and Simone Warzel. Absence of mobility edge for the Anderson random potential on tree graphs at weak disorder. *EPL (Europhys. Lett.)*, 96:37004, 2011. doi: 10.1209/0295-5075/96/37004. arXiv:1109.2210.
- [11] Michael Aizenman and Simone Warzel. Extended states in a Lifshitz tail regime for random Schrödinger operators on trees. *Phys. Rev. Lett.*, 106(13):136804, 2011. doi: 10.1103/PhysRevLett.106.136804. arXiv:1010.2673.
- [12] Michael Aizenman and Simone Warzel. Resonant delocalization for random Schrödinger operators on tree graphs. *J. Eur. Math. Soc. (JEMS)*, 15(4):1167–1222, 2013. doi: 10.4171/JEMS/389. arXiv:1104.0969.

- [13] Michael Aizenman and Simone Warzel. *Random Operators: Disorder Effects on Quantum Spectra and Dynamics*, volume 168 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [14] Michael Aizenman, Robert Sims, and Simone Warzel. Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs. *Probab. Theory Related Fields*, 136(3):363–394, 2006. arXiv:math-ph/0502006.
- [15] David Aldous and Russell Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12:Paper No. 54, 1454–1508, 2007. arXiv:math/0603062.
- [16] David Aldous and J. Michael Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In Harry Kesten, editor, *Probability on Discrete Structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 1–72. Springer, Berlin, 2004. doi: 10.1007/978-3-662-09444-0_1.
- [17] David J. Aldous and Antar Bandyopadhyay. A survey of max-type recursive distributional equations. *Ann. Appl. Probab.*, 15(2):1047–1110, 2005. doi: 10.1214/105051605000000142. arXiv:math/0401388.
- [18] Johannes Alt, Raphaël Ducatez, and Antti Knowles. Delocalization transition for critical Erdős–Rényi graphs. *Comm. Math. Phys.*, 388(1):507–579, 2021. arXiv:2005.14180.
- [19] Johannes Alt, Raphaël Ducatez, and Antti Knowles. Extremal eigenvalues of critical Erdős–Rényi graphs. *Ann. Probab.*, 49(3):1347–1401, 2021. doi: 10.1214/20-AOP1483. arXiv:1905.03243.
- [20] Johannes Alt, Raphaël Ducatez, and Antti Knowles. The completely delocalized region of the Erdős–Rényi graph. *Electron. Commun. Probab.*, 27:Paper No. 10, 9 pp., 2022. doi: 10.1214/22-ECP450. arXiv:2109.03227.
- [21] Johannes Alt, Raphaël Ducatez, and Antti Knowles. Poisson statistics and localization at the spectral edge of sparse Erdős–Rényi graphs. *Ann. Probab.*, 51(1):277–358, 2023. doi: 10.1214/22-AOP1596. arXiv:2106.12519.
- [22] Johannes Alt, Raphaël Ducatez, and Antti Knowles. Localized phase for the Erdős–Rényi graph. *Comm. Math. Phys.*, 405(1):Paper No. 9, 2024. doi: 10.1007/s00220-023-04918-z. arXiv:2305.16294.
- [23] Nalini Anantharaman and Etienne Le Masson. Quantum ergodicity on large regular graphs. *Duke Math. J.*, 164(4):723–765, 2015. doi: 10.1215/00127094-2881592. arXiv:1304.4343.
- [24] Nalini Anantharaman and Mostafa Sabri. Quantum ergodicity for the Anderson model on regular graphs. *J. Math. Phys.*, 58(9):091901, 2017. doi: 10.1063/1.5000962. arXiv:1704.02765.

- [25] Nalini Anantharaman and Mostafa Sabri. Quantum ergodicity on graphs: from spectral to spatial delocalization. *Ann. of Math. (2)*, 189(3):753–835, 2019. doi: 10.4007/annals.2019.189.3.3. arXiv:1704.02766.
- [26] Philip W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109(5):1492–1505, 1958. doi: 10.1103/PhysRev.109.1492.
- [27] Nachman Aronszajn. On a problem of Weyl in the theory of singular Sturm–Liouville equations. *Amer. J. Math.*, 79:597–610, 1957.
- [28] Adam Arras and Charles Bordenave. Existence of absolutely continuous spectrum for Galton–Watson random trees. *Comm. Math. Phys.*, 403(1):495–527, 2023. arXiv:2105.10177.
- [29] Victor Bapst. The large connectivity limit of the Anderson model on tree graphs. *J. Math. Phys.*, 55(9):092101, 2014. doi: 10.1063/1.4894055. arXiv:1303.4908.
- [30] Victor Bapst and Guilhem Semerjian. Lifshitz tails on the Bethe lattice: a combinatorial approach. *J. Stat. Phys.*, 145:51, 2011. arXiv:1104.5637.
- [31] Joseph W. Baron and Tim Rogers. Local density of states distribution and multifractal eigenvectors of weighted random networks via the cavity approach. *Phys. Rev. B*, 114:074204, 2026. arXiv:2606.10915.
- [32] Gérard Ben Arous and Alice Guionnet. The spectrum of heavy tailed random matrices. *Comm. Math. Phys.*, 278(3):715–751, 2008. doi: 10.1007/s00220-007-0389-x. arXiv:0707.2159.
- [33] Florent Benaych-Georges, Charles Bordenave, and Antti Knowles. Largest eigenvalues of sparse inhomogeneous Erdős–Rényi graphs. *Ann. Probab.*, 47(3):1653–1676, 2019. arXiv:1704.02953.
- [34] Florent Benaych-Georges, Charles Bordenave, and Antti Knowles. Spectral radii of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(3):2141–2161, 2020. doi: 10.1214/19-AIHP1033. arXiv:1704.02945.
- [35] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:Paper No. 23, 13 pp., 2001. doi: 10.1214/EJP.v6-96. arXiv:math/0011019.
- [36] Shankar Bhamidi, Steven N. Evans, and Arnab Sen. Spectra of large random trees. *J. Theoret. Probab.*, 25(3):613–654, 2012. doi: 10.1007/s10959-011-0360-9. arXiv:0903.3589.
- [37] Patrick Billingsley. *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. John Wiley & Sons, New York, second edition, 1999.
- [38] Giulio Biroli and Rémi Monasson. A single defect approximation for localized states on random lattices. *J. Phys. A*, 32(24):L255, 1999. arXiv:cond-mat/9902032.

- [39] Giulio Biroli and Marco Tarzia. Delocalization and ergodicity of the Anderson model on Bethe lattices, 2018. Preprint, arXiv:1810.07545.
- [40] Giulio Biroli and Marco Tarzia. Anomalous dynamics on the ergodic side of the many-body localization transition and the glassy phase of directed polymers in random media. *Phys. Rev. B*, 102:064211, 2020. arXiv:2003.09629.
- [41] Giulio Biroli and Marco Tarzia. Lévy–Rosenzweig–Porter random matrix ensemble. *Phys. Rev. B*, 103:104205, 2021. arXiv:2012.12841.
- [42] Giulio Biroli, Jean-Philippe Bouchaud, and Marc Potters. On the top eigenvalue of heavy-tailed random matrices. *EPL*, 78(1):10001, 2007. arXiv:cond-mat/0609070.
- [43] Giulio Biroli, Guilhem Semerjian, and Marco Tarzia. Anderson model on Bethe lattices: density of states, localization properties and isolated eigenvalue. *Prog. Theor. Phys. Suppl.*, 184:187–199, 2010. arXiv:1005.0342.
- [44] Giulio Biroli, A. C. Ribeiro-Teixeira, and Marco Tarzia. Difference between level statistics, ergodicity and localization transitions on the Bethe lattice, 2012. Preprint, arXiv:1211.7334.
- [45] Giulio Biroli, Alexander K. Hartmann, and Marco Tarzia. Critical behavior of the Anderson model on the Bethe lattice via a large-deviation approach. *Phys. Rev. B*, 105:094202, 2022. arXiv:2110.01524.
- [46] Charles Bordenave. On quantum percolation in finite regular graphs. *Ann. Henri Poincaré*, 16(11):2465–2497, 2015. doi: 10.1007/s00023-014-0382-9. arXiv:1409.2000.
- [47] Charles Bordenave. Spectral measures of random graphs. In *Advanced Topics in Random Matrices*, volume 53 of *Panor. Synthèses*, pages 91–150. Soc. Math. France, Paris, 2017.
- [48] Charles Bordenave and Alice Guionnet. Localization and delocalization of eigenvectors for heavy-tailed random matrices. *Probab. Theory Related Fields*, 157(3-4): 885–953, 2013. doi: 10.1007/s00440-012-0473-9. arXiv:1201.1862.
- [49] Charles Bordenave and Alice Guionnet. Delocalization at small energy for heavy-tailed random matrices. *Comm. Math. Phys.*, 354(1):115–159, 2017. doi: 10.1007/s00220-017-2914-x. arXiv:1603.08845.
- [50] Charles Bordenave and Marc Lelarge. Resolvent of large random graphs. *Random Structures Algorithms*, 37:332–352, 2010. arXiv:0801.0155.
- [51] Charles Bordenave, Pietro Caputo, and Djalil Chafaï. Spectrum of large random reversible Markov chains: heavy-tailed weights on the complete graph. *Ann. Probab.*, 39(4):1544–1590, 2011. doi: 10.1214/10-AOP587. arXiv:0903.3528.
- [52] Charles Bordenave, Marc Lelarge, and Justin Salez. The rank of diluted random graphs. *Ann. Probab.*, 39(3):1097–1121, 2011. doi: 10.1214/10-AOP567. arXiv:0907.4244.

- [53] Charles Bordenave, Arnab Sen, and Bálint Virág. Mean quantum percolation. *J. Eur. Math. Soc. (JEMS)*, 19(12):3679–3707, 2017. doi: 10.4171/JEMS/750. arXiv:1308.3755.
- [54] Shimon Brooks and Elon Lindenstrauss. Non-localization of eigenfunctions on large regular graphs. *Israel J. Math.*, 193:1–14, 2013. arXiv:0912.3239.
- [55] Thomas Buc-d’Alché and Antti Knowles. Semilocalization for inhomogeneous random graphs, 2026. Preprint, arXiv:2604.11682.
- [56] Pierre Cizeau and Jean-Philippe Bouchaud. Theory of Lévy matrices. *Phys. Rev. E*, 50:1810, 1994.
- [57] Simon Coste and Justin Salez. Emergence of extended states at zero in the spectrum of sparse random graphs. *Ann. Probab.*, 49(4):2012–2030, 2021. doi: 10.1214/20-AOP1499. arXiv:1809.07587.
- [58] Leticia F. Cugliandolo, Grégory Schehr, Marco Tarzia, and Davide Venturelli. Multifractal phase in the weighted adjacency matrices of random Erdős–Rényi graphs. *Phys. Rev. B*, 110:174202, 2024. doi: 10.1103/PhysRevB.110.174202. arXiv:2404.06931.
- [59] Andrea De Luca, Boris L. Altshuler, Vladimir E. Kravtsov, and Antonello Scardicchio. Anderson localization on the Bethe lattice: nonergodicity of extended states. *Phys. Rev. Lett.*, 113:046806, 2014. arXiv:1403.7817.
- [60] Klaus Deimling. *Nonlinear Functional Analysis*. Springer-Verlag, Berlin, 1985. doi: 10.1007/978-3-662-00547-7.
- [61] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications*, volume 38 of *Applications of Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [62] Bernard Derrida and Herbert Spohn. Polymers on disordered trees, spin glasses, and traveling waves. *J. Stat. Phys.*, 51:817–840, 1988.
- [63] Reuben Drogin and Charles K. Smart. The regular tree Anderson model at low disorder, 2025. Preprint, arXiv:2511.10564.
- [64] Rick Durrett. *Probability: Theory and Examples*, volume 49 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fifth edition, 2019.
- [65] Eleftherios N. Economou and Morrel H. Cohen. Existence of mobility edges in Anderson’s model for random lattices. *Phys. Rev. B*, 5(8):2931, 1972.
- [66] Konstantin B. Efetov. Anderson transition on a Bethe lattice (the symplectic and orthogonal ensembles). *Sov. Phys. JETP*, 66(3):634, 1987. [*Zh. Eksp. Teor. Fiz.* 93, 1125 (1987)].
- [67] László Erdős, Antti Knowles, Horng-Tzer Yau, and Jun Yin. Spectral statistics of Erdős–Rényi graphs I: local semicircle law. *Ann. Probab.*, 41(3B):2279–2375, 2013. doi: 10.1214/11-AOP734. arXiv:1103.1919.

- [68] Paul Erdős and Alfréd Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl. (Publ. Math. Inst. Hungar. Acad. Sci.)*, 5:17–61, 1960.
- [69] S. N. Evangelou. A numerical study of sparse random matrices. *J. Stat. Phys.*, 69: 361–383, 1992.
- [70] S. N. Evangelou and Eleftherios N. Economou. Spectral density singularities, level statistics, and localization in a sparse random matrix ensemble. *Phys. Rev. Lett.*, 68 (3):361, 1992.
- [71] Ferdinand Evers and Alexander D. Mirlin. Anderson transitions. *Rev. Mod. Phys.*, 80:1355, 2008. arXiv:0707.4378.
- [72] Richard Froese, David Hasler, and Wolfgang Spitzer. Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein’s theorem. *Comm. Math. Phys.*, 269(1):239–257, 2007. doi: 10.1007/s00220-006-0120-3. arXiv:math-ph/0511050.
- [73] Yan V. Fyodorov and Alexander D. Mirlin. Localization in ensemble of sparse random matrices. *Phys. Rev. Lett.*, 67(15):2049, 1991.
- [74] Yan V. Fyodorov, Alexander D. Mirlin, and Hans-Jürgen Sommers. A novel field theoretical approach to the Anderson localization: sparse random hopping model. *J. Phys. I France*, 2(8):1571–1605, 1992.
- [75] Georges Glaeser. Racine carrée d’une fonction différentiable. *Ann. Inst. Fourier (Grenoble)*, 13(2):203–210, 1963. doi: 10.5802/aif.146.
- [76] Yukun He, Antti Knowles, and Matteo Marcozzi. Local law and complete eigenvector delocalization for supercritical Erdős–Rényi graphs. *Ann. Probab.*, 47(5): 3278–3302, 2019. doi: 10.1214/19-AOP1339. arXiv:1808.09437.
- [77] Yukun He, Jiaoyang Huang, and Xiaoyu Wang. Bulk universality of random regular graphs, 2026. URL <https://arxiv.org/abs/2610.08127>.
- [78] Joscha Henheik and Antti Knowles. Bernoulli flow for Erdős–Rényi graphs, 2026. URL <https://arxiv.org/abs/2609.15566>.
- [79] Ella Hiesmayr and Theo McKenzie. The spectral edge of constant degree Erdős–Rényi graphs, 2023. Preprint, arXiv:2309.11007.
- [80] Vojkan Jakšić, Yoram Last, and Simone Warzel. A note on extended states on the Bethe lattice, 2026. Preprint, arXiv:2609.06987.
- [81] Olav Kallenberg. *Foundations of Modern Probability*. Probability and its Applications. Springer-Verlag, New York, second edition, 2002.
- [82] Tosio Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

- [83] Matthias Keller. Absolutely continuous spectrum for multi-type Galton Watson trees. *Ann. Henri Poincaré*, 13(8):1745–1766, 2012. doi: 10.1007/s00023-012-0169-9. arXiv:1109.1959.
- [84] Matthias Keller, Daniel Lenz, and Simone Warzel. Absolutely continuous spectrum for random operators on trees of finite cone type. *J. Anal. Math.*, 118(1):363–396, 2012. arXiv:1108.0057.
- [85] O. Khorunzhy, Mariya Shcherbina, and Valentin Vengerovsky. Eigenvalue distribution of large weighted random graphs. *J. Math. Phys.*, 45(4):1648–1672, 2004. doi: 10.1063/1.1667610.
- [86] Scott Kirkpatrick and Thomas P. Eggarter. Localized states of a binary alloy. *Phys. Rev. B*, 6:3598, 1972.
- [87] Abel Klein. Absolutely continuous spectrum in the Anderson model on the Bethe lattice. *Math. Res. Lett.*, 1(4):399–407, 1994.
- [88] Abel Klein. Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, 133(1):163–184, 1998.
- [89] Vladimir E. Kravtsov, Boris L. Altshuler, and Lev B. Ioffe. Non-ergodic delocalized phase in Anderson model on Bethe lattice and regular graph. *Ann. Phys.*, 389: 148–191, 2018. arXiv:1712.00614.
- [90] Mark G. Kreĭn and Mark A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspekhi Mat. Nauk*, 3(1(23)):3–95, 1948. In Russian. English translation: Amer. Math. Soc. Translation 1950, no. 26.
- [91] Reimer Kühn. Spectra of sparse random matrices. *J. Phys. A*, 41(29):295002, 2008. arXiv:0803.2886.
- [92] Reimer Kühn. Disentangling giant component and finite cluster contributions in sparse random matrix spectra. *Phys. Rev. E*, 93:042110, 2016. arXiv:1601.04690.
- [93] Hervé Kunz and Bernard Souillard. The localization transition on the Bethe lattice. *J. Physique Lett.*, 44(11):L411–L414, 1983. doi: 10.1051/jphyslet:019830044011041100.
- [94] Günter Last and Mathew Penrose. *Lectures on the Poisson Process*, volume 7 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2018.
- [95] Ilya M. Lifshitz. The energy spectrum of disordered systems. *Adv. Phys.*, 13(52): 483–536, 1964.
- [96] Suhan Liu and Patrick Lopatto. Mobility edge for the Anderson model on random regular graphs, 2026. Preprint, arXiv:2603.14230.
- [97] Joseph Mecke. Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 9(1):36–58, 1967. doi: 10.1007/BF00535466.

- [98] Fernando L. Metz, Izaak Neri, and Désiré Bollé. Localization transition in symmetric random matrices. *Phys. Rev. E*, 82:031135, 2010. arXiv:1005.3712.
- [99] Jeffrey D. Miller and Bernard Derrida. Weak-disorder expansion for the Anderson model on a tree. *J. Stat. Phys.*, 75:357–388, 1994. arXiv:cond-mat/9312023.
- [100] Alexander D. Mirlin and Yan V. Fyodorov. Localization transition in the Anderson model on the Bethe lattice: spontaneous symmetry breaking and correlation functions. *Nucl. Phys. B*, 366(3):507–532, 1991.
- [101] Alexander D. Mirlin and Yan V. Fyodorov. Universality of level correlation function of sparse random matrices. *J. Phys. A*, 24(10):2273, 1991.
- [102] Cécile Monthus and Thomas Garel. Anderson transition on the Cayley tree as a traveling wave critical point for various probability distributions. *J. Phys. A*, 42:075002, 2009. arXiv:0811.1422.
- [103] Nevill F. Mott. Electrons in disordered structures. *Adv. Phys.*, 16(61):49–144, 1967.
- [104] OpenAI. Absolutely Continuous Spectrum for Weak-Disorder Anderson Models in Dimensions at Least Three. OpenAI Math Release preprint, September 2026. URL <https://github.com/openai/math/blob/main/preprints/Absolutely-Continuous-Spectrum-for-Weak-Disorder-Anderson-Models-in-Dimensions-at-least-three.pdf>.
- [105] OpenAI. Fixed-energy universality for weak Anderson disorder on random regular graphs. OpenAI Math Release preprint, October 2026. URL <https://github.com/openai/math/blob/main/preprints/Fixed-energy-universality-for-weak-Anderson-disorder-on-random-regular-graphs-October-2026.pdf>.
- [106] OpenAI. GOE bulk universality for fixed-degree random regular graphs. OpenAI Math Release preprint, September 2026. URL <https://github.com/openai/math/blob/main/preprints/GOE-bulk-universality-for-fixed-degree-random-regular-graphs-September-23-2026/paper.pdf>.
- [107] OpenAI. Pure-Point Spectrum for the Two-Dimensional Anderson Model at Every Positive Disorder. OpenAI Math Release preprint, September 2026. URL <https://github.com/openai/math/blob/main/preprints/Pure-Point-Spectrum-for-the-Two-Dimensional-Anderson-Model-at-Every-Positive-Disorder.pdf>.
- [108] Raymond E. A. C. Paley and Antoni Zygmund. A note on analytic functions in the unit circle. *Proc. Cambridge Philos. Soc.*, 28(3):266–272, 1932.
- [109] Giorgio Parisi, Saverio Pascazio, Francesca Pietracaprina, Valentina Ros, and Antonello Scardicchio. Anderson transition on the Bethe lattice: an approach with real

- energies. *J. Phys. A: Math. Theor.*, 53:014003, 2020. doi: 10.1088/1751-8121/ab56e8. arXiv:1812.03531.
- [110] Francesca Pietracaprina, Valentina Ros, and Antonello Scardicchio. Forward approximation as a mean-field approximation for the Anderson and many-body localization transitions. *Phys. Rev. B*, 93:054201, 2016. arXiv:1508.05097.
- [111] Dafne Prado Bandeira, Angelica Vecchiarelli, and Marco Tarzia. Anderson localization on the Bethe lattice, 2026. Lecture notes, arXiv:2606.24671.
- [112] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975.
- [113] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics. I. Functional Analysis*. Academic Press, New York, second edition, 1980.
- [114] Tommaso Rizzo and Marco Tarzia. The localized phase of the Anderson model on the Bethe lattice, 2024. Preprint, arXiv:2406.18748.
- [115] Geoff J. Rodgers and Alan J. Bray. Density of states of a sparse random matrix. *Phys. Rev. B*, 37(7):3557, 1988.
- [116] Geoff J. Rodgers and C. De Dominicis. Density of states of sparse random matrices. *J. Phys. A*, 23(9):1567, 1990.
- [117] Tim Rogers, Isaac Pérez Castillo, Reimer Kühn, and Koujin Takeda. Cavity approach to the spectral density of sparse symmetric random matrices. *Phys. Rev. E*, 78(3):031116, 2008. arXiv:0803.1553.
- [118] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, third edition, 1987.
- [119] Justin Salez. Every totally real algebraic integer is a tree eigenvalue. *J. Combin. Theory Ser. B*, 111:249–256, 2015. arXiv:1302.4423.
- [120] Justin Salez. Spectral atoms of unimodular random trees. *J. Eur. Math. Soc. (JEMS)*, 22(2):345–363, 2020. arXiv:1609.09374.
- [121] Guilhem Semerjian and Leticia F. Cugliandolo. Sparse random matrices: the eigenvalue spectrum revisited. *J. Phys. A*, 35(23):4837, 2002. arXiv:cond-mat/0202406.
- [122] Jeferson D. Silva and Fernando L. Metz. Analytic solution of the resolvent equations for heterogeneous random graphs: spectral and localization properties. *J. Phys. Complex.*, 3:045012, 2022. arXiv:2209.06805.
- [123] Barry Simon. Spectral analysis of rank one perturbations and applications. In J. Feldman, R. Froese, and L. M. Rosen, editors, *Mathematical Quantum Theory. II. Schrödinger Operators (Vancouver, BC, 1993)*, volume 8 of CRM Proc. Lecture Notes, pages 109–149. Amer. Math. Soc., Providence, RI, 1995.

- [124] Barry Simon and Tom Wolff. Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math.*, 39(1): 75–90, 1986. doi: 10.1002/cpa.3160390105.
- [125] František Slanina. Localization of eigenvectors in random graphs. *Eur. Phys. J. B*, 85:361, 2012. arXiv:1401.1681.
- [126] Vito A. R. Susca, Pierpaolo Vivo, and Reimer Kühn. Cavity and replica methods for the spectral density of sparse symmetric random matrices. *SciPost Phys. Lect. Notes*, 33, 2021. arXiv:2101.08029.
- [127] Elena Tarquini, Giulio Biroli, and Marco Tarzia. Level statistics and localization transitions of Lévy matrices. *Phys. Rev. Lett.*, 116:010601, 2016. arXiv:1507.00296.
- [128] Elena Tarquini, Giulio Biroli, and Marco Tarzia. Critical properties of the Anderson localization transition and the high-dimensional limit. *Phys. Rev. B*, 95:094204, 2017. arXiv:1612.04753.
- [129] Marco Tarzia. Fully localized and partially delocalized states in the tails of Erdős–Rényi graphs in the critical regime. *Phys. Rev. B*, 105:174201, 2022. arXiv:2112.11560.
- [130] Konstantin S. Tikhonov and Alexander D. Mirlin. From Anderson localization on random regular graphs to many-body localization. *Ann. Phys.*, 435:168525, 2021. arXiv:2102.05930.
- [131] Konstantin S. Tikhonov, Alexander D. Mirlin, and Mikhail A. Skvortsov. Anderson localization and ergodicity on random regular graphs. *Phys. Rev. B*, 94:220203, 2016. arXiv:1604.05353.
- [132] Pietro Valigi, Joseph W. Baron, Izaak Neri, Giulio Biroli, and Chiara Cammarota. Eigenvalue spectral tails and localization properties of asymmetric networks. *J. Phys. A*, 58(45):455002, 2025. arXiv:2507.20225.
- [133] Remco van der Hofstad. *Random Graphs and Complex Networks. Vol. 1*, volume 43 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2017.
- [134] Remco van der Hofstad. *Random Graphs and Complex Networks. Vol. 2*, volume 54 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2024.
- [135] Simone Warzel. Surprises in the phase diagram of the Anderson model on the Bethe lattice. In *XVIIth International Congress on Mathematical Physics*, pages 239–253. World Scientific, Hackensack, NJ, 2013. doi: 10.1142/9789814449243_0014. arXiv:1212.4367.
- [136] Martin R. Zirnbauer. Localization transition on the Bethe lattice. *Phys. Rev. B*, 34(9): 6394, 1986.