

The Sharp Commutator Bound for the Polymath14 Inequality

We present a simplified proof of the Polymath14 inequality, along with a new sharpness result.

Let G be a group. Write $\text{cl}_G(g)$ for the commutator length of g , with $\text{cl}_G(g) = \infty$ if $g \notin [G, G]$, and set $\text{scl}_G(g) := \inf_{n \geq 1} \text{cl}_G(g^n)/n$.

Theorem. *Let $f: G \rightarrow \mathbb{R}$ satisfy*

$$f(g^2) \geq 2f(g) - 1 \quad \text{and} \quad f(gh) \leq f(g) + f(h)$$

for all $g, h \in G$. Then $f(g) \leq 1 + 2\text{scl}_G(g)$ whenever $\text{scl}_G(g) < \infty$. In particular, $f([g, h]) \leq 2$ for all $g, h \in G$.

Moreover, there exists $F: G \rightarrow \mathbb{R}$ satisfying the same two hypotheses and $F(g) = 1 + 2\text{scl}_G(g)$ whenever $\text{scl}_G(g) < \infty$.

We call the two hypotheses the *doubling inequality* and *subadditivity*.

Lemma 1. *For every $g \in G$ and $k \geq 0$,*

$$f(g) \leq \frac{f(g^{2^k})}{2^k} + 1.$$

Proof. Iterate $f(g) \leq \frac{1}{2}f(g^2) + \frac{1}{2}$. □

Lemma 2. *Assume that f is conjugation invariant. If g is conjugate to h , then, for every $x \in G$,*

$$f(g) = f(h) \leq \frac{f(x^{-1}g) + f(hx)}{2} + 1.$$

Proof. Choose $y \in G$ such that $g = yhy^{-1}$. Since

$$g^{n+1}yh^{n+1} \sim (x^{-1}g)g^nyh^n(hx),$$

repeated use of conjugation invariance and subadditivity gives

$$f(g^N y h^N) \leq f(y) + N(f(x^{-1}g) + f(hx)).$$

Also $g^{2N} = g^N y h^N y^{-1}$. Taking $2N = 2^k$, applying lemma 1, and letting $k \rightarrow \infty$ gives the result. □

Lemma 3. *Assume that f is conjugation invariant. Then $f([g, h]) \leq 3$ for all $g, h \in G$.*

Proof. This is the random-walk argument of [1, Proposition 2.1]. Fix $g, h \in G$ and put $q = [g, h]$. We have

$$\begin{aligned} f(g^m q^k) &\leq \frac{f(g^{-1}g^m q^k) + f(h^{-1}g^m q^k h g)}{2} + 1 && \text{(Lemma 2)} \\ &= \frac{f(g^{m-1} q^k) + f(h^{-1}g^m q^{k-1} g h)}{2} + 1 \\ &= \frac{f(g^{m-1} q^k) + f(g^{m+1} q^{k-1})}{2} + 1 && \text{(conjugation invariance).} \end{aligned}$$

Let S_{2n} be the sum of $2n$ independent random signs. Iterating the inequality above from $(m, k) = (0, n)$ gives

$$f(q^n) \leq \mathbb{E}f\left(g^{S_{2n}} q^{-S_{2n}/2}\right) + 2n.$$

By subadditivity, there is a constant $C = C(g, h)$ such that $f(g^s q^{-s/2}) \leq C(|s| + 1)$ for every even integer s . Since $\mathbb{E}|S_{2n}| \leq \sqrt{\mathbb{E}(S_{2n}^2)} = \sqrt{2n}$, we obtain $f(q^n) \leq C(\sqrt{2n} + 1) + 2n$. Taking $n = 2^k$, applying lemma 1, and letting $k \rightarrow \infty$ gives $f(q) \leq 3$. □

Proof of the theorem. We first reduce to the conjugation-invariant case. By lemma 1 and subadditivity, $f(xgx^{-1}) \leq f(g) + (f(x) + f(x^{-1}))/2^k + 1$, so $f(xgx^{-1}) \leq f(g) + 1$. Thus $\bar{f}(g) := \sup_{x \in G} f(xgx^{-1})$ is finite, dominates f , is conjugation invariant, and satisfies the same two hypotheses. Replacing f by $\bar{f}$, we may assume that f is conjugation invariant.

Set $M := \sup_{g,h \in G} f([g,h])$. By lemma 3, $M \leq 3$. We claim that $f(g) \leq 1 + M \operatorname{scl}_G(g)$ whenever $\operatorname{scl}_G(g) < \infty$. Let r be the order of the image of g in G^{ab} . lemma 1 and subadditivity give

$$f(g) \leq 1 + \frac{\max_{0 \leq a < r} f(g^a) + M \max\{1, \operatorname{cl}_G(g^{r \lfloor 2^k/r \rfloor})\}}{2^k}.$$

Letting $k \rightarrow \infty$ proves the claim.

For a commutator q , Culler's identities[2] give $\operatorname{cl}_G(q^N) \leq \lfloor N/2 \rfloor + 1$, hence $\operatorname{scl}_G(q) \leq 1/2$. The claim gives $f([g,h]) \leq 1 + M/2$, so $M \leq 1 + M/2$ and hence $M \leq 2$. Applying the claim again gives $f(g) \leq 1 + 2 \operatorname{scl}_G(g)$ whenever $\operatorname{scl}_G(g) < \infty$.

For the second assertion, choose $(g_i)_{i \in I}$ in G whose images form a basis of $G^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $\mathcal{Q}$ be the set of homogeneous quasimorphisms $\phi: G \rightarrow \mathbb{R}$ with defect $D(\phi) \leq 1$ such that $\phi(g_i) = 0$ for all $i \in I$, where

$$D(\phi) := \sup_{g,h \in G} |\phi(gh) - \phi(g) - \phi(h)|.$$

Set $F(g) := 1 + \sup_{\phi \in \mathcal{Q}} |\phi(g)|$. Since the images of the g_i form a basis of $G^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$, there exist $m \geq 1$, indices $i_1, \dots, i_k$, and integers $n_1, \dots, n_k$ such that $z := g^m g_{i_1}^{-n_1} \dots g_{i_k}^{-n_k} \in [G, G]$. Writing $\ell = \operatorname{cl}_G(z)$, every $\phi \in \mathcal{Q}$ satisfies

$$m |\phi(g)| \leq |\phi(z)| + k \leq 2\ell + k.$$

Thus $F(g) < \infty$.

The defect condition gives $F(gh) \leq F(g) + F(h)$, while homogeneity gives $F(g^2) = 2F(g) - 1$. Suppose $\operatorname{scl}_G(g) < \infty$, and let r be the order of the image of g in G^{ab} . Bavard duality [3, Theorem 2.70] for g^r , together with homogeneity, gives

$$2 \operatorname{scl}_G(g) = \sup_{D(\phi) \leq 1} |\phi(g)|.$$

Given such a ϕ , let $\chi: G \rightarrow \mathbb{R}$ be the homomorphism determined by $\chi(g_i) = \phi(g_i)$ for all $i \in I$. Since g has torsion image in G^{ab} , we have $\chi(g) = 0$. Thus $\phi - \chi \in \mathcal{Q}$ has the same defect and value at g as ϕ . Hence the supremum may be restricted to $\mathcal{Q}$, giving $F(g) = 1 + 2 \operatorname{scl}_G(g)$. $\square$

References

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