

Multi-coverings by lattice translates of convex bodies

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Abstract

We show that for every convex body $K \subset \mathbb{R}^n$ ($n \geq 2$), every $0 < \varepsilon < 1$, and every prescribed density $\rho \geq A_\varepsilon n \log n$, there is an arrangement of translates of K along a single lattice with mean multiplicity exactly ρ and multiplicity at least $(1 - \varepsilon)\rho$ everywhere. Consequently, lattice k -fold coverings exist with density at most $\max\{(1 + \varepsilon)k, C_\varepsilon n \log n\}$, where C_ε depends only on ε . In sufficiently large dimension n , a version with prescribed common covolume holds simultaneously for up to $\exp(n^{c_0})$ bodies whose uniform distributions have a common covariance matrix, with $c_0 > 0$ absolute. The proof builds on OpenAI’s horizontal–vertical construction for single-lattice coverings and uses Gaussian layer averaging and Laplace bounds for covering counts.

1 Introduction

Let $K \subset \mathbb{R}^n$ be a *convex body*, i.e., a compact convex set with nonempty interior. For a positive integer k , a k -fold covering (or *multi-covering*) of $\mathbb{R}^n$ by translates of K is a family in which every point of $\mathbb{R}^n$ belongs to at least k distinct translates. We study *lattice k -fold coverings*: the family is exactly $\{K + \lambda : \lambda \in \Lambda\}$ for a full-rank lattice $\Lambda \subset \mathbb{R}^n$, with one translate for each lattice point.

The *multiplicity function* and *density* of this lattice arrangement are

$$N_{\Lambda, K}(x) = \sum_{\lambda \in \Lambda} \mathbf{1}_K(x - \lambda), \quad \rho = \frac{\text{vol}_n(K)}{\det \Lambda},$$

where the *covolume* $\det \Lambda$ is the volume of a fundamental parallelepiped of Λ . The density ρ is the mean of $N_{\Lambda, K}$ on the *lattice torus* $\mathbb{R}^n/\Lambda$, equipped with normalized Haar measure. Thus the covering condition $N_{\Lambda, K}(x) \geq k$ for every $x \in \mathbb{R}^n$ implies $\rho \geq k$. We ask how close the density of a lattice k -fold covering can come to this lower bound.

We write $\theta_{\text{lat}, k}(K)$ for the infimum of the densities of such lattice k -fold coverings. Thus $\theta_{\text{lat}, k}(K) \geq k$. Moreover, the infimum is attained, as the compactness argument in Appendix A shows.

OpenAI [8, Theorem 1.1] proved that $\theta_{\text{lat}, 1}(K) = O(n \log n)$ uniformly over convex bodies. If $K + \Lambda_0$ covers space, then an index- k superlattice of Λ_0 gives a k -fold covering, as each of its k cosets supplies one covering; this yields density $O(kn \log n)$. We show however that it is possible to do much better: the dimensional cost need only be paid once.

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Theorem 1.1 (Lattice multi-coverings). *For every $\varepsilon > 0$ there is a constant C_ε such that, for every $n \geq 2$, every convex body $K \subset \mathbb{R}^n$, and every positive integer k ,*

$$\theta_{\text{lat},k}(K) \leq \max\{(1 + \varepsilon)k, C_\varepsilon n \log n\}.$$

Thus $\theta_{\text{lat},k}(K)/k \rightarrow 1$ whenever $k/(n \log n) \rightarrow \infty$, uniformly in the dimension and the body. Fixed-dimensional asymptotic efficiency follows from elementary lattice refinement, as we recall below.

The proof gives more: We may prescribe the mean count and make the minimum count nearly as large, simultaneously for many bodies with the same covariance matrix.

For a convex body K , the *uniform distribution on K* is normalized Lebesgue measure, with density $\mathbf{1}_K/|K|$, where $|K| = \text{vol}_n(K)$. Its *barycenter* and *covariance matrix* are

$$b_K = \frac{1}{|K|} \int_K x \, dx, \quad \Sigma_K = \frac{1}{|K|} \int_K (x - b_K)(x - b_K)^\top dx,$$

respectively.

Theorem 1.2 (Prescribed covolume). *There is an absolute constant $c_0 > 0$ such that, for every $0 < \varepsilon < 1$, there are constants $A_\varepsilon, n_\varepsilon$ with the following property. Suppose $n \geq n_\varepsilon$, $1 \leq N \leq \exp(n^{c_0})$, and $K_1, \dots, K_N \subset \mathbb{R}^n$ are convex bodies whose uniform distributions have a common covariance matrix. If $\Delta > 0$ satisfies*

$$\rho_j := \frac{\text{vol}_n(K_j)}{\Delta} \geq A_\varepsilon n \log n \quad (1 \leq j \leq N),$$

then there is a full-rank lattice $\Lambda = \Lambda(\Delta)$ with

$$\det \Lambda = \Delta, \quad N_{\Lambda, K_j}(x) \geq (1 - \varepsilon)\rho_j \quad (x \in \mathbb{R}^n, 1 \leq j \leq N).$$

When $N = 1$, the conclusion holds for every $n \geq 2$.

The covariance hypothesis in Theorem 1.2 allows a common linear normalization; it holds, for example, for orthogonal images of a body with scalar covariance matrix. The bodies may be nonsymmetric and need not have comparable volumes.

Proof of Theorem 1.1 from Theorem 1.2. Fix $\varepsilon > 0$ and $k \geq 1$. Put $\zeta = \varepsilon/(1 + \varepsilon)$, and prescribe

$$\rho = \max\{(1 + \varepsilon)k, A_\zeta n \log n\}, \quad \Delta = \frac{|K|}{\rho}.$$

The one-body clause of Theorem 1.2 applies in every dimension $n \geq 2$, giving a lattice of density exactly ρ with

$$N_{\Lambda, K}(x) \geq (1 - \zeta)\rho = \frac{\rho}{1 + \varepsilon} \geq k \quad (x \in \mathbb{R}^n).$$

This proves Theorem 1.1 with $C_\varepsilon = A_{\varepsilon/(1+\varepsilon)}$. □

More generally, for prescribed positive integers $k_1, \dots, k_N$ and $\varepsilon > 0$, set $\zeta = \varepsilon/(1 + \varepsilon)$ and choose

$$\Delta = \min_{1 \leq j \leq N} \frac{\text{vol}_n(K_j)}{\max\{(1 + \varepsilon)k_j, A_\zeta n \log n\}}. \quad (1.1)$$

For $n \geq n_\zeta$, this gives $N_{\Lambda, K_j} \geq \rho_j/(1 + \varepsilon) \geq k_j$ for every j . The common covolume couples the densities: for some bodies, ρ_j may greatly exceed $\max\{(1 + \varepsilon)k_j, A_\zeta n \log n\}$.

Note that Theorem 1.2 supplies a lattice that may depend on Δ ; it does not assert that one lattice shape works at all larger densities. We do not optimize the dependence of $A_\varepsilon, C_\varepsilon$, or n_ε on the accuracy.

The order of the bound is best-possible in the worst case.

Corollary 1.3 (Worst-case order). *Uniformly for $n \geq 2$ and positive integers k ,*

$$\sup_{K=-K} \theta_{\text{lat},k}(K) \asymp \sup_K \theta_{\text{lat},k}(K) \asymp k + n \log n,$$

where the suprema range over origin-symmetric convex bodies and all convex bodies in $\mathbb{R}^n$, respectively.

Proof. Li and Liu [5, Theorem 1.3] construct, for every sufficiently large n , an origin-symmetric body K_n with $\theta_{\text{lat},1}(K_n) \geq c_{\text{LL}} n \log n$, for an absolute $c_{\text{LL}} > 0$. Since $\theta_{\text{lat},k}(K) \geq \max\{k, \theta_{\text{lat},1}(K)\}$, this gives the lower bound; the finitely many smaller dimensions follow from $\theta_{\text{lat},k}(K) \geq k$. Theorem 1.1 with $\varepsilon = 1$ gives the upper bound. $\square$

Related work. Multiple lattice coverings have a classical literature, see, for example, the work of Cohn [1] and Groemer [4]. Asymptotic efficiency in fixed dimension already follows from an elementary refinement. If an ordinary covering lattice for K has density ρ_0 , the argument in Section 5.4 gives

$$\theta_{\text{lat},k}(K) \leq (k^{1/n} + \rho_0^{1/n})^n = k + O_n(k^{(n-1)/n}), \quad (1.2)$$

where a bound for ρ_0 depending only on n makes the estimate uniform over K . The quantitative issue is how large k must be as the dimension grows. To guarantee density at most $(1+\varepsilon)k$, the bound (1.2) requires $k \geq \rho_0[(1+\varepsilon)^{1/n} - 1]^{-n}$; for fixed ε , this sufficient lower bound on k is comparable to $\rho_0(n/\log(1+\varepsilon))^n$, with comparison constants depending only on ε . Theorem 1.1 guarantees the same relative accuracy as soon as $k \geq C_\varepsilon n \log n$; the improvement is that a multiplicity of order $n \log n$ suffices for near-optimal density, uniformly over convex bodies.

Rogers [12] proved that ordinary coverings by arbitrary translates have density $O(n \log n)$. Under the single-lattice restriction, his general bound was $n^{\log_2 \log n + O(1)}$ [14]. Ordentlich, Regev, and Weiss [10] obtained the universal bound $O(n^2)$; Li and Liu [5] then reduced this to $n \log n$ times an iterated-logarithmic factor, which OpenAI [8] was able to remove. Our proof uses local Gaussian block estimates from the OpenAI construction [8, Section 4]. A companion to the OpenAI preprint [8] states [9, Theorem 1.1] that there are centrally symmetric bodies requiring density $\Omega(n \log n)$ even for ordinary coverings by arbitrary translates; we do not use that result here, as the lattice lower bound of Li and Liu already suffices for Corollary 1.3.

For multiple coverings by arbitrary translates, Frankl, Nagy, and Naszódi [3, Theorem 1.3] obtained density $O(n \log n)$ for $k \leq n(\log n + \log \log n)$. Naszódi and Polyanskii [7, Theorem 1.4] proved the bound $3.153(1+o(1)) \max\{n \log n, k\}$. Retaining the free parameter in their greedy bound for multiple transversals [7, Theorem 1.1] also gives $\max\{(1+\varepsilon)k, C_\varepsilon^{\text{tr}} n \log n\}$ for arbitrary translates. Appendix B gives the deduction, including the geometric reduction for nonsymmetric bodies. Thus near-unit efficiency at this scale is already available for arbitrary translates—Theorem 1.1 realizes that efficiency *with a single lattice*.

An ε -smooth covering has multiplicity between $(1-\varepsilon)\rho$ and $(1+\varepsilon)\rho$ everywhere, where ρ is its mean. Ordentlich, Regev, and Weiss [11, Theorem 1.7] construct such coverings at density $O_\varepsilon(n \log n)$ using a *periodic arrangement*, whose centers form a finite union of translates of one lattice. For a single lattice, [11, Theorem 1.2] gives a sufficient density bound cubic in n ; Lund [6, Theorem 1.7] obtains $O(\varepsilon^{-4} n^2)$ in sufficiently large dimension. Both single-lattice results also bound

the failure probability for a Haar-random lattice; for each fixed failure probability, the sufficient density thresholds have the stated orders. In particular, Lund’s theorem implies the analogue of Theorem 1.1 in sufficiently large dimension with n^2 in place of $n \log n$, by taking smoothness error $\varepsilon/(1 + \varepsilon)$ and a fixed failure probability. Our construction establishes the lower multiplicity inequality at every prescribed density $\rho \geq A_\varepsilon n \log n$. Note that we show existence of a specially constructed lattice; we give neither the upper inequality nor the same random-lattice conclusion.

Proof strategy. We describe the proof of Theorem 1.2 for one body in large dimension, at prescribed mean ρ . Put $\delta = \varepsilon/10$. We first work at a slightly smaller mean ρ' , reserving the density cost of a final dilation. The aim is to make the count at least $(1 - 4\delta)\rho'$ outside a sufficiently small set, and then remove that exceptional set.

We use a version of the horizontal–vertical construction of OpenAI [8, Sections 5–6], which builds on the method of Li and Liu [5]. Decompose the ambient space as $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^D$, where $n = m + D$, with a large horizontal dimension m and a small vertical dimension D ; and write its points as $(x, y) \in \mathbb{R}^m \times \mathbb{R}^D$. We use the term *target* informally for the point currently under consideration, either in the full space or in the vertical coordinates being treated.

Put $K[y] = \{x \in \mathbb{R}^m : (x, y) \in K\}$ for its *horizontal section* at height y . Let $\Lambda_h \subset \mathbb{R}^m$ and $L \subset \mathbb{R}^D$ be lattices, let $M : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be invertible, and let $Z : \mathbb{R}^D \rightarrow \mathbb{R}^m$ be linear. The lattice

$$\Gamma = \{(h + Z\ell, M\ell) : h \in \Lambda_h, \ell \in L\}$$

has covering count

$$N_{\Gamma, K}(x, y) = \sum_{\ell \in L} N_{\Lambda_h, K[y - M\ell]}(x - Z\ell). \quad (1.3)$$

Each vertical layer therefore contributes a horizontal section count; Figure 1 illustrates the decomposition in two dimensions. The map Z determines a *shear*: it translates the horizontal arrangement in layer ℓ by $Z\ell$, keeping all centers in one lattice. Each translate retains the shape and orientation of K . The covolume $\det \Gamma = \det \Lambda_h \det(ML)$ is independent of Z , so the shear redistributes overlap between layers while preserving the mean multiplicity. Figure 2 shows the corresponding covering translates. Counts from different layers add, with distinct centers in each layer.

After choosing suitable normalized coordinates, the section volumes satisfy $|K[w]| \approx |K|\gamma_D(w)$ in the range of residual heights we will use, where γ_D is the standard Gaussian density. Put $\Delta_h = \det \Lambda_h$ and $\Delta_v = \det(ML)$, so that $\rho' = |K|/(\Delta_h \Delta_v)$. The mean contribution of layer ℓ is therefore approximately

$$\frac{|K[y - M\ell]|}{\Delta_h} \approx \rho' w_y(\ell), \quad w_y(\ell) = \Delta_v \gamma_D(y - M\ell).$$

The full sum $\sum_{\ell \in L} w_y(\ell)$ has mean 1 over the vertical torus. We need selected layers whose weights total at least $1 - \delta$ at every height, with each individual weight small. This is a lower bound relative to the torus mean; the full sum at a particular height may be much larger.

The vertical construction retains nearly the full mean. We group the vertical coordinates into *blocks*, and use the local Gaussian-weighted lattice constructions of [8, Section 4] in successively smaller blocks. The choices in one block shift the target in the preceding block, averaging its available weight. The last block has no successor, so its weight still depends on the target. Averaging over all its lattice cosets replaces that weight by its mean. We keep the last block bounded in dimension, depending only on the accuracy, so this average has a bounded number of components. Halving the dimensions makes the accumulated local losses summable. The original one-fold argument only

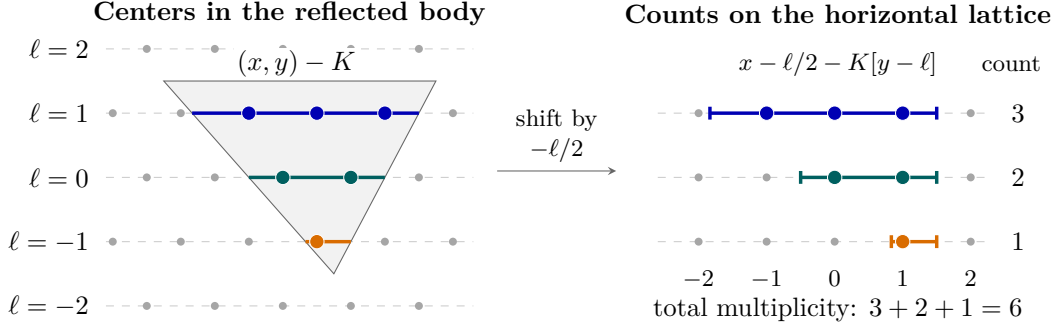


Figure 1: *Counting by horizontal layers.* A translate $K + \lambda$ contains (x, y) precisely when $\lambda \in (x, y) - K$. The downward-pointing triangle is this *reflected translate*: reflect K through the origin, then translate by (x, y) . Here the lattice is $\{(h + \ell/2, \ell) : h, \ell \in \mathbb{Z}\}$. Subtracting the shear $\ell/2$ in each row reduces its contribution to a count on $\mathbb{Z}$. Filled colored dots are the counted centers. Figure 2 shows the actual covering translates.

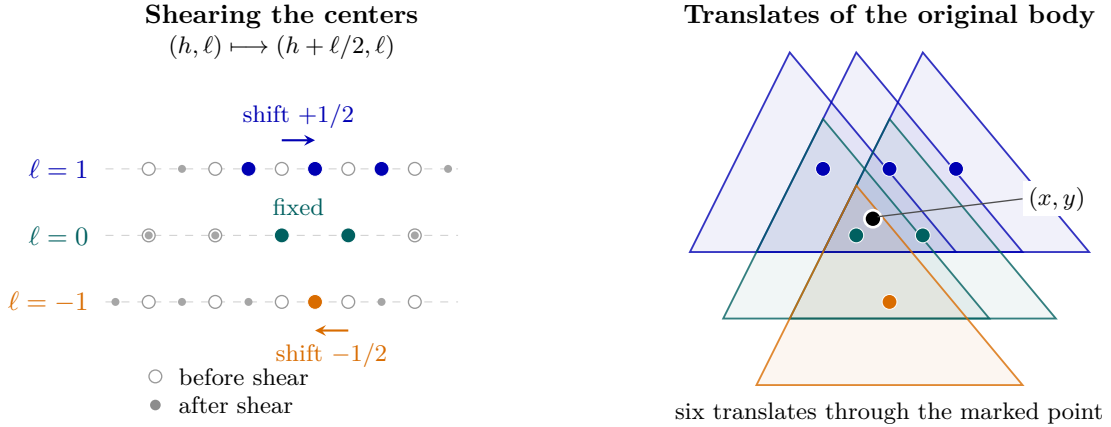


Figure 2: *From the centers in Figure 1 to covering translates.* The shear slides rows of centers; each translate retains the original upward-pointing triangle. The reflection in Figure 1 serves only to count centers. All six displayed translates contain the same marked target, with colors recording their rows. A k -fold covering requires at least k translates from the full arrangement through every point.

needs the positive fraction of the mean supplied by [8, Proposition 5.1, equation (5.10)]; a certificate of near-mean multiplicity based on selected layers requires their mean to be nearly the full mean.

We also make each selected weight small. A direct use of the horizontal mean-hole estimate has an additive error of order e^{-cm} , whereas removing the deficient set at the end requires a proportion of order $e^{-n \log n}$. Many small layers allow us to combine moderate estimates into a much stronger one. The largest block controls how small the individual weights are; its dimension can grow slowly while the last block stays bounded.

To make one horizontal lattice and shear work at every height, we replace the sections by finitely many slightly smaller translated representatives. For ordinary coverings, the shear estimate in [8, Lemma 6.1] makes an intersection of hole sets small. For multiple coverings we instead control the sum of the counts through negative exponentials, which are large where counts are small. The exponential converts the sum into a product, and spatial thinning lets the Rogers–Schmidt estimate

bound the average of each factor over horizontal lattices.

Within each terminal coset, the selected indices are *binary by suffixes*: once the later coordinates are fixed, each coordinate has at most two values, differing by 1 when both occur. A uniform shear column gives the two branches a uniform relative translation. A fractional moment lets Hölder's inequality handle their shared earlier columns. One choice of horizontal lattice and shear then works for all required section patterns. Combining the coset components and applying exponential Markov's inequality bounds the deficient set. Adjusting the exponential parameter keeps the individual tests in the mean-hole estimate's range, while the small layer weights keep the combined bound effective at arbitrarily large prescribed densities.

The final *completion* step removes the exceptional set by enlarging K slightly. If the deficient proportion is small enough, every target $z \in \mathbb{R}^n$ has some $w \in K/q$ for which $z - w$ is sufficiently covered. The same distinct centers then cover z with $K + K/q = (1 + 1/q)K$. Put $q = \lceil n/\delta \rceil$. The dilation factor $s = 1 + 1/q$ costs at most $s^n \leq e^\delta$ in density, so we reserve that cost by starting at mean $\rho' = \rho/s^n$. After completion, rescaling the lattice restores mean ρ exactly.

The count budget is

$$\begin{aligned} \text{selected mean} &\geq (1 - \delta)^2 \rho', \\ \text{count off the deficient set} &\geq (1 - 4\delta) \rho', \\ \text{final count everywhere} &\geq (1 - 5\delta) \rho \geq (1 - \varepsilon) \rho. \end{aligned}$$

The two losses in the selected mean come from the section representatives and the vertical weights. The Laplace estimate allows for its approximation error and slack in Markov's inequality; the final dilation accounts for the remaining loss. Choosing ρ as in the deduction of Theorem 1.1 then makes the final lower count at least k .

Completion explains the scale $n \log n$. It requires a deficient proportion smaller than approximately q^{-n} , and $\log(q^n) = n \log n + O_\delta(n)$. The combined Laplace bound makes that proportion at most n^{-2n} once the density exceeds a sufficiently large accuracy-dependent multiple of $n \log n$. The number of section patterns contributes only a lower-order cost.

With common covariance, the same coordinates, horizontal lattice, and shear work for all bodies. Each body gives its own covering by Λ -translates, with multiplicity compared with its mean $|K_j|/\det \Lambda$; these means need not agree. Figure 3 illustrates the simultaneous covering condition using the same lattice as Figure 2: it gives a 3-fold covering by the triangle and a 5-fold covering by an ellipse with the same covariance.¹

Roadmap and conventions. Sections 2–4 establish the thinning estimate, section comparisons, and vertical patterns; Section 5 assembles the covering. Section 6 shows that the count is close to its mean in L^1 but has large pointwise peaks. The appendixes establish attainment and the translative comparison.

All logarithms are natural unless a base is displayed. The notation $|A|$ denotes volume, cardinality, or normalized torus measure, according to context. Named constants retain their stated values. Implied constants are absolute unless otherwise indicated; asymptotics in the covering construction keep the accuracy fixed.

¹We verify that the illustrated coverings have the claimed multiplicities everywhere in the plane. For the triangle in Figures 1–2, the inverse shear $(u, v) \mapsto (u - v/2, v)$ sends Γ to $\mathbb{Z}^2$ and the triangle to a translate of $\text{conv}\{(0, 0), (4, 0), (0, 3)\}$. This contains the three disjoint half-open unit squares with lower-left corners $(0, 0)$, $(1, 0)$, and $(0, 1)$, each a fundamental domain, giving three distinct covering centers everywhere. For the ellipse in Figure 3, a horizontal section at vertical displacement t from its center has length $2\sqrt{(2/3)(4 - 2t^2)}$. The two nearest distinct lattice rows to the center of any reflected translate have distances at most $1/2$ and 1 . Their sections therefore have lengths greater than 3 and 2, and hence contain at least five lattice points altogether.

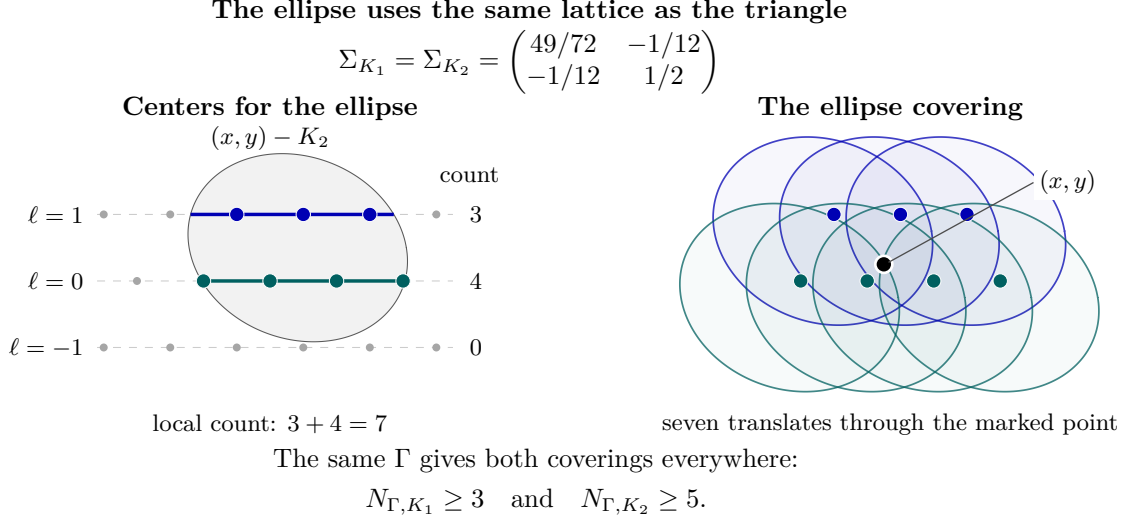


Figure 3: *Simultaneous coverings using the same lattice Γ and shear as in Figure 2.* The seven colored centers give the ellipse translates through $(x, y) = (1/4, 1/4)$ shown on the right. Here $K_1 = \text{conv}\{(-2, -5/4), (2, -5/4), (-1/2, 7/4)\}$, $b = (-1/6, -1/4)$, and $K_2 = \{u : (u - b)^\top \Sigma^{-1}(u - b) \leq 4\}$, so their uniform distributions have the displayed covariance Σ . The full families $\{K_j + \lambda : \lambda \in \Gamma\}$ have respective global multiplicities at least 3 and 5; the displayed local counts are 6 and 7.

2 Thinning and Laplace bounds

To bound small lattice counts, we convert the Rogers–Schmidt mean-hole estimate into a Laplace bound by *independent thinning*. Retaining each of r points independently with probability $1 - e^{-t}$, where $t > 0$, leaves none with probability e^{-tr} . We realize this operation through a random region independent of the lattice; small spatial cells make its tests at distinct lattice points independent.

Write μ_d for invariant probability measure on the space of *unimodular lattices* (i.e., those of covolume 1) in $\mathbb{R}^d$, and $\pi_\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d/\Lambda$ for the quotient map. For a Borel set $A \subset \mathbb{R}^d$, its *uncovered proportion* is $h_\Lambda(A) = 1 - |\pi_\Lambda(A)|$. We use the mean-hole estimate of Rogers [13] (see also Schmidt [15]), in the form recorded in [10, Theorem 2.3]. With

$$\eta_d = \frac{d}{4} \log \frac{27}{16} - 3 \log d,$$

there is an absolute constant C_{RS} such that

$$\left| \int h_\Lambda(A) d\mu_d(\Lambda) - e^{-|A|} \right| \leq C_{\text{RS}} e^{-\eta_d} \quad (|A| \leq \eta_d). \quad (2.1)$$

The estimate applies to arbitrary Borel sets, as the thinning argument requires. We assume $\eta_d > 0$ throughout this section; in the applications $d \rightarrow \infty$ and $\eta_d \asymp d$.

For a bounded nonnegative integrable Borel function f , set

$$S_{\Lambda, f}(x) = \sum_{\lambda \in \Lambda} f(x - \lambda).$$

For $f = \mathbf{1}_K$, each summand records whether $x \in K + \lambda$, so $S_{\Lambda, \mathbf{1}_K} = N_{\Lambda, K}$, the case used for horizontal sections in the covering argument. Integration over a fundamental domain gives

$$\int_{\mathbb{R}^d/\Lambda} S_{\Lambda, f}(x) dx = \frac{1}{\det \Lambda} \int_{\mathbb{R}^d} f. \quad (2.2)$$

Thus the sum is finite almost everywhere; it is finite everywhere when f has compact support.

For $t > 0$, we call $x \mapsto e^{-tS_{\Lambda,f}(x)}$ an *exponential test* for small values of the sum. Its integral over the normalized torus is the *Laplace transform* of the sum at a uniform point. For any threshold $a > 0$,

$$|\{x : S_{\Lambda,f}(x) < a\}| \leq e^{ta} \int_{\mathbb{R}^d/\Lambda} e^{-tS_{\Lambda,f}(x)} dx.$$

A small Laplace transform thus bounds the proportion of points at which the sum falls below a .

For a weighted sum, retain a point y with probability $1 - e^{-tf(y)}$. The integral v_t of this probability is the expected volume of the thinned region. The following lemma transfers the mean-hole estimate whenever $v_t < \eta_d$. Since $v_t \leq t \int f$, this condition can be met by choosing t sufficiently small, even when $\int f$ is large.

Lemma 2.1 (Functional thinning). *Let f be as above and $t > 0$. If $v_t = \int_{\mathbb{R}^d} (1 - e^{-tf}) < \eta_d$, then*

$$\left| \int \int_{\mathbb{R}^d/\Lambda} e^{-tS_{\Lambda,f}(x)} dx d\mu_d(\Lambda) - e^{-v_t} \right| \leq C_{\text{RSE}} e^{-\eta_d}. \quad (2.3)$$

Proof. First suppose f is compactly supported. Put $q = 1 - e^{-tf}$ and partition space into half-open cubes of side h . In each cube meeting the support, choose an independent uniform random variable $U_Q \in [0, 1]$. Define

$$A_h = \bigcup_Q \{y \in Q : q(y) > U_Q\}.$$

This is a bounded Borel set sampled independently of the lattice. All averages below are measurable: writing $\Lambda = g\mathbb{Z}^d$ and $x = gz$ for $z \in [0, 1)^d$, avoidance is a countable intersection of Borel conditions in $(U_Q)_Q$, g , and z . Thus Fubini applies.

In this proof, $\mathbb{E}$, $\mathbb{P}$, and Var refer to the independent random variables $(U_Q)_Q$ used to construct A_h ; averages over lattices are written explicitly as integrals against μ_d . The volumes $V_Q = |A_h \cap Q|$ are independent, with $\mathbb{E}V_Q = \int_Q q$ and $V_Q^2 \leq h^d V_Q$. Hence

$$\mathbb{E}|A_h| = v_t, \quad \text{Var}|A_h| \leq h^d v_t.$$

Thus $|A_h| \rightarrow v_t$ in probability. Since $v_t < \eta_d$, $\mathbb{P}\{|A_h| > \eta_d\} \rightarrow 0$, and $\mathbb{E}e^{-|A_h|} \rightarrow e^{-v_t}$.

Fix a unimodular lattice Λ . Once $\sqrt{d}h$ is smaller than the length of its shortest nonzero vector, distinct points of $x - \Lambda$ lie in distinct cubes for every x . Their retention tests are then independent, giving

$$\mathbb{E}h_{\Lambda}(A_h) = \int_{\mathbb{R}^d/\Lambda} \prod_{\lambda \in \Lambda} (1 - q(x - \lambda)) dx = \int_{\mathbb{R}^d/\Lambda} e^{-tS_{\Lambda,f}(x)} dx.$$

Only finitely many factors differ from 1. The hole measures lie in $[0, 1]$, so dominated convergence handles the lattice-dependent cutoff in h when averaging over μ_d . On $\{|A_h| \leq \eta_d\}$ we can apply (2.1); on its complement the absolute difference in that estimate is at most 1. Averaging over A_h therefore gives

$$\left| \mathbb{E} \int h_{\Lambda}(A_h) d\mu_d(\Lambda) - \mathbb{E}e^{-|A_h|} \right| \leq C_{\text{RSE}} e^{-\eta_d} + \mathbb{P}\{|A_h| > \eta_d\}.$$

Letting $h \rightarrow 0$ proves (2.3).

For general f , apply the compactly supported result to $f_R = f\mathbf{1}_{\{\|y\| \leq R\}}$. The sums increase to $S_{\Lambda,f}$ and the corresponding $v_t(R)$ increase to $v_t < \eta_d$. Bounded convergence then proves the assertion. $\square$

For an indicator $f = \mathbf{1}_A$, the comparison term in (2.3) is $e^{-(1-e^{-t})|A|}$, the Laplace transform of a Poisson random variable of mean $|A|$. More generally, e^{-v_t} is the Laplace transform of $\sum_{z \in \mathcal{X}} f(z)$ for a unit-intensity Poisson point process $\mathcal{X}$. Thus the lemma compares exponential tests averaged over the lattice and a uniform point of its torus, in the range $v_t < \eta_d$. The covering proof combines these transforms before estimating an exceptional set; it requires no independence between section counts.

3 Common Gaussian sections

The preceding Laplace bound concerns a fixed region. To treat all required heights and bodies in one probabilistic choice, we refine [8, Lemma 2.1] to find common coordinates in which the section volumes, divided by the body volumes, have an approximately Gaussian profile. Finitely many representative bodies then fit inside the sections after translation, with arbitrarily small relative volume loss. Their counts bound the actual section counts from below.

Write $\gamma_D(y) = (2\pi)^{-D/2} e^{-\|y\|^2/2}$ for the *standard Gaussian density*. A probability distribution is *isotropic* if its mean is zero and its covariance matrix is the identity. Uniform distributions on convex bodies are log-concave. The pointwise marginal theorem of Eldan and Klartag [2, Theorem 1] supplies absolute positive constants c_1, c_2, c_3, c_4, C_E such that, for an isotropic log-concave probability distribution on $\mathbb{R}^n$ and $1 \leq D \leq n^{c_1}$, the orthogonal projection onto a Haar-random D -dimensional subspace E has a density f_E satisfying

$$\left| \frac{f_E(y)}{\gamma_D(y)} - 1 \right| \leq C_E n^{-c_3} \quad (\|y\| \leq n^{c_4}) \quad (3.1)$$

with probability at least $1 - C_E e^{-n^{c_2}}$. Fix an absolute constant

$$0 < c_0 < \min(c_2, 1/2). \quad (3.2)$$

For $K \subset \mathbb{R}^m \times \mathbb{R}^D$, write $K[y] = \{x \in \mathbb{R}^m : (x, y) \in K\}$. For a family of bodies, $K_j[y]$ denotes the section of the j -th body.

Lemma 3.1 (Common Gaussian sections). *Fix $0 < \delta < 1/10$ and $A_{\text{sec}} > 0$. For all sufficiently large n , depending only on these two parameters, the following holds. Let $K_1, \dots, K_N \subset \mathbb{R}^n$ be convex bodies whose uniform distributions have a common covariance matrix, with $1 \leq N \leq e^{n^{c_0}}$. Choose a vertical dimension D and a radius R satisfying*

$$1 \leq D \leq A_{\text{sec}}(\log \log n)^2, \quad 1 \leq R, \quad R^2 \leq A_{\text{sec}} D \log(D+1).$$

After centering the bodies separately and applying one common linear map, there are coordinates $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^D$, with $m = n - D$, common to all the bodies. Write $V_j = |K_j|$ in these coordinates. For each body K_j there are:

- an affine map $T_j : \mathbb{R}^D \rightarrow \mathbb{R}^m$, written $T_j(y) = A_j y + t_j$;
- a family of at most $(3m^2)^D$ representative convex bodies $J_{j,a} \subset \mathbb{R}^m$.

For each j and every height y with $\|y\| \leq R$, we can choose a label a such that the translated representative lies in the section and has the required volume:

$$\begin{aligned} J_{j,a} + T_j(y) &\subset K_j[y], \\ (1 - \delta)V_j \gamma_D(y) &\leq |J_{j,a}| \leq V_j \gamma_D(y). \end{aligned} \quad (3.3)$$

The label may depend on the body and the height; the finite families and affine maps are chosen once.

Whenever label a serves height y , the containment transfers covering counts for any horizontal lattice:

$$N_{\Lambda_h, K_j[y]}(x) \geq N_{\Lambda_h, J_{j,a}}(x - T_j(y)).$$

Every center counted on the right is also counted on the left.

Proof of Lemma 3.1. Center each body and apply the inverse square root of their common covariance matrix, making every uniform distribution isotropic. Put $R_{\text{out}} = 2DR + 2$. For sufficiently large n , we have $D \leq n^{c_1}$, $R_{\text{out}} < n^{c_4}$, and $C_E e^{n^{c_0} - n^{c_2}} < 1$. A union bound in (3.1) therefore gives one subspace for which the marginal estimate holds for every body.

Use orthonormal coordinates $E^\perp \times E$ and put $\varepsilon_n = C_E n^{-c_3}$. For the uniform distribution on K_j , the projection density is $f_{j,E}(y) = |K_j[y]|/V_j$ almost everywhere. The section-volume function is upper semicontinuous: compactness gives $\limsup_\nu \mathbf{1}_{K_j}(x, y_\nu) \leq \mathbf{1}_{K_j}(x, y)$ when $y_\nu \rightarrow y$, and all sections have a common bounded horizontal support, so reverse Fatou applies. Its almost-everywhere Gaussian lower bound therefore extends to every height:

$$|K_j[y]| \geq (1 - \varepsilon_n) V_j \gamma_D(y) \quad (\|y\| \leq R_{\text{out}}). \quad (3.4)$$

To compare nearby sections, choose affine centers as in [8, Lemma 2.1]. Let $v_0, \dots, v_D$ be the vertices of a regular simplex centered at 0 with inradius $2R$. Their norms are $2DR < R_{\text{out}}$, so (3.4) lets us choose $z_{j,i} \in K_j[v_i]$. Define the affine map T_j by $T_j(v_i) = z_{j,i}$. If $y = \sum_i \alpha_i v_i$ is in the simplex, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$, convexity gives

$$(T_j(y), y) = \sum_i \alpha_i (z_{j,i}, v_i) \in K_j.$$

Since the simplex contains the ball of radius $2R$, we obtain

$$0 \in \widehat{K}_j[y] := K_j[y] - T_j(y) \quad (\|y\| \leq 2R).$$

Segments toward those origins let us move between nearby heights while shrinking the horizontal section by a controlled factor.

To control both the contraction loss and the Gaussian variation, choose a maximal R/m^2 -separated set $\{y_a\}$ in the ball of radius R , using the same net for every body. It has at most $(3m^2)^D$ elements, so its logarithmic cardinality is $O(D \log n)$. Every height in the ball lies within R/m^2 of a net point. Convexity gives

$$(1 - m^{-2}) \widehat{K}_j[y_a] \subset \widehat{K}_j[y] \quad (\|y - y_a\| \leq R/m^2).$$

Indeed, $y = (1 - m^{-2})y_a + m^{-2}z$ for $z = y_a + m^2(y - y_a)$, where $\|z\| \leq 2R$. The contraction retains a fraction $(1 - m^{-2})^m \geq 1 - m^{-1}$ of the volume. By (3.4), its volume is therefore at least $(1 - \varepsilon_n)(1 - m^{-1})V_j \gamma_D(y_a)$. For sufficiently large n , we may shrink it further about zero to obtain a contained convex body $J_{j,a}$ of volume exactly $(1 - \delta/2)V_j \gamma_D(y_a)$. For y and y_a in the radius- R ball with $\|y - y_a\| \leq R/m^2$,

$$\left| \log \frac{\gamma_D(y_a)}{\gamma_D(y)} \right| \leq R^2/m^2 \leq \delta/8.$$

Consequently $|J_{j,a}|/(V_j \gamma_D(y))$ lies between $(1 - \delta/2)e^{-\delta/8} \geq 1 - \delta$ and $(1 - \delta/2)e^{\delta/8} \leq 1$, as required. $\square$

Separate body translations preserve the covering requirements, and the common linear map multiplies all volumes and covolumes by the same factor. The body-dependent maps T_j are used only to compare sections; they do not change the ambient coordinates of the covering problem.

4 Averaging the last Gaussian block

We now choose the vertical lattice and, at each height, finitely many layers whose normalized Gaussian weights total at least $1 - \delta$, with each weight small. Multiplying the Gaussian weights by the vertical covolume makes their full sum have torus mean 1.

A *block* is a group of coordinates, viewed as one factor $\mathbb{R}^b$ in a product decomposition of the vertical space. The standard Gaussian density factors over these groups. We call the local lattice construction in one factor, weighted by γ_b , a *Gaussian block*. We assemble the block estimates of [8, Section 4] in successively smaller dimensions and finish with an exact average over the last block's cosets.

One block. For a block of dimension b , put $h(b) = \sqrt{(9/4) \log b}$ and measure heights in units of $h(b)$. The block lattice will have the form $L_b = \mathbb{Z}^b + \mathcal{C}_b/p(b)$, where $p(b)$ is a prime and $\mathcal{C}_b \subset \mathbb{F}_{p(b)}^b$ is a one-dimensional subspace. Since $[L_b : \mathbb{Z}^b] = p(b)$, the lattice $h(b)L_b$ in the original height units has covolume $h(b)^b/p(b)$. For a target $t \in \mathbb{R}^b$, choose a cube base point $\lambda \in L_b$ with $\tau = t - \lambda \in [0, 1]^b$. The 2^b corners $\lambda + \sigma$, $\sigma \in \{0, 1\}^b$, of the cube $\lambda + [0, 1]^b$ are candidate layers. Their residuals lie in $[-1, 1]^b$, and their Gaussian weights split into a total weight and a product probability law. Specifically, define

$$\begin{aligned} f_b(v) &= h(b)\gamma_1(h(b)v) + h(b)\gamma_1(h(b)(v-1)) \quad (0 \leq v \leq 1), \\ F_b(\tau) &= \prod_{s=1}^b f_b(\tau_s), \quad P_{b,\tau}(\sigma) = \prod_{s=1}^b \frac{h(b)\gamma_1(h(b)(\tau_s - \sigma_s))}{f_b(\tau_s)}. \end{aligned} \quad (4.1)$$

Then

$$\frac{h(b)^b}{p(b)} \gamma_b(h(b)(\tau - \sigma)) = \frac{F_b(\tau)}{p(b)} P_{b,\tau}(\sigma).$$

The factor $F_b(\tau)/p(b)$ is the normalized *folded weight* of the cube; $P_{b,\tau}$ is its *Gaussian-weighted bit law*. A coordinate is *balanced* if both bit probabilities are at most $e/(1+e)$, equivalently $h(b)^2|\tau_s - 1/2| \leq 1$. Many balanced coordinates spread the folded weight over many corners.

We write

$$\mathbf{m}(b) = \frac{\int_0^1 f_b(v) \log f_b(v) dv}{\int_0^1 f_b(v) dv}.$$

The denominator is $\mathbb{P}\{|X| \leq h(b)\}$ for standard normal X ; its b -th power is $1 - O(b^{-1/8})$, so truncation to $[-h(b), h(b)]^b$ retains nearly all the Gaussian mass. The quantity $\mathbf{m}(b)$ is the mean logarithmic folded density; it determines the prime window below.

We now record the precise consequences of the Gaussian block estimates in [8, Lemmas 4.1, 4.2, and 4.4] that we use. The following lemma constructs a block lattice and a function g_b recording the normalized Gaussian weight available in the current block. Its mean is close to 1, although it may vanish at some targets. Wherever g_b is positive, the lemma selects a cube with small individual corner weights and fixes a bit law at a nearby grid point, with a small comparison loss at the actual residual. The same bits sample the preceding block's function g_0 , preserving almost all of its mean uniformly over translates.

Lemma 4.1 (Gaussian block input). *There are absolute constants $B_{\text{blk}}, C_{\text{blk}} > 0$ and $c_{\text{at}} > 0$ with the following property. For every integer $b \geq B_{\text{blk}}$, put $h(b) = \sqrt{(9/4) \log b}$. There is a prime $p(b)$ such that*

$$b\mathbf{m}(b) - 4b^{14/25} \leq \log p(b) \leq b\mathbf{m}(b) - 3b^{14/25}. \quad (4.2)$$

Moreover, $\mathbf{m}(b) = \log h(b) + O(1)$, and hence

$$\log p(b) = b \log h(b) + O(b) = O(b \log \log b), \quad (4.3)$$

with absolute implied constants. Put $\mathcal{G}_b = \mathbb{F}_{p(b)}^b$ and $\xi(b) = bh(b)^2/p(b)$. Suppose H is a finite abelian group with $\log |H| \leq b^4$ and $g_0 : H \rightarrow [0, \infty)$ satisfies

$$\frac{1}{2} \leq \bar{g}_0 \leq 2, \quad g_0 \leq \exp(5b^{28/45}),$$

where bars denote uniform means. Then there are shifts $w_1, \dots, w_b \in H$, a one-dimensional subspace $\mathcal{C}_b \subset \mathcal{G}_b$, and a function $g_b : \mathcal{G}_b \rightarrow [0, \infty)$, constant on $\mathcal{C}_b$ -cosets, such that

$$|\bar{g}_b - 1| \leq C_{\text{blk}} b^{-1/100}, \quad g_b \leq \exp(5b^{14/25}). \quad (4.4)$$

Write $L_b = \mathbb{Z}^b + \mathcal{C}_b/p(b)$, using integer representatives for $\mathcal{C}_b$. For every $t \in \mathbb{R}^b$ with $g_b(x) > 0$, where $x = \lfloor p(b)t \rfloor \bmod p(b)$, there are $\lambda \in L_b$ and a probability law π_t on $\{0, 1\}^b$ such that, for every $\sigma \in \{0, 1\}^b$,

$$\|t - \lambda - \sigma\|_\infty \leq 1, \quad (4.5)$$

$$e^{-\xi(b)} g_b(x) \pi_t(\sigma) \leq \frac{h(b)^b}{p(b)} \gamma_b(h(b)(t - \lambda - \sigma)) \leq e^{-c_{\text{cat}} b^{7/10}}, \quad (4.6)$$

and, simultaneously for every $u \in H$,

$$\sum_{\sigma \in \{0, 1\}^b} \pi_t(\sigma) g_0 \left(u - \sum_{s=1}^b \sigma_s w_s \right) \geq (1 - C_{\text{blk}} b^{-1/100}) \bar{g}_0. \quad (4.7)$$

There is also a base version with no H, g_0 , or shifts, for which (4.4)–(4.6) hold.

Derivation from the block estimates in [8, Section 4]. Grid selection. The prime window and the estimate for $\mathbf{m}(b)$ are given by [8, Lemma 4.1, equation (4.3)]. For a grid point $z \in \mathcal{G}_b$, use its representative in $\{0, \dots, p(b) - 1\}^b$ and put $t_z = z/p(b)$. It is *eligible* if

$$|\log F_b(t_z) - b\mathbf{m}(b)| \leq b^{14/25}, \quad \#\{s : h(b)^2 |t_{z,s} - 1/2| \leq 1\} \geq b^{7/10}.$$

Thus eligibility controls the total folded weight and guarantees many balanced bits.

The construction in [8, Lemma 4.2] selects common shifts w_s and a subset U_b of eligible points at which these bits sample g_0 uniformly over its translate:

$$\sum_{\sigma \in \{0, 1\}^b} P_{b, t_z}(\sigma) g_0 \left(u - \sum_{s=1}^b \sigma_s w_s \right) \geq (1 - C_{\text{blk}} b^{-1/100}) \bar{g}_0 \quad (z \in U_b, u \in H);$$

these are the *usable* grid points of [8, Lemma 4.2]. The hypothesis on g_0 in [8, Lemma 4.2] allows the stated cap, as $(10/9)(14/25) = 28/45$.

The construction in [8, Lemma 4.2] also supplies a line $\mathcal{C}_b$ such that

$$g_b(x) = \max_{c \in \mathcal{C}_b} \frac{F_b(t_{x-c})}{p(b)} \mathbf{1}_{U_b}(x - c) \quad (4.8)$$

satisfies (4.4); see [8, equations (4.11)–(4.12)]. For the base version, take U_b to be the full eligible set and use the final clause of [8, Lemma 4.2].

The negative offset in (4.2) is deliberate. For a suitable line, cosets containing competing usable points contribute little to the total folded weight. The coset maximum therefore retains nearly the mean of the coset sum; this is the argument in [8, equations (4.15)–(4.17)]. The resulting near-1 estimate is an average over cosets, not a pointwise estimate for g_b .

From grid points to targets. For a target t with $g_b(x) > 0$, put $\nu_t = \lfloor p(b)t \rfloor$ and $x = \nu_t \bmod p(b)$. Choose $z \in x - \mathcal{C}_b$ attaining the maximum in (4.8); then $z \in U_b$. Define

$$\begin{aligned}\lambda &= \frac{\nu_t - z}{p(b)} \in L_b, & \pi_t &= P_{b,t_z}, \\ \tau &= t - \lambda = \frac{z + \{p(b)t\}}{p(b)} \in t_z + [0, p(b)^{-1})^b.\end{aligned}$$

Here fractional parts are taken coordinatewise. These choices agree with [8, Lemma 4.4, equations (4.18)–(4.19)], and $g_b(x) = F_b(t_z)/p(b)$. The law π_t is evaluated at the grid point t_z , not at the actual residual τ ; it depends on t but not on u . Its sampling bound is therefore simultaneous for every u . Since $\tau \in [0, 1)^b$, all corners have residuals in $[-1, 1]^b$.

Moving from t_z to τ changes the logarithm of each Gaussian corner weight by at most $\xi(b)$; see [8, equation (4.6)]. Consequently

$$e^{-\xi(b)} g_b(x) \pi_t(\sigma) \leq \frac{h(b)^b}{p(b)} \gamma_b(h(b)(\tau - \sigma)) \leq e^{\xi(b)} g_b(x) \pi_t(\sigma) \quad (\sigma \in \{0, 1\}^b).$$

Eligibility and the prime window give $g_b(x) \leq \exp(5b^{14/25})$. The balanced coordinates give $\pi_t(\sigma) \leq (e/(1+e))^{b^{7/10}}$ for every σ . Hence

$$e^{\xi(b)} g_b(x) \pi_t(\sigma) \leq \exp(\xi(b) + 5b^{14/25}) \left(\frac{e}{1+e} \right)^{b^{7/10}} \leq e^{-c_{\text{at}} b^{7/10}},$$

with fixed absolute $c_{\text{at}} > 0$ after choosing B_{blk} sufficiently large. Here $14/25 < 7/10$, and (4.3) makes $\xi(b)$ smaller than every fixed negative power of b . This proves the residual, weight, and sampling conclusions, including the base version. $\square$

We will assemble these blocks into a vertical lattice of the form ML , where points of L encode the successive choices and the invertible map M places them in Euclidean space. The shear argument requires the following branching structure on these indices.

A finite nonempty subset of $\mathbb{R}^d$ is *binary by suffixes* if, on reading coordinates from last to first, each coordinate has at most two possible values once the later coordinates are fixed, and these values differ by 1 when both occur. Thus the two choices may depend on the later coordinates. Such a set has at most 2^d points.² Nonempty subsets and slices obtained by fixing and deleting final coordinates retain the property. In dimension zero the pattern consists of the empty tuple.

Proposition 4.2 (Vertical patterns). *For every $\delta \in (0, 1/10)$ there are integers $B_\delta, I_\delta \geq 1$ with the following property. For every integer $b_1 \geq B_\delta$ there are a dimension D , a radius $R \geq 1$, a full-rank lattice $L \subset \mathbb{R}^D$ containing $\mathbb{Z}^D$, and an invertible linear map M such that, writing $\Delta_v = \det(ML)$,*

$$b_1 \leq D \leq 2b_1, \quad R^2 \leq 5b_1 \log(b_1 + 1),$$

²For example, $\{(0, 0), (1, 0), (3, 1), (4, 1)\}$ is binary by suffixes: the last coordinate is 0 or 1, and the first-coordinate choices are $\{0, 1\}$ or $\{3, 4\}$ accordingly.

and every $y \in \mathbb{R}^D$ has a finite nonempty pattern $\mathcal{P}_y \subset L$ satisfying

$$\|y - M\ell\| \leq R \quad (\ell \in \mathcal{P}_y), \quad (4.9)$$

$$\sum_{\ell \in \mathcal{P}_y} \Delta_v \gamma_D(y - M\ell) \geq 1 - \delta, \quad (4.10)$$

$$\max_{\ell \in \mathcal{P}_y} \Delta_v \gamma_D(y - M\ell) \leq \exp(-c_{\text{at}} b_1^{7/10}). \quad (4.11)$$

The pattern is a disjoint union of at most I_δ patterns that are binary by suffixes. There are fixed finite sets $\mathcal{A}_1, \dots, \mathcal{A}_D$ with $\log |\mathcal{A}_s| = O(D \log(D+2))$ such that $\mathcal{P}_y \subset \prod_{s=1}^D \mathcal{A}_s$ whenever $y \in M[0, 1]^D$. The implied constant in the alphabet bound is absolute.

Proof. Iterating Lemma 4.1 gives a lower bound for the total selected weight that still depends on the last block's target. We first establish this recursive bound, then remove the dependence by averaging over the last block's cosets.

Block construction. For a sufficiently large cutoff B , set $b_{i+1} = \lfloor b_i/2 \rfloor$ while this is at least B , and stop at b_r . Then

$$B \leq b_r < 2B, \quad b_i \geq 2b_{i+1}, \quad D := \sum_{i=1}^r b_i \leq 2b_1. \quad (4.12)$$

A large initial block gives small individual weights, while a bounded final block limits the number of terminal cosets. Halving connects these scales with summable losses and, as we check next, permits successive applications of the block input.

Fix one prime $p(b)$ as in Lemma 4.1 for each dimension $b \geq B$. In block i , of dimension b_i , abbreviate $h_i = h(b_i)$, $p_i = p(b_i)$, and $\mathcal{G}_i = \mathbb{F}_{p_i}^{b_i}$. Apply Lemma 4.1 successively, starting with its base version and then using $H = \mathcal{G}_{i-1}$ and $g_0 = g_{i-1}$. For sufficiently large B , its hypotheses hold at every step. Equation (4.4) places the preceding mean in $[1/2, 2]$. Together with (4.3) and the fact that $b_{i-1} \leq 2b_i + 1 \leq 3b_i \leq b_i^{10/9}$, we thus obtain

$$g_{i-1} \leq \exp(5b_{i-1}^{14/25}) \leq \exp(5b_i^{28/45}),$$

$$\log |\mathcal{G}_{i-1}| = O(b_{i-1}^2 \log \log b_{i-1}) \leq b_i^4.$$

We obtain lines $\mathcal{C}_i \subset \mathcal{G}_i$, functions g_i , and, for $i < r$, coupling matrices W_i of size $b_i \times b_{i+1}$: represent the shifts in $\mathcal{G}_i$ by columns in $\{0, \dots, p_i - 1\}^{b_i}$ and divide by p_i . Thus $p_i W_i$ is integral, while every entry of W_i lies in $[0, 1)$.

Use block coordinates $\mathbb{R}^D = \prod_{i=1}^r \mathbb{R}^{b_i}$; subscripts on vectors denote blocks, and floors and reductions modulo a prime are taken coordinatewise. The block lattices $L_i^0 = \mathbb{Z}^{b_i} + \mathcal{C}_i/p_i$ are the usual *Construction A* preimages of the lines $\mathcal{C}_i$ over $\mathbb{F}_{p_i}$, scaled by p_i^{-1} ; they have covolume p_i^{-1} . Put $L^0 = \prod_i L_i^0$ and define

$$(M\ell)_i = h_i(\ell_i + W_i \ell_{i+1}) \quad (i < r), \quad (M\ell)_r = h_r \ell_r, \quad \Delta_v^0 = \det(ML^0) = \prod_i \frac{h_i^{b_i}}{p_i}.$$

The map is triangular: changing ℓ_i moves only its own block and, when present, the preceding block. We can therefore choose points from the last block backwards, leaving the later choices undisturbed and absorbing each new block's effect into a translation of the earlier target. The bits of block i sample the preceding weight g_{i-1} at shifted grid points, preserving nearly its mean.

The recursive weight estimate. For $i \leq r$, retain only the first i blocks of M , omitting their interaction with block $i + 1$. Denote the resulting map by $M^{[i]}$ and put $d_i = \sum_{j \leq i} b_j$. The lattice $M^{[i]}(\prod_{j \leq i} L_j^0)$ has covolume $\Delta_v^{[i]} = \prod_{j \leq i} h_j^{b_j}/p_j$.

We claim that, for every target $y \in \mathbb{R}^{d_i}$, there is a possibly empty pattern $P_y^{[i]} \subset \prod_{j \leq i} L_j^0$ with

$$\sum_{\ell \in P_y^{[i]}} \Delta_v^{[i]} \gamma_{d_i}(y - M^{[i]} \ell) \geq \alpha_i g_i(x_i), \quad x_i = \lfloor p_i y_i / h_i \rfloor \bmod p_i.$$

The pattern is binary by suffixes when nonempty, and each of its points satisfies the block residual bounds $\|(y - M^{[i]} \ell)_j\|_\infty \leq h_j$. The coefficients are defined by

$$\alpha_1 = e^{-\xi_1}, \quad \alpha_i = \alpha_{i-1} e^{-\xi_i} (1 - C_{\text{blk}} b_i^{-1/100}) \bar{g}_{i-1}, \quad \xi_i = \xi(b_i) = b_i h_i^2 / p_i. \quad (4.13)$$

If $g_i(x_i) = 0$, we take the empty pattern. Otherwise, we apply the block input at $t = y_i/h_i$, obtaining λ_i and π_t . For $i = 1$, take all $\lambda_1 + \sigma$ and sum (4.6).

For $i > 1$, attach to each last-block choice $\ell_i = \lambda_i + \sigma$ the preceding pattern with modified target

$$\tilde{y}_{i-1} = y_{i-1} - h_{i-1} W_{i-1}(\lambda_i + \sigma),$$

leaving the earlier target blocks unchanged. Its last grid index is

$$x - p_{i-1} W_{i-1} \sigma \pmod{p_{i-1}}, \quad x = \lfloor p_{i-1} y_{i-1} / h_{i-1} - p_{i-1} W_{i-1} \lambda_i \rfloor \pmod{p_{i-1}}. \quad (4.14)$$

Here $p_{i-1} W_{i-1} \sigma$ is integral, so it commutes with the coordinatewise floor. The vector involving λ_i need not be integral; no compatibility between consecutive primes is required.

By induction and (4.6), the total weight is at least

$$\alpha_{i-1} e^{-\xi_i} g_i(x_i) \sum_{\sigma \in \{0,1\}^{b_i}} \pi_t(\sigma) g_{i-1}(x - p_{i-1} W_{i-1} \sigma) \geq \alpha_i g_i(x_i),$$

where the last step is (4.7). Thus the recurrence has no further normalization loss.

Different bit choices give different last blocks. Once a suffix of later blocks is fixed, each current block lies in one translate $\lambda_j + \{0,1\}^{b_j}$. This proves the scalar suffix property. The residual bounds follow from (4.5); the normalized Gaussian factors multiply over the blocks, so (4.6) gives an individual weight at most

$$\prod_{j \leq i} e^{-c_{\text{at}} b_j^{7/10}} \leq e^{-c_{\text{at}} b_1^{7/10}}.$$

Taking $i = r$ and writing $P_y^0 = P_y^{[r]}$ gives

$$\sum_{\ell \in P_y^0} \Delta_v^0 \gamma_D(y - M \ell) \geq \alpha_r g_r(\lfloor p_r y_r / h_r \rfloor \bmod p_r). \quad (4.15)$$

Accumulated loss. Equation (4.3) makes ξ_i smaller than every fixed negative power of b_i . Together with (4.4) and $b_{i-1} \geq b_i$, this gives a fixed absolute $C_{\text{step}} > 0$ such that $\alpha_1 \geq 1 - C_{\text{step}} b_1^{-1/100}$ and $\alpha_i / \alpha_{i-1} \geq 1 - C_{\text{step}} b_i^{-1/100}$.

Set $C_{\text{sum}} = C_{\text{step}} / (1 - 2^{-1/100})$. The dyadic sizes give $\sum_i b_i^{-1/100} \leq B^{-1/100} / (1 - 2^{-1/100})$. With B large enough all factors below are nonnegative, so

$$\alpha_r \geq \prod_i (1 - C_{\text{step}} b_i^{-1/100}) \geq 1 - C_{\text{step}} \sum_i b_i^{-1/100} \geq 1 - C_{\text{sum}} B^{-1/100} =: \kappa_B.$$

Averaging over cosets. Choose $B = B_\delta$ sufficiently large that

$$\kappa_B(1 - C_{\text{blk}}B^{-1/100}) \geq 1 - \delta.$$

Although $\bar{g}_r$ is close to 1, the value of g_r at a particular target need not be. The last block has bounded dimension, so retaining all its cosets requires only a bounded number of binary components. Enlarge only the last factor of L^0 :

$$L = \left(\prod_{i < r} L_i^0 \right) \times p_r^{-1} \mathbb{Z}^{b_r}, \quad \iota = [L : L^0] = p_r^{b_r-1}, \quad \Delta_v = \iota^{-1} \Delta_v^0.$$

Let $\mathcal{V}$ be integer representatives for $\mathcal{G}_r/\mathcal{C}_r$ and write $q_v = (0, \dots, 0, v/p_r)$. For each target put

$$P_v = q_v + P_{y-Mq_v}^0, \quad \mathcal{P}_y = \bigcup_{v \in \mathcal{V}} P_v.$$

Distinct q_v represent distinct cosets of L^0 in L , so the components are disjoint. Translation preserves their residuals and suffix property.

Although q_v is supported in the last block, Mq_v also changes block $r-1$ when $r > 1$. This causes no loss: (4.15) holds for every earlier target, and its right side depends only on the last block. In that block the integer vector v changes the grid index from $x_r = \lfloor p_r y_r / h_r \rfloor \bmod p_r$ to $x_r - v$. Hence

$$\sum_{\ell \in \mathcal{P}_y} \Delta_v \gamma_D(y - M\ell) \geq \frac{\kappa_B}{\iota} \sum_{v \in \mathcal{V}} g_r(x_r - v) = \kappa_B \bar{g}_r \geq 1 - \delta. \quad (4.16)$$

The equality holds because g_r is constant on $\mathcal{C}_r$ -cosets.

The union is therefore nonempty. Each normalized individual weight is an inherited weight divided by ι , which proves (4.11). Omitting empty components leaves at most ι components. Since $B_\delta \leq b_r < 2B_\delta$, we may fix

$$I_\delta = \max_{B_\delta \leq b < 2B_\delta} p(b)^{b-1},$$

so that $\iota \leq I_\delta$ uniformly in the initial block size. Also L contains $\mathbb{Z}^D$, and the inherited residual bounds give

$$R^2 = \sum_i b_i h_i^2 \leq \frac{9}{4} D \log b_1 \leq \frac{9}{2} b_1 \log b_1 \leq 5b_1 \log(b_1 + 1).$$

The finite coordinate sets. To choose one shear for all heights, we need finitely many possible patterns when $y \in M[0, 1)^D$. For $y = Mz$ with $z \in [0, 1)^D$, put $e_i = z_i - \ell_i$. The residual bounds imply

$$\|e_r\|_\infty \leq 1, \quad \|e_i\|_\infty \leq 1 + D\|e_{i+1}\|_\infty.$$

Hence each coordinate of ℓ is bounded by $Q_{\text{coord}} = 1 + (D+2)^D$. Since $\ell_i \in p_i^{-1} \mathbb{Z}^{b_i}$, including in the enlarged last block, each scalar coordinate has at most $2Q_{\text{coord}}p_i + 1$ possible values. Since $b_i \leq D$, equation (4.3) gives the claimed bound. $\square$

5 Coverings with prescribed covolume

We now assemble the covering from the vertical patterns of Proposition 4.2 and the representative bodies $J_{j,a}$ for the horizontal sections of K_j supplied by Lemma 3.1. We choose one horizontal lattice and shear for these representatives, then use completion to remove the deficient set. The shear estimate extends [8, Lemma 6.1] to nonnegative functions. For a pattern P , a fractional moment of order $1/|P|$ lets Hölder's inequality handle dependence between branches; we later apply Hölder again to average over the common horizontal lattice.

Lemma 5.1 (Functional shear). *Let $\Lambda_h \subset \mathbb{R}^m$ be a full-rank lattice, and equip $\mathbb{T} = \mathbb{R}^m / \Lambda_h$ with normalized Haar measure. Let $P \subset \mathbb{R}^D$ be binary by suffixes, and let $\phi_\ell : \mathbb{T} \rightarrow [0, \infty)$, $\ell \in P$, be bounded measurable functions. Choose the columns of a linear map $Z : \mathbb{R}^D \rightarrow \mathbb{R}^m$ independently and uniformly in a fixed fundamental parallelepiped of Λ_h , and put*

$$G_P(Z) = \int_{\mathbb{T}} \prod_{\ell \in P} \phi_\ell(x - \pi_{\Lambda_h}(Z\ell)) dx.$$

Then

$$\mathbb{E} G_P^{1/|P|} \leq \prod_{\ell \in P} \left(\int_{\mathbb{T}} \phi_\ell \right)^{1/|P|}. \quad (5.1)$$

Proof. We induct on the number of coordinates. In dimension zero P is a singleton and equality holds. If the last coordinate has only one value, its column only translates the product, and induction applies. Otherwise let c and $c+1$ be its two values. Deleting the last coordinate from the points whose last coordinate is $c+i$, and retaining their functions, gives slices $P_0, P_1 \subset \mathbb{R}^{D-1}$. They are binary by suffixes, and $|P_0| + |P_1| = |P|$. Write z for the last column of Z and Z' for the others, and put

$$\psi_i(x) = \prod_{\ell' \in P_i} \phi_{(\ell', c+i)}(x - \pi_{\Lambda_h}(Z'\ell')) \quad (i = 0, 1),$$

so that $G_{P_i}(Z') = \int_{\mathbb{T}} \psi_i$. Translation invariance of the torus integral gives, for real c ,

$$\begin{aligned} G_P(Z) &= \int_{\mathbb{T}} \psi_0(x - cz) \psi_1(x - (c+1)z) dx \\ &= \int_{\mathbb{T}} \psi_0(x) \psi_1(x - z) dx, \end{aligned}$$

where translations are taken modulo Λ_h . Since $\pi_{\Lambda_h}(z)$ is uniform on $\mathbb{T}$ and independent of Z' , Fubini's theorem gives $\mathbb{E}[G_P \mid Z'] = G_{P_0}(Z') G_{P_1}(Z')$. Conditional Jensen then gives

$$\mathbb{E}[G_P^{1/|P|} \mid Z'] \leq (G_{P_0} G_{P_1})^{1/|P|} = (G_{P_0}^{1/|P_0|})^{|P_0|/|P|} (G_{P_1}^{1/|P_1|})^{|P_1|/|P|}.$$

The weights $|P_0|/|P|$ and $|P_1|/|P|$ sum to 1. Taking expectations over Z' and applying Hölder's inequality with exponents $|P|/|P_0|$ and $|P|/|P_1|$ gives

$$\mathbb{E} G_P^{1/|P|} \leq \left(\mathbb{E} G_{P_0}^{1/|P_0|} \right)^{|P_0|/|P|} \left(\mathbb{E} G_{P_1}^{1/|P_1|} \right)^{|P_1|/|P|};$$

the inductive hypothesis then proves (5.1). $\square$

Below, the shear estimate and Lemma 2.1 will give a lattice whose covering count meets the required lower bound outside a small exceptional set. To complete the argument, we slightly enlarge K to remove this exceptional set. If $x - w$ has enough centers for some $w \in K/q$, those same centers cover x after the dilation $K + K/q = (1 + 1/q)K$. The following volume condition guarantees such a displacement for every x , extending [8, Lemma 6.2] to multiplicity with the same quantitative hypothesis.

Lemma 5.2 (Completion with multiplicity). *Let $K \subset \mathbb{R}^n$ be a convex body and Γ a full-rank lattice. Let $\kappa > 0$ be real and $q \geq 1$ an integer. If*

$$\omega = |\{x \in \mathbb{R}^n / \Gamma : N_{\Gamma, K}(x) < \kappa\}| < (1 - \omega)q^{-n},$$

then $N_{\Gamma, (1+1/q)K}(x) \geq \kappa$ for every x .

Proof. Every x with $N_{\Gamma,K}(x) \geq \kappa > 0$ lies in $K + \Gamma$, so the image $\pi_\Gamma(K)$ has measure at least $1 - \omega$. Multiplication by q on the torus maps $\pi_\Gamma(K/q)$ onto this image and increases measure by at most q^n : partition a fundamental parallelepiped into q^n pieces on which this map is injective. Thus $|\pi_\Gamma(K/q)| \geq q^{-n}(1 - \omega) > \omega$. For every x , some $w \in K/q$ therefore has $N_{\Gamma,K}(x - w) \geq \kappa$. Choose distinct $\lambda_1, \dots, \lambda_{\lceil \kappa \rceil} \in \Gamma$ with $x - w - \lambda_i \in K$. Then $x - \lambda_i \in K + K/q = (1 + 1/q)K$ for every i , by convexity. $\square$

5.1 Parameters and mean section counts

This subsection fixes the parameters for Lemma 5.3; the large-dimensional proof of Theorem 1.2 is completed in Section 5.3.

Fix $0 < \varepsilon < 1$ and convex bodies $K_1, \dots, K_N \subset \mathbb{R}^n$ whose uniform distributions have a common covariance matrix, where $1 \leq N \leq \exp(n^{c_0})$, and $\Delta > 0$. Assume $n \geq n_\varepsilon$ and $\rho_j = |K_j|/\Delta \geq A_\varepsilon^{\text{large}} n \log n$ for every j . The constant $A_\varepsilon^{\text{large}}$ is specified below; all dimension requirements depend only on ε and the fixed input constants. Section 5.4 then enlarges the density constant to cover the remaining one-body cases.

Scales. Put $\delta = \varepsilon/10$ and

$$q = \lceil n/\delta \rceil, \quad s = 1 + 1/q, \quad s^n \leq e^\delta.$$

We first construct a lattice of covolume $s^n \Delta$ with a small deficient set, then complete the covering and divide the lattice by s . This reserves the final dilation's density cost.

Use Proposition 4.2 at accuracy δ with $b_1 = \lceil (\log \log n)^2 \rceil$. Write its lattice and map as L, M , and put $\Delta_v = \det(ML)$, $m = n - D$, and $I = I_\delta$. Then

$$D \leq 2b_1, \quad \tau_n = e^{-c_{\text{at}} b_1^{7/10}}, \quad 2^D = n^{o(1)}, \quad \frac{\eta_m}{\tau_n n \log n} \longrightarrow \infty. \quad (5.2)$$

Indeed $b_1^{7/10} \asymp (\log \log n)^{7/5}$ grows faster than $\log \log n$, so $\tau_n \log n \rightarrow 0$. Fix

$$A_\varepsilon^{\text{large}} = \frac{3e^\delta I_\delta}{\delta^2}.$$

For all sufficiently large n , Lemma 3.1 applies with $A_{\text{sec}} = 5$ and gives common horizontal-vertical coordinates and affine section maps $T_j(y) = A_j y + t_j$. Multiply Δ by the absolute determinant of the common linear map; the ratios $|K_j|/\Delta$ are unchanged.

Keep the notation K_j, Δ for the normalized data, and put

$$V_j = |K_j|, \quad \Delta_h = \frac{s^n \Delta}{\Delta_v}, \quad \rho'_j = \frac{V_j}{s^n \Delta}, \quad u_{j,a} = \frac{|J_{j,a}|}{\Delta_h}.$$

Here ρ'_j is the mean count of the temporary arrangement of covolume $s^n \Delta$, whereas $\rho_j = s^n \rho'_j$ is the prescribed final mean. The quantity $u_{j,a}$ is the mean lattice count for the representative body $J_{j,a}$.

Mean section counts. Fix a body j and a height y . For each $\ell \in \mathcal{P}_y$, choose the label $a_y(\ell)$ from (3.3) at height $y - M\ell$. This residual has norm at most R by (4.9). Writing $w_y(\ell) = \Delta_v \gamma_D(y - M\ell)$ for the normalized vertical weight, the bounds on the representative volumes give

$$(1 - \delta)\rho'_j w_y(\ell) \leq u_{j,a_y(\ell)} \leq \rho'_j w_y(\ell).$$

Thus $w_y(\ell)$ gives the representative's fraction of the full mean up to the displayed $1 - \delta$ loss. Combining these inequalities with (4.10) and (4.11) gives

$$\sum_{\ell \in \mathcal{P}_y} u_{j,a_y(\ell)} \geq (1 - \delta)^2 \rho'_j, \quad u_{j,a_y(\ell)} \leq \tau_n \rho'_j. \quad (5.3)$$

For body j , retain only labels with $u_{j,a} \leq \tau_n \rho'_j$. Every selected residual still has a retained label by (5.3), and the uniform upper bound on these mean counts allows us to apply thinning to every labeled pattern considered below.

Labeled patterns. A *labeled pattern* for body j consists of a nonempty set $P \subset \mathcal{A}_1 \times \cdots \times \mathcal{A}_D$, binary by suffixes, and a retained label $a(\ell)$ at each point. Here the finite coordinate sets come from Proposition 4.2. Every nonempty component of $\mathcal{P}_y$ for $y \in M[0, 1)^D$, labeled by a_y , belongs to this family.

For a fixed body, a labeled point has at most $\exp(O(D \log n + D^2 \log(D+2)))$ possible descriptions. Each pattern has at most 2^D points, so the total number Q of labeled patterns, counting all N bodies, satisfies

$$\log Q \leq \log N + O\left(2^D[D \log n + D^2 \log(D+2)]\right) = o(\sqrt{m}). \quad (5.4)$$

Laplace parameters. For each body put

$$\beta_j = \delta \min \left\{ 1, \frac{\eta_m}{\tau_n \rho'_j} \right\}. \quad (5.5)$$

Then $\beta_j \leq \delta$ and $\beta_j u_{j,a} \leq \delta \eta_m$ for every retained label, keeping each exponential test within the range of thinning. The total exponent satisfies $\beta_j \rho'_j = \delta \min\{\rho'_j, \eta_m/\tau_n\}$.

5.2 A Laplace bound at every height

Lemma 2.1 controls a single function's Laplace transform on average over lattices. Here we apply it to the representative indicators and use the shear estimate to select one horizontal lattice and shear that work for all labeled patterns. The section containments and vertical weight bounds then give the following estimate for the full covering count, at every height and for every body.

Lemma 5.3 (Uniform Laplace bound). *With the setup and parameters of Section 5.1, for all sufficiently large n there are a lattice $\Lambda_h \subset \mathbb{R}^m$ of covolume Δ_h and a linear map $Z : \mathbb{R}^D \rightarrow \mathbb{R}^m$ such that the lattice*

$$\Gamma = \{(\lambda + Z\ell, M\ell) : \lambda \in \Lambda_h, \ell \in L\},$$

of covolume $\Delta_h \Delta_v = s^n \Delta$, satisfies

$$\int_{\mathbb{R}^m / \Lambda_h} e^{-(\beta_j/I) N_{\Gamma, K_j}(x, y)} dx \leq \exp \left[2^D \sqrt{m} - (1 - \delta)^3 \frac{\beta_j}{I} \rho'_j \right] \quad (5.6)$$

for every j and every $y \in \mathbb{R}^D$. The integral uses normalized Haar measure on the horizontal torus.

The three factors $1 - \delta$ come from the representative volumes, the selected vertical weight, and the linearization of $1 - e^{-\beta_j}$. The coset components need not form one binary pattern, so Hölder combines their bounds at parameter β_j/I . The parameter β_j contains a factor δ , and the Markov step uses the gap of at least $\delta \rho'_j$ between $(1 - \delta)^3 \rho'_j$ and the deficient-count cutoff $(1 - 4\delta) \rho'_j$. Together these give the factor I_δ/δ^2 in the density constant.

Proof of Lemma 5.3. One representative. Choose Λ_h with the distribution obtained by scaling μ_m by $\Delta_h^{1/m}$. Conditional on Λ_h , choose the columns of Z independently and uniformly in a fundamental parallelepiped, using a measurable choice of lattice bases. For each retained label a of body j , put

$$F_{j,a}(\Lambda_h) = \int_{\mathbb{R}^m / \Lambda_h} e^{-\beta_j N_{\Lambda_h, J_{j,a}}(x)} dx.$$

Apply Lemma 2.1 to $\mathbf{1}_{\Delta_h^{-1/m} J_{j,a}}$. Since

$$(1 - e^{-\beta_j})u_{j,a} \geq (1 - \beta_j/2)\beta_j u_{j,a} \geq (1 - \delta)\beta_j u_{j,a}, \quad (1 - e^{-\beta_j})u_{j,a} \leq \delta\eta_m < \eta_m,$$

we obtain

$$\begin{aligned} \mathbb{E}_{\Lambda_h} F_{j,a}(\Lambda_h) &\leq e^{-(1-\delta)\beta_j u_{j,a}} + C_{\text{RS}} e^{-\eta_m} \\ &\leq (1 + C_{\text{RS}}) e^{-(1-\delta)\beta_j u_{j,a}}, \end{aligned} \tag{5.7}$$

where the last step uses $(1 - \delta)\beta_j u_{j,a} \leq \delta\eta_m \leq \eta_m$.

One pattern. The affine section centers satisfy

$$T_j(y - M\ell) + Z\ell = T_j(y) + (Z - A_j M)\ell.$$

For a labeled pattern P of body j , define

$$G_P(\Lambda_h, Z) = \int_{\mathbb{R}^m / \Lambda_h} \exp \left[-\beta_j \sum_{\ell \in P} N_{\Lambda_h, J_{j,a(\ell)}}(x - (Z - A_j M)\ell) \right] dx.$$

Conditional on Λ_h , apply Lemma 5.1 to $\phi_\ell(x) = e^{-\beta_j N_{\Lambda_h, J_{j,a(\ell)}}(x + A_j M\ell)}$. Translation preserves its integral $F_{j,a(\ell)}(\Lambda_h)$. All factors $F_{j,a(\ell)}$ depend on the same lattice. Hölder's inequality with $|P|$ equal exponents nevertheless gives

$$\begin{aligned} \mathbb{E}_{\Lambda_h, Z} G_P^{1/|P|} &\leq \mathbb{E}_{\Lambda_h} \prod_{\ell \in P} F_{j,a(\ell)}(\Lambda_h)^{1/|P|} \\ &\leq \prod_{\ell \in P} (\mathbb{E}_{\Lambda_h} F_{j,a(\ell)})^{1/|P|} \\ &\leq (1 + C_{\text{RS}}) \exp \left[-\frac{(1 - \delta)\beta_j}{|P|} \sum_{\ell \in P} u_{j,a(\ell)} \right]. \end{aligned} \tag{5.8}$$

Markov's inequality, applied to $G_P^{1/|P|}$, shows that each labeled pattern violates

$$G_P(\Lambda_h, Z) \leq e^{|P|\sqrt{m}} \exp \left[-(1 - \delta)\beta_j \sum_{\ell \in P} u_{j,a(\ell)} \right] \tag{5.9}$$

with probability at most $(1 + C_{\text{RS}})e^{-\sqrt{m}}$. By (5.4), $(1 + C_{\text{RS}})Qe^{-\sqrt{m}} < 1$, so one pair (Λ_h, Z) satisfies every bound. Fix such a pair, and define Γ as in the statement.

All components at one height. Fix a body j and a height $y \in M[0, 1]^D$. Write $\mathcal{P}_y = P_1 \sqcup \dots \sqcup P_I$, allowing empty components, and define the contribution of a component by

$$S_i(x) = \sum_{\ell \in P_i} N_{\Lambda_h, J_{j,a_y(\ell)}}(x - T_j(y) - (Z - A_j M)\ell).$$

By (1.3) and the section inclusions, $N_{\Gamma, K_j}(x, y) \geq \sum_i S_i(x)$. All counted centers are distinct: different ℓ have different vertical coordinates because M is invertible.

Translation invariance and (5.9) give

$$\int e^{-\beta_j S_i(x)} dx \leq \exp \left[|P_i|\sqrt{m} - (1 - \delta)\beta_j \sum_{\ell \in P_i} u_{j,a_y(\ell)} \right].$$

For an empty component both sides are 1.

Hölder's inequality over the I components, the bound $|P_i| \leq 2^D$, and (5.3) now give

$$\begin{aligned} \int e^{-(\beta_j/I)N_{\Gamma,K_j}(x,y)} dx &\leq \prod_{i=1}^I \left(\int e^{-\beta_j S_i(x)} dx \right)^{1/I} \\ &\leq \exp \left[2^D \sqrt{m} - (1-\delta)^3 \frac{\beta_j}{I} \rho'_j \right]. \end{aligned}$$

Here all integrals are over the normalized horizontal torus.

All heights. Let $y \in \mathbb{R}^D$ be arbitrary, and write $y = y' + Mz$ with $y' \in M[0,1)^D$ and $z \in \mathbb{Z}^D \subset L$. Since $(Zz, Mz) \in \Gamma$, we have $N_{\Gamma,K_j}(x,y) = N_{\Gamma,K_j}(x - Zz, y')$, and translation invariance of the horizontal integral transfers the bound from y' to y . $\square$

5.3 Completion

Proof of Theorem 1.2 in sufficiently large dimension. Take Λ_h , Z , and Γ from Lemma 5.3. Call a point (x, y) *deficient* for body j if $N_{\Gamma,K_j}(x, y) < (1-4\delta)\rho'_j$. Since $(1-\delta)^3 - (1-4\delta) \geq \delta$, exponential Markov and (5.6) give, for every j and every y ,

$$\left| \{x : N_{\Gamma,K_j}(x, y) < (1-4\delta)\rho'_j\} \right| \leq \exp \left[2^D \sqrt{m} - \frac{\delta^2}{I} \min \left\{ \rho'_j, \frac{\eta_m}{\tau_n} \right\} \right]. \quad (5.10)$$

By (5.2), $2^D \sqrt{m} = o(n \log n)$ and $\delta^2 \eta_m / (I \tau_n) \geq 3n \log n$ for sufficiently large n . The hypothesis $\rho_j \geq A_\varepsilon^{\text{large}} n \log n$ and our choice of $A_\varepsilon^{\text{large}}$ also give $\delta^2 \rho'_j / I \geq 3n \log n$. Thus (5.10) is at most n^{-2n} .

Let F_h, F_v be fundamental parallelepipeds for Λ_h, ML . Their product is a fundamental domain for Γ : reduce vertically, including the shear, and then horizontally. The count is a locally finite sum of indicators of compact translates, hence measurable. Fubini therefore converts the heightwise estimate into a full deficient proportion at most n^{-2n} , without requiring the auxiliary patterns to vary measurably with height.

Since $q = O_\delta(n)$, for sufficiently large n we have $n^{-2n} < (1-n^{-2n})q^{-n}$. Lemma 5.2 applies with $\kappa = (1-4\delta)\rho'_j$ and this same q . The identity $N_{s^{-1}\Gamma,K_j}(x) = N_{\Gamma,sK_j}(sx)$ shows that $\Lambda = s^{-1}\Gamma$ has exactly the prescribed covolume Δ and

$$N_{\Lambda,K_j} \geq (1-4\delta)\rho'_j \geq (1-4\delta)e^{-\delta} \frac{V_j}{\Delta} \geq (1-5\delta) \frac{V_j}{\Delta} \geq (1-\varepsilon) \frac{V_j}{\Delta}.$$

Undo the common linear normalization. Choose $n_\varepsilon \geq 3$ large enough for all the preceding dimension requirements. They depend only on ε and the fixed constants from the input lemmas, so this completes the large-dimensional case. $\square$

5.4 The one-body case in all dimensions

It remains to prove the one-body assertion of Theorem 1.2 in the finitely many dimensions $2 \leq n < n_\varepsilon$. We start from a *1-fold covering lattice* Λ_0 , meaning that $K + \Lambda_0 = \mathbb{R}^n$. Such a lattice always exists: the elementary affine normalization below gives one of density at most $(2n)^n$. The subsequent contraction argument works for any initial covering lattice.

Proof of Theorem 1.2 in the remaining dimensions. Choose a maximum-volume simplex contained in K , which exists by compactness, and map it to $\text{conv}(0, e_1, \dots, e_n)$. Write $\widetilde{K}$ for the image of K . Replacing the simplex's i -th nonzero vertex by any $x \in \widetilde{K}$ shows that $|x_i| \leq 1$, so $\widetilde{K} \subset [-1, 1]^n$.

Also $[0, 1/n]^n \subset \widetilde{K}$. This cube is a fundamental cell of $(1/n)\mathbb{Z}^n$, so the translates of $\widetilde{K}$ along that grid cover space, with density at most $2^n/(1/n)^n = (2n)^n$; pulling the grid back by the linear part of the normalization gives a covering lattice Λ_0 for the original body: the translation part does not affect whether the translates cover all of space. Volume and covolume scale by the same factor, so $\rho_0 = |K|/\det \Lambda_0 \leq (2n)^n$.

To contract Λ_0 , we use the identity

$$N_{\Lambda_0/s, K}(x) = N_{\Lambda_0, sK}(sx), \quad s > 1.$$

Thus we may equivalently enlarge the body and count along the original lattice. Choose a measurable fundamental domain $F \subset K$ for Λ_0 : enumerate the lattice and, for each point in a half-open fundamental parallelepiped, use the first translate placing it in K . The translates $F + \lambda$ partition space. For $s > 1$, every such tile meeting $x - (s-1)K$ has $\lambda \in x - sK$: a point in the intersection can be written as $f + \lambda = x - (s-1)u$, with $f, u \in K$, and convexity gives $f + (s-1)u \in sK$. These tiles cover $x - (s-1)K$, each has volume $\det \Lambda_0$, and all their indices are counted by $N_{\Lambda_0, sK}(x)$. Comparing volumes gives

$$N_{\Lambda_0, sK}(x) \geq \rho_0(s-1)^n.$$

Given the prescribed density $\rho = |K|/\Delta > \rho_0$, set $s = (\rho/\rho_0)^{1/n}$ and $\Lambda = \Lambda_0/s$. Then $\det \Lambda = \Delta$ and

$$N_{\Lambda, K}(x) \geq \rho(1 - (\rho_0/\rho)^{1/n})^n \geq (1 - \varepsilon)\rho$$

provided

$$\rho \geq \rho_0[1 - (1 - \varepsilon)^{1/n}]^{-n}.$$

The sufficient lower bound on ρ depends only on n and ε after replacing ρ_0 by $(2n)^n$. To express it in the form $\rho \geq A_\varepsilon n \log n$ simultaneously in all remaining dimensions, we set

$$A_\varepsilon^{\text{small}} = \max_{2 \leq d < n_\varepsilon} \frac{(2d)^d}{d \log d} [1 - (1 - \varepsilon)^{1/d}]^{-d}, \quad A_\varepsilon = \max\{A_\varepsilon^{\text{large}}, A_\varepsilon^{\text{small}}\}.$$

This finite maximum depends only on ε , and its $d = n$ term ensures that $\rho \geq A_\varepsilon n \log n$ satisfies the density condition for the contraction argument in every remaining small dimension. Since $A_\varepsilon \geq A_\varepsilon^{\text{large}}$, the large-dimensional proof, including its simultaneous conclusion, also applies. $\square$

6 Upper multiplicity and smooth coverings

Theorem 1.2 keeps the multiplicity everywhere above nearly its mean. A *smooth covering* also requires an upper bound: for $0 < \varepsilon < 1$, an arrangement of density ρ is ε -smooth if

$$(1 - \varepsilon)\rho \leq N_{\Lambda, K}(x) \leq (1 + \varepsilon)\rho \quad \text{for every } x \in \mathbb{R}^n.$$

For the lattices constructed here, Proposition 6.1 below shows that the maximum normalized count grows with dimension. This is a limitation of their vertical structure; it does not exclude smooth coverings by other lattices.

The lower bound does, however, control the average deviation. Write $\mathcal{N} = N_{\Lambda, K}$ and $\rho = |K|/\det \Lambda$. Since $\mathcal{N}/\rho$ has mean 1, the bound $\mathcal{N} \geq (1 - \varepsilon)\rho$ gives

$$\int_{\mathbb{R}^n/\Lambda} \left| \frac{\mathcal{N}(x)}{\rho} - 1 \right| dx = 2 \int_{\mathbb{R}^n/\Lambda} \left(1 - \frac{\mathcal{N}(x)}{\rho} \right)_+ dx \leq 2\varepsilon, \quad (6.1)$$

where the integrals use normalized Haar measure and $a_+ = \max\{a, 0\}$. This L^1 bound allows large counts on small sets. In our construction, one horizontal layer can already have mean count far above the mean of the full arrangement. A shear translates that layer's covering pattern without changing its horizontal mean, so some point must still have high multiplicity. The small-atom bound in Proposition 4.2 controls only the selected pattern and is compatible with such a peak in the full sum.

Proposition 6.1 (Upper multiplicity from a single layer). *Let $K \subset \mathbb{R}^m \times \mathbb{R}^D$ be a convex body, $\Lambda_h \subset \mathbb{R}^m$ and $L \subset \mathbb{R}^D$ full-rank lattices, $M : \mathbb{R}^D \rightarrow \mathbb{R}^D$ invertible linear, and $Z : \mathbb{R}^D \rightarrow \mathbb{R}^m$ linear. Put*

$$\Gamma = \{(\lambda + Z\ell, M\ell) : \lambda \in \Lambda_h, \ell \in L\},$$

$$\Delta_h = \det \Lambda_h, \quad \Delta_v = \det(ML), \quad \rho = \frac{|K|}{\Delta_h \Delta_v}.$$

Then

$$\frac{\sup_{x,y} N_{\Gamma,K}(x,y)}{\rho} \geq \Delta_v \frac{\sup_y |K[y]|}{|K|}. \quad (6.2)$$

For the vertical lattices in Proposition 4.2, with cutoff B fixed and $b_1 \rightarrow \infty$,

$$\log(\Delta_v \gamma_D(0)) = (\tfrac{1}{2} + o(1))D. \quad (6.3)$$

Proof. For every height y , normalized horizontal integration gives

$$\int_{\mathbb{R}^m/\Lambda_h} N_{\Gamma,K}(x,y) dx = \frac{1}{\Delta_h} \sum_{\ell \in L} |K[y - M\ell]| \geq \frac{|K[y]|}{\Delta_h}.$$

The sum is finite because K is compact, and translation invariance removes Z from each integral. Divide by ρ and take the supremum over y to obtain (6.2).

It remains to compute the vertical covolume of our construction. Recall the folded density f_b and its logarithmic mean $\mathbf{m}(b)$ from Section 4. If X is standard normal conditioned on $|X| \leq h(b)$, then $\psi_b(X) = X/h(b) \pmod{1}$, represented in $[0, 1)$, has density $f_b/\int_0^1 f_b$. Thus

$$\begin{aligned} \mathbf{m}(b) &= \mathbb{E} \log f_b(\psi_b(X)) \\ &= \log h(b) - \tfrac{1}{2} \log(2\pi) - \tfrac{1}{2} \mathbb{E} X^2 + \mathbb{E} \log(1 + e^{-h(b)^2/2 + h(b)|X|}) \\ &= \log h(b) - \tfrac{1}{2} \log(2\pi e) + o(1). \end{aligned}$$

Indeed $\mathbb{E} X^2 \rightarrow 1$, and the last logarithmic summand tends to zero pointwise. The inequality $-h(b)^2/2 + h(b)|X| \leq X^2/2$ bounds it by $\log 2 + X^2/2$. Dominated convergence under the unconditioned normal law, divided by $\mathbb{P}(|X| \leq h(b)) \rightarrow 1$, proves the last equality.

The prime window (4.2) gives $\log p(b) = b\mathbf{m}(b) + O(b^{14/25})$. Since $\Delta_v = \iota^{-1} \prod_i h_i^{b_i}/p_i$ with $\iota = p_r^{b_r-1}$, it follows that

$$\begin{aligned} \log(\Delta_v \gamma_D(0)) &= \sum_i b_i \left(\log h_i - \mathbf{m}(b_i) - \tfrac{1}{2} \log(2\pi) \right) + O\left(\sum_i b_i^{14/25} \right) - \log \iota \\ &= \tfrac{D}{2} + o(D). \end{aligned}$$

Here $\log \iota = O_B(1)$ and the dyadic sizes give $\sum_i b_i^{14/25} = O(b_1^{14/25}) = o(D)$. For the remaining errors, put $e(b) = \log h(b) - \mathbf{m}(b) - \frac{1}{2} \log(2\pi e)$, so $e(b) \rightarrow 0$. Given $\xi > 0$, choose $b_* > B$ such that $|e(b)| \leq \xi$ for $b \geq b_*$. The blocks with $b_i < b_*$ have total dimension at most $2b_*$, hence

$$\left| \sum_i b_i e(b_i) \right| \leq \xi D + 2b_* \max_{B \leq b < b_*} |e(b)|.$$

First let $b_1 \rightarrow \infty$ with B, b_* fixed, and then let $\xi \rightarrow 0$. This proves that their sum is $o(D)$. $\square$

The covolume is therefore $(2\pi e)^{D/2+o(D)}$. A Gaussian residual with squared norm $D + o(D)$ has normalized weight $e^{o(D)}$, whereas the zero residual has weight $e^{D/2+o(D)}$. The exponential peak reflects the difference between the Gaussian density at its mode and its density at typical radius.

Fix the accuracy, so the block cutoff B is fixed as $n \rightarrow \infty$. By (3.4), the normalized bodies in the large-dimensional construction of Section 5 satisfy $|K[0]|/|K| \geq (1 - o(1))\gamma_D(0)$. Thus their supremum multiplicity divided by density is at least $\exp((1/2 + o(1))D)$, independently of the horizontal lattice and shear. Replacing Γ by Γ/s multiplies the right side of (6.2) by s^{-D} . In the completion step $s = 1 + O_\varepsilon(1/n)$ and $D = o(n)$, so this factor tends to 1. Consequently the final covering lattice Λ satisfies

$$\frac{\sup_x N_{\Lambda, K}(x)}{|K|/\det \Lambda} \geq \exp((1/2 + o(1))D) \rightarrow \infty.$$

Undoing the linear normalization preserves this ratio. Thus these lattices eventually violate every fixed upper smoothness bound, even though their lower counts remain close to the mean everywhere.

It remains open whether a different construction gives two-sided ε -smooth lattice coverings at density $O_\varepsilon(n \log n)$, matching the periodic benchmark discussed in the introduction. Another question is whether the density constant A_ε in Theorem 1.2 can be chosen to grow at most polynomially in $1/\varepsilon$. The present proof does not provide such a bound.

A Attainment of the optimal lattice density

Proposition A.1 (Attainment of the infimum). *For every convex body $K \subset \mathbb{R}^n$ and positive integer k , there is a lattice k -fold covering by translates of K with density $\theta_{\text{lat}, k}(K)$.*

Proof. A cube contained in K gives a one-fold covering on a sufficiently fine lattice; passing to an index- k superlattice gives a k -fold covering. Choose a sequence Λ_j of k -fold covering lattices whose densities tend to $\theta_{\text{lat}, k}(K)$. The densities are bounded above by a finite constant, so their covolumes, equal to $|K|$ divided by those densities, are bounded below by some $\Delta_* > 0$. Fix $R > 0$ with $\|z\| \leq R$ for all $z \in K$. Since $K + \Lambda_j = \mathbb{R}^n$, every point of $\mathbb{R}^n$ is within distance R of Λ_j .

Each Λ_j contains n linearly independent vectors of norm at most $3R$. Indeed, given their current span, choose a unit vector u perpendicular to it. A lattice point within distance R of $2Ru$ lies outside that span and has norm at most $3R$. Let U_j have the selected vectors as columns. Then

$$\Delta_* \leq |\det U_j| \leq (3R)^n, \quad [\Lambda_j : U_j \mathbb{Z}^n] \leq (3R)^n / \Delta_*.$$

There are only finitely many superlattices of $\mathbb{Z}^n$ of bounded index, as a superlattice of index r lies in $r^{-1}\mathbb{Z}^n$. The matrices U_j are bounded, with determinants bounded away from zero. After passing to a subsequence, therefore,

$$U_j \rightarrow U, \quad U_j^{-1} \Lambda_j = \Lambda_*,$$

where U is invertible and Λ_* is fixed.

Set $\Lambda = U\Lambda_*$. For each fixed x , boundedness of K and of U_j^{-1} means that only finitely many $v \in \Lambda_*$ can satisfy $x - U_j v \in K$, uniformly in j . Along a subsequence attaining the limsup of the counts, the same finite set of covering vectors occurs infinitely often, by the pigeonhole principle. Closedness of K preserves each of these vectors in the limit, giving

$$N_{\Lambda, K}(x) \geq \limsup_{j \rightarrow \infty} N_{\Lambda_j, K}(x) \geq k.$$

Thus $K + \Lambda$ is a k -fold covering. Continuity of the covolume shows that its density is $\theta_{\text{lat}, k}(K)$. $\square$

B The comparison with arbitrary translates

In its set-cover formulation, the greedy bound of Naszodi and Polyanskii [7, Theorem 1.1] controls the size of an integral multicover in terms of the fractional one-fold covering number, with a free parameter $0 < \lambda < 1$. Their geometric bound [7, Theorem 1.4] uses a fixed parameter; choosing it according to ε gives the near-unit bound below. We record the deduction for nonsymmetric bodies and distinct translation vectors. The density of a periodic arrangement is its number of centers in one period, multiplied by $|K|$ and divided by the period volume.

Proposition B.1 (Periodic multi-coverings). *For every $\varepsilon > 0$ there is $C_\varepsilon^{\text{tr}} > 0$ such that every convex body $K \subset \mathbb{R}^n$, $n \geq 2$, admits a periodic k -fold covering by distinct translates of density at most*

$$\max\{(1 + \varepsilon)k, C_\varepsilon^{\text{tr}} n \log n\} \quad (k \geq 1).$$

Proof. We first show that, for $0 < t, \lambda < 1$, the infimum of the densities of such coverings is at most

$$(1 - t)^{-n} \frac{1 - \lambda^k}{1 - \lambda} \left[1 + n \log \frac{4}{t} - (k - 1) \log \lambda \right]. \quad (\text{B.1})$$

A suitable translate of K has $S = K \cap (-K)$ with $|S| \geq 2^{-n}|K|$. Indeed,

$$\int_{2K} |K \cap (z - K)| dz = |K|^2, \quad |2K| = 2^n |K|,$$

so some z gives an intersection of at least this volume; replace K by $K - z/2$. Thus S is a symmetric convex body contained in K .

A finite covering problem. Let $C_R = [-R, R]^n$, and choose a maximal packing of translates of $(t/2)S$ with centers in a finite set $\mathcal{A}_R \subset C_R$. Maximality gives $C_R \subset \mathcal{A}_R + tS$. If centers of this packing lie in $x + (1 - t)K$, their disjoint interiors lie in $x + (1 - t/2)K$. Consequently,

$$\sup_x \#(\mathcal{A}_R \cap [x + (1 - t)K]) \leq \frac{(1 - t/2)^n |K|}{(t/2)^n |S|} \leq (4/t - 2)^n < (4/t)^n. \quad (\text{B.2})$$

We first cover the finite set $\mathcal{A}_R$ fractionally: assign nonnegative weights to translates of $(1 - t)K$ so that the total weight covering each point is at least 1. Giving translation vectors in $C_R - (1 - t)K$ Lebesgue weight $1/|(1 - t)K|$ does this: the vectors covering a fixed $a \in \mathcal{A}_R$ form $a - (1 - t)K$, so their total weight is exactly 1. The total weight of all translation vectors is

$$\frac{|C_R - (1 - t)K|}{(1 - t)^n |K|} = (1 + o_{R \rightarrow \infty}(1)) \frac{|C_R|}{(1 - t)^n |K|}.$$

There are only finitely many *incidence patterns*, namely the subsets $\mathcal{A}_R \cap (z + (1-t)K)$. Thus this is a finite covering problem: combine the weights of all translation vectors with the same pattern, discard empty patterns, and cap weights at 1 if necessary. Its largest set has cardinality bounded by (B.2). Applying [7, Theorem 1.1] in the dual covering formulation therefore bounds the number of selected translates by this fractional weight times

$$\frac{1 - \lambda^k}{1 - \lambda} \left[1 + n \log \frac{4}{t} - (k-1) \log \lambda \right].$$

The greedy construction may select a translate more than once. Since $(1-t)K + tS \subset K$, the corresponding translates of K cover C_R at least k times, counting these repetitions. Repeat the finite list with period lattice $2R\mathbb{Z}^n$. This gives a periodic covering of the whole space whose density, counting repetitions, is at most $(1 + o_{R \rightarrow \infty}(1))$ times (B.1).

Distinct translates. Repetitions can be removed with arbitrarily small density cost. Apply the construction to $(1-\alpha)K$, where $0 < \alpha < 1$, and choose $r > 0$ with $rB_2^n \subset K$. Perturb each translation vector in the finite period list by a vector e with $\|e\| < \alpha r$, choosing distinct representatives modulo the period lattice. This is possible by avoiding finitely many congruence classes at each choice. The inclusion

$$(1-\alpha)K - e \subset (1-\alpha)K + \alpha K = K$$

shows that the new, distinct translates of K retain every covering provided by the old list.

Their density, measured using K , is $(1-\alpha)^{-n}$ times the old density measured using $(1-\alpha)K$. Letting $R \rightarrow \infty$ and then $\alpha \downarrow 0$ proves the claimed infimum bound for distinct translates.

Choice of parameters. For $0 < \varepsilon \leq 1$, set $a = \varepsilon/16$, $t = a/n$, and $\lambda = 1 - a$. Bernoulli's inequality and $-\log(1-a) \leq a/(1-a)$ bound (B.1) by

$$\frac{k}{(1-a)^2} + \frac{1 + n \log(64n/\varepsilon)}{a(1-a)} \leq (1 + \varepsilon/2)k + \frac{32}{\varepsilon} [1 + n \log(64n/\varepsilon)].$$

The last inequality uses $(1-a)^{-2} \leq 1 + 8a$ for $0 \leq a \leq 1/16$.

Since $n \geq 2$, the second term is bounded by a constant depending only on ε times $n \log n$. Increasing that constant also absorbs the arbitrarily small approximation to the infimum. Thus there is an actual periodic covering by distinct translates of density at most

$$(1 + \varepsilon/2)k + C_\varepsilon^{\text{add}} n \log n$$

for some $C_\varepsilon^{\text{add}} > 0$.

Split into the cases $k \geq 2C_\varepsilon^{\text{add}} n \log n / \varepsilon$ and $k < 2C_\varepsilon^{\text{add}} n \log n / \varepsilon$. The desired maximum follows with $C_\varepsilon^{\text{tr}} = 2(1 + \varepsilon)C_\varepsilon^{\text{add}} / \varepsilon$. For $\varepsilon > 1$, use the case $\varepsilon = 1$. $\square$

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