

AN EXPONENTIAL IMPROVEMENT FOR BORSUK'S PROBLEM

ABSTRACT. We present a method for improving the classical exponential base $\sqrt{3/2}$ in the upper bound for Borsuk's partition number. The emphasis is on explaining the method rather than optimizing the bound. The problem is reduced to constructing directions which make an acute angle with every member of a prescribed spherical set. These play the same role as the illuminating directions in Schramm's method. The geometric ingredients are an antipodal pairing, a lift by one coordinate, and reflection in a self-dual cone. This reflection folds Euclidean space into the cone. A translation opposite to the horizontal part of its Gaussian mean makes the required supporting inequalities hold with high probability. The Gaussian Poincaré inequality controls the mean, and the formula for translating a Gaussian density bounds the measure cost of this construction.

1. THE RESULT AND THE IDEAS

Let $b(n)$ be the least integer such that every bounded set of positive diameter in $\mathbb{R}^n$ can be partitioned into at most $b(n)$ sets of strictly smaller diameter. Schramm [8] and Bourgain–Lindenstrauss [2] established the classical bound

$$b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1) \right)^n.$$

The purpose of this note is to improve its exponential base with simple constants and a fully explained proof¹. For further background on Borsuk's problem, see Kalai's 2015 survey [5].

Theorem 1.1. *As $n \rightarrow \infty$,*

$$b(n) \leq (C + o(1))^n, \quad C = \sqrt{\frac{9536}{6375}} = 1.22304666\dots$$

In particular, $b(n) \leq 1.224^n$ for all sufficiently large n , whereas $\sqrt{3/2} = 1.22474487\dots$

For a compact nonempty $A \subset \mathbb{S}^{n-1}$, write

$$A_+ = \{v \in \mathbb{S}^{n-1} : \langle u, v \rangle > 0 \text{ for every } u \in A\}.$$

This is the open positive polar: the directions making an acute angle with every member of A . Let ν_j be probability surface measure on $\mathbb{S}^j$. The estimate behind the theorem is the following stable form.

Proposition 1.2. *For every $\beta < C^{-1}$ with $\beta > 0$, there are $\delta > 0$ and n_0 such that*

$$\nu_{n-1}(A_+) \geq \beta^n \quad \text{whenever } n \geq n_0, \quad \text{diam}_{\mathbb{S}} A \leq \frac{\pi}{3} + \delta. \quad (1.1)$$

Date: October 6, 2026.

2020 Mathematics Subject Classification. Primary 52A37; Secondary 52A23, 52A55.

Key words and phrases. Borsuk's partition problem, asymptotic convex geometry, spherical polarity, self-dual cones, Gaussian measure, Gaussian Poincaré inequality.

Origin and curation. The main proof arguments were primarily generated using *ChatGPT-6 Astra*, Sept.–Oct. 2026 with discussion and prompts by Andrii Arman, Andriy Bondarenko, Andriy Prymak and Danylo Radchenko. Andriy Prymak curates this note but does not claim authorship of its mathematical results. The primary context provided to AI was the study of small spherical bodies of constant width by Hall, Prymak and Sujsuntinukul arXiv:2606.30960v2. Sources of the key proof ingredients are credited in the text.

Verification: fully checked. The complete mathematical argument has been checked by Andriy Prymak, who believes the statements and proofs to be correct. This does not assert formal verification or independent peer review.

¹A sharper bound, which is not yet verified as of October 6, 2026, can be found at <https://prymak.net/aiomn/>.

Here diam_S denotes angular diameter, and the choices are uniform over all compact nonempty A .

Why this is relevant to partitions. Fix a small $\varepsilon > 0$ and join two points of a diameter-one set whenever their distance is at least $1 - \varepsilon$. A direction v selects a point x if $\langle v, x - y \rangle > 0$ for every neighbour y : viewed along v , the point x lies strictly ahead of all its neighbours. Two neighbours cannot both be selected, because this would require both $\langle v, x - y \rangle > 0$ and $\langle v, y - x \rangle > 0$. The class selected by v therefore has diameter at most $1 - \varepsilon$. A collection of directions selecting every point gives a covering by such classes, and assigning each point to one class gives the desired partition.

Within a small neighborhood in the set, collect all unit vectors pointing from nearly diametral neighbours towards points of that neighborhood. Their angular diameter is close to $\pi/3$, and any direction in their open positive polar selects the whole neighborhood. Proposition 1.2 gives a uniform lower bound for the probability of this event. Only exponentially many neighborhoods are needed; a union bound then requires an additional factor of order n in the number of sampled directions.

This is a direct analogue of Schramm's approach to illuminating bodies of constant width [8]. At a boundary point, the outward unit normals have angular diameter at most $\pi/3$, and the negative of a direction in their open positive polar illuminates that point. Schramm used spherical cap-containment and an antipodal pairing argument to estimate the measure of admissible directions. Discretization and random sampling then gave the exponential base $C = \sqrt{3/2}$, both for illumination and, as a consequence, for Borsuk's problem.

How the polar estimate improves the exponent. We first prove the polar estimate for the set A in Proposition 1.2 under the exact assumption $\text{diam}_S A \leq \pi/3$; the small increase of the diameter is handled in Section 6. Central projection (also known as gnomonic projection) writes A as

$$A = \left\{ \frac{p + x}{\sqrt{1 + |x|^2}} : x \in L \right\},$$

where p is a suitable pole and L lies in the unit ball of $p^\perp$. If a unit vector $u \in p^\perp$ satisfies $h_L(u) := \sup_{x \in L} \langle x, u \rangle \leq a$, then

$$v_\theta = \cos \theta p - \sin \theta u \in A_+, \quad 0 \leq \theta < \arctan(1/a).$$

This sector supplies the exponential measure factor $(1 + a^2)^{-n/2}$. The value $a = 1/\sqrt{2}$ corresponds to the classical base $\sqrt{3/2}$, so we seek many directions satisfying the stronger inequality with $a = 7/10$.

An antipodal pairing first reduces the task to the outer shell $E = \{x \in L : |x|^2 \geq 3/4\}$, losing only a factor two in measure. The lift

$$x \mapsto \left(x, \frac{7}{6} - |x|^2 \right)$$

makes the shell acute: every pair of lifted vectors has nonnegative scalar product. A self-dual cone D containing the lift has a reflection R which is even, preserves length, and is nonexpansive, meaning $|R(z) - R(w)| \leq |z - w|$. Evenness and Gaussian Poincaré force $\mu = \mathbb{E}R(G)$ to have length at least $\sqrt{n/2}$. The cone also ensures $R(G) - G \in D$, giving supporting inequalities for every lifted point at once.

We translate G opposite to the horizontal part of μ , by a fraction $k = 1/16$ of that mean. The supporting inequalities and the mean bound then show that the direction of the translated horizontal Gaussian is good for the entire shell with probability tending to one. Returning to ordinary Gaussian measure costs a factor at worst $\exp(-n/512 - o(n))$. Together with the sector factor, this gives the reciprocal exponential base $e^{-k^2/2}/\sqrt{1 + a^2}$. Once the support bound is established, these two factors improve the classical exponent provided that

$$(1 + a^2)e^{k^2} < \frac{3}{2}. \tag{1.2}$$

For $a = 7/10$ and $k = 1/16$, the reciprocal base is $e^{-1/512}/\sqrt{149/100}$. Its reciprocal is less than the value C in Theorem 1.1, hence less than $\sqrt{3/2}$.

Sources of the ideas. The study of small spherical bodies of constant width by Hall, Prymak and Sujsuntinukul [3], including Gaussian methods near the central width $\pi/2$, provided a crucial context for this work. The central projection and antipodal pairing follow Schramm's method [8]. The outer-shell pairing used here is a generalization of that idea. The key geometric fact about self-dual cones comes from Moreau's decomposition [7]: every vector z is the difference of two orthogonal vectors $P_D z$ and $P_D(-z)$ in the cone D , where P_D is the nearest-point projection onto D . Replacing their difference by their sum defines the length-preserving folding R and ensures $R(z) - z \in D$. The additional ingredient in the present argument is the radial lift which makes this cone geometry available for the outer shell.

2. FROM GOOD DIRECTIONS TO SMALLER-DIAMETER PIECES

We first prove Theorem 1.1 from Proposition 1.2. This is Schramm's discretization and random-direction strategy, expressed directly in terms of partition classes. It uses packing, random sampling, and the observation that two points cannot both lie ahead of each other.

Fix $0 < \beta < C^{-1}$ and choose δ from the proposition. Choose a fixed $0 < \varepsilon < 1/4$ so small that

$$2 \arcsin \frac{1 + 4\varepsilon}{2} < \frac{\pi}{3} + \delta. \quad (2.1)$$

After taking a closure and rescaling, it suffices to partition a compact set $K \subset \mathbb{R}^n$ of diameter one. Take a maximal ε -separated family $x_1, \dots, x_M \in K$ and put $F_i = K \cap \overline{B}(x_i, \varepsilon)$. These neighborhoods cover K . Disjoint balls of radius $\varepsilon/2$ around their centres lie in a ball of radius $1 + \varepsilon/2$, so

$$M \leq (1 + 2/\varepsilon)^n. \quad (2.2)$$

For each neighborhood consider all directions from a nearly diametral neighbour towards a point in that neighborhood:

$$A_i = \left\{ \frac{x - y}{|x - y|} : x \in F_i, y \in K, |x - y| \geq 1 - \varepsilon \right\}.$$

If nonempty, A_i is compact. For $u = (x - y)/|x - y|$ and $u' = (x' - y')/|x' - y'|$ in A_i ,

$$|u - u'| \leq |x - x'| + |y - y'| + (1 - |x - y|) + (1 - |x' - y'|) \leq 1 + 4\varepsilon.$$

Thus (2.1) and Proposition 1.2 give $\nu_{n-1}((A_i)_+) \geq \beta^n$ for all sufficiently large n .

Choose

$$J = \lceil (\log M + 1)\beta^{-n} \rceil$$

independent directions uniformly distributed on $\mathbb{S}^{n-1}$. The probability of missing at least one of the polars $(A_i)_+$ with $A_i \neq \emptyset$ is at most $M(1 - \beta^n)^J \leq M \exp(-J\beta^n) < 1$. Some choice $v_1, \dots, v_J$ therefore meets them all.

For a direction v define

$$B_v = \{x \in K : \langle v, x - y \rangle > 0 \text{ for every } y \in K \text{ with } |x - y| \geq 1 - \varepsilon\}.$$

If $v \in (A_i)_+$, then $F_i \subset B_v$. If A_i is empty, this inclusion holds for every v . Consequently the B_{v_j} cover K . If $x, y \in B_v$ had $|x - y| \geq 1 - \varepsilon$, the definition would give both $\langle v, x - y \rangle > 0$ and $\langle v, y - x \rangle > 0$, an impossibility. Hence

$$\text{diam } B_v \leq 1 - \varepsilon < 1.$$

Assigning each point to the first covering set containing it gives a partition. By (2.2), $b(n) \leq O_\varepsilon(n)\beta^{-n}$. Letting $\beta \uparrow C^{-1}$ proves the theorem.

3. CENTRAL PROJECTION AND AN OUTER SHELL

We now prove Proposition 1.2, starting with the exact diameter assumption $\text{diam}_{\mathbb{S}} A \leq \pi/3$, or equivalently $\langle z, z' \rangle \geq 1/2$ for all $z, z' \in A$.

3.1. Flattening the spherical set. Let q be the nearest point of $\text{conv } A$ to the origin and set $p = q/|q|$. For any convex combination $q = \sum_i \alpha_i z_i$,

$$|q|^2 = \sum_{i,j} \alpha_i \alpha_j \langle z_i, z_j \rangle \geq \frac{1}{2}.$$

The nearest-point inequality gives $\langle z, q \rangle \geq |q|^2$ for $z \in A$, so $\langle z, p \rangle \geq 1/\sqrt{2}$. Central projection onto the tangent hyperplane $p + p^\perp$ therefore writes

$$A = \left\{ \frac{p + x}{\sqrt{1 + |x|^2}} : x \in L \right\}, \quad L \subset \mathbb{B}^{n-1} \subset p^\perp, \quad (3.1)$$

where $\mathbb{B}^{n-1}$ is the unit ball in $p^\perp$. The spherical diameter bound $\text{diam}_{\mathbb{S}} A \leq \pi/3$ says that any two points of A have scalar product at least $1/2$. Substituting their representations from (3.1) therefore gives

$$\frac{1 + \langle x, y \rangle}{\sqrt{(1 + |x|^2)(1 + |y|^2)}} \geq \frac{1}{2} \quad (x, y \in L). \quad (3.2)$$

For a compact set F put $h_F(u) = \sup_{x \in F} \langle x, u \rangle$; for the empty set use $h_\emptyset = -\infty$. Fix throughout

$$a = \frac{7}{10}, \quad r^2 = \frac{3}{4}, \quad k = \frac{1}{16}. \quad (3.3)$$

Let $E = \{x \in L : |x| \geq r\}$ and $\mathcal{H}_F = \{u \in \mathbb{S}^{n-2} : h_F(u) \leq a\}$.

3.2. Support bounds give a spherical sector. We first relate the measure of $\mathcal{H}_L$ to that of the positive polar A_+ . For every $u \in \mathcal{H}_L$ and $0 \leq \theta < \theta_a := \arctan(1/a)$, the direction $v_\theta = \cos \theta p - \sin \theta u$ belongs to A_+ , because

$$\langle (p + x)/\sqrt{1 + |x|^2}, v_\theta \rangle = \frac{\cos \theta - \sin \theta \langle x, u \rangle}{\sqrt{1 + |x|^2}} > 0 \quad (x \in L).$$

Figure 1 illustrates the projection and this sector. Polar integration gives

$$\nu_{n-1}(A_+) \geq \nu_{n-2}(\mathcal{H}_L) \frac{\int_0^{\theta_a} \sin^{n-2} \theta \, d\theta}{\int_0^\pi \sin^{n-2} \theta \, d\theta}. \quad (3.4)$$

The ratio is at least $(\sin \theta_a)^n \exp(-o(n)) = (1 + a^2)^{-n/2} \exp(-o(n))$. For an elementary verification, bound the denominator by π and integrate the numerator only over $[\theta_a - 1/n, \theta_a]$; this loses only a polynomial factor and an absolute constant. Thus the remaining task is to find enough support directions with threshold $a < 1/\sqrt{2}$ to compensate for their probability cost, as in (1.2).

3.3. Pairing opposite directions.

Lemma 3.1. *The outer shell controls the whole set up to a factor two:*

$$\nu_{n-2}(\mathcal{H}_L) \geq \frac{1}{2} \nu_{n-2}(\mathcal{H}_E). \quad (3.5)$$

Proof. We first exclude $x, y \in L$ and a unit vector u with

$$|x| \leq r, \quad \langle x, u \rangle \geq a, \quad \langle y, u \rangle \leq -a.$$

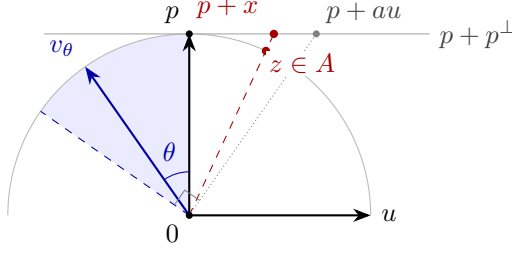


FIGURE 1. A two-dimensional section through p and u . Central projection follows the ray through z to $p + x$. When $\langle x, u \rangle \leq a$, directions with $0 \leq \theta < \theta_a$ lie in the positive polar. The sector's limiting boundary ray is orthogonal to $p + au$.

Put $z = \sqrt{|x|^2 - a^2}$ and $w = \sqrt{|y|^2 - a^2}$. Decomposing along u bounds $\langle x, y \rangle$ above by $-a^2 + zw$. By (3.2), $1 + \langle x, y \rangle > 0$, so this upper bound can be squared after adding one. For the next step, apply Cauchy–Schwarz to the two vectors

$$\left(\frac{1 - a^2}{\sqrt{1 + a^2}}, z \right), \quad (\sqrt{1 + a^2}, w).$$

Their scalar product is $1 - a^2 + zw$, so

$$(1 - a^2 + zw)^2 \leq \left(\frac{(1 - a^2)^2}{1 + a^2} + z^2 \right) (1 + a^2 + w^2).$$

Consequently,

$$\begin{aligned} \frac{(1 + \langle x, y \rangle)^2}{(1 + |x|^2)(1 + |y|^2)} &\leq \frac{(1 - a^2 + zw)^2}{(1 + a^2 + z^2)(1 + a^2 + w^2)} \\ &\leq \frac{(1 - a^2)^2 + (1 + a^2)z^2}{(1 + a^2)(1 + a^2 + z^2)} \\ &\leq \frac{1 - 3a^2 + (1 + a^2)r^2}{(1 + a^2)(1 + r^2)} < \frac{1}{4}. \end{aligned}$$

The expression on the second line increases with z^2 , and $z^2 \leq r^2 - a^2$. The strict last inequality is exactly

$$13a^2 - 3 - 3(1 + a^2)r^2 = \frac{7}{400} > 0. \quad (3.6)$$

This contradicts (3.2).

Now suppose $u \in \mathcal{H}_E \setminus \mathcal{H}_L$. A point witnessing $h_L(u) > a$ must lie inside the shell. The excluded configuration shows that $h_L(-u) \leq a$. Thus $\mathcal{H}_E \subset \mathcal{H}_L \cup (-\mathcal{H}_L)$, and antipodal symmetry proves (3.5). $\square$

The point of this pairing is that we need not control the inner part of L probabilistically. If an inner point “spoils” a direction already good for the shell, the pairing argument shows that the opposite direction is good for all of L .

3.4. A lift with one formula. Set $g(t) = 7/6 - t$ and lift each $x \in E$ to

$$\lambda_x = (x, g(|x|^2)) \in \mathbb{R}^n.$$

Geometrically, these points lie on the graph of the paraboloid $s = 7/6 - |x|^2$, restricted to $x \in E$. The last coordinate is its height above the horizontal space $p^\perp$. These lifted vectors have positive

pairwise scalar products. Indeed, for $t = |x|^2$ and $s = |y|^2$ in $[3/4, 1]$, (3.2) gives

$$\langle \lambda_x, \lambda_y \rangle \geq g(t)g(s) + \frac{1}{2}\sqrt{(1+t)(1+s)} - 1. \quad (3.7)$$

The right-hand side is concave in each variable separately, so its minimum occurs at a corner. The three distinct corner values are

$$\frac{7}{144}, \quad \frac{5}{72} + \sqrt{\frac{7}{8}} - 1, \quad \frac{1}{36}.$$

All are positive: for the middle value, $7/8 - (67/72)^2 = 47/5184 > 0$.

The extra coordinate compensates for possible negative products $\langle x, y \rangle$. Its height decreases towards the outside, where (3.2) already gives better products. This is why a radius-dependent lift is useful.

4. REFLECTION AND ITS GAUSSIAN MEAN

For a closed convex cone $D \subset \mathbb{R}^n$ let $D^* = \{y : \langle x, y \rangle \geq 0 \ (x \in D)\}$. A cone is *acute* when $D \subset D^*$ and *self-dual* when $D = D^*$.

We use the classical completion result of Barker and Foran [1], which is an easy consequence of Zorn's lemma.

Lemma 4.1 (Barker–Foran completion). *Every closed acute convex cone is contained in a closed self-dual cone in the same Euclidean space.*

By (3.7), the closed convex cone generated by the λ_x is acute. Complete it to a self-dual cone $D \subset \mathbb{R}^n$. If E is empty, $\mathcal{H}_E$ is the whole sphere and this and the next section may be skipped.

Let P_D be nearest-point projection onto D and put $R = 2P_D - I$. Moreau's decomposition reads

$$z = P_D z - P_D(-z), \quad P_D z \perp P_D(-z).$$

For completeness, if $p_0 = P_D z$, the projection inequality gives $z - p_0 \in -D$ and $\langle z - p_0, p_0 \rangle = 0$. Writing $q_0 = p_0 - z \in D$, the same inequality shows $q_0 = P_D(-z)$. Thus $R(z) = p_0 + q_0 \in D$: the reflection folds the space into the cone. It also follows that

$$R(-z) = R(z), \quad |R(z)| = |z|, \quad R(z) - z \in D. \quad (4.1)$$

The folding is illustrated in Figure 2.

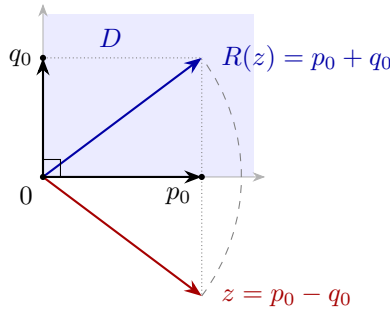


FIGURE 2. Moreau's orthogonal pair $p_0 = P_D z$, $q_0 = P_D(-z)$. Replacing their difference by their sum folds z into D without changing its length. The dashed arc records the equal lengths; the shaded quadrant, in the plane spanned by p_0, q_0 , lies in D .

Projection also satisfies

$$|P_D z - P_D w|^2 \leq \langle P_D z - P_D w, z - w \rangle.$$

Expanding a square gives $|R(z) - R(w)| \leq |z - w|$. For $D = \mathbb{R}_+^n$, we have $R(z) = (|z_1|, \dots, |z_n|)$: this folds each coordinate to the positive half-line and preserves length. The general reflection has the same three properties in (4.1).

Let G be a standard Gaussian vector in $\mathbb{R}^n$ and $\mu = \mathbb{E}R(G)$. The coordinates of G are independent real Gaussians of mean zero and variance one. We use the ordinary Gaussian Poincaré inequality and its even version:

$$\text{Var } f(G) \leq \mathbb{E}|\nabla f(G)|^2, \quad \text{Var } f(G) \leq \frac{1}{2}\mathbb{E}|\nabla f(G)|^2 \quad (f(-z) = f(z)). \quad (4.2)$$

The first inequality is the standard Gaussian Poincaré inequality; see [4]. For the second, apply [6, Lemma 2.3] to $f - \mathbb{E}f(G)$: an even function is orthogonal to every linear function under Gaussian measure. Both inequalities extend to Lipschitz functions by smooth approximation, preserving evenness when needed. Each coordinate function R_i is even and 1-Lipschitz. The second inequality therefore gives $\text{Var } R_i(G) \leq 1/2$. Summing these scalar inequalities yields

$$\mathbb{E}|R(G) - \mu|^2 = \sum_{i=1}^n \text{Var } R_i(G) \leq \frac{n}{2}.$$

Since $\mathbb{E}|R(G)|^2 = n$, we obtain

$$\frac{n}{2} \leq |\mu|^2 \leq n. \quad (4.3)$$

This is the source of the gain: an even contraction preserving Gaussian length cannot have a small mean.

5. A TRANSLATION MAKES GOOD DIRECTIONS TYPICAL

Let e be the added coordinate direction and let Π be orthogonal projection onto $e^\perp$. We use the horizontal part and the height of the mean:

$$m = \Pi\mu, \quad d = \frac{|\langle \mu, e \rangle|}{\sqrt{n}}.$$

The projection Π keeps the original horizontal coordinates and discards the coordinate added in the lift. Thus m is the average horizontal position of the folded Gaussian $R(G)$, while d is the absolute value of its average height, on the scale $\sqrt{n}$. Now define the shifted horizontal Gaussian and its direction:

$$X = \frac{\Pi G - km}{\sqrt{n}}, \quad U = \frac{X}{|X|}.$$

The vector X is shifted opposite to m and rescaled to have typical length of order one. Its direction U is the candidate good support direction; $X \neq 0$ almost surely.

The cone projection supplies the auxiliary vector

$$W = \Pi P_D G - km.$$

It is the horizontal part of $P_D G$, shifted by $-km$. Since $P_D G = (G + R(G))/2$, the candidate splits as

$$\sqrt{n} X = W + \frac{1}{2}\Pi(G - R(G)). \quad (5.1)$$

The full vector $G - R(G)$ belongs to $-D$, so its scalar product with every lifted point is nonpositive. Thus the second term has an upper support bound supplied by the added coordinate. It remains to control the length of W .

5.1. Three scalar estimates. Because R is even, $\mathbb{E}\langle \Pi G, \Pi R(G) \rangle = 0$. Also $\mathbb{E}P_D G = \mu/2$. Consequently

$$\begin{aligned} \mathbb{E}|W|^2 &= \frac{1}{4}(\mathbb{E}|\Pi G|^2 + \mathbb{E}|\Pi R(G)|^2) - (k - k^2)|m|^2 \\ &\leq \frac{n}{2} - \frac{|\langle \mu, e \rangle|^2}{4} - (k - k^2) \left(\frac{n}{2} - |\langle \mu, e \rangle|^2 \right) \\ &= n(A_0 - B_0 d^2), \end{aligned}$$

where (4.3) was used and

$$A_0 = \frac{1 - k + k^2}{2} = \frac{241}{512}, \quad B_0 = (1/2 - k)^2 = \frac{49}{256}. \quad (5.2)$$

Notice the term linear in k in A_0 : the geometric improvement starts at first order in the translation, while its measure cost will be quadratic.

The ordinary inequality in (4.2) and Chebyshev's inequality now imply, with probability $1 - o(1)$,

$$|X| \geq 1 - o(1), \quad \frac{|W|}{\sqrt{n}} \leq \sqrt{A_0 - B_0 d^2} + o(1), \quad \frac{\langle R(G) - G, e \rangle}{\sqrt{n}} \leq d + o(1). \quad (5.3)$$

Here are the details needed for uniformity. The functions $|\Pi G - km|$, $|W|$, and $\langle R(G) - G, e \rangle$ have Lipschitz constants at most 1, 1, 2, respectively, so their variances are bounded by 1, 1, 4. Each differs from its mean by at most $n^{1/4}$ with probability $1 - O(n^{-1/2})$. Moreover,

$$\mathbb{E}|\Pi G - km|^2 = n - 1 + k^2|m|^2 \geq n - 1,$$

so $\mathbb{E}|\Pi G - km| \geq \sqrt{n - 2}$; and $\mathbb{E}|W| \leq \sqrt{\mathbb{E}|W|^2}$. These observations prove (5.3), with errors independent of D .

5.2. Controlling the whole shell at once. For $x \in E$, put $t = |x|^2$. Since $\lambda_x \in D = D^*$ and $R(G) - G \in D$,

$$\langle x, \Pi(G - R(G)) \rangle \leq g(t) \langle R(G) - G, e \rangle.$$

Combining this with (5.1), on the single event (5.3) we obtain, simultaneously for all $x \in E$,

$$\begin{aligned} \langle x, U \rangle &\leq \frac{\sqrt{t}|W| + \frac{1}{2}g(t)\langle R(G) - G, e \rangle}{\sqrt{n}|X|} \\ &\leq \sqrt{t(A_0 - B_0 d^2)} + \frac{1}{2}g(t)d + o(1) \\ &\leq \sqrt{A_0 \left(t + \frac{64}{49}(7/6 - t)^2 \right)} + o(1). \end{aligned} \quad (5.4)$$

The last inequality is Cauchy-Schwarz applied to $(\sqrt{A_0 - B_0 d^2}, \sqrt{B_0}d)$ and $(\sqrt{t}, g(t)/(2\sqrt{B_0}))$. This step handles every possible orientation of the mean.

Only a quadratic on $[3/4, 1]$ remains. It is convex, and its endpoint values are

$$\left[t + \frac{64}{49}(7/6 - t)^2 \right]_{t=3/4} = \frac{1723}{1764} < 1, \quad \left[t + \frac{64}{49}(7/6 - t)^2 \right]_{t=1} = \frac{457}{441} < \frac{26}{25}.$$

Therefore the square of the main term in (5.4) is less than

$$\frac{241}{512} \frac{26}{25} = \frac{3133}{6400} < \frac{3136}{6400} = a^2. \quad (5.5)$$

It follows that $h_E(U) \leq a$ with probability $1 - o(1)$. There is no union bound over E : the same three scalar estimates control every point of this possibly infinite set.

5.3. Paying for the translation. Let $v = km$ and define

$$\mathcal{E} = \{z \in \mathbb{R}^n : \Pi z \neq 0, h_E(\Pi z / |\Pi z|) \leq a\}.$$

We have just proved $\mathbb{P}(G - v \in \mathcal{E}) = 1 - o(1)$, and $|v|^2 \leq k^2 n$ by (4.3). Write γ_n for standard Gaussian measure, whose density is $(2\pi)^{-n/2} e^{-|z|^2/2}$. This density gives the exact change-of-variables identity

$$\gamma_n(\mathcal{E}) = \mathbb{E} \left[\mathbf{1}_{\mathcal{E}}(G - v) \exp \left(\langle v, G \rangle - \frac{|v|^2}{2} \right) \right]. \quad (5.6)$$

Since $\text{Var} \langle v, G \rangle = |v|^2 = O(n)$, Chebyshev gives $\mathbb{P}(\langle v, G \rangle < -n^{3/4}) = o(1)$. Restricting the expectation to the intersection of the two good events yields

$$\gamma_n(\mathcal{E}) \geq \exp(-k^2 n/2 - o(n)).$$

The direction of ΠG is uniform on $\mathbb{S}^{n-2}$, so

$$\nu_{n-2}(\mathcal{H}_E) \geq \exp(-n/512 - o(n)). \quad (5.7)$$

This calculation is the whole measure comparison. In probabilistic terms, we first make the desired event typical by shifting a Gaussian, then read its probability under the original Gaussian from the density.

6. THE FINAL BOUND AND STABILITY

Combining (3.5), (5.7), and (3.4) proves the uniform estimate

$$\nu_{n-1}(A_+) \geq \left(\frac{e^{-1/512}}{\sqrt{149/100}} \right)^n \exp(-o(n)). \quad (6.1)$$

If E is empty, (3.5) gives $\nu_{n-2}(\mathcal{H}_L) \geq 1/2$, which is stronger than needed. Finally, $e^x < (1 - x)^{-1}$ for $0 < x < 1$, and therefore

$$\frac{149}{100} e^{1/256} < \frac{149}{100} \frac{256}{255} = \frac{9536}{6375} < \frac{3}{2}, \quad \frac{3}{2} - \frac{9536}{6375} = \frac{53}{12750}. \quad (6.2)$$

6.1. Handling a small increase of the diameter. To apply Proposition 1.2 simultaneously to all points in each small neighborhood in Section 2, the polar bound must remain valid for angular diameters slightly larger than $\pi/3$. The strict gaps in the pairing and lift estimates allow this enlargement. Suppose now $\langle z, z' \rangle \geq c$, where $c < 1/2$ is sufficiently close to $1/2$. The same nearest-point argument writes the image under central projection as $s_c L$, with

$$s_c = \sqrt{\frac{1-c}{c}}, \quad L \subset \mathbb{B}^{n-1}.$$

For $x, y \in L$ its constraint is

$$\langle x, y \rangle \geq \frac{c \sqrt{(1 + s_c^2 |x|^2)(1 + s_c^2 |y|^2)} - 1}{s_c^2}. \quad (6.3)$$

On the compact square $0 \leq |x|^2, |y|^2 \leq 1$, the right-hand side converges uniformly to that of (3.2) solved for $\langle x, y \rangle$. The exclusion in Lemma 3.1 has the fixed gap (3.6), and the lift inequality (3.7) has strictly positive corner minima. Both therefore hold for c close enough to $1/2$, with exactly the same a, r, g . The Gaussian part of the argument is unchanged. Only the sector factor changes, to $(1 + s_c^2 a^2)^{-n/2}$, so

$$\nu_{n-1}(A_+) \geq \left(\frac{e^{-1/512}}{\sqrt{1 + s_c^2 a^2}} \right)^n \exp(-o(n)).$$

All errors are uniform. For any $\beta < C^{-1}$, (6.2) and continuity allow a fixed $c < 1/2$ for which the displayed base is greater than β . Taking $\delta = \arccos c - \pi/3$ proves Proposition 1.2.

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