

THE CROSS-ENERGY OF DIRICHLET PARTIAL SUMS AND THE ZEROS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. Let $X_n(s) = \sum_{k \leq n} k^{-s}$ and let $C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma}$ be the cross-energy of consecutive partial sums. At every point of the critical strip the limit of $C_2(n, s)$ is $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$; this trichotomy is the trace of the pole at $s = 1$. On the critical line the associated helix has a natural orientation whose signed radius is Hardy's $Z(t)$, and the zeros of the truncated functions spiral into the zeros of ζ at the rate $n^{-\beta}$. We prove that the mean energy of the normalized error $E(u)$ in the prime number theorem equals $\sum_{\rho} \lim_n C_2(n, \rho)$ in $[0, \infty]$; it is at least $2 + \gamma_0 - \log 4\pi$, with equality if and only if the Riemann hypothesis (RH) holds and all zeros are simple, and an unconditional local version expresses the energy in a Gaussian window as a double sum over zeros. We then study the pointwise inequality $S(\sigma, t) = \partial_{\sigma} \log |\xi(\sigma + it)| > 0$, which is equivalent to RH: an off-line pair lowers S exactly inside a disc, the failure set at abscissa σ up to height T has measure at most $N(\sigma, T + 1)$, and S is Weil's functional at a Poisson kernel, whose truncation is an exact finite expression in the primes up to e^T . Every criterion is tested against the Davenport–Heilbronn function, which has zeros off the critical line, and against its Euler-product twin $L(s, \chi) \bmod 5$: the former violates each criterion in the predicted way, the latter behaves like ζ . For ζ , the primes up to 10^8 give mean energy 0.0453 against 0.0462. Under RH, simple zeros and a spacing condition we determine the next term of the energy, $\int_1^X (\psi(x) - x)^2 x^{-2} dx = C_B \log X + K + \Phi(\log X) + o(1)$ with $K = 2.5125\dots$ and Φ the almost periodic interference of pairs of zeros, which the primes up to 10^7 follow with correlation 0.999; random models of the primes and of μ carry 20 to 150 times more energy. On the circle of the Euler factor of a prime p the zeros lean toward the angle π , and toward $\pi + \arg \chi(p)$ for $L(s, \chi)$. With the endpoint term $\frac{1}{2}N^{-s}$ restored, the zeros of the truncations converge to those of ζ along helices of radius $|\rho|N^{-\beta-1}/(12|\zeta'(\rho)|)$, which we prove with explicit constants; for the Davenport–Heilbronn function the two zeros of an off-line pair converge at rates that differ by $2(\beta - \frac{1}{2})$. Weil's form with two windows at the ends of its support detects an isolated zero with $\beta = \frac{1}{2} + \delta$ at height γ with the prime powers up to $(C \log \gamma)^{1/\delta}$; the corresponding positivity holds for ζ up to $3 \cdot 10^6$ but fails for a random multiplicative model. We do not prove the Riemann hypothesis.

1. INTRODUCTION

Let $X_n(s) = \sum_{k \leq n} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma} = 2 \operatorname{Re} (X_{n-1}(s) \overline{n^{-s}}).$$

It records how the newest term n^{-s} interacts with all previous ones. In a series of preprints [C1]–[C8] the author studied this quantity at the zeros of ζ and related it to several criteria for the Riemann hypothesis (RH). The present paper collects the results of that series which, in our view, add something to the existing literature, states them with complete proofs, and tests every criterion against the Davenport–Heilbronn function f , which satisfies a functional equation of the same type as ζ but has zeros off the critical line.

Date: Version 3, October 18, 2026. Version 1: [doi:10.5281/zenodo.23243641](https://doi.org/10.5281/zenodo.23243641); Hexagon, hexagon:2610.00161v2.

2020 *Mathematics Subject Classification.* Primary 11M26; Secondary 11M06, 11N05, 11Y35.

Key words and phrases. Riemann hypothesis, cross-energy, mean square of the prime number theorem error, partial sums, explicit formula, Weil positivity, monotonicity of ξ , zero density, Davenport–Heilbronn function, Landau's formula.

Throughout, $\rho = \beta + i\gamma$ denotes a nontrivial zero of ζ , $\Theta = \sup \beta$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

where γ_0 is Euler's constant and the sum, unconditional, counts multiplicity and is taken in symmetric order [Da, C4]. Under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$.

Main results.

1. **Theorem A (Section 2).** For $0 < \sigma < 1$,

$$X_{n-1}(s) \overline{n^{-s}} = \frac{n^{1-2\sigma}}{1-s} + \zeta(s) n^{-\sigma} e^{it \log n} - \frac{1}{2} n^{-2\sigma} + O(|s| n^{-2\sigma-1}).$$

Consequently $\lim_n C_2(n, s)$ equals $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$, at every point of the strip; the zeros of ζ appear only in the second term. The first term is the trace of the pole at $s = 1$. On the critical line the helix described by $X_{n-1}(s) \overline{n^{-s}}$ has a natural orientation, and its signed radius is Hardy's function $Z(t)$; each zero on the line is a crossing of the axis, and an off-line pair is a missing crossing (Turing's method). At every zero, uniformly, $C_2(\cdot, \rho)$ changes sign exactly once beyond $|\gamma|^{3/2}$, at $n^* = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$ when $\beta = \frac{1}{2}$ and at distance $(2\beta-1)\gamma^2/(2(1-\beta)) + O(1)$ from it otherwise (Proposition 2.2), and the zeros of the truncated function $X_{n-1}(s) - n^{1-s}/(1-s)$ spiral into their limits at the rate $n^{-\beta}$ (Proposition 2.3). As applications we give a zero detector on the critical line, and we show that the zeros, placed on the circle $|p^{-s}| = p^{-1/2}$ of the Euler factor of a prime p , are biased toward the angle π , whereas on the circle of e used in [E24] they are uniformly distributed; for a Dirichlet L -function the bias points toward $\pi + \arg \chi(p)$. For the finite sums $\sum_{k \leq N} k^{-s}$, the real parts of the zeros are governed by a polygon condition in $\pi(\sqrt{N})$ angles (Proposition 2.6); adding a prime never moves their right edge to the left (Proposition 2.5), adding a power of 2 moves it to the left for $2^k \leq 16$ (Proposition 2.7), and 11^2 is an exception to the corresponding rule for prime powers. At a zero on the line, $C_2(n, \rho) = 1/(\gamma^2 + \frac{1}{4}) - \frac{1}{n} - \frac{1}{12n^2} + O(|\gamma|^5 n^{-5})$ increases monotonically to its limit, and RH is equivalent to the boundedness of $C_2(\cdot, \rho)$ at every zero (Corollary 2.1).

2. **Theorem B (Section 3).** Let $E(u)$ be the normalized error term of the prime number theorem. In $[0, \infty]$,

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du = \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho)$$

when all zeros are simple; in general both sides are finite or both are infinite, and RH holds if and only if they are finite. The exponential growth rate of the energy is $2\Theta - 1$ (Theorem 3.2). An unconditional local version expresses the energy in a Gaussian window as an absolutely convergent double sum over zeros (Theorem 3.4); under RH it stays in a band of explicit width around C_B . The energy is at least C_B unconditionally, with equality if and only if RH holds and all zeros are simple (Corollary 3.7). Under RH, simple zeros and a spacing condition, the energy up to X is $C_B \log X + K + \Phi(\log X) + o(1)$, with an explicit constant $K = 2.5125\dots$ and an almost periodic interference term Φ from pairs of zeros (Proposition 3.8); the Gram matrix of the zeros has the C_2 limits on its diagonal and as its asymptotic spectrum, and random models of the primes carry far more energy (Remark 3.9). Each zero contributes an orbit of the prime signal, a closed ellipse if $\beta = \frac{1}{2}$ and an expanding spiral if $\beta > \frac{1}{2}$, whose phase also determines β .

3. **Theorem C (Section 4).** Let $S(\sigma, t) = \partial_{\sigma} \log |\xi(\sigma + it)|$, so that RH is equivalent to $S > 0$ for $\sigma > \frac{1}{2}$ [La, SD]. An off-line pair $\rho, 1 - \bar{\rho}$ contributes negatively to $S(\sigma, t)$ exactly inside the disc $(\sigma - \frac{1}{2})^2 + (t - \gamma)^2 < (\beta - \frac{1}{2})^2$.

4. **Theorem D (Section 4).** For $\frac{1}{2} < \sigma < 1$, the set of $t \in [0, T]$ where $S(\sigma, t) \leq 0$ has measure at most $2\sqrt{\sigma(1-\sigma)}N(\sigma, T+1)$, so zero-density estimates make it sparse.
5. **Theorem E (Section 5).** $S(\sigma, t)$ is Weil's functional evaluated at a Poisson kernel of width $\sigma - \frac{1}{2}$. Truncating the kernel to a window of length L gives an exact finite expression in the prime powers $n \leq e^L$, which under RH lies between $(1 - e^{-\delta L})^2$ and $(1 + e^{-\delta L})^2$ times $2S(\sigma, t)$, $\delta = \sigma - \frac{1}{2}$. Consequently, under RH, primes up to P resolve S at distance δ from the critical line once $\delta \log P \gtrsim \log(1/\varepsilon)$ (Corollary 5.1); for the Davenport–Heilbronn function, an off-line zero at horizontal distance $\beta - \sigma$ from the probe becomes visible in the truncated functional at $L \approx 4.5/(\beta - \sigma)$.

Section 6 treats the Euler product inside the strip, identifies each partial Euler product with the C_2 energy of the partial sums restricted to smooth integers, and shows that twisting by characters cannot remove the pole from linear positivity arguments: a corrected partial product converges to ζ in $\frac{1}{2} < \sigma < 1$ exactly when RH holds, and at a zero on the line it decays like $|\zeta'(\rho)|e^{-\gamma_0}/\log x$. On the critical line the corrected partial products, read along $u = \log x$, are a superposition of helices, one for each zero, whose mean energy at $\frac{1}{2} + it$ is the curvature $-(\log |\xi|)''(t)$ (Remark 6.5). Computed with care, the helix built from the primes up to 10^8 fixes the real parts of known zeros to within about $4 \cdot 10^{-5}$, a precision that improves with x ; symmetric off-line pairs enter only at second order (Remark 6.6). The amplification that proves the analogue of RH for curves becomes over $\mathbb{Q}$ a one-sided upper bound for $\psi(x)$; Selberg's sieve gives such bounds, and on short intervals near 10^8 it loses a factor close to 2 exactly along Liouville's function, the parity barrier (Remark 6.7). On the heat-flow deformation of ξ , the mean energy of the Euler helix, which is the first Laguerre expression, changes sign exactly at the first collision of zeros; its variance is the second Laguerre expression, and on the side of the primes the whole Laguerre hierarchy reduces to the finiteness of the helix energy (Remark 6.8). In the square $v = (\sigma - \frac{1}{2})^2$ of the distance to the line, RH is equivalent to the concavity of $\log |\xi|$ in v at every height, a criterion that combines the functional equation, the C_2 energy and the Euler helix (Proposition 4.3). In the variable $\lambda = s(1-s)$ the crossing law reads $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} + \dots$, a universal series in λ with rational coefficients. The X – Y energies obey the same rule: at a zero on the line $|X_N^C|^2 = N/\lambda - \frac{1}{6N} + (\frac{1}{90} + \frac{\lambda}{240})N^{-3} + \dots$ to all orders, two zeros interact only through $\lambda_1 - \lambda_2$, and at a zero ρ the normalised energy gap tends to $-\frac{1}{6} \operatorname{Re} \lambda(\rho)/|1 - \rho|^2$, which equals $-B_2$ exactly when $\beta = \frac{1}{2}$. In the Jensen moment, mollifying by the Euler–Cáceres factor instead of a Dirichlet polynomial removes the pole barrier 0.0386: numerically the resulting bound for the off-line excess decays like $1.828 W(s_0)/\log X$ for ζ and grows with X for the Davenport–Heilbronn function (Proposition 6.1'''); an attained off-line supremum of the real parts forces it to grow like a power of X (Theorem 6.1'''), while under RH it tends to 0 (Theorem 6.1⁽⁵⁾). Hence RH is equivalent to its convergence to 0, and, when the supremum of the real parts is attained, to the boundedness of a single mean square of ζ/E^* on the critical line (Corollary 6.1⁽⁶⁾). Evaluated at its own crossing, the cross-energy reduces to $Z(t) \cos \varphi(t)$ on the line, and the zeros of ζ are the dislocations of its nodal pattern in the strip. In energy-slope form, the argument of de la Vallée Poussin gives $S > 0$ in a region of width $c/\log t$ to the left of $\sigma = 1$ (Proposition 4.4). Finally, the sums over the zeros that see β are non-holomorphic and are reached from the primes only through energies; the damped energy of the prime signal has its first singularity at $2 \sup \beta - 1$ and detects an off-line zero at height T only with the primes up to about $T^{1/\delta}$ (Remark 6.9). Sieving the wave $C_2 + iS_2$ by the primes up to y and removing the pole correction turns its free amplitude into the Euler helix, so the partial sums of Section 2 and the partial products of Section 6 describe the same object (Remark 6.10). With a smooth weight in $\log x$ the identity becomes local (Proposition 6.1'): ζ is reproduced to 10^{-12} by the primes and the zeros within distance 20 of t , the radius of the C_2 helix factors into a prime part and a local zero part, and $\zeta'(\rho)$ at every simple zero is given exactly, without the Deep Riemann Hypothesis. On the critical line each prime acts on the wave by the real operator $1 - p^{-1}T_p$, the same at

every height, and the sieved functional equation shows why this rigidity does not combine with the reflection: it maps the half of the strip where the Euler product converges onto the half where it diverges (Remark 6.11). We also show that no combination of cosines without a constant term can be nonnegative at every $\log p$ (Proposition 6.12): positivity prime by prime requires the pole, and without it positivity can only be quadratic.

The Davenport–Heilbronn control. Each criterion above is equivalent to RH or implied by it, and none uses the Euler product beyond what is visible in the numerics. We therefore test each of them on f , whose off-line zeros below height 400 are known [C5]. For f the energy grows as predicted by its off-line zeros; the failure set of the pointwise inequality consists of one interval per off-line zero and fills 86–90% of the bound of Theorem D; the truncated Weil functional becomes negative, but outside the discs of Theorem C; and the spectrum of its zeros has spikes of both signs. The zeros of the minimiser of Weil’s form, which approximate the zeros of ζ to dozens of digits on short windows [Cn26], approximate those of f equally well until shortly after its form loses positivity at $L^* \approx 3.44$, which high precision locates below the double-precision value 3.66; beyond that threshold all the approximations degrade at once and one zero of the minimiser locks onto the ordinate of the off-line pair. The L -function of the character mod 5, which has the same conductor and gamma factor as f and an Euler product, behaves like ζ in every test. For ζ the corresponding quantities behave as RH predicts.

Additions in this version. Proposition 2.3’ gives the spiral of a zero with explicit constants for the truncation with the endpoint term, verified for the first 100 zeros, and measures on the Davenport–Heilbronn function the different rates of the two zeros of an off-line pair. Section 2 adds a scan of Hardy’s function up to height 10^5 without an extremum that fails to cross, against eight such extrema for f , one at each of its off-line pairs. Proposition 5.3 and the measurements after it treat Weil’s form at large height, where an isolated off-line zero is detected by the prime powers up to a power of $\log \gamma$ but the corresponding positivity is not a property of the statistics of the primes. Section 7 adds two structural remarks, and the compendium the finite parts of $1 - \zeta - \zeta'/\zeta$ at the zeros.

What is classical. Backlund’s bound for the Euler–Maclaurin remainder, Trudgian’s explicit bound for $S(T)$, Hilberdink’s theorem on periodic integer counts and the dynamics of the de Bruijn–Newman flow are known. Cramér’s theorem, the equivalence of bounded energy with RH, Hardy’s function and Turing’s method, Lagarias’s monotonicity criterion, Weil’s criterion, Landau’s formula, the density theorems and the positivity argument behind Proposition 6.4 are known, and we cite them as such. Proposition 6.1’ and Corollary 6.1’’ are exact forms of the hybrid products of Gonek, Hughes and Keating [GHK] and of Bui, Gonek and Milinovich [BGM]. The disc lemma, the crossing law and the spiral of a zero are elementary consequences of Euler–Maclaurin summation and Lagarias’s pairing argument. The asymptotic $I(X) \sim C_B \log X$ is Cramér’s, and the random models of Remark 3.9 are those of Cramér, Denjoy and Wintner. To our knowledge the formulation of Theorem B, the constant K of Proposition 3.8, the local energy law (Theorem 3.4) and the characterization of equality in Corollary 3.7, the exceptional-set bound of Theorem D, the identification and truncation of Theorem E, the reading of Landau’s formula on the circles of the Euler factors (including the twisted circles), and the systematic tests against the Davenport–Heilbronn function and its Euler-product twin $L(s, \chi)$ are new. We do not prove the Riemann hypothesis. Appendix A collects the formulas used in the paper, with their status.

2. THE CROSS-ENERGY AND ITS LIMITS

2.1. The expansion. Theorem A. Let $s = \sigma + it$ with $0 < \sigma < 1$ and put $Z_n(s) = X_{n-1}(s)\overline{n^{-s}}$, so that $C_2(n, s) = 2 \operatorname{Re} Z_n(s)$. As $n \rightarrow \infty$,

$$Z_n(s) = \frac{n^{1-2\sigma}}{1-s} + \zeta(s) n^{-\sigma} e^{it \log n} - \frac{1}{2} n^{-2\sigma} + O(|s| n^{-2\sigma-1}). \quad (2.1)$$

In particular

$$C_2(n, s) = \frac{2(1-\sigma)}{|1-s|^2} n^{1-2\sigma} + 2|\zeta(s)| n^{-\sigma} \cos(t \log n + \arg \zeta(s)) - n^{-2\sigma} + O(|s| n^{-2\sigma-1}),$$

and $\lim_n C_2(n, s)$ equals $1/(\frac{1}{4} + t^2)$ if $\sigma = \frac{1}{2}$, 0 if $\sigma > \frac{1}{2}$, and $+\infty$ if $\sigma < \frac{1}{2}$.

Proof. By Euler–Maclaurin summation, $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$ with $\zeta(s, n) = \frac{n^{1-s}}{s-1} + \frac{1}{2}n^{-s} + O(|s|n^{-\sigma-1})$. Multiply by $\overline{n^{-s}} = n^{-\sigma} e^{it \log n}$ and use $\operatorname{Re} \frac{1}{1-s} = \frac{1-\sigma}{|1-s|^2}$. $\square$

Two consequences shape the rest of the paper.

The limit sees the line, not the zeros. The limit of C_2 is decided by the first term of (2.1), which depends only on s . At $s = 0.7 + 20i$, where $\zeta(s) \neq 0$, $C_2 \rightarrow 0$; at $0.3 + 20i$, $C_2 \rightarrow \infty$. A zero changes only the second term, of lower order. The first term is the tail $n^{1-s}/(1-s)$ of the harmonic series, the trace of the pole of ζ at $s = 1$. The Davenport–Heilbronn function has coefficients of mean zero, no pole, and partial sums that converge throughout the strip, so its analogue of C_2 tends to 0 for every $\sigma > 0$: it has no trichotomy.

At a zero the spiral collapses. Geometrically, Z_n winds around the centre $n^{1-2\sigma}/(1-s)$ with radius $|\zeta(s)|n^{-\sigma}$ and angle $t \log n$; on the critical line $\sqrt{n}(Z_n - \frac{1}{1-s})$ is a helix of radius $|\zeta(\frac{1}{2} + it)|$ (Figure 1). At a zero the helix collapses and $\arg Z_n \rightarrow \arg \frac{1}{1-\rho}$. RH is the statement that every collapse occurs on a vertical axis, $\sigma = \frac{1}{2}$. The two properties come from different terms of (2.1), and the expansion alone does not link them: at a hypothetical zero with $\beta > \frac{1}{2}$ the sequence $Z_n(\rho)$ converges to 0 along the tilted axis without any inconsistency. For $t = 17, 20, 14.1347$ and 21.0220 the expansion agrees with the computed Z_n to within $2 \cdot 10^{-4}$ for $100 < n \leq 2 \cdot 10^4$.

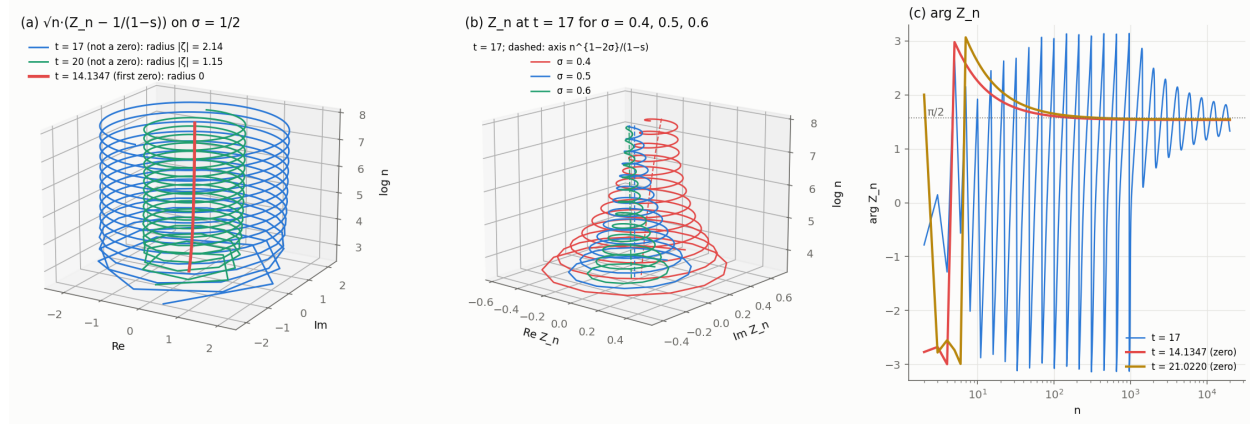


FIGURE 1. The C_2 wave $Z_n = X_{n-1}(s)\overline{n^{-s}}$ (Theorem A). (a) On the critical line $\sqrt{n}(Z_n - \frac{1}{1-s})$ is a helix of radius $|\zeta(s)|$, which degenerates to the axis at a zero. (b) At $t = 17$ the axis $n^{1-2\sigma}/(1-s)$ (dashed) is vertical only for $\sigma = \frac{1}{2}$. (c) $\arg Z_n$ converges to $\arctan(2\gamma)$ at zeros and rotates otherwise.

With the imaginary partner $S_2(n, s) = 2\operatorname{Im}(X_{n-1}(s)\overline{n^{-s}})$ the full wave $W = C_2 + iS_2 = 2Z_n$ shows the three cases at once (Figure 2). By (2.1), $W(n, s) = \frac{2n^{1-2\sigma}}{1-s} + 2\zeta(s)n^{-\sigma}e^{it \log n} - n^{-2\sigma} + O(|s|n^{-2\sigma-1})$: a centre, a rotating term of radius $2|\zeta(s)|n^{-\sigma}$, and a drift. At a zero the rotating term vanishes and W approaches the centre $2/(1-\rho)$ along a smooth curve, crossing the imaginary axis ($C_2 = 0$) at $n^* = \gamma^2 + \frac{1}{3}$ (Proposition 2.2). On the critical line away from a zero, W is a spiral of radius $2|\zeta(s)|n^{-1/2}$ around a fixed centre. Off the line the centre $2n^{1-2\sigma}/(1-s)$ moves toward 0 and the radius decays like $n^{-\sigma}$. Only for $\sigma = \frac{1}{2}$ is the centre fixed. This is a property of the pole and not of RH: for the Davenport–Heilbronn function, whose series has no pole, the

centre is 0 on and off the line. What distinguishes a zero is the rate of collapse. At a zero ρ , $|W(n, \rho) - 2n^{1-2\beta}/(1-\rho)|$ decays like $n^{-2\beta}$, and RH states that every collapse has rate n^{-1} .

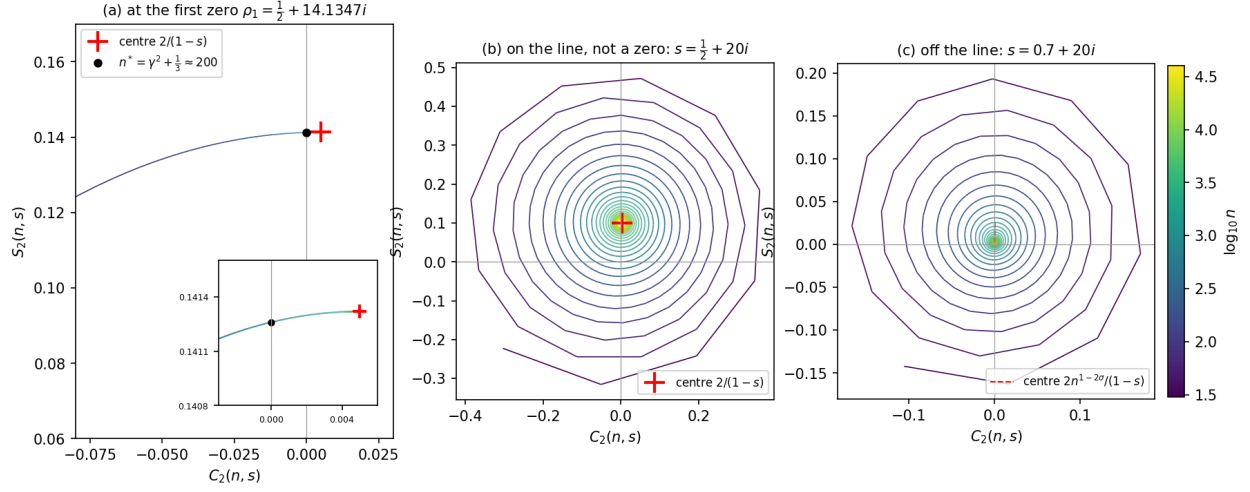


FIGURE 2. The complex wave $W(n, s) = C_2(n, s) + iS_2(n, s)$ for $3 \leq n \leq 40000$, coloured by $\log_{10} n$. (a) At $\rho_1 = \frac{1}{2} + 14.1347i$: no rotation; W reaches the centre $2/(1-\rho_1)$ (cross) and crosses $C_2 = 0$ at $n^* \approx 200$ (dot; inset). (b) At $s = \frac{1}{2} + 20i$: a spiral of radius $2|\zeta(s)|n^{-1/2}$ around the fixed centre $2/(1-s)$. (c) At $s = 0.7 + 20i$: the centre $2n^{1-2\sigma}/(1-s)$ (dashed) moves toward 0.

For the Davenport–Heilbronn function, with $W_f(n, s) = 2X_{n-1}^f(s)\overline{n^{-s}}$ and X_{n-1}^f its partial sums, the collapse at a zero is a shrinking pentagon, from the period 5 of the coefficients, both at the zero $\frac{1}{2} + 43.0813i$ on the line and at the zero $0.80852 + 85.69935i$ off it (Figure 3a,b). The two pictures look alike; the distance to the centre decays with slopes -1.000 and $-1.617 = -2\beta$ in $\log n$, the latter as for ζ at its own zeros with slope -1.000 (Figure 3c).

The wave also counts zeros. Put $F_n(s) = W(n, s) - \frac{2n^{1-2\sigma}}{1-s} + n^{-2\sigma} + \frac{s}{6}n^{-2\sigma-1}$, which by (2.1) equals $2\zeta(s)n^{-\bar{s}} + O(|s|^3n^{-2\sigma-3})$. Along the boundary of a rectangle the factor $n^{-\bar{s}}$ has no net winding, since n^{it} turns by opposite amounts on the two vertical sides, so the winding number of F_n is the number of zeros of ζ inside, computed from the partial sums alone. With $n = 2000$ and the rectangles $0.1 < \sigma < 0.9$, $10 < t < T$, the winding number equals the number of zeros for all 21 values of T tested in $[13, 60.5]$, and is an integer to 10^{-14} (Figure 4a). For the Davenport–Heilbronn function, where $F_n = W_f$, the winding with $n = 20000$ on cells of a grid over $[0.1, 0.95] \times [80, 90]$ locates its seven zeros there, five on the line and the two at $0.19152 + 85.69935i$ and $0.80852 + 85.69935i$ off it (Figure 4b); with $n = 10^5$ the winding on the whole rectangle is 7, as for f itself. This is the argument principle read on the cross-energy; it finds every zero in the region it is applied to, on or off the line, and therefore decides RH only up to the height computed. Status: the expansion and the winding identity are proved; the counts are numerical.

Corollary 2.1 (limits at zeros, monotonicity and boundedness). Let $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$, $0 < \beta < 1$.

- (i) $C_2(n, \rho) = \frac{2(1-\beta)}{|1-\rho|^2} n^{1-2\beta} - n^{-2\beta} + O(|\rho|n^{-2\beta-1})$, so $\lim_n C_2(n, \rho)$ is $1/(\frac{1}{4} + \gamma^2)$, 0 or $+\infty$ according as $\beta = \frac{1}{2}$, $\beta > \frac{1}{2}$ or $\beta < \frac{1}{2}$. On the critical line this is the straight-line law $|X_n(\rho)|^2 = (n + \frac{1}{2})/(\frac{1}{4} + \gamma^2) + O(n^{-1})$ [C1].

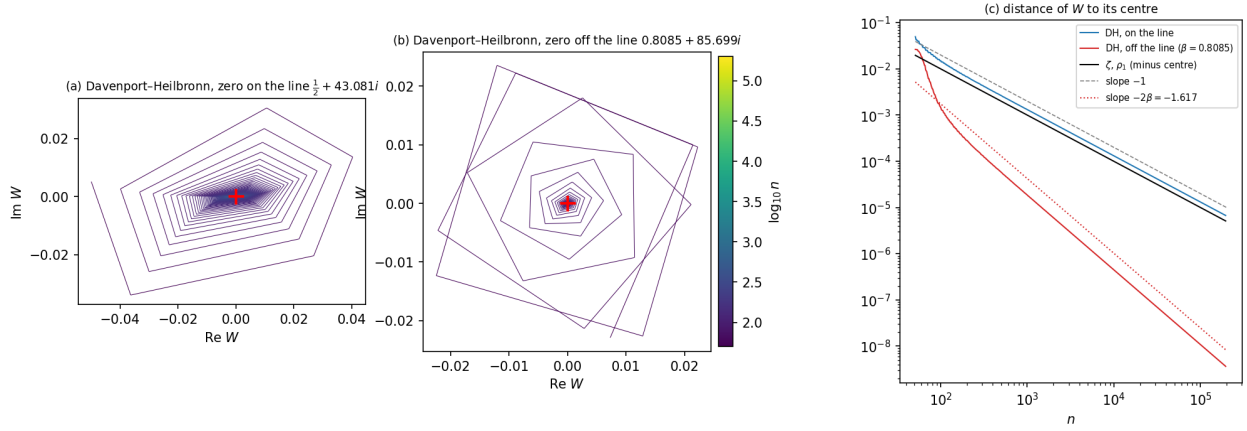


FIGURE 3. The wave W_f of the Davenport–Heilbronn function. (a) At the zero $\frac{1}{2} + 43.0813i$ on the line. (b) At the zero $0.80852 + 85.69935i$ off the line. (c) Distance to the centre (maximum over blocks of five consecutive n) for these two zeros and for ζ at ρ_1 , with slopes -1 and -2β .

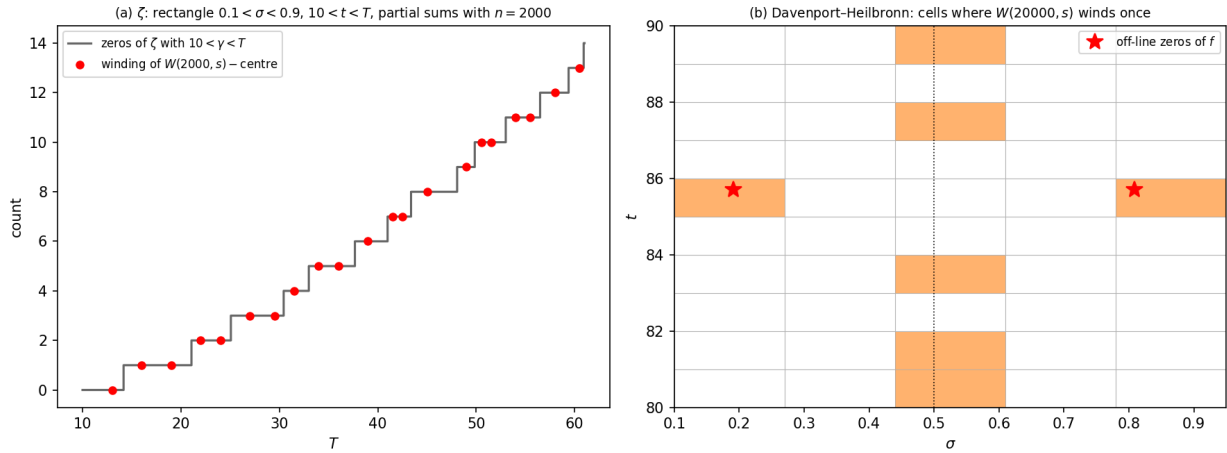


FIGURE 4. Counting zeros with the wave. (a) Winding number of F_{2000} on $0.1 < \sigma < 0.9$, $10 < t < T$ (dots) and the number of zeros of ζ with $10 < \gamma < T$. (b) Cells of $[0.1, 0.95] \times [80, 90]$ on which $W_f(20000, \cdot)$ winds once, for the Davenport–Heilbronn function; stars: its two off-line zeros.

(ii) If $\beta = \frac{1}{2}$, then for every integer $n \geq 1$

$$C_2(n, \rho) = \frac{1}{\gamma^2 + \frac{1}{4}} - \frac{1}{n} - \frac{1}{12n^2} + \frac{\text{Re}(\rho)_3}{360n^4} + R_n, \quad |R_n| \leq 2\kappa |(\rho)_5| n^{-5},$$

where $(\rho)_k = \rho(\rho+1) \cdots (\rho+k-1)$ and $\kappa = \|B_5\|_\infty / 540 < 4.6 \cdot 10^{-5}$. Consequently $C_2(n+1, \rho) > C_2(n, \rho)$ for every n with

$$\frac{n^4}{n+1} > K(\rho) := \frac{|(\rho)_3|}{90} + 4\kappa |(\rho)_5|,$$

so $C_2(\cdot, \rho)$ is eventually strictly increasing, bounded, and converges to $1/(\gamma^2 + \frac{1}{4})$ from below; the threshold is of order $(4\kappa)^{1/3} |\gamma|^{5/3} \approx 0.057 |\gamma|^{5/3}$.

- (iii) The Riemann hypothesis holds if and only if $\sup_n C_2(n, \rho) < \infty$ for every zero ρ , and if and only if $\lim_n C_2(n, \rho)$ is finite and positive for every zero ρ .
- (iv) Unconditionally: (a) $\lim_n C_2(n, \rho) = 1/(\gamma^2 + \frac{1}{4})$ for every zero with $0 < \gamma \leq 3 \cdot 10^{12}$; (b) the same holds for a proportion greater than $\frac{5}{12}$ of all zeros, counted by height; (c) for every $\delta > 0$ the number of zeros with $0 < \gamma \leq T$ at which $C_2(n, \rho) \neq O(n^{2\delta})$ is $O_\delta(T) = o(N(T))$; (d) at every zero, $C_2(n, \rho) \leq (A_\rho + o(1)) n^{1-2/(R_0 \log \gamma)}$ with $A_\rho = 2(1 - \beta)/|1 - \rho|^2$ and $R_0 = 5.573412$.

Proof. (i) is Theorem A with $\zeta(\rho) = 0$. (ii) By Euler–Maclaurin, $X_{n-1}(\rho) = -\zeta(\rho, n)$ and $\zeta(\rho, n) = \frac{n^{1-\rho}}{\rho-1} + \frac{1}{2}n^{-\rho} + \frac{\rho}{12}n^{-\rho-1} - \frac{(\rho)_3}{720}n^{-\rho-3} + R$, with $|R| \leq \frac{\|B_5\|_\infty}{5!} \frac{|(\rho)_5|}{\beta+4} n^{-\beta-4}$. Multiply by $-2n^{-\rho} = -2n^{-\bar{\rho}}$ and take real parts; since $n^{-\rho}n^{-\bar{\rho}} = n^{-1}$ no oscillating factor remains, $-2\operatorname{Re} \frac{1}{\rho-1} = \frac{1}{\gamma^2+1/4}$ and $\operatorname{Re} \rho = \frac{1}{2}$, which gives the expansion with $|R_n| \leq 2n^{-1/2}|R|$. Then $C_2(n+1, \rho) - C_2(n, \rho) \geq \frac{1}{n(n+1)} - \frac{|(\rho)_3|}{360}(n^{-4} - (n+1)^{-4}) - 4\kappa|(\rho)_5|n^{-5}$, and $n^{-4} - (n+1)^{-4} \leq 4n^{-5}$. (iii) If $\beta < \frac{1}{2}$ the limit is infinite by (i); if $\beta > \frac{1}{2}$, then $1 - \bar{\rho}$ is a zero with real part $1 - \beta < \frac{1}{2}$. So all limits are finite (and then equal to $1/(\gamma^2 + \frac{1}{4}) > 0$) exactly when every zero has $\beta = \frac{1}{2}$. (iv) By (i) the limit is $1/(\gamma^2 + \frac{1}{4})$ whenever $\beta = \frac{1}{2}$. (a) All zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the line [PT]. (b) More than five twelfths of the zeros lie on the line [PRZZ]. (c) By (i), $C_2(n, \rho) \neq O(n^{2\delta})$ requires $\beta < \frac{1}{2} - \delta$, and then $1 - \bar{\rho}$ has real part $> \frac{1}{2} + \delta$; the number of such zeros up to height T is $O_\delta(T)$ (Proposition 4.5), while $N(T) \sim \frac{T}{2\pi} \log T$. (d) There are no zeros with $\sigma \geq 1 - 1/(R_0 \log |t|)$, $|t| \geq 2$ [MoT]; applied to $1 - \bar{\rho}$ this gives $\beta \geq 1/(R_0 \log \gamma)$, and (i) gives the bound. $\square$

For the 341 zeros with $\gamma < 600$, $C_2(20000, \rho)$ agrees with $1/(\gamma^2 + \frac{1}{4}) - \frac{1}{N} - \frac{1}{12N^2}$ to $1.4 \cdot 10^{-12}$. The last decrease of $C_2(\cdot, \rho)$ occurs at $n \leq 0.952 \gamma/2\pi$ for every one of them (median $0.932 \gamma/2\pi$), the threshold of the approximate functional equation, while the proven threshold in (ii) is $n_0 = 6, 39, 513, 2413$ for $\gamma = 14.13, 49.77, 236.52, 599.55$. Part (iv) is the unconditional part of (iii): the boundedness is proved for all zeros up to height $3 \cdot 10^{12}$ and for a positive proportion of all zeros, an exponent 2δ is proved for almost all of them, and for every zero only the exponent $1 - 2/(R_0 \log \gamma)$, barely below 1, is known. The step from these statements to every zero is RH. Part (iii) is a reformulation and not a constraint: at a zero, $C_2(n, \rho) = -2\operatorname{Re}[n^{-\bar{\rho}}\zeta(\rho, n)]$ depends only on the position of ρ through the Hurwitz function, and boundedness at ρ is equivalent to $\beta \geq \frac{1}{2}$. A proof of RH along these lines would need an independent reason for the boundedness of $C_2(\cdot, \rho)$ at the zeros.

Proposition 2.2 (crossing law). Extend C_2 to real $n \geq 1$ by $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$, with $\zeta(s, n)$ the Hurwitz zeta function, and for $s = \sigma + it$ put

$$g_s(n) = -\operatorname{Re}[n^s \zeta(s, n)], \quad A(s) = \frac{|1 - s|^2}{2(1 - \sigma)}, \quad c(\sigma) = (1 - \sigma) \left(\frac{1 - \sigma^2}{30} - \frac{\sigma^2}{18} \right).$$

There is an absolute constant t_0 such that for $0 < \sigma < 1$ and $|t| \geq t_0$:

- (i) g_s has exactly one zero $n^*(s)$ in $n \geq |t|^{3/2}$, it lies in $[0.99A, 1.01A]$, and g_s changes sign there from negative to positive;
- (ii) uniformly in σ ,

$$n^*(s) = A(s) + \frac{\sigma}{6} + \frac{c(\sigma)}{t^2} + O(t^{-4}).$$

If $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$, then $C_2(n, \rho) = 2n^{-2\beta}g_\rho(n)$ for every real $n \geq 1$. Hence, for every zero with $|\gamma| \geq t_0$, $C_2(\cdot, \rho)$ has exactly one sign change in $n \geq |\gamma|^{3/2}$, at $n^*(\rho)$, and

$$n^*(\rho) - \left(\gamma^2 + \frac{1}{3} \right) = (2\beta - 1) \left(\frac{\gamma^2}{2(1 - \beta)} - \frac{1}{6} \right) + \frac{c(\beta)}{\gamma^2} + O(\gamma^{-4}). \quad (2.2)$$

In particular $n^*(\rho) = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$ if $\beta = \frac{1}{2}$, while for $\beta \neq \frac{1}{2}$ the deviation is $(2\beta - 1)\gamma^2/(2(1 - \beta)) + O(1)$. For an off-line pair the two deviations add up to $(2\beta - 1)^2\gamma^2/(2\beta(1 - \beta)) + O(\gamma^{-2}) \geq 0$.

Proof. Euler–Maclaurin summation of x^{-s} over $x = n, n + 1, \dots$ gives, for every $m \geq 1$,

$$\zeta(s, n) = \frac{n^{1-s}}{s-1} + \frac{1}{2}n^{-s} + \sum_{j=1}^m \frac{B_{2j}}{(2j)!} (s)_{2j-1} n^{-s-2j+1} + R_m, \quad |R_m| \leq \frac{\|B_{2m+1}\|_\infty}{(2m+1)!} \frac{|(s)_{2m+1}|}{\sigma + 2m} n^{-\sigma-2m},$$

where $(s)_k = s(s+1)\cdots(s+k-1)$ and $\|B_{2m+1}\|_\infty$ is the maximum of the Bernoulli polynomial on $[0, 1]$. With $m = 2$ and $\operatorname{Re} \frac{1}{1-s} = \frac{1}{2A}$,

$$g_s(n) = \frac{n}{2A} - \frac{1}{2} - \frac{\sigma}{12n} + \frac{\operatorname{Re}(s)_3}{720n^3} + E(n), \quad |E(n)| \leq \kappa |(s)_5| n^{-4}, \quad \kappa = \frac{\|B_5\|_\infty}{480} < 5.1 \cdot 10^{-5}.$$

For $n \geq |t|^{3/2}$ the last three terms are bounded by $\frac{1}{12}|t|^{-3/2} + (|t|+3)^3|t|^{-9/2}/720 + \kappa(|t|+5)^5|t|^{-6}$, which is less than 0.002 for $|t| \geq 16$. Hence $g_s < 0$ for $|t|^{3/2} \leq n \leq 0.99A$ and $g_s > 0$ for $n \geq 1.01A$ (note $A \geq t^2/2$). On $[0.99A, 1.01A]$ we have $g'_s(n) = \frac{1}{2A} + \frac{\sigma}{12n^2} + O(|t|^3n^{-4} + |t|^6n^{-5})$, and since $n \asymp A \geq t^2/2$ the error is $O(t^{-2})$ times $\frac{1}{2A}$; so g_s is increasing there and has one zero, which proves (i). For (ii) take $m = 4$, so that the remainder contributes $O(|t|^9n^{-8}) = O(t^{-7})$ to g_s near $n = A$, and the Bernoulli terms with $j = 3, 4$ contribute $O(t^{-6})$. Write $n^* = A + \frac{\sigma}{6} + \varepsilon$. Expanding $\frac{\sigma}{12n} = \frac{\sigma}{12A}(1 - \frac{\sigma}{6A} + O(\frac{1+|\varepsilon|}{A^2}))$ and using $\operatorname{Re}(s)_3 = -3(\sigma+1)t^2 + O(1)$, the equation $g_s(n^*) = 0$ becomes

$$\frac{\varepsilon}{2A} + \frac{\sigma^2}{72A^2} - \frac{(\sigma+1)t^2}{240A^3} = O(t^{-6}),$$

so $\varepsilon = -\frac{\sigma^2}{36A} + \frac{(\sigma+1)t^2}{120A^2} + O(t^{-4})$, and $A^{-1} = 2(1-\sigma)t^{-2} + O(t^{-4})$ gives $\varepsilon = c(\sigma)t^{-2} + O(t^{-4})$. All constants are independent of $\sigma \in (0, 1)$, because $A \geq t^2/2$. At a zero, $X_{n-1}(\rho) = -\zeta(\rho, n)$, and multiplying by $\overline{n^{-\rho}} = n^{\rho-2\beta}$ gives $C_2(n, \rho) = 2n^{-2\beta}g_\rho(n)$. Finally $A(\rho) - \gamma^2 - \frac{1}{4} = (2\beta - 1)(\frac{\gamma^2}{2(1-\beta)} - \frac{1}{4})$ and $\frac{\beta}{6} - \frac{1}{12} = \frac{2\beta-1}{12}$, which gives (2.2); $c(\frac{1}{2}) = \frac{1}{180}$. Adding (2.2) for β and $1-\beta$ gives the statement on pairs. $\square$

The constant $\frac{1}{3}$ is the sum of $\frac{1}{4}$, from the leading term of Theorem A, and $\frac{1}{12}$, from the first Bernoulli term; the next term, $\frac{1}{180\gamma^2}$, comes from the first two Bernoulli terms together. For the first eight zeros of ζ the root of $C_2(\cdot, \frac{1}{2} + i\gamma)$ agrees with $\gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2}$ to within $3 \cdot 10^{-7}$, against $3 \cdot 10^{-5}$ for $\gamma^2 + \frac{1}{3}$ (at $\gamma = 14.1347$ the root is 200.1238157, and $\gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} = 200.1238160$); with integer n , $C_2(n, \rho_1)$ is negative for $n \leq 200$ and positive for $n \geq 201$.

The range $n \geq |t|^{3/2}$ in (i) comes from the use of a fixed number of Bernoulli terms. Numerically the sign change is unique from about $n = |\gamma|/2\pi$ on, the threshold of the approximate functional equation: for $\gamma = 37.59, 79.34$ and 165.54 the function $C_2(\cdot, \rho)$ has 9, 28 and 98 sign changes in $1 < n < |\gamma|/2\pi$, the last at $n = 4.02, 9.72$ and 22.07 , and then only the one at $n^*(\rho)$. Below that threshold $X_{n-1}(\rho)$ is a short partial sum and oscillates.

Proposition 2.2 holds for all zeros at once because it is a statement about the Hurwitz tail $\zeta(s, n)$ at an arbitrary point s . For $s = 0.7 + 20i, 0.6 + 30i, 0.8 + 15i$ and $0.3 + 40i$, which are not zeros, the root of g_s agrees with $A + \frac{\sigma}{6} + c(\sigma)t^{-2}$ to within $3 \cdot 10^{-8}$. The zero condition enters only through $X_{n-1}(\rho) = -\zeta(\rho, n)$, which removes the oscillating term $2|\zeta(s)|n^{-\sigma} \cos(t \log n + \arg \zeta(s))$ of Theorem A. The law therefore characterizes the critical line, on which $n^*(s) - t^2 - \frac{1}{3} \rightarrow 0$, and not the set of zeros: the location of the crossing determines β once ρ is known, but it does not exclude $\beta \neq \frac{1}{2}$. The function $t \mapsto C_2(t^2 + \frac{1}{3}, \frac{1}{2} + it)$ vanishes at every zero on the critical line (within $9 \cdot 10^{-5}$ on a grid of step 0.002), but it has 110 sign changes on $10 \leq t \leq 50$, against 10 zeros, because of that oscillating term. The crossing is a trace of the pole: for the Davenport–Heilbronn function there is no term $n^{1-2\beta}/(1-s)$, and its analogue of C_2 has no such sign change.

The last sign change measures $|\zeta(s)|$. The logarithmic derivative C'_2/C_2 in n tends to $1 - 2\sigma$ at every point of the strip, because the centre term $2(1 - \sigma)n^{1-2\sigma}/|1 - s|^2$ of Theorem A eventually dominates the oscillating term $2|\zeta(s)|n^{-\sigma}$; it therefore carries only σ . Its poles carry $|\zeta(s)|$: they are the sign changes of $C_2(\cdot, s)$, which persist while the oscillating term is the larger and stop near the point where the two terms are equal,

$$n_{\text{last}}(s) \approx \left(\frac{|\zeta(s)| |1 - s|^2}{1 - \sigma} \right)^{1/(1-\sigma)},$$

of order $4|\zeta(s)|^2 t^4$ on the critical line. At a zero the oscillating term is absent and only the sign change at n^* of Proposition 2.2 remains. At eight points with $\frac{1}{2} \leq \sigma \leq 0.7$ and $20 \leq t \leq 40$ the last sign change lies between 0.75 and 1.01 times n_{last} , for values from $2.5 \cdot 10^5$ to $3 \cdot 10^{10}$; at $\frac{1}{2} + 20i$ there are 82 sign changes, the last at $6.8 \cdot 10^5$ against $8.4 \cdot 10^5$ (Figure 5). This is a reading of Theorem A: the position of the last sign change is a measure of $|\zeta(s)|$, and the crossing law is its limit as $|\zeta(s)| \rightarrow 0$. Status: the asymptotic statement follows from Theorem A; the comparison is numerical.

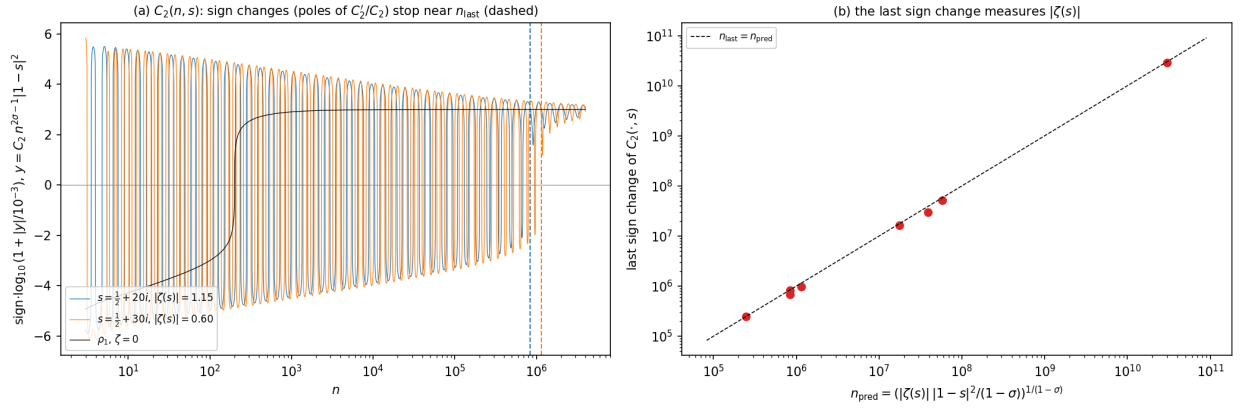


FIGURE 5. (a) $C_2(n, s) n^{2\sigma-1} |1-s|^2$ in a signed logarithmic scale for $s = \frac{1}{2} + 20i$, $\frac{1}{2} + 30i$ and ρ_1 ; dashed: the predicted n_{last} . (b) The last sign change of $C_2(\cdot, s)$ against $n_{\text{last}}(s)$ at eight points of the strip.

The spectral variable. Put $\lambda = s(1 - s)$. At a zero on the critical line $\lambda = \frac{1}{4} + \gamma^2$, and Proposition 2.2 reads

$$n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + O(\lambda^{-2}).$$

The constant $\frac{1}{12} = \frac{B_2}{2} = -\zeta(-1)$ comes from the first Bernoulli term. The variable λ is the one in which the Hilbert-Pólya picture places the zeros as eigenvalues: it is real for every zero on the line, while an off-line zero has $\text{Im } \lambda = \gamma(1 - 2\beta) \neq 0$; the crossing point $n^*(\rho)$ is real in both cases. Since $\xi(s) = \xi(1 - s)$, there is an entire function Ψ with $\xi(s) = \Psi(s(1 - s))$, and with Riemann's kernel $\Phi(u) = 2 \sum_{n \geq 1} (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}$,

$$\Psi(\lambda) = \sum_{k \geq 0} \frac{(-1)^k b_k}{(2k)!} \left(\lambda - \frac{1}{4} \right)^k, \quad b_k = \int_{-\infty}^{\infty} \Phi(u) u^{2k} du > 0.$$

The numbers $n^* - \frac{1}{12}$ are therefore, up to $O(\lambda^{-1})$, the real zeros of an entire function defined by positive moments, without reference to the zeros. We computed $b_0, \dots, b_{400}$ by the trapezoidal rule on the whole line (step $\frac{1}{400}$, decimal arithmetic with 260 digits); $\Psi(\frac{1}{4}) = 0.4971207781883141 = \xi(\frac{1}{2})$. The truncation of the series after $K = 50, 100, 200, 400$ terms has its first 3, 10, 24 and 25

zeros (all those with $\lambda < 8000$) equal to $\frac{1}{4} + \gamma^2$ within 10^{-8} , the accuracy of our table of zeros, and $n^* - \frac{1}{12} - \lambda$ is $2.8 \cdot 10^{-5}$ at the first zero, the term $1/(180\lambda)$. RH is equivalent to Ψ having only real zeros; in Pólya's formulation [Po], to the hyperbolicity of all the Jensen polynomials $J^{d,n}(X) = \sum_{j \leq d} \binom{d}{j} \hat{\gamma}(n+j) X^j$, $\hat{\gamma}(j) = j! b_j / (2j)!$, which Griffin, Ono, Rolin and Zagier proved for each fixed d and all sufficiently large n [GORZ]. All $J^{d,n}$ with $d \leq 150$ and $n \in \{0, 1, 2, 5, 10\}$ are hyperbolic, as expected from the verification of RH to large height. Proposition 2.2 thus says that the crossing of the cross-energy measures the spectral variable λ , with an explicit correction. It is a reading of a classical reformulation, not a new constraint. Status: the identity is proved; the computations are numerical.

The crossing law to all orders. On the critical line the expansion in the proof of Proposition 2.2 can be carried to any order, and every coefficient is a polynomial in λ . At a zero $\rho = \frac{1}{2} + i\gamma$ we have $C_2(n, \rho) = 2n^{-1}g_\rho(n)$ and $A(\rho) = \lambda$, and Euler–Maclaurin summation gives, for every $m \geq 1$,

$$C_2(n, \rho) = \frac{1}{\lambda} - \frac{1}{n} - 2 \sum_{j=1}^m \frac{B_{2j}}{(2j)!} P_j(\lambda) n^{-2j} + O_m(\lambda^m n^{-2m-2}), \quad P_j(\lambda) = \text{Re}(\rho)_{2j-1}.$$

Since $\text{Re}(\rho)_{2j-1}$ is a polynomial in $\gamma^2 = \lambda - \frac{1}{4}$, each P_j is a polynomial of degree $j-1$ in λ with rational coefficients: $P_1 = \frac{1}{2}$, $P_2 = 3 - \frac{9}{2}\lambda$, $P_3 = 60 - 125\lambda + \frac{25}{2}\lambda^2$. Near $n = \lambda$ the j -th term is $O(\lambda^{-j-1})$, so the equation $C_2(n^*, \rho) = 0$ can be solved term by term in $1/\lambda$, in exact rational arithmetic, and the argument of Proposition 2.2 (one sign change, increasing) identifies the root:

$$n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} - \frac{683}{129600\lambda^3} + \frac{232349}{39916800\lambda^4} + O(\lambda^{-5}).$$

On the critical line the crossing point therefore depends on the zero only through λ : $n^*(\rho) = \mathcal{N}(\lambda)$ for a universal asymptotic series whose coefficients are fixed by the Bernoulli numbers. Off the line the coefficients involve β separately, as in (2.2). At the first four zeros the remainder after the term $1/(180\lambda)$, multiplied by λ^2 , is -0.01054 , -0.01053 , -0.01052 , -0.01052 , against $-\frac{53}{5040} = -0.010516$; at the first three, the next remainder multiplied by λ^3 is -0.00524 , -0.00526 , -0.00525 , against $-\frac{683}{129600} = -0.005270$. Status: the expansion is proved and its coefficients are exact (`crossing/nstar.series.py`); the checks are numerical.

The X – Y energies to all orders, and two zeros. With X_N^C , Y_N^C and $D_N = X_N^C - Y_N^C$ as in (A.4), $|X_N^C|^2 - |Y_N^C|^2 = 2 \text{Re}(Y_N^C \overline{D_N}) + |D_N|^2$, and Euler–Maclaurin summation gives $D_N(s) = \zeta(s) - \sum_{k \leq K} c_k(s)_{2k-1} N^{-s-2k+1} + O_{s,K}(N^{-\sigma-2K-1})$ with $c_k = B_{2k}/(2k)!$. Three consequences:

(i) At every zero $\rho = \beta + i\gamma$ of ζ , trivial or not,

$$N^{2\beta} (|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2) = \kappa(\rho) + O_\rho(N^{-2}), \quad \kappa(\rho) = -\frac{1}{6} \frac{\text{Re} \lambda(\rho)}{|1-\rho|^2} = -\frac{1}{6} - \frac{(1-\beta)(2\beta-1)}{6|1-\rho|^2}.$$

Indeed $Y_N^C \overline{D_N} = -\frac{1}{12} N^{-2\beta} \bar{\rho}/(1-\rho) + O(N^{-2\beta-2})$ and $\bar{\rho}/(1-\rho) = \overline{\lambda(\rho)}/|1-\rho|^2$. For a nontrivial zero, $\kappa(\rho) = -B_2 = -\frac{1}{6}$ if and only if $\beta = \frac{1}{2}$, with $\kappa < -\frac{1}{6}$ for $\beta > \frac{1}{2}$ and $\kappa > -\frac{1}{6}$ for $\beta < \frac{1}{2}$. At the trivial zeros $\kappa(-2k) = k/(3(2k+1))$; for $\rho = -2$ everything is exact, since $X_N^C(-2) = \frac{N^3}{3} + \frac{N}{6}$ and $Y_N^C(-2) = \frac{N^3}{3}$ give $N^{-4}(|X_N^C|^2 - |Y_N^C|^2) = \frac{1}{9} + \frac{1}{36N^2}$.

(ii) On the critical line $|Y_N^C|^2 = N/\lambda$ exactly, with $\lambda = \frac{1}{4} + t^2$. If $\zeta(\frac{1}{2} + it) = 0$, then for every $K \geq 1$

$$|X_N^C|^2 = \frac{N}{\lambda} + \sum_{m=1}^K Q_m(\lambda) N^{1-2m} + O_K(N^{-2K-1}),$$

$$Q_m = -2c_m \text{Re}(s+1)_{2m-2} + \sum_{k+l=m} c_k c_l \text{Re}[(s)_{2k-1} \overline{(s)_{2l-1}}],$$

because $s + \bar{s} = 1$ removes every oscillating factor and $\overline{(s)_{2k-1}} = (1-s)\overline{(s+1)_{2k-2}}$. Each Q_m is even in t , hence a polynomial of degree $m-1$ in λ with rational coefficients: $Q_1 = -\frac{1}{6}$, $Q_2 = \frac{1}{90} + \frac{\lambda}{240}$, $Q_3 = -\frac{1}{210} + \frac{19\lambda}{7560} + \frac{\lambda^2}{6048}$, $Q_4 = \frac{1}{210} - \frac{1439\lambda}{302400} + \frac{17\lambda^2}{151200} + \frac{\lambda^3}{172800}$. If $\zeta(\frac{1}{2} + it) \neq 0$ and $t \neq 0$, then $N(|X_N^C|^2 - N/\lambda) = 2N^{3/2}|\zeta(s)||1-s|^{-1}\cos(\varphi - t\log N) + O(N)$ with φ fixed, and $t\log N$ is dense modulo 2π , so $\limsup_N N^{-1/2}||X_N^C|^2 - N/\lambda| = 2|\zeta(s)|/|1-s|$. Hence $\lim_N N(|X_N^C|^2 - N/\lambda)$ exists if and only if $\zeta(s) = 0$, and it is then $-\frac{1}{6}$. This is the counterpart for the energies of the expansion of C_2 above, and the all-orders form of the straight-line law of Corollary 2.1(i).

- (iii) On the line the diagonal part of $|X_N^C|^2$ is the harmonic number H_{N-1} , the same for every t ; all dependence on the height sits in the off-diagonal sum. Subtracting (ii) at two zeros $\rho_j = \frac{1}{2} + i\gamma_j$, $\lambda_j = \frac{1}{4} + \gamma_j^2$,

$$|X_N^C(\rho_1)|^2 - |X_N^C(\rho_2)|^2 = N\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{\lambda_1 - \lambda_2}{240N^3} + O(N^{-5}),$$

and with $m = \frac{1}{2}(\gamma_1 + \gamma_2)$, $h = \gamma_1 - \gamma_2$, $\ell = \log(k/j)$, the off-diagonal part of the left side is $-4\sum_{j < k < N} (jk)^{-1/2} \sin(m\ell) \sin(\frac{h}{2}\ell)$, the mean-height and sine structure of the vertical displacement $\zeta(\frac{1}{2} + i\gamma_1) - \zeta(\frac{1}{2} + i\gamma_2)$. This is the all-orders form of the relation between two zeros of [C20], which obtained only $\lim_N |X_N^C(\rho)|^2/N = 1/\lambda$. In the ratio form of (A.6), $N^2[X_N^C/Y_N^C(\rho_1) - X_N^C/Y_N^C(\rho_2)] \rightarrow -(\lambda_1 - \lambda_2)/12$, while the same quantity is of exact order $N^{3/2}$ along a subsequence if one of the two points is not a zero.

Like the crossing point, the X - Y energies see a zero on the line only through λ , to all orders, and two zeros interact only through $\lambda_1 - \lambda_2$. Off the line the first correction $\kappa(\rho)$ would differ from $-B_2$; as for Corollary 2.1(iii), this reads the abscissa of a point already known to be a zero and does not constrain it. At the first five zeros $N(|X_N^C|^2 - N/\lambda)$ is -0.16666666 , -0.16666665 , -0.16666664 , -0.16666663 , -0.16666662 for $N = 10^4$; $N^5(|X_N^C|^2 - N/\lambda + \frac{1}{6N} - Q_2N^{-3})$ at $N = 100$ is 7.11946 and 33.48647 at ρ_1, ρ_2 , against $Q_3 + Q_4N^{-2} = 7.11941$ and 33.48655 ; the two-zero residual times N^3 is -1.0089 at $N = 10^3$ against $(\lambda_1 - \lambda_2)/240 = -1.0089$; at $\frac{1}{2} + 17.2i$, which is not a zero, $N(|X_N^C|^2 - N/\lambda)$ is 272 , 3251 , 305240 for $N = 10^2, 10^3, 10^4$. Status: (i)–(iii) are proved and the coefficients Q_m are exact (xy_energy/xy_energy.py); the checks are numerical.

The cross-energy at its own crossing. Let $0 < \sigma < 1$, $|t| \geq t_0$, let $n^*(s)$ be the crossing point of Proposition 2.2, and put $K(s) = C_2(n^*(s), s)$, with C_2 extended to real n as there. Then:

- (i) $K(s) = 2n^*(s)^{-\sigma} \operatorname{Re} [\zeta(s) n^*(s)^{it}]$;
- (ii) on the critical line $K(\frac{1}{2} + it)\sqrt{n^*}/2 = Z(t) \cos \varphi(t)$, with $\varphi(t) = t \log n^*(\frac{1}{2} + it) - \theta(t)$;
- (iii) $\varphi'(t) = \frac{3}{2} \log t + O(1)$, so the zeros of K on the critical line up to height T are the zeros of Z together with $\varphi(T)/\pi + O(1) = \frac{3}{2\pi} T \log T + O(T)$ further zeros, asymptotically three for each zero of ζ ;
- (iv) the zeros of ζ in the strip are the common zeros of K and of

$$K^\perp(s) = 2n^*(s)^{-\sigma} \operatorname{Im} [\zeta(s) n^*(s)^{it}],$$

and at a zero of multiplicity m the phase of $K + iK^\perp$ turns by $2\pi m$ along a small circle.

Proof. With $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$ we have $C_2(n, s) = 2\operatorname{Re}[\zeta(s)n^{-\bar{s}}] - 2\operatorname{Re}[\zeta(s, n)n^{-\bar{s}}]$, and $\operatorname{Re}[\zeta(s, n)n^{-\bar{s}}] = n^{-2\sigma} \operatorname{Re}[n^s \zeta(s, n)] = -n^{-2\sigma} g_s(n)$, which vanishes at $n = n^*(s)$; since $n^{-\bar{s}} = n^{-\sigma} n^{it}$, this is (i). On the line $\zeta(\frac{1}{2} + it) = e^{-i\theta(t)} Z(t)$, which gives (ii). By Proposition 2.2, $n^*(\frac{1}{2} + it) = t^2 + O(1)$, so $\frac{d}{dt}[t \log n^*] = 2 \log t + 2 + O(t^{-1})$, while $\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-2})$; thus $\varphi'(t) = \frac{3}{2} \log t + O(1) > 0$ for large t , $\cos \varphi$ has $\varphi(T)/\pi + O(1)$ zeros up to T , and $N(T) \sim \frac{T}{2\pi} \log T$ gives the ratio 3, which is (iii). For (iv), $K + iK^\perp = 2n^{*-\sigma} n^{*it} \zeta(s)$, and $n^*(s)$ is a positive smooth

function of s , so the factor in front of ζ is smooth and does not vanish and the index of $K + iK^\perp$ at a zero equals that of ζ . $\square$

K can be read from the partial sums: at $s = \frac{1}{2} + 20i$, $n^* = 400.33$, and $C_2(400, s) = 0.0849$ from $X_{399}(s)$ against $K(s) = 0.0861$; the difference is the smooth part at the integer point. Identity (ii) holds numerically to 10^{-13} , and on $10 < t < 100$ there are 278 zeros of K on the line against 29 zeros of ζ (Figure 6). In the strip the sign of K forms bands, the nodal lines of K , and by (iv) each zero of ζ is a dislocation of this pattern, where a band begins or ends. In this picture RH says that all dislocations lie on $\sigma = \frac{1}{2}$. The function K is real but not the real part of a holomorphic function, because $n^*(s)$ is real; this is why $K = 0$ alone does not locate the zeros. Status: (i)–(iv) are proved; the figures are numerical.

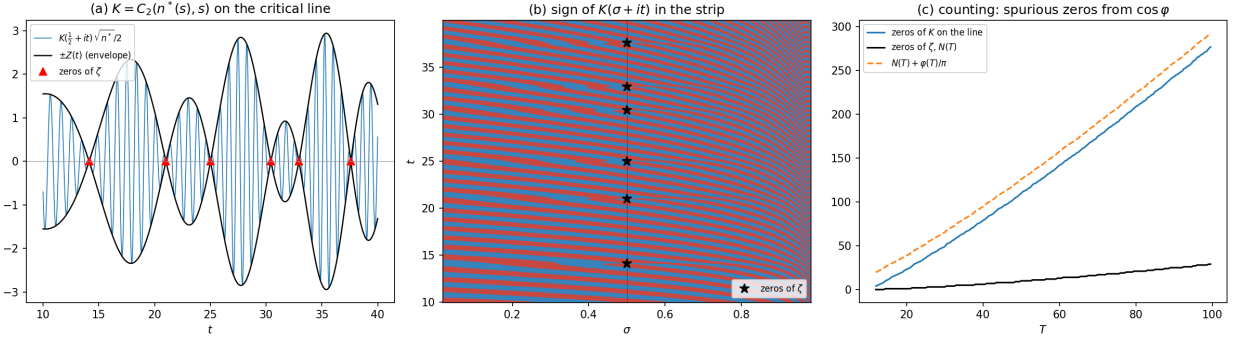


FIGURE 6. The cross-energy at its own crossing, $K(s) = C_2(n^*(s), s)$. (a) $K(\frac{1}{2} + it)\sqrt{n^*}/2$ and the envelope $\pm Z(t)$; triangles: zeros of ζ . (b) Sign of $K(\sigma + it)$ on $0 < \sigma < 1$, $10 < t < 40$; stars: zeros of ζ , at the dislocations of the band pattern. (c) Number of zeros of K on the critical line up to T , of Z , and $N(T) + \varphi(T)/\pi$.

2.2. The orientation of the helix. On the critical line the helix of Theorem A carries a natural orientation. Let $\theta(t) = \text{Im} \log \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$, so that $Z(t) = e^{i\theta(t)} \zeta(\frac{1}{2} + it)$ is real (Hardy's function). By (2.1),

$$e^{i(\theta(t) - t \log n)} \sqrt{n} \left(Z_n(\frac{1}{2} + it) - \frac{1}{\frac{1}{2} - it} \right) = Z(t) + O(n^{-1/2}),$$

so rotating the helix back by the angle $t \log n - \theta(t)$ gives a real number, its *signed radius*, which is $Z(t)$ for every n . Each zero on the critical line is then a crossing of the axis by the signed radius. A pair of zeros $\rho, 1 - \bar{\rho}$ off the line adds two zeros to the Riemann–von Mangoldt count $N(T) = \theta(T)/\pi + 1 + S(T)$ but no crossing, so RH holds if and only if the number of crossings up to T equals $N(T)$ for every T . This is Turing's method for verifying RH, read on the C_2 helix; it locates zeros on the line but cannot exclude an off-line pair above the computed height.

The signed radius computed from the partial sums with $n = 4000$ agrees with $Z(t)$, with imaginary part at most 0.008, the size of the term $\frac{1}{2}n^{-1/2}$ (Figure 7a). On $0 < t < 400$ it has 202 sign changes, equal to the number of zeros of ζ in that range. The Davenport–Heilbronn function has an analogous real function $Z_f(t) = e^{i\theta_f(t)} f(\frac{1}{2} + it)$, with $\theta_f(t) = \text{Im}[\frac{s}{2} \log \frac{5}{\pi} + \log \Gamma(\frac{s+1}{2})]$ at $s = \frac{1}{2} + it$, and the same construction applies. It has 287 sign changes on $0 < t < 400$, against 303 zeros: the deficit is exactly 16, the eight off-line pairs. Near the pair at height 85.699 the signed radius approaches the axis and turns back without crossing (Figure 7b), and the difference between the number of crossings and the smooth count drops by 2 at each of the eight heights (Figure 7c).

Extrema without crossing. An off-line pair appears on the critical line as an extremum of the signed radius that does not reach the axis, a positive local minimum or a negative local

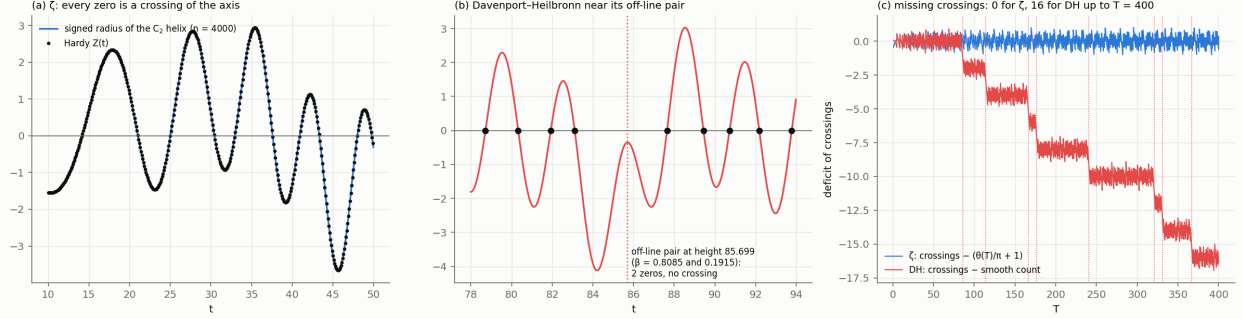


FIGURE 7. The orientation of the C_2 helix. (a) For ζ , the signed radius of the helix computed from the partial sums ($n = 4000$, line) and Hardy's $Z(t)$ (dots). (b) The same for the Davenport–Heilbronn function near its off-line pair at height 85.699; dots mark the crossings. (c) Number of crossings up to T minus the smooth count, for ζ ($\theta(T)/\pi + 1$) and for the Davenport–Heilbronn function; dotted lines at the heights of its off-line zeros.

maximum. We scanned Hardy's function $Z(t)$ with the Riemann–Siegel formula (checked against Euler–Maclaurin to 10^{-6}) on $10 < t < 10^5$ with step 0.004: there are 138069 sign changes, against $\theta(10^5)/\pi + 1 = 138068.6$, and none of its 138069 extrema is without crossing. The lowest extremum is $Z = -5.5 \cdot 10^{-4}$ at $t = 71732.909$, between the zeros 71732.9015 and 71732.9159, a Lehmer pair; the next ones are at 78974.81, 66678.09 and 63137.22, and the classical pairs at 7005.06 and 17143.79 appear among the lowest twelve. For f the scan of Z_f on $t < 400$ finds exactly eight extrema without crossing, each within 0.025 of the ordinate of one of its eight off-line pairs, and the height missing to reach the axis follows $|Z_f| \approx 2.81 \delta^2 \cdot \frac{1}{2} \log(\gamma/2\pi)$, $\delta = \beta - \frac{1}{2}$, within -5% and $+13\%$ for all eight (Figure 8). Near a pair, $Z \approx A((t - \gamma)^2 + \varepsilon)$ with $\varepsilon = \delta^2$ for an off-line pair and $\varepsilon = -g^2/4$ for two zeros on the line at distance g ; RH is the statement that $\varepsilon \leq 0$ for every pair, and the Lehmer pairs show that the margin is arbitrarily small. This is the picture of the de Bruijn–Newman flow, in which such pairs collide under the backward heat flow [RT, Pm].

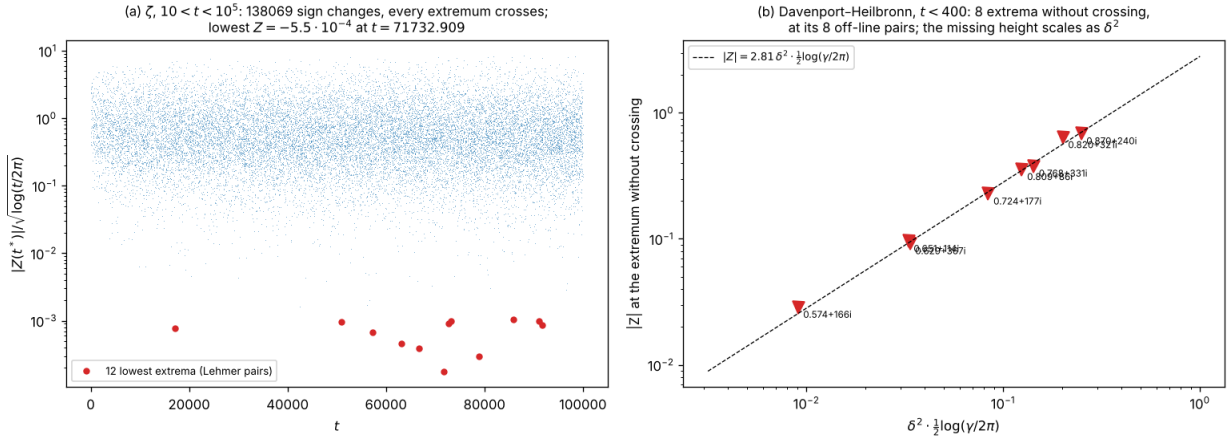


FIGURE 8. Extrema of Hardy's function. (a) $|Z(t^*)|/\sqrt{\log(t^*/2\pi)}$ at the extrema t^* for $10 < t < 10^5$ (a random subsample of 20000); dots: the twelve lowest, at Lehmer pairs. (b) The eight extrema without crossing of Z_f for $t < 400$, at the eight off-line pairs of f , against $\delta^2 \cdot \frac{1}{2} \log(\gamma/2\pi)$.

2.3. The spiral of a zero. The truncated function $F_n(s) = X_{n-1}(s) - n^{1-s}/(1-s)$ has its own zeros, and by (2.1) they converge to those of ζ . The way they converge carries the real part of the limit.

Proposition 2.3. Let $\rho = \beta + i\gamma$ be a simple zero of ζ with $0 < \beta < 1$. For n large, F_n has a unique zero ρ_n near ρ , and

$$\rho_n = \rho + \frac{n^{-\rho}}{2\zeta'(\rho)} + O_\rho(n^{-2\beta} + |\rho|n^{-\beta-1}).$$

Thus ρ_n moves along a logarithmic spiral around ρ , with angle $\pi - \gamma \log n - \arg \zeta'(\rho)$ and radius $n^{-\beta}/(2|\zeta'(\rho)|)$; the rate of convergence is $n^{-\beta}$.

Proof. By Euler–Maclaurin, $F_n(s) = \zeta(s) - \frac{1}{2}n^{-s} + \frac{s}{12}n^{-s-1} + O(|s|^3n^{-\sigma-3})$, uniformly near ρ . Since $\zeta(\rho) = 0 \neq \zeta'(\rho)$, Rouché’s theorem gives a unique zero ρ_n in a small disc around ρ , and one Newton step from ρ gives $\rho_n - \rho = -F_n(\rho)/\zeta'(\rho) + O(|F_n(\rho)|^2) = \frac{1}{2}n^{-\rho}/\zeta'(\rho) + O_\rho(n^{-2\beta} + |\rho|n^{-\beta-1})$. $\square$

A zero on the critical line is its own reflection, and its spiral closes at the rate $n^{-1/2}$, circling the line. An off-line pair $\rho, 1 - \bar{\rho}$ gives two spirals with the different rates $n^{-\beta}$ and $n^{-(1-\beta)}$; their product is n^{-1} whether or not RH holds, so the functional equation imposes no constraint here. The proposition is another detector: it measures β for a zero that has been located, but does not force $\beta = \frac{1}{2}$. It complements the helix of Theorem A: at a point s the helix has radius $|\zeta(s)|n^{-\sigma}$, at a zero this radius vanishes, and what remains turning with the same angle $t \log n$ is the spiral of the approximate zero.

We followed ρ_n by Newton’s method for $100 \leq n \leq 2 \cdot 10^5$ (Figure 9). For ρ_1 the distance $|\rho_n - \rho_1|$ decays like $n^{-0.495}$ and equals $1.4 \cdot 10^{-3}$ at $n = 2 \cdot 10^5$, as predicted by $\frac{1}{2}n^{-1/2}/|\zeta'(\rho_1)|$; for ρ_2 the exponent is -0.485 . For the Davenport–Heilbronn function, with F_n replaced by its partial sums (there is no pole term), the same perturbation argument applies. Its on-line zero at $\gamma = 43.08$ converges like $n^{-0.486}$, the off-line zero $0.8085 + 85.699i$ like $n^{-0.797}$, and its partner $0.1915 + 85.699i$ like $n^{-0.175}$, against the predicted exponents -0.5 , -0.8085 and -0.1915 . At $n = 2 \cdot 10^5$ the right zero is within $3 \cdot 10^{-5}$ of its limit and the left one still at distance $1.2 \cdot 10^{-2}$.

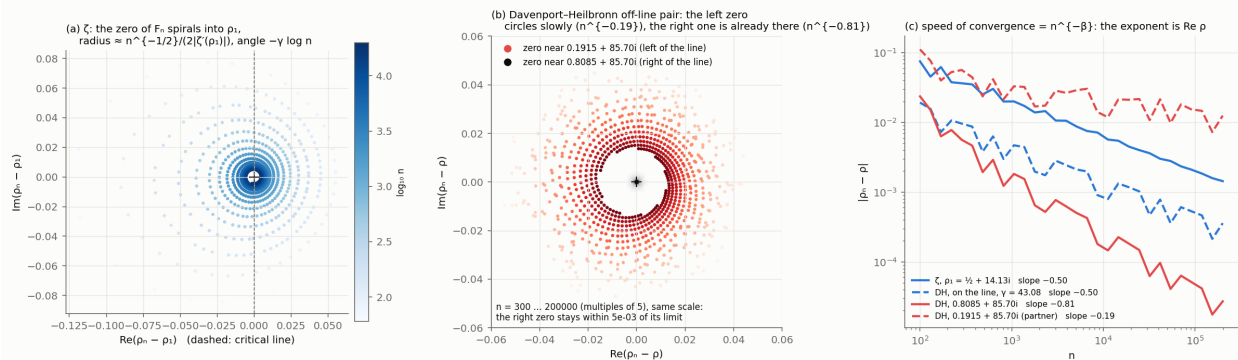


FIGURE 9. The spiral of a zero (Proposition 2.3). (a) The zero ρ_n of F_n near ρ_1 , as $\rho_n - \rho_1$, for $60 \leq n \leq 20000$, coloured by $\log_{10} n$. (b) The two zeros of the Davenport–Heilbronn partial sums near its off-line pair at height 85.699, on the same scale ($300 \leq n \leq 2 \cdot 10^5$, multiples of 5): the zero left of the critical line circles slowly, the zero right of it stays within $5 \cdot 10^{-3}$ of its limit. (c) $|\rho_n - \rho|$ against n ; the slopes are $-\text{Re } \rho$.

Crossings of the spiral. The spiral of Proposition 2.3 can be read without knowing its centre. If $\beta = \frac{1}{2}$, then $\text{Re } \rho_n - \frac{1}{2} = -\cos(\gamma \log n + \arg \zeta'(\rho)) n^{-1/2}/(2|\zeta'(\rho)|) + O_\rho(n^{-1})$ changes sign twice per turn, so for $N \leq n \leq 2N$ the trajectory crosses the critical line $\gamma \log 2/\pi + O(1)$ times.

If $\beta \neq \frac{1}{2}$, it stops crossing once the radius $n^{-\beta}/(2|\zeta'(\rho)|)$ is smaller than $|\beta - \frac{1}{2}|$, that is, for $n > n_0 = (2|\zeta'(\rho)| |\beta - \frac{1}{2}|)^{-1/\beta}$. For the first 18 zeros of ζ and $10^4 \leq n \leq 2 \cdot 10^4$ the number of crossings differs from $\gamma \log 2/\pi$ by at most 1 (3 against 3.12 at ρ_1 , 16 against 15.90 at $\gamma = 72.07$). For the Davenport–Heilbronn function, with n a multiple of 5, the on-line zero at $\gamma = 80.33$ gives 18 crossings against 17.7 for $N = 500, 2000$ and 8000, and the two zeros of the off-line pair at height 85.699 give none (Figure 10). Summed over the zeros up to height T , the count is a version of Turing’s method in which each zero carries the weight γ , and an off-line pair removes about $2\gamma \log 2/\pi$ crossings; like Turing’s method it verifies RH up to the computed height and no further. The threshold $\log n_0 \approx 2 \log (1/(2|\zeta'(\rho)| |\beta - \frac{1}{2}|))$ is much shorter than the detection length of the prime signal (Section 5), because F_n evaluates ζ itself. For a close off-line pair $|\zeta'(\rho)|$ is of order $|\beta - \frac{1}{2}|$, and n_0 grows like $|\beta - \frac{1}{2}|^{-4}$.

The spiral with explicit constants. Proposition 2.3 uses the truncation F_n , whose error $-\frac{1}{2}n^{-s}$ fixes the rate $n^{-\beta}$. Restoring the endpoint term gives the function $D_N(s) = X_N(s) - \frac{1}{2}N^{-s} - N^{1-s}/(1-s)$ of (A.4), whose zeros converge faster and with a leading term that can be controlled explicitly. Put $T_k(s, N) = \frac{B_{2k}}{(2k)!} s(s+1) \cdots (s+2k-2) N^{-s-2k+1}$. Euler–Maclaurin summation with Backlund’s bound for the remainder [Ed, §6.4] gives, for $\sigma > -2K-1$,

$$\zeta(s) = D_N(s) + \sum_{k=1}^K T_k(s, N) + R_K, \quad |R_K| \leq \frac{|s+2K+1|}{\sigma+2K+1} |T_{K+1}(s, N)|,$$

so that $\zeta - D_N = T_1(1+\eta)$ with $T_1 = \frac{s}{12}N^{-s-1}$ and

$$|\eta| \leq \bar{\eta}_N(s) = \frac{|(s+1)(s+2)|}{60N^2} + \frac{|s+5|}{\sigma+5} \cdot \frac{|(s+1)(s+2)(s+3)(s+4)|}{2520N^4}.$$

Proposition 2.3' (explicit spiral). Let $\rho = \beta + i\gamma$ be a simple zero of ζ , $d = \zeta'(\rho)$, $A_N = |\rho|N^{-\beta-1}/12$, $r_N = 3A_N/|d|$ and $M_2 = \max_{|s-\rho| \leq r_N} |\zeta''(s)|$. Assume (H1) $M_2 r_N \leq |d|/2$, (H2) $\bar{\eta}_N \leq \frac{1}{4}$ on $|s-\rho| \leq r_N$, and (H3) $(1+r_N/|\rho|)N^{r_N} \leq \frac{3}{2}$. Then D_N has exactly one zero ρ_N in $|s-\rho| < r_N$, and

$$\rho_N = \rho + \frac{\rho N^{-\rho-1}}{12 \zeta'(\rho)} (1 + \theta_N), \quad |\theta_N| \leq \left(1 + \frac{r_N}{|\rho|}\right) N^{r_N} (1 + \bar{\eta}_N) - 1 + \frac{9M_2 A_N}{2|d|^2}.$$

The three hypotheses are monotone in N : once they hold, they hold for all larger N .

Proof. On $|s-\rho| = r_N$, Taylor’s formula and (H1) give $|\zeta(s)| \geq |d|r_N - M_2 r_N^2/2 \geq \frac{3}{4}|d|r_N = \frac{9}{4}A_N$, while (H2) and (H3) give $|\zeta(s) - D_N(s)| \leq |T_1(s)|(1+\bar{\eta}_N) \leq A_N(1+r_N/|\rho|)N^{r_N} \cdot \frac{5}{4} \leq \frac{15}{8}A_N$. By Rouché’s theorem D_N and ζ have the same number of zeros in the disc, and ζ has exactly one, since $|\zeta(s) - d(s-\rho)| \leq M_2 r_N^2/2 < |d|r_N$ on the circle. Writing $\varepsilon = \rho_N - \rho$, the equation $\zeta(\rho_N) = T_1(\rho_N)(1+\eta(\rho_N))$ becomes $d\varepsilon + q = T_1(\rho)(1+\varepsilon/\rho)N^{-\varepsilon}(1+\eta)$ with $|q| \leq M_2|\varepsilon|^2/2$, and $|\varepsilon| < r_N$ gives the bound for θ_N . As N grows, A_N , r_N , $\bar{\eta}_N$ and the disc decrease. $\square$

For the first 100 zeros, with M_2 sampled on the circle with a margin of 5% (not interval arithmetic), (H1)–(H3) hold from $N_0 = 0.36|\rho|$ to $0.64|\rho|$ on, and for $N_0 \leq N \leq \max(4N_0, 3000)$ the measured $|\theta_N|$ is at most 0.32 times the bound. The normalized displacement $(\rho_N - \rho)N^{3/2}$ runs on a circle of radius $|\rho|/(12|\zeta'(\rho)|)$, measured 1.4856 against 1.4860 at ρ_1 , and $|\rho_N - \rho|$ decays with slopes between -1.502 and -1.508 for the seven zeros with $\gamma < 42$ (Figure 11). In the space $(\sigma, t, \log N)$ the zeros of D_N thus form helices around the vertical lines $(\frac{1}{2}, \gamma)$, while the zeros of X_N alone, without the pole and endpoint corrections, wander over the strip.

An off-line pair would give two helices with the rates $N^{-(\beta+1)}$ and $N^{-(2-\beta)}$, whose product is N^{-3} whatever β is; as for Proposition 2.3 the functional equation imposes no constraint, and the proposition measures β without forcing it. For the Davenport–Heilbronn function we use the same

correction on each Hurwitz piece,

$$F_M(s) = \sum_{b=1}^4 a(b)5^{-s} \left[\sum_{k=0}^M \left(k + \frac{b}{5}\right)^{-s} - \frac{1}{2} \left(M + \frac{b}{5}\right)^{-s} - \frac{(M + b/5)^{1-s}}{1-s} \right],$$

where $a(b)$ are the periodic coefficients of f . Since $\sum_b a(b) = 0$ the terms of order M^{-s-1} cancel and $\rho_M - \rho \approx -5^{-\rho} \rho(\rho+1) \left(\sum_b a(b) \frac{b}{5}\right) M^{-\rho-2} / (12f'(\rho))$, with rate $M^{-(\beta+2)}$. The measured slopes are -2.8065 and -2.1900 for the pair $0.8085, 0.1915 + 85.699i$ (predicted $-2.8085, -2.1915$), -2.6490 and -2.3477 for the pair at height 114.163 (predicted $-2.6508, -2.3492$), and -2.4982 for the on-line zero at 83.109 ; at $M = 4000$ the modulus of the displacement agrees with the predicted one to $3 \cdot 10^{-4}$. The two zeros of a pair converge at rates that differ by $2(\beta - \frac{1}{2})$, measured 0.6165 against 0.617 .

2.4. A detector of zeros on the critical line. By Corollary 2.1, at a zero $\frac{1}{2} + i\gamma$ the quantity $D_n(t) = |X_n(\frac{1}{2} + it)|^2 - (n + \frac{1}{2})/(\frac{1}{4} + t^2)$ is $O(1/n)$ for every n , whereas at other points (2.1) shows that it oscillates in n with amplitude about $2|\zeta(\frac{1}{2} + it)|\sqrt{n/(\frac{1}{4} + t^2)}$. A single value of n is not enough: the equation $D_n(t) = 0$ has many spurious solutions, because $D_n(t)$ oscillates in t with period about $2\pi/\log n$. Requiring smallness for several n removes them. With $\max_{n \in \{2000, 2500, 3000, 3500, 4000\}} |D_n(t)|$ on a grid of step 0.002 in $14 \leq t \leq 31$, the local minima below 10^{-2} occur exactly at $14.134, 21.022, 25.010$ and 30.424 , with values between $2 \cdot 10^{-4}$ and $5 \cdot 10^{-3}$ (limited by the grid), while the median elsewhere is 7.7 . The detector is valid but not efficient: each evaluation costs $O(n)$ operations, against $O(\sqrt{t})$ for the Riemann–Siegel formula, which is itself a truncated C_2 sum [C1]. It finds zeros on the critical line only, and each zero is detected by its own identity; a known zero gives no information about the next one [Er].

The same mechanism is visible in C_2^2 . Along a zero, $C_2(n, \rho)^2$ decreases like n^{-2} , vanishes at the crossing $n^* \approx \gamma^2 + \frac{1}{3}$ of Proposition 2.2 and increases monotonically to $(\gamma^2 + \frac{1}{4})^{-2}$; along the critical line at fixed n , by (2.1), $C_2(n, \frac{1}{2} + it)^2$ oscillates under the envelope $((\frac{1}{4} + t^2)^{-1} + 2|\zeta(\frac{1}{2} + it)|n^{-1/2})^2$, which drops to $(\gamma^2 + \frac{1}{4})^{-2}$ exactly at the zeros (Figure 12). The detector is confined to the critical line: off it, $C_2'/C_2 \rightarrow 1 - 2\sigma$ at every point (Section 2, the last sign change), so $C_2(\cdot, s)$ is eventually increasing at every point with $\sigma < \frac{1}{2}$ and eventually decreasing at every point with $\sigma > \frac{1}{2}$, zero or not, and monotonicity alone cannot distinguish an off-line zero from its neighbours. What distinguishes it is the absence of the oscillating term, that is $\zeta(s) = 0$. Status: follows from (2.1) and Corollary 2.1; the figure is numerical.

2.5. The circle of a prime. The construction of [E24] placed each zero $\frac{1}{2} + i\gamma$ on the circle $|w| = e^{-1/2}$, $w = e^{-s}$, at the angle $-\gamma \bmod 2\pi$, which is the circle of the Euler factor $(1 - e^{-s})^{-1}$ of a “prime” of size e . By Hlawka’s theorem the angles $\gamma \bmod 2\pi$ are uniformly distributed [HI], so the circle carries no arithmetic information [Er]. With a true prime the situation changes. Landau’s formula [Lan] states that for fixed $x > 1$,

$$\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O_x(\log T). \quad (2.3)$$

Under RH, $x^\rho = \sqrt{x} e^{i\gamma \log x}$, and (2.3) says that the angles $\gamma \log p \bmod 2\pi$ of the zeros on the circle $|p^{-s}| = p^{-1/2}$ are biased toward π , with mean cosine about $-\Lambda(p)/(\sqrt{p} \log \frac{T}{2\pi e})$, while for bases that are not prime powers there is no bias. Results on the distribution of $\gamma \alpha \bmod 1$ in this direction are due to Ford and Zaharescu [FZ].

For the 202 zeros with $0 < \gamma < 400$ the mean of $\cos(\gamma \log b)$ is:

base b	2	3	5	7	e	6	10
measured	-0.147	-0.194	-0.226	-0.222	-0.003	+0.014	+0.009
from (2.3)	-0.155	-0.200	-0.227	-0.232	0	0	0

The full spectrum $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$, $2 \leq n \leq 40$, has spikes exactly at the prime powers, all negative and within a few percent of $-(T/2\pi)\Lambda(n)/\sqrt{n}$ (Figure 13). For the Davenport–Heilbronn function the same sum over its zeros, $\sum \operatorname{Re} n^{\rho-1/2}$, reproduces $-(T/2\pi)b(n)/\sqrt{n}$, where $b(n)$ are the Dirichlet coefficients of $-f'/f$; these take both signs and are not supported on prime powers (for example +46.6 at $n = 4$, -50.7 at $n = 6$, +49.1 at $n = 9$). The one-signed spectrum is the Euler product seen from the zeros.

The arithmetic information lies in the exact positions of the zeros, not in their local statistics. Displacing each zero independently by a Gaussian amount of standard deviation ε times the local mean spacing keeps a fraction 0.96, 0.89, 0.73 and 0.32 of the prime spikes ($p \leq 13$) for $\varepsilon = 0.02, 0.1, 0.2$ and 0.5 . Points with the GUE statistics and the same density keep 0.04 ± 0.06 .

Twisted circles. The same holds for Dirichlet L -functions, with the character visible in the angle. Let χ be the character mod 5 with $\chi(2) = i$. Its L -function has 303 zeros with $0 < \gamma < 400$ and 303 with $-400 < \gamma < 0$, all on the critical line, against 303.2 for each sign predicted by the Riemann–von Mangoldt formula. Fitting $\sum_{|\gamma| < 400} n^{i\gamma}$, $2 \leq n \leq 40$, by $c\Lambda(n)\chi(n)/\sqrt{n}$ gives $c = -126.6$, against $-T/\pi = -127.3$, with relative residual 3%; at integers that are not prime powers the sum is at most 7.9 in modulus, against a median of 76 at prime powers. On the circle of p the zeros therefore lean toward the angle $\pi + \arg \chi(p)$. The measured mean directions are $-0.496\pi, 0.504\pi, -0.503\pi$ and -1.000π for $p = 2, 3, 7, 11$, against $-\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}, \pi$.

Remark 2.4 (the angles do not see β). The angle of a zero $\rho = \beta + i\gamma$ on the circle of p is $\gamma \log p$, which does not depend on β ; an off-line zero differs from an on-line zero at the same height only through the modulus $x^{\beta-1/2}$ of its term in (2.3). The joint distribution of the angles $(\gamma \log p)_p$ has Fourier coefficients $\sum_{\gamma} (m/n)^{i\gamma}$, which Landau’s formula determines unconditionally. Statistics of the angles alone are therefore consistent with off-line zeros. The quadratic version of the question, how small a Dirichlet polynomial $\sum a_n n^{i\gamma}$ can be at all zeros simultaneously, is Weil’s form in a basis of Dirichlet polynomials (Section 5).

Zeros of the sections. Remark 2.4 has a counterpart for the finite sums $S_N(s) = \sum_{k \leq N} k^{-s}$. With $x_p = p^{-s}$ for $p \leq N$, S_N becomes a polynomial P_N in $\pi(N)$ variables, and since the numbers $\log p$ are linearly independent, Kronecker’s theorem makes $(p^{-it})_{p \leq N}$ dense in the torus $\mathbb{T}^{\pi(N)}$. Hence the closure of the set of real parts of the zeros of S_N is the set A_N of a for which $P_N(p^{-a}w_p) = 0$ for some $w \in \mathbb{T}^{\pi(N)}$. A zero is a closed polygon with sides of lengths k^{-a} in which only the directions of the prime sides are free. For $N = 2$ the zeros are the points $i(2m+1)\pi/\log 2$. For $N = 3$ the condition is that $1, 2^{-a}$ and 3^{-a} form a triangle, and $A_3 = [-1, 0.7879]$, the right end being the root of $2^{-a} + 3^{-a} = 1$. Let $e(N) = \sup A_N$ and $m_N(a) = \min_w |P_N(p^{-a}w_p)|$.

Proposition 2.5 (the prime step). If p is prime, then $e(p) = \sup\{a \geq 0 : m_{p-1}(a) \leq p^{-a}\}$; in particular $e(p) \geq e(p-1)$.

Proof. Let $a \geq 0$. The image of the torus under $w \mapsto P_{p-1}(q^{-a}w_q)$ is connected, and at $w = 1$ it takes the value $\sum_{k < p} k^{-a} \geq 1 \geq p^{-a}$. If $m_{p-1}(a) \leq p^{-a}$, it contains a point z with $|z| = p^{-a}$, and the free coordinate w_p with $p^{-a}w_p = -z$ gives a zero of P_p . If $m_{p-1}(a) > p^{-a}$, then $|P_{p-1}| > |p^{-a}w_p|$ on the torus and P_p has no zero. Since $m_{p-1} = 0$ on A_{p-1} , $e(p) \geq e(p-1)$. $\square$

For a composite n the direction of n^{-s} is fixed by those of its prime factors, and no such statement holds. We computed $e(N)$ for $3 \leq N \leq 60$ by minimizing $|P_N|$ on the torus; each value is a lower bound, since the zero found is genuine. In all 15 cases where N is prime, $e(N) > e(N-1)$, as Proposition 2.5 requires. In all 8 cases where N is a prime power (4, 8, 9, 16, 25, 27, 32, 49), $e(N) < e(N-1)$; for the other composites e rose 13 times and fell 21 times (Figure 14). The edge exceeds 1

for the first time at $N = 19$ (1.0070), falls below 1 again at $N = 20, 21$ and 28, and lies between 1.01 and 1.03 for $29 \leq N \leq 60$. Turán showed that $S_N(s) \neq 0$ for $\sigma > 1$ and all large N would imply RH [Tu]; Montgomery showed that S_N has zeros with $\sigma > 1 + (\frac{4}{\pi} - 1 - \varepsilon) \log \log N / \log N$ [Mo83], and Montgomery and Vaughan gave the matching bound from the other side [MV]. The configuration in which all primes have direction π , which turns P_N into the Liouville sum $\sum_{n \leq N} \lambda(n) n^{-a}$ of Turán's problem, is not extremal: its largest root is 0.901 at $N = 19$, against $e(19) = 1.007$.

The dimension of the problem can be reduced in general. Two primes larger than $\sqrt{N}$ have product larger than N , so every $n \leq N$ contains at most one of them, to the first power.

Proposition 2.6 (polygon reduction). For $N \geq 2$ and $a \geq 0$ write

$$P_N = R_N(w') + \sum_{\sqrt{N} < p \leq N} w_p p^{-a} C_p(w'),$$

where $w' = (w_p)_{p \leq \sqrt{N}}$, R_N collects the terms $n^{-a} w^{\alpha(n)}$ with n free of primes larger than $\sqrt{N}$, and $C_p = \sum m^{-a} w^{\alpha(m)}$ over the $m \leq N/p$ free of such primes (so $C_p = 1$ for $p > N/2$). Then $a \in A_N$ if and only if there is $w' \in \mathbb{T}^{\pi(\sqrt{N})}$ for which the largest of the numbers $|R_N(w')|$ and $p^{-a} |C_p(w')|$, $\sqrt{N} < p \leq N$, is at most the sum of the others.

Proof. The coordinates w_p with $p > \sqrt{N}$ appear in no R_N and no C_q , and each appears in exactly one term. For fixed w' , the sums $\sum_p w_p \ell_p$ with free $|w_p| = 1$ and lengths $\ell_p = p^{-a} |C_p(w')|$ fill the annulus $\max(0, 2 \max_p \ell_p - \sum_p \ell_p) \leq |z| \leq \sum_p \ell_p$. Hence P_N vanishes for some choice of these coordinates if and only if $|R_N(w')|$ lies in that annulus, which is the polygon condition. $\square$

By Bertrand's postulate there is always a prime in $(N/2, N]$, a side of length p^{-a} independent of w' ; Proposition 2.5 is the case in which this is the only change from $N - 1$ to N . For $4 \leq N \leq 8$ the problem has one variable, w_2 , and for $9 \leq N \leq 24$ two. For $N = 4$ the sides are $|1 + z + z^2|$, $z = 2^{-a} w_2$, and 3^{-a} ; the minimum of $|1 + z + z^2|$ on $|w_2| = 1$ is $\frac{\sqrt{3}}{2}(1 - 4^{-a})$ when $2^{-a} \geq 2 - \sqrt{3}$, so $e(4) = 0.6263$ is the root of $3^{-a} = \frac{\sqrt{3}}{2}(1 - 4^{-a})$, and $e(4) < e(3)$ reduces to $2^{-e(3)} > 2/\sqrt{3} - 1$, which holds.

Proposition 2.7. $e(2^k) < e(2^k - 1)$ for $k = 2, 3, 4$. More precisely $e(8) < 0.93 < 0.97 \leq e(7)$ and $e(16) < 0.975 < 0.99 \leq e(15)$.

Proof. $k = 2$ is the computation above. For $N = 7$ at $a = 0.97$ and $N = 15$ at $a = 0.99$ we exhibit points w' at which the polygon of Proposition 2.6 closes. For $N = 8$ on $0.93 \leq a \leq 3$ (variables θ_2, a) and $N = 16$ on $0.975 \leq a \leq 3$ (variables θ_2, θ_3, a) we cover the domain by boxes, evaluate the polygon defect $2 \max - \sum$ at the centres and discard a box when the value exceeds an explicit Lipschitz bound times its half-widths, subdividing the others; all boxes are discarded. For $a \geq 3$, $|R_N| \geq 1 - \sum_{n > 1} n^{-a}$ exceeds the sum of the other sides (0.857 against 0.053 for $N = 8$, 0.814 against 0.014 for $N = 16$). $\square$

The computation is in double precision with explicit error bounds; a version in interval arithmetic would make it a computer-assisted proof in the strict sense. With the reduction we extended the computation of $e(N)$ to $N \leq 128$. The powers 64, 81, 125 and 128 again lower the edge, from 1.0320, 1.0328, 1.0392, 1.0383 to 1.0277, 1.0300, 1.0380, 1.0366, but $121 = 11^2$ raises it slightly, from 1.03714 to 1.03720. At $a = 1.03719$ the polygon for $N = 121$ closes (defect $-2 \cdot 10^{-5}$), while 1600 independent minimizations for $N = 120$ return the same positive defect $1.8 \cdot 10^{-4}$; this last statement is numerical. The regularity of prime powers is therefore not a general law.

The powers of 2 behave differently because of a mechanism that can be read off the extremal configurations. In P_{2^k-1} the powers of 2 form the geometric sum $1 + z + \dots + z^{k-1} = (1 - z^k)/(1 - z)$, and at the edge the direction of w_2 sits near a k -th root of unity, where this sum nearly cancels: $k\theta_2 \bmod 2\pi$ is 0.008π , 0.067π , 0.128π , 0.227π for $k = 3, \dots, 6$. The new term z^k completes the geometric sum and undoes the cancellation. In the form of Proposition 2.6 the edge of $N - 1$ is the point where the minimum of the polygon defect reaches 0. If this minimum is attained at a

single point w^* , where the side R_{N-1} is the largest, adding $\varepsilon q^{-a} w_p^k$, $q = p^k$, changes the defect by $\varepsilon q^{-a} \operatorname{Re}(\overline{R_{N-1}(w^*)} w_p^{*k}) / |R_{N-1}(w^*)| + O(\varepsilon^2)$, so the edge moves to the left when this projection is positive. For $q = 2^k$ the normalized projection is 0.93, 0.92, 0.94, 0.99 for $k = 3, \dots, 6$. For 11^2 the powers of 11 give only $1 + z_{11}$, the projection is 0.10, and the edge moves to a second local minimum. We expect $e(2^k) < e(2^k - 1)$ for every $k \geq 2$; it holds for $k \leq 4$ by Proposition 2.7 and numerically for $k \leq 7$, and the powers 9, 27, 81 of 3 behave in the same way.

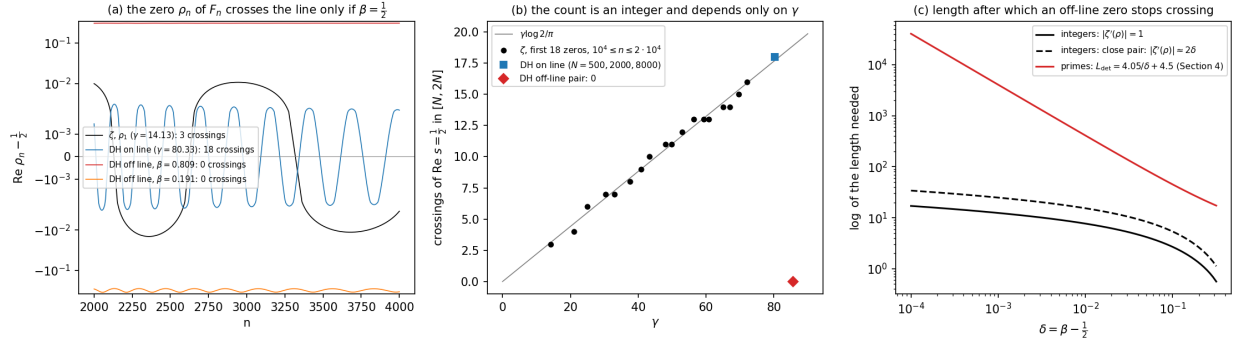


FIGURE 10. Crossings of the spiral. (a) $\text{Re } \rho_n - \frac{1}{2}$ for $2000 \leq n \leq 4000$: the first zero of ζ , an on-line zero of the Davenport–Heilbronn function, and the two zeros of its off-line pair at height 85.699. (b) Crossings of the critical line for $10^4 \leq n \leq 2 \cdot 10^4$ against $\gamma \log 2/\pi$. (c) The length $\log n_0$ after which an off-line zero at distance δ from the line stops crossing, for $|\zeta'(\rho)| = 1$ and for a close pair, against the detection length of the prime signal.

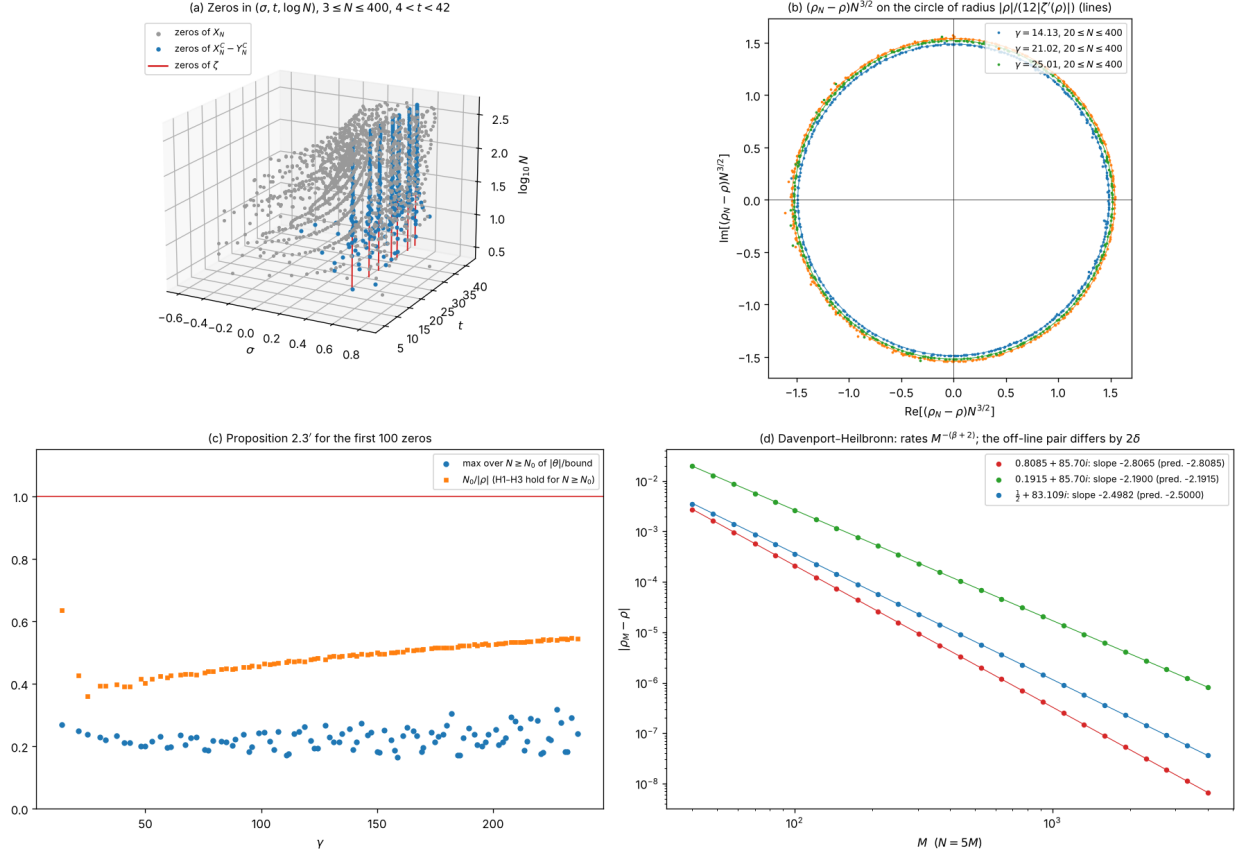


FIGURE 11. The explicit spiral (Proposition 2.3'). (a) Zeros of X_N (grey) and of $D_N = X_N - \frac{1}{2}N^{-s} - N^{1-s}/(1-s)$ (blue) in $(\sigma, t, \log_{10} N)$; vertical lines at the zeros of ζ . (b) $(\rho_N - \rho)N^{3/2}$ for the first three zeros and the circles of radius $|\rho|/(12|\zeta'(\rho)|)$. (c) For the first 100 zeros, the measured error divided by the bound of Proposition 2.3', and the ratio $N_0/|\rho|$ from which the hypotheses hold. (d) Davenport–Heilbronn function: $|\rho_M - \rho|$ for the two zeros of the off-line pair at height 85.699 and for an on-line zero, with the predicted rates $M^{-(\beta+2)}$ (lines).

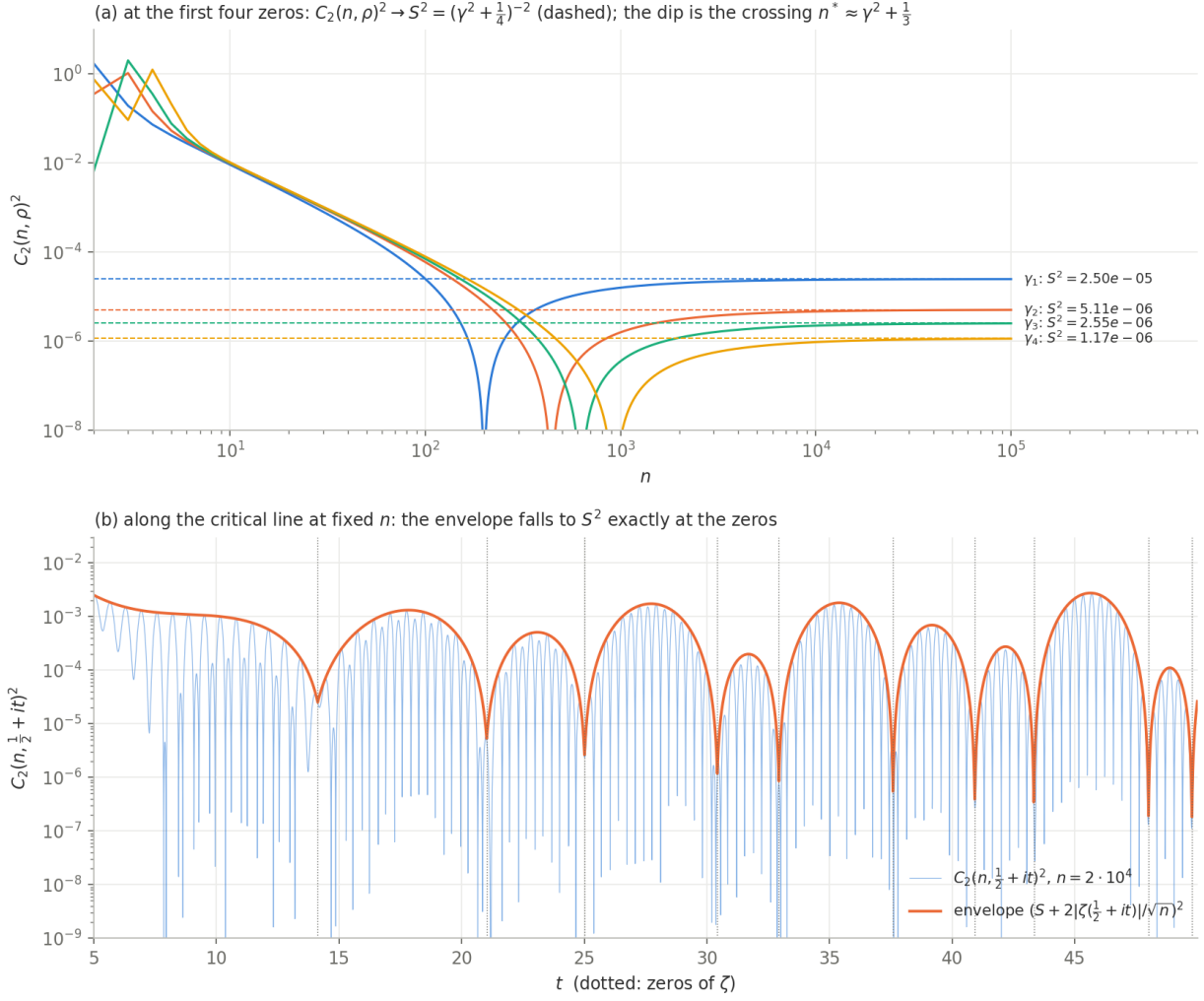


FIGURE 12. (a) $C_2(n, \rho)^2$ at the first four zeros, with the limits $(\gamma^2 + \frac{1}{4})^{-2}$ (dashed).
 (b) $C_2(n, \frac{1}{2} + it)^2$ for $n = 2 \cdot 10^4$ and its envelope $((\frac{1}{4} + t^2)^{-1} + 2|\zeta(\frac{1}{2} + it)|n^{-1/2})^2$;
 dotted: zeros of ζ .

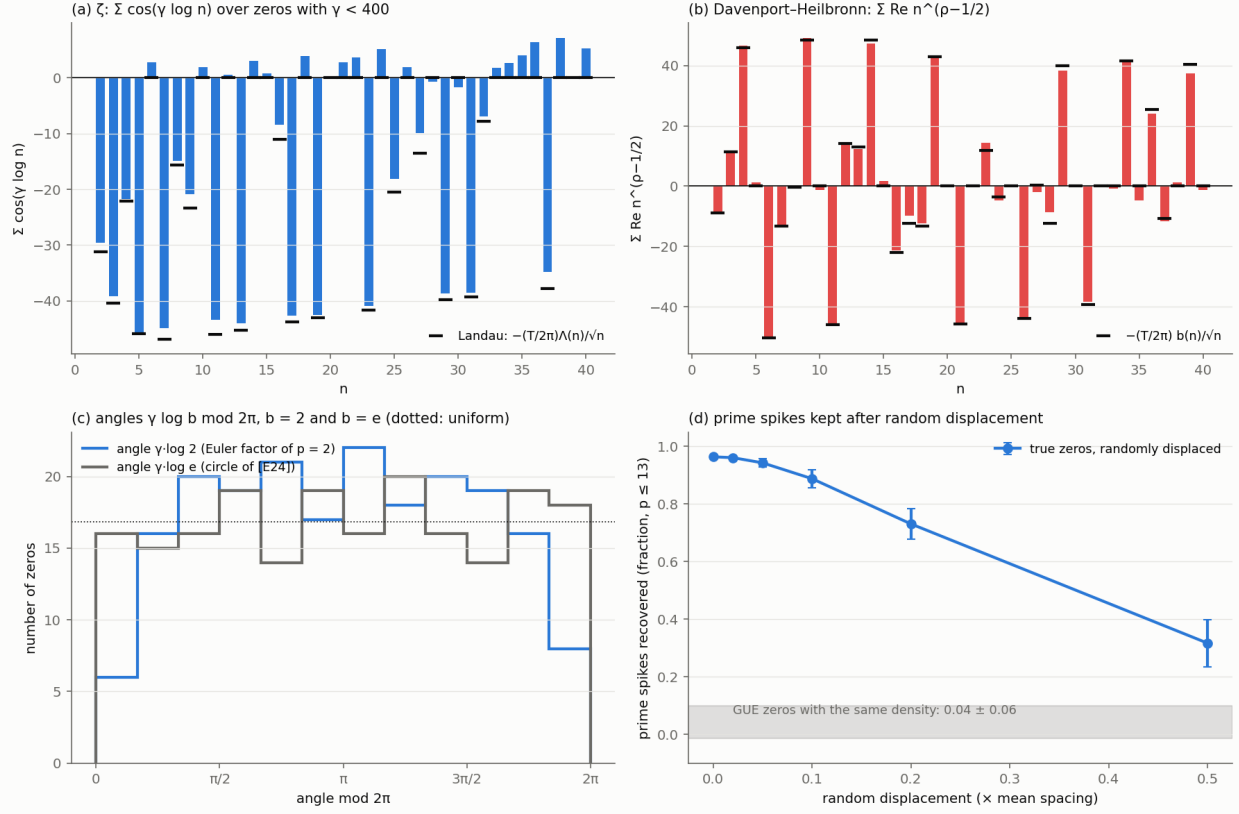


FIGURE 13. (a) $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$ for $2 \leq n \leq 40$ (bars) and Landau's prediction $-(T/2\pi)\Lambda(n)/\sqrt{n}$ (marks). (b) The same sum for the Davenport–Heilbronn function and $-(T/2\pi)b(n)/\sqrt{n}$. (c) Histogram of $\gamma \log b \bmod 2\pi$ for $b = 2$ and for $b = e$, the circle of [E24]. (d) Fraction of the prime spikes ($p \leq 13$) kept after an independent Gaussian displacement of each zero, in units of the mean spacing; shaded band: GUE points of the same density.

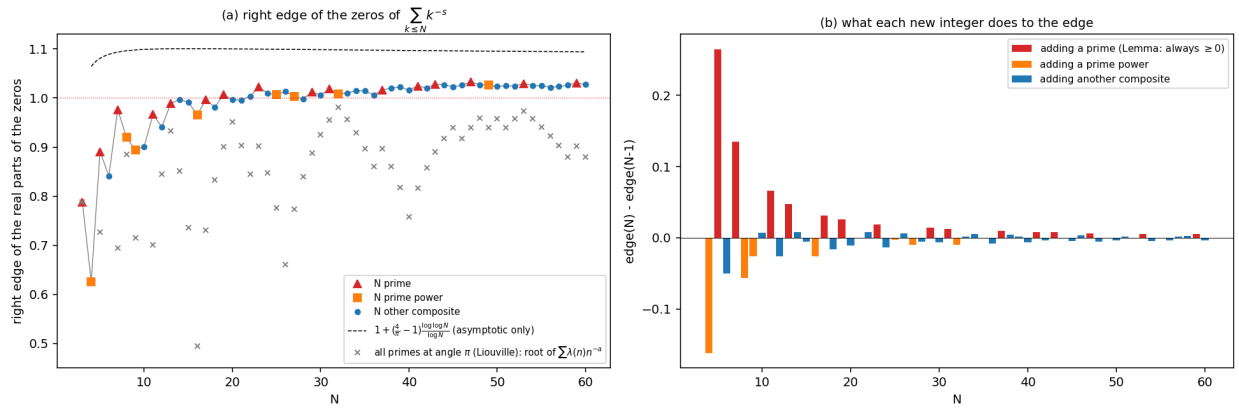


FIGURE 14. Zeros of the sections $\sum_{k \leq N} k^{-s}$. (a) The right edge $e(N)$ of the real parts of the zeros, $3 \leq N \leq 60$, marked by the type of N ; dashed: the asymptotic $1 + (\frac{4}{\pi} - 1) \log \log N / \log N$; crosses: the largest root of the Liouville sum. (b) The change $e(N) - e(N-1)$: positive for every prime (Proposition 2.5), negative for every prime power up to 60.

3. THE C_2 ENERGY LAW OF THE PRIMES

Weil positivity (Section 5) is a sign problem. There is a classical way to make the sign automatic: exhibit the quadratic form as the energy of a real signal. For ζ , the natural signal is the normalized error in the prime number theorem. With $u = \log x$, put

$$E(u) = e^{-u/2} \left(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}) \right).$$

By the explicit formula [In, T], $E(u) = -\sum_{\rho} e^{i\gamma_{\rho}u}/\rho$ whenever all γ_{ρ} are real. The signal is built from the primes alone, and if RH holds its frequencies are the zeros.

Proposition 3.1 (Cramér).

(a) Assume RH and that the zeros are simple. Then the autocorrelation

$$R(a) = \lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)E(u+a) du = \sum_{\rho} \frac{e^{i\gamma_{\rho}a}}{|\rho|^2} = 2 \sum_{\gamma > 0} \frac{\cos \gamma a}{\frac{1}{4} + \gamma^2}$$

exists. It is positive-definite, because it is an autocorrelation. For every test function φ ,

$$\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U |(\varphi * E)(u)|^2 du = \sum_{\rho} \frac{|\hat{\varphi}(\gamma_{\rho})|^2}{\frac{1}{4} + \gamma_{\rho}^2} \geq 0.$$

In particular $R(0) = \sum_{\rho} |\rho|^{-2} = 2 + \gamma_0 - \log 4\pi = C_B$, the constant of the introduction [C4, §8.3].

(b) Conversely, if $\frac{1}{U} \int_0^U E(u)^2 du$ is bounded as $U \rightarrow \infty$, then RH holds.

Proof. (a) Under RH, E is a Besicovitch almost periodic function with Fourier–Bohr coefficients $-1/\rho$ at the frequencies γ_{ρ} (Cramér [Cr]; see also [In, Ch. V]). Parseval’s identity for almost periodic functions gives R and the filtered energy. For simple zeros, $\sum_{\rho} |\rho|^{-2} = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi$.

(b) Suppose the running energy is bounded. By summation by parts, $\int_0^{\infty} E(u)^2 e^{-2\varepsilon u} du < \infty$ for every $\varepsilon > 0$. By Cauchy–Schwarz, the Laplace transform $\int_0^{\infty} E(u) e^{-(s-1/2)u} du$ then converges absolutely for $\operatorname{Re} s > \frac{1}{2}$ and defines an analytic function there. For $\operatorname{Re} s > 1$ it equals

$$\int_1^{\infty} (\psi(x) - x) x^{-s-1} dx + (\text{terms analytic in } \operatorname{Re} s > 0) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} + (\text{analytic}).$$

So ζ'/ζ continues analytically to $\operatorname{Re} s > \frac{1}{2}$, and ζ has no zeros there. $\square$

So a Weil-type positivity, with the weight $1/(\frac{1}{4} + \gamma^2)$, holds *by construction* as soon as the prime signal has finite energy. The sign problem becomes a size problem, and the size problem is exactly RH. The weight $1/(\frac{1}{4} + \gamma^2)$ is the limit of C_2 at the zeros and the harmonic measure seen from the pole [C4, §§2, 4]. So the C_2 weight is precisely what converts Weil’s distribution into the autocorrelation of a real arithmetic signal.

The growth rate of the energy. Proposition 3.1(b) has a quantitative form. Let $\Theta = \sup_{\rho} \operatorname{Re} \rho$, so that $\frac{1}{2} \leq \Theta \leq 1$ and RH is the statement $\Theta = \frac{1}{2}$, and write $F(U) = \int_0^U E(u)^2 du$.

Theorem 3.2. If $\Theta > \frac{1}{2}$, then

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \log F(U) = 2\Theta - 1.$$

If $\Theta = \frac{1}{2}$, then $F(U) \sim C_B U$. In particular RH holds if and only if $F(U) = O(U)$, and if and only if F grows subexponentially.

Proof. The case $\Theta = \frac{1}{2}$ is Proposition 3.1(a). Let $\Theta > \frac{1}{2}$ and let δ_c be the abscissa of convergence of $I(\delta) = \int_0^{\infty} E(u)^2 e^{-2\delta u} du$. Since F is nondecreasing and unbounded, the Laplace–Stieltjes abscissa formula gives $\limsup_U \frac{1}{U} \log F(U) = 2\delta_c$, and it remains to show $\delta_c = \Theta - \frac{1}{2}$. The classical

bound $\psi(x) - x \ll x^\Theta \log^2 x$ [In, Ch. IV] gives $E(u) \ll u^2 e^{(\Theta-1/2)u}$, so $I(\delta) < \infty$ for $\delta > \Theta - \frac{1}{2}$. Conversely, if $I(\delta) < \infty$ for some $\delta < \Theta - \frac{1}{2}$, the Cauchy–Schwarz argument in the proof of Proposition 3.1(b) shows that ζ'/ζ continues analytically to $\operatorname{Re} s > \frac{1}{2} + \delta$. This contradicts the existence of a zero with $\beta > \frac{1}{2} + \delta$. $\square$

In the language of the series, the energy of the prime signal is a sum of C_2 slopes. At a zero ρ on the critical line the partial sums satisfy the straight-line law $|X_n(\rho)|^2 = (n + \frac{1}{2})/|\rho|^2 + O(n^{-1})$ of [C1] (Corollary 2.1), so $1/|\rho|^2 = 1/(\frac{1}{4} + \gamma^2)$ is the slope of the C_2 energy at ρ . Proposition 3.1(a) states that the mean energy of the primes is the sum of these slopes over all zeros, and Theorem 3.2 states that the exponential growth rate of the same energy is $2\Theta - 1$, twice the distance from the critical line of the zero farthest from it.

The same argument applies to the Davenport–Heilbronn signal E_f , whose Laplace transform is $-f'(s)/(sf(s))$ without a pole term: the growth rate of its energy is $2\Theta_f - 1$. Davenport and Heilbronn proved that f has zeros with real part greater than 1 [DH], so $2\Theta_f - 1 > 1$. These zeros lie high in the strip and have small weight $1/|\rho|^2$, so in any computable range the growth of E_f is governed by the low off-line zeros listed in [C5].

The C_2 energy law. Proposition 3.1 and Theorem 3.2 combine with Corollary 2.1 into a single identity.

Theorem B (C_2 energy law). In $[0, \infty]$,

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du = \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho) \quad \text{if all zeros are simple,}$$

and in general both sides are finite or both are infinite. RH holds if and only if they are finite, and then the right side equals $\sum_{\rho} |\rho|^{-2} = C_B$.

Proof. Suppose RH fails, and let ρ be a zero with $\beta > \frac{1}{2}$. Then $\Theta > \frac{1}{2}$, so $\limsup_U \frac{1}{U} \log F(U) > 0$ by Theorem 3.2 and the left side is $+\infty$. The reflected zero $1 - \bar{\rho}$ has real part $1 - \beta < \frac{1}{2}$, so by Corollary 2.1 its term on the right is $+\infty$; all other terms are nonnegative. Suppose RH holds. By Corollary 2.1 the right side is $\sum_{\rho} (\frac{1}{4} + \gamma^2)^{-1} = C_B$, counted with multiplicity. By Cramér’s theorem (Proposition 3.1(a)) the left side is a limit, equal to $\sum_{\rho} \text{distinct } m_{\rho}^2/|\rho|^2$, where m_{ρ} is the multiplicity. This is finite because $m_{\rho} \ll \log |\gamma|$, and it equals C_B when all zeros are simple. $\square$

Thus the mean energy of the primes is the sum over the zeros of the limiting values of C_2 , and RH is the statement that no zero has an infinite C_2 limit. This is the trichotomy of Theorem A read through the primes. The identity is a reformulation, not a new constraint: proving that the right side is finite is RH.

The Nyman–Beurling distance. The same constant appears in an approximation problem in which the positivity comes from a Gram matrix built from the integers alone. Let $\rho_k(x) = \{1/(kx)\}$, χ the indicator of $(0, 1)$, and

$$d_N^2 = \inf_{c_1, \dots, c_N} \int_0^1 \left| \chi(x) - \sum_{k \leq N} c_k \rho_k(x) \right|^2 dx = 1 - b^T G^{-1} b,$$

with $b_k = \langle \chi, \rho_k \rangle = (\log k + 1 - \gamma_0)/k$ and $G_{jk} = \langle \rho_j, \rho_k \rangle$, given in closed form by Vasyunin’s formula [Va]. The following are known, written with Corollary 2.1.

- (i) RH holds if and only if $d_N \rightarrow 0$ [Ny, Be, BD]. If RH fails, d_N is bounded below by a positive constant, so $d_N^2 \log N \rightarrow \infty = \sum_{\rho} \lim_n C_2(n, \rho)$.
- (ii) Unconditionally, $\liminf_N d_N^2 \log N \geq \sum_{\rho: \beta=1/2} m_{\rho}^2 \lim_n C_2(n, \rho)$, where m_{ρ} is the multiplicity [Bu02].
- (iii) It is conjectured [BBLs] that $d_N^2 \log N \rightarrow \sum_{\rho} |\rho|^{-2} = C_B$ under RH with simple zeros; in the notation of Corollary 2.1 this reads $\lim_N d_N^2 \log N = \sum_{\rho} m_{\rho}^2 \lim_n C_2(n, \rho)$, which by (i) also holds when RH fails.

We computed G from Vasyunin's formula, checked against direct integration to 10^{-9} , and d_N^2 in double precision for $N \leq 1200$; the condition number of G is $5 \cdot 10^6$ at $N = 1200$. The values of $d_N^2 \log N$ are 0.0464, 0.0469, 0.0462, 0.0453 at $N = 50, 100, 400, 1000$, and lie in $[0.0449, 0.0464]$ with mean 0.0456 for $300 \leq N \leq 1200$ (Figure 15). The sum of the limits of C_2 over the 341 zeros with $\gamma < 600$ is 0.04324, and with the tail from the zero density 0.046193, against $C_B = 2 + \gamma_0 - \log 4\pi = 0.046191$; the mean square of the prime error up to 10^8 is 0.0466 (Figure 17). The three quantities coincide because each is $\sum_\rho |\rho|^{-2}$, by Parseval applied to the explicit formula, so the coincidence is not a new constraint. It shows, however, that $\lim_n C_2(n, \rho)$ is the contribution of ρ to the Nyman–Beurling distance, in a problem where the positivity is that of a Gram matrix on the integers and the open part, $d_N \rightarrow 0$, is an upper bound. Proving it requires good approximating coefficients c_k ; the natural ones, of Möbius type, lead back to the behaviour of $1/\zeta$ on the critical line.

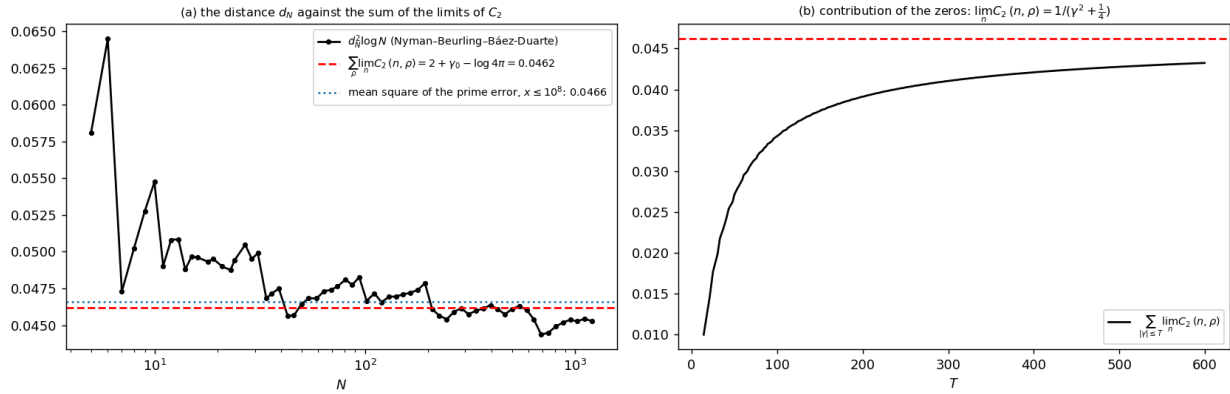


FIGURE 15. The Nyman–Beurling distance and the limits of C_2 . (a) $d_N^2 \log N$ for $5 \leq N \leq 1200$; dashed: $\sum_\rho \lim_n C_2(n, \rho) = C_B$; dotted: mean square of the prime error for $x \leq 10^8$. (b) $\sum_{|\gamma| \leq T} \lim_n C_2(n, \rho)$ over the zeros with $\gamma < 600$.

The optimal coefficients. The minimizing coefficients $c = G^{-1}b$ have a simple structure (Figure 16a, b). For $N = 1200$ their sign is $-\mu(k)$ at every squarefree k , 95.6% of $\sum c_k^2$ lies on the squarefree k , and the profile $w(u) = -c_k \mu(k)$, $u = \log k / \log N$, decreases from about 1 at $u = 0$ to about 0.2 at $u = 1$; its means over the ten intervals of u agree within 0.02 for $N = 200, 600, 1200$. Restricting c to the squarefree k raises d_N^2 by 2 to 5%. Coefficients built from the primes alone come close to the optimum when the profile is chosen in the norm of the distance: with $c_k = \mu(k)P(\log k / \log N)$ and P a polynomial of degree 10 optimized for each N , $d_N^2 \log N$ lies between 0.051 and 0.058 for $50 \leq N \leq 1200$, within a factor 1.13 to 1.24 of the optimum, while $c_k = \mu(k)$ and $c_k = \mu(k)(1 - \log k / \log N)$, each with the best multiple, are worse by factors 3.5 to 16 (Figure 16c). A profile fitted pointwise to w and used without reoptimization is worse by a factor 5.6 to 10; the quadratic form is sensitive to the fine structure of the coefficients.

The reason is the Mellin form of the distance. Since the Mellin transform of $\rho(1/(kx))$ is $-k^{-s}\zeta(s)/s$ for $0 < \sigma < 1$, Plancherel's theorem gives

$$d_N^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |1 + \zeta(s)C_N(s)|^2 \frac{dt}{|s|^2}, \quad s = \frac{1}{2} + it, \quad C_N(s) = \sum_{k \leq N} c_k k^{-s},$$

which we checked numerically: for $N = 50$ the integral over $|t| < 3000$ is 0.011686 against $d_N^2 = 0.011869$, the difference being the omitted tail. The approximation problem is therefore a mollification problem: C_N is a Dirichlet polynomial approximating $-1/\zeta$ on the critical line, and the structure of the optimal coefficients is that of the mollifiers of Levinson and Conrey.

At each zero $\zeta C_N = 0$, so the integrand rises to $1/|s|^2$ (Figure 16d); a zero contributes about $1/|\rho|^2 = \lim_n C_2(n, \rho)$ times a width of order $1/\log N$, which is the content of (ii) and (iii). An estimate $d_N^2 \ll 1/\log N$ for $c_k = \mu(k)P(\log k/\log N)$ amounts to a mean value theorem for $\int |1 + \zeta C_N|^2$ over $|t| \leq N^{1/\theta}$ with mollifiers of length t^θ for arbitrarily large θ . Asymptotic formulas of this kind are known for $\theta < \frac{4}{7}$ [Co89]; for large θ they would require the zeros to lie on the line, since a zero off the line leaves $|1 + \zeta C_N| = 1$ on a window of fixed width. Status: the Mellin identity is classical; the rest is numerical.

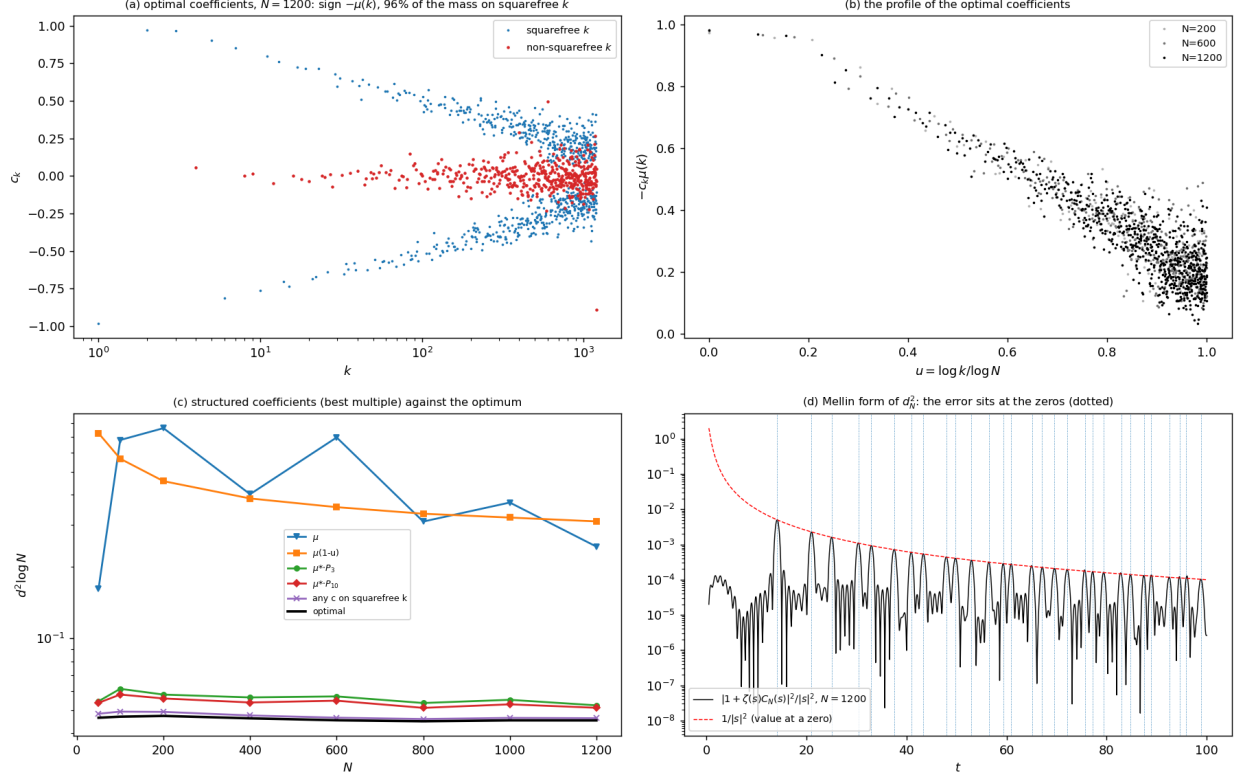


FIGURE 16. The coefficients of the Nyman–Beurling approximation. (a) Optimal c_k for $N = 1200$, on squarefree and non-squarefree k . (b) The profile $-c_k \mu(k)$ against $\log k / \log N$ for $N = 200, 600, 1200$. (c) $d_N^2 \log N$ for coefficients $\mu(k)$, $\mu(k)(1-u)$, $\mu(k)P_3(u)$, $\mu(k)P_{10}(u)$, arbitrary coefficients on the squarefree k , and the optimum. (d) The integrand $|1 + \zeta(s)C_N(s)|^2/|s|^2$ of the Mellin form for $N = 1200$, and $1/|s|^2$; dotted: zeros.

The law on vertical lines. The Laplace transform of E gives an equivalent form in terms of ζ'/ζ , the logarithmic gradient of the C_2 energy $|\zeta|^2$. Put $s = \frac{1}{2} + \delta + it$ and

$$\mathcal{J}(\delta) = \delta \int_{-\infty}^{\infty} \left| \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \right|^2 \frac{dt}{|s|^2}.$$

Proposition 3.3. If RH holds and the zeros are simple, then $\mathcal{J}(\delta) \rightarrow \pi C_B$ as $\delta \rightarrow 0^+$. If $\rho = \beta + i\gamma$ is a zero with $\frac{1}{2} < \beta < 1$, then $\mathcal{J}(\beta - \frac{1}{2}) = \infty$.

Proof. For $\text{Re } s > \Theta$ the Laplace transform $L(s) = \int_0^\infty E(u) e^{-(s-1/2)u} du$ equals $-\frac{1}{s} \left(\frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \right) + H(s)$, where $H(s) = O(\log |s|/|s|)$ comes from the lower-order terms of E . By Plancherel, $I(\delta) = \int_0^\infty E(u)^2 e^{-2\delta u} du = \frac{1}{2\pi} \int |L(\frac{1}{2} + \delta + it)|^2 dt$. Under RH, $F(U) \sim C_B U$, and an Abelian argument gives $I(\delta) \sim C_B/(2\delta)$. The contribution of H changes $\mathcal{J}$ by $O(\delta^{1/2})$, so $\mathcal{J}(\delta) = 2\pi\delta I(\delta) +$

$o(1) \rightarrow \pi C_B$. If the line $\operatorname{Re} s = \beta$ contains a zero, the integrand behaves like $|t - \gamma|^{-2}$ near $t = \gamma$ and the integral diverges. $\square$

Near each zero on the critical line, $\zeta'/\zeta(s) \approx (s - \rho)^{-1}$, and $\delta \int |s - \rho|^{-2} |s|^{-2} dt \rightarrow \pi/|\rho|^2$: each zero contributes π times its C_2 slope through a Poisson kernel of width δ , the kernel that reappears in Sections 4 and 5. The mean square of ζ'/ζ near the critical line is also the object that Goldston, Gonek and Montgomery relate to primes in short intervals [GGM]. We computed $\mathcal{J}(\delta)$ from ζ on $|t| \leq T$ ($T = 600$, and 400, 300 for the two smallest δ), adding the estimate $(\log \frac{T}{2\pi} + 1)/T$ for the zeros above T :

δ	0.2	0.1	0.05	0.025	0.0125	0.00625
$\mathcal{J}(\delta)$	2.220	1.407	0.849	0.518	0.337	0.243

The differences halve with δ , consistent with $\mathcal{J}(\delta) = \pi C_B + O(\delta)$. Richardson extrapolation from the last two values gives 0.148, against $\pi C_B = 0.1451$.

Numerics. We computed $E(u)$ from all prime powers up to 10^8 and took time averages over $1 < u < \log 10^8$ (Figure 17). The autocorrelation obtained from the primes alone matches $2 \sum_{0 < \gamma < 600} \cos(\gamma a) / (\frac{1}{4} + \gamma^2)$ with correlation 0.9997 over $0 \leq a \leq 3$:

a	0	0.10	0.50	0.69	1.00	1.10	2.00	3.00
primes $\leq 10^8$	0.04533	-0.00694	0.00627	-0.01681	0.00227	-0.01991	-0.00713	0.00648
zeros $\gamma < 600$	0.04324	-0.00693	0.00603	-0.01672	0.00220	-0.02051	-0.00713	0.00649

At $a = 0$ the empirical energy 0.0453 lies between the truncated zero sum 0.0432 and $C_B = 0.0462$.

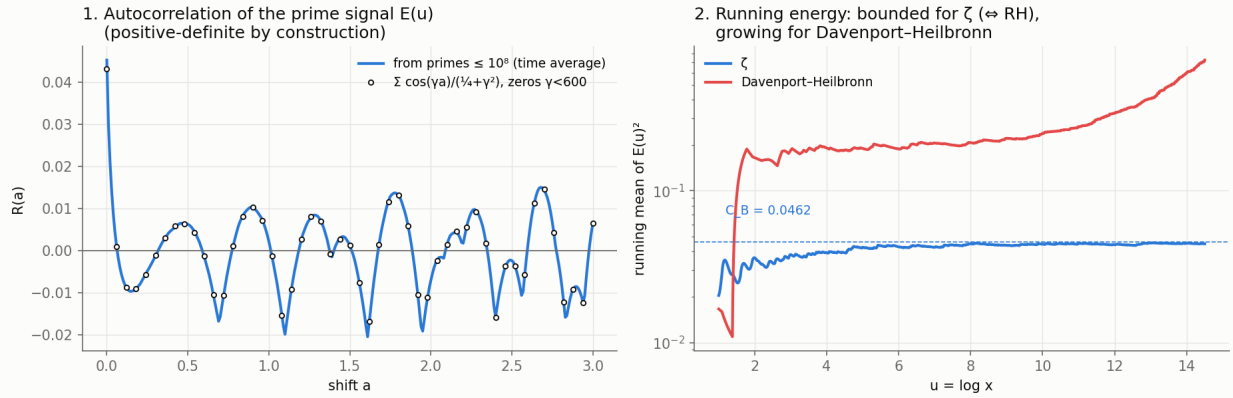


FIGURE 17. Left: autocorrelation of the prime signal $E(u)$ from the primes up to 10^8 (line) against $2 \sum_{0 < \gamma < 600} \cos(\gamma a) / (\frac{1}{4} + \gamma^2)$ (circles). Right: running mean of $E(u)^2$ for ζ , tending to C_B , and of the analogous signal for the Davenport–Heilbronn function, which grows.

For the Davenport–Heilbronn function, the analogous signal $E_f(u) = e^{-u/2} \sum_{n \leq e^u} b(n)$ has no pole term. Its running energy grows:

$x = e^U$	55	403	2981	22026	162755	$1.2 \cdot 10^6$
running energy, ζ	0.041	0.044	0.045	0.045	0.045	0.045
running energy, f	0.21	0.20	0.22	0.25	0.34	0.61

The growth is driven by the off-line zeros, which contribute modes of size $e^{(\beta-1/2)u}$. By Proposition 3.1(b) applied to f , bounded energy would force f to satisfy RH, which it does not. To test Theorem 3.2 we extended E_f to $x = 2 \cdot 10^7$ and computed the energy over windows $U - 1 < u \leq U$. Each off-line zero ρ contributes on average $2e^{(2\beta-1)u}/|\rho|^2$ to E_f^2 . Adding these contributions for the eight off-line zeros with $\gamma < 400$ to the base level 0.200 observed for $U \leq 8$ gives the prediction in the third row:

U	6	8	10	11	12	13	14	15	16
window energy of E_f	0.183	0.222	0.395	0.526	0.952	1.449	2.729	5.256	9.225
base + off-line zeros $\gamma < 400$	0.213	0.244	0.356	0.496	0.764	1.281	2.282	4.231	8.038

The prediction follows the measured growth within 10–25%; the remainder is consistent with off-line zeros above height 400. The local growth rate on $12 \leq U \leq 16$ is 0.58, between the exponents $2\beta - 1 = 0.617$ and 0.739 of the dominant zeros $0.8085 + 85.70i$ and $0.8695 + 240.40i$. For ζ the window energy stays between 0.036 and 0.060 throughout.

A local energy law. Theorem B concerns an average over all u . A version over a finite window holds unconditionally. For $U \in \mathbb{R}$ and $H > 0$ put

$$\mathcal{E}(U, H) = \frac{1}{H} \int_{-\infty}^{\infty} E(u)^2 e^{-\pi(u-U)^2/H^2} du.$$

Theorem 3.4 (local energy law). For all U and $H > 0$,

$$\mathcal{E}(U, H) = \sum_{\rho, \rho'} \frac{e^{zU + z^2 H^2 / (4\pi)}}{\rho \bar{\rho}'}, \quad z = \rho + \bar{\rho}' - 1,$$

where the sum runs over all pairs of nontrivial zeros, counted with multiplicity, and converges absolutely.

Proof. By the truncated explicit formula [In, T], $E_T(u) = -\sum_{|\gamma| < T} e^{(\rho - \frac{1}{2})u} / \rho$ converges to E in L^2 on every bounded interval as $T \rightarrow \infty$, and the Gaussian weight makes the contribution of $|u - U| > R$ negligible uniformly in T . For finite T , $\int e^{zu} e^{-\pi(u-U)^2/H^2} du / H = e^{zU + z^2 H^2 / (4\pi)}$ gives the identity with the sum restricted to $|\gamma|, |\gamma'| < T$. Since $\operatorname{Re} z \leq 1$, each term is at most $e^{U + H^2 / (4\pi)} e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'|$, and $\sum_{\rho, \rho'} e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'| < \infty$ because $N(T+1) - N(T) \ll \log T$. Letting $T \rightarrow \infty$ gives the result. $\square$

Corollary 3.5. Let $H > 0$ and $B(H) = \sum_{\rho \neq \rho'} e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'|$, the sum over pairs of distinct indices.

(a) If RH holds, then $|\mathcal{E}(U, H) - C_B| \leq B(H)$ for all U .

(b) RH holds if and only if $\sup_U \mathcal{E}(U, H) < \infty$.

Proof. (a) Under RH, $z = i(\gamma - \gamma')$; the terms with $\rho = \rho'$ give $\sum_{\rho} |\rho|^{-2} = C_B$, and the others have modulus at most $e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'|$. (b) If $\sup_U \mathcal{E}(U, H) \leq M$, then $\int_{U-H/2}^{U+H/2} E^2 \leq e^{\pi/4} H M$ for all U , so $\int_0^U E^2 = O(U)$, and Theorem 3.2 gives $\Theta = \frac{1}{2}$. The converse is (a). $\square$

A zero with $\beta > \frac{1}{2}$ enters $\mathcal{E}$ through its diagonal term $e^{(2\beta-1)U + (2\beta-1)^2 H^2 / (4\pi)} / |\rho|^2 > 0$, which no other term can cancel for large U , since all other terms involving ρ grow at most like $e^{(\beta-1/2)U}$. It becomes visible only for $U \gtrsim 2 \log |\gamma| / (2\beta - 1)$, so the law does not by itself exclude zeros above the computed range.

We computed $\mathcal{E}(U, H)$ from the prime powers up to 10^8 and from the double sum over the 682 zeros with $|\gamma| < 600$, adding the diagonal contribution $C_B - \sum_{|\gamma| < 600} |\rho|^{-2} = 0.00295$ of the higher zeros; the two agree to within $3.3 \cdot 10^{-4}$ for $H = 1, 2$ and $6 \leq U \leq 14$. On $5 \leq U \leq 14.5$, $|\mathcal{E}(U, H) - C_B|$ is at most 0.0074 for $H = 1$ and 0.0023 for $H = 2$, against $B(1) = 0.044$ and

$B(2) = 0.0123$ computed from the same zeros. For the Davenport–Heilbronn signal ($H = 1$) the window energy grows from 0.22 at $U = 6$ to 2.06 at $U = 13$; the double sum over its zeros with $\gamma < 400$ accounts for 84–96% of it, the diagonal terms of its eight off-line zeros grow from 0.02 to 1.52, and the remainder increases with U , as expected from off-line zeros above height 400. The Euler-product twin $L(s, \chi)$ of Section 4, which has the same conductor and gamma factor as f , behaves like ζ : the mean of $|E_\chi(u)|^2$, with $E_\chi(u) = e^{-u/2} \sum_{n \leq e^u} \Lambda(n) \chi(n)$, is 0.2052 on $3 \leq u \leq 18.4$, against $\sum_\rho |\rho|^{-2} = 0.2032$ over its zeros (those with $|\gamma| < 400$ plus a density estimate for the rest), and its window energy with $H = 2$ stays between 0.189 and 0.221, inside the band 0.203 ± 0.077 of Corollary 3.5.

Corollary 3.5(c) (energy cost of a spiral). Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ with $\delta > 0$, let $\Gamma > 0$, and suppose that the zeros with $|\gamma - \gamma_0| \leq \Gamma$ other than ρ_0 and $1 - \bar{\rho}_0$ lie on the critical line. Then for all U and $H > 0$,

$$\sqrt{\mathcal{E}(U, H)} \geq \frac{e^{\delta U + \delta^2 H^2 / (2\pi)}}{|\rho_0|} - D_{\gamma_0}(H) - T_{\gamma_0}(U, H, \Gamma),$$

where

$$D_{\gamma_0}(H) = \sum_{\rho \neq \rho_0} \frac{e^{-(\gamma - \gamma_0)^2 H^2 / (4\pi)}}{|\rho|},$$

in D the zeros other than ρ_0 are counted as if on the line, and

$$T_{\gamma_0}(U, H, \Gamma) = e^{U/2 + 3H^2 / (16\pi)} \sum_{|\gamma - \gamma_0| > \Gamma} \frac{e^{-(\gamma - \gamma_0)^2 H^2 / (4\pi)}}{|\rho|}.$$

Proof. $\mathcal{E}(U, H) = \|E\|^2$ in $L^2(w)$, $w(u) = H^{-1} e^{-\pi(u-U)^2 / H^2}$, so $\mathcal{E} \geq |\langle E, h \rangle|^2 / \|h\|^2$ for $h(u) = e^{(\rho_0 - \frac{1}{2})u}$. By the computation in the proof of Theorem 3.4, $\langle E, h \rangle = -\sum_\rho G_{\rho\rho_0} / \rho$ with $G_{\rho\rho_0} = e^{zU + z^2 H^2 / (4\pi)}$, $z = \rho + \bar{\rho}_0 - 1$, and $\|h\|^2 = G_0 = e^{2\delta U + \delta^2 H^2 / \pi}$. The term $\rho = \rho_0$ gives $\sqrt{G_0} / |\rho_0|$. For a zero on the line, $|G_{\rho\rho_0}| / \sqrt{G_0} = e^{-(\delta^2 + (\gamma - \gamma_0)^2) H^2 / (4\pi)}$; for $1 - \bar{\rho}_0$, $z = 0$ and the ratio is $G_0^{-1/2} \leq 1$; for any zero with $|\gamma - \gamma_0| > \Gamma$ the ratio is at most $e^{U/2 + 3H^2 / (16\pi) - (\gamma - \gamma_0)^2 H^2 / (4\pi)}$, since $\beta - \frac{1}{2} \leq \frac{1}{2}$. The triangle inequality gives the claim. $\square$

The off-diagonal terms do not grow with U , so a spiral of drift δ at height γ_0 costs at least $e^{2\delta U} / |\rho_0|^2$ in window energy, up to a bounded interference. With the bound $\mathcal{E}(U, 1) \leq C_B + 0.0074$ measured above for $U \leq 14.5$ (primes up to 10^8), $\Gamma = 20$ and the zeros below 2000, the corollary excludes an isolated off-line pair at height γ_0 unless $\beta \leq \frac{1}{2} + \log(|\rho_0|(\sqrt{\mathcal{E}} + D + T)) / U$: this gives $\beta \leq 0.600$ at $\gamma_0 = 14.13$, 0.727 at 100, 0.877 at 1000 and 0.921 at 1900, with $D = 0.072, 0.036, 0.006, 0.004$ and $T < 10^{-13}$. To exclude $\delta = 0.01$ at the same heights one needs $U = 146, 329, 547$ and 610. For the Davenport–Heilbronn zero $0.8085 + 85.699i$ the diagonal term alone forces $\mathcal{E}(13, 1) \geq 0.43$, against the measured 2.06. In general, the primes up to x exclude $\beta > \frac{1}{2} + \delta$ only up to heights of order x^δ . This is a region of the Landau–Turán type, far weaker than the direct verification of RH, and it quantifies Corollary 3.5(b): every finite window of primes controls a finite range of heights, and RH requires the bound for all U . Status: the inequality is elementary given Theorem 3.4; the numbers are from `anc/spiral_bound/spiral_bound.py`.

Proposition 3.5' (the Jensen moment [BSY]). Put μ for the probability measure $dt / (2\pi(\frac{1}{4} + t^2))$ on the critical line and, for a function f , $J_f(T) = \int_{-T}^T \log |f(\frac{1}{2} + it)| d\mu(t)$. Then

$$J_\zeta(\infty) = \sum_{\beta > \frac{1}{2}} \log \left| \frac{\rho}{1 - \rho} \right|,$$

and RH holds if and only if $J_\zeta(\infty) = 0$. Each term equals $\log(1/|w_\rho|)$ with $w_\rho = 1 - 1/\rho$, the Möbius map that also enters the identity for $\mathcal{Q}$ below, and is approximately $(\beta - \frac{1}{2})/|\rho|^2 > 0$. For the Davenport–Heilbronn function f , $J_f(\infty) = \log|f(1)| + \sum_{\beta > \frac{1}{2}} \log|\rho/(1 - \rho)|$.

Proof. The map $w = 1 - 1/s$ sends the half-plane $\operatorname{Re} s > \frac{1}{2}$ onto the unit disc and $s = 1$ to its centre, and μ is the Poisson measure of the half-plane at $s = 1$. Poisson–Jensen for a function F of bounded type in the half-plane with no singular inner factor gives $\int \log|F(\frac{1}{2} + it)| d\mu = \log|F(1)| + \sum_{\beta > \frac{1}{2}} \log(1/|w_\rho|)$. Apply it to $F(s) = (s - 1)\zeta(s)/s$, for which $|F| = |\zeta|$ on the line and $F(1) = 1$, and to f . $\square$

Equivalently, $-J_\zeta(\infty) = \log|B(1)|$ for the Blaschke product $B(s) = \prod_{\beta > \frac{1}{2}} (s - \rho)/(s - 1 + \bar{\rho})$ of the half-plane, and RH states that F is an outer function. We computed J by quadrature between consecutive zeros on the line. For ζ , $J_\zeta(1000) = 3.1 \cdot 10^{-8}$ (Figure 18a). For the Davenport–Heilbronn function, $\log|f(1)| = -0.0803407$ and $J_f(400) - \log|f(1)| = 1.52477 \cdot 10^{-4}$, against $1.52687 \cdot 10^{-4}$ from its eight off-line zeros below height 400 and their conjugates; the value without off-line zeros, 0, is excluded by a factor 700 of the discrepancy. The partial integral rises in steps at the heights of the off-line zeros (Figure 18b), so the moment localizes them.

Proposition 3.5'' (a barrier for mollified Jensen bounds). Let $M(s) = \sum_{n \leq N} a_n n^{-s}$ with $M(1) \neq 0$. Then

$$\sum_{\beta > \frac{1}{2}} \log \left| \frac{\rho}{1 - \rho} \right| \leq \frac{1}{2} \log R(M), \quad R(M) = \frac{\int |\zeta M(\frac{1}{2} + it)|^2 d\mu}{|M(1)|^2} \geq 1 + c_0,$$

$$c_0 = \sum_{k \geq 1} \left(1 - k(k+1) \log^2 \left(1 + \frac{1}{k} \right) \right) = 0.0803270 \dots$$

Proof. Poisson–Jensen for FM gives $\int \log|FM| d\mu \geq \log|M(1)| + \sum_{\beta > \frac{1}{2}} \log(1/|w_\rho|)$, since the zeros of M and a possible singular factor at infinity only increase the left side, and $\int \log|FM| d\mu \leq \frac{1}{2} \log \int |FM|^2 d\mu$ by Jensen’s inequality. For the lower bound, $\zeta(s)/s = -\int_0^\infty \{1/x\} x^{s-1} dx$ on $0 < \sigma < 1$, so by Plancherel $\int |\zeta M|^2 d\mu = \int_0^\infty |h(u)|^2 u^{-2} du$ with $h(u) = \sum_n a_n \{u/n\} = M(1)u - S(u)$, $S(u) = \sum_{m \leq u} (a * 1)(m)$. On $(0, 1)$, $h = M(1)u$ contributes $|M(1)|^2$; on $[k, k+1)$, S is constant and $\int_k^{k+1} |M(1)u - S|^2 u^{-2} du \geq |M(1)|^2 \min_c \int_k^{k+1} (1 - c/u)^2 du = |M(1)|^2 (1 - k(k+1) \log^2(1 + \frac{1}{k}))$. $\square$

With $M(1) = 1$ the mollifier must approximate the line u by a step function with jumps at the integers, and the sawtooth costs c_0 . Jensen’s inequality with Dirichlet mollifiers therefore cannot give a bound below $\frac{1}{2} \log(1 + c_0) = 0.0386$ for $\sum_{\beta > \frac{1}{2}} \log|\rho/(1 - \rho)|$, whose value under RH is 0. The loss comes from the passage from $\log|\cdot|$ to $|\cdot|^2$ and from the discreteness of the integers, not from the arithmetic of ζ ; it is the analogue, for the full critical line, of the constant $c(\theta) > 1$ that limits the method of Levinson and Conrey [Co89]. The coefficients $a = \mu * b$ with $b_m = S_m - S_{m-1}$, $S_m = m(m+1) \log(1 + \frac{1}{m})$, attain the minimum on each interval up to N , and the truncated choice gives $R - 1 = 0.102, 0.088, 0.0995, 0.0894, 0.0872, 0.0861$ for $N = 10, 30, 100, 300, 1000, 3000$ (Figure 18c). The mollifiers $\mu(n)(1 - \log n / \log N)$ of the Nyman–Beurling criterion [Ny, BD] give R growing with N : they approximate $1/\zeta$, while the normalization at $s = 1$ requires $1/F$. Status: Proposition 3.5' is in [BSY]; Proposition 3.5'' is elementary; the computations are numerical (`anc/jensen/`). The floor is a property of Dirichlet polynomials: the mollifier $1/P$ built from the Euler–Cáceres factor of Section 6 is not one, and with it the bound falls to 0.0015 at $X = 10^4$ (Proposition 6.1''').

Remark 3.6 (the harmonic offset of [E24]). The harmonic-series construction of [E24, §7.2] compares, at two zeros, the quantity $H(\frac{1}{2}, \gamma, n) = H_n - |X_n(\frac{1}{2} + i\gamma)|^2 + n/(\frac{1}{4} + \gamma^2)$, where H_n is the harmonic number, and claimed that its difference at two zeros tends to 0. By the straight-line

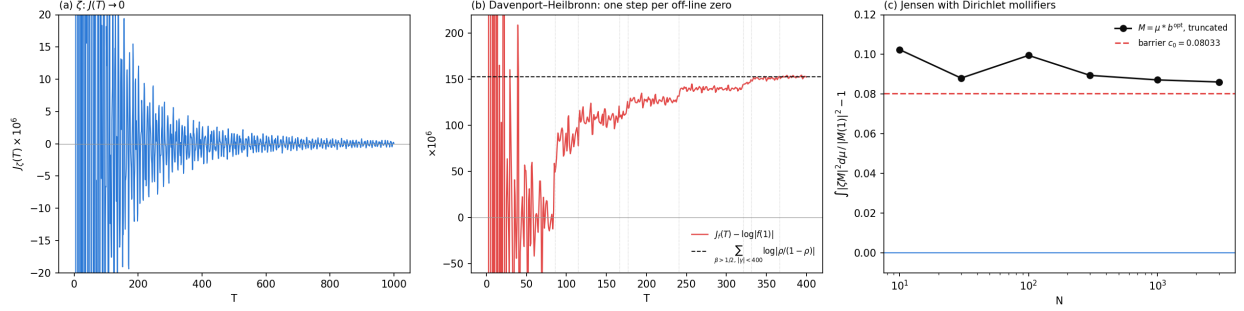


FIGURE 18. The Jensen moment (Propositions 3.5' and 3.5''). (a) $J_\zeta(T)$, which tends to 0. (b) $J_f(T) - \log|f(1)|$ for the Davenport–Heilbronn function; dashed: the sum over its off-line zeros below height 400; dotted: their heights. (c) $R(M) - 1$ for the truncated optimal mollifier against the barrier c_0 ; RH would need 0.

law of Corollary 2.1, at each zero on the critical line

$$H_n - H(\tfrac{1}{2}, \gamma, n) \longrightarrow \frac{1}{2} \cdot \frac{1}{\frac{1}{4} + \gamma^2} = \frac{1}{2} \lim_{n \rightarrow \infty} C_2(n, \rho),$$

so the difference at two zeros tends to $\frac{1}{2}[L(\gamma_2) - L(\gamma_1)]$, $L(\gamma) = (\frac{1}{4} + \gamma^2)^{-1}$, which is -0.001369 for the first two zeros [Er]. The harmonic offset of each zero is half its C_2 limit and depends only on its own height, so it does not relate one zero to another. The offsets are related collectively: by Theorem B their sum over all zeros is half the energy of the prime signal, $C_B/2 = 0.0231$ under RH, and an off-line pair would make the sum infinite.

We checked the limit with $n = 2 \cdot 10^5$, writing the offset for any s as $H(s, n) = \sum_{k \leq n} |a_k|^2 k^{-2\sigma} - |X_n(s)|^2 + n/(\sigma^2 + t^2)$. At the zeros $\frac{1}{2} + 14.1347i$ and $\frac{1}{2} + 21.0220i$, $H - \log n = 0.5747$ and 0.5761 , against $\gamma_0 - \frac{1}{2}(\frac{1}{4} + \gamma^2)^{-1} = 0.57472$ and 0.57608 (Figure 19). At a point of the line that is not a zero the cross term between $\zeta(s)$ and the pole survives, and H oscillates with amplitude $2\sqrt{n}|\zeta(s)|/|1-s|$ (± 44 at $\frac{1}{2} + 20i$); off the line the correction $n/(\sigma^2 + t^2)$ no longer matches the pole and H grows like n (489 at $0.7 + 20i$), zero or not. For the Davenport–Heilbronn function, which has no pole, we drop the correction and subtract $c \log n$ with c the mean of $|a_k|^2$: the partial sums converge, so $H - c \log n$ tends to a constant at every point of the line, zero or not (0.62 at the zero $\frac{1}{2} + 43.0813i$, -0.15 at $\frac{1}{2} + 60i$), and decreases like $-c \log n$ at the off-line zero $0.8085 + 85.699i$, where the weighted harmonic sum converges. The offset thus recognises the zeros of ζ on the line through the vanishing of the cross term with the pole, and off the line it records σ , not the zero (anc/harmonic/).

Corollary 3.7 (the energy attains its lower bound exactly under RH). Let

$$\mathcal{E} = \limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du \in [0, \infty].$$

Then

$$\mathcal{E} \geq 2 + \gamma_0 - \log 4\pi = 0.0461914 \dots,$$

with equality if and only if RH holds and all zeros are simple.

Proof. The identity $\sum_\rho \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi$, with multiplicity and in symmetric order, holds unconditionally [Da, Ch. 12]. If RH fails, $\mathcal{E} = \infty$ by Theorem B. If RH holds, $\rho(1-\rho) = |\rho|^2$, and $E(u) = -\sum_\gamma m_\gamma e^{i\gamma u}/\rho$ over the distinct zeros, m_γ denoting the multiplicity; as in Proposition 3.1, the mean of the cross terms vanishes and $\mathcal{E} = \sum_\gamma m_\gamma^2/|\rho|^2 \geq \sum_\gamma m_\gamma/|\rho|^2 = 2 + \gamma_0 - \log 4\pi$, with equality exactly when every $m_\gamma = 1$. $\square$

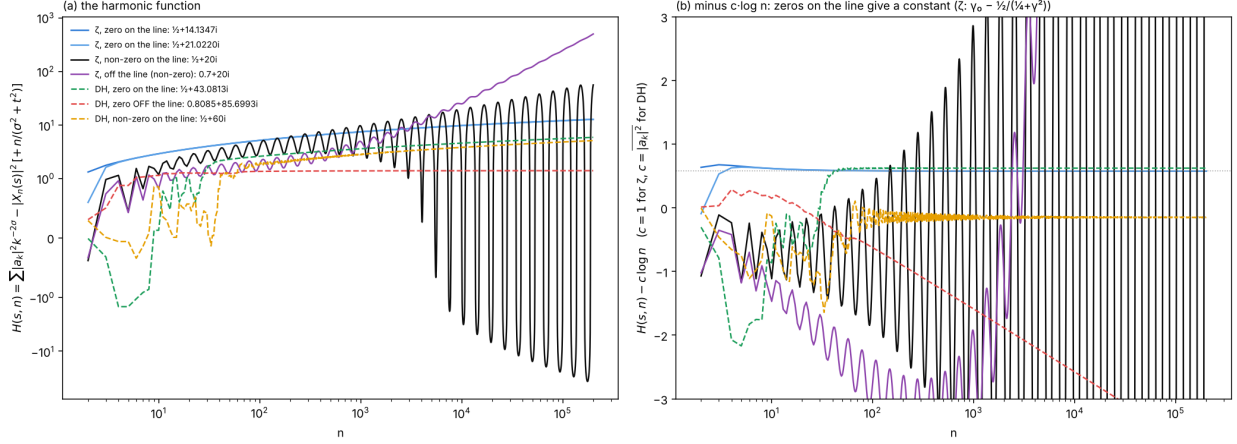


FIGURE 19. The harmonic offset (Remark 3.6) for $n \leq 2 \cdot 10^5$. (a) $H(s, n)$ for zeros of ζ on the line, a point of the line that is not a zero, a point off the line, and for the Davenport–Heilbronn function (dashed) at a zero on the line, at its off-line zero $0.8085 + 85.6993i$ and at a point of the line that is not a zero. (b) $H(s, n) - c \log n$; dotted: γ_0 .

Thus one side of the criterion is an explicit constant and the other is computed from the primes alone: the measured value 0.0453 (primes up to 10^8) lies slightly below the limit, by an amount consistent with the finite range of u . A finite computation cannot decide the equality, since an off-line zero at height γ affects the energy only for $u \gtrsim 2 \log \gamma / (2\beta - 1)$.

The non-holomorphic gap. The constant C_B is the sum over the zeros of a holomorphic function of ρ ; the diagonal of Theorem B is $\sum_\rho |\rho|^{-2}$, which is not. The difference is a positive form in the abscissae. Pair ρ with $1 - \bar{\rho}$, a bijection of the zeros preserving multiplicity, and put $a = |\rho|^2$, $b = |1 - \rho|^2 = a + 1 - 2\beta$. Then $\frac{\beta}{a} + \frac{1-\beta}{b} - \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) = -\frac{2(\beta - \frac{1}{2})^2}{ab}$, and summing over the zeros, with multiplicity and in symmetric order,

$$\sum_\rho \frac{1}{|\rho|^2} = 2 + \gamma_0 - \log 4\pi + \mathcal{Q}, \quad \mathcal{Q} = \sum_\rho \frac{(\beta - \frac{1}{2})^2}{|\rho|^2 |1 - \rho|^2} \geq 0.$$

Both series converge absolutely, since $\sum_\rho |\rho|^{-2} < \infty$, and $\mathcal{Q} = 0$ if and only if RH holds. The Möbius map $w_\rho = 1 - 1/\rho$ sends the critical line onto the unit circle, the functional equation gives $|w_{1-\bar{\rho}}| = |w_\rho|^{-1}$, and $\frac{(\beta - \frac{1}{2})^2}{|\rho|^2 |1 - \rho|^2} = \frac{1}{4}(|w_\rho| - |w_\rho|^{-1})^2$; an off-line pair has reciprocal radii $r, 1/r$ and contributes $\frac{1}{2}(r - r^{-1})^2 > 0$, so the symmetry does not cancel the defect. By Corollary 2.1 and Theorem A, $1 - 2\beta = \lim_n \log C_2(n, \rho) / \log n$ for $0 < \beta < 1$, so $\mathcal{Q}$ is also a sum of squared growth exponents of the cross-energy, weighted by $\frac{1}{4}|\rho|^{-2}|1 - \rho|^{-2}$. The identity explains why the criterion cannot be closed from the primes alone. The explicit formula evaluates $\sum_\rho F(\rho)$ only for F holomorphic in a neighbourhood of the critical strip, and gives $\sum_\rho \frac{1}{\rho(1-\bar{\rho})} = C_B$ without information on β ; the gap $\mathcal{Q}$ is the part of $\sum_\rho |\rho|^{-2}$ that depends on $|\rho|$ and not on ρ , and it is reached only through a positivity statement on the spectral side, such as the mean square of Theorem B. This is the obstruction of Proposition 6.21 in its simplest form. Status: the identity is elementary, and the equivalence of RH with $\sum_\rho |\rho|^{-2} = 2 + \gamma_0 - \log 4\pi$ that it contains is known; we include it for the interpretation of Theorem B.

The holomorphic and the modulus sides. The complex cross-energy of s and $1 - \bar{s}$ gives the holomorphic side directly. Since $\overline{X_N(1 - \bar{s})} = X_N(1 - s)$, the expansion (2.1) gives, for every s

in the strip,

$$\lim_{N \rightarrow \infty} \frac{X_N(s) \overline{X_N(1-\bar{s})}}{N} = \lim_{N \rightarrow \infty} \frac{X_N(s) X_N(1-s)}{N} = \frac{1}{s(1-s)},$$

the terms with $\zeta(s)$ and $\zeta(1-s)$ decaying like $N^{\sigma-1}$ and $N^{-\sigma}$; on the critical line the quotient is $|X_N|^2/N$, real for every N . The limit is the spectral variable $\lambda = s(1-s)$ of Section 2 and does not depend on whether s is a zero (at $s = \frac{1}{2} + 20i$ and $N = 10^6$ it is 0.002389 against $1/\lambda = 0.002498$; at a zero the convergence is faster), so the statement that it is real at every zero is RH restated. Summed over the zeros it gives $\sum_{\rho} 1/(\rho(1-\rho)) = C_B$, the side seen by the primes, while the C_2 limits give $\sum_{\rho} |\rho|^{-2}$ under RH; their difference is the gap $\mathcal{Q}$ above. Status: follows from (2.1); the computation is numerical.

The energy itself is shown in Figure 20: the normalized error $E(u)$ oscillates around zero with stable amplitude, its accumulated energy follows a straight line after the initial transient, and the slope $(I(X) - K)/\log X$ settles at $C_B = \sum_{\rho} \lim_n C_2(n, \rho)$.

The constant term of the energy. Corollary 3.7 fixes the slope of $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx = \int_0^{\log X} E(u)^2 du$. The next term can be computed. Write the Mellin transform of the prime error as

$$F(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = -\sum_{\rho} \frac{1}{\rho(s-\rho)} + R(s), \quad R(s) = \frac{1}{s} \left(\frac{1}{2} \frac{\Gamma'}{\Gamma} \left(1 + \frac{s}{2} \right) + \frac{\gamma_0}{2} - \log 2\pi \right),$$

where R collects the constant $\log 2\pi$ of the explicit formula and the trivial zeros, and let δ_{ρ} be the distance from γ to the nearest other ordinate.

Proposition 3.8 (second-order energy law). Assume RH, that the zeros are simple, and that $\sum_{\rho} |\rho|^{-2} \delta_{\rho}^{-1} < \infty$. Then, with $u = \log X$,

$$I(X) = C_B u + K + \Phi(u) + o(1), \quad \Phi(u) = \sum_{\rho \neq \rho'} \frac{e^{i(\gamma-\gamma')u}}{i(\gamma-\gamma') \rho \bar{\rho}'},$$

where Φ is a Bohr almost periodic function of mean zero with $|\Phi(u)| \leq \frac{3\pi}{2} \sum_{\rho} |\rho|^{-2} \delta_{\rho}^{-1}$, and

$$K = \frac{1}{2} \left(C_B^2 - \sum_{\rho} \frac{1}{|\rho|^4} \right) - 2 \sum_{\rho} \frac{R(1-\rho)}{\rho} + \frac{1}{2\pi} \int_{-\infty}^{\infty} |R(\tfrac{1}{2} + it)|^2 dt.$$

Proof (sketch). Under RH the explicit formula gives $E(u) = S(u) + r(u)$ with $S(u) = -\sum_{\rho} e^{i\gamma u}/\rho$ and $r(u) = -e^{-u/2} (\log 2\pi + \frac{1}{2} \log(1-e^{-2u}))$, whose transforms $\int_0^{\infty} (\cdot) e^{-(s-1/2)u} du$ are the two parts of $F(s)$. Since $r \in L^2(0, \infty)$, $\int_0^U r^2 \rightarrow \frac{1}{2\pi} \int |R|^2$ by Plancherel. The cross term converges absolutely, because the window energy of S is bounded (Theorem 3.4) and r decays like $e^{-u/2}$; termwise, $\int_0^{\infty} e^{i\gamma u} r(u) du = R(1-\rho)$, which gives $-2 \sum_{\rho} R(1-\rho)/\rho$ up to $O(e^{-u/2})$. Finally

$$\int_0^U S^2 = C_B U + \sum_{\rho \neq \rho'} \frac{e^{i(\gamma-\gamma')U} - 1}{i(\gamma-\gamma') \rho \bar{\rho}' }.$$

The oscillating part is $\Phi(U)$; the generalized Hilbert inequality of Montgomery and Vaughan [MV74] bounds it, and its truncations, by $\frac{3\pi}{2} \sum |\rho|^{-2} \delta_{\rho}^{-1}$, so Φ is a uniform limit of trigonometric polynomials without constant term. In the constant part, the terms (ρ, ρ') and (ρ', ρ) add up to exactly $|\rho|^{-2} |\rho'|^{-2}$, since $\text{Im}(1/\rho \bar{\rho}') = (\gamma' - \gamma)/(2|\rho|^2 |\rho'|^2)$; their sum is $\frac{1}{2}(C_B^2 - \sum |\rho|^{-4})$. The interchange of the sums over zeros with the limits is justified by the same inequality. $\square$

Status: the formula for K is derived under the stated hypotheses; the spacing condition is not known, but it follows from any bound $\delta_{\rho} \gg |\gamma|^{-1+\varepsilon}$, far weaker than the gaps expected from random matrix theory. The asymptotic $I(X) \sim C_B \log X$ is Cramér's [Cr]; related double sums over zeros for $\int_X^{2X} (\psi - x)^2$ are studied in [BPT]; we have not found the constant K in the literature.

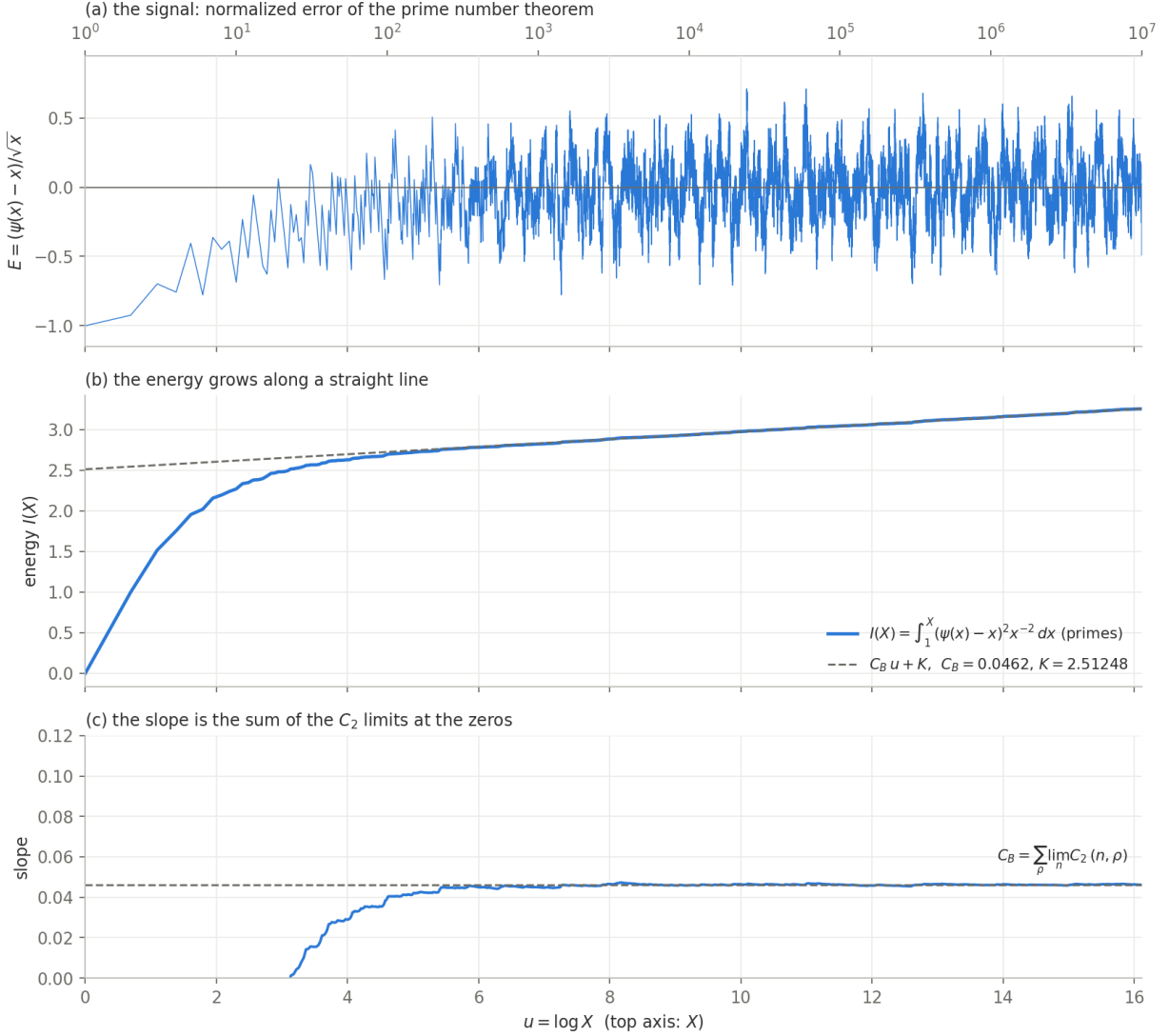


FIGURE 20. The energy of the primes up to 10^7 . (a) $E = (\psi(x) - x)/\sqrt{x}$ against $u = \log x$. (b) $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx$ and the line $C_B \log X + K$. (c) $(I(X) - K)/\log X$ and C_B .

The three terms are 0.00103, -0.0100 and 2.5216 , so that

$$K = 2.51248 \dots$$

(the last two computed from $\sum_{\rho} \frac{1}{\rho(a-\rho)} = \frac{1}{a} \left(\frac{C_B}{2} + \frac{\xi'(a)}{\xi(a)} \right)$ at $a = 1 + 2k$ and from the poles of R , with truncation extrapolated). K is almost entirely the contribution of the trivial part: the interval $1 \leq x < 2$, where $\psi = 0$, the constant $\log 2\pi$ and the trivial zeros; the nontrivial zeros contribute -0.009 . From the primes up to 10^7 , the mean of $I(X) - C_B \log X$ over $u \in [\log 10^4, \log 10^7]$ is 2.51255 , and $I(X) - C_B \log X - K$ follows $\Phi(\log X)$ computed from the 341 zeros with $0 < \gamma < 600$: the correlation is 0.979, 0.995, 0.999, 0.999 on the decades from $[10^3, 10^4]$ to $[10^6, 10^7]$, and the root-mean-square residual decreases from $1.7 \cdot 10^{-3}$ to $1.2 \cdot 10^{-4}$ (Figure 21). Thus, under RH, the slope of the energy is $\sum_{\rho} \lim_n C_2(n, \rho)$, the intercept is fixed by the trivial part, and the oscillation is the interference of pairs of zeros at the frequencies $\gamma - \gamma'$.

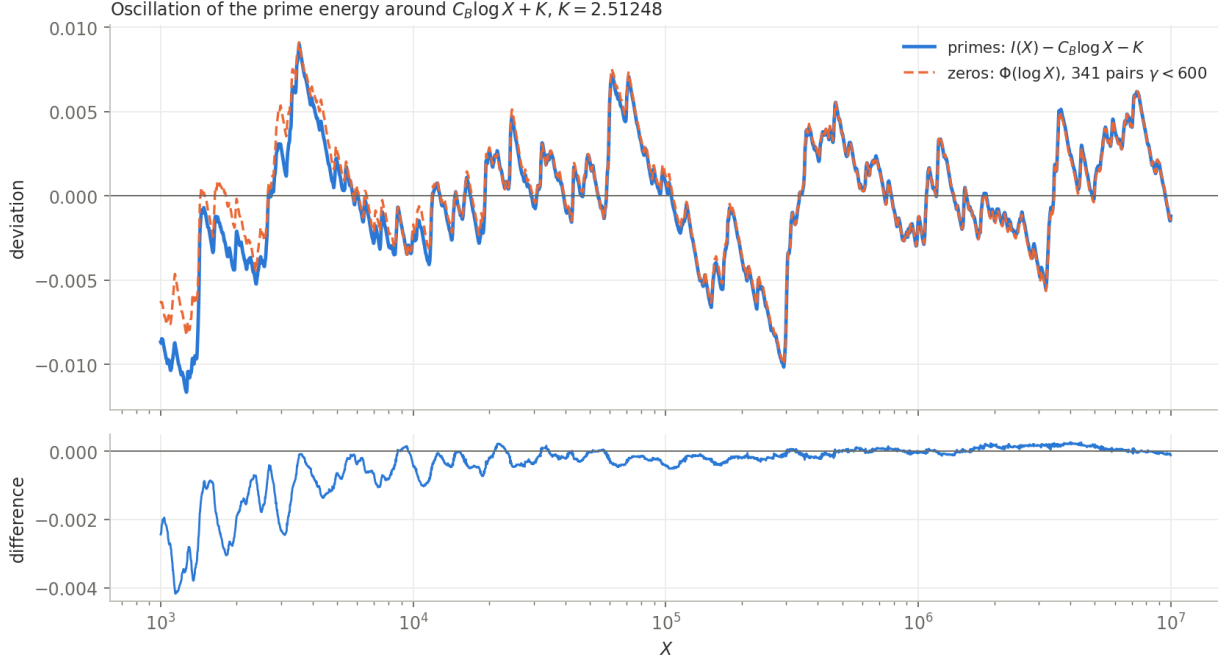


FIGURE 21. The oscillation of the prime energy. Top: $I(X) - C_B \log X - K$ from the primes up to 10^7 (line) and $\Phi(\log X)$ from the zeros with $0 < \gamma < 600$ (dashed). Bottom: their difference.

The energy matrix of the zeros. The quadratic structure behind Proposition 3.8 is the Gram matrix of the orbits $e^{i\gamma u}/\rho$ on $[0, U]$,

$$M(U)_{\rho\rho'} = \frac{1}{\rho\bar{\rho}'} \cdot \frac{1}{U} \int_0^U e^{i(\gamma-\gamma')u} du, \quad M(U)_{\rho\rho} = \frac{1}{|\rho|^2} = \lim_n C_2(n, \rho),$$

a Hermitian positive semidefinite matrix whose diagonal is the C_2 limit of each zero and whose total sum $\mathbf{1}^T M(U) \mathbf{1}$ is the mean energy of S on $[0, U]$. Its trace is C_B ; the off-diagonal entries, the interference, decay like $1/U$, and the constant they leave in Proposition 3.8 is the second elementary symmetric function $\sum_{\rho < \rho'} \lim_n C_2(n, \rho) \lim_n C_2(n, \rho')$ of the C_2 limits. As $U \rightarrow 0$, $M(U)$ tends to the rank-one matrix vv^* , $v_\rho = 1/\rho$, with the single eigenvalue C_B ; as $U \rightarrow \infty$ its eigenvalues tend to the diagonal, each C_2 limit appearing twice (ρ and $\bar{\rho}$). With the 682 zeros $|\gamma| < 600$ the eigenvalues agree with $\lim_n C_2(n, \rho)$ to four digits from $U = 10$, and the von Neumann entropy of $M/\text{tr } M$ rises from 0 to 4.391, the entropy of the distribution $\{\lim_n C_2(n, \rho)/C_B\}$ (Figure 22). The asymptotic spectrum of the energy of the primes is therefore the set of C_2 limits at the zeros. The matrix is positive semidefinite whatever the location of the zeros, as every Gram matrix is; an off-line zero would show only in the size of its diagonal entry, which diverges with its C_2 limit (Theorem A). Status: elementary; the computations are numerical.

Remark 3.9 (random models). The boundedness in Theorem B is not a property of typical arithmetic-like sequences. We replaced the primes by Cramér’s model [Cr36] (each $n \geq 3$ is “prime” with probability $1/\log n$, with weight $\log n$), and μ by independent signs on the squarefree integers (Denjoy [Dj]) and by a random multiplicative function supported on them (Wintner [Wi]; see [Hp]), with 100 realizations of each up to 10^7 . The logarithmic mean of the energy on $[100, X]$ at $X = 10^7$ is 0.0488 for the primes and 0.0289 for $(M(x) + 2)/\sqrt{x}$, against a median of 7.24 for Cramér’s model (smallest of 100: 2.23), $0.627 \approx 6/\pi^2$ on average for independent signs (smallest: 0.175), and a median of 0.242 for the random multiplicative function (smallest: 0.050); no realization reaches the

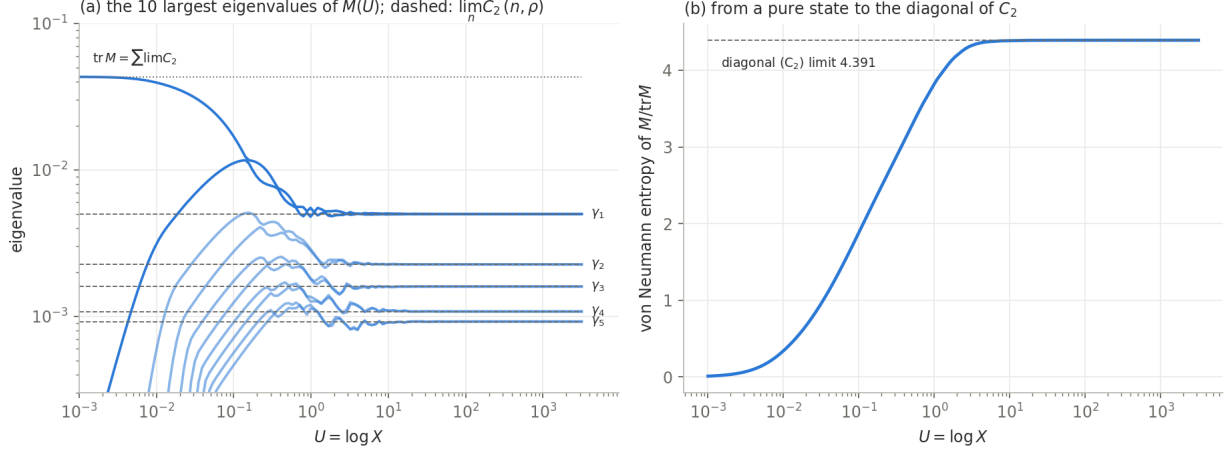


FIGURE 22. (a) The ten largest eigenvalues of $M(U)$ for the zeros with $|\gamma| < 600$; dashed: $\lim_n C_2(n, \rho)$ for the first five zeros; dotted: $\text{tr } M = \sum \lim_n C_2$. (b) The von Neumann entropy of $M(U)/\text{tr } M(U)$.

value of the primes (Figure 23). For Cramér's model the energy grows like $\frac{1}{2}(u + \log 100) - 2$, as its variance $\sum_{n \leq x} (\log n - 1)$ predicts, although the model satisfies $\psi(x) - x = O(x^{1/2+\varepsilon})$ almost surely. Square-root cancellation is therefore weaker than Theorem B: bounded mean energy requires the discrete spectrum supplied by the zeros, while a random model has a continuous spectrum. The primes are 50 to 150 times more rigid than Cramér's model, and μ about 20 times more than its random models. Status: numerical.

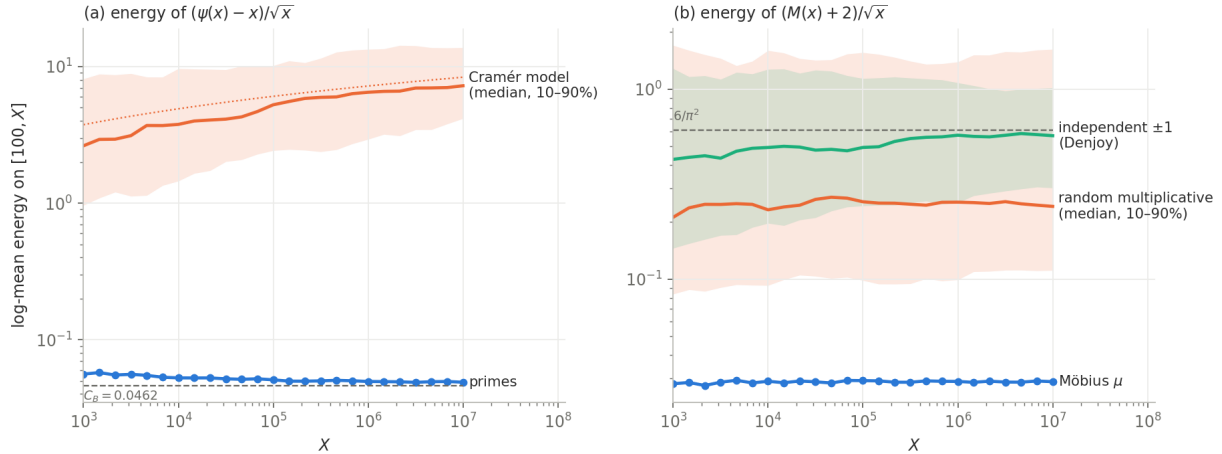


FIGURE 23. Logarithmic mean of the energy on $[100, X]$. (a) $(\psi(x) - x)/\sqrt{x}$ for the primes and for Cramér's model (median and 10–90% band of 100 realizations; dotted: $\frac{1}{2}(u + \log 100) - 2$). (b) $(M(x) + 2)/\sqrt{x}$ for μ , for independent signs and for a random multiplicative function on the squarefree integers.

The orbit of each zero. The energy can be resolved zero by zero. For a frequency γ put

$$A_\gamma(u) = \int E(v) e^{-i\gamma v} w_s(u-v) dv, \quad E_\gamma(u) = 2 \operatorname{Re} (A_\gamma(u) e^{i\gamma u}),$$

with w_s a Gaussian window of width s , wide enough that $s\Delta \gg 1$, where Δ is the distance from γ to the neighbouring zeros. If $\frac{1}{2} + i\gamma$ is a simple zero and the neighbouring zeros are resolved by the window, the explicit formula gives $A_\gamma(u) \approx -1/\rho$, so the curve $u \mapsto (E_\gamma(u), E_\gamma(u + \frac{\pi}{3\gamma}))$ is a closed ellipse of semi-amplitude $2/|\rho|$, traversed once for every increase of u by $2\pi/\gamma$. A zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ contributes $-e^{(\beta-\frac{1}{2})u}e^{i\gamma u}/\rho$ to E , and the same curve is a spiral whose amplitude grows at the rate $\beta - \frac{1}{2}$. In this language RH states that every orbit of the prime signal is closed, and Theorem B states that the mean squares of the orbits, $2|A_\gamma|^2 = 2/|\rho|^2$ for each $\gamma > 0$, add up to the energy $\sum_\rho \lim_n C_2(n, \rho)$.

For ζ at $\gamma = 14.135$ ($s = 1$, primes up to 10^8) the measured mean of $|A_\gamma|$ is 0.0706, against $1/|\rho| = 0.0707$, and $\log |A_\gamma|$ has slope 0.0001 on $4.6 \leq u \leq 14.5$: the ellipse is traversed about 22 times and closes on itself (Figure 24a). For the Davenport–Heilbronn signal ($s = 1.5$, coefficients up to $2 \cdot 10^7$) the isolated on-line zero at $\gamma = 43.08$ gives a closed ellipse with $|A_\gamma| = 0.0233$, against 0.0232, and slope 0.007 (Figure 24b). The off-line zero $0.8085 + 85.699i$ gives a spiral with growth rate 0.3057, against $\beta - \frac{1}{2} = 0.3085$ (Figure 24c,d), and the zero $0.8695 + 240.40i$ gives 0.3716, against 0.3695. The curve depends on β through the growth of its amplitude, in contrast with the circles of Section 2, which record only γ .

The phase of each orbit. The complex amplitude $A_\gamma(u) \approx -e^{(\beta-\frac{1}{2})u}/\rho$ carries β twice: in the growth of its modulus and in its phase, $\arg(-A_\gamma) = -\arg \rho = -\arctan(\gamma/\beta)$. On the critical line all orbits are phase-locked, and

$$E(u) = - \sum_{\gamma>0} \frac{\cos(\gamma u) + 2\gamma \sin(\gamma u)}{\frac{1}{4} + \gamma^2},$$

so the prime signal is close to a sine series. The phase can be read in a finite window, without waiting for any growth. From the primes up to 10^8 it gives $\beta = 0.499 \pm 0.023$, 0.480 ± 0.029 and 0.505 ± 0.059 for the first three zeros of ζ , where the error is γ times the standard deviation of the measured phase along the window. The sensitivity decreases with the height, since the phase difference between β and $\frac{1}{2}$ is about $(\beta - \frac{1}{2})/\gamma$: the error is 0.23 at $\gamma = 30.4$ and 0.44 at $\gamma = 49.8$, and for the off-line zero $0.8085 + 85.699i$ of the Davenport–Heilbronn function the phase gives 0.93 ± 0.05 , while the growth rate gives 0.8057. The two readings are complementary: the phase sees β in a short window but only at low height, and the growth sees it at every height but only over a long range of u . Neither constrains a zero that has not been located.

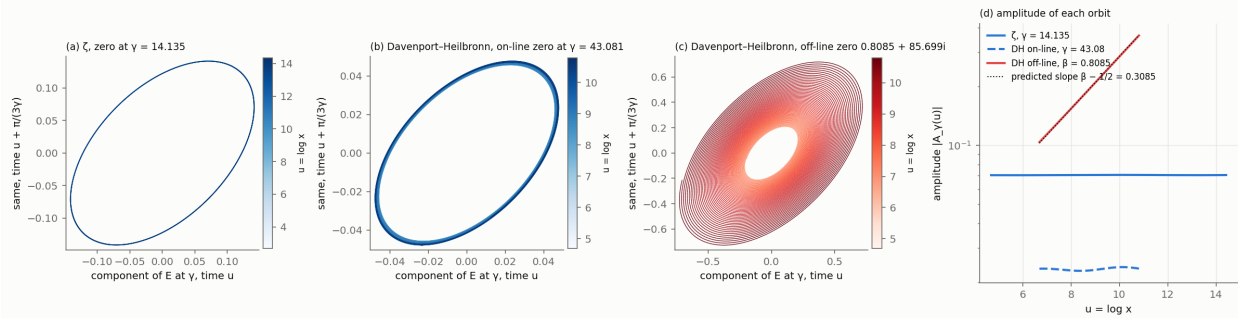


FIGURE 24. Orbits of single zeros in the prime signal: the component of E at frequency γ at time u against its value at $u + \pi/(3\gamma)$, coloured by u . (a) ζ , zero at $\gamma = 14.135$, from the primes up to 10^8 : a closed ellipse. (b) Davenport–Heilbronn function, on-line zero at $\gamma = 43.081$: a closed ellipse. (c) Its off-line zero $0.8085 + 85.699i$: an expanding spiral. (d) Amplitude $|A_\gamma(u)|$; the dotted line has the predicted slope $\beta - \frac{1}{2}$.

Spiral and harmonic. Proposition 2.3 and the orbit amplitude are two readings of the same exponential: the zero of the truncated function satisfies $\rho_n - \rho \approx n^{-\rho}/(2\zeta'(\rho))$, and the component of the prime signal at frequency γ is $h_\rho(u) = A_\gamma(u)e^{i\gamma u} \approx -e^{(\rho-\frac{1}{2})u}/\rho$. Their product at $u = \log n$ does not depend on β :

$$-2\sqrt{n}(\rho_n - \rho)h_\rho(\log n) \rightarrow \frac{1}{\rho\zeta'(\rho)},$$

the residue of $1/(s\zeta(s))$ at ρ , which is the coefficient of ρ in the explicit formula for $M(x) = \sum_{n \leq x} \mu(n)$. The integers supply $1/\zeta'(\rho)$ and the primes supply $1/\rho$. For the first three zeros of ζ the product, computed from F_n with $n \leq 2 \cdot 10^5$ and from the primes up to 10^8 , agrees with $1/(\rho\zeta'(\rho))$ to within 1–3% (Figure 25a). Summed over the zeros, the modes rebuild $(M(x) + 2)/\sqrt{x}$: with the 341 zeros with $0 < \gamma < 600$ the correlation with the values computed from $\mu(n)$ up to 10^8 is 0.995 (Figure 25b). The energy is the Möbius analogue of Theorem B: under RH, with simple zeros and a hypothesis on the sums $\sum_{0 < \gamma < T} |\zeta'(\rho)|^{-2}$, the logarithmic mean of $(M(x) + 2)^2/x$ is $2 \sum_{\gamma > 0} |\rho\zeta'(\rho)|^{-2}$ [Ng]. We obtain 0.0290 from $\mu(n)$ up to 10^8 and 0.0287 from the zeros with $\gamma < 600$. In the variable of Theorem B the constant reads $\sum_\rho \lim_n C_2(n, \rho)/|\zeta'(\rho)|^2$: the slope of the Möbius energy is the C_2 energy law with each zero weighted by $|\zeta'(\rho)|^{-2}$. With the 1517 zeros with $0 < \gamma < 2000$ and the tail $\sum_{\gamma > T} |\rho\zeta'(\rho)|^{-2} \approx 6/(\pi^3 T)$ suggested by the conjecture of Gonek and Hejhal, the constant is 0.02914, and a least-squares fit of $\int_1^X M(x)^2 x^{-2} dx = C_M \log X + K_M$ on $10^4 \leq X \leq 10^8$ gives the slope 0.02891 and, with $C_M = 0.02914$, the constant term $K_M = 1.2603 \pm 0.0023$ (mean and spread of the oscillation); a closed formula for K_M , analogous to Proposition 3.8, is not derived here. For $L(s, \chi)$ with the character mod 5 the slope on $10^3 \leq X \leq 10^6$ is 0.089, against $\sum_{|\gamma| < 400} |\rho L'(\rho)|^{-2} = 0.090$. The same signal gives Littlewood's criterion in convergence form: $\sum_{n \leq N} \mu(n)n^{-s} \rightarrow 1/\zeta(s)$ for $\text{Re } s > \sigma_0$ if and only if there are no zeros with $\beta > \sigma_0$. For the inverse of the Davenport–Heilbronn coefficients at height 85.7, the distance to $1/f(s)$ decreases at $\sigma = 0.95$ and 0.85 and grows at $\sigma = 0.75$ and 0.65 , by the factors 1.40 and 2.47 between $N = 10^3$ and $3 \cdot 10^5$, which are the factors $N^{\beta-\sigma}$ predicted by the off-line zero $\beta = 0.8085$; for ζ at $\sigma = 0.55, 0.6, 0.7, 0.9$ and height 20 it decreases, slowly near the line. For the Davenport–Heilbronn function, the summatory function of the Dirichlet inverse of its coefficients, computed up to 10^7 , has window energies that grow at the rate 0.61 in u , against $2\beta - 1 = 0.617$ for its off-line zero (Figure 25c). The product reads $1/\zeta'(\rho)$ at a located zero; it does not involve β and adds no constraint on it.

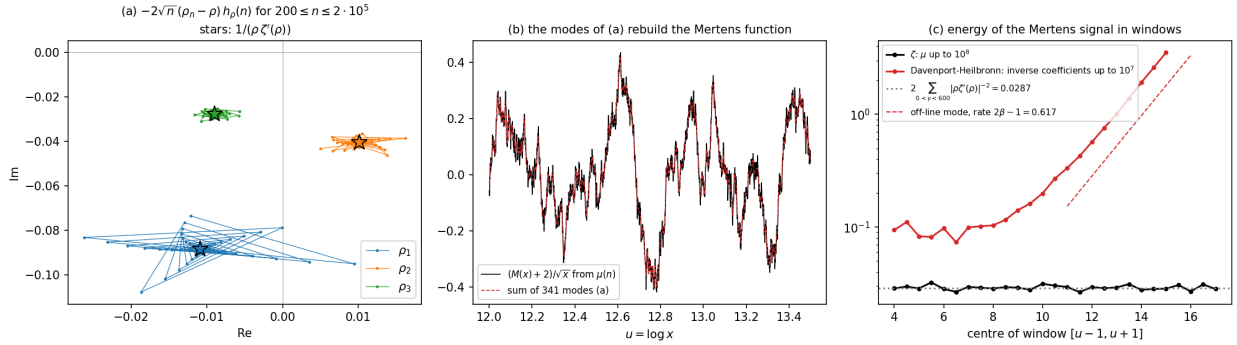


FIGURE 25. Spiral times harmonic. (a) $-2\sqrt{n}(\rho_n - \rho)h_\rho(\log n)$ for $200 \leq n \leq 2 \cdot 10^5$ at the first three zeros of ζ ; stars: $1/(\rho\zeta'(\rho))$. (b) $(M(x) + 2)/\sqrt{x}$ from $\mu(n)$ and the sum of the 341 modes. (c) Energy of the Mertens signal in windows of length 2, for ζ and for the Davenport–Heilbronn function; the dashed line has the slope $2\beta - 1 = 0.617$.

4. THE ENERGY SLOPE: A POINTWISE C_2 INEQUALITY

Theorem B is a statement about an average over $u = \log x$. This section formulates an inequality at each fixed t , describes its local structure, and determines which parts of it can be proved.

The criterion. Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and define the *energy slope*

$$S(\sigma, t) = \partial_\sigma \log |\xi(\sigma + it)| = G(\sigma, t) + \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it), \quad G(\sigma, t) = \operatorname{Re} \left[\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \psi\left(\frac{s}{2}\right) \right],$$

where $\psi = \Gamma'/\Gamma$. The term G , which we call the gamma pressure, equals $\frac{1}{2} \log \frac{t}{2\pi} + O(t^{-1})$. The term $\log |\zeta|$ is half the logarithm of the C_2 energy: for $\sigma > 1$, $|\zeta(s)|^2 = \zeta(2\sigma) + \sum_{n \geq 2} C_2(n, s)$, and for $0 < \sigma < 1$, $|\zeta(s)|$ is the radius of the spiral of Section 2. The slope can be evaluated from a finite C_2 sum: with $E_N(s) = |X_N(s) + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + \frac{s}{12}N^{-s-1}|^2$, the quantity $\frac{1}{2}\partial_\sigma \log E_N + G$ agrees with S to six digits at $N = 200$ for the points we tested ($\sigma = 0.55, 0.7, 0.9$; $t = 27.7, 100$).

Theorem 4.1 (Lagarias [La]; Sondow and Dumitrescu [SD]). RH holds if and only if $S(\sigma, t) > 0$ for all $\sigma > \frac{1}{2}$ and all real t , that is, if and only if $|\xi(\sigma + it)|$ is strictly increasing in σ on $\sigma > \frac{1}{2}$ for every fixed t .

The proof rests on the Hadamard product, which gives $S(\sigma, t) = \sum_\rho \operatorname{Re} \frac{1}{s-\rho}$ with the zeros summed in symmetric order [La]. The next lemma makes the contribution of each zero explicit.

Theorem C (disc lemma). Let $\sigma > \frac{1}{2}$. A zero on the critical line contributes $(\sigma - \frac{1}{2})/((\sigma - \frac{1}{2})^2 + (t - \gamma)^2) > 0$ to $S(\sigma, t)$. A pair of zeros $\rho = \beta + i\gamma$ and $\iota(\rho) = 1 - \beta + i\gamma$ with $\beta > \frac{1}{2}$ contributes a negative amount if and only if

$$\left(\sigma - \frac{1}{2}\right)^2 + (t - \gamma)^2 < \left(\beta - \frac{1}{2}\right)^2,$$

that is, if and only if s lies in the open disc D_ρ with centre $\frac{1}{2} + i\gamma$ passing through ρ and $\iota(\rho)$.

Proof. Put $a = \beta - \sigma$, $b = \sigma - 1 + \beta$ and $d = t - \gamma$. The pair contributes

$$-\frac{a}{a^2 + d^2} + \frac{b}{b^2 + d^2} = \frac{(a-b)(ab-d^2)}{(a^2+d^2)(b^2+d^2)}.$$

Here $a - b = 1 - 2\sigma < 0$ and $ab = (\beta - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2$, so the contribution is negative exactly when $ab > d^2$. $\square$

Corollary 4.2. If $\sigma > \frac{1}{2}$ and $s = \sigma + it$ lies in no disc D_ρ , then $S(\sigma, t) > 0$. In particular, the inequality $S > 0$ can fail only within distance $\beta - \frac{1}{2} < \frac{1}{2}$ of the ordinate of an off-line zero.

The criterion is therefore local: each off-line pair can make S negative only inside its own disc. Near an off-line zero, $S \rightarrow -\infty$ as $s \rightarrow \rho$ from the left, so no positive contribution from other zeros can compensate there. A proof of $S > 0$ inside a disc is equivalent to the absence of the zero.

Proposition 4.3 (concavity in the square of the distance to the line). For real t and $v \geq 0$ put $\ell_t(v) = \log |\xi(\frac{1}{2} + \sqrt{v} + it)|$. RH holds if and only if ℓ_t is concave on $(0, \infty)$ for every t at which it is finite. Moreover $\ell'_t(v) = S(\sigma, t)/(2(\sigma - \frac{1}{2}))$ with $\sigma = \frac{1}{2} + \sqrt{v}$, and under RH

$$\lim_{v \rightarrow 0^+} \ell'_t(v) = \frac{1}{2} \sum_\rho \frac{1}{(t - \gamma)^2}, \quad \ell'_t(v) \downarrow 0 \quad (v \rightarrow \infty).$$

Proof. By the functional equation $|\xi(\frac{1}{2} + a + it)| = |\xi(\frac{1}{2} - a + it)|$, so ℓ_t is a function of $v = a^2$. Write $\xi(\frac{1}{2} + w) = \xi(\frac{1}{2}) \prod (1 + w^2/z^2)$ over the zeros $\rho = \frac{1}{2} + iz$ with $\operatorname{Re} z > 0$, the product converging absolutely since $\sum |z|^{-2} < \infty$. For $w = a + it$ and a real $z = \gamma$, $|\gamma^2 + w^2|^2 = (v + (t - \gamma)^2)(v + (t + \gamma)^2)$, so under RH

$$2\ell_t(v) = \text{const} + \sum_{\gamma > 0} \left[\log(v + (t - \gamma)^2) + \log(v + (t + \gamma)^2) - 4 \log \gamma \right],$$

a convergent sum of concave functions of v ; the derivative at $v = 0$ is $\frac{1}{2} \sum_{\rho} (t - \gamma)^{-2}$ and every term of ℓ'_t decreases to 0. Conversely, let $z = \gamma - ib$ with $0 < b < \frac{1}{2}$, so that $\beta = \frac{1}{2} + b$. At height $t = \gamma$ the four zeros $\pm z, \pm \bar{z}$ contribute $|(z^2 + w^2)(\bar{z}^2 + w^2)|^2/|z|^8$, and $z^2 + w^2 = (z + iw)(z - iw)$ with $z + iw = i(a - b)$ and $\bar{z} + iw = i(a + b)$; the contribution contains the factor $(a^2 - b^2)^2 = (v - b^2)^2$. Hence $\ell_{\gamma}(v) \rightarrow -\infty$ as $v \rightarrow b^2$ while ℓ_{γ} is finite on both sides, which is impossible for a concave function. $\square$

The proposition combines the three structures used in this paper. The functional equation makes $v = (\sigma - \frac{1}{2})^2$ the natural variable; ℓ_t is, up to the gamma factor, half the logarithm of the C_2 energy; the Euler product makes $\ell'_t > 0$ for $\sigma > 1$; and the derivative at the line is half the energy of the Euler helix (Remark 6.5). Under RH the normalized slope $S/(2(\sigma - \frac{1}{2}))$ starts at this energy on the line and decreases to 0 without changing sign. Numerically, on $10 \leq t \leq 100$ (step 0.05) and $0 < v \leq 0.36$, the largest value of ℓ''_t computed by second differences is -0.0017 . For the completed Davenport–Heilbronn function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ the same quantity reaches $1.9 \cdot 10^5$, and it is positive exactly for $84.98 \leq t \leq 86.40$, around the height 85.699 of its off-line pair. The criterion is equivalent to RH and local in t : one off-line pair breaks the concavity only near its own height. It is a direct consequence of the Hadamard product, and is presumably known in equivalent forms (the Laguerre–Pólya property in the variable y^2 , Remark 6.8). Status: proved; the computations are numerical.

What is proved. The following statements hold unconditionally.

- $\sigma > 1$. By the Euler product, $\operatorname{Re} \frac{\xi'}{\zeta}(\sigma + it) = -\sum_n \Lambda(n) n^{-\sigma} \cos(t \log n) \geq \frac{\xi'}{\zeta}(\sigma)$. Hence $S(\sigma, t) \geq G(\sigma, t) + \frac{\xi'}{\zeta}(\sigma)$, and $S > 0$ whenever $G(\sigma, t) > -\frac{\xi'}{\zeta}(\sigma) = \frac{1}{\sigma-1} - \gamma_0 + O(\sigma-1)$. We checked numerically that $-\frac{\xi'}{\zeta}(\sigma) < \frac{1}{\sigma-1}$ on $1 < \sigma \leq 12$ (the difference is at most -0.09), so $S > 0$ for $\sigma \geq 1 + 1/G(\sigma, t)$ in that range. In the same region the Euler product bounds the spiral radius from below, $|\zeta(s)| \geq \zeta(2\sigma)/\zeta(\sigma)$, so that $\sum_{n \geq 2} C_2(n, s) \geq (\zeta(2\sigma)/\zeta(\sigma))^2 - \zeta(2\sigma)$. For $\sigma = 1.2$ and $0.5 < t < 200$ the minimum of $|\zeta|$ is 0.424, against the bound 0.247.
- *Near $\sigma = 1$.* The energy slope gives a zero-free region directly; this is the classical argument of de la Vallée Poussin written for S (Proposition 4.4 below). With the Vinogradov–Korobov region, Corollary 4.2 gives $S(\sigma, t) > 0$ for $\sigma \geq 1 - c(\log t)^{-2/3}(\log \log t)^{-1/3}$ and large t . Section 6 explains why this type of argument does not extend deeper into the strip: the pole at $s = 1$ dominates any linear inequality with nonnegative coefficients.
- *Bounded height.* All zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the critical line [PT]. By Corollary 4.2, $S(\sigma, t) > 0$ for all $\sigma > \frac{1}{2}$ and $0 < t \leq 3 \cdot 10^{12} - \frac{1}{2}$.

Proposition 4.4 (energy-slope form of the zero-free region). Let $P(\theta) = \sum_{k=0}^K a_k \cos k\theta \geq 0$ with $a_k \geq 0$ and $a_1 > a_0 > 0$. For $\sigma > 1$ and real t ,

$$\sum_{k=0}^K a_k [G(\sigma, kt) - S(\sigma, kt)] = \sum_{n \geq 2} \frac{\Lambda(n)}{n^{\sigma}} P(t \log n) \geq 0. \quad (4.1)$$

Consequently there is $t_0(P)$ such that $\zeta(\beta + i\gamma) \neq 0$ for

$$\beta > 1 - \frac{c(P) - \varepsilon}{\log |\gamma|}, \quad c(P) = \frac{2(\sqrt{a_1} - \sqrt{a_0})^2}{\sum_{k \geq 1} a_k},$$

for every $\varepsilon > 0$ and $|\gamma| \geq t_0(P, \varepsilon)$, and $S(\sigma, t) > 0$ for $\sigma \geq 1 - (c(P) - \varepsilon)/\log(|t| + 1)$ and $|t| \geq t_0 + 1$.

Proof. By definition $-\operatorname{Re} \frac{\xi'}{\zeta} = G - S$, and for $\sigma > 1$, $-\operatorname{Re} \frac{\xi'}{\zeta}(\sigma + ikt) = \sum \Lambda(n) n^{-\sigma} \cos(kt \log n)$, which gives (4.1). For $\sigma > 1$ every term of $S(\sigma, t) = \sum_{\rho} \operatorname{Re} \frac{1}{s-\rho}$ is positive, so $S(\sigma, kt) \geq 0$ for all k , and if $\rho = \beta + it$ is a zero then $S(\sigma, t) \geq \frac{1}{\sigma-\beta}$. The term $k = 0$ is $a_0[G(\sigma, 0) - S(\sigma, 0)] \leq a_0 G(\sigma, 0) =$

$\frac{a_0}{\sigma-1} + O(1)$, the pole. For $k \geq 1$, $G(\sigma, kt) = \frac{1}{2} \log |t| + O_K(1)$ by Stirling's formula. Hence

$$\frac{a_1}{\sigma - \beta} \leq \frac{a_0}{\sigma - 1} + \frac{\log |t|}{2} \sum_{k \geq 1} a_k + O_P(1).$$

Put $\sigma = 1 + \frac{\eta}{\log |t|}$ and $L = \sum_{k \geq 1} a_k$. Then $(1 - \beta) \log |t| \geq \frac{a_1 \eta}{a_0 + \eta L/2} - \eta + o(1)$, and the maximum over $\eta > 0$ of the right side is $2(\sqrt{a_1} - \sqrt{a_0})^2/L$, attained at $\eta = 2(\sqrt{a_0 a_1} - a_0)/L$. The statement on S follows from Corollary 4.2, since every disc D_ρ meeting the line of abscissa σ at height t has $|\gamma - t| < \frac{1}{2}$ and $\beta > \sigma$. $\square$

The classical choice $P = 3 + 4 \cos \theta + \cos 2\theta$ gives $c = 0.02872 = 1/34.8$. Optimizing over nonnegative cosine polynomials of degree K (numerically, by sequential quadratic programming with random starts) gives $1/c = 26.6, 18.5, 17.45, 17.27$ and 17.25 for $K = 2, 3, 4, 8, 16$. These are the constants of the elementary method; refinements of the same inequality give $1/c = 9.65$ [Ste], 5.70 [Ka], 5.573 [MoT] and, in a 2026 preprint, 4.896 [BTY]. Equation (4.1) is the form of the argument in the language of this section: a nonnegative combination of energy slopes at the heights $0, t, 2t, \dots$ is bounded by the gamma pressures and the pole, and the pole term $a_0 > 0$, which cannot be avoided, limits the method to regions of width $c/\log t$ (Section 6). Status: proved; the optimal constants for each K are numerical.

In the remaining region only a statement in measure is available.

Theorem D (exceptional set). Let $\frac{1}{2} < \sigma < 1$ and $T \geq 2$, and let $N(\sigma, T)$ be the number of zeros with $\beta > \sigma$ and $0 < \gamma \leq T$. Then

$$\text{meas}\{t \in [0, T] : S(\sigma, t) \leq 0\} \leq \sum_{\substack{\beta > \sigma \\ 0 < \gamma \leq T+1/2}} 2\sqrt{(\beta - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2} \leq 2\sqrt{\sigma(1 - \sigma)} N(\sigma, T + 1).$$

Proof. Since ζ has zeros on the critical line, the zero sum for S contains positive terms, and by Corollary 4.2 the set $\{t : S(\sigma, t) \leq 0\}$ is contained in the union of the sections of the discs D_ρ at abscissa σ . The section of D_ρ is an interval of length $2\sqrt{(\beta - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2}$ if $\beta > \sigma$ and is empty otherwise. Since $\beta < 1$, this length is less than $2\sqrt{\sigma(1 - \sigma)}$. Only zeros with $\gamma < T + \frac{1}{2}$ meet $[0, T]$. $\square$

Combined with zero-density estimates, Theorem D shows that the proportion of $t \in [0, T]$ at which the energy slope fails is $O(T^{-\theta(\sigma)+\varepsilon})$ for some $\theta(\sigma) > 0$. Selberg's bound $N(\sigma, T) \ll T^{1-(\sigma-1/2)/4} \log T$ [Sel] gives this for every $\sigma > \frac{1}{2}$. The bounds of Huxley, $N(\sigma, T) \ll T^{12(1-\sigma)/5+\varepsilon}$ [Hu], and of Guth and Maynard, $N(\sigma, T) \ll T^{30(1-\sigma)/13+\varepsilon}$ [GM], give the stronger rates $\theta(\sigma) = 1 - \frac{12}{5}(1-\sigma)$ and $\theta(\sigma) = 1 - \frac{30}{13}(1-\sigma)$ where these are positive. By contrast, functions of Davenport–Heilbronn type have $\gg T$ zeros in strips $\sigma_1 < \sigma < \sigma_2$ inside $\frac{1}{2} < \sigma < 1$ [KK], so their exceptional sets have positive proportion. The Euler product therefore makes the failures of the pointwise inequality rare, and RH is the statement that they do not occur.

The energy slope counts the zeros. Theorem D bounds the set where $S \leq 0$ pointwise; integrated in t , the energy slope counts the zeros to its right exactly. Let $\sigma < 2$ and $T_1 < T_2$, and suppose that ξ has no zeros on the boundary of $R = [\sigma, 2] \times [T_1, T_2]$. Since $\log \xi$ is holomorphic away from the zeros, $\partial_\sigma \log |\xi| = \partial_t \arg \xi$, so $\int_{T_1}^{T_2} S(\sigma, t) dt$ is the variation of $\arg \xi$ along the left side of R , and the argument principle on R gives

$$\int_{T_1}^{T_2} S(\sigma, t) dt = \arg_h \xi(\sigma + iT_2) - \arg_h \xi(\sigma + iT_1) - 2\pi N(\sigma; T_1, T_2),$$

where $\arg_h$ is continued along the horizontal sides from $\sigma = 2$ and $N(\sigma; T_1, T_2)$ counts the zeros with $\beta > \sigma$ and $T_1 < \gamma < T_2$. Each zero to the right of σ lowers the integrated energy slope by exactly 2π , and the boundary terms are $O(\log T_2)$ [T, §9.4]. Integrating in σ gives Littlewood's lemma, used

in the proof of Proposition 4.5 below; in the form $2\pi \sum_{\beta > \sigma} \min(\beta - \sigma, d)$ it also measures how far the zeros lie to the right of σ . The identity holds for every function holomorphic on R ; we checked it with ζ and with f , by numerical integration of $\operatorname{Re}(f'/f)$ and continuation of the argument from $\sigma = 3$: for ζ , $N = 4.6 \cdot 10^{-14}$ at $\sigma = 0.6$, $10 \leq t \leq 30$, and $N = 1.0$ at $\sigma = 0.4$, $13 \leq t \leq 15$ (the zero ρ_1); for f , $N = 1.0$ at $\sigma = 0.75$ and $N = -3 \cdot 10^{-10}$ at $\sigma = 0.85$, on $85 \leq t \leq 86.5$, around $\rho_f = 0.8085 + 85.6993i$. On the same rectangle the Littlewood sum $\sum_{\beta > \sigma} \min(\beta - \sigma, d)$ with $\sigma = 0.75$, $d = 0.1$ is 0.058517098, against $\beta_f - 0.75 = 0.05851718$, and the argument principle on $[0.75, 0.85] \times [85, 86.5]$ counts one zero, on $[0.55, 0.65] \times [85, 86.5]$ none. Status: the identity is the argument principle; the checks are numerical (zerocount/).

A density estimate from the square of the cross-energy. The energy $|X_N(s)|^2 = \sum_{n \leq N} n^{-2\sigma} + \sum_{2 \leq n \leq N} C_2(n, s)$ is a sum of squares, and its mean over t keeps only the diagonal: each $C_2(n, s) = 2n^{-\sigma} \sum_{j < n} j^{-\sigma} \cos(t \log \frac{n}{j})$ has mean 0 over long ranges of t . This quadratic positivity has no pole term, and it is the source of the density theorems used in Theorem D. Its simplest consequence is the following.

Proposition 4.5 (Bohr–Landau, in energy form). Let $\frac{1}{2} < \sigma < 1$ and let $N(\sigma; T, 2T)$ be the number of zeros with $\beta > \sigma$ and $T < \gamma \leq 2T$. Then

$$N(\sigma; T, 2T) \leq \frac{\log \zeta(\sigma + \frac{1}{2})}{2\pi(\sigma - \frac{1}{2})} T + o(T),$$

so that the proportion of zeros with $\beta > \sigma$ among those with $T < \gamma \leq 2T$ is $O(1/((\sigma - \frac{1}{2}) \log T))$.

Proof. Let $\sigma_0 = \frac{1}{2}(\sigma + \frac{1}{2})$. Littlewood’s lemma applied to ζ on the rectangle $[\sigma_0, 2] \times [T, 2T]$ gives

$$2\pi \int_{\sigma_0}^2 N(u; T, 2T) du = \int_T^{2T} \log |\zeta(\sigma_0 + it)| dt - \int_T^{2T} \log |\zeta(2 + it)| dt + O(\log T),$$

the error coming from $\arg \zeta$ on the horizontal sides, which is $O(\log T)$ [T, §9.4]. The second integral is $O(1)$, since $\log \zeta(2 + it) = \sum \Lambda(n) n^{-2-it} / \log n$. By the concavity of the logarithm,

$$\int_T^{2T} \log |\zeta(\sigma_0 + it)| dt \leq \frac{T}{2} \log \left(\frac{1}{T} \int_T^{2T} |\zeta(\sigma_0 + it)|^2 dt \right) = \frac{T}{2} \log \zeta(2\sigma_0) + o(T),$$

by the mean value $\frac{1}{T} \int_T^{2T} |\zeta(\sigma_0 + it)|^2 dt \rightarrow \zeta(2\sigma_0)$ for $\sigma_0 > \frac{1}{2}$ [T, Thm 7.2], which is the vanishing of the mean of the cross terms C_2 . Since $N(u; T, 2T)$ is nonincreasing in u , the left side is at least $2\pi(\sigma - \sigma_0)N(\sigma; T, 2T)$, and $2\sigma_0 = \sigma + \frac{1}{2}$, $\sigma - \sigma_0 = \frac{1}{2}(\sigma - \frac{1}{2})$. The proportion follows from $N(2T) - N(T) \sim \frac{T}{2\pi} \log T$. $\square$

This is the classical theorem of Bohr and Landau, written with the energy: the zeros off the line have density zero among all zeros because the mean energy at $\sigma_0 > \frac{1}{2}$ is finite, while the energy at $\sigma = \frac{1}{2}$, where $\zeta(2\sigma)$ has its pole, grows like $\log T$. Numerically, on $1000 \leq t \leq 2000$ the mean of $\sum_{2 \leq n \leq 200} C_2(n, \sigma + it)$ is 0.0012, 0.0005, -0.0004 for $\sigma = 0.6, 0.75, 0.9$, against $\sum_{n \leq 200} n^{-2\sigma} = 3.86, 2.47, 1.86$, and $\frac{1}{T} \int_T^{2T} |\zeta|^2 dt = 4.10, 2.52, 1.87$ approaches $\zeta(2\sigma) = 5.59, 2.61, 1.88$ slowly, the deficit at $\sigma = 0.6$ being the tail $\sum_{n > t/2\pi} n^{-2\sigma}$. At $T = 1000$ the bound of Proposition 4.5 is 3755, 971 and 451 for these σ , against 868 zeros in $[T, 2T]$; it becomes nontrivial only for large T , as the factor $1/\log T$ shows. The stronger estimates cited after Theorem D use higher moments and mollifiers, but the mechanism is the same quadratic positivity, which can show that zeros off the line are rare and cannot show that there are none: a single off-line zero changes the mean energy over $[T, 2T]$ by an amount that the average does not see. Status: classical, proved; the computations are numerical.

Remark 4.6 (mollified energy and the Jensen gap). The bound of Proposition 4.5 is $T \log \bar{\mathcal{E}}$ divided by $2\pi(\sigma - \sigma_0)$, where $\bar{\mathcal{E}}$ is the mean energy, and it improves when the energy is lowered by

a mollifier, as in Selberg's method. Let

$$M_y(s) = \sum_{d \leq y} \mu(d) d^{-s} \left(1 - \frac{\log d}{\log y}\right),$$

a smoothed truncation of $1/\zeta$; in the language of Section 6 this sieves the wave by the primes up to y (Remark 6.10). The zeros of ζM_y include those of ζ , so the proof of Proposition 4.5 applied to ζM_y gives

$$N(\sigma; T, 2T) \leq \frac{T}{2\pi(\sigma - \sigma_0)} \log \left(\frac{1}{T} \int_T^{2T} |\zeta M_y(\sigma_0 + it)|^2 dt \right) + o(T).$$

On $1000 \leq t \leq 2000$, which contains 868 zeros, the mean of $|\zeta M_y|^2$ at $\sigma_0 = 0.625$ is 3.72, 1.80, 1.34, 1.19 for $y = 1, 10, 100, 1000$, and the bound for $\sigma = 0.75$ is 1672, 745, 376, 224; at $\sigma_0 = 0.55$ the means are 5.12, 2.33, 1.66, 1.42 and the bound for $\sigma = 0.6$ is still above 868. What limits the method is the gap in Jensen's inequality. The mean of $\log |\zeta M_y(\sigma_0 + it)|$ is 0 to within 10^{-3} , since there are no zeros to the right of σ_0 in this range, and the log of the mean of $|\zeta M_y|^2$ is close to twice the variance of $\log |\zeta M_y|$: the variances are 0.70, 0.34, 0.18, 0.11, and $e^{2\text{var}} = 4.05, 1.99, 1.43, 1.24$. To prove that there is no zero in $[T, 2T]$ the variance would have to be below $\pi(\sigma - \sigma_0)/T$, about 10^{-4} here. Heuristically, $\log |\zeta M_y|$ behaves like $\sum_{p > y} p^{-\sigma_0} \cos(t \log p + \dots)$, whose variance is about $\frac{1}{2} \sum_{p > y} p^{-2\sigma_0} \asymp y^{1-2\sigma_0}/\log y$ (the measured values are larger by a factor between 1 and 3, the smoothing of M_y being imperfect). A variance of order $1/T$ needs $y \gtrsim T^{1/(2\sigma_0-1)}$, longer than T , while the mean value theorems on which the method rests control mollifiers only up to length T . With $y = T$ the heuristic gives $N(\sigma, T) \ll T^{2-2\sigma+\varepsilon}$, the form of the density hypothesis, which is known only in part of the strip. Quadratic means of the cross-energy, however mollified, can therefore show that off-line zeros are rare but not that they are absent; the obstruction is the variance of $\log |\zeta M_y|$, which a mollifier of admissible length cannot make smaller than about $T^{1-2\sigma_0}$. Status: the inequality is proved (Littlewood's lemma and Jensen's inequality); the variance law and the resulting exponent are heuristic; the computations are numerical.

Numerical results (Figure 26). For ζ we computed $S(\sigma, t)$ by numerical differentiation of $\log |\xi|$ on $5 \leq t \leq 600$ (step 0.01) and checked it against the zero sum (3.4526 against 3.4520 at $\sigma = 0.7$, $t = 14$). For $t \geq 20$ the minimum over t is attained at $t \approx 27.7$, in the gap between the zeros 25.011 and 30.425, for every σ tested:

σ	0.51	0.55	0.6	0.7	0.8	0.9	1.0
$\min_t S(\sigma, t)$	0.0039	0.0195	0.0389	0.0776	0.1159	0.1534	0.1901
$\max_t (-\text{Re } \frac{\zeta'}{\zeta})/G$	0.995	0.974	0.948	0.895	0.844	0.793	0.744

The margin is linear near the line, $\min_t S \approx 0.39(\sigma - \frac{1}{2})$, because $S(\frac{1}{2}, t) = 0$ by the functional equation. The second row shows that the Euler-product term uses almost all of the gamma pressure near $\sigma = \frac{1}{2}$. For the Davenport–Heilbronn function f of [C5] we used the completed function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ and its eight off-line zeros with $\gamma < 400$. The minimum of its slope over $10 \leq t \leq 400$ is negative for every $\sigma \leq 0.85$, with values between -30 and -273 , and positive for $\sigma \geq 0.9$, beyond its largest $\beta = 0.8695$. Near $0.8085 + 85.699i$, 99.6% of the grid points with negative slope lie in the disc of Theorem C, and 91% of the disc is negative. The failure set at abscissa σ consists of exactly one interval for each off-line zero with $\beta > \sigma$:

σ	0.55	0.65	0.75	0.85
measure of $\{t \leq 400 : S < 0\}$	3.220	2.295	1.315	0.205
bound of Theorem D (sum of chords)	3.568	2.589	1.498	0.237
number of intervals = zeros with $\beta > \sigma$	8	6	4	1

σ	0.55	0.65	0.75	0.85
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The bound of Theorem D is attained up to a factor of about 0.9 in this example.

An Euler-product twin. The Davenport–Heilbronn function is $f = c_1 L(s, \chi) + c_2 L(s, \bar{\chi})$ with χ the character mod 5 of Section 2, so $L(s, \chi)$ has the same conductor, the same gamma factor and a functional equation of the same type, and differs from f only by its Euler product. Since the zeros of $L(s, \chi)$ are symmetric under $\rho \mapsto 1 - \bar{\rho}$, the proofs of Theorems 4.1, C and D apply to $S_\chi = \partial_\sigma \log |(5/\pi)^{(s+1)/2} \Gamma(\frac{s+1}{2}) L(s, \chi)|$ without change. On $10 \leq |t| \leq 400$ we find $\min_t S_\chi = 0.0068, 0.034, 0.068, 0.135$ at $\sigma = 0.51, 0.55, 0.6, 0.7$, that is, about $0.68(\sigma - \frac{1}{2})$, and no point with $S_\chi < 0$. On the same range the slope of f is negative on sets of measure 6.7, 6.5, 5.6 and 3.8. The Euler-product term of $L(s, \chi)$ uses a fraction $\max_t (-\operatorname{Re} \frac{L'}{L})/G = 0.994$ of the gamma pressure at $\sigma = 0.51$ and 0.968 at $\sigma = 0.55$, as for ζ (0.995 and 0.974). The absence of a pole does not create slack: the tightness near the line is set by the density of the zeros.

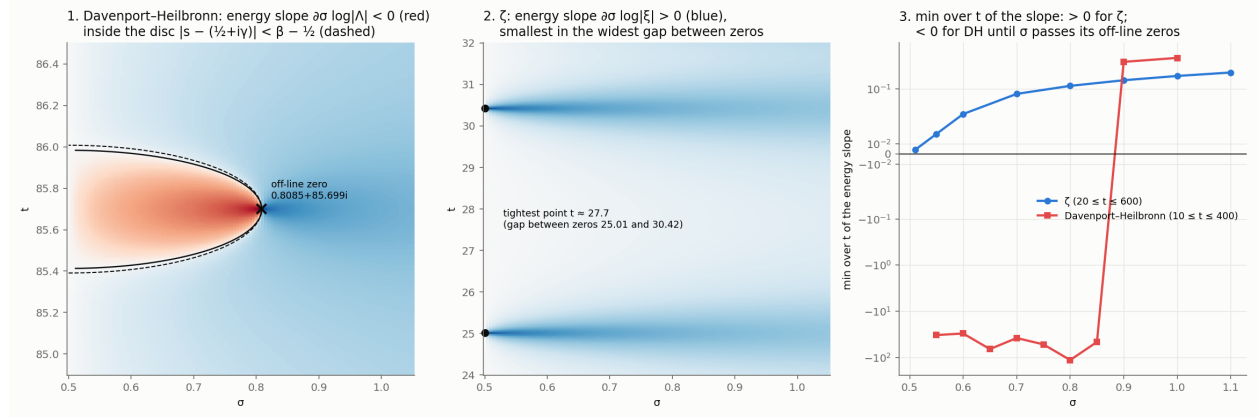


FIGURE 26. The energy slope $S(\sigma, t)$. Left: Davenport–Heilbronn function near its zero $0.8085 + 85.699i$; $S < 0$ (red) inside the disc of Theorem C (dashed), zero level solid. Centre: ζ near $t = 27.7$, where the minimum over t is attained; $S > 0$ throughout. Right: $\min_t S(\sigma, t)$ for ζ ($20 \leq t \leq 600$) and for the Davenport–Heilbronn function ($10 \leq t \leq 400$), symmetric logarithmic scale.

The margin near the critical line and random matrices. Write $\Xi(t) = \xi(\frac{1}{2} + it)$, which is real. Expanding in $\sigma - \frac{1}{2}$ gives $S(\sigma, t) = (\sigma - \frac{1}{2}) L(t) + O((\sigma - \frac{1}{2})^2)$ with $L = (\Xi'^2 - \Xi\Xi'')/\Xi^2 = -(\Xi'/\Xi)'$. Under RH, $L(t) = \sum_\gamma (t - \gamma)^{-2}$. The inequality $\Xi'^2 - \Xi\Xi'' \geq 0$ is the first Laguerre inequality; it is implied by RH, and the full family of generalized Laguerre inequalities is equivalent to RH [CV]. The minimum of L over a gap between consecutive zeros measures how close S comes to 0 there. We unfolded the 341 zeros with $0 < \gamma < 600$ by the Riemann–von Mangoldt formula, computed $M_j = \min_u \sum_{|k-j| \leq 20} (u - x_k)^{-2}$ over each gap (x_j, x_{j+1}) , and compared the 300 values with 6360 values from 40 bulk spectra of 400×400 GUE matrices and with a Poisson sequence (Figure 27). The empirical quantiles at 1%, 5%, 25% and 50% are 3.86, 4.53, 7.11, 10.68 for ζ and 3.39, 4.28, 6.88, 10.75 for GUE. The Kolmogorov–Smirnov test gives $p = 0.35$ against GUE and $p = 3 \cdot 10^{-34}$ against Poisson. The margin is determined by the gap: M_j has correlation 0.9998 with Δ_j^{-2} , and an isolated gap of length Δ gives $8/\Delta^2$. The tightest points of the inequality are therefore the largest gaps, and their distribution is the large-gap distribution predicted by random matrix theory [Mo, Od].

Remark 4.7 (statistical mechanics). For real $\beta > 1$, $\zeta(\beta) = \sum_n e^{-\beta \log n}$ is the partition function of a system with energy levels $\log n$ [Ju]; Bost and Connes realize it as a quantum statistical system

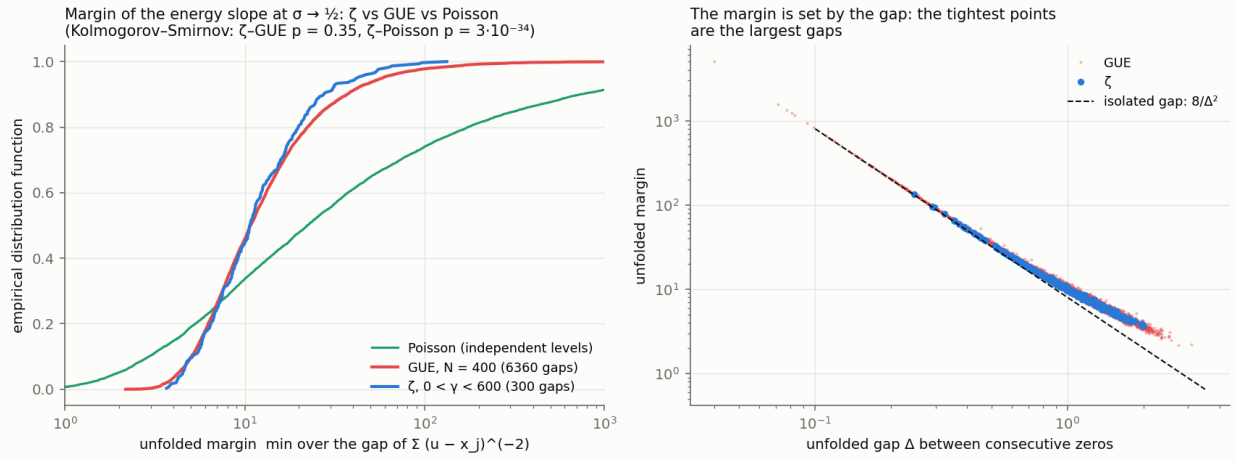


FIGURE 27. Left: empirical distribution of the unfolded margin M_j for ζ (300 gaps), GUE (6360 gaps) and a Poisson sequence. Right: M_j against the unfolded gap Δ_j ; the dashed line is $8/\Delta^2$.

with a phase transition at $\beta = 1$ [BC]. In this language $-\zeta'(\beta)$ is the mean energy, the bound in the case $\sigma > 1$ above compares the mean energy with the gamma pressure, and the zeros of ζ are the zeros of the continued partition function at complex temperature. The Lee–Yang theorem places the zeros of ferromagnetic Ising partition functions on a circle by means of a positivity property that is preserved under products and limits [LY]. The analogous property for ζ would have to hold in $\frac{1}{2} < \sigma < 1$, which lies beyond the transition at $\beta = 1$, where the partition sum diverges; this is the pole barrier of Section 6 in another form. The heat-flow deformation H_λ of Ξ studied by de Bruijn and Newman [Ne] has only real zeros for $\lambda \geq \Lambda$, RH is equivalent to $\Lambda \leq 0$, and $0 \leq \Lambda \leq 0.2$ [RT, PT]. Under the backward flow, zeros leave the real line first at pairs of unusually close zeros (Lehmer pairs). These are the points where the margin M_j is largest, not smallest, so the heat-flow picture and the energy-slope margin measure different aspects of the zero distribution.

5. THE ENERGY SLOPE AS A WEIL FUNCTIONAL

5.1. The explicit formula and Weil’s criterion. Let $h(r) = \int g(u)e^{iru} du$ with g smooth and compactly supported. For a zero ρ put $\gamma_\rho = -i(\rho - \frac{1}{2})$, so that γ_ρ is real exactly when ρ lies on the critical line. The Riemann–Weil explicit formula [We2], [IK, Thm. 5.12] reads

$$\begin{aligned} \sum_{\rho} h(\gamma_\rho) &= h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) - g(0) \log \pi + \frac{1}{2\pi} \int h(r) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) dr \\ &\quad - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (g(\log n) + g(-\log n)). \end{aligned} \quad (5.1)$$

For the Davenport–Heilbronn function the completed function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ is entire and symmetric under $s \mapsto 1-s$; its explicit formula has no pole terms, the constant is $+g(0) \log(5/\pi)$, the gamma term is $\operatorname{Re} \psi(\frac{3}{4} + \frac{ir}{2})$, and $\Lambda(n)$ is replaced by the coefficients $b(n)$ of $-f'/f$ [C5]. With $h(r) = e^{-(r/a)^2}$, $2 \leq a \leq 5$, the two sides of each formula agree to 10^{-14} for ζ (682 zeros) and to 10^{-16} for f [C7].

Take $g = \varphi * \tilde{\varphi}$, $\tilde{\varphi}(u) = \overline{\varphi(-u)}$, so that $h = |\hat{\varphi}|^2$ on the real line, and write $W(\varphi)$ for the right side of (5.1). Under RH every γ_ρ is real and $W(\varphi) = \sum_{\rho} |\hat{\varphi}(\gamma_\rho)|^2 \geq 0$. Weil proved the converse: RH holds if and only if $W(\varphi) \geq 0$ for every such φ [We2, Bo]. If φ is supported in an interval of

length L , only $n \leq e^L$ enters, and by Parseval the prime part of W is the cross-energy of φ with its translates by $\log p^k$, weighted by $\Lambda(p^k)p^{-k/2}$ [C7]:

$$W(\varphi) = 2 \operatorname{Re} \left[\hat{\varphi}\left(\frac{i}{2}\right) \overline{\hat{\varphi}\left(-\frac{i}{2}\right)} \right] + \frac{1}{2\pi} \int |\hat{\varphi}(r)|^2 \left(\operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) - \log \pi - 2 \operatorname{Re} \sum_{n \leq e^L} \frac{\Lambda(n)}{n^{1/2+ir}} \right) dr.$$

5.2. The Poisson kernel and its truncation. Theorem E. Let $\sigma > \frac{1}{2}$, $\delta = \sigma - \frac{1}{2}$ and $h_{\delta,t}(r) = \delta/(\delta^2 + (r-t)^2)$.

- (a) $S(\sigma, t) = \sum_{\rho} h_{\delta,t}(\gamma_{\rho})$, with the zeros summed in symmetric order. The function $h_{\delta,t}$ is the Fourier transform of the positive-definite function $g_{\delta,t}(u) = \frac{1}{2}e^{-\delta|u|}e^{-itu}$. For an off-line pair, $h_{\delta,t}(\gamma_{\rho}) + h_{\delta,t}(\bar{\gamma}_{\rho}) = 2 \operatorname{Re} h_{\delta,t}(\gamma_{\rho})$, which is negative exactly in the disc of Theorem C.
- (b) For $L > 0$ let $\varphi_L(u) = \sqrt{2\delta} e^{-\delta u} e^{-itu} \mathbf{1}_{[0,L]}(u)$ and $g_L = \varphi_L * \tilde{\varphi}_L$. Then $g_L(u) = e^{-itu}(e^{-\delta|u|} - e^{-\delta(2L-|u|)})$ for $|u| \leq L$ and $g_L = 0$ otherwise, and its Fourier transform is the entire function

$$h_L(r) = \frac{2\delta (1 - e^{-\delta L} e^{i(r-t)L})(1 - e^{-\delta L} e^{-i(r-t)L})}{\delta^2 + (r-t)^2}.$$

Let $W_L(\sigma, t) = \sum_{\rho} h_L(\gamma_{\rho}) = W(\varphi_L)$. Then:

1. $W_L(\sigma, t)$ is given exactly by (5.1) with the prime powers $n \leq e^L$:

$$\begin{aligned} W_L &= h_L\left(\frac{i}{2}\right) + h_L\left(-\frac{i}{2}\right) - g_L(0) \log \pi + \frac{1}{2\pi} \int h_L(r) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) dr \\ &\quad - 2 \sum_{n \leq e^L} \frac{\Lambda(n)}{\sqrt{n}} \operatorname{Re} g_L(\log n). \end{aligned}$$

2. If all zeros lie on the critical line, then $h_L(\gamma) = 2h_{\delta,t}(\gamma) |1 - e^{-\delta L} e^{i(\gamma-t)L}|^2$ for every zero, and therefore

$$(1 - e^{-\delta L})^2 \leq \frac{W_L(\sigma, t)}{2S(\sigma, t)} \leq (1 + e^{-\delta L})^2.$$

3. A zero with $\beta > \sigma$ contributes to $W_L(\sigma, t)$ a term of size $e^{(\beta-\sigma)L}$, at every t , with a phase that rotates with L .

Proof. (a) By the Hadamard product, $S = \sum_{\rho} \operatorname{Re} \frac{1}{s-\rho}$ in symmetric order [La], and $s - \rho = \delta + i(t - \gamma_{\rho})$. The maps $\rho \mapsto 1 - \bar{\rho}$ and $\rho \mapsto 1 - \rho$ act on γ_{ρ} as $\gamma \mapsto \bar{\gamma}$ and $\gamma \mapsto -\gamma$, so the multiset $\{\gamma_{\rho}\}$ is invariant under conjugation. Hence

$$\sum_{\rho} \operatorname{Re} \frac{1}{\delta + i(t - \gamma_{\rho})} = \frac{1}{2} \sum_{\rho} \left[\frac{1}{\delta + i(t - \gamma_{\rho})} + \frac{1}{\delta - i(t - \gamma_{\rho})} \right] = \sum_{\rho} h_{\delta,t}(\gamma_{\rho}).$$

The Fourier transform of $g_{\delta,t}$ is elementary, and the last assertion is Theorem C written in the variable γ_{ρ} .

- (b) The formula for g_L follows by direct integration, and $h_L = |\hat{\varphi}_L|^2$ on the real line, which gives the displayed expression and its continuation. Since g_L is compactly supported and h_L is entire, (5.1) applies, with $g_L(\log n) + g_L(-\log n) = 2 \operatorname{Re} g_L(\log n)$; this is (1). For real γ the factor $|1 - e^{-\delta L} e^{i(\gamma-t)L}|^2$ lies between $(1 - e^{-\delta L})^2$ and $(1 + e^{-\delta L})^2$, which with (a) gives (2). For $\gamma_{\rho} = \gamma - i(\beta - \frac{1}{2})$ the factor $e^{-\delta L} e^{i(\gamma_{\rho}-t)L}$ has modulus $e^{(\beta-\sigma)L}$, which gives (3). $\square$

Part (a) identifies the pointwise inequality $S > 0$ of Theorem 4.1 with Weil positivity along the family $h_{\delta,t}$, and the disc of Theorem C with the region where $h_{\delta,t}$ is negative at complex arguments. The test function $g_{\delta,t}$ is not admissible in (5.1): $h_{\delta,t}$ has poles at $r = t \pm i\delta$, inside the strip $|\operatorname{Im} r| < \frac{1}{2}$, and the prime sum $\sum \Lambda(n)n^{-1/2}g_{\delta,t}(\log n)$ converges only for $\delta > \frac{1}{2}$, that is, for $\sigma > 1$. This is the pole barrier again. Part (b) is the admissible version. It contains no remainder term:

W_L is an exact finite expression in the primes up to e^L . Under RH it is positive for every L and determines S to relative precision about $2e^{-\delta L}$, so recovering the sign of S at $\sigma = \frac{1}{2} + \delta$ with a fixed margin requires the primes up to $e^{C/\delta}$.

Corollary 5.1 (prime-resolution scale). Assume RH. Let $0 < \varepsilon < 1$, $\sigma = \frac{1}{2} + \delta$ and $P \geq 2$. If

$$\delta \log P \geq \log \frac{3}{\varepsilon},$$

then the finite quantity $W_{\log P}(\sigma, t)$, which involves only the prime powers $n \leq P$, satisfies, for every t ,

$$\left| \frac{W_{\log P}(\sigma, t)}{2S(\sigma, t)} - 1 \right| \leq \varepsilon.$$

Equivalently, primes up to P determine S to relative precision ε at every distance δ from the critical line with $\delta \geq \delta_{\text{res}}$, where $\delta_{\text{res}} = \delta_{\text{res}}(P, \varepsilon) = \log(3/\varepsilon)/\log P$.

Proof. By Theorem E(b2), $|W_L/2S - 1| \leq \max\{(1+a)^2 - 1, 1 - (1-a)^2\} = 2a + a^2 \leq 3a$ with $a = e^{-\delta L} \leq 1$. Take $L = \log P$. $\square$

For example, primes up to 10^8 give $\delta_{\text{res}} = 0.18$ at $\varepsilon = 0.1$ and 0.31 at $\varepsilon = 0.01$; resolving $\delta = 0.01$ to 1% requires $P \geq 10^{248}$. The corollary is conditional, and this cannot be removed by a better estimate. An unconditional bound for $W_L - 2S$ at abscissa σ must control the terms $e^{(\beta-\sigma)L}$ of Theorem E(b3), that is, it requires knowing that no zero lies to the right of σ in the relevant range of heights. Such knowledge is available only inside the classical zero-free region and below the height $3 \cdot 10^{12}$ to which RH has been verified [PT], where $S > 0$ already follows from Corollary 4.2. A finite certificate of the form “ $W_L > B_L \Rightarrow S > 0$ ” therefore certifies nothing beyond what is known.

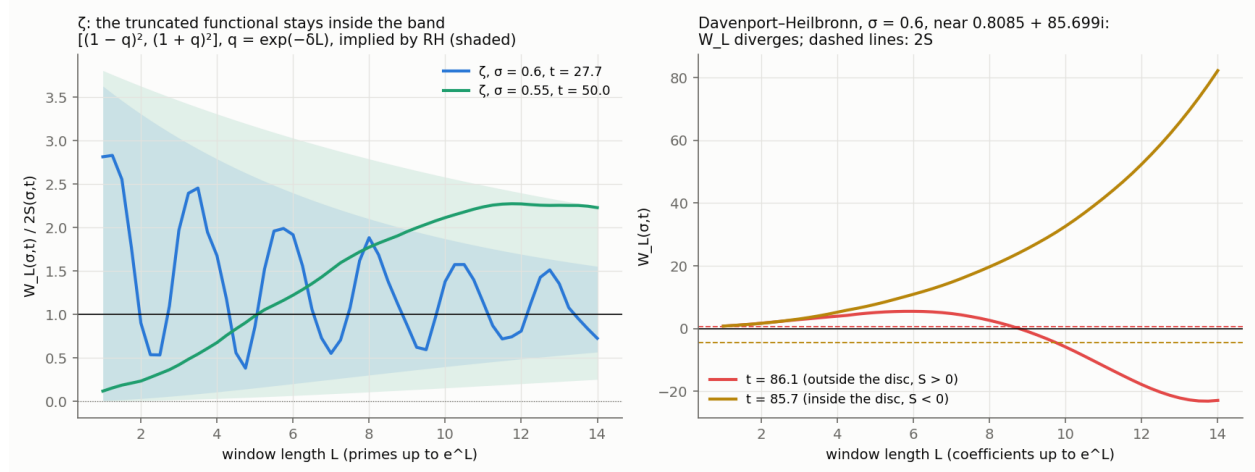


FIGURE 28. The truncated Poisson functional $W_L(\sigma, t)$ of Theorem E(b), computed from the primes (or coefficients) up to e^L . Left: ζ ; the ratio $W_L/2S$ and the band $[(1 - e^{-\delta L})^2, (1 + e^{-\delta L})^2]$ implied by RH. Right: Davenport–Heilbronn function at $\sigma = 0.6$, inside ($t = 85.7$) and outside ($t = 86.1$) the disc of its zero $0.8085 + 85.699i$; dashed lines show $2S$.

5.3. Numerical results. The truncated functional. We computed W_L from the primes and, independently, from the zeros. The two agree to within $3 \cdot 10^{-3}$, both for ζ and for the Davenport–Heilbronn function with $b(n)$ in place of $\Lambda(n)$; the difference is the contribution of the zeros beyond the computed range. For ζ at $(\sigma, t) = (0.6, 27.7)$ and $(0.55, 50)$, $W_L/2S$ stays inside the band of (b2) for $1 \leq L \leq 14$ (Figure 28, left). For the Davenport–Heilbronn function at $\sigma = 0.6$ (Figure 28,

right) the behaviour is the opposite of the disc picture. At $t = 85.7$, inside the disc, where $S < 0$, W_L is positive and grows exponentially, reaching 82 at $L = 14$. At $t = 86.1$, outside the disc, where $S > 0$, W_L becomes negative at $L = 9$ and reaches -23 at $L = 13.75$. The truncated functional therefore detects the off-line zero from the coefficients, but not locally: by (b3) every zero with $\beta > \sigma$ affects $W_L(\sigma, t)$ at every t once L is large. The locality of Theorem C holds only in the limit $L \rightarrow \infty$, and that limit is what the Euler product does not control in $\frac{1}{2} < \sigma < 1$.

Detection length. The same mechanism, read from the side of an off-line zero, can be measured on f , whose zeros are known. For each off-line zero ρ of f with $\gamma < 400$ we computed, at $\sigma = 0.55$, the smallest L (step 0.25) at which the contribution of the pair $\rho, 1 - \bar{\rho}$ to $W_L(\sigma, t)$ exceeds in modulus the contribution of all zeros on the critical line, for some t with $8 \leq |t - \gamma| \leq 12$, that is, outside the disc of Theorem C:

β	γ	$\beta - \sigma$	L_{det}	$L_{\text{det}}(\beta - \sigma)$
0.8695	240.40	0.320	16.5	5.3
0.8195	320.88	0.270	16.0	4.3
0.8085	85.70	0.259	19.3	5.0
0.7682	331.05	0.218	18.5	4.0
0.7243	176.70	0.174	22.8	4.0
0.6508	114.16	0.101	49.3	5.0
0.6285	366.64	0.079	69.5	5.5
0.5744	166.48	0.024	167.0	4.1

A least-squares fit gives $L_{\text{det}} \approx 4.05/(\beta - \sigma) + 4.5$, with correlation 0.993 against $1/(\beta - \sigma)$ (0.986 against $1/(\beta - \frac{1}{2})$). The product $L_{\text{det}}(\beta - \sigma)$ stays between 4.0 and 5.5, its variation coming from the logarithm of the kernel factor $2\delta/|\delta^2 + (\gamma_\rho - t)^2|$ in (b3). The detection is not local in t : at finite L the zero is seen at heights well outside its disc. The zero at $\beta - \sigma = 0.024$ requires $L \approx 167$, that is, the coefficients up to $e^{167} \approx 10^{72}$.

Weil matrices on windows. The infimum of W over test functions on a window,

$$\lambda_{\min}(L) = \min_{\text{supp } \varphi \subset [-L/2, L/2]} \frac{W(\varphi)}{\|\varphi\|_2^2},$$

involves only $n \leq e^L$. We computed it with the basis $\varphi_m(u) = \sin(m\pi(u + L/2)/L)$, $1 \leq m \leq [10 + 12L]$, in two independent ways: from the explicit side of (5.1), and from the zero side, as the smallest singular value squared of $(\hat{\varphi}_m(\gamma_\rho))$ over the zeros with $|\gamma| < 600$, scaled by the Gram matrix [C7].

L	from the primes	from the zeros
0.25	$3.305 \cdot 10^{-1}$	$3.305 \cdot 10^{-1}$
0.50	$3.719 \cdot 10^{-2}$	$3.718 \cdot 10^{-2}$
0.75	$5.798 \cdot 10^{-4}$	$5.798 \cdot 10^{-4}$
1.00	$1.079 \cdot 10^{-6}$	$1.079 \cdot 10^{-6}$
1.25	$3.47 \cdot 10^{-10}$	$3.40 \cdot 10^{-10}$
1.50	—	$6.3 \cdot 10^{-15}$
2.00	—	$1.7 \cdot 10^{-26}$
2.50	—	$1.4 \cdot 10^{-36}$

The dashes mark values below the double-precision floor of the explicit side. The data fit $\log_{10} \lambda_{\min}(L) \approx -6.3L^2$ (Figure 35, left). These values belong to a fixed finite basis, and enlarging the basis can only lower the smallest eigenvalue. With $K = 42, 51, 60$ cosine functions $\cos(m\pi x/L)$, m odd, and 70 to 120 digits, the explicit side gives $\lambda_{\min} = 8.0 \cdot 10^{-30}, 3.9 \cdot 10^{-54}$

and $6.5 \cdot 10^{-82}$ at $L = 2$, 2.5, 3, and $3.0 \cdot 10^{-10}$ at $L = 1.25$, which fit $\log_{10} \lambda_{\min} \approx -5.0 L^{2.53}$ (Figure 32a). The law $10^{-6.3L^2}$ is therefore an upper bound for the true margin, which decreases at least like $10^{-5.0L^{2.53}}$ on $1.25 \leq L \leq 3$. The transform of the minimizer vanishes near the zeros of ζ : at the first 2 for $L = 1$ and the first 11 for $L = 2$. The primes create the positivity. At $L = 1.25$ only $p = 2, 3$ enter; without them (pole and archimedean terms only) $\lambda_{\min} = -0.571$ and the minimizer vanishes at 8.50, 13.49, 18.52, 23.56; with them $\lambda_{\min} > 0$ and the minimizer vanishes at 14.134, 21.022, 25.012, against the zeros 14.135, 21.022, 25.011 (Figure 35, centre). Positivity is therefore decided by a cancellation whose depth grows at least like $6.3L^2$ decimal digits, which explains why certified lower bounds are so small [Liu]; earlier results establish positivity on small windows [Yo, CC].

Why the margin is so small. The collapse of $\lambda_{\min}(L)$ is not caused by the primes. It is already present for $L < \log 2$, before any prime enters, and it is a property of the density of the zeros, as a comparison with curves over finite fields shows. Let E be an elliptic curve over $\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = p + 1 - a$ and $a = 2\sqrt{p} \cos \varphi$. Its zeta function depends on s only through p^{-s} , and its zeros, all on the line by Hasse's theorem, are $\frac{1}{2} + i\gamma$ with $\gamma \in \pm\varphi/\log p + \Omega\mathbb{Z}$, $\Omega = 2\pi/\log p$. We use the same test functions on $[-\frac{L}{2}, \frac{L}{2}]$ as for ζ .

Proposition 5.2. If $f \in L^2$ is supported in an interval of length $L < \log p$ and $F(r) = \int f(u)e^{iru} du$, then

$$\sum_{\gamma} |F(\gamma)|^2 = 2 \log p \|f\|_2^2,$$

the sum running over the zeros of the zeta function of E . The smallest value of Weil's form on such windows is therefore $2 \log p$, independent of L and of a .

Proof. Let $g = f * \bar{f}$, which is continuous, supported in $(-L, L)$, and satisfies $\hat{g} = |F|^2$. By Poisson summation, for each θ ,

$$\sum_{k \in \mathbb{Z}} \hat{g}(\theta + k\Omega) = \frac{2\pi}{\Omega} \sum_{n \in \mathbb{Z}} g\left(\frac{2\pi n}{\Omega}\right) e^{-2\pi i n \theta / \Omega} = \log p \sum_n g(n \log p) e^{-in\theta \log p}.$$

Since $L < \log p$, only $n = 0$ contributes, and the sum is $\log p g(0) = \log p \|f\|_2^2$. The zeros form two such progressions, $\theta = \pm\varphi/\log p$. $\square$

The terms $n \neq 0$, which enter when $L \geq \log p$, are the closed points of E at $u = n \log p$; they play the role of the primes. Numerically, the smallest value is $9.23 = 2 \log 101$ for $p = 101$ up to $L = 4$, $2.20 = 2 \log 3$ and $1.39 = 2 \log 2$ for $p = 3$ and $p = 2$ up to $L \approx \log p$ (Figure 29). The zeros of E have constant density $\log p/\pi$; those of ζ have density $\frac{1}{2\pi} \log \frac{t}{2\pi}$, which comes from the factor $\Gamma(\frac{s}{2})$, and none below height 14.13. A function supported in a window of length L can be made small on a set of points only where the points are sparser than $L/2\pi$, and it hides its mass there. Two experiments confirm that this is what happens. Removing the zeros of E (for $p = 101$) with $|t| < 14$ makes its smallest value collapse like that of ζ : $1.2 \cdot 10^{-4}$ at $L = 1$, $5.9 \cdot 10^{-11}$ at $L = 2$. Adding these zeros to those of ζ raises the value of ζ from $1.7 \cdot 10^{-26}$ to $2.0 \cdot 10^{-3}$ at $L = 2$.

The positions of the zeros of ζ play no role beyond their density. Replacing them by the points defined by $N_0(\gamma_k) = k - \frac{1}{2}$, where $N_0(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}$ is the smooth counting function, gives $\lambda_{\min} = 1.3 \cdot 10^{-6}$, $6.2 \cdot 10^{-15}$ and $1.4 \cdot 10^{-26}$ at $L = 1, 1.5, 2$, against $1.1 \cdot 10^{-6}$, $6.3 \cdot 10^{-15}$ and $1.7 \cdot 10^{-26}$ for the zeros of ζ ; random displacements of 25% of the spacing change nothing, and a Poisson process with the same density gives smaller values, because of its larger gaps. Scaling the density by c gives $\log_{10} \lambda_{\min} \approx -11.9L^2, -6.6L^2, -3.2L^2$ for $c = \frac{1}{2}, 1, 2$, that is,

$$\log_{10} \lambda_{\min}(L) \approx -6.5 L^2 / c \quad (L \leq 2)$$

(Figure 30). A heuristic for the form of this law: a function of exponential type $L/2$ has at most about LR/π zeros in $[-R, R]$, and the zeros of ζ in $[-R, R]$ number about $c \frac{R}{\pi} \log \frac{R}{2\pi e}$, so the minimizer can follow them up to the height R with $\log R \approx L/c$; if its residual beyond that height

decays like a power $R^{-\alpha L}$, then $\log \lambda_{\min} \approx -\alpha L^2/c$, and the data give $\alpha \approx 6.5 \log 10 \approx 15$. We have not proved this. Its consequence is that the size of the margin carries no arithmetic information: it is the same for points with the density of the zeros and no relation to the primes, and no argument based on it can distinguish the zeros of ζ from such points.

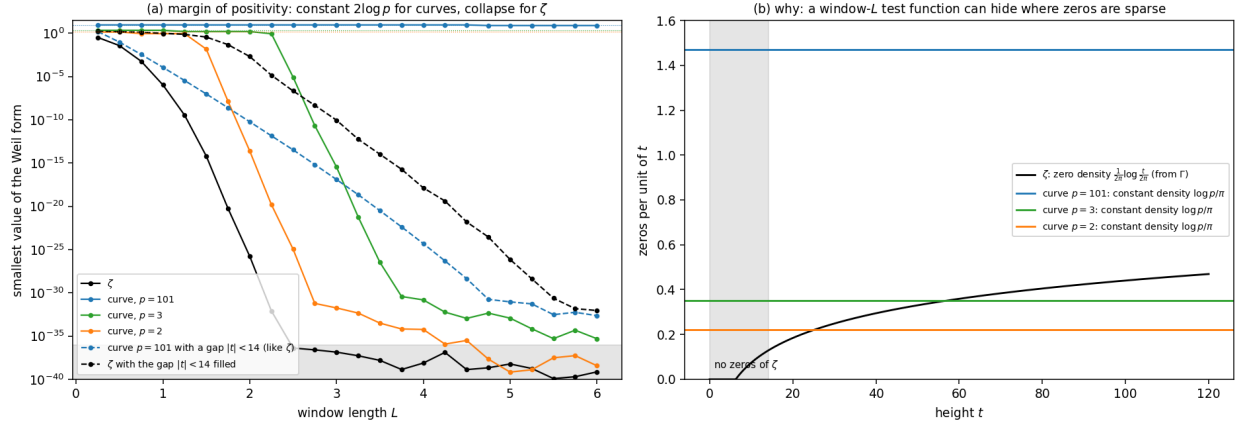


FIGURE 29. (a) Smallest value of Weil's form on windows of length L , from the zeros: ζ , curves over $\mathbb{F}_p$ (constant value $2 \log p$, Proposition 5.2), a curve with its zeros below height 14 removed, and ζ with that gap filled. (b) Density of the zeros: constant for the curves, $\frac{1}{2\pi} \log \frac{t}{2\pi}$ for ζ , with no zeros below 14.13.

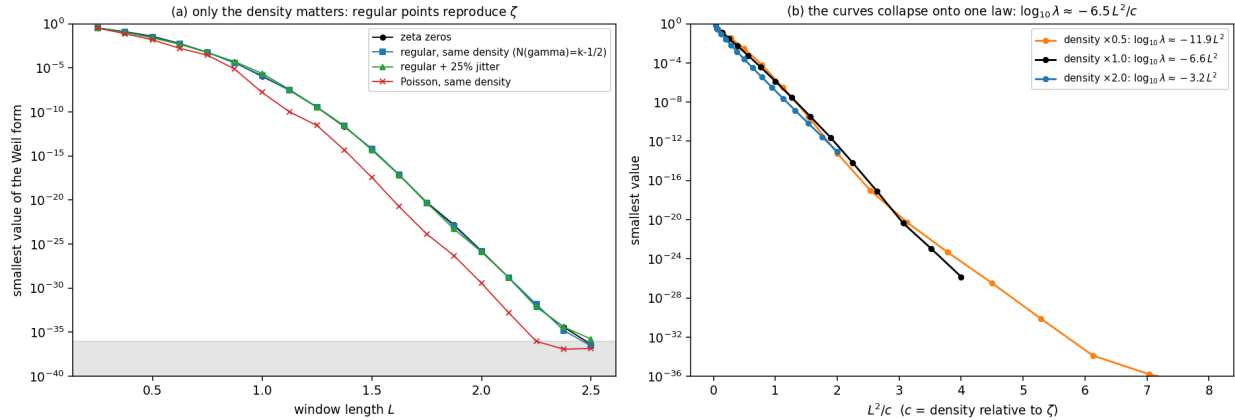


FIGURE 30. (a) The smallest value for the zeros of ζ , for regular points with the same density, for these points displaced at random by 25% of the spacing, and for a Poisson process with the same density. (b) Regular point sets with c times the density of the zeros of ζ , against L^2/c .

For the Davenport–Heilbronn function the same construction gives a form that is positive up to a threshold L^* and negative beyond. In double precision the negative eigenvalue first appears at $L \approx 3.6$: $\lambda_{\min} = -1.8 \cdot 10^{-9}$, -0.017 , -0.226 , -0.730 at $L = 3.6$, 3.7 , 3.8 , 4.0 . With 50 digits the threshold is earlier: $\lambda_{\min} = +1.2 \cdot 10^{-28}$ at $L = 3.40$ (the same with 67 and with 91 basis functions), $+1.5 \cdot 10^{-29}$ at 3.42 , $-7.2 \cdot 10^{-30}$ at 3.45 , $-1.7 \cdot 10^{-23}$ at 3.50 , $-2.3 \cdot 10^{-18}$ at 3.55 , $-1.7 \cdot 10^{-7}$ at 3.60 and -0.228 at 3.80 , so $3.42 < L^* \leq 3.45$, and between 3.45 and 3.6 the negative eigenvalue lies below double precision. At $L = 3.8$ the pair of the off-line zero $\rho_f = 0.8085 + 85.6993i$ contributes -0.786

in the direction of the minimizer, and the other seven off-line zeros below height 400 less than 0.006 together; at $L = 4$ the minimizer concentrates at $r \approx 84.5$, and that pair contributes -0.486 (Figure 35, right). An independent discretization by cubic B-splines of mesh 0.03, for which the Weil matrix is Toeplitz, finds the negativity once it exceeds double precision: the smallest eigenvalue is $-5 \cdot 10^{-7}$ at $L = 3.66$, -0.006 at 3.75 and -1.2 at 3.9, with the eigenvector peaked at $r = 84.5$. In this sense Weil positivity separates ζ from f once the coefficients up to $n \approx e^{3.44} \approx 31$ are included. The same discretization for ζ gives positive values on the spline space up to $L \approx 3.3$, where double precision is exhausted, and with 120 digits the form of ζ is positive at $L = 3$, with the margin above.

Sensitivity to an off-line pair. The threshold depends on the height of the off-line zero and only weakly on its distance to the line. We remove from the form of f its eight off-line pairs below height 400, replacing each by a double zero on the line, which leaves a positive form (smallest eigenvalue $2.5 \cdot 10^{-30}$ at $L = 3.45$, $1.6 \cdot 10^{-36}$ at 3.6), and then split a double zero at height γ into $\frac{1}{2} \pm \delta + i\gamma$; the change is a finite sum over zeros and is computed exactly. For $\gamma = 85.7$ the first window length at which the form becomes negative is 3.45, 3.55, 3.60 and at most 3.80 for $\delta = 0.3$, 0.1, 0.03, 0.01, and at $L = 3.8$ the smallest eigenvalue is negative down to $\delta = 10^{-4}$, scaling roughly like $-\delta^{11}$ until it meets the margin of the clean form (Figure 32b,c). For $\delta = 0.3$ the threshold is 2.5, 3.0, 3.0, 3.4, 3.45 and above 3.8 for $\gamma = 10, 20, 40, 60, 85.7$ and 100, on the grid of L we computed. So $L^* \approx L_0(\gamma) + b \log(1/\delta)$ with a small b : an off-line pair at height γ makes the form negative once the coefficients up to a power of $1/\delta$ with a small exponent enter, against $x \approx \gamma^{1/\delta}$ for the window energy of Corollary 3.5(c). The price is precision, since the margin to be resolved decreases superexponentially in L . This is a statement about sensitivity, not a certificate: positivity on a fixed window is compatible with off-line zeros, as f shows for $L < L^*$.

Large heights: two windows at the ends. The thresholds above are at small heights, where an off-line pair is detected only once the window resolves the neighbouring zeros. At large heights a simple family suffices. Let $\varphi_0(x) = \cos^2(\pi x/w)$ for $|x| \leq w/2$, $c = (L - w)/2$, $L' = L - w$, and

$$g(x) = (a\varphi_0(x - c) + b\varphi_0(x + c))e^{-i\gamma_0 x}, \quad \text{supp } g \subset [-L/2, L/2].$$

For $h = g * \tilde{g}$ the form (5.1) is $Q(a, b) = (|a|^2 + |b|^2)D - 2\text{Re}(a\bar{b}Y)$, where D is the archimedean term at height γ_0 , of size $\|\varphi_0\|_2^2 \log(\gamma_0/2\pi)$, minus the terms of the prime powers $n \leq e^w$, and

$$Y = \sum_n \frac{\Lambda(n)}{\sqrt{n}} K(\log n - L') n^{-i\gamma_0}, \quad K = \varphi_0 * \varphi_0,$$

is a sum over the prime powers in a window around $e^{L'}$. The form is indefinite on the family if and only if $|Y| > D$. At $\gamma_0 = 300, 800$ and 1200 the zero side and the prime side of Q agree to four digits.

Proposition 5.3 (detection of an isolated zero). Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ be a zero with $0.2 \leq \delta < \frac{1}{2}$, and assume that no other zero off the critical line has $|\gamma - \gamma_0| \leq (\log \gamma_0)^2$. Let $w = 5$, $S_5 = \sum_{k \in \mathbb{Z}} \sup_{[k, k+1]} |\hat{\varphi}_0|^2 = 18.03$, $c(T) = \frac{\log T}{2\pi} + 2(0.111 \log T + 0.275 \log \log T + 2.45) + 1$, and let L' satisfy $\sinh^2(\delta L'/2) \geq 1.05 \cdot (2S_5/w^2) c(\gamma_0)$. Then, for γ_0 large, $(a, b) = (1, -1)$ and $L = L' + w$ make Weil's form negative. In particular $L = (\log \log \gamma_0 + O(1))/\delta$, and the prime powers up to $(C \log \gamma_0)^{1/\delta}$ suffice.

Proof. On the zero side, with $\Delta = \gamma - \gamma_0$, a zero on the line contributes $4|\hat{\varphi}_0(\Delta)|^2 \sin^2(\Delta L'/2) \leq 4|\hat{\varphi}_0(\Delta)|^2$, and an off-line pair $\frac{1}{2} \pm \delta' + i(\gamma_0 + \Delta)$ contributes $-8 \text{Re}\{\hat{\varphi}_0(\Delta - i\delta')^2 \sinh^2((\delta' + i\Delta)L'/2)\}$. For ρ_0 this is $-8\hat{\varphi}_0(-i\delta)^2 \sinh^2(\delta L'/2) \leq -2w^2 \sinh^2(\delta L'/2)$, since $\hat{\varphi}_0(-i\delta) = \int \varphi_0 \cosh \delta x \geq w/2$. Trudgian's bound $|S(T)| \leq 0.111 \log T + 0.275 \log \log T + 2.450$ [Tr] gives at most $c(T)$ zeros in $[T, T + 1]$, so the zeros on the line contribute at most $4c(\gamma_0)S_5(1 + o(1))$. The zeros with $|\Delta| > (\log \gamma_0)^2$, including those of negative ordinate, contribute $O(e^{L/2}(\log \gamma_0)^{-9})$, because $|\hat{\varphi}_0(z)| \leq e^{|\text{Im } z|w/2} k^2 / (|z||z^2 - k^2|)$ with $k = 2\pi/w$ and $\delta' < \frac{1}{2}$; for $\delta \geq 0.2$ this is $o(w^2 e^{\delta L'})$. By hypothesis there are no other off-line zeros. $\square$

The hypothesis of isolation prevents the phases of nearby off-line zeros from cancelling the negative term; we do not know how to remove it, and Turán's power sum method is the natural tool. The bound is a worst case: for $\delta = 0.2$ it gives $L = 26.8, 28.6, 30.3$ at $\gamma_0 = 10^6, 10^{12}, 10^{20}$. Measured thresholds are smaller. Inserting a pair $\frac{1}{2} \pm \delta + i\gamma$ into the zeros of ζ (rescaled to the density $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$ for $\gamma > 2000$; the results do not change when the window of zeros or the band of test functions is widened), the general form becomes negative at $L^* \approx \log(\gamma/2\pi) + 0.8$ at small heights, where L^* depends little on δ , as above, and at $L^* \approx \log \log(\gamma/2\pi)/\delta$ at large heights for $\delta = 0.3$ and 0.4 , with $L^*\delta$ about 0.5 below this for $\delta = 0.2$; at $\gamma = 10^{12}$ and $\delta = 0.2$, $L^* = 13.65$ for the general form and 16.05 for the two windows (Figure 31a). The two scales balance in the expected way: a zero at distance δ makes the prime powers in the window resonate with amplitude $e^{\delta L'}$, against the archimedean term $\log \gamma$. An isolated zero with $\beta \geq 0.7$ at height γ is thus seen by the prime powers up to a power of $\log \gamma$, against $\gamma^{1/\delta}$ for the window energy of Corollary 3.5(c).

The converse direction asks for positivity of the two-window form at every height with the window that detects $\delta = 0.2$. For ζ we computed $|Y|/D$ at 2400 heights in $[2 \cdot 10^3, 3 \cdot 10^6]$, with $L \approx 11$ to 13 (prime powers up to $5.6 \cdot 10^5$): the mean is $0.43, 0.38, 0.32$ and the maximum $0.94, 0.90, 0.85$ on $[2 \cdot 10^3, 2 \cdot 10^4)$, $[2 \cdot 10^4, 2 \cdot 10^5)$, $[2 \cdot 10^5, 3 \cdot 10^6)$, with 99th percentile 0.84 . Replacing $p^{-i\gamma}$ by independent uniform phases, the random multiplicative model has the same mean, but maximum $1.53, 1.32, 1.13$, 99th percentile 1.07 , and $|Y| > D$ at 1.5% of the heights (Figure 31b). At this window, positivity is not a consequence of the statistics of the primes; it is carried by the exact positions of the zeros, as the spikes of Section 2. High moments of Y , which the mean value theorem gives exactly since the polynomial is short, bound only the measure of the heights where $|Y| > D$, a statement of density type. Status: Proposition 5.3 is proved; the rest is numerical (anc/weil_height/).

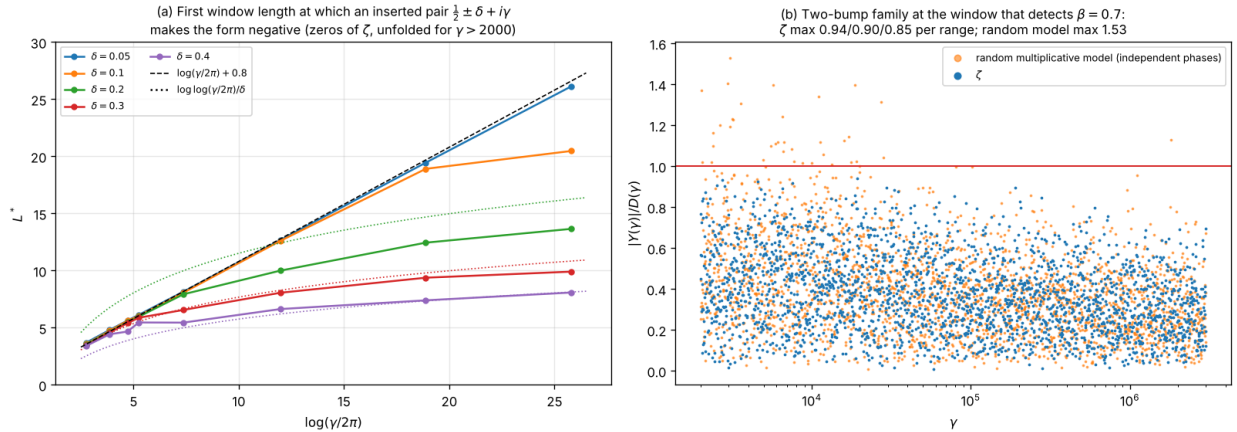


FIGURE 31. Weil's form at large height. (a) First window length L^* at which a pair $\frac{1}{2} \pm \delta + i\gamma$ inserted into the zeros of ζ makes the form negative, against $\log(\gamma/2\pi)$; dashed: $\log(\gamma/2\pi) + 0.8$; dotted: $\log \log(\gamma/2\pi)/\delta$. (b) $|Y|/D$ for the two-window family at the window that detects $\beta = 0.7$: ζ and the random multiplicative model.

Who pays for positivity. In the direction of the minimizer the form splits into the pole term, the archimedean term with the constant c_0 , and one term for each n . For ζ at $L = 3$ these are $+1.5413, -1.4722$ and -0.0691 , the last made of -0.0682 from $n = 2$, $-8.5 \cdot 10^{-4}$ from $n = 3$ and terms decreasing to $-4.8 \cdot 10^{-65}$ at $n = 19$, with total $6.5 \cdot 10^{-82}$; at $L = 2$ the split is $+1.4694, -1.4186, -0.0509$. Every prime power term is negative, because $\Lambda(n) \geq 0$ and the extremal h is positive at every $\log n$: positivity is a coherent balance of the pole against the archimedean term and all the prime powers together. For f , which has no pole, the archimedean term is $+0.0072$ at

$L = 2$ and -0.0297 at $L = 3.42$, and the coefficients carry the rest with both signs: at $L = 3.42$, -0.160 from $n = 2$, $+0.084$ from $n = 3$, $+0.120$ from $n = 4$, -0.013 from $n = 6$, and further terms at composite n (12, 14, 18, ...) that exist only because f has no Euler product. Positivity for f rests on cancellation between coefficients of opposite sign, and it fails at L^* . Status: numerical ([anc/weil_hp/](#)).

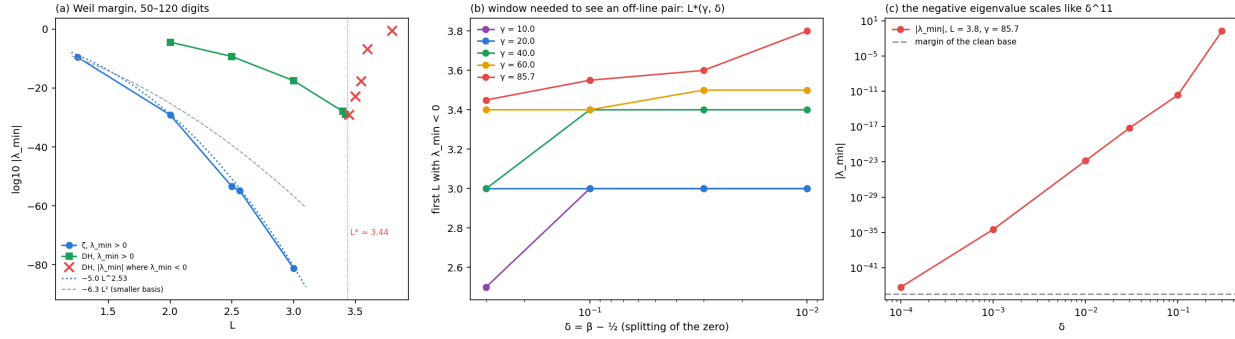


FIGURE 32. Weil's form in high precision. (a) Smallest eigenvalue for ζ (70 to 120 digits) and for the Davenport-Heilbronn function (50 to 70 digits; crosses: $|\lambda_{\min}|$ where negative), against the law $-6.3L^2$ of a smaller basis and the fit $-5.0L^{2.53}$; dotted: $L^* \approx 3.44$. (b) First window length at which the clean form of f with a pair split to $\frac{1}{2} \pm \delta + i\gamma$ becomes negative. (c) Smallest eigenvalue at $L = 3.8$, $\gamma = 85.7$, against δ .

The minimiser as an approximate spectrum. Connes observed that the zeros of the Fourier transform of the minimiser of Weil's form on short windows reproduce the zeros of ζ with remarkable accuracy: with the primes up to 13, the first 50 zeros agree to between $2.6 \cdot 10^{-55}$ and 10^{-3} [Cn26], and Groskin has pushed this to several hundred digits [Gk]; whether the approximation converges as $L \rightarrow \infty$ is open. We repeated the construction in the even sector of the sine basis above, for ζ and for f , in double precision and, at two points, in high precision (mpmath). For ζ the median error of the first ten zeros decreases steadily, from 0.8 at $L = 1$ to the double-precision floor 10^{-12} for $L \geq 2.4$; at $L = \log 13$ with 100 digits the first zero is reproduced to $1.0 \cdot 10^{-51}$ and the tenth to $3.4 \cdot 10^{-33}$. For f the construction works equally well while its form is positive: at $L = 3.4$, with 50 digits, the first zero of f is reproduced to $2.8 \cdot 10^{-27}$ and the tenth to $7 \cdot 10^{-16}$, although f violates RH. The off-line pair ρ_f is projected onto the line: at $L = 3.4$ the minimiser has two real zeros, 85.569 and 85.852, on either side of the ordinate 85.6993. Beyond the threshold, as the negative eigenvalue grows from 10^{-30} at $L = 3.45$ to 10^{-7} at 3.6 and 0.2 at 3.8, the picture changes abruptly. The median error of the first ten zeros of f rises from $3 \cdot 10^{-13}$ at $L = 3.55$ to $3 \cdot 10^{-5}$ at 3.65, $8 \cdot 10^{-3}$ at 3.70 and about 10^{-1} for $L \geq 3.8$, and the zero of the minimiser next to ρ_f converges to its ordinate, 85.7092 at $L = 4.0$ and 85.6993 at $L = 4.25$: the minimiser becomes the mode of the off-line pair (Figure 33). Two conclusions follow. The accuracy of the construction on short windows does not by itself say anything about RH, since a function that violates RH reproduces its zeros just as well up to and slightly beyond L^* . And an off-line zero shows up as a positivity threshold at a finite L , beyond which all the approximations degrade at once and one zero of the minimiser locks onto the ordinate of the pair. For ζ , RH is the statement that no such threshold occurs. Status: numerical (scripts in [connes/](#)).

The prime powers one at a time. On a window of length L only the $n \leq e^L$ enter (5.1), and the prime powers enter one at a time, at $L = \log n$. Let $\lambda_N(L)$ be the smallest eigenvalue of the form in which only the prime powers $n \leq N$ are kept; for $N \geq e^L$ it is $\lambda_{\min}(L)$. We computed $\lambda_N(L)$ in the sine basis in double precision on a grid of step 0.02 in L , and on cubic splines of mesh

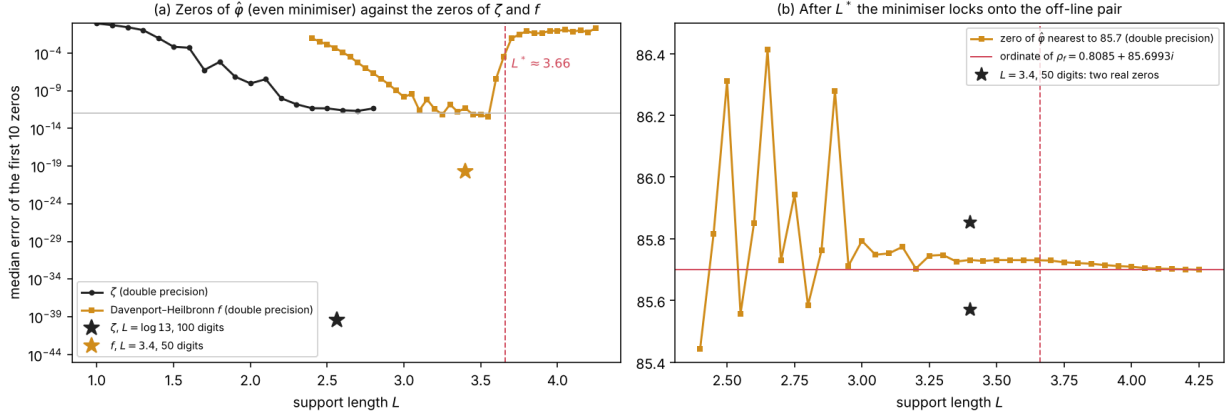


FIGURE 33. The minimiser of Weil’s form as an approximate spectrum. (a) Median error of the first ten zeros of $\hat{\varphi}$, for the even minimiser on a window of length L , against the zeros of ζ and of the Davenport–Heilbronn function f ; stars: high-precision values. (b) The zero of $\hat{\varphi}$ nearest to 85.7 for f ; beyond the threshold ($L^* \approx 3.44$; the negative eigenvalue exceeds double precision near 3.6) it converges to the ordinate of the off-line zero ρ_f . Stars: at $L = 3.4$, with 50 digits, the minimiser has two real zeros on either side of that ordinate.

0.02 in mpmath with 36 to 79 digits for $1 \leq L \leq 2.8$. With the pole and archimedean terms only, the form is positive up to $L = 0.755$, consistent with [CC], which covers $L \leq \log 2$. On the subspace $\hat{\varphi}(\pm \frac{i}{2}) = 0$, where the pole term vanishes, it is positive up to $L = 1.207$, beyond $\log 3$, and adding the prime 2 alone lowers this to 1.146. In every window the form becomes negative when the most recent prime power is removed, however short the time since it entered: at $L = 2.0$, removing 7 (which entered at 1.946) gives -0.0031 ; at $L = 2.6$, removing 13 (2.565) gives $-2.5 \cdot 10^{-4}$; at $L = 2.8$, removing 16 (2.773) gives $-1.4 \cdot 10^{-8}$ (Figure 34a). The deficit grows roughly like δ^6 in $\delta = L - \log q$ (fitted exponents between 4.8 and 7.2, Figure 34b). Since $\lambda_{\min}(L) \leq 10^{-6.3L^2}$, a truncated form survives only for a length of about 10^{-L^2} after the next prime power enters: positivity is maintained prime power by prime power, with no reserve. The spline values below 10^{-14} are a discretization floor of order h^8 , not the true $\lambda_{\min}$, which the table above gives.

Dirichlet polynomials as test functions. The joint question of Remark 2.4 leads to the test functions $\varphi = \sum_{n \leq X} a_n \delta_{\log n}$, smoothed by a Gaussian. With $w(r) = e^{-(r/T)^2}$ put

$$A_{mn} = \sum_{\rho} w(\gamma_{\rho}) \left(\frac{m}{n} \right)^{i\gamma_{\rho}}, \quad 1 \leq m, n \leq X.$$

Then $\sum_{m,n} a_m \bar{a}_n A_{mn} = \sum_{\rho} w(\gamma_{\rho}) \left| \sum_n a_n n^{i\gamma_{\rho}} \right|^2$ under RH, and (5.1) with $h(r) = w(r)(m/n)^{ir}$ expresses each entry through the primes k with $|\log k - \log \frac{m}{n}| \lesssim 1/T$, the pole and the archimedean term. For the Davenport–Heilbronn function the same holds with its explicit formula. With $T = 60$ the entries computed from the primes (or the coefficients $b(n)$) and from the zeros agree to six digits or better. The smallest eigenvalue is:

X	16	32	64	128	256	384	512
ζ	0.28	$1.9 \cdot 10^{-3}$	$2.6 \cdot 10^{-7}$	$7 \cdot 10^{-15}$	0*	0*	0*
Davenport–Heilbronn	18.9	8.0	0.11	$3 \cdot 10^{-11}$	$-3.8 \cdot 10^{-4}$	-1.6	-5.1

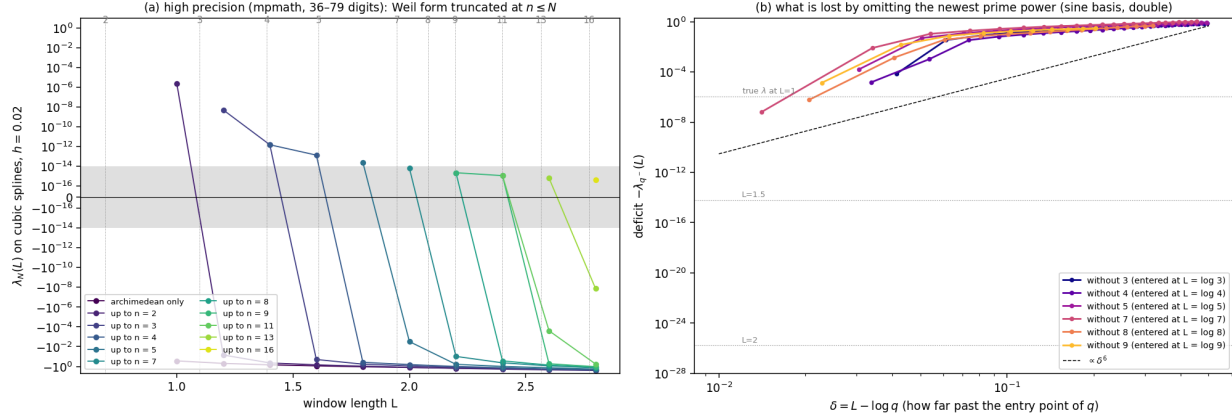


FIGURE 34. Weil positivity prime power by prime power. (a) $\lambda_N(L)$ for the form truncated at $n \leq N$, on cubic splines of mesh 0.02 in high precision; dotted lines at $L = \log q$; grey band: discretization floor. (b) The deficit $-\lambda_{q-}(L)$ caused by omitting the most recent prime power q , against $\delta = L - \log q$ (sine basis); dotted lines: $\lambda_{\min}$ at $L = 1, 1.5, 2$.

(0*: below 10^{-13} in modulus, the rounding level.) For f the form becomes negative between $X = 192$ and $X = 256$; at $X = 512$ the off-line zeros contribute -9.1 to the minimal value and the zeros on the line $+4.0$. For ζ the margin collapses as on windows. At $X = 128$ the computed minimizer, normalized as $c_n = a_n \sqrt{n}/a_1$, alternates between 1 at odd n and -1.14 at even n : the polynomial $\sum c_n n^{-s}$ is of the type of the alternating series $\eta(s) = (1 - 2^{1-s})\zeta(s)$, has modulus below 10^{-4} at the first three zeros, and lies between 0.2 and 2.3 at the other points tested. This vector is not typical, however. The eigenvalues below $10^{-9}\lambda_{\max}$ span a space of dimension 11 for ζ and 6 for f at $X = 128$, and 87 and 74 at $X = 256$. On average 78% ($X = 128$) and 88% ($X = 256$) of the mass of these vectors lies on $X/2 < n \leq X$ for ζ , and 100% and 93% for f ; after division by ζ (Möbius inversion), resp. by f , at most 4% of the mass lies on $n \leq 10$. The collapse of the margin is therefore a counting effect, present in the same form for f : Dirichlet polynomials on the top octave have more free coefficients than there are zeros of non-negligible weight, and can vanish at all of them. It is not produced by multiples of ζ by short Euler factors, and it does not involve the Euler product. What separates ζ from f in this basis is only the sign of the minimum, decided by the off-line zeros.

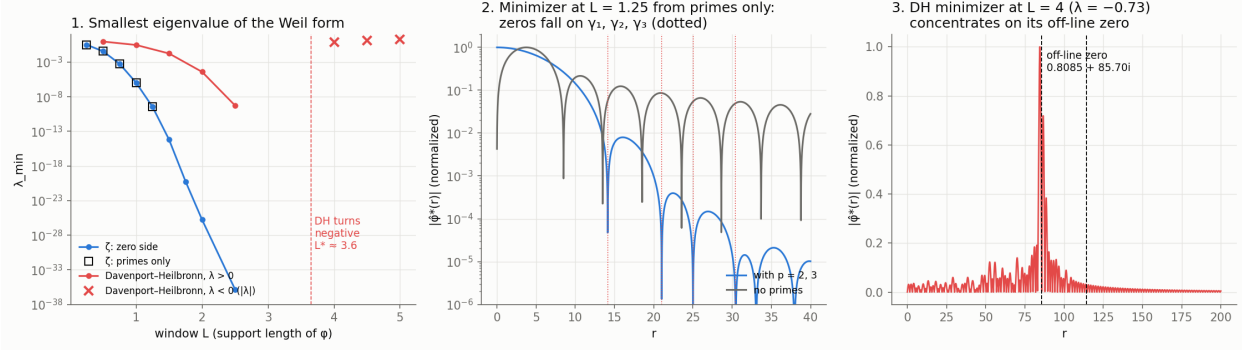


FIGURE 35. Left: $\lambda_{\min}(L)$ for ζ (zero side: line; primes only: squares) and for the Davenport–Heilbronn function (positive values: line; $|\lambda_{\min}|$ where negative: crosses). Centre: the minimizer at $L = 1.25$ with and without the primes 2, 3; dotted lines at the zeros of ζ . Right: the Davenport–Heilbronn minimizer at $L = 4$ and its off-line zeros.

6. THE EULER PRODUCT INSIDE THE STRIP

Sections 3–5 use the primes through $\Lambda(n)$ in an additive or quadratic form. This section asks what the partial Euler product itself does in the strip. Let

$$E_x(s) = \prod_{p \leq x} (1 - p^{-s})^{-1}, \quad \log E_x(s) = \sum_{p \leq x} \sum_{k \geq 1} \frac{p^{-ks}}{k}.$$

For $\sigma < 1$ the product diverges because of the pole. The sum $\sum_{p \leq x} p^{-s}$ contains the smooth term $\int_x^u u^{-s} du / \log u$, which has size $x^{1-\sigma} / (|1-s| \log x)$. On the critical line at $t = 20$ the ratio E_x/ζ takes the values 0.92, 1.60, 0.35, 0.58, 84.5 at $x = 10^3, 10^4, 10^5, 10^6, 2 \cdot 10^6$. We remove this term with the exponential integral $E_1(w) = \int_1^\infty e^{-wv} dv/v$, continued to $\mathbb{C} \setminus (-\infty, 0]$; for $\sigma > 1$, $E_1((s-1) \log x) = \int_x^\infty u^{-s} du / \log u$. Put

$$E_x^*(s) = E_x(s) \exp(E_1((s-1) \log x)).$$

Proposition 6.1 (explicit formula in product form). Let $t \neq 0$ and let s not be a zero of ζ . Then

$$\log E_x^*(s) = \log \zeta(s) + \sum_{\rho} E_1((s-\rho) \log x) + R_x(s) + O(x^{-2-\sigma}), \quad R_x(s) = \sum_{k \geq 2} \sum_{x^{1/k} < p \leq x} \frac{p^{-ks}}{k},$$

with the zeros in order of increasing $|\operatorname{Im} \rho|$ and the logarithms continued from $\operatorname{Re} s > 1$.

Proof. $\log E_x(s) = \sum_{n \leq x} \Lambda(n) n^{-s} / \log n + R_x(s)$. Write the sum as $\int_{2-}^x u^{-s} d\psi(u) / \log u$ and insert $\psi(u) = u - \sum_{\rho} u^{\rho} / \rho - \log 2\pi - \frac{1}{2} \log(1 - u^{-2})$ [T, §3.8], [In]. For $\sigma > 1$, letting $x \rightarrow \infty$ gives $\log \zeta(s)$; the tail of the main term is $E_1((s-1) \log x)$, the tail of each zero is $-E_1((s-\rho) \log x)$, and the trivial zeros give $O(x^{-2-\sigma})$. Both sides are analytic away from the zeros, the pole and the branch cuts of E_1 , and the sum over ρ converges locally uniformly in the stated order by the truncated explicit formula, so the identity extends by analytic continuation. $\square$

The term of a zero is $E_1((s-\rho) \log x) = x^{\rho-s} / ((s-\rho) \log x) (1 + O(1/(|s-\rho| \log x)))$. It decays like $x^{\beta-\sigma} / \log x$ if $\beta < \sigma$ and grows if $\beta > \sigma$: on the line $\operatorname{Re} s = \sigma$ the corrected product sees exactly the zeros to its right. The identity is a reorganization of the classical explicit formula; a smoothed form with the zeros truncated is the hybrid Euler–Hadamard product of Gonek, Hughes and Keating [GHK].

Corollary 6.2. The following are equivalent: (i) RH; (ii) $\log E_x^*(s) \rightarrow \log \zeta(s)$ locally uniformly on $\{\frac{1}{2} < \operatorname{Re} s < 1, t \neq 0\}$; (iii) for $\frac{1}{2} < \sigma < 1, t \neq 0$, $\log E_x^*(s) - \log \zeta(s) = O_s(x^{1/2-\sigma} \log^2 x)$ locally uniformly.

Proof. (i) $\Rightarrow$ (iii): under RH each mode is $O(x^{1/2-\sigma}/(|s-\rho|\log x))$, the truncated explicit formula bounds their sum by $O_s(x^{1/2-\sigma} \log^2 x)$ [T, Ch. XIV], and $R_x(s) = O(x^{1/2-\sigma})$. (iii) $\Rightarrow$ (ii) is immediate. (ii) $\Rightarrow$ (i): each $\log E_x^*$ is analytic on the region, so a locally uniform limit is an analytic logarithm of ζ there, and ζ has no zeros with $\frac{1}{2} < \beta < 1$. $\square$

This is the product form of the classical equivalence of RH with the analyticity of $\log \zeta$ in $\operatorname{Re} s > \frac{1}{2}$. We computed E_x^* from all primes $p \leq 2 \cdot 10^6$ and the right side of Proposition 6.1 from the 682 zeros with $|\gamma| < 600$:

The Euler product as a C_2 energy. The partial product and the cross-energy are related by an exact identity, valid inside the strip. For $P \geq 2$ let $1 = n_1 < n_2 < \dots$ be the P -smooth integers, those whose prime factors are all at most P , and put $X_k^{(P)}(s) = \sum_{j \leq k} n_j^{-s}$ and $C_2^{(P)}(n_k, s) = 2 \operatorname{Re} (X_{k-1}^{(P)}(s) \overline{n_k^{-s}})$.

Proposition 6.3. For $\sigma > 0$ and $P \geq 2$,

$$|E_P(s)|^2 = \prod_{p \leq P} (1 - p^{-2\sigma})^{-1} + \sum_{k \geq 2} C_2^{(P)}(n_k, s),$$

the series converging absolutely. For $\sigma > 1$, letting $P \rightarrow \infty$ gives $|\zeta(s)|^2 = \zeta(2\sigma) + \sum_{n \geq 2} C_2(n, s)$.

Proof. For $\sigma > 0$ the product $E_P(s) = \prod_{p \leq P} \sum_{m \geq 0} p^{-ms}$ of finitely many absolutely convergent geometric series equals $\sum_k n_k^{-s}$, with absolute convergence. Telescoping $|X_k^{(P)}|^2 - |X_{k-1}^{(P)}|^2 = n_k^{-2\sigma} + C_2^{(P)}(n_k, s)$ and letting $k \rightarrow \infty$ gives the identity, since $\sum_k n_k^{-2\sigma} = \prod_{p \leq P} (1 - p^{-2\sigma})^{-1}$. For $\sigma > 1$ the P -smooth integers exhaust all integers as $P \rightarrow \infty$, with dominated convergence. $\square$

With $P = 7$ and the 14672 seven-smooth integers up to 10^{12} , the right side equals 0.886590 at $s = 1.5 + 20i$, as does $|E_7|^2$, and 0.761394 at $s = 0.6 + 20i$, against $|E_7|^2 = 0.761404$; the difference is the contribution of the smooth integers above 10^{12} . Each partial Euler product is therefore the cross-energy of the partial sums of ζ restricted to the P -smooth integers. Inside the strip the limit $P \rightarrow \infty$ does not exist without the correction of Proposition 6.1: the energy over the smooth integers inherits the drift of size $x^{1-\sigma}/\log x$ coming from the pole, which in the ordinary C_2 is the term $n^{1-2\sigma}/(1-s)$ of Theorem A.

s	x	measured $\log E_x^* - \log \zeta$	predicted
$0.8 + 20i$	$2 \cdot 10^6$	$-0.00068 - 0.00019i$	$-0.00070 - 0.00015i$
$0.7 + 20i$	$2 \cdot 10^6$	$-0.00285 - 0.00113i$	$-0.00292 - 0.00095i$
$0.6 + 20i$	10^4	$-0.01512 - 0.04052i$	$-0.01427 - 0.03802i$
$0.6 + 20i$	$2 \cdot 10^6$	$-0.01161 - 0.00629i$	$-0.01194 - 0.00552i$
$0.7 + 50i$	$2 \cdot 10^6$	$-0.00422 + 0.01193i$	$-0.00403 + 0.01198i$

The remaining discrepancy is the contribution of the zeros above height 600. The error of the Euler product is, point by point, the superposition of the modes of the zeros (Figure 36, left). Conversely, removing the computed modes from E_x^* reproduces ζ to about 1% at $0.6 + 20i$, $0.75 + 50i$, and at $\frac{1}{2} + 20i$ on the critical line.

At a zero. Let $\rho_* = \frac{1}{2} + i\gamma_*$ be a simple zero. Isolating its own term and letting $s \rightarrow \rho_*$, with $E_1(w) = -\gamma_0 - \log w + O(w)$, gives

$$|E_x^*(\rho_*)| \log x = |\zeta'(\rho_*)| e^{-\gamma_0} \left| \exp \left(\sum_{\rho \neq \rho_*} E_1((\rho_* - \rho) \log x) + R_x(\rho_*) \right) \right|.$$

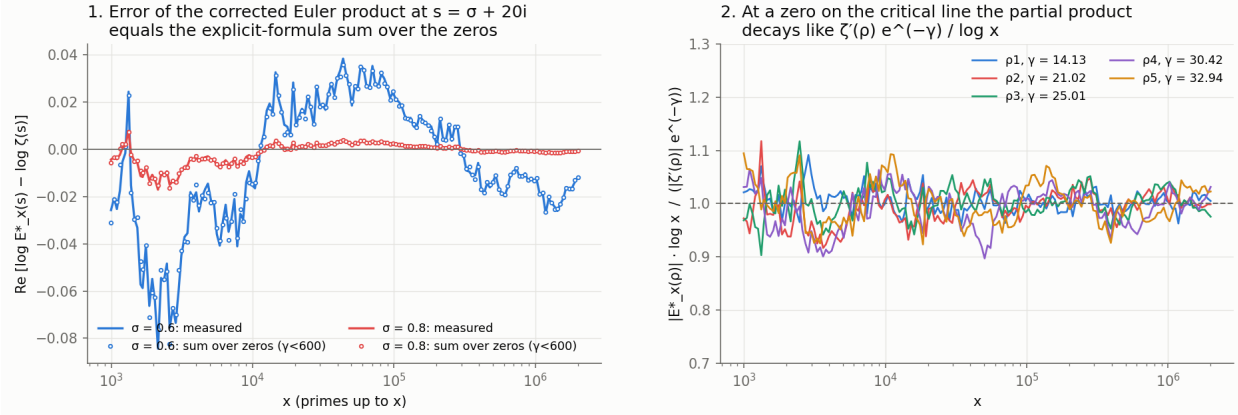


FIGURE 36. Left: the error $\text{Re}[\log E_x^*(s) - \log \zeta(s)]$ at $s = \sigma + 20i$ for $\sigma = 0.6$ and 0.8 (lines), against the sum over the zeros with $|\gamma| < 600$ plus R_x (circles). Right: $|E_x^*(\rho)| \log x / (|\zeta'(\rho)| e^{-\gamma})$ at the first five zeros.

The partial product decays like $|\zeta'(\rho_*)| e^{-\gamma_0} / \log x$ provided the sum over the other zeros tends to 0, which is the Deep Riemann Hypothesis at ρ_* ; the constant agrees with Conrad [Co, Thm. 5.11], see also [Sh]. For the first zero, $\zeta'(\rho_1) = 0.7833 + 0.1247i$ and $|\zeta'(\rho_1)| e^{-\gamma_0} = 0.4453$. The ratio $|E_x^*(\rho)| \log x / (|\zeta'(\rho)| e^{-\gamma_0})$ at the first five zeros stays within 1 ± 0.1 for $10^3 \leq x \leq 2 \cdot 10^6$, and its fluctuations are reproduced by the other 681 zeros to about 0.005 (Figure 36, right). For ρ_1 :

x	10^3	10^4	10^5	10^6	$2 \cdot 10^6$
ratio	1.021	1.001	1.017	0.975	1.005
predicted	1.035	1.002	1.013	0.979	1.007

The uncorrected product behaves differently: $|E_x(\rho_1)|$ is 0.023, 0.013, 0.417 and $1.4 \cdot 10^{-4}$ at $x = 10^4, 10^5, 10^6, 2 \cdot 10^6$, because the pole term oscillates with amplitude $x^{1/2} / (|1 - \rho_1| \log x)$. The logarithmic increment of E_x at a prime is $-2 \log |1 - p^{-s}| = 2 \text{Re } p^{-s} + O(p^{-2\sigma})$, the prime analogue of C_2 [C2]; at a zero these increments add up to $-2 \log \log x + 2 \log |\zeta'(\rho)| - 2\gamma_0$ plus the other modes. The constant $|\zeta'(\rho)|$ is the one that governs the second-order behaviour of C_2 at a zero [C3].

In logarithmic form the same statement concerns the prime sum alone. Let $D_x(s) = \sum_{p \leq x} p^{-s} - \int_2^x u^{-s} du / \log u$. Removing the prime powers $Q(s) = \sum_p \sum_{k \geq 2} p^{-ks} / k$, which converge at ρ_* , and using $\int_2^x u^{-s} du / \log u = E_1((s-1) \log 2) - E_1((s-1) \log x)$, the formula above gives

$$\text{Re } D_x(\rho_*) = -\log \log x + C_{\rho_*} + o(1), \quad C_{\rho_*} = \log |\zeta'(\rho_*)| - \gamma_0 - \text{Re } Q(\rho_*) - \text{Re } E_1((\rho_* - 1) \log 2),$$

under the same proviso on the other zeros. The term $-\log \log x$ is the resonance of the zero with itself, $x^{\rho_* - s} = 1$ at $s = \rho_*$. For the first three zeros $C_\rho = -1.142, -0.406, -0.076$, against $\text{Re } D_x(\rho) + \log \log x = -1.131, -0.436, -0.095$ at $x = 10^7$; the differences oscillate with the other zeros and decrease like $1/\log x$. At the point $\frac{1}{2} + i\gamma$ below a hypothetical zero $\frac{1}{2} + \delta + i\gamma$ the same term becomes $-\int_2^x u^{\delta-1} du / \log u \sim -x^\delta / (\delta \log x)$, the amplification of Remark 6.9.

A smoothed form: the zeros act locally. With the sharp cut-off at x each zero contributes a term that decays only like $1/\log x$ on the critical line, and in the tables above the zeros below height 600 leave errors of order 10^{-3} to 10^{-2} . Averaging the identity over $\log x$ removes this.

Proposition 6.1' (smoothed form). Let $g \geq 0$ be piecewise continuous with compact support in $(0, \infty)$ and $\int g = 1$, and put

$$\psi_g(v) = \int_v^\infty g(u) du, \quad K_g(w) = \int_0^\infty g(u) E_1(wu) du.$$

Let $t \neq 0$, and let s be neither a zero of ζ nor of the form $\rho - r$ with $r > 0$. Then

$$\log \zeta(s) = \sum_{n \geq 2} \frac{\Lambda(n) \psi_g(\log n)}{n^s \log n} + K_g(s-1) - \sum_{\rho} K_g(s-\rho) - \sum_{k \geq 1} K_g(s+2k),$$

with the zeros in order of increasing $|\operatorname{Im} \rho|$. Equivalently $\zeta = E_g^* Z_g$, where $Z_g(s) = \prod_{\rho} e^{-K_g(s-\rho)}$ and E_g^* collects the primes, the pole and the trivial zeros.

Proof. For $x > 1$ not a prime power, the proof of Proposition 6.1 with the term $\frac{1}{2} \log(1-u^{-2})$ of $\psi(u)$ kept exactly gives

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s \log n} + E_1((s-1) \log x) = \log \zeta(s) + \sum_{\rho} E_1((s-\rho) \log x) + \sum_{k \geq 1} E_1((s+2k) \log x).$$

Multiply by $g(\log x)$ and integrate over $\log x$. On the left the sum becomes the finite sum with weights $\psi_g(\log n)$. On the right the partial sums over $|\gamma| \leq T$ are bounded uniformly for $\log x$ in the support of g , by the truncated explicit formula, so the sum over the zeros can be integrated term by term. $\square$

If g is smooth, the zeros act locally. Write $E_1(i\tau u) = e^{-i\tau u} a(\tau u)$ with $a(v) = e^{iv} E_1(iv)$, so that $a(v) \sim 1/(iv)$ and each derivative of $a(\tau u)$ in u is $O(1/|\tau|)$ on the support of g . Integrating by parts m times in u gives $K_g(i\tau) = O_{g,m}(|\tau|^{-m})$ for every m , and the same holds for $K_g(\delta + i\tau)$ with δ real and bounded. On the line $\operatorname{Re} s = \sigma$ a zero at distance $|t - \gamma|$ therefore contributes a term that decays faster than any power of $|t - \gamma|$; for a Gaussian of width σ_g the decay is like $e^{-\sigma_g^2(t-\gamma)^2/2}$. We take g Gaussian with $\sigma_g = 0.4$, centred at $\log X$, $X = 1000.5$, and truncated at nine widths (prime powers up to $3.7 \cdot 10^4$), and keep only the zeros with $|\gamma - t| \leq W$. The relative error $|E_g^* Z_g / \zeta - 1|$ is:

s	$W = 6$	$W = 10$	$W = 15$	$W = 20$	sharp, 1517 zeros
$\frac{1}{2} + 20i$	$2.3 \cdot 10^{-6}$	$2.3 \cdot 10^{-6}$	$2.9 \cdot 10^{-13}$	$3.9 \cdot 10^{-13}$	$8.9 \cdot 10^{-4}$
$\frac{1}{2} + 33i$	$2.2 \cdot 10^{-4}$	$2.8 \cdot 10^{-6}$	$1.4 \cdot 10^{-10}$	$3.0 \cdot 10^{-12}$	$8.9 \cdot 10^{-4}$
$\frac{1}{2} + 100i$	$1.7 \cdot 10^{-4}$	$1.2 \cdot 10^{-6}$	$7.8 \cdot 10^{-11}$	$2.7 \cdot 10^{-13}$	$8.9 \cdot 10^{-4}$
$\frac{1}{2} + 1000i$	$2.4 \cdot 10^{-4}$	$2.4 \cdot 10^{-6}$	$1.1 \cdot 10^{-11}$	$3.8 \cdot 10^{-12}$	$1.3 \cdot 10^{-3}$
$0.6 + 20i$	$1.1 \cdot 10^{-6}$	$1.1 \cdot 10^{-6}$	$1.6 \cdot 10^{-13}$	$2.0 \cdot 10^{-13}$	$4.5 \cdot 10^{-4}$
$0.7 + 50i$	$2.8 \cdot 10^{-4}$	$2.8 \cdot 10^{-7}$	$2.7 \cdot 10^{-11}$	$2.9 \cdot 10^{-13}$	$2.2 \cdot 10^{-4}$
$0.3 + 30i$	$7.4 \cdot 10^{-4}$	$3.7 \cdot 10^{-6}$	$6.1 \cdot 10^{-11}$	$4.1 \cdot 10^{-12}$	$3.5 \cdot 10^{-3}$

The last column is Proposition 6.1 at $x = 1000.5$ with the 1517 zeros below height 2000. With the 6 to 33 zeros within distance 20 of t the smoothed identity holds to the accuracy of the computation of ζ itself (Figure 37b).

Corollary 6.1'' (the derivative at a zero). Let ρ_* be a simple zero, and suppose no other zero has ordinate γ_* and real part larger than β_* . Then

$$\begin{aligned} \zeta'(\rho_*) = \exp \Big(\sum_{n \geq 2} \frac{\Lambda(n) \psi_g(\log n)}{n^{\rho_*} \log n} + K_g(\rho_* - 1) + \gamma_0 + \int_0^\infty g(u) \log u du \\ - \sum_{\rho \neq \rho_*} K_g(\rho_* - \rho) - \sum_{k \geq 1} K_g(\rho_* + 2k) \Big). \end{aligned}$$

Proof. Since $E_1(w) = -\gamma_0 - \log w + O(w)$, uniformly for u in the support of g , $e^{-K_g(s-\rho_*)} = (s - \rho_*) e^{\gamma_0 + \int g \log u} (1 + O(|s - \rho_*|))$. Divide $\zeta(s) = E_g^*(s)Z_g(s)$ by $s - \rho_*$ and let $s \rightarrow \rho_*$; the other factors are continuous at ρ_* . $\square$

This is an exact and unconditional form of the hybrid formula $\zeta'(\rho) = P_X(\rho)Z_X'(\rho)(1 + o(1))$ of Bui, Gonek and Milinovich [BGM], who prove it under RH with explicit error terms. Compared with the sharp cut-off of the previous paragraphs, the resonance $-\log \log x$ of the zero with itself is absent: averaging over $\log x$ turns the term of ρ_* into the constant $\gamma_0 + \int g \log u$, and no hypothesis on the other zeros is needed. With the Gaussian above and the zeros within distance 25, the formula gives $\zeta'(\rho_1) = 0.7832965119 + 0.1246998297i$, and at the first five zeros it agrees with ζ' computed directly to a relative error between $3 \cdot 10^{-11}$ and $9 \cdot 10^{-11}$.

The radius of the helix. Theorem A writes C_2 on the critical line as a centre, a drift and a helix of radius $|\zeta(\frac{1}{2} + it)|$. By Proposition 6.1' the radius factorizes as $|E_g^*(s)||Z_g(s)|$: the first factor comes from the primes, the pole and the trivial zeros, and the second is in effect a finite product over the zeros near t . Figure 37(a) shows the oscillating part of C_2 at $N = 2000$, rescaled by $\sqrt{N}$, inside the envelope $\pm 2|E_g^*||Z_g|$ computed with the zeros within distance 20. At a zero the collapse of the helix is carried by the single factor $e^{-K_g(s-\rho)} \approx e^{\gamma_0 + \int g \log u}(s - \rho)$, and the remaining factors give $\zeta'(\rho)$, the constant that governs the second-order behaviour of C_2 at a zero [C3]. Status: Proposition 6.1' and Corollary 6.1'' are proved; they are reorganizations of the explicit formula in the line of [GHK] and [BGM]; the Gaussian decay rate and the tables are numerical (euler_local/).

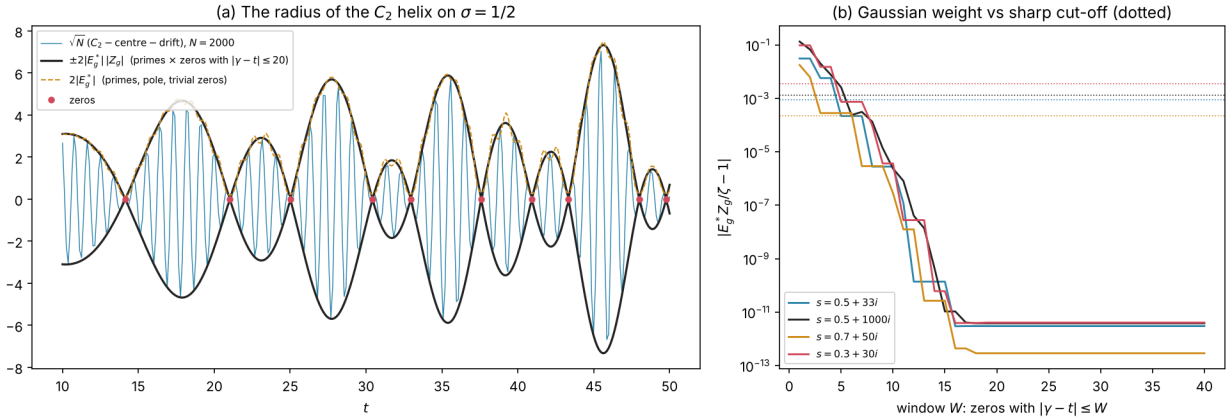


FIGURE 37. The smoothed Euler product (Proposition 6.1'), Gaussian weight of width 0.4 around $\log X$, $X = 1000.5$. (a) $C_2(N, \frac{1}{2} + it)$ minus its centre and Bernoulli terms, multiplied by $\sqrt{N}$, for $N = 2000$, against $\pm 2|E_g^*||Z_g|$ with the zeros $|\gamma - t| \leq 20$ and against the prime factor $2|E_g^*|$ alone; dots: zeros. (b) Relative error of $E_g^*Z_g$ against the window W of zeros used, at four points; dotted: the sharp cut-off of Proposition 6.1 at $x = 1000.5$ with the 1517 zeros below height 2000.

The zero factor from C_2 . Since E_g^* has no zeros, the factor Z_g is determined by $\zeta = E_g^*Z_g$. It can nevertheless be computed from the partial sums alone, without dividing by E_g^* and without a list of zeros. Put $W_N(w) = C_2 + iS_2 = 2N^{-\bar{w}}X_{N-1}(w)$. By Euler–Maclaurin, for $\text{Re } w > 0$,

$$\zeta(w) = \lim_{N \rightarrow \infty} \left[\frac{1}{2} N^{\bar{w}} W_N(w) - \frac{N^{1-w}}{1-w} + \frac{1}{2} N^{-w} \right].$$

On the critical line $\bar{w} = 1 - w$, and this becomes $\zeta(w) = \lim_N N^{1-w} \left[\frac{1}{2} W_N(w) - \frac{1}{1-w} \right] + \frac{1}{2} N^{-w}$: at a zero on the line $W_N \rightarrow 2/(1 - \rho)$ and $C_2 \rightarrow 1/|\rho|^2$, while off the line the factor $N^{\bar{w}} = N^{1-w} N^{2\sigma-1}$

produces the growth or decay of Corollary 2.1. The potential $\Phi = \log |\zeta|$ is harmonic in the strip except at the zeros, $\Delta\Phi = 2\pi \sum_\rho \delta_\rho - 2\pi\delta_1$ in the sense of distributions, and therefore

$$Z_g(s) = \exp \left(-\frac{1}{2\pi} \iint K_g(s-w) \Delta\Phi(w) dA(w) \right),$$

the integral being over a part of the strip around height t wide enough for the decay of K_g . In this form Z_g is the field of the charges of the potential built from C_2 , and the zeros are not an input. We computed Φ from W_N with $N = 200$ and five Bernoulli corrections on a grid of step $h = 0.01$ in $0.2 < \sigma < 0.8$, $5 < t < 55$, shifted so that no node is a zero. The five-point Laplacian $h^2 \Delta_h \Phi / 2\pi$ is concentrated in eleven charges, each of total 1 within $1.2 \cdot 10^{-5}$, with centres on $\sigma = \frac{1}{2}$ and at the ordinates of the zeros within $6 \cdot 10^{-7}$; outside the charges it is the discretization error. The factor $Z_g(\frac{1}{2} + it) = \exp(-\sum_j Q_j K_g(s - w_j))$ built from these charges and their conjugates agrees with the product over the same zeros within $3.4 \cdot 10^{-4}$ for $10 \leq t \leq 50$ (Figure 38). The construction locates the zeros as charges; it does not say where they must be. Status: the identities are exact; the figure is numerical (`anc/zero_charges/`).

Primes and charges in the Jensen moment. Proposition 3.5' splits along Proposition 6.1'. Write $F(s) = (s-1)\zeta(s)/s = P(s)Z_g(s)$ with $P(s) = (s-1)E_g^*(s)/s$. Since $e^{K_g(w)} = e^{-\gamma_0 - \int g \log u} w^{-1} \exp \int g(u) \text{Ein}(wu) du$ with Ein entire, the factor $e^{K_g(s-1)}$ is single-valued with a simple pole at $s = 1$, and P has neither zeros nor poles in $\text{Re } s > \frac{1}{2}$; each factor $e^{-K_g(s-\rho)}$ of Z_g is entire with a single simple zero at ρ . The Poisson–Jensen formula of the proof of Proposition 3.5' gives

$$J_P(\infty) = \log |P(1)|, \quad J_{Z_g}(\infty) = \log |Z_g(1)| + \sum_{\beta > \frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|, \quad \log |P(1)| + \log |Z_g(1)| = 0,$$

with $\log P(1) = \sum_{n \geq 2} \Lambda(n) \psi_g(\log n) / (n \log n) - \gamma_0 - \int g \log u - \sum_{k \geq 1} K_g(1+2k)$. The prime factor is outer (Proposition 6.1'''(iii) below): its moment is a number computed from the primes, and all the information on the position of the zeros is in the factor of the charges. Hence RH holds if and only if $J_{Z_g}(\infty) = -\log |P(1)|$, and each off-line pair adds $\log |\rho/(1-\rho)| > 0$ to the left side. The weight of μ , $1/(\frac{1}{4} + t^2)$, is the limit $1/|\rho|^2$ of C_2 at a zero on the line: both are the Poisson kernel of the half-plane at $s = 1$. For the Gaussian above, $\log |P(1)| = -6.857 \cdot 10^{-11}$ and $\log |Z_g(1)| = 6.855 \cdot 10^{-11}$ from the 1517 zeros below height 2000, which sum to $-1.1 \cdot 10^{-14}$. This is a reformulation of Proposition 3.5', not a new criterion: it separates the part of the Jensen moment fixed by the primes from the part carried by the charges of C_2 . Status: exact given Proposition 6.1' and [BSY]; the numbers are from `anc/zero_charges/jensen_split.py`.

Proposition 6.1''' (the Euler–Cáceres mollifier). Let $s_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$, let $P_{s_0}(t) = (\sigma_0 - \frac{1}{2}) / (\pi((\sigma_0 - \frac{1}{2})^2 + (t - t_0)^2))$ be the Poisson kernel of the half-plane $\text{Re } s > \frac{1}{2}$ at s_0 , and $b_\rho(s) = (s - \rho) / (s - 1 + \bar{\rho})$.

$$(i) \quad \int \log |\zeta(\frac{1}{2} + it)| P_{s_0}(t) dt = \log \left| \frac{(s_0 - 1)\zeta(s_0)}{s_0} \right| + \sum_{\beta > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|}.$$

(ii) For every g as in Proposition 6.1', with $Z_g = \zeta / E_g^*$,

$$\sum_{\beta > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|} \leq B_g(s_0) := \frac{1}{2} \log \int |Z_g(\frac{1}{2} + it)|^2 P_{s_0}(t) dt - \log |Z_g(s_0)|.$$

Consequently, if $\liminf B_g(s_0) = 0$ along a family of weights g for a single s_0 , the Riemann hypothesis holds.

(iii) The identity $\zeta = E_g^* Z_g$ of Proposition 6.1' holds as an identity of meromorphic functions on $\mathbb{C}$, and $\log |Z_g(s)| = -\sum_\rho \text{Re } K_g(s - \rho)$ for every s that is not a zero (in particular at points $\rho - r$, $r > 0$, excluded in Proposition 6.1'). $\log |P|$ is continuous and bounded on

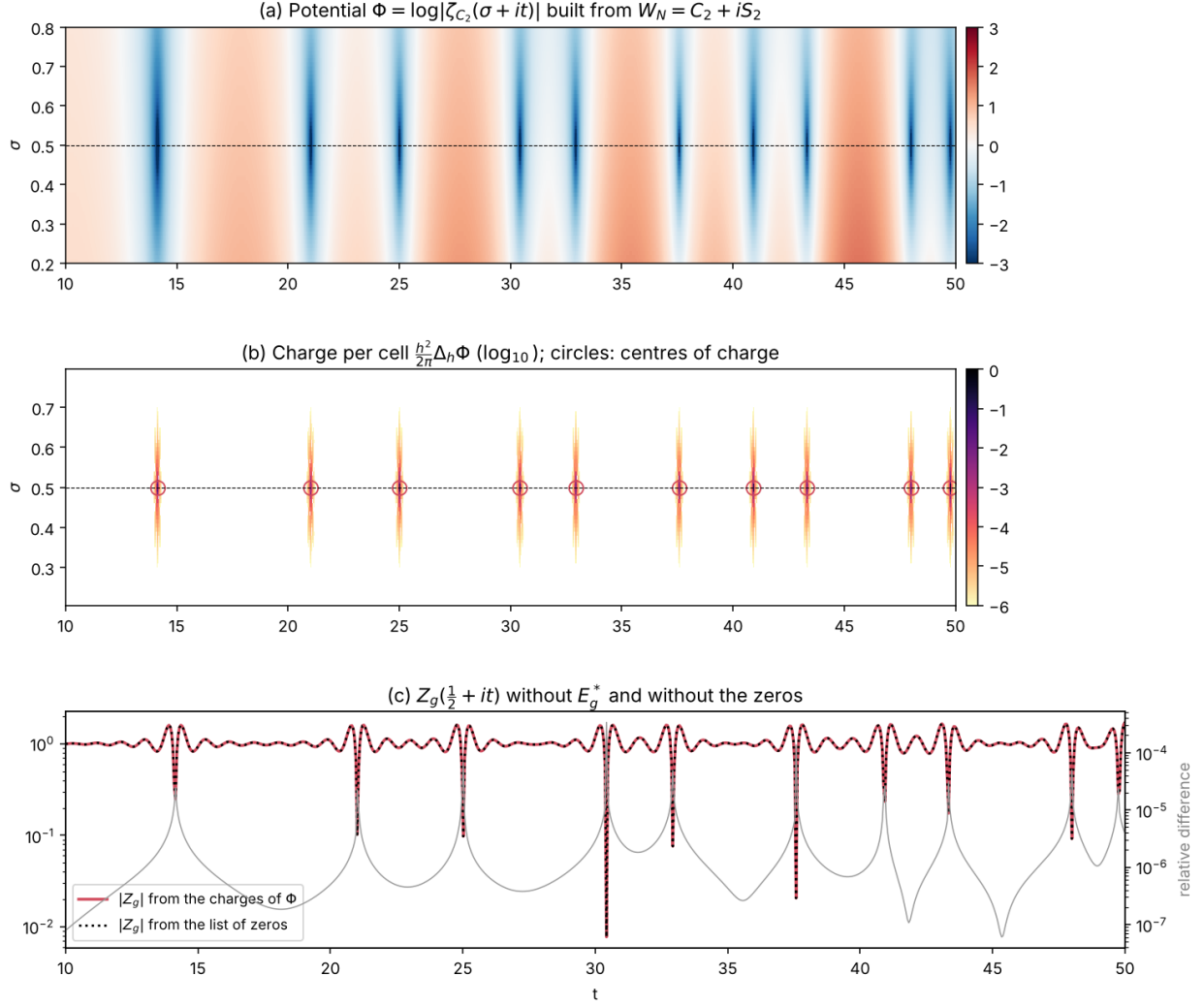


FIGURE 38. The zero factor of Proposition 6.1' computed from C_2 . (a) The potential $\Phi = \log |\zeta|$ built from $W_N = C_2 + iS_2$, $N = 200$. (b) Its charge density $h^2 \Delta_h \Phi / 2\pi$, $h = 0.01$, in $\log_{10}$; circles: centres of the eleven charges, each of total 1. (c) $|Z_g(\frac{1}{2} + it)|$ from these charges against the product over the zeros, Gaussian weight of width 0.4 around $\log X$, $X = 1000.5$; grey, right axis: relative difference.

$\operatorname{Re} s \geq \frac{1}{2}$; hence P is outer, Z_g lies in the Smirnov class N^+ of $\operatorname{Re} s > \frac{1}{2}$, and $\log |Z_g(s)| \leq \int \log |Z_g(\frac{1}{2} + it)| P_s(t) dt$ for $\operatorname{Re} s > \frac{1}{2}$.

Proof. (i) is the Poisson–Jensen formula of Proposition 3.5' at the point s_0 instead of 1, applied to $F(s) = (s-1)\zeta(s)/s$; for $s_0 = 1$ it is Proposition 3.5'. (ii) $P = (s-1)E_g^*/s$ has neither zeros nor poles in $\operatorname{Re} s > \frac{1}{2}$, so $F/P = Z_g$ has the same zeros there as ζ ; Poisson–Jensen for Z_g and Jensen's inequality $\int \log |Z_g| P_{s_0} \leq \frac{1}{2} \log \int |Z_g|^2 P_{s_0}$ give the bound, exactly as in Proposition 3.5'' with the mollifier $M = 1/P$; that P contributes no singular part is (iii). (iii) Across the cut of E_1 the kernel K_g jumps by $2\pi i \int g = 2\pi i$, so $e^{\pm K_g}$ is single-valued and $\operatorname{Re} K_g$ is continuous; the product Z_g converges locally uniformly (absolutely for $g = g_X$ with $g_0 \in C_c^\infty$, by (L1)–(L3) below), so both sides of $\zeta = E_g^* Z_g$ are meromorphic and the identity extends from the open set of Proposition 6.1' to $\mathbb{C}$. For the outer property write $\log |P(s)| = -\log |s| + (\log |s-1| + \operatorname{Re} K_g(s -$

1)) + $\operatorname{Re} D(s) - \sum_k \operatorname{Re} K_g(s + 2k)$, with D the finite prime sum, $|\operatorname{Re} D| \leq \sum \Lambda(n)n^{-1/2}/\log n$. Since $\log |z| + \operatorname{Re} E_1(z) = -\gamma_0 + \operatorname{Re} \operatorname{Ein}(z)$, the bracket is continuous and bounded on $|s - 1| \leq R$; for $|s - 1| \geq R$, $E_1(z) = e^{-z}\varphi(z)$ with $\varphi(z) = \int_0^\infty e^{-\tau}(z + \tau)^{-1}d\tau$ and

$$|\varphi^{(k)}(z)| \leq k! d(z)^{-k-1}, \quad d(z) = \operatorname{dist}(z, (-\infty, 0]),$$

give $|E_1((s - 1)u)| \leq e^{u/2}/d((s - 1)u)$ with $d \geq u\sqrt{R^2 - 1}$, so $K_g(s - 1)$ and $\log |(s - 1)/s|$ are bounded; and $|E_1(z)| \leq e^{-\operatorname{Re} z}/|z|$ bounds the trivial-zero terms. A zero-free function with $\log |P|$ bounded, harmonic and continuous up to the line is the Poisson integral of its boundary values, i.e. outer. By [BSY] F is a Blaschke product times an outer function (no singular inner factor, also at ∞ , since $F(\sigma) \rightarrow 1$), so $Z_g = F/P \in N^+$, and the last inequality is the subharmonic inequality in N^+ . $\square$

The mollifier $1/P$ is not a Dirichlet polynomial: P carries the pole through the factor $e^{K_g(s-1)}$, so the normalization at $s = 1$ no longer forces the step-function approximation of Proposition 3.5'', and the sawtooth constant c_0 disappears. With the Gaussian weight of width 0.4 around $\log X$ and $X = 10, 10^2, 10^3, 10^4$, $B_g(1) = 0.0049, 0.0026, 0.0018, 0.0015$, against the floor $\frac{1}{2} \log(1 + c_0) = 0.0386$ of every Dirichlet mollifier, and $B_g(1 + 100i) = 0.131, 0.101, 0.066, 0.048$, against 0.080 for the optimal Dirichlet polynomial of length 500; identity (i) holds to 10^{-4} at $s_0 = 1, 1 + 85.7i, 1.5 + 100i$ and $2 + 20i$. The decay follows, heuristically, a law in $\log X$. Near a simple zero on the line, $-\operatorname{Re} K_g(i(t - \gamma)) \rightarrow \operatorname{Ci}(|v|)$ with $v = (t - \gamma) \log X$, so $|Z_g|$ has the universal profile $e^{\operatorname{Ci}(|v|)}$ and tends to 1 away from the zeros (Figure 39a). Since $\int_0^\infty \operatorname{Ci}(v) dv = 0$ the moment of $\log |Z_g|$ is unchanged, while each zero adds $\frac{2}{\log X} \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv$ to the energy, whence

$$B_g(s_0) \approx \frac{\kappa W(s_0)}{\log X}, \quad \kappa = \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv = 1.828\dots, \quad W(s_0) = \sum_\rho P_{s_0}(\gamma).$$

For $s_0 = 1$, $W = \frac{1}{2\pi} \sum_\rho |\rho|^{-2} = 0.00735$ and $\kappa W / \log 10^4 = 0.00146$ against 0.0015; for $s_0 = 1 + 100i$, $W = 0.2305$ and the ratio of the measured value to $\kappa W / \log X$ is 1.07 at $X = 10^3$ and 1.05 at $X = 10^4$ (Figure 39b). The Davenport–Heilbronn function separates sharply. Its analogue $Z_f = f/E_f^*$, built from the coefficients of $-f'/f$ and its trivial zeros at $-1, -3, \dots$, satisfies identity (i) with excess 1.4408 at $s_0 = 1 + 85.7i$ against 1.4407 from its off-line zeros, but $B_f(1 + 85.7i) = 1.65, 2.55, 4.49, 7.58$ and $B_f(1 + 100i) = 0.34, 0.31, 1.26, 5.91$ for the same X : near an off-line zero the factor $e^{-K_g(s-\rho)}$ grows with X on the line, since $\operatorname{Re}(s-\rho) < 0$ there, and the bound diverges instead of decaying (Figure 39c). The upper bound $\int |\zeta/E_g^*(\frac{1}{2} + it)|^2 P_{s_0}(t) dt \leq |\zeta/E_g^*(s_0)|^2 (1 + o(1))$ as $X \rightarrow \infty$, a Poisson-weighted form of the splitting of ζ into an Euler and a Hadamard factor [GHK, BGM], is itself equivalent to RH, unconditionally (Corollary 6.1⁽⁶⁾(a)); it holds for ζ in the computed range and fails numerically for f . Status: (i) and (ii) are proved; the law for B_g is a heuristic derivation for simple, isolated zeros on the line; the numbers are computed (`euler_mollifier`).

Theorem 6.1'''' (an off-line zero makes the bound explode). Let $g_0 \in C_c^\infty(\mathbb{R})$, $g_0 \geq 0$, $\int g_0 = 1$, $\operatorname{supp} g_0 \subset [-a, a]$, put $g_X(u) = g_0(u - \log X)$ and $B_X(s_0) = B_{g_X}(s_0)$. Suppose that the real parts of the zeros of ζ attain their supremum Θ , and that $\Theta > \frac{1}{2}$. Then for every s_0 with $\operatorname{Re} s_0 > \Theta$ (in particular every s_0 with $\operatorname{Re} s_0 > 1$) there are $r \in (0, \Theta - \frac{1}{2})$, C and X_0 such that

$$B_X(s_0) \geq \frac{1}{2} \operatorname{Ei}(r(\log X - a)) - C \log \log X \geq \frac{X^r}{3r e^{ra} \log X} \quad (X \geq X_0).$$

Consequently, if $\liminf_{X \rightarrow \infty} \int |\zeta/E_{g_X}^*(\frac{1}{2} + it)|^2 P_{s_0}(t) dt < \infty$ for one s_0 with $\operatorname{Re} s_0 > 1$, then either the Riemann hypothesis holds or the supremum of the real parts of the zeros is not attained.

Proof. Write $Z = Z_{g_X}$, $L = \log X$, $\hat{g}_0(z) = \int g_0(v)e^{-zv} dv$.

(a) *Reduction to a point.* By Proposition 6.1'''(iii) and Jensen's inequality, $\log |Z(s_1)| \leq \int \log |Z| P_{s_1} \leq \frac{1}{2} \log \int |Z|^2 P_{s_1}$ for every s_1 in the half-plane (also when the last integral

is infinite), and since $P_{s_1} \leq H P_{s_0}$ with $H = \sup_t P_{s_1}(t)/P_{s_0}(t) < \infty$, independent of X ,

$$B_X(s_0) \geq \log |Z(s_1)| - \frac{1}{2} \log H - \log |Z(s_0)|.$$

- (b) *The kernel.* Let $w = -x + iy$, $|x| \leq x_{\max}$, $X \geq e^{a+1}$. For every integer $A \geq 0$ there is C depending on A , $x_{\max}$ and g_0 only such that: (L1) if $\operatorname{Re} w \geq \eta > 0$, $|K_{g_X}(w)| \leq C_\eta X^{-\operatorname{Re} w} (1 + |y|)^{-A-1}/L$; (L2) if $\operatorname{Re} w \geq 0$, $w \neq 0$, $|K_{g_X}(w)| \leq C(1 + |\log(|w|L)|)$, and if moreover $|y| \geq 1$, $|K_{g_X}(w)| \leq C|y|^{-A-1}/L$; (L3) if $x > 0$, $y \neq 0$, $|K_{g_X}(w)| \leq e^{x(L+a)}/(|y|(L-a))$, and if moreover $|y| \geq 1$, $|K_{g_X}(w)| \leq Ce^{xa} X^x |y|^{-A-1}/L$; (L4) for $x > 0$, $-\operatorname{Re} K_{g_X}(-x) = \int g_X(u) \operatorname{Ei}(xu) du \geq \operatorname{Ei}(x(L-a))$. Indeed $K_{g_X}(w) = e^{-wL} \int g_0(v) e^{-wv} \varphi(w(v+L)) dv$ with $u = v + L \in [L-a, L+a]$ and $d(wu) = u d(w)$, where $d(w) = |w|$ if $x \leq 0$ and $d(w) = |y|$ if $x > 0$. With no integration by parts the bound on φ in the proof of Proposition 6.1'''(iii) gives $|K_{g_X}(w)| \leq e^{x(L+a)}/(d(w)(L-a))$, the first part of (L3). Integrating by parts A times against e^{-wv} (no boundary terms) gives $K_{g_X}(w) = w^{-A} e^{-wL} \int e^{-wv} \partial_v^A [g_0(v) \varphi(w(v+L))] dv$, and the m -th v -derivative of $\varphi(w(v+L))$ is at most $m! (|w|/d(w))^m / (u d(w))$, with $|w|/d(w) = 1$ if $x \leq 0$ and $\leq 1 + x_{\max}$ if $x > 0$, $|y| \geq 1$. Hence $|K_{g_X}(w)| \leq Ce^{x(L+a)} |w|^{-A} / (d(w)(L-a))$, which gives the second parts of (L2), (L3), and (L1). For the first part of (L2), $|E_1(wu)| \leq |\varphi(wu)| \leq 1$ if $|wu| \geq 1$, and $|E_1(wu)| \leq \gamma_0 + |\log |wu|| + \pi + e$ if $|wu| < 1$. For (L4), $E_1(-\xi \pm i0) = -\operatorname{Ei}(\xi) \mp i\pi$ and Ei is increasing.
- (c) *The point s_1 .* Take a zero $\rho_0 = \Theta + i\gamma_0$ and let $d \in (0, 1]$ be at most the distance from γ_0 to the ordinates of the other zeros of real part Θ (finitely many lie in any bounded window); d depends only on ρ_0 . Let $S = \sum' \min(1/|y|, C|y|^{-3})$ over the zeros $\rho \neq \rho_0$ with $\beta = \Theta$, $y = \gamma_0 - \gamma$, with C the constant of (L3) for $A = 2$, $x_{\max} = \frac{1}{2}$; $S < \infty$ does not depend on r . Choose $r \in (0, \Theta - \frac{1}{2})$ with $2re^{2ra}S \leq \frac{1}{4}$ and such that $s_1 = \rho_0 - r$ is not a zero (a countable set of r is excluded), and let $\delta > 0$ be the distance from s_1 to the zeros. By (iii), $\log |Z(s_1)| = -\sum_\rho \operatorname{Re} K_{g_X}(s_1 - \rho)$, with $w = s_1 - \rho$, $x = r - (\Theta - \beta) \leq r$. *The zero ρ_0 :* $w = -r$, and the term is at least $\operatorname{Ei}(r(L-a))$ by (L4). *The other zeros with $\beta = \Theta$:* $x = r$, $|y| \geq d$, and by (L3) their sum is at most $X^r e^{ra} S / (L-a) \leq \frac{1}{2} X^r e^{-ra} / (r(L-a))$ by the choice of r . *The zeros with $\Theta - r < \beta < \Theta$:* if $y = 0$ (finitely many) the term is $\int g_X \operatorname{Ei}(xu) du \geq 0$ for $X \geq X_0$; if $0 < |y| < 1$ (finitely many) each term is $O(e^{x(L+a)}/L) = o(X^r/L)$; if $|y| \geq 1$, by (L3) each term divided by X^r/L is at most $Ce^{ra} X^{x-r} |y|^{-3} L / (L-a)$, which tends to 0 and is dominated by the summable $2Ce^{ra} |y|^{-3}$; together $o(X^r/L)$. *The zeros with $\beta \leq \Theta - r$:* $\operatorname{Re} w \geq 0$, $|w| \geq \delta$, and by (L2) the finitely many with $|y| < 1$ contribute $O(\log L)$, the others $O(1/L)$. Since $\operatorname{Ei}(\xi) \geq e^\xi/\xi$ for $\xi \geq 1.35$, the main term is at least $X^r e^{-ra} / (r(L-a))$ for $X \geq X_0$, and so $\log |Z(s_1)| \geq \frac{1}{2} \operatorname{Ei}(r(L-a)) - C \log L$.
- (d) *The centre.* $\operatorname{Re}(s_0 - \rho) \geq \operatorname{Re} s_0 - \Theta =: \eta > 0$ for every zero, so by (L1) with $A = 2$, $|\log |Z(s_0)|| \leq C_\eta X^{-\eta} L^{-1} \sum_\rho (1 + |t_0 - \gamma|)^{-3} \rightarrow 0$. Combine (a), (c), (d). $\square$

The theorem does *not* apply as stated to the Davenport–Heilbronn function f , whose analogue E_f^* is built from $-f'/f$: f has infinitely many zeros in $\sigma > 1$, so $\Theta_f > 1$, its attainment is unknown, and the zero ρ_f below is not of maximal real part; nor do we prove the factorization and the Smirnov property for f . Numerically the dominant term (L4) is nevertheless visible for moderate X . At its zero $\rho_f = 0.808517 + 85.699348i$ and $s_1 = \rho_f - r$, $\log |Z_f(s_1)|$ is 1.0332 and 1.6702 for $X = 10^3, 10^4$ with $r = 0.1$, against the main term $\int g \operatorname{Ei}(ru) du = 1.0373$ and 1.6800, and 2.9489, 4.3721 with $r = 0.2$, against 2.9564, 4.3976. At the centre $s_0 = 2 + 85.7i$ of the theorem, B_X is 0.241, 0.121, 0.090, 0.071 for ζ and 0.775, 1.779, 3.822, 7.081 for f , for $X = 10, 10^2, 10^3, 10^4$. The bound B_X separates the two alternatives by its order of growth: under RH it tends to 0 (Theorem 6.1⁽⁵⁾; the rate $\kappa W / \log X$ is heuristic), and it grows like a power of X as soon as a zero of maximal real part lies off the line and Θ is attained. What remains open is described in Remark 6.1^{''''}. Status: Theorem 6.1^{''''} is proved under the stated hypothesis; the statements about f are numerical; the numbers are computed (`euler_mollifier/explosion.py`).

Theorem 6.1⁽⁵⁾ (the converse under RH). Let g_X be as in Theorem 6.1^{'''} and

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} \left| \frac{\zeta}{E_{g_X}^*} \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{1+t^2},$$

which is finite for every $X > e^a$, since by the proof of Proposition 6.1^{'''}(iii) $\log |E_{g_X}^*|$ is bounded below on the line and $|\zeta(\frac{1}{2} + it)| \ll (|t| + 2)^{1/4}$. Assume RH. Then:

- (i) for every $\varepsilon > 0$ there are X_ε and C_ε such that $\log |Z_{g_X}(\frac{1}{2} + it)| \leq \varepsilon \log(|t| + 3) + C_\varepsilon$ for all real t and all $X \geq X_\varepsilon$;
- (ii) $\mathcal{E}(X) \rightarrow \pi$, and $B_X(s_0) \rightarrow 0$ for every s_0 with $\operatorname{Re} s_0 > \frac{1}{2}$.

Proof. Under RH, $\operatorname{Re}(s - \rho) = 0$ for $s = \frac{1}{2} + it$, and $\operatorname{Re} E_1(ix) = -\operatorname{Ci}(|x|)$, so, for t not an ordinate and with multiplicity,

$$\log |Z_{g_X}(\frac{1}{2} + it)| = \sum_{\gamma} c_X(t - \gamma), \quad c_X(y) = \int g_X(u) \operatorname{Ci}(|y|u) du.$$

Three bounds for c_X , with $L = \log X$: (c1) $c_X(y) \leq \max \operatorname{Ci} = \operatorname{Ci}(\frac{\pi}{2}) = 0.4720$, since $g_X \geq 0$ and $\int g_X = 1$; (c2) $|c_X(y)| \leq 2/(|y|(L - a))$, from $\operatorname{Ci}(x) = \frac{\sin x}{x} - \int_x^\infty \frac{\sin \tau}{\tau^2} d\tau$, so $|\operatorname{Ci}(x)| \leq 2/x$ (indeed $\leq 1/x$); (c3) $|c_X(y)| \leq C|y|^{-3}/L$ for $|y| \geq 1$, by (L2) of Theorem 6.1^{'''}. Under RH, $S(T) \ll \log T / \log \log T$ (Littlewood), so an interval of length $h \leq 1$ centred at t contains at most $\frac{h}{2\pi} \log(|t| + 3) + C \log(|t| + 3) / \log \log(|t| + 3)$ ordinates. By (c1) the zeros with $|t - \gamma| < 1/L$ contribute at most $0.472(\frac{1}{\pi L} + \frac{C}{\log \log(|t| + 3)}) \log(|t| + 3)$; by (c2) on dyadic shells $2^k/L \leq |t - \gamma| < 2^{k+1}/L$ the zeros with $|t - \gamma| < 1$ contribute $\ll (\frac{\log L}{L} + \frac{1}{\log \log(|t| + 3)}) \log(|t| + 3)$; by (c3) the others contribute $\ll \log(|t| + 3)/L$; treating $|t| \leq T_\varepsilon$ separately gives (i). For (ii), for each t that is not an ordinate, (c2) and (c3) give $\sum_{\gamma} |c_X(t - \gamma)| \leq (L - a)^{-1} (\sum_{|t - \gamma| < 1} 2/|t - \gamma| + C \sum_{|t - \gamma| \geq 1} |t - \gamma|^{-3}) \rightarrow 0$, so $\log |Z_{g_X}(\frac{1}{2} + it)| \rightarrow 0$ for almost every t ; by (i) with $\varepsilon = \frac{1}{4}$, $|Z_{g_X}|^2 \leq C(|t| + 3)^{1/2}$, which is integrable against $dt/(1 + t^2)$, and dominated convergence gives $\mathcal{E}(X) \rightarrow \int dt/(1 + t^2) = \pi$. Finally $\operatorname{Re}(s_0 - \rho) = \operatorname{Re} s_0 - \frac{1}{2} > 0$ for every zero, so $\log |Z_{g_X}(s_0)| \rightarrow 0$ by (L1) of Theorem 6.1^{'''}, and $\int |Z_{g_X}|^2 P_{s_0} \rightarrow 1$ in the same way. $\square$

Corollary 6.1⁽⁶⁾ (criteria). (a) For every s_0 with $\operatorname{Re} s_0 > \frac{1}{2}$ and $\zeta(s_0) \neq 0$, unconditionally,

$$\text{RH} \iff \lim_{X \rightarrow \infty} B_X(s_0) = 0.$$

(b) If the supremum Θ of the real parts of the zeros is attained (for instance, if there are only finitely many zeros off the line),

$$\text{RH} \iff \limsup_{X \rightarrow \infty} \mathcal{E}(X) < \infty \iff \liminf_{X \rightarrow \infty} \mathcal{E}(X) < \infty.$$

Proof. (a) If RH fails, by Proposition 6.1^{'''}(ii) $B_X(s_0)$ is bounded below by the off-line contribution, a positive number independent of X ; if RH holds, $B_X(s_0) \rightarrow 0$ by Theorem 6.1⁽⁵⁾. (b) RH gives $\mathcal{E}(X) \rightarrow \pi$ by Theorem 6.1⁽⁵⁾. Conversely, if $\Theta > \frac{1}{2}$ is attained, fix s_0 with $\operatorname{Re} s_0 > \max(\Theta, 1)$; since $P_{s_0}(t) \leq C/(1 + t^2)$, $B_X(s_0) \leq \frac{1}{2} \log(C\mathcal{E}(X)) - \log |Z_{g_X}(s_0)|$, with $\log |Z_{g_X}(s_0)| \rightarrow 0$ by part (d) of the proof of Theorem 6.1^{'''}; as $B_X(s_0) \rightarrow \infty$, $\mathcal{E}(X) \rightarrow \infty$, so already $\liminf \mathcal{E}(X) < \infty$ excludes $\Theta > \frac{1}{2}$. $\square$

Part (a) is a Jensen-type reformulation, since its lower bound does not depend on X ; the new content is (b), where a single weighted mean square, with no normalization at a point, detects an attained off-line supremum by blowing up. Without the attainment hypothesis, $\limsup \mathcal{E}(X) < \infty$ gives only “RH or Θ not attained”. With the weight of Remark 6.1^{'''}(iii), the ratio

$$R_X = \max_{t < 600} \frac{\log |Z_{g_X}(\frac{1}{2} + it)|}{\log(t + 2)}$$

is 0.185, 0.169, 0.170 for $X = 10^2, 10^3, 10^4$, and on $200 \leq t < 600$ it decreases, 0.179, 0.161, 0.146, consistent with (i). Status: Theorem 6.1⁽⁵⁾ is proved assuming RH; Corollary 6.1⁽⁶⁾(a) is unconditional and (b) is proved under the stated hypothesis on Θ ; the numbers are computed (`euler_mollifier/largeX_bump.py`).

Remark 6.1^{''''} (what is left). (i) *A single centre*. The theorem needs one s_0 , and for fixed s_0 with $\operatorname{Re} s_0 > 1$ the kernel $P_{s_0}(t)$ is comparable to $1/(1+t^2)$ while $\log |Z_{g_X}(s_0)| \rightarrow 0$. The height of s_0 therefore plays no role: the question is the limit $X \rightarrow \infty$ of the single weighted mean square

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} \left| \frac{\zeta}{E_{g_X}^*} \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{1+t^2},$$

and Theorem 6.1^{''''} says that $\liminf_X \mathcal{E}(X) < \infty$ implies RH unless the supremum Θ of the real parts is larger than $\frac{1}{2}$ and not attained; by Theorem 6.1⁽⁵⁾ the converse holds. (ii) *The case Θ not attained* (a heuristic outline). For each X choose ρ^* maximizing $\beta \log X - A \log(2 + |\gamma|)$ (finitely many candidates, $|\gamma^*| \leq X^{2/A}$ for large X). The zeros at distance ≥ 1 from ρ^* then contribute at most $rS X^r / \log X$ at $s_1 = \rho^* - r + i\tau$, and a vertical shift τ aligns the common phase $e^{-i\tau \log X}$. What remains is the cluster amplitude $\sum_{|\gamma - \gamma^*| < 1} X^{\beta - \beta^*} e^{i(\gamma - \gamma^*) \log X} \hat{g}_0(w)/w$, a sum of Cauchy kernels in τ . It is small only if off-line zeros of nearly maximal real part close to ρ^* cancel each other; for two zeros of equal real part at distance Δ we expect the cancellation to be incomplete, with a remainder of order $X^r \Delta / (r^2 \log X)$, so a lower bound $\Delta \geq X^{-o(1)}$ for the separation of such zeros would plausibly close this case. No such bound is known, and averaging over X , simultaneous Dirichlet approximation of the phases and the positivity of the Gram matrix of the Cauchy kernels all meet the same obstruction, clusters of nearly coincident zeros. (iii) *The limit $X \rightarrow \infty$ numerically*. With a weight $g_0 \in C_c^\infty$ on $[-1.2, 1.2]$, for which the prime sum is exactly finite, at $s_0 = 2$: $B_X = 0.0138, 0.0074, 0.0052, 0.0039, 0.0032, 0.0026$ for $X = 10, \dots, 10^6$, and $B_X \log X = 0.032, 0.034, 0.036, 0.036, 0.036, 0.037$ against $\kappa W(2) = 0.040$; the Jensen identity holds to $1.3 \cdot 10^{-4}$. With the Gaussian weight truncated at 4.5 standard deviations the neglected tail of the prime sum grows like $X^{1/2}$ and inflates B_X from $X = 10^5$ on, which is why the compactly supported weight is used. (iv) *What a proof of RH along this line requires*: two things, both open. First, the bound $\limsup_X \mathcal{E}(X) < \infty$, a mean-value statement related to the hybrid Euler–Hadamard products of Gonek–Hughes–Keating, uniform in the length of the Euler factor; by Corollary 6.1⁽⁶⁾(b) it is equivalent to RH when Θ is attained, so it is at least as hard as RH in that case. Second, the case of (ii), $\Theta > \frac{1}{2}$ not attained. The factor $E_{g_X}^*$ is built from the prime powers up to $e^a X$ and the explicit pole and trivial-zero terms only; it has no zeros and its definition involves none, and the zeros enter only through the quotient $Z_{g_X} = \zeta / E_{g_X}^*$. The quantity $\mathcal{E}(X)$ therefore compares $|\zeta|$ on the critical line with a finite, zero-free factor built from the primes, and the explicit formula makes its boundedness a statement about zeros. Heuristically, a proof of the bound cannot rest on mean-value estimates valid for every Dirichlet series with a functional equation, since the bound fails numerically for the Davenport–Heilbronn function (we have not proved that $\mathcal{E}_f(X)$ is unbounded); it must use a local property of the prime coefficients, such as $\Lambda(n) \geq 0$ supported on prime powers (a single Euler product), which the coefficients $b(n)$ of $-f'/f$ lack. Status: (i) is a reformulation of Theorem 6.1^{''''}; (ii) is a heuristic reduction, not a proof; (iii) is numerical (`euler_mollifier/largeX_bump.py`); (iv) is partly heuristic, as stated.

The geometry of Z_X . Figure 40 shows what the factorization $\zeta = E_{g_X}^* Z_{g_X}$ does, with the weight of Remark 6.1^{''''}(iii). (a) On the critical line the truncated Euler product $E_X = \prod_{p \leq X} (1 - p^{-s})^{-1}$ does not converge and oscillates around $|\zeta|$, while the smoothed factor $E_{g_X}^*$, built from the prime powers up to $e^a X$ and the explicit pole and trivial-zero terms only, follows $|\zeta(\frac{1}{2} + it)|$ closely and never vanishes: at γ_1 it equals 0.097 for $X = 10^2$ and 0.048 for $X = 10^4$. It is an envelope made of primes that passes over each zero without touching it. (b) The quotient $Z_{g_X} = \zeta / E_{g_X}^*$, what is left after the primes are removed, is close to 1 on the line except for a narrow well of

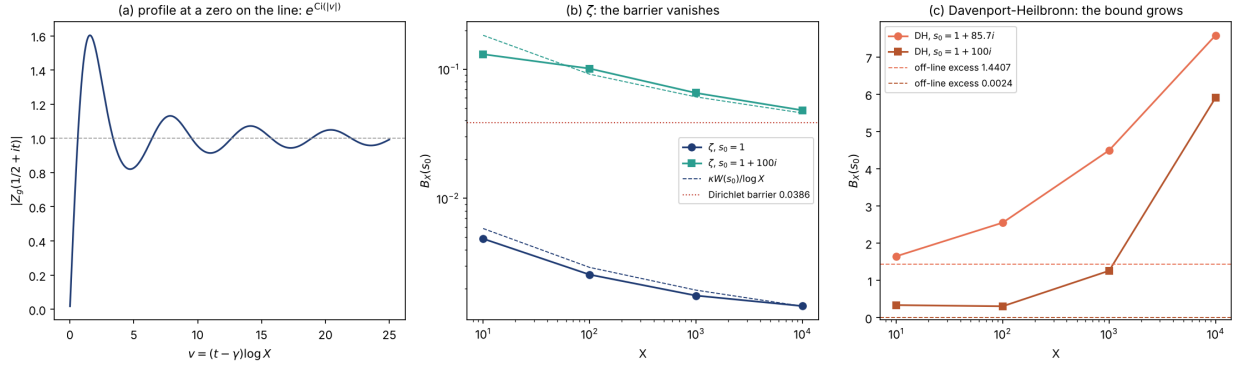


FIGURE 39. The Euler–Cáceres mollifier (Proposition 6.1'''). (a) Profile $e^{\text{Ci}(|v|)}$ of $|Z_g|$ at a zero on the line, $v = (t - \gamma) \log X$. (b) $B_g(s_0)$ for ζ at $s_0 = 1$ and $1 + 100i$, the heuristic law $\kappa W(s_0)/\log X$ (dashed), and the floor 0.0386 of Dirichlet mollifiers (dotted). (c) $B_f(s_0)$ for the Davenport–Heilbronn function, which grows with X ; dashed: its off-line excess.

width of order $1/\log X$ at each zero, flanked by an overshoot of height $e^{\max \text{Ci}} = e^{0.472} = 1.60$; this is the profile $e^{\text{Ci}(|v|)}$, $v = (t - \gamma) \log X$, of Proposition 6.1''', and the narrowing of the wells as X grows is behind the (heuristic) decay $B_X \approx \kappa W/\log X$. (c) In the strip, $\log |Z_{g_X}|$ is flat to the right of the line. The factor $e^{-K_g(s-\rho)}$ of a zero grows like $\exp(X^{\beta-\sigma}/((\beta-\sigma)\log X))$ where $\text{Re}(s-\rho) < 0$, so every zero has a *volcano* to its left and is seen from the right only as a *crater*. For a zero on the line the volcano lies in $\sigma < \frac{1}{2}$, outside the half-plane of the Poisson–Jensen formula. (d) For the Davenport–Heilbronn function the volcano of the off-line zero $0.8085 + 85.6993i$ starts at $\sigma = 0.81$ and crosses the critical line, where it produces the growth of $|Z_f|$ seen in Figure 39c, the mechanism of Theorem 6.1'''. In these terms Corollary 6.1⁽⁶⁾(b) says that, when Θ is attained, RH holds if and only if only craters reach the critical line, and the open problem is to prove this from properties of the primes, without knowing in advance where the zeros are. Status: numerical illustration (`euler_mollifier/geometry_ZX.py`).

The reflection asymmetry. For $s^* = 1 - \bar{s}$ put $D_X(s) = \log |Z_{g_X}(s)| - \log |Z_{g_X}(s^*)|$. By the functional equation $|\zeta(s^*)| = |\zeta(s)|/|\chi(s)|$ the values of ζ cancel, and

$$D_X(s) = \log |\chi(s)| - \log |E_{g_X}^*(s)| + \log |E_{g_X}^*(s^*)|,$$

a quantity built from the primes and the gamma factor alone; the analogue for f uses $f(s) = X_f(s)f(1-s)$. The two expressions agree to $2 \cdot 10^{-13}$ on $\sigma = 0.7$ for ζ and for f (Figure 41d). D_X looked like the place where the two ingredients meet, the primes through E^* and the functional equation through χ , but it does not separate ζ from the controls. Reindexing the zeros by $\rho \mapsto 1 - \bar{\rho}$, which the functional equation allows, gives

$$D_X(s) = \sum_{\rho} h(s - \rho), \quad h(w) = -\text{Re} [K_g(w) - K_g(-\bar{w})],$$

the same kernel for every zero, on the line or off it, so that D_X is a prime-side form of the explicit formula in which only the positions of the zeros matter. Accordingly the three pictures look alike (Figure 41a–c): ζ ; the Davenport–Heilbronn function; and a Beurling-type toy with the nonnegative prime measure $(1 - 2 \text{Re } x^{\rho_B - 1}) dx / \log x$ on $[16, \infty)$, $\rho_B = 0.75 + 20i$, whose zeta function $\log \zeta_B(s) = E_1((s-1)\log 16) - E_1((s-\rho_B)\log 16) - E_1((s-\bar{\rho}_B)\log 16)$ has the zeros $\rho_B, \bar{\rho}_B$, a positive prime measure and no functional equation. In ζ the bands come from the volcanoes of the zeros on the line, reflected. The prime side of D_X also contains $E_{g_X}^*$ at $\sigma < \frac{1}{2}$, where the prime sum is larger than $\sqrt{X}$, so a bound for D_X from the positivity of Λ meets the pole barrier of the

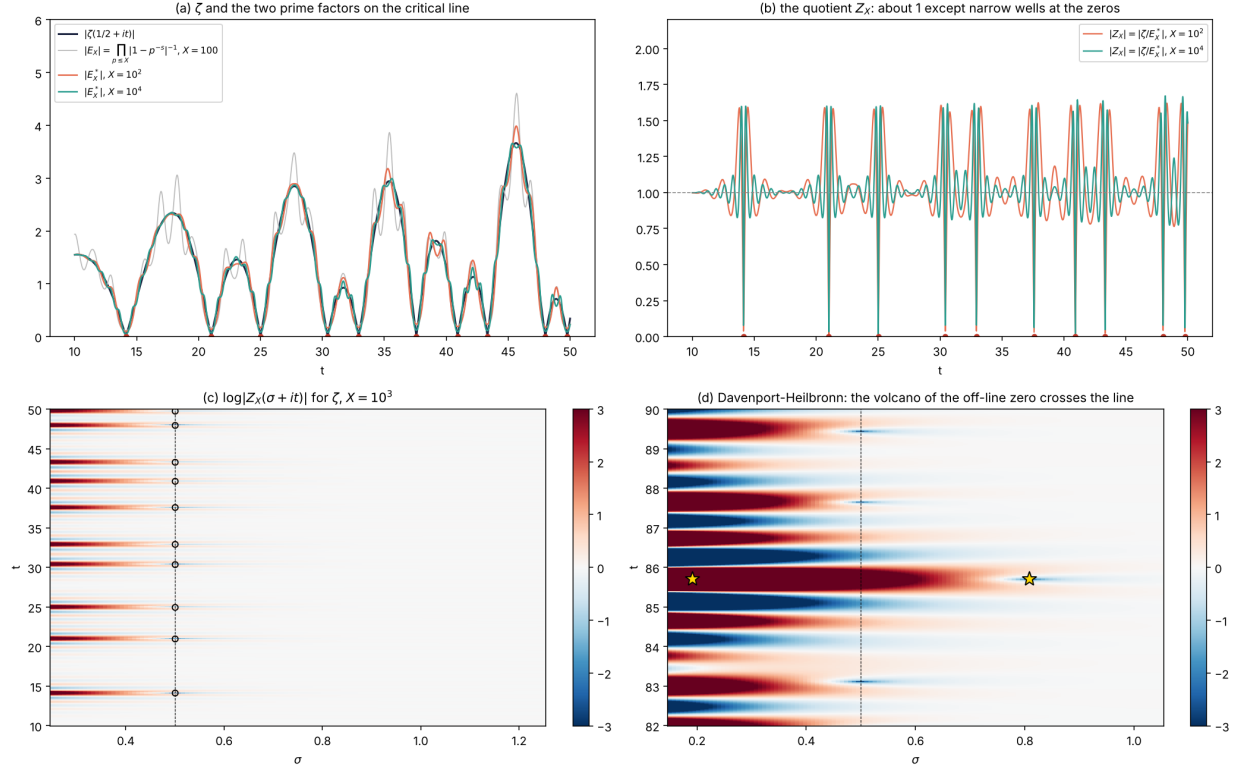


FIGURE 40. The geometry of Z_X . (a) $|\zeta(\frac{1}{2} + it)|$, the truncated Euler product $|E_X|$ with $X = 100$, and the Euler-Cáceres factor $|E_X^*|$ for $X = 10^2, 10^4$; dots: zeros. (b) $|Z_X| = |\zeta/E_X^*|$ on the line: wells at the zeros with the overshoot $e^{0.472}$. (c) $\log|Z_X(\sigma + it)|$ for ζ , $X = 10^3$: craters on the line, volcanoes to the left. (d) The same for the Davenport-Heilbronn function near its off-line zero (stars: the zero and its reflection); its volcano crosses the line.

paragraph *Positivity and the pole* below, in reflected form. Status: the identity is exact; the figures are numerical (`euler_mollifier/reflection_asymmetry.py`).

The Davenport-Heilbronn control. The analogue of Proposition 6.1 holds for f with $\log f = \sum b(n)n^{-s}/\log n$, without a pole term. At height 85.6993 and $x = 2 \cdot 10^5$ the relative error

$$\left| \exp \left(\sum_{n \leq x} \frac{b(n)}{n^s \log n} \right) / f(s) - 1 \right|$$

is 0.075 and 0.200 for $\sigma = 0.95$ and 0.90 , to the right of $\rho_f = 0.8085 + 85.6993i$, and 1.334 and 1.062 for $\sigma = 0.75$ and 0.70 , to its left, with no decrease from $x = 10^4$. The equivalence of convergence with the absence of zeros to the right is a property of logarithmic Dirichlet series in general. What distinguishes ζ is that its coefficients $\Lambda(n)/\log n$ are nonnegative and supported on prime powers; for f , 72% of $\sum_{n \leq 2 \cdot 10^5} |b(n)|/n$ lies on integers that are not prime powers [C5].

The apparent abscissa. The smoothed form measures, from the primes alone, the real parts of the zeros near a given height. Differentiating Proposition 6.1' with g the normal density of mean $\Lambda = \log X$ and variance σ_g^2 gives, for $F = \zeta$ or $F = f$ and coefficients $c(n) = \Lambda(n)$ or $b(n)$,

$$R(s; \Lambda) := -\frac{F'}{F}(s) - \sum_n c(n) \psi_g(\log n) n^{-s} + \epsilon_F G(s-1) = \sum_\rho G(s-\rho) + (\text{trivial zeros}),$$

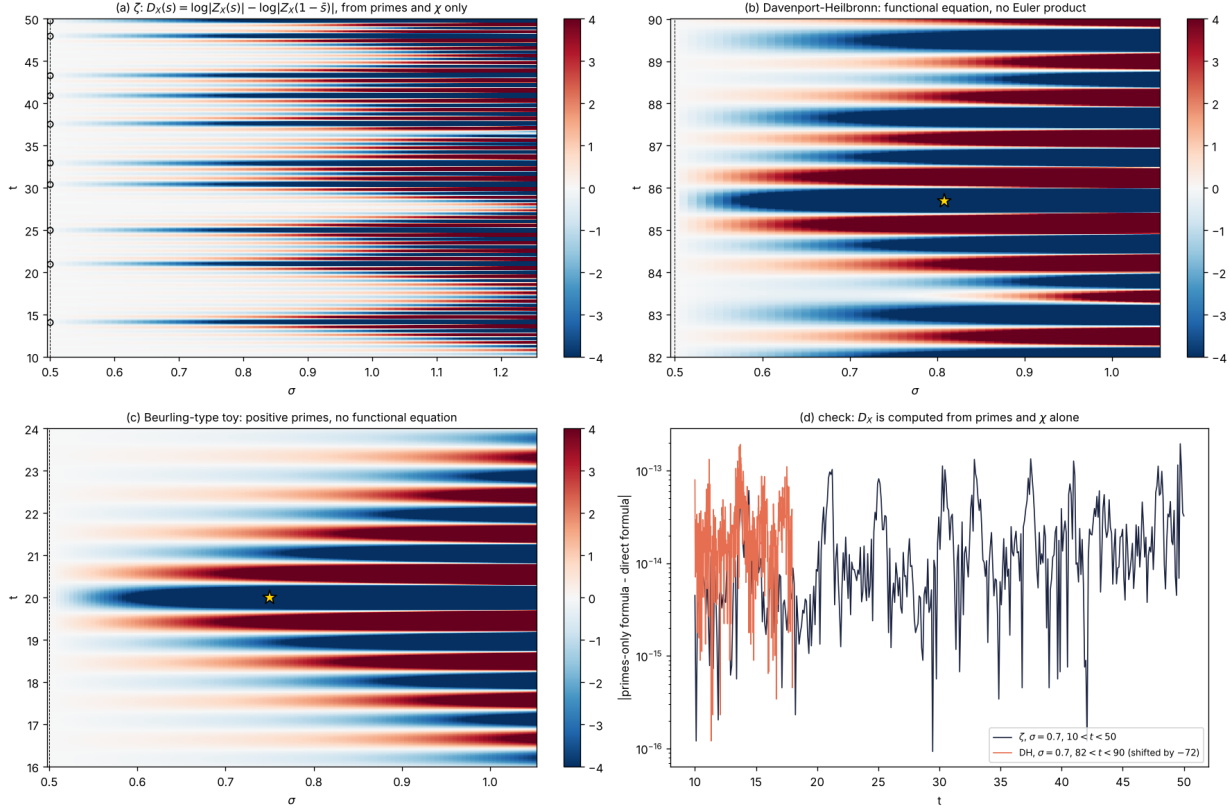


FIGURE 41. The reflection asymmetry $D_X(s) = \log |Z_X(s)| - \log |Z_X(1 - \bar{s})|$, $X = 10^3$, computed from the primes and χ only. (a) ζ ; circles: zeros. (b) The Davenport–Heilbronn function; star: its off-line zero. (c) The Beurling-type toy with positive prime measure and no functional equation; star: its zero $0.75 + 20i$. (d) Difference between the primes-only formula and the direct one, on $\sigma = 0.7$.

with $\epsilon_\zeta = 1$, $\epsilon_f = 0$ and

$$G(w) = \frac{e^{-w\Lambda + w^2\sigma_g^2/2}}{w}.$$

Each zero contributes $|G(s - \rho)| \approx e^{-(\sigma - \beta)\Lambda} e^{-(t - \gamma)^2\sigma_g^2/2} / |s - \rho|$, so the growth rate of $\log |R(\sigma + it; \Lambda)|$ in Λ is $-(\sigma - \beta)$ for the largest β among the zeros within about $1/\sigma_g$ of the height t , and $\Theta(t) = \sigma + \frac{d}{d\Lambda} \log |R|$ is an apparent abscissa at height t . For ζ the residual computed from the prime powers agrees with $\sum_\rho G(s - \rho)$ over the zeros below 2000 to 10^{-10} . With $\sigma = 1$, $\sigma_g = 0.8$, $4 \leq \Lambda \leq 11$, the coefficients up to $7 \cdot 10^6$, and the slope fitted to the root mean square of R over windows of length 2 in Λ , the apparent abscissa of ζ on $10 \leq t \leq 120$ has median 0.500, with 90% of the values in $[0.47, 0.54]$. For the Davenport–Heilbronn function it has a plateau at 0.806 around $t = 85.7$, against $\beta = 0.8085$, and one at 0.631 around $t = 114.2$, against $\beta = 0.6508$ (Figure 42). Two zeros on the line closer than about 1 produce a beat whose period in Λ exceeds the window, and the fit reads it as growth: such pairs give narrow false peaks, at $t \approx 40.6$ for f , whose zeros 40.163 and 40.683 are 0.52 apart, and at $t \approx 88$ and 95 for ζ . They are separated from off-line zeros by their width, and would be removed by a longer range of Λ or by demodulating at the frequency γ as for the prime signal in Section 3. The apparent abscissa is a local detector computed from the primes, not a bound valid at all heights. Status: the identity is Proposition 6.1'; the computations are numerical (`anc/theta/`, Hurwitz zeta functions by Euler–Maclaurin in double precision).

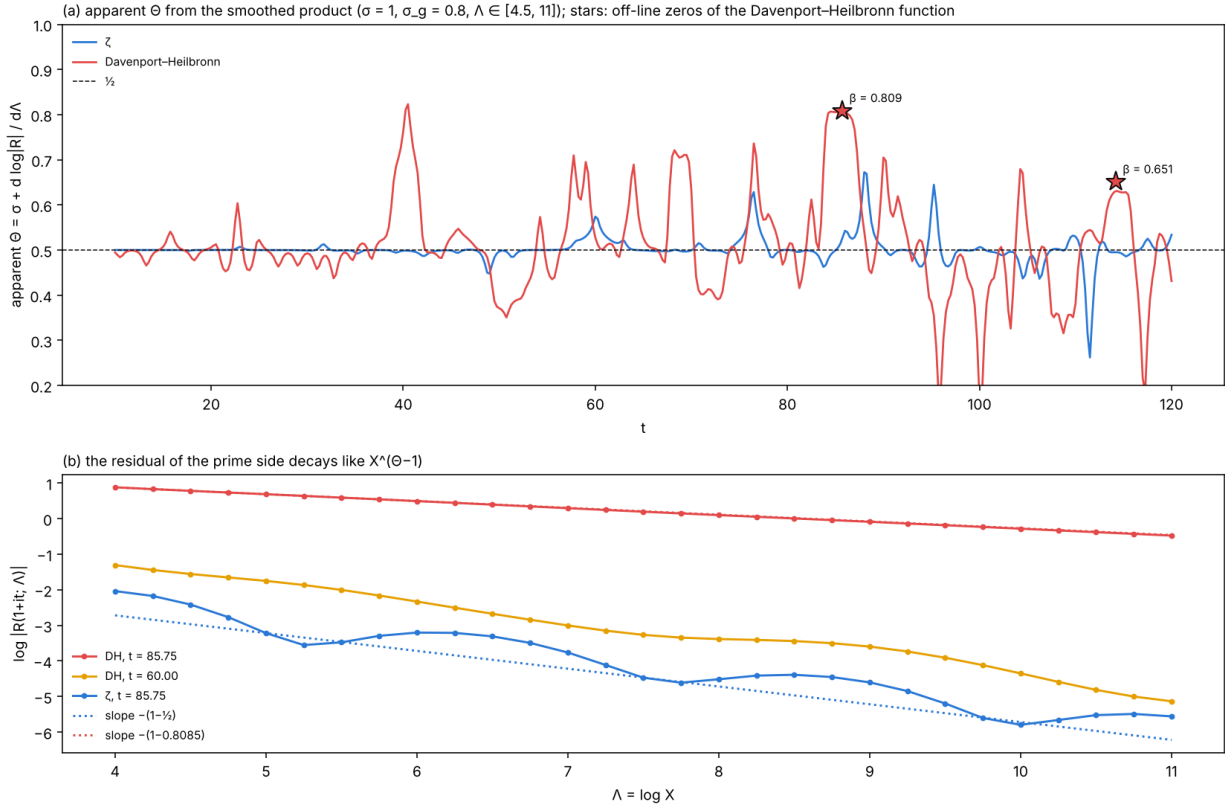


FIGURE 42. The apparent abscissa from the smoothed product. (a) $\Theta(t) = \sigma + \frac{d}{d\Lambda} \log |R(\sigma + it; \Lambda)|$ for ζ and for the Davenport-Heilbronn function, $\sigma = 1$, $\sigma_g = 0.8$; stars: the off-line zeros of f below height 120. (b) $\log |R|$ against Λ at $t = 85.75$ for both functions and at $t = 60$ for f , with the slopes $-(1 - \frac{1}{2})$ and $-(1 - 0.8085)$.

Resonance of prime polynomials. Let $R_N(s) = \sum_{n \leq N} (\Lambda(n) - 1) n^{-s}$. For $s \neq 1$ in the strip it differs from $\sum_{n \leq N} \Lambda(n) n^{-s} - N^{1-s}/(1-s)$ by $\zeta(s) + O(N^{-\sigma})$, so it is the prime sum with the wave of the pole removed, and by the explicit formula what remains is a superposition of waves $N^{\rho-s}/(\rho-s)$, one for each zero, of the same form $N^{\beta-\sigma} \cos((\gamma-t) \log N + \phi)$ as the wave $N^{1-\sigma} \cos(t \log N + \phi)$ of the pole. RH is equivalent to $R_N(\frac{1}{2} + i\gamma_0) = O(N^\varepsilon)$ for every $\varepsilon > 0$ at a single real γ_0 . Under RH, $\psi(x) - [x] = O(x^{1/2} \log^2 x)$ and partial summation give $R_N(\frac{1}{2} + i\gamma_0) = O((1 + |\gamma_0|) \log^3 N)$. Conversely, the bound makes the Dirichlet series $\sum_n (\Lambda(n) - 1) n^{-s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s)$, in which the poles at $s = 1$ cancel, convergent for $\sigma > \frac{1}{2} + \varepsilon$, so ζ'/ζ has no poles there. One frequency controls every height, but only in the limit: at finite N a zero at height γ is seen at γ_0 with amplitude $N^{\beta-1/2}/|\rho - \frac{1}{2} - i\gamma_0|$. The computation therefore measures the resonance exponent

$$E_T(N) = \frac{\log \max_{10 \leq \gamma \leq T} |R_N(\frac{1}{2} + i\gamma)|}{\log N}, \quad T = 100,$$

with the pole wave removed up to a bounded term, and the same quantity for $L(s, \chi) \bmod 5$ (coefficients $\chi(n)\Lambda(n)$, no pole) and for the Davenport-Heilbronn function (coefficients $c(n)$ of $-f'/f$, obtained by Dirichlet inversion and checked against $-f'/f$ at $s = 2$ to 10^{-9}). For ζ and $L(s, \chi)$ with $N \leq 2 \cdot 10^7$ the maximum follows $\log N$ (16.4 at $N = 2 \cdot 10^7$, where $\log N = 16.8$) and $E_T(N)$ follows $\log \log N / \log N$. For f with $N \leq 4 \cdot 10^6$ the maximum sits at $\gamma = 85.7$ from $N \approx 100$ on, and $E_T(4 \cdot 10^6) = 0.385$, against $0.3085 + \log(1/0.3085)/\log N = 0.386$ predicted by

the zero $0.8085 + 85.699i$ (Figure 43). A windowed version separates the waves one by one. With a Hann window w of width 6 in $u = \log n$, the height of the peak of $|\sum_n a(n)n^{-1/2-i\gamma w(\log n - u_0)}|$ at an ordinate γ , after the pole wave is removed, is $3e^{(\beta-1/2)u_0}$ up to the leakage of neighbouring zeros. For the 21 isolated zeros of ζ and the 19 of $L(s, \chi)$ below height 100 the fitted slopes are below 0.003 and 0.01 in absolute value; for f the peak at 85.70 grows by a factor 8 with slope 0.303, against $\beta - \frac{1}{2} = 0.3085$. As for the apparent abscissa, two zeros on the line closer than the resolution $2\pi/6$ beat; the beat stays below twice the height of one zero, and the off-line peak exceeds that bound from the first window on. The same zero is seen in the approximate functional equation. At $x = y = \sqrt{5\gamma/2\pi} = 8.26$ the halves $\sum_{n \leq x} a(n)n^{-\rho}$ and $\chi_f(\rho) \sum_{n \leq y} a(n)n^{\rho-1}$ have moduli 0.343 and 0.276, whereas on the critical line the two halves always have equal moduli and only a phase has to match. The truncated sum alone already vanishes at $0.724 + 85.704i$, and after $n = 1$ the largest cancelling terms are $n = 4$ and $n = 6$, where $a(4) = -1 \neq a(2)^2$ and $a(6) = 1 \neq a(2)a(3)$. If a is replaced by the completely multiplicative function with the same values at the primes, the truncated sum has no zeros off the line for $80 \leq t \leq 91$, and its value at ρ rises from 0.09 to 0.82. At ρ the two Euler products are nonzero, $|L(\rho, \chi)| = |L(\rho, \bar{\chi})| = 0.702$, and in opposite phase; their truncations to the primes up to 10^7 reproduce them to $2 \cdot 10^{-3}$. The zero of f is an interference of two Euler products [C5], and in the truncated sum it is carried by the coefficients at which multiplicativity fails. What this leaves is the question whether a Dirichlet polynomial with multiplicative coefficients can resonate at one frequency with amplitude $N^{\beta-1/2}$, $\beta > \frac{1}{2}$; by the criterion above it is equivalent to RH, and it is not answered here. Status: the criterion is classical (von Koch, Landau [In]); the computations are numerical (`anc/resonance/`, double precision for the sums, `mpmath` for the values of L and f).

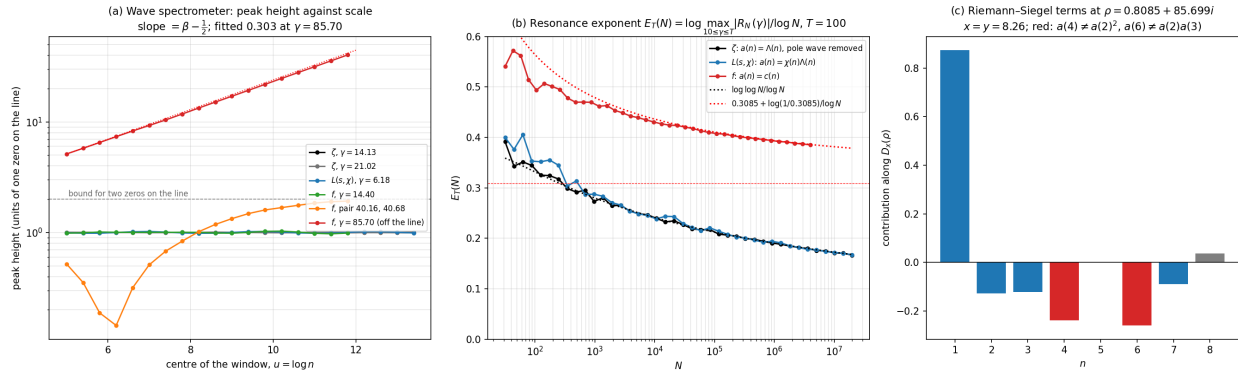


FIGURE 43. Resonance of prime polynomials. (a) Height of the spectral peak at an ordinate against the centre of a Hann window of width 6 in $\log n$, in units of the height of one zero on the line; dotted: slope 0.3085. (b) The resonance exponent $E_T(N)$, $T = 100$, for ζ , $L(s, \chi) \bmod 5$ and the Davenport–Heilbronn function f , with the predictions $\log \log N / \log N$ and $0.3085 + \log(1/0.3085) / \log N$. (c) Contributions of the terms of the approximate functional equation of f at its zero $0.8085 + 85.699i$, projected on the direction of the first half.

The C-transform of the primes. For coefficients $a(n)$ let $C_N(s; a) = \sum_{n \leq N} a(n)n^{-s} - \epsilon_a \int_1^N x^{-s} dx$, with $\epsilon_a = 1$ if $\sum a(n)n^{-s}$ has a simple pole of residue 1 at $s = 1$ and $\epsilon_a = 0$ otherwise. For $a = 1$ this is $X_N(s) - (N^{1-s} - 1)/(1 - s)$, which tends to $\zeta(s) + 1/(1 - s)$ for $\sigma > 0$, $s \neq 1$, with error $\frac{1}{2}N^{-s} + O(N^{-\sigma-1})$. For $a = \Lambda$ the transform at $s = 0$ is the error term of the prime number theorem, $C_N(0; \Lambda) = \psi(N) - N + 1$, and its abscissa of convergence is $\Theta = \sup \beta$. Partial summation and $\psi(x) - x = O(x^{\Theta+\varepsilon})$ give convergence to $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ for $\sigma > \Theta$; convergence at

a point with $\sigma < \Theta$ would make this function analytic to the right of that point, which the zeros with $\beta > \sigma$ forbid. RH is therefore the statement that the C-transform of the primes converges in $\sigma > \frac{1}{2}$, as the transform of 1 converges in $\sigma > 0$. The same argument applies to the multiplicative transform $\sum_{p \leq N} -\log(1-p^{-s}) - \int_2^N \frac{-\log(1-x^{-s})}{\log x} dx$, the transform of $\log \zeta$. Numerically, at $t = 85.7$ with $N \leq 10^7$, the exponent of $|C_N(s; a) - \lim_N C_N(s; a)|$ fitted on dyadic blocks from 2^{12} is $-\sigma$ to three decimals for $a = 1$ and within 0.02 of $\frac{1}{2} - \sigma$ for $a = \Lambda$, for $0.55 \leq \sigma \leq 1.1$. For the coefficients $c(n)$ of $-f'/f$, with $N \leq 4 \cdot 10^6$, it is $0.8085 - \sigma$ to three decimals ($+0.209$ at $\sigma = 0.6$, $+0.009$ at $\sigma = 0.8$, -0.092 at $\sigma = 0.9$), the wave of the zero $0.8085 + 85.699i$ (Figure 44). This is the measured exponent near one height, not the abscissa of $\sum c(n)n^{-s}$ itself, which is at least 1 because f has zeros in $\sigma > 1$ [DH]. The measurement does not extend: all zeros below height $3 \cdot 10^{12}$ are on the line [PT], so an off-line zero is seen at a fixed $s_0 = \frac{1}{2} + i\gamma_0$ with amplitude $N^{\beta-1/2}/|\rho - s_0|$ and $|\rho - s_0| \geq 3 \cdot 10^{12} - |\gamma_0|$, and its wave exceeds the level $\log N$ only for $N \gtrsim 10^{29}$ if $\beta = 0.99$, $N \gtrsim 10^{150}$ if $\beta = 0.6$ and $N \gtrsim 10^{1600}$ if $\beta = 0.51$. Status: the abscissa statement is classical; the computations are numerical (anc/ctransform/).

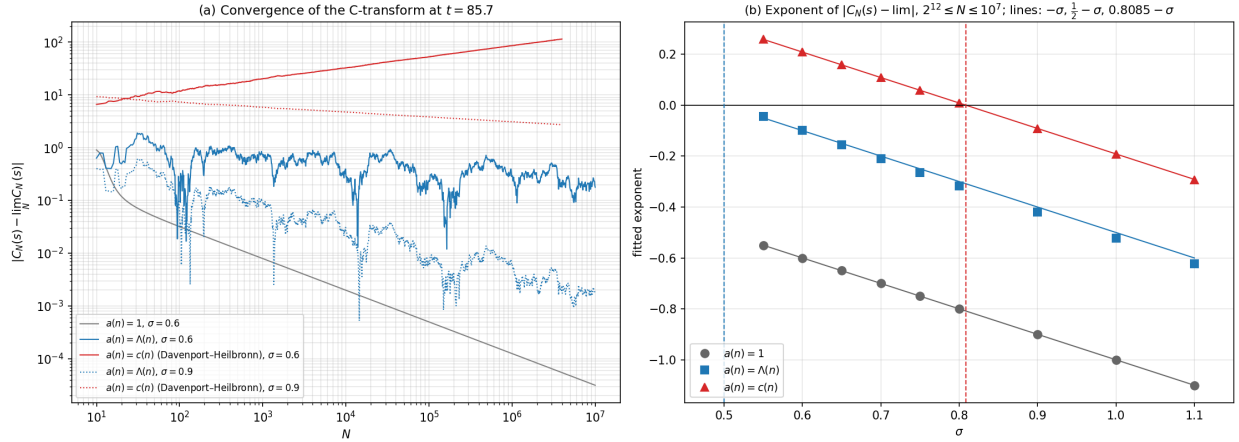


FIGURE 44. The C-transform $C_N(s; a)$. (a) Distance to the limit at $t = 85.7$ for $a = 1$, $a = \Lambda$ and the coefficients $c(n)$ of $-f'/f$, at $\sigma = 0.6$ and 0.9 . (b) Fitted exponent against σ , with the lines $-\sigma$, $\frac{1}{2} - \sigma$ and $0.8085 - \sigma$; the abscissae $\frac{1}{2}$ and 0.8085 are marked.

A Beurling control. The Davenport–Heilbronn function has the functional equation and no Euler product. The opposite control is a Beurling system of generalized primes, which has unique factorization and an Euler product but whose integers are not $\mathbb{Z}$. Diamond, Montgomery and Vorhauer [DMV] constructed such systems with well-distributed integers whose zeta function has zeros as close to $\sigma = 1$ as the region of de la Vallée Poussin allows, so unique factorization and a regular count of integers do not by themselves exclude zeros off the line. We built a simplified example with one prescribed zero. The prime-power density is $1 - 2 \operatorname{Re} x^{\rho-1}$ for $x \geq 20$, with $\rho = 0.75 + 20i$, so that $\psi_P(x) = x - 2 \operatorname{Re}(x^\rho/\rho) + \dots$, and the primes p_k are the points where the counting function $\Pi(x) = \int f(u) du / \log u$ equals $k - \frac{1}{2}$. This gives 663 888 generalized primes below 10^7 (against 664 579 primes), beginning 1.739, 2.462, 3.546, $\dots$. Their products, 2 269 793 of them below 10^6 , are counted by $N_P(x) = 2.2715x - 1.65\sqrt{x} + O(10^2)$ on $10^3 \leq x \leq 10^6$, against $[x] = x + O(1)$ for the integers. All three diagnostics separate the system from the primes (Figure 45). The error $|\psi_P(x) - x|$ follows $2x^{0.75}/|\rho|$; its fitted exponent on dyadic blocks up to 10^7 is 0.61 , against 0.50 for the primes. The exponent of the increments of the C-transform at $t = 20$ is $0.75 - \sigma$ to within 0.005 for $0.55 \leq \sigma \leq 1$, so the abscissa is 0.75 , while for the primes it stays negative

for $\sigma \geq 0.55$ and lies within 0.05 of $\frac{1}{2} - \sigma$. The prime sum $R_N(\frac{1}{2} + i\gamma_0)$ grows like $N^{0.25}/0.25$ at $\gamma_0 = 20$ and stays between 1 and 10 at $\gamma_0 = 30$, as it does for the primes at both frequencies. Together with the Davenport–Heilbronn function this gives a control for each half of the structure of ζ . A proof that uses only unique factorization and the density of the integers would apply to the Beurling system, and one that uses only the functional equation would apply to f . What ζ has and neither control has is that the integers generated by the primes are exactly $1, 2, 3, \dots$, equally spaced, which is the source of the functional equation through Poisson summation. Status: numerical (`anc/beurling/`); the construction is a simplified form of [DMV], and no error bound is claimed for its zeta function beyond the measured exponents.

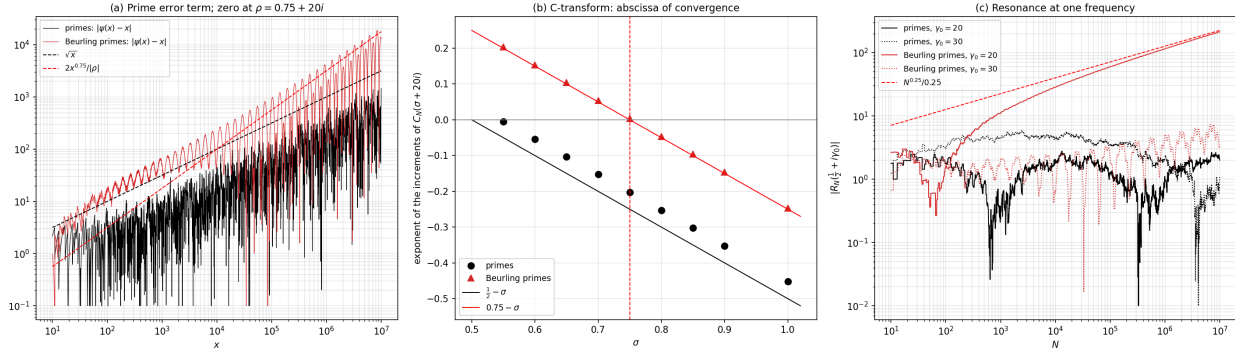


FIGURE 45. A Beurling system with a zero at $\rho = 0.75 + 20i$, against the primes. (a) $|\psi(x) - x|$, with $\sqrt{x}$ and $2x^{0.75}/|\rho|$. (b) Exponent of the increments of the C-transform at $t = 20$ on dyadic blocks up to 10^7 , with the lines $\frac{1}{2} - \sigma$ and $0.75 - \sigma$. (c) $|R_N(\frac{1}{2} + i\gamma_0)|$ at $\gamma_0 = 20$ and 30 , with $N^{0.25}/0.25$.

Positivity and the pole. For every x , $\sigma > 0$ and real t , the inequality of Mertens and de la Vallée Poussin [T, §3.3],

$$3 \log |E_x(\sigma)| + 4 \log |E_x(\sigma + it)| + \log |E_x(\sigma + 2it)| = \sum_{p \leq x} \sum_k \frac{p^{-k\sigma}}{k} (3 + 4 \cos \theta + \cos 2\theta) \geq 0,$$

with $\theta = kt \log p$, holds exactly. By Proposition 6.1 the corrected products satisfy the same identity with the pole corrections subtracted. At $(\sigma, t) = (0.7, 14.13)$ and $x = 2 \cdot 10^6$ the positive sum is 70.77, the pole correction -74.45 , and the total -3.69 , against the limit $3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| = -3.77$. Inside the strip the positive sum grows like $3x^{1-\sigma}/((1-\sigma) \log x)$ and the correction from the real point σ cancels it. This cannot be avoided: a nonnegative trigonometric polynomial $\sum a_j \cos j\theta$ has $a_0 > 0$, and the a_0 term sits at $t = 0$, at the pole. Every positivity argument of this type pays a correction of size $x^{1-\sigma}$, which is why such arguments give zero-free regions only near $\sigma = 1$. The same happens for every kernel tested: with $P(\theta) = 3 + 4 \cos \theta + \cos 2\theta$ and the Fejér kernels of degree 2, 4, 8 and 16 (for which $a_1/a_0 = 2n/(n+1)$ stays below 2), the combination $\sum_k a_k \operatorname{Re}(-\zeta'/\zeta)(\sigma + ikt)$ at $t = \gamma_1$ is positive for $\sigma > 1$ and already negative at $\sigma = 0.98$, and with the pole term removed it still tends to $-\infty$ as $\sigma \rightarrow \frac{1}{2}$, because of the zero $\frac{1}{2} + i\gamma_1$ on the line. Below $\sigma = 1$ the sum over prime powers diverges, and the positivity of its terms says nothing about the value of its continuation. Counting only the leading terms, a zero at $\beta + it$ lowers the left side by about $4x^{\beta-\sigma}/((\beta-\sigma) \log x)$, while the margin left by the pole is about $3x^{1-\sigma}/((1-\sigma) \log x)$. Optimizing over σ , the inequality can exclude exactly the zeros with $\beta > 1 - 0.427/\log x$, with the same constant for x from 10^3 to 10^{100} ; with x a power of t this is the shape of the zero-free region of de la Vallée Poussin [T, Ch. III].

Twisting by characters does not remove this term.

Proposition 6.4. Let $F(\theta) = \sum_{|j| \leq J} c_j e^{ij\theta}$ be a nonnegative trigonometric polynomial, not identically zero, let ψ be a Dirichlet character mod q , and put

$$\Phi_x(\sigma, t) = \sum_{|j| \leq J} c_j \log E_x(\sigma + ijt, \psi^j), \quad \log E_x(s, \psi) = \sum_{p \leq x} \sum_{k \geq 1} \frac{\psi(p)^k p^{-ks}}{k}.$$

Then $\operatorname{Re} \Phi_x(\sigma, t) \geq 0$ for all x , $\sigma > 0$ and t , and its $j = 0$ term is $c_0 \log E_x(\sigma, \psi^0)$ with $c_0 > 0$ and ψ^0 the principal character mod q . For $\sigma < 1$ this term is $c_0 x^{1-\sigma} / ((1-\sigma) \log x) (1 + o(1))$.

Proof. The prime power p^k , $p \nmid q$, contributes $p^{-k\sigma} k^{-1} F(k(\arg \psi(p) - t \log p)) \geq 0$. The constant coefficient c_0 is the mean of F , which is positive. The asymptotic is the prime number theorem in the form $\sum_{p \leq x} p^{-\sigma} \sim x^{1-\sigma} / ((1-\sigma) \log x)$. $\square$

The same argument applies to several characters at once, with F a nonnegative trigonometric polynomial on a torus. A twist rotates the phase of each prime by $\arg \psi(p)$ but leaves the mean of F attached to the principal character, which carries the pole. For a single nonprincipal character there is no pole: the uncorrected product $E_x(s, \chi)$ for the character mod 5 of Section 2 approaches $L(s, \chi)$ in the strip, with $|E_x/L - 1| = 0.070, 0.025, 0.017, 0.009, 0.003$ at $s = 0.75 + 85.7i$ for $x = 10^3, 10^4, 10^5, 10^6, 2 \cdot 10^6$, where the logarithmic series of f stays above 1.3. But no linear inequality with nonnegative coefficients can be built from it alone. The quadratic analogue, averaging $|\sum_p \chi(p) a_p|^2$ over a family of characters, is the large sieve, which is how the density estimates used in Theorem D are proved. **The Euler helix.** Proposition 6.1 has a geometric form. For $s = \sigma + it$ with $t \neq 0$, not a zero, and $u = \log x$, put

$$H_s(u) = u \left(\sum_{n \leq e^u} \frac{\Lambda(n)}{n^s \log n} + E_1((s-1)u) - \log \zeta(s) \right),$$

the corrected partial product without the term R_x , minus its limit, multiplied by u . Since $E_1(w) = e^{-w} w^{-1} (1 + O(|w|^{-1}))$, Proposition 6.1 gives

$$H_s(u) = \sum_{\rho} \frac{e^{(\rho-s)u}}{s-\rho} \left(1 + O\left(\frac{1}{|s-\rho|u}\right) \right).$$

Each zero contributes a vector of length $e^{(\beta-\sigma)u} / |s-\rho|$ that turns with angular velocity $\gamma - t$. With u as a third coordinate this is a helix of constant radius if $\beta = \sigma$, an expanding spiral if $\beta > \sigma$ and a contracting one if $\beta < \sigma$. On the critical line, in the frame that turns with e^{itu} , the pair $\rho, \bar{\rho}$ with $\rho = \frac{1}{2} + i\gamma$ contributes

$$-i \left(\frac{e^{i\gamma u}}{t-\gamma} + \frac{e^{-i\gamma u}}{t+\gamma} \right),$$

two circles turning in opposite directions, that is, an ellipse with semi-axes $|t-\gamma|^{-1} \pm |t+\gamma|^{-1}$. The pairs of zeros on the line are ellipses, as for the prime signal of Section 3, now centred at the point s .

Remark 6.5 (energy of the Euler helix). Assume RH, and let $s = \frac{1}{2} + it$ with t not an ordinate. If the mean square of H_s is computed term by term, one obtains

$$\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U |H_s(u)|^2 du = \sum_{\gamma} \frac{m_{\gamma}^2}{(t-\gamma)^2} \geq \sum_{\gamma} \frac{m_{\gamma}}{(t-\gamma)^2} = -\frac{d^2}{dt^2} \log |\xi(\tfrac{1}{2} + it)|,$$

where the sums run over the distinct ordinates and m_{γ} is the multiplicity. The first equality is Parseval's identity for the almost periodic sum above; the second follows from the Hadamard product, in which each zero contributes $m_{\gamma} \log |t-\gamma|$ to $\log |\xi(\frac{1}{2} + it)|$. Equality holds if and only if the zeros are simple, as in Theorem B. If a zero has $\beta > \frac{1}{2}$, its spiral grows like $e^{(\beta-\frac{1}{2})u}$ and the mean square is infinite. We do not justify the interchange of the sum over the zeros with the mean; the computation below supports it.

At $t = 20$, H_s computed from the prime powers up to 10^8 agrees with the sum over the 682 zeros with $|\gamma| < 600$ to a root mean square of 0.056 on $8 < u < 18.4$ when both are sampled at the points of a grid of step 0.005, against a root mean square of 1.02 for H_s itself; the discrepancy is a sampling effect (Remark 6.6). The running mean of $|H_s|^2$ is 1.045 at $u = 18.4$. The sum over the zeros is 1.074, or 1.077 with the tail estimate for $|\gamma| > 600$, and $-(\log |\xi|)''(20) = 1.077$ computed directly from ξ . At $\sigma = 0.3$ and 0.7 , $|H_s|$ grows and decays like $e^{\pm 0.2u}$ (Figure 46). Seen from $\sigma = 0.3$ every zero lies to the right and produces an expanding spiral, as an off-line zero would on the critical line. The construction is the explicit formula and Parseval's identity in product form. It shows the Euler product inside the strip as a superposition of circles, one per zero, but it does not exclude spirals.

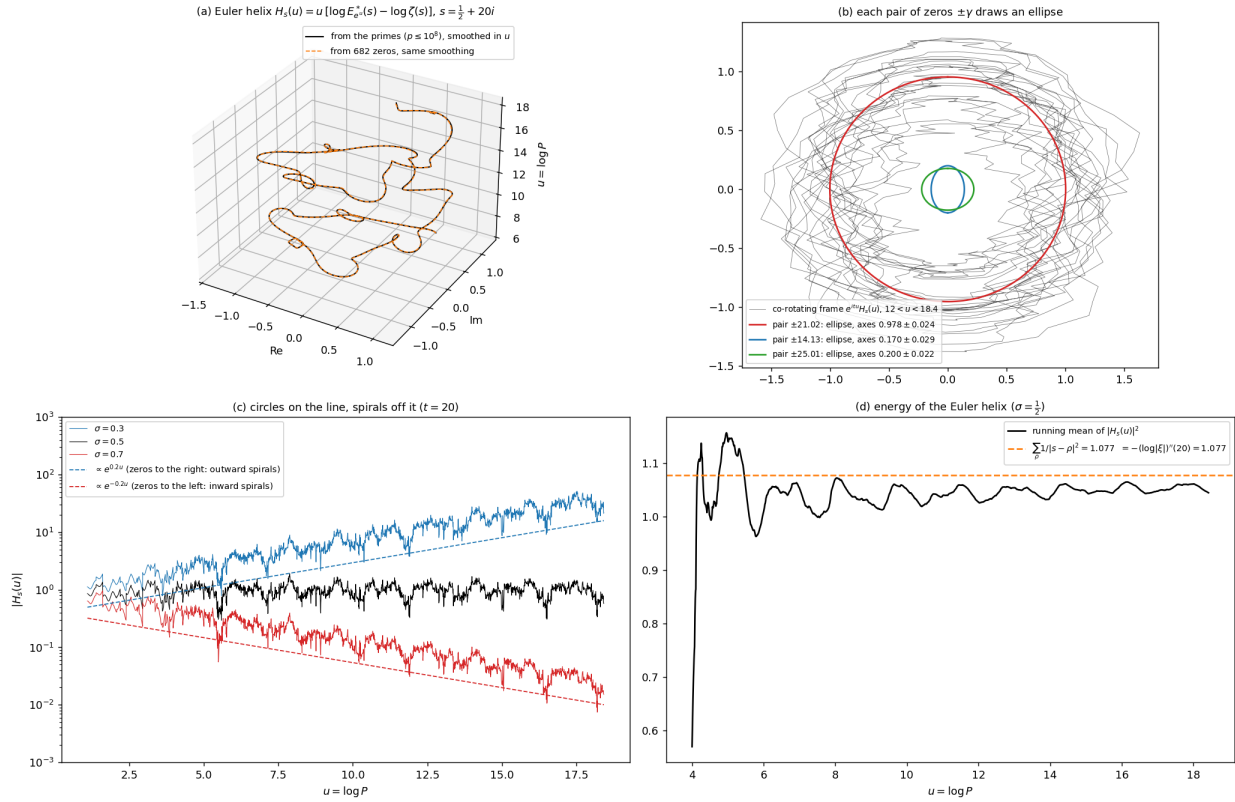


FIGURE 46. The Euler helix (Remark 6.5). (a) $H_s(u)$ at $s = \frac{1}{2} + 20i$ from the prime powers up to 10^8 (solid) and from the 682 zeros with $|\gamma| < 600$ (dashed), both smoothed in u . (b) In the frame turning with e^{itu} , the curve $e^{itu}H_s(u)$ for $12 < u < 18.4$ and the ellipses of the pairs ± 21.02 , ± 14.13 , ± 25.01 . (c) $|H_s(u)|$ at $t = 20$ for $\sigma = 0.3, 0.5, 0.7$. (d) Running mean of $|H_s|^2$ at $s = \frac{1}{2} + 20i$ against $\sum_p |s - \rho|^{-2} = -(\log |\xi|)''(20)$.

Remark 6.6 (rigidity of the Euler helix). Proposition 6.1 is an identity, so the agreement between primes and zeros in Remark 6.5 is limited by the computation and not by the primes. H_s is a step function in u , and the zeros with $|\gamma| > 600$, which are not in the sum, contribute oscillations of every frequency; sampled at isolated points they alias onto the grid and produce the root mean square 0.056. We therefore average H_s exactly over bins of width $h = 5 \cdot 10^{-4}$, using for

the pole term and for each zero the primitive

$$\int u E_1(cu) du = \frac{u^2}{2} E_1(cu) - \frac{e^{-cu}}{2} \left(\frac{u}{c} + \frac{1}{c^2} \right),$$

and convolve with a Gaussian of width 0.02 in u , which removes the contribution of the zeros with $|\gamma \mp t| > 300$. For 45 heights $16.5 \leq t \leq 117.7$ and the prime powers up to 10^8 , the smoothed H_s then agrees with the smoothed sum over the zeros to a relative root mean square between $2 \cdot 10^{-7}$ and $8 \cdot 10^{-6}$ (median $2.3 \cdot 10^{-6}$), with no growth on $10 \leq u \leq 18.3$ (Figure 47a). The remaining error is numerical.

Replace a zero $\rho = \frac{1}{2} + i\gamma$ and its conjugate by the pairs $\frac{1}{2} \pm \delta + i\gamma$ and $\frac{1}{2} \pm \delta - i\gamma$, each of multiplicity $\frac{1}{2}$, which respects the functional equation. The first-order terms in δ cancel, and the sum over the zeros changes by

$$\frac{\delta^2}{2} \partial_\beta^2 [uE_1((s - \rho)u) + uE_1((s - \bar{\rho})u)] + O(\delta^4), \quad \partial_\beta^2 uE_1((s - \rho)u) = u^3 E_1''((s - \rho)u).$$

A symmetric pair is therefore seen by the helix only at second order. To measure it we fit the residual $H_s - \sum_\rho$ by least squares on the second derivatives in β of the three zeros nearest to t , together with their first derivatives in γ , after the same smoothing; the coefficient of the second derivative estimates $\delta^2/2$. On the actual data these coefficients are noise, and twice their standard deviation over the 45 heights defines a threshold $\delta_{\min}$. With the primes up to x we obtain

$\delta_{\min} = 1.3 \cdot 10^{-4}, 0.9 \cdot 10^{-4}, 0.5 \cdot 10^{-4}, 0.6 \cdot 10^{-4}, 0.4 \cdot 10^{-4}$ for $\log x = 10, 12, 14, 16, 18.3$, a decrease close to $(\log x)^{-1.9}$ (Figure 47b). When the zero at 21.022 is split by $\delta = 0.003, 0.01, 0.03$ in synthetic data that keep the true residual, the fit returns 0.0030, 0.0100 and 0.0291 for every x between e^{10} and 10^8 ; the last value is low by the term of order δ^4 (Figure 47c). Without smoothing, a direct comparison of root mean squares could not separate the split zeros from the true ones below $\delta \approx 0.03$. That threshold came from the sampling and not from the second-order nature of the effect.

The Euler product with primes up to 10^8 thus fixes the real parts of the zeros near a given height to within a few units of 10^{-5} , and this precision improves as x grows. The computation does not approach RH, for three reasons. (i) A measurement bounds $|\beta - \frac{1}{2}|$ but never gives $\beta = \frac{1}{2}$. Symmetry closes this gap locally: if a disc centred on the critical line contains exactly one zero counted with multiplicity, then ρ and $1 - \bar{\rho}$ both lie in it, so $\rho = 1 - \bar{\rho}$ and $\beta = \frac{1}{2}$. This is the logic of Turing's method (Section 2). With interval arithmetic and rigorous bounds for the zeros outside the model, the computation would verify from the primes alone that the zeros near height t lie on the line. (ii) This verification is local: the number of primes needed grows with t , and no finite computation covers all heights. (iii) The fit compares the helix with a list of known zeros. A zero outside the list appears only as a residual, and excluding it requires control of all zeros at once, which is RH itself or a density estimate. The smoothed H_s is Weil's explicit formula evaluated on Gaussian test functions; using it to locate zeros is standard in numerical work, and RH is equivalent to the positivity of Weil's functional on all test functions (Section 5), which no finite family reaches. Status: numerical.

Remark 6.7 (amplification and parity). For curves over finite fields the analogue of RH is proved by amplification. If C has genus g over $\mathbb{F}_q$, then $N_k - q^k - 1 = -\sum_{i \leq 2g} \alpha_i^k$, and a bound $|N_k - q^k - 1| \leq Cq^{k/2}$ valid for all k forces $|\alpha_i| = \sqrt{q}$, because a root with $|\alpha| > \sqrt{q}$ produces a deviation that grows geometrically in k . Figure 48a shows this for $q = 101$: the numerator $1 - 20T + 101T^2$ of an elliptic curve stays within the Weil bound, while $1 - 25T + 101T^2$, whose counts N_k and numbers of closed points are nonnegative integers but which is not the numerator of a curve, leaves it by a factor $4.5 \cdot 10^5$ at $k = 20$. Stepanov's method [St, Bo73] proves the upper bound $N_k \leq q^k + O(q^{k/2})$ with an auxiliary polynomial; the lower bound needs an auxiliary family. In the example the upper bound holds for every k and the analogue of RH fails: the large root is

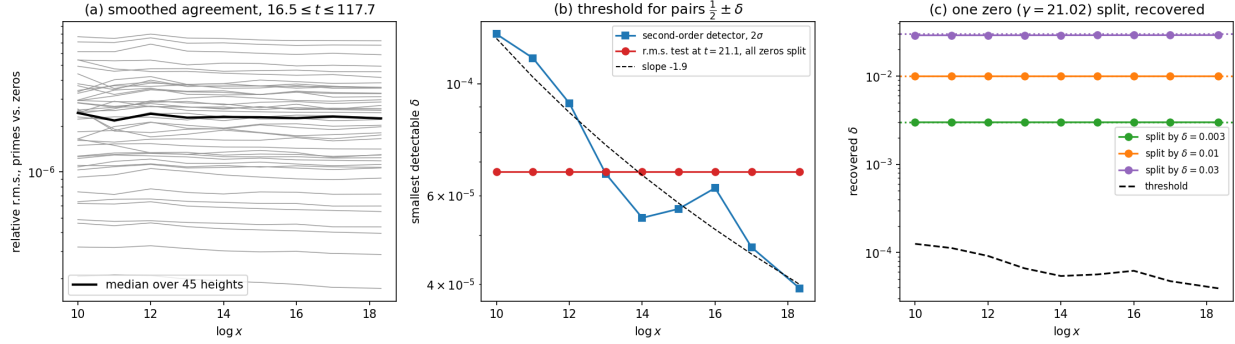


FIGURE 47. Rigidity of the Euler helix (Remark 6.6), with the prime powers up to $x \leq 10^8$. (a) Relative root mean square of the difference between the smoothed H_s computed from the primes and from the zeros, at 45 heights $16.5 \leq t \leq 117.7$ (grey) and their median (black). (b) Threshold $\delta_{\min}$ of the second-order detector for symmetric pairs $\frac{1}{2} \pm \delta$ (squares), with a fitted power law, and the threshold of a direct root-mean-square comparison when every zero is split, at $t = 21.1$ (circles). (c) The zero at $\gamma = 21.022$ split by $\delta = 0.003, 0.01, 0.03$ in synthetic data, and the values of δ recovered by the detector.

real, the analogue of a real zero. Deligne [De] extended the amplification to all varieties through tensor powers.

Over $\mathbb{Q}$ the powers α^k become x^ρ , and $N_k - q^k - 1$ becomes $x - \psi(x)$. Here one side suffices. If $\psi(x) \leq x + Cx^\theta$ for all large x , the Mellin transform of the nonnegative function $x + Cx^\theta - \psi(x)$ is $C/(s - \theta) + 1/(s - 1) + \zeta'(s)/(s\zeta(s))$ up to an entire function, which is regular on the real axis for $s > \theta$ since ζ has no zeros in $(0, 1)$. By Landau's theorem on transforms of nonnegative functions its abscissa of convergence is at most θ , and ζ has no zeros with $\beta > \theta$ [In, Ch. V]. RH is therefore equivalent to an upper bound for the primes, $\psi(x) \leq x + O(x^{1/2} \log^2 x)$, which is the form the amplification would take over $\mathbb{Q}$.

Upper bounds for primes are what sieves provide, and Selberg's Λ^2 sieve is a positivity method: it minimizes the quadratic form $\sum_n \left(\sum_{d|n, d \leq R} \lambda_d \right)^2$ with $\lambda_1 = 1$, as Stepanov's method minimizes the size of an auxiliary polynomial. On the intervals $(10^8, 10^8 + y]$, with $R = \sqrt{y}$ and the optimal weights, the bound exceeds the number of primes by the factors 3.07, 2.61, 2.24 for $y = 10^4, 10^5, 10^6$ (Figure 48b). Splitting the sum according to Liouville's function gives $859 + 832, 7159 + 6966$ and $62341 + 59012$ for $\lambda(n) = -1$ and $\lambda(n) = +1$. The sieve data $\#\{n \equiv 0 \pmod d : \lambda(n) = \pm 1\}$ agree within the statistical fluctuation for every $d \leq R$, so the sieve cannot distinguish the half that contains the primes from the half that contains none. A factor close to 2 is the parity barrier [FI10]; the remaining factor 1.15 at $y = 10^6$ is the ratio $\log x / \log y$ of the level of distribution, and $2y / \log y$ is the Brun–Titchmarsh bound [MoV].

The barrier is not a numerical gap. With no zeros at all, the estimate $\psi(x + y) - \psi(x) \approx y$ is correct to 0.1%, 0.3% and 1.5% for the three intervals, and the 341 zeros below 600 do not change this materially (Figure 48c). What is missing is an inequality. The information that separates the primes from the products of two primes in the sifted set is the single sum $\sum \lambda(n)$ over the sifted n , and bounding it is the cancellation of Liouville's function in short intervals. Matomäki and Radziwiłł [MR] proved this cancellation for almost all intervals using factorizations $n = pm$, that is, bilinear information of type II, the same kind of input that gave primes of the forms $x^2 + y^4$ and $x^3 + 2y^3$ [FI98, HB]. The cross term $C_2(n, s) = 2 \operatorname{Re} (X_{n-1}(s) \overline{n^{-s}})$ is bilinear in the additive sense: its coefficients are all 1 and it does not split n into factors, so its arithmetic content comes

only through the zeros. It does not supply the type II input. Status: the reductions are classical; the computations are ours.

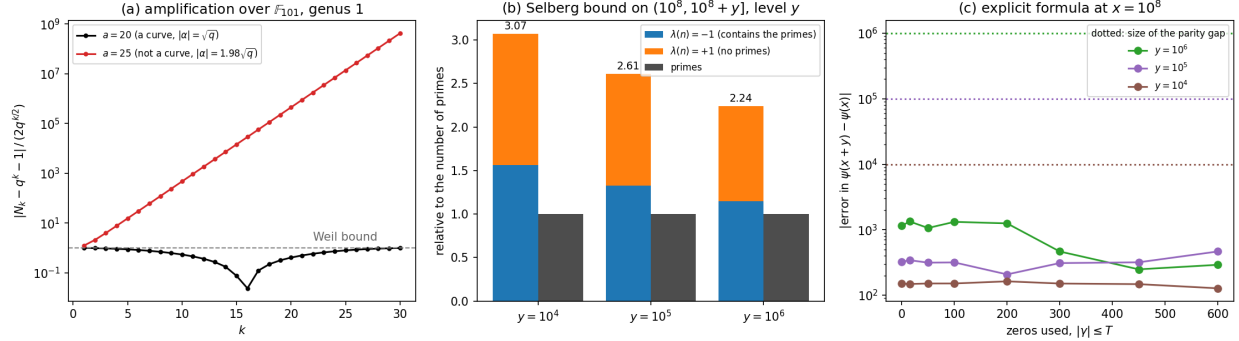


FIGURE 48. Amplification and parity (Remark 6.7). (a) Normalized deviation $|N_k - q^k - 1|/(2q^{k/2})$ for $q = 101$ and the numerators $1 - 20T + 101T^2$ (an elliptic curve) and $1 - 25T + 101T^2$ (not a curve). (b) Selberg's Λ^2 upper bound for the primes in $(10^8, 10^8 + y]$ with $R = \sqrt{y}$, split according to $\lambda(n) = -1$ and $\lambda(n) = +1$, relative to the number of primes. (c) Error of the explicit formula for $\psi(x+y) - \psi(x)$ at $x = 10^8$ when the zeros with $|\gamma| \leq T$ are used; dotted lines mark y , the size of the parity gap in this normalization.

Remark 6.8 (heat flow and the Laguerre hierarchy). The properties of the cross-energy that could force the zeros onto the line can be tested on the heat-flow deformation

$$H_t(z) = \int_{-\infty}^{\infty} e^{tu^2} \Phi(u) \cos(zu) du, \quad H_0(z) = \xi\left(\frac{1}{2} + iz\right).$$

There is a constant Λ , the de Bruijn–Newman constant, such that H_t has only real zeros if and only if $t \geq \Lambda$ [Ne]. Rodgers and Tao proved $\Lambda \geq 0$ [RT], the Polymath project $\Lambda \leq 0.22$ [Pm], and RH is the statement $\Lambda \leq 0$. Hence H_t has non-real zeros for every $t < 0$, and a property that implies RH must hold at $t = 0$ and fail for every $t < 0$. We computed H_t and its derivatives exactly by the trapezoidal rule (step $\frac{1}{150}$, 60 digits) on $2 < x < 58$. All 12 zeros there are real for $t \geq -0.45$. The first collision is that of the closest pair, 48.005 and 49.774, at $t_c = -0.469477$ and $x = 49.67$; for smaller t the number of real zeros falls to 10, 7 and 5 at $t = -0.5, -1, -1.5$ (Figure 49a).

Three properties were compared. (i) The first Laguerre expression $L_1 = H_t'^2 - H_t H_t''$, which on the critical line is ξ^2 times the mean energy of the Euler helix (Remark 6.5), is positive for $t > t_c$ and changes sign at the collision point linearly in $t - t_c$ (Figure 49b). (ii) The energy slope of Theorems C and D, in the form $\partial_y |H_t(x + iy)|^2 \geq 0$ for $y > 0$, fails exactly where non-real zeros exist. (iii) The Jensen polynomials $J^{d,0}$ of H_t with $d \leq 120$ remain hyperbolic at $t = -0.6, -1$ and -1.5 , although H_t has non-real zeros at height about 50. The local energy properties detect the collision exactly, and the low-degree global ones do not see it. Inside any finite window ξ has a margin, here $|t_c| = 0.47$. Since $\Lambda \geq 0$, for every $t < 0$ some pair collides at a greater height, so collisions occur arbitrarily close to $t = 0$, and no finite computation can decide the question.

The first Laguerre expression alone is not sufficient. The generating identity $|f(x + iy)|^2 = \sum_{n \geq 0} L_n(f)(x) y^{2n}$ with $L_n(f) = \sum_{k=0}^{2n} (-1)^{k+n} f^{(k)} f^{(2n-k)} / (k! (2n-k)!)$ shows that f has only real zeros if and only if $L_n(f) \geq 0$ for all n and x [CV]. A pair $x_0 \pm iy_0$ contributes $-2/y_0^2$ to $-(\log |f|)''$ at x_0 , while real zeros with mean spacing Δ contribute about π^2/Δ^2 , so pairs with $y_0 > (\sqrt{2}/\pi)\Delta$ pass the test $L_1 \geq 0$. For ζ this would apply to off-line zeros above height about 2000, where $\Delta < 1$. On the deformation, the non-real zeros $47.07 + 4.10i$ and $55.66 + 5.01i$ at

$t = -4$ are first detected by L_2 , and $49.89 + 6.53i$ at $t = -6$ by L_3 ; all others are detected by L_1 (Figure 49c).

On the side of the primes the hierarchy is a hierarchy of moments of the Euler helix. If all zeros are on the line, then $L_n/\xi^2 = e_n(w)$, the elementary symmetric function of the numbers $w = (t \mp \gamma)^{-2}$, by the Hadamard product. If moreover the ordinates are linearly independent over $\mathbb{Q}$, the moments M_{2m} of $|H_s|$ over u determine the power sums of w , and L_n/ξ^2 is a polynomial in $M_2, \dots, M_{2n}$. In particular

$$\frac{L_1}{\xi^2} = M_2, \quad \frac{L_2}{\xi^2} = \frac{1}{2}(M_4 - M_2^2) = \frac{1}{2} \text{Var } |H_s|^2.$$

At $s = \frac{1}{2} + 20i$ the exact value $L_2/\xi^2 = 0.12026$ agrees with $e_2(w) = 0.12025$ from the zeros and with $\frac{1}{2}(M_4 - M_2^2) = 0.115$ to 0.120 computed from the prime powers up to 10^8 on windows $[u_0, 18.4]$, $8 \leq u_0 \leq 12$. These moments are automatically nonnegative: the helix of a function with all zeros on the line is a sum of circles, and a variance is not negative. An off-line zero does not make L_2 negative on the helix side; it turns a circle into a growing spiral and makes the moments infinite. In the language of the primes the whole Laguerre hierarchy therefore reduces to the finiteness of the energy of the Euler helix, which is Remark 6.5 and Theorem B, and the higher orders add no constraint. Status: the identities are classical or follow from the Hadamard product (the moment identity uses linear independence); the computations are numerical.

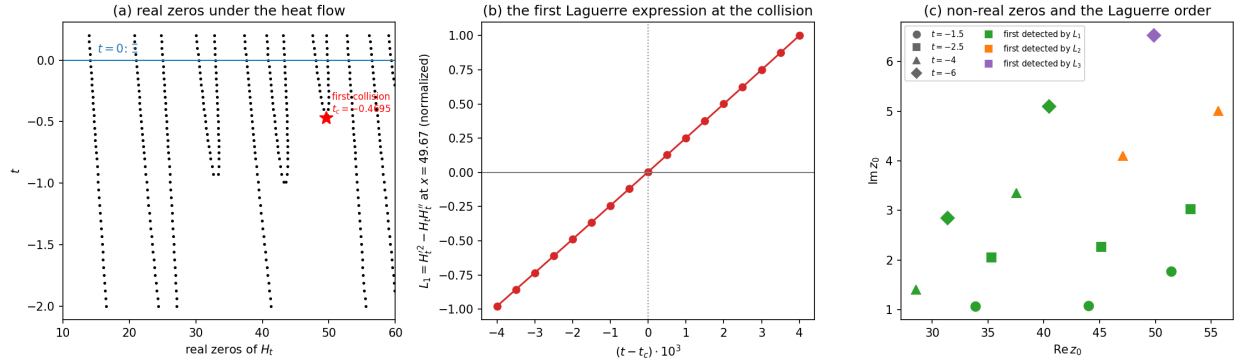


FIGURE 49. Heat flow and the Laguerre hierarchy (Remark 6.8). (a) Real zeros of H_t in $10 < x < 60$ for $-2 \leq t \leq 0.2$; the star marks the first collision, at $t_c = -0.4695$. (b) $L_1 = H_t'^2 - H_t H_t''$ at $x = 49.67$ for $|t - t_c| \leq 0.004$, normalized. (c) Non-real zeros z_0 of H_t in the upper half plane for $t = -1.5, -2.5, -4, -6$, coloured by the lowest n for which $L_n < 0$ near $\text{Re } z_0$.

Remark 6.9 (holomorphic and non-holomorphic sums; the damped energy). Proposition 2.2 can be written as

$$n^*(\rho) = \frac{1}{2 \text{Re } \frac{1}{1-\rho}} + \frac{\beta}{6} + \frac{c(\beta)}{\gamma^2} + O(\gamma^{-4}),$$

since $A(\rho) = |1 - \rho|^2 / (2(1 - \beta))$ and $\text{Re } \frac{1}{1-\rho} = (1 - \beta) / |1 - \rho|^2$. The crossing is a non-holomorphic function of ρ , which is why it sees β . The same holds for sums over the zeros. Summing over the zeros in symmetric order,

$$\sum_{\rho} \frac{1}{\rho(1 - \rho)} = 2 \sum_{\rho} \frac{1}{\rho} = 2 + \gamma_0 - \log 4\pi = 0.0461914 \dots,$$

by the Hadamard product, unconditionally, while for a pair $\rho = \beta + i\gamma$, $1 - \bar{\rho}$,

$$\left(\frac{1}{|\rho|^2} + \frac{1}{|1 - \rho|^2} \right) - \left(\frac{2}{|\rho|^2} \beta + \frac{2}{|1 - \rho|^2} (1 - \beta) \right) = \frac{(2\beta - 1)^2}{|\rho|^2 |1 - \rho|^2} \geq 0.$$

Hence $\sum_{\rho} |\rho|^{-2} \geq 2 + \gamma_0 - \log 4\pi$, with equality if and only if RH holds. With the 341 zeros below 600 and the density estimate for the rest, $\sum_{\rho} |\rho|^{-2} = 0.0461947$, the difference being the accuracy of the tail estimate. An off-line pair at height T changes this sum by about $2(2\beta - 1)^2/T^4$, while it moves the crossing points by $(2\beta - 1)^2 T^2 / (2\beta(1 - \beta))$: $2 \cdot 10^{-6}$ against 17 for $\beta = 0.6$ at $T = 14$, and $8 \cdot 10^{-54}$ against $8 \cdot 10^{18}$ for $\beta = 0.501$ at $T = 10^{12}$.

This is a general dichotomy. Holomorphic sums $\sum_{\rho} h(\rho)$ are given by the explicit formula as linear expressions in the primes; they are finite and computable, but they do not see β . Non-holomorphic sums such as $\sum_{\rho} |\rho|^{-2}$ or the crossing points see β , and from the primes they are reached only through quadratic quantities, the energies of Theorem B and Remark 6.5, which are infinite exactly when a zero leaves the line. In the coordinate $w = 1 - 1/\rho$ of Li's criterion [Li], RH says $|w_{\rho}| = 1$ for every zero. The functional equation acts by $w \mapsto 1/w$, which is holomorphic, while the reflection in the critical line acts by $w \mapsto 1/\bar{w}$; the distance of a zero from the line is the distance of $|w|$ from 1, and any quantity that measures it must contain $w\bar{w}$. The functional equation provides the inversion, and RH concerns the radius.

The damped energy interpolates between the two. Let $S(u) = e^{-u/2}(\psi(e^u) - e^u)$ and $E(\varepsilon) = \int_2^{\infty} e^{-\varepsilon u} S(u)^2 du$. By the explicit formula, $S(u) = -\sum_{\rho} e^{(\rho - \frac{1}{2})u} / \rho - e^{-u/2} \log 2\pi + O(e^{-5u/2})$, so $E(\varepsilon)$ is a double sum over the zeros with denominators $\varepsilon + 1 - \rho - \bar{\rho}'$. An off-line zero gives a pole at $\varepsilon = 2\beta - 1$, at a position that does not depend on its height, and since the integrand is nonnegative, Landau's theorem places the first singularity of E on the real axis, at $2\sup \beta - 1$. RH holds if and only if $E(\varepsilon) < \infty$ for every $\varepsilon > 0$. With the prime powers up to 10^8 , $E(\varepsilon)$ agrees with the double sum over the 682 zeros with $|\gamma| < 600$, the constant term and the diagonal contribution of the higher zeros to within 0.3 to 0.6% for $0 \leq \varepsilon \leq 1$ (Figure 50a).

The price is the size of the primes needed. We added to the zeros a pair $\frac{1}{2} \pm \delta + iT$ (with conjugates) and compared the mean energy on $[2, U]$ with that of the same pair placed on the line. The difference reaches 10% at $U = 13, 39$ and 115 for $T = 27.5$ and $\delta = 0.1, 0.03, 0.01$; at $U = 25.5, 81$ and 240 for $T = 100.3$; and at $U = 43.5$ and 145 for $T = 500.7$ and $\delta = 0.1, 0.03$ (Figure 50b,c). This follows $x^* = e^{U^*} \approx (cT^2)^{1/(2\delta)}$, roughly $T^{1/\delta}$: an energy detects an off-line zero at any height, but only with the primes up to about $T^{1/\delta}$. Since RH is verified up to $T = 3 \cdot 10^{12}$ [PT], detecting there a zero with $\delta = 0.01$ would need x of order 10^{1250} . That an off-line zero influences $\psi(x)$ only for x beyond a power $1/\delta$ of its height is the classical heuristic; here it is measured on the energy. Status: the inequality and the location of the singularity are proved; the computations are numerical.

Remark 6.10 (the sieved wave is the Euler helix). The wave $W(n, s) = 2X_{n-1}(s)\overline{n^{-s}}$ of Section 2 and the Euler helix of Remark 6.5 are one object. Remove from the partial sums the multiples of the primes $p \leq y$. By multiplicativity,

$$X_{n-1}^{(y)}(s) = \sum_{\substack{k < n \\ (k, P(y))=1}} k^{-s} = \sum_{d|P(y)} \mu(d) d^{-s} X_{\lceil n/d \rceil - 1}(s), \quad P(y) = \prod_{p \leq y} p,$$

an identity that has no analogue for the Davenport–Heilbronn function. By (2.1) applied to each term, the sieved wave $W_y = 2X_{n-1}^{(y)} \overline{n^{-s}}$ has the centre $2c_y n^{1-2\sigma} / (1 - s)$ with $c_y = \prod_{p \leq y} (1 - \frac{1}{p})$, a rotating term of amplitude $2\zeta_y(s) n^{-\sigma}$ with

$$\zeta_y(s) = \zeta(s) \prod_{p \leq y} (1 - p^{-s}),$$

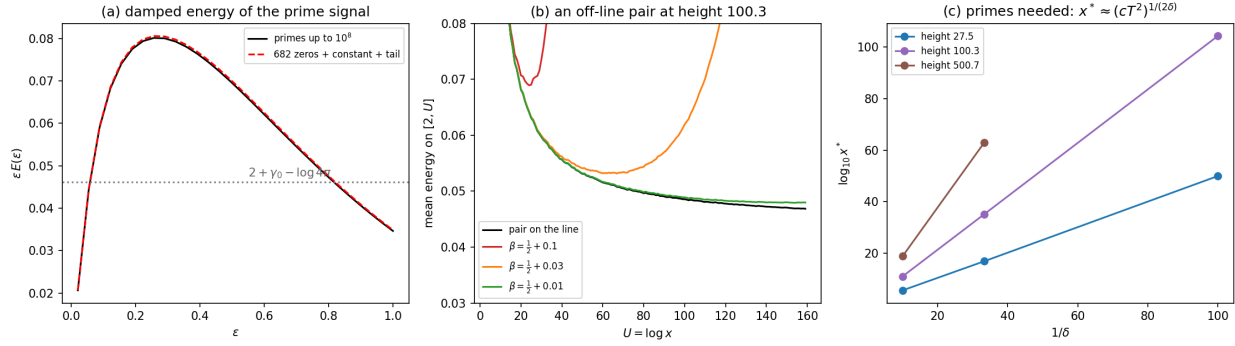


FIGURE 50. The damped energy (Remark 6.9). (a) $\varepsilon E(\varepsilon)$ on $2 \leq u \leq \log 10^8$ from the prime powers (solid) and from the zeros with the constant term and the tail (dashed); the dotted line is $2 + \gamma_0 - \log 4\pi$, the limit as $\varepsilon \rightarrow 0$ when the range of u is infinite. (b) Mean energy on $[2, U]$ with a pair at height 100.3 on the line (black) and at $\beta = \frac{1}{2} + \delta$. (c) The size $x^* = e^{U^*}$ of the primes at which the mean energy exceeds its value under RH by 10%, against $1/\delta$.

and a correction of order $n^{-2\sigma}$, periodic modulo $P(y)$. At $s = \frac{1}{2} + 20i$ the radius of W_y near $n = 10^5$ gives 1.155, 1.924, 1.352 for $y = 1, 3, 7$, against $|\zeta_y(s)| = 1.148, 1.914, 1.345$ (Figure 51a). A finite Euler product does not remove zeros: at ρ_1 the sieved waves collapse like n^{-1} for every y (Figure 51b). Removing the pole correction from the free amplitude leaves the helix: by Proposition 6.1,

$$A_y^*(s) := \zeta_y(s) e^{-E_1((s-1)\log y)} = \exp\left(-\frac{H_s(\log y)}{\log y} - R_y(s)\right),$$

where R_y collects the prime powers $p^k > y$ with $p \leq y$. With the primes up to 10^6 the two sides agree to about 1%, the accuracy of the grid on which H_s was stored. At $t = 20$, $|A_y^* - 1|$ is 0.035 and 0.005 for $y = 10^3$ and 10^6 at $\sigma = 0.7$, decreases like $1/\log y$ at $\sigma = \frac{1}{2}$, and grows at $\sigma = 0.3$, where every zero produces an expanding spiral (Figure 51c). Hence RH holds if and only if $A_y^*(s) \rightarrow 1$ for every s with $\sigma > \frac{1}{2}$, which is the statement on corrected partial products of this section. Sieving the wave by the Euler product turns its free amplitude into the Euler helix and adds no new information; the obstruction is again the pole correction, which has to be removed by hand. Status: the identities are proved; the computations are numerical.

Remark 6.10' (the cross-energy of the prime zeta function). The construction of Theorem A can be carried out on the primes alone. Put $P_x(s) = \sum_{p \leq x} p^{-s}$ and, for a prime p ,

$$C_2^P(p, s) = |P_p(s)|^2 - |P_{p-1}(s)|^2 - p^{-2\sigma} = 2p^{-\sigma} \operatorname{Re}(p^{it} P_{p-1}(s)).$$

For $0 < \sigma < 1$ and s not a zero, the explicit formula for $\pi(x)$ gives $P_x(s) = P(s) + \operatorname{Li}(x^{1-s}) - \sum_{\rho} \operatorname{Li}(x^{\rho-s}) + O(x^{\frac{1}{2}-\sigma}/\log x)$, where $P(s) = \sum_k \mu(k)k^{-1} \log \zeta(ks)$ is the continuation of the prime zeta function and the error collects the prime squares. Since $\operatorname{Li}(x^w) = x^w/(w \log x)(1 + O(1/|w| \log x))$,

$$C_2^P(x, s) = \frac{2(1-\sigma)x^{1-2\sigma}}{|1-s|^2 \log x} + 2x^{-\sigma} \operatorname{Re}(x^{it} P(s)) - \frac{2}{\log x} \operatorname{Re} \sum_{\rho} \frac{x^{\rho-2\sigma}}{\rho-s} + O\left(\frac{x^{1-2\sigma}}{|1-s|^3 \log^2 x} + \frac{x^{\frac{1}{2}-2\sigma}}{\log x}\right),$$

the analogue of Theorem A: a pole term, a wave term carrying $P(s)$ in place of $\zeta(s)$, and a term from the zeros which, up to the factor $x^{\frac{1}{2}-2\sigma}/\log x$, is the Euler helix of Remark 6.5 with amplitudes $1/(\rho-s)$. By the prime number theorem the sum over the zeros is $o(x^{1-2\sigma})$, so $\log x \cdot C_2^P(x, s)$ tends to $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$: the trichotomy of Theorem A holds

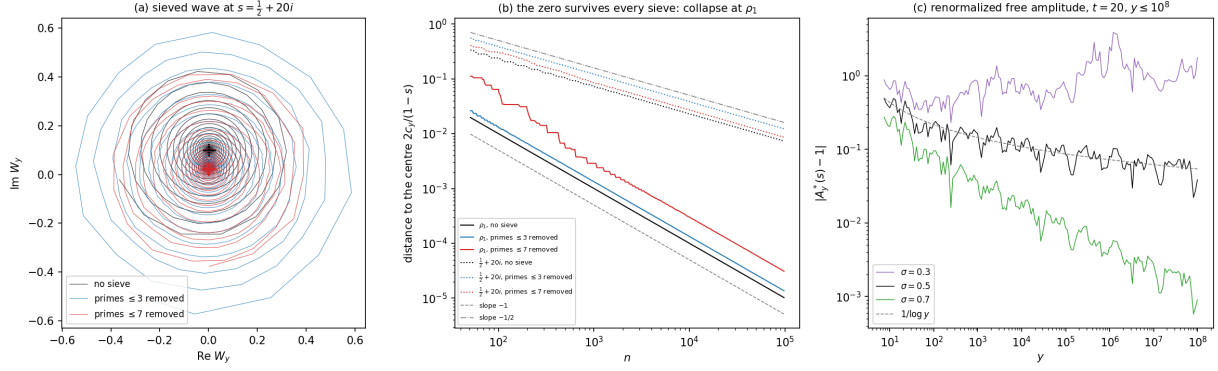


FIGURE 51. The sieved wave (Remark 6.10). (a) W_y at $s = \frac{1}{2} + 20i$ with no sieve and with the primes ≤ 3 and ≤ 7 removed; crosses: the centres $2c_y/(1-s)$. (b) Distance of W_y to its centre (maximum over blocks of 210 consecutive n) at ρ_1 (solid) and at $\frac{1}{2} + 20i$ (dotted). (c) $|A_y^*(s) - 1| = |e^{-H_s(\log y)/\log y} - 1|$ at $t = 20$ for $\sigma = 0.3, 0.5, 0.7$, from the prime powers up to 10^8 .

on the primes with the normalization $\log x$, the density of the primes. The roles of the zeros are reversed. At a zero of ζ the wave term of C_2 vanishes and the pole term appears alone, whereas P has a logarithmic singularity at ρ , so near a zero the wave term of C_2^P is largest. With the primes up to $2 \cdot 10^7$, the mean of $(\frac{1}{4} + t^2) \log p C_2^P(p, \frac{1}{2} + it)$ over $\log p \in [13.8, 16.8]$ is 0.856, 0.825 and 1.104 for $t = 5, 10$ and 20 , and 1.101, with much larger oscillations, at the ordinate $t = 14.1347$ (Figure 52a). The wave term dominates the pole term until x is of the order of $(t^2 |P(s)| \log x)^2$, about 10^8 for $t = 20$, which explains the slow convergence (Figure 52b). Off-line zeros contribute $x^{\beta-1}/\log x$ at $\sigma = \frac{1}{2}$, which tends to 0, so the pointwise limit does not see them; the energy of the zero term is that of the Euler helix, and Remark 6.5 and Theorem B apply unchanged. Status: the decomposition is the explicit formula and the limit follows from the prime number theorem; the numbers are from `anc/c2_primezeta/c2_primezeta.py`.

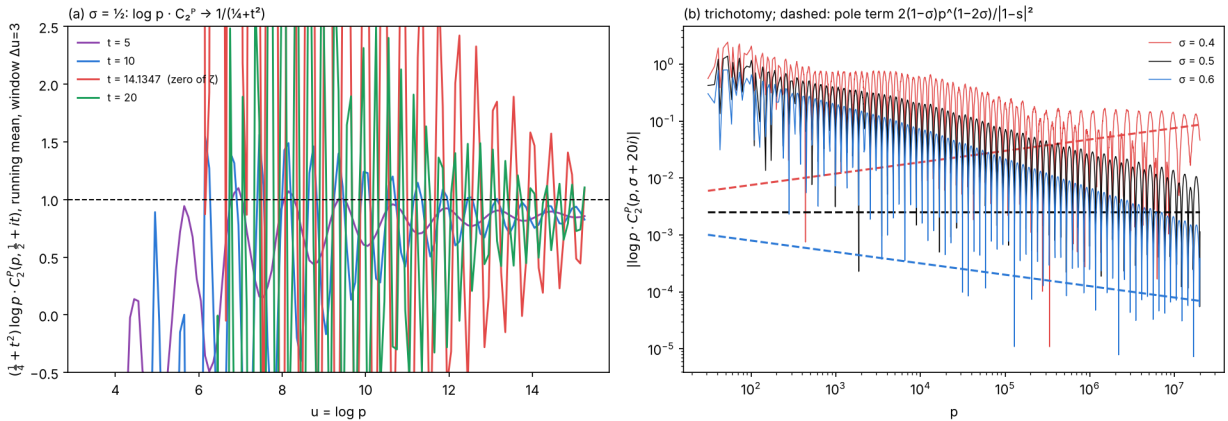


FIGURE 52. The cross-energy of the prime zeta function (Remark 6.10'), primes up to $2 \cdot 10^7$. (a) Running mean, over windows of length 3 in $u = \log p$, of $(\frac{1}{4} + t^2) \log p C_2^P(p, \frac{1}{2} + it)$ for $t = 5, 10, 20$ and at the ordinate $t = 14.1347$; the limit is 1. (b) $|\log p C_2^P(p, \sigma + 20i)|$ for $\sigma = 0.4, 0.5, 0.6$ (solid) and the pole term $2(1-\sigma)p^{1-2\sigma}/|1-s|^2$ (dashed), still below the wave term in this range.

Remark 6.10'' (zeros of $\log \zeta$, of ζ' and of the prime zeta function). The Euler product has no zeros, and the zeros of ζ are singularities of its logarithm. The zeros of $\log \zeta$ are the 1-points of ζ , the solutions of $\zeta(s) = 1$; those of $(\log \zeta)' = \zeta'/\zeta$ are the zeros of ζ' , which by Speiser's theorem [Sp] lie in $\sigma < \frac{1}{2}$ only if RH fails, and whose distance to the line is $\log \log T / \log T$ on average [LM]. Their numbers satisfy $N_1(T) = N(T) - \frac{T}{2\pi} \log 2 + O(\log T)$ and $N'(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T)$ [GS, Bt], so there are as many 1-points as zeros of ζ' , and fewer than zeros of ζ by one every $2\pi/\log 2 = 9.06$ units of height. Below $t = 350$ there are 169 zeros of ζ , 132 1-points and 132 zeros of ζ' . Of the 168 gaps between consecutive zeros of ζ , 38 contain no 1-point, 129 one, and one gap, $(317.73, 318.85)$, contains two, at $0.614 + 318.149i$ and $1.283 + 318.833i$; the counting formula predicts 38.6 empty gaps. The 1-points and the zeros of ζ' are paired one to one at almost the same height: $|t' - t_w| \leq 0.19$ (median 0.083), the zero s' of ζ' lies to the right of its 1-point w in all 132 cases, and $|s' - w|$ lies between 0.32 and 1.06, with mean 0.77, 0.63 and 0.56 on $t < 100$, $100 < t < 200$ and $200 < t < 350$ (Figure 53). At the zeros of ζ' with $t < 220$, ζ is close to a positive real number, with $|\arg \zeta| \leq 0.62$ and median $|\zeta| = 0.84$. The mechanism is the first term of the Dirichlet series. Write $\zeta(s) - 1 = 2^{-s}H(s)$ with $H(s) = 1 + (2/3)^s + (2/4)^s + \dots$; then $\zeta' = (\zeta - 1)' = 2^{-s}(H' - H \log 2)$, so $\zeta'(s) = 0$ if and only if $H'/H = \log 2$, and near a zero w of H , with R the regular part of H'/H , this gives $s' - w \approx 1/(\log 2 - R(s'))$: a displacement that is nearly real and positive. The zeros of the prime zeta function $P(s) = \log \zeta(s) - \frac{1}{2} \log \zeta(2s) - \dots$ follow the 1-points: below $t = 100$ we located 18, each within 0.08 to 0.59 of a 1-point (one more, next to the 1-point $0.19 + 95.3i$, is expected at $\sigma < 0.13$, near the natural boundary $\sigma = 0$ of P , where we did not locate it). The singularities of P are the points ρ/k and $1/k$ for squarefree k ; RH is equivalent to all of them lying on the lines $\sigma = 1/(2k)$, a restatement of RH by dilation. None of these families sees the zeros of ζ : they are determined by the first terms of the Dirichlet series, and the Davenport–Heilbronn function shows the same structure. Below $t = 200$ it has 108 1-points and 109 zeros of f' (the extra one at $2.881 + 199.99i$, whose 1-point lies above 200), paired one to one at almost the same height, $|t' - t_w| \leq 0.28$ (median 0.052), with the zero of f' to the right of its 1-point in every pair and $|s' - w|$ between 0.24 and 1.12 (mean 0.60 for $t < 100$, 0.47 for $100 < t < 200$). Its first coefficient after $a_1 = 1$ is $a_2 = \kappa = 0.284\dots$, so $f - 1 = \kappa 2^{-s} H_f(s)$ with $H_f(s) \rightarrow 1$, and $N_f(T) - N_1(T) = \frac{T}{2\pi} \log 2 + O(\log T)$ as for ζ : 22.1 predicted and 22 observed at $T = 200$. Because its zeros are denser, 20 of its 126 gaps contain two 1-points. The pairing is therefore a property of the Dirichlet series, not of the Euler product or of RH. Both families are visible in a single sum over the primes. With the pole term removed, $P_x(s) - \text{Li}(x^{1-s})$ is $P(s)$ plus the term from the zeros of Remark 6.10'; computed from the primes up to 10^6 on the critical line, its logarithm has a peak at each of the 21 ordinates of zeros of ζ below 80, the logarithmic singularities of P , and its deepest minima at the ordinates of the zeros of P closest to the line, such as $0.51 + 68.5i$ and $0.59 + 65.6i$ (Figure 54). At finite x every peak is finite, and a zero at distance δ from the line would also produce a peak, lower by an amount that depends on δ ; the picture is the explicit formula and does not separate ζ from the Davenport–Heilbronn function. Status: the counting formulas and Speiser's theorem are known; the pairing, its mechanism and the numbers are observations (`anc/onepoints/`; for f , `dh_np.py` evaluates the Hurwitz zeta function by Euler–Maclaurin in double precision, checked against `mpmath`).

Remark 6.11 (prime operators and the functional equation). Extend the wave to real arguments by $W(x, s) = 2X_{[x]-1}(s)x^{-s}$. Removing the multiples of a prime p from the partial sums acts on the sieved wave by

$$W_{y \cup \{p\}}(n, s) = W_y(n, s) - p^{-2\sigma} W_y(n/p, s),$$

because the multiples of p below n contribute $p^{-s} X_{[n/p]-1}^{(y)}(s)$ and $\overline{n^{-s}} = \overline{p^{-s}} \overline{(n/p)^{-s}}$. The identity holds to 10^{-15} at $s = \frac{1}{2} + 20i$, $0.7 + 33i$ and ρ_1 for $p = 2, 3, 5, 7$. The phase t cancels: on the critical line each prime acts by the real operator $1 - p^{-1}T_p$, $T_p W(n) = W(n/p)$, the same at every height, so the primes act on the waves of all zeros at once by the same operators. On a mode n^λ

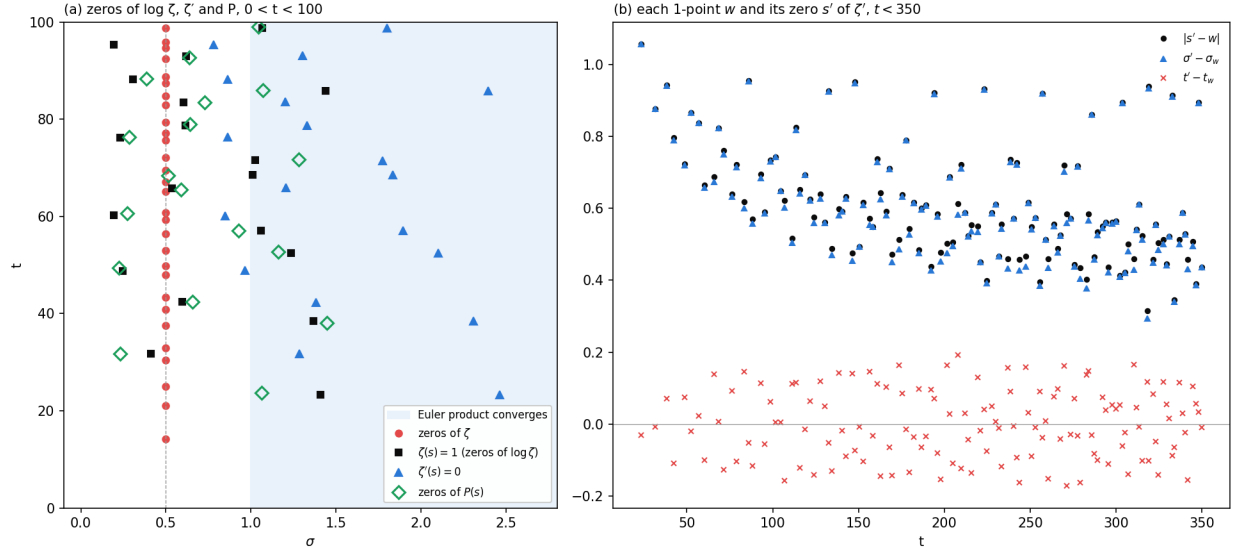


FIGURE 53. Zeros of $\log \zeta$, ζ' and P (Remark 6.10''). (a) $0 < t < 100$: zeros of ζ (circles), 1-points (squares), zeros of ζ' (triangles), zeros of the prime zeta function (diamonds); shaded: $\sigma > 1$. (b) For each 1-point w with $t < 350$ and its zero s' of ζ' : distance $|s' - w|$, horizontal and vertical displacements.

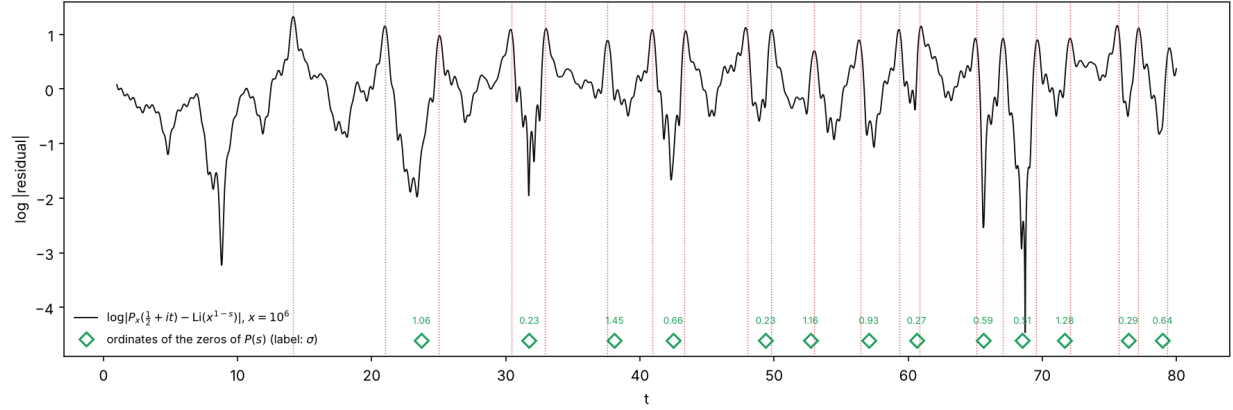


FIGURE 54. Both families in one prime sum (Remark 6.10''). $\log |P_x(\frac{1}{2} + it) - \text{Li}(x^{1-s})|$ from the primes up to $x = 10^6$: peaks at the ordinates of the zeros of ζ (dotted), minima at the ordinates of the zeros of P (diamonds, labelled by their abscissa).

the operator multiplies by $1 - p^{-2\sigma-\lambda}$, and all primes together by $1/\zeta(2\sigma+\lambda)$. For the three modes of W this gives $1/\zeta(1) = 0$ on the centre ($\lambda = 1 - 2\sigma$), $1/\zeta(s)$ on the rotating term ($\lambda = -\sigma + it$), and 0 on the drift ($\lambda = -2\sigma$); equivalently $W = \sum_m m^{-2\sigma} T_m[2n^{-\bar{s}}]$ is a positive combination of dilates of the elementary wave $2n^{-\bar{s}}$. At a zero, the rotating amplitude 0 meets the eigenvalue $1/\zeta(\rho) = \infty$. For the Davenport–Heilbronn function no such operator exists: the best fit of the sieved wave by $W_f - cT_p W_f$ leaves a relative residual 1.00 for $p = 2, 3, 7$. For $L(s, \chi)$ the operator is $1 - \chi(p)p^{-2\sigma}T_p$; the structure is that of every Euler product, so an argument for RH resting on it alone would apply to the whole Selberg class.

Combined with the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, the sieve gives the sieved factor

$$\chi_y(s) = \frac{\zeta_y(s)}{\zeta_y(1-s)} = \chi(s) \prod_{p \leq y} \frac{1-p^{-s}}{1-p^{s-1}}.$$

On the critical line $|\chi_y(s)| = 1$ for every y , since $1-s = \bar{s}$; at $s = \frac{1}{2} + 20i$ the computed values are 1 for $1 \leq y \leq 10^6$. Off the line the factor is erratic: at $s = 0.7 + 20i$, $|\chi_y| = 0.79, 0.70, 1.53, 0.72, 19.5, < 10^{-3}, 3 \cdot 10^{-4}$ for $y = 1, 10, \dots, 10^6$. The reason is that the reflected point $1-\sigma = 0.3$ lies to the left of the critical line, where the Euler product diverges whether or not RH holds, because the zeros on the line lie to its right. The helix measures this: $|H_s(u)/u|$ decreases from 0.147 to 0.0025 as $y = e^u$ goes from 10^2 to 10^8 at $s = 0.7 + 20i$, and grows from 0.79 to 2.94, like $y^{0.2}/\log y$, at $s = 0.3 + 20i$. The functional equation therefore pairs the half of the strip where the Euler product converges, provided there are no zeros to the right, with the half where it always diverges. The compatibility of the prime operators with the reflection on $\sigma = \frac{1}{2}$ is a symmetry and not a constraint, and at an off-line pair the relation reads $0 = \chi_y \cdot 0$. Combining the two structures would require control of the Euler product on the divergent side. Status: the identities are proved; the computations are numerical.

Proposition 6.12 (no linear polarization without the pole). Let $F(x) = \sum_{j=1}^K c_j \cos(t_j x + \phi_j)$ with real c_j, ϕ_j and distinct $t_j > 0$, so that F has no constant term. If $F(\log p) \geq 0$ for all but finitely many primes p , then $F \equiv 0$.

Proof. For real $u \neq 0$ the partial sums $\sum_{p \leq x} p^{-1-iu}$ are bounded as $x \rightarrow \infty$; this is the prime number theorem in the form $\zeta(1+iu) \neq 0$ [T, Ch. III], [IK]. Since $\sum_{p \leq x} p^{-1} \rightarrow \infty$, the weighted mean $\langle G \rangle_x = \sum_{p \leq x} G(\log p) p^{-1} / \sum_{p \leq x} p^{-1}$ tends to 0 for every $G(x) = \cos(ux + \phi)$ with $u \neq 0$, and to 1 for $G = 1$. Hence $\langle F \rangle_x \rightarrow 0$. Expanding the square,

$$F^2 = \frac{1}{2} \sum_j c_j^2 + \frac{1}{2} \sum_j c_j^2 \cos(2t_j x + 2\phi_j) + \sum_{j < k} c_j c_k [\cos((t_j - t_k)x + \phi_j - \phi_k) + \cos((t_j + t_k)x + \phi_j + \phi_k)],$$

in which every non-constant term has a nonzero frequency, so $\langle F^2 \rangle_x \rightarrow \frac{1}{2} \sum c_j^2$. If $F(\log p) \geq 0$ for all large p , then $F(\log p)^2 \leq \|F\|_\infty F(\log p)$ there, and the finitely many exceptions do not affect the limits; hence $\frac{1}{2} \sum c_j^2 \leq \|F\|_\infty \lim \langle F \rangle_x = 0$. $\square$

The proposition explains the term $a_0 > 0$ in (4.1) and Proposition 6.4: a combination of the values $\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it_j)$ that is positive prime by prime must contain the point $t = 0$, that is, the pole. The obstruction is the uniform distribution of the angles $t \log p$, which follows from the absence of zeros on $\sigma = 1$. Finitely many primes can be polarized: maximizing $\min_p F(\log p)$ by linear programming over 80 frequencies in $[0.5, 40]$ with $|c_j| \leq 1$ gives a positive minimum 12.6, 5.5 and 1.6 on the primes up to 30, 100 and 1000, but the optimized F is then negative on 57%, 52% and 59% of the primes from there to 10^6 , and its mean over x is 0 to within 10^{-3} . A positivity without the pole term can only be quadratic, $|\sum a_n n^{-s}|^2 \geq 0$; this is the route of Weil’s functional (Section 5), where the margin decreases at least like $10^{-6.3L^2}$, and of Selberg’s sieve, where it meets the parity barrier (Remark 6.7). Status: the proposition is proved; the optimization is numerical.

Remark 6.13 (the wave over a curve and the degree form). The constructions of Section 2 can be carried out for an elliptic curve $E/\mathbb{F}_q$, where the missing positivity is known. With $T = q^{-s}$ one has $\zeta_E(s) = Z(T) = (1 - aT + qT^2)/((1 - T)(1 - qT))$, $a = q + 1 - N_1$, and the coefficient of T^n is the number of effective divisors of degree n , $b_0 = 1$ and $b_n = N_1(q^n - 1)/(q - 1)$ for $n \geq 1$. Define the partial sums $X_{N-1}(s) = \sum_{n < N} b_n T^n$ and the wave $W(N, s) = 2X_{N-1}(s)\overline{T^N}$. Summing the geometric tail,

$$W(N, s) = 2Z(T)\overline{T^N} - \frac{2N_1}{q-1} \left[\frac{q^{N(1-2\sigma)}}{1-qT} - \frac{q^{-2\sigma N}}{1-T} \right],$$

the exact analogue of the decomposition of Section 2: a centre $-2N_1q^{N(1-2\sigma)}/((q-1)(1-qT))$ coming from the pole at $T = 1/q$, a term rotating with $\overline{T^N}$ and proportional to $\zeta_E(s)$, and a remainder. At a zero $\rho = \beta + i\gamma$ the rotating term vanishes and

$$|W(N, \rho) - \text{centre}| = \frac{2N_1}{(q-1)|1-q^{-\rho}|} q^{-2\beta N}$$

for every $N \geq 1$, so the collapse rate $n^{-2\beta}$ of Section 2 holds here as an identity, without Euler–Maclaurin corrections. The identity uses only the values of b_n , and these depend on a only through N_1 . It therefore holds equally for the numerator $1 - 25T + 101T^2$ of Remark 6.7, whose coefficients b_n are nonnegative integers but which is not the numerator of a curve. Its zeros are real, with $\beta = 0.3516$ and 0.6484 , and the computed slopes of $\log_q |W - \text{centre}|$ are -0.7032 and -1.2968 , against -1.0000 for the curves with $a = \pm 20$ (Figure 55b). The wave detects the zeros and measures β , but it does not distinguish a curve from a numerator that is not one.

The positivity that does is geometric. If F is the Frobenius endomorphism, with roots $\alpha, \bar{\alpha}$ of $X^2 - aX + q$, then for integers m, n

$$\deg(m - nF) = (m - n\alpha)(m - n\bar{\alpha}) = m^2 - amn + qn^2,$$

and the degree of an isogeny is a nonnegative integer. The quadratic form is positive semidefinite if and only if $a^2 \leq 4q$, which is the Riemann hypothesis for E . Among the 10100 curves $y^2 = x^3 + Ax + B$ over $\mathbb{F}_{101}$ the trace satisfies $|a| \leq 20 < 2\sqrt{101} = 20.10$ (Figure 55a). For $a = 25$ the form takes negative values, $m^2 - 25mn + 101n^2 < 0$ at 58 points with $|m| \leq 25$, $|n| \leq 2$ (Figure 55c), and its minimum on the line of slope $\frac{25}{2}$ is $-\frac{221}{4}n^2$. The identity $\deg(m - nF) = |m - n\alpha|^2$ shows where the conjugation comes from: the dual isogeny $\hat{F}$ satisfies $F\hat{F} = q$, so $\bar{\alpha} = q/\alpha$ is supplied by the geometry, and positivity of the degree, the case $g = 1$ of the Rosati positivity and of the Castelnuovo–Severi inequality used by Weil, contains the product $w\bar{w}$ that Remark 6.9 identifies as necessary. Over $\mathbb{Q}$ the functional equation provides the inversion $w \mapsto 1/w$, the Dirichlet series provides the wave and its collapse, and there is no known object whose degree is $|w|^2$. Status: the identities are proved; the counts are exact computations.

Remark 6.14 (graphs and Dirichlet L -functions over $\mathbb{F}_q[T]$). Two further settings separate the roles of the wave and of the positivity.

- (i) *Graphs*. For a connected $(q+1)$ -regular graph G with adjacency matrix A , n vertices and $|E|$ edges, the Ihara zeta function [Ih, Ha] is $Z_G(u) = \prod_P (1 - u^{\ell(P)})^{-1}$ over the primitive closed non-backtracking cycles, with $Z_G(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + qu^2I)$, $r - 1 = |E| - n$. Each eigenvalue λ contributes the factor $1 - \lambda u + qu^2$, the numerator of an elliptic curve with $a = \lambda$, and the analogue of RH, that the nontrivial poles satisfy $|u| = q^{-1/2}$, holds exactly when $|\lambda| \leq 2\sqrt{q}$ for every $\lambda \neq \pm(q+1)$ (Ramanujan graphs, [LPS]). Unlike the numerator $1 - 25T + 101T^2$ of Remark 6.7, a graph that is not Ramanujan is a genuine object with an Euler product and a functional equation. We took $q = 2$, $n = 40$: a random cubic graph with $\lambda_2 = 2.798$, $\lambda_{\min} = -2.812$, and a dumbbell made of two random cubic graphs on 20 vertices joined by exchanging one edge, with $\lambda_2 = 2.918 > 2\sqrt{2} = 2.828$. The dumbbell has a pair of real poles off the circle, at $\beta = 0.861$ and 0.139 (Figure 56a). The numbers

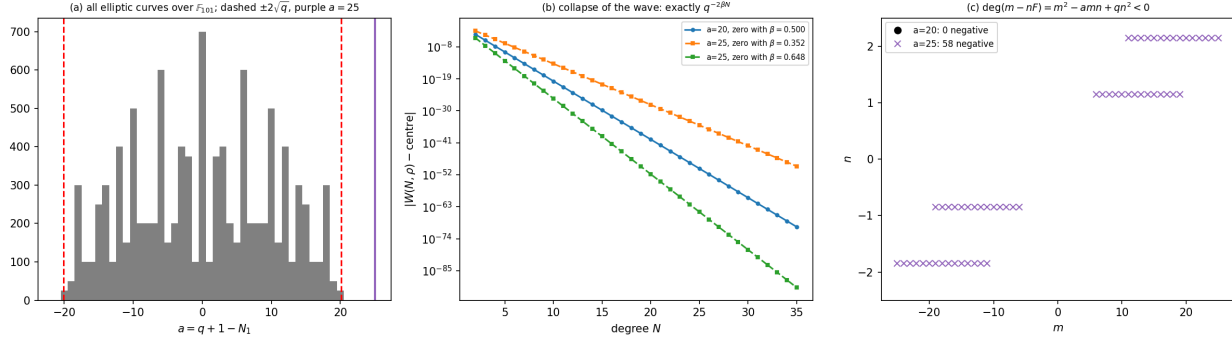


FIGURE 55. The wave over a curve (Remark 6.13), $q = 101$. (a) Traces $a = q+1 - N_1$ of the 10100 elliptic curves $y^2 = x^3 + Ax + B$ over $\mathbb{F}_{101}$; dashed lines $\pm 2\sqrt{q}$, and the value $a = 25$ of the numerator that is not a curve. (b) $|W(N, \rho) - \text{centre}|$ at the zeros of $1 - 20T + 101T^2$ ($\beta = \frac{1}{2}$) and of $1 - 25T + 101T^2$ ($\beta = 0.352$ and 0.648), computed in 160-digit arithmetic. (c) Lattice points with $\deg(m - nF) = m^2 - amn + qn^2 < 0$ for $a = 20$ (none) and $a = 25$.

N_m of closed non-backtracking cycles, computed from the Hashimoto matrix, agree with $q^m + 1 + 2(r-1)[m \text{ even}] + \sum(\alpha^m + \bar{\alpha}^m)$, and the normalized deviation stays bounded for the Ramanujan graph and grows geometrically for the dumbbell (b). The wave built from the partial sums of $\sum_n \text{tr}(A_n)u^n = \text{tr}(1 - u^2)(I - Au + qu^2I)^{-1}$, where $\text{tr} A_n$ counts the closed non-backtracking walks of length n , has the centre $-2(1 + q^{-1})q^N|u|^{2N}/(1 - qu)$ coming from the trivial pole, of constant modulus on $\sigma = \frac{1}{2}$ as for ζ . In 200-digit arithmetic, $|W - \text{centre}|$ at two points of the critical circle decays with slopes -0.50 and -0.47 in $\log_q$ per step for the Ramanujan graph, the second affected by the beating of the 78 nontrivial frequencies, and $-0.139 = \beta - 1$ for the dumbbell (c). The Toeplitz matrix $[s_{j-k}]$ built from the primes, $s_m = q^{-m/2}(N_m - q^m - 1 - 2(r-1)[m \text{ even}])$ and $s_0 = 2(n-1)$, which is the analogue of the Weil matrices of Section 5, is positive definite up to size 40 for the Ramanujan graph and has a negative eigenvalue from size 13 for the dumbbell (d). The structural positivity of a graph is the symmetry of A : the eigenvalues are real, so a nontrivial pole lies either on the circle or on the real axis, and a graph that is not Ramanujan uses the second possibility. Over $\mathbb{Q}$ this alternative is closed, since $\zeta(\sigma) < 0$ for $0 < \sigma < 1$; a self-adjoint operator in the sense of Hilbert and Pólya would therefore suffice for ζ , while for graphs it does not. The explicit Ramanujan graphs of [LPS] obtain the bound $|\lambda| \leq 2\sqrt{q}$ from the Ramanujan–Petersson conjecture, proved by Deligne, that is, from geometry.

- (ii) *Dirichlet L -functions over $\mathbb{F}_q[T]$.* For a nonprincipal character χ modulo a monic irreducible M of degree d , $L(u, \chi) = \sum_f \chi(f)u^{\deg f}$ over monic f is a polynomial of degree at most $d-1$, its zeros satisfy $|u| = q^{-1/2}$ by Weil’s theorem [Ro], and it satisfies $L(u, \chi) = \epsilon(\chi)(\sqrt{q}u)^{d-1}L(1/(qu), \bar{\chi})$. The combination $F = L(\chi) + \epsilon(\chi)L(\bar{\chi})$ satisfies $F(u) = (\sqrt{q}u)^{d-1}F(1/(qu))$, the analogue of the Davenport–Heilbronn construction, but its coefficients are not multiplicative. For $q = 3$, $M = T^7 + 2T^6 + T^5 + 1$ and the character $\chi(g) = e^{2\pi i/2186}$ on a generator g , the six zeros of L lie on the circle, while F has a real pair off it with $\beta = -0.120$ and 1.120 , outside the strip, as is possible without an Euler product (Figure 56e). The prime sums $\psi_n = \sum_{\deg P=j=n} \deg P \chi(P)^j$, computed by enumerating the irreducibles for $n \leq 8$, agree with $-\sum \alpha^n$ to 10^{-12} and satisfy $|\psi_n| \leq 6q^{n/2}$, while $\sum \alpha^n$ for F grows geometrically (f). Here the wave carries no information: the Dirichlet series is a polynomial, so $W(N, \rho) = 2L(\rho)\rho^N = 0$ for $N \geq d$ at every zero, of L and of F alike

(g). The Toeplitz matrix built from $s_m = q^{-m/2} \sum \alpha^m$ is positive semidefinite for L , with smallest eigenvalue exactly 0 (to 10^{-13}) from size 7 on, the rank being $d - 1 = 6$, and is indefinite for F from size 4, with smallest eigenvalue $-5 \cdot 10^8$ at size 30 (h). For L the positivity is that of a Gram matrix: $\sqrt{q}^{-1} \alpha_j$ are the eigenvalues of a unitary operator, the normalized Frobenius on the first cohomology of the corresponding curve.

In the three settings of Remarks 6.13 and 6.14 the wave measures β when the Dirichlet series is infinite and is blind when it is finite, and in none of them does it decide the analogue of RH. What decides it is in each case Weil's quadratic positivity, and where it holds it comes from a unitary or self-adjoint operator: the dual isogeny, the symmetric adjacency matrix together with a bound on its spectrum, or Frobenius on H^1 . Status: the identities are classical; the computations are exact or carried out to 10^{-12} .

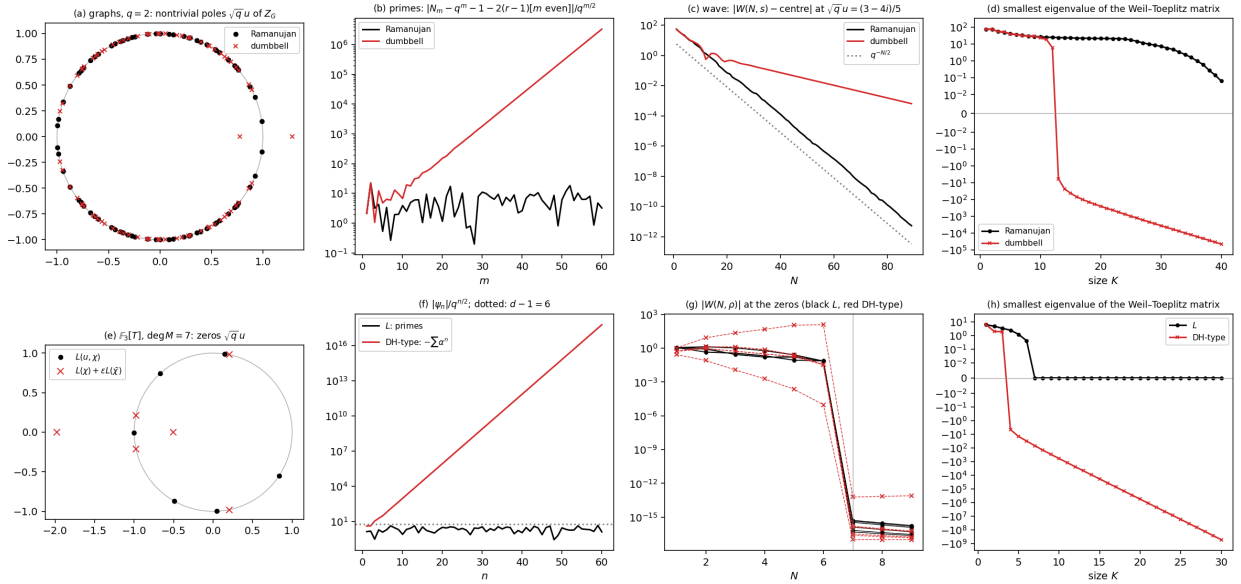


FIGURE 56. Graphs and $\mathbb{F}_q[T]$ (Remark 6.14). Top, cubic graphs on 40 vertices ($q = 2$), a Ramanujan graph and a dumbbell with $\lambda_2 = 2.918$: (a) nontrivial poles $\sqrt{q}u$ of Z_G ; (b) normalized deviation of the counts N_m of closed non-backtracking cycles; (c) distance of the wave to its centre at $\sqrt{q}u = (3 - 4i)/5$; (d) smallest eigenvalue of the Toeplitz matrix built from the N_m . Bottom, $q = 3$, $\deg M = 7$: (e) zeros $\sqrt{q}u$ of $L(u, \chi)$ and of the Davenport–Heilbronn-type combination; (f) $|\psi_n|/q^{n/2}$; (g) $|W(N, \rho)|$ at the zeros, identically 0 for $N \geq 7$; (h) smallest eigenvalue of the Toeplitz matrix.

Remark 6.15 (the wave as a quantum state). Sample $W(n, s)$ at 64 points with $\log n$ equally spaced in $[7, 9]$, form the 3×62 trajectory matrix $H_{jk} = W(n_{j+k}, s)$ and the density matrix $\rho = HH^*/\text{tr } HH^*$. We record the von Neumann entropy $S(\rho)/\log 3 \in [0, 1]$, the ℓ_1 -coherence, and the drift κ of the dominant eigenvector v , defined by $v_{j+1}/v_j = e^{\kappa h}$ with h the step in $\log n$, so that a single mode $W \propto n^\kappa$ gives a pure state with drift κ .

The recursion $W(n+1, s) = (n/(n+1))^{\bar{s}}[W(n, s) + 2n^{-2\sigma}]$ splits exactly into a transient and a steady part,

$$W(n, s) = 2\zeta(s) n^{-\bar{s}} - 2\zeta(s, n) n^{-\bar{s}},$$

with the Hurwitz function $\zeta(s, n)$ (verified to 10^{-15}). The steady part is $-2n^{1-2\sigma}/(1-s) + O(n^{-2\sigma})$, the centre of Section 2, and $\zeta(s)$ is the amplitude of the transient. At a zero the transient is absent

and the state is pure with drift $\kappa = 1 - 2\beta$; elsewhere the second eigenvalue of ρ decays like $n^{2\sigma-2}$ as the window moves, and at a zero like n^{-2} . Measured slopes of $\log \lambda_2$ are -0.962 at $\frac{1}{2} + 20i$ (predicted -1), -0.587 at $0.7 + i\gamma_1$ (-0.6) and -2.000 at ρ_1 (Figure 58a).

On $\sigma = \frac{1}{2}$ the entropy falls from 0.1–0.5 between zeros to 10^{-5} – 10^{-3} at every zero, with $\kappa = 10^{-4}$ (Figure 57a, b). For the Davenport–Heilbronn function the zeros are instead maximally mixed, $S = 0.96$ – 0.99 at the five zeros on the line with $80 < t < 90$ and $S = 0.990$ at $0.8085 + 85.699i$ (Figure 57c): without a pole there is no steady part, and at a zero only the periodic remainder is left. The purity at the zeros of ζ comes from the pole, not from the line. To separate the two, take $\zeta + 5f$, which has the pole, no functional equation, and zeros off the line. At its zeros the state is nearly pure ($S = 0.03$ – 0.05 , the residue of the period 5), and the drift measures β : $\kappa = -0.2916$, $+0.1216$, $+0.1779$, -0.0494 against $1 - 2\beta = -0.2966$, $+0.1200$, $+0.1765$, -0.0526 at the zeros $0.648 + 85.569i$, $0.440 + 80.234i$, $0.412 + 82.019i$, $0.526 + 89.376i$ (Figure 57d). In this language a zero of a function with a pole is a pure state, stationary exactly when $\beta = \frac{1}{2}$, and RH says that every zero of ζ is a stationary pure state.

The family of pure states $s \mapsto W(\cdot, s)/\|W\|$ on $12 \leq n \leq 90$ has a Berry curvature, computed with gauge-invariant plaquettes on a grid of steps 0.0125 in σ and 0.025 in t (Figure 58b–d). Near a zero the state passes once around the Bloch sphere between the transient and the steady mode, so each zero carries one unit of flux: the flux/ 2π through $0.2 \leq \sigma \leq 0.95$, $10 \leq t \leq 50$ is 10.35, against 10 zeros, and rises in steps of 0.88 to 1.26 at the individual zeros. The zero of $\zeta + 5f$ at $\beta = 0.648$ carries its flux off the line. The flux counts zeros, as the argument principle does, but does not place them. Status: the splitting is an identity; the rest is numerical.

Remark 6.16 (coherence of the Euler state and coherence between characters). Let L be the operator of multiplication by $\log n$ on $\ell^2(\mathbb{N})$ and, for $\sigma > \frac{1}{2}$, $\psi_s = \zeta(2\sigma)^{-1/2} \sum_n n^{-s} |n\rangle$. Then

$$\psi_s = e^{-itL} \psi_\sigma,$$

and e^{-itL} is diagonal in the basis $|n\rangle$ and a product of diagonal unitaries over the primes. Hence every coherence measure with respect to the basis $|n\rangle$, every entropy of the populations and every entanglement between groups of primes is a function of σ alone; the zeros are invisible to all of them. For the truncation $n \leq 4096$ on $\sigma = \frac{1}{2}$ the relative entropy of coherence 6.066379389410, the ℓ_1 -coherence 1799.345 and the entanglement between the prime 2 and the odd primes 0.233336 coincide to twelve digits at $t = 0, 14.13, 20$ and 37.59 . The relative entropy of coherence of ψ_s is the entropy of the populations $n^{-2\sigma}/\zeta(2\sigma)$,

$$C(\sigma) = \log \zeta(2\sigma) - 2\sigma \frac{\zeta'}{\zeta}(2\sigma) \sim \frac{1}{2\sigma - 1} \quad (\sigma \rightarrow \tfrac{1}{2}^+),$$

the thermal entropy of the gas of primes [Ju, BC] at inverse temperature 2σ ; ψ_s is normalizable exactly for $\sigma > \frac{1}{2}$, and the critical line is the boundary of the Hilbert space (Figure 59a).

The zeros appear in the dynamics. With the alternating state $|a\rangle = N^{-1/2} \sum_{n \leq N} (-1)^{n+1} |n\rangle$, the frequency $\frac{1}{2}$ of the additive Fourier basis, one has $\langle a | e^{-itL} | \psi_{1/2} \rangle \propto (1 - 2^{1/2-it}) \zeta(\frac{1}{2} + it)$ up to the truncation, so the zeros on the line are the times at which the evolved state becomes orthogonal to $|a\rangle$. For $N = 4096$ the fidelity drops to 10^{-9} – 10^{-10} at the ten zeros with $t < 50$, while at $\sigma = 0.6$ it stays above 10^{-6} (Figure 59b). For the other frequencies a/q the overlap is $F(s, a/q) = \sum_n e^{2\pi i n a/q} n^{-s}$, and every zero found with $t < 40$ lies off the line, with $\max |\beta - \frac{1}{2}| = 0.66, 0.73, 0.79$ for $q = 3, 4, 6$ and $0.77, 0.78, 0.81$ for $\frac{1}{5}, \frac{2}{5}, \frac{1}{8}$ (Figure 59c), in accordance with [DH]. The additive character $e^{2\pi i n a/q}$ is a superposition of $\sum_{d|q} \#\{\text{primitive characters mod } d\}$ primitive L -functions, which is 1 only for $q \leq 2$, 2 for $q = 3, 4, 6$ and 4 for $q = 5, 8$ (Figure 59d). Measured against a single multiplicative character the orthogonality times lie on the line; measured against a coherent superposition of characters they do not.

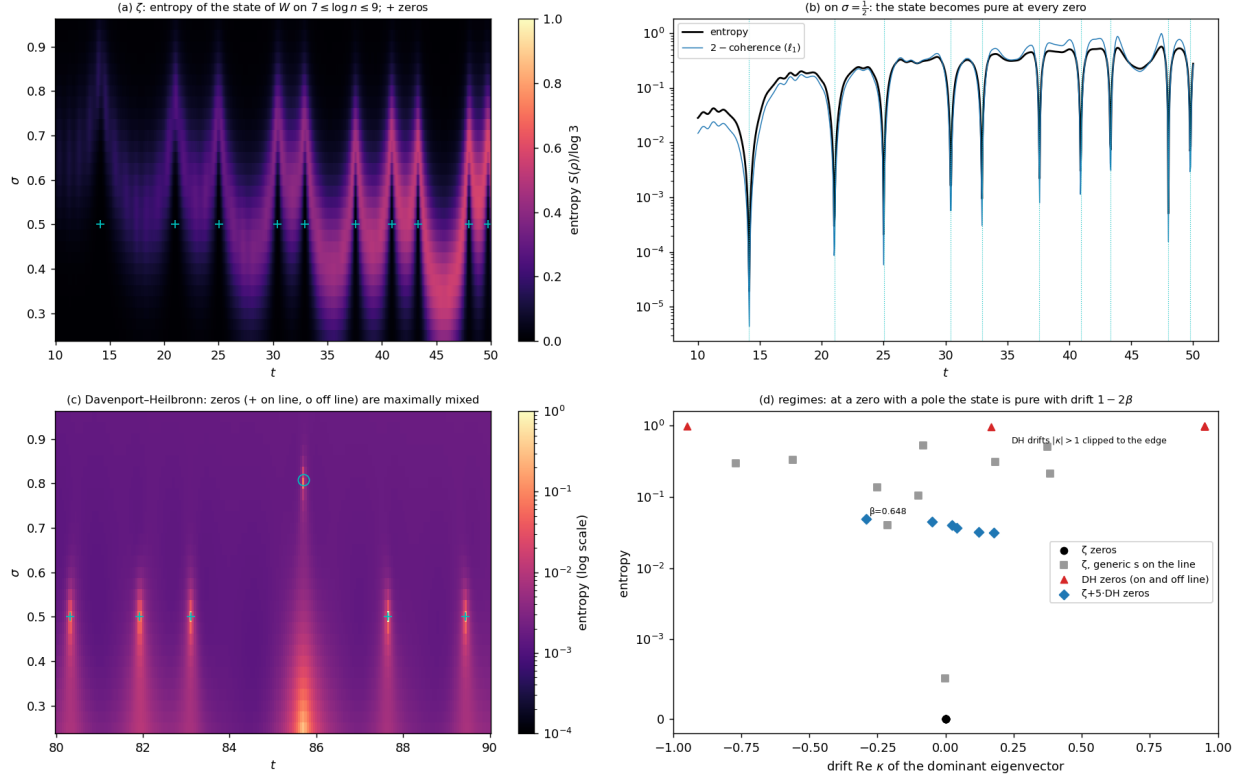


FIGURE 57. The wave as a quantum state (Remark 6.15), $7 \leq \log n \leq 9$. (a) Entropy $S(\rho)/\log 3$ for ζ in the (σ, t) plane; crosses: zeros. (b) Entropy and $2 - \text{coherence}$ on $\sigma = \frac{1}{2}$. (c) Entropy for the Davenport–Heilbronn function near its zero $0.8085 + 85.699i$ (circle) and its zeros on the line (crosses), logarithmic scale. (d) Entropy against the drift $\text{Re } \kappa$ at the zeros of ζ , at generic points of the line, at zeros of the Davenport–Heilbronn function (drifts beyond ± 1 drawn at the edge), and at the zeros of $\zeta + 5f$.

The dependence on the amount of coherence can be measured. For the real character χ_5 modulo 5 let

$$\Lambda(s, \chi_5) = \frac{1}{2}s(s-1)(5/\pi)^{s/2}\Gamma(\frac{s}{2})L(s, \chi_5),$$

and $G_\varepsilon = \xi + \varepsilon\Lambda(\cdot, \chi_5)$, which satisfies $G_\varepsilon(s) = G_\varepsilon(1-s)$ and is real on the line for every real ε . For $t < 100$, ξ has 29 zeros and $\Lambda(\cdot, \chi_5)$ has 54, all on the line. For every $\varepsilon > 0$, G_ε has the larger number, and the 13 additional pairs with $t < 100$ lie where the two terms have equal size, at

$$\beta \approx \frac{2\log(1/\varepsilon)}{\log 5},$$

spaced by $4\pi/\log 5 = 7.81$ in t (Proposition 6.17): the median β is 8.585, 5.733, 3.795 for $\varepsilon = 10^{-3}$, 10^{-2} , 0.05, against 8.584, 5.723, 3.723 (Figure 60a, b). As $\varepsilon \rightarrow 0$ these zeros recede to infinity, while the zeros of ξ stay on the line apart from one pair that leaves it for $0.1 \leq \varepsilon \leq 0.15$. As ε grows the distant zeros enter the strip and land on the line in pairs, and the number of zeros on the line increases from 29 to 54 (Figure 60c). Each landing is a square-root bifurcation: the pair that lands at $t = 57.69$ for $\varepsilon_c = 0.6197$ satisfies $\beta - \frac{1}{2} \propto (\varepsilon_c - \varepsilon)^{0.495}$ over three decades (Figure 60d), the same collision law as in the heat flow (Remark 6.8), proved in Proposition 6.18. The two pure functions have their zeros on the line, every superposition has zeros off it, and these come either from infinity, at distance of order $\log(1/\varepsilon)$, or from collisions on the line. In this language RH says

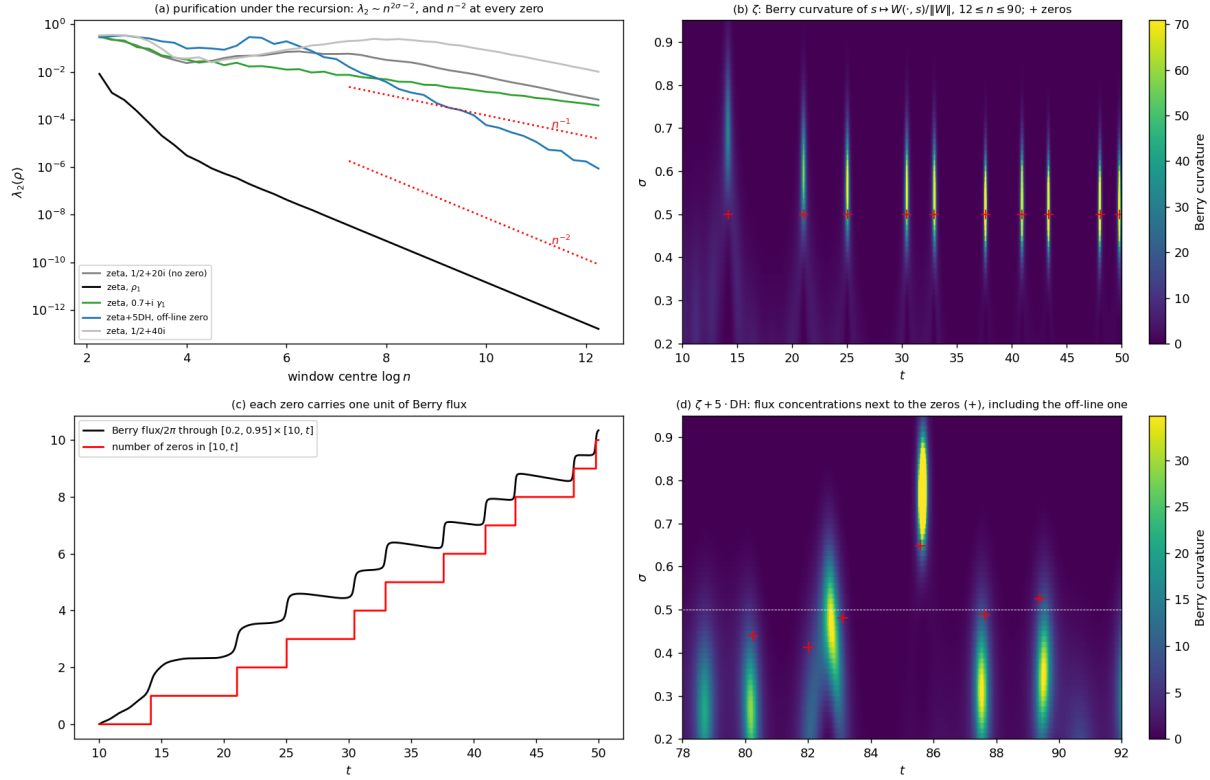


FIGURE 58. The recursion as a channel and the Berry curvature (Remark 6.15). (a) Second eigenvalue of ρ on windows of width 1.5 in $\log n$, at points with and without a zero. (b) Berry curvature of $s \mapsto W(\cdot, s)/\|W\|$, $12 \leq n \leq 90$, for ζ . (c) Flux/2π through $[0.2, 0.95] \times [10, t]$ and the number of zeros. (d) The same curvature for $\zeta + 5f$; crosses: its zeros.

that ζ contains no second component, however small. Status: the invariance and the formula for $C(\sigma)$ (Proposition 6.19), the location of the distant zeros (Proposition 6.17) and the collision law (Proposition 6.18) are proved; the rest is numerical.

Proposition 6.17 (distant zeros of a superposition). Let χ_5 be the real character modulo 5, $G_\varepsilon = \xi + \varepsilon\Lambda(\cdot, \chi_5)$ as in Remark 6.16 with $0 < \varepsilon \leq 5^{-5/2}$, and put

$$\sigma_\varepsilon = \frac{2 \log(1/\varepsilon)}{\log 5} \quad (\geq 5), \quad s_k = \sigma_\varepsilon + \frac{2\pi i(2k+1)}{\log 5},$$

$$R_k = \left\{ |\sigma - \sigma_\varepsilon| \leq 1, |t - \operatorname{Im} s_k| \leq \frac{2\pi}{\log 5} \right\} \quad (k \in \mathbb{Z}).$$

Then: (i) G_ε has no zeros with $\sigma \geq \frac{5}{2}$ outside $\bigcup_k R_k$; (ii) each R_k contains exactly one zero ρ_k of G_ε , and it is simple; (iii) with $\delta = \zeta(\sigma_\varepsilon - 1)^2 - 1$,

$$\rho_k = s_k - \frac{2}{\log 5} \operatorname{Log} A(\rho_k), \quad |\rho_k - s_k| \leq \frac{2}{\log 5} \frac{\delta}{1 - \delta}, \quad \rho_k = s_k + \frac{4}{\log 5} (2^{-s_k} + 3^{-s_k}) + O(4^{-\sigma_\varepsilon}),$$

where $A(s) = L(s, \chi_5)/\zeta(s)$. By the functional equation and $G_\varepsilon(\bar{s}) = \overline{G_\varepsilon(s)}$, the points $1 - \rho_k$ are zeros as well.

Proof. The factors $\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})$ are common to ξ and $\Lambda(\cdot, \chi_5)$, so $G_\varepsilon = \xi H$ with $H(s) = 1 + \varepsilon 5^{s/2} A(s)$, and $\xi \neq 0$ for $\sigma > 1$. The Dirichlet series $A(s) = \sum a(n)n^{-s}$ has $a = \chi_5 * \mu$, so $a(1) = 1$,

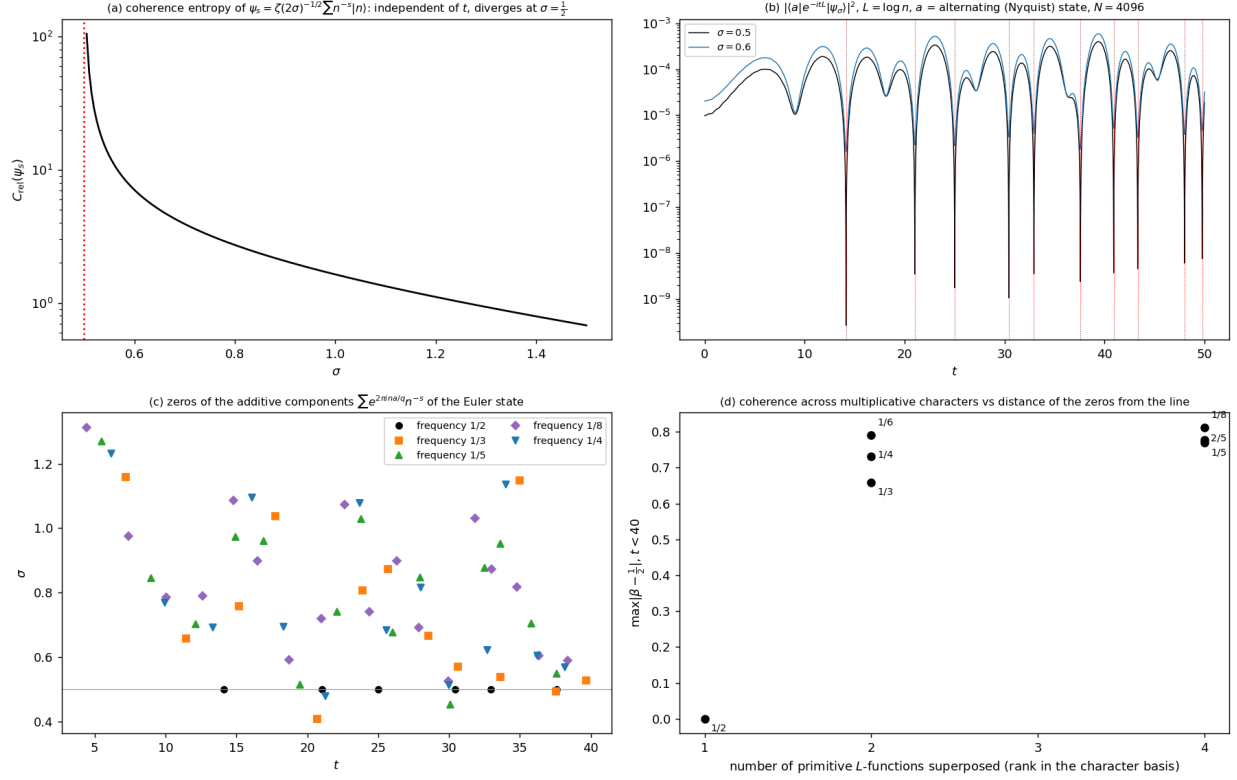


FIGURE 59. Coherence of the Euler state (Remark 6.16). (a) Relative entropy of coherence $C(\sigma)$ of ψ_s . (b) $|\langle a|e^{-itL}|\psi_s\rangle|^2$ for the alternating state, $N = 4096$; dotted: zeros. (c) Zeros of $\sum e^{2\pi i n a/q} n^{-s}$ with $t < 40$, $\beta \geq \frac{1}{2}$ shown, except the zeros of the local factors on $\sigma = 1$. (d) Number of primitive L -functions in the additive character against $\max|\beta - \frac{1}{2}|$.

$a(2) = a(3) = -2$ and $|a(n)| \leq 2^{\omega(n)} \leq d(n)$; hence $|A(s)| \leq \zeta(\sigma)^2$ and $|A(s) - 1| \leq \zeta(\sigma)^2 - 1$ for $\sigma > 1$. Note $|\varepsilon 5^{s/2}| = 5^{(\sigma - \sigma_\varepsilon)/2}$.

- (i) If $\frac{5}{2} \leq \sigma \leq \sigma_\varepsilon - 1$, then $|\varepsilon 5^{s/2} A(s)| \leq 5^{-1/2} \zeta(\frac{5}{2})^2 = 0.804$, so $|H| \geq 0.196$. If $\sigma \geq \sigma_\varepsilon + 1 \geq 6$, then $|\varepsilon 5^{s/2} A(s)| \geq 5^{1/2} (2 - \zeta(6)^2) > 2$, so $|H| > 1$.
- (ii) Let $H_0(s) = 1 + \varepsilon 5^{s/2}$, whose zeros are the points s_k , all simple, one in each R_k . On the vertical sides of R_k , $|\varepsilon 5^{s/2}| = 5^{\mp 1/2}$ and $|H_0| \geq 1 - 5^{-1/2} = 0.553$. On the horizontal sides, $\frac{t}{2} \log 5 \equiv 0 \pmod{2\pi}$, so $\varepsilon 5^{s/2} > 0$ and $H_0 \geq 1$. On R_k , $|H - H_0| = |\varepsilon 5^{s/2}| |A - 1| \leq 5^{1/2} \delta \leq 5^{1/2} (\zeta(4)^2 - 1) = 0.383 < 0.553$. By Rouché's theorem H and H_0 have the same number of zeros in R_k , namely one.
- (iii) At ρ_k we have $\varepsilon 5^{\rho_k/2} A(\rho_k) = -1 = \varepsilon 5^{s_k/2}$, so $5^{(\rho_k - s_k)/2} A(\rho_k) = 1$. Since $|\text{Im}(\rho_k - s_k)| \leq 2\pi / \log 5$ and $|A(\rho_k) - 1| \leq \delta < 1$, the principal branch gives the first identity, and $|\text{Log}(1 + w)| \leq |w| / (1 - |w|)$ gives the bound. Finally $\text{Log } A(\rho) = -2 \cdot 2^{-\rho} - 2 \cdot 3^{-\rho} + O(4^{-\beta})$, and $2^{-\rho_k} - 2^{-s_k} = O(2^{-\sigma_\varepsilon} |\rho_k - s_k|) = O(4^{-\sigma_\varepsilon})$. $\square$

The proof applies to any two primitive Dirichlet characters of the same parity and conductors $q_1 < q_2$, with 5 replaced by q_2/q_1 and the constants adjusted. The distant zeros, about $\frac{T \log 5}{4\pi}$ with $0 < t < T$ on each side of the line, account for the difference $\frac{T}{2\pi} \log 5$ between the counting functions of $L(s, \chi_5)$ and ζ . Numerically, with $\varepsilon = 10^{-3}, 10^{-4}, 10^{-6}$ and $0 \leq k \leq 5$, the zeros obtained by Newton's method satisfy $|\rho_k - s_k| \leq 6.5 \cdot 10^{-3}, 8.9 \cdot 10^{-4}, 1.7 \cdot 10^{-5}$ (bounds $1.6 \cdot 10^{-2}$,

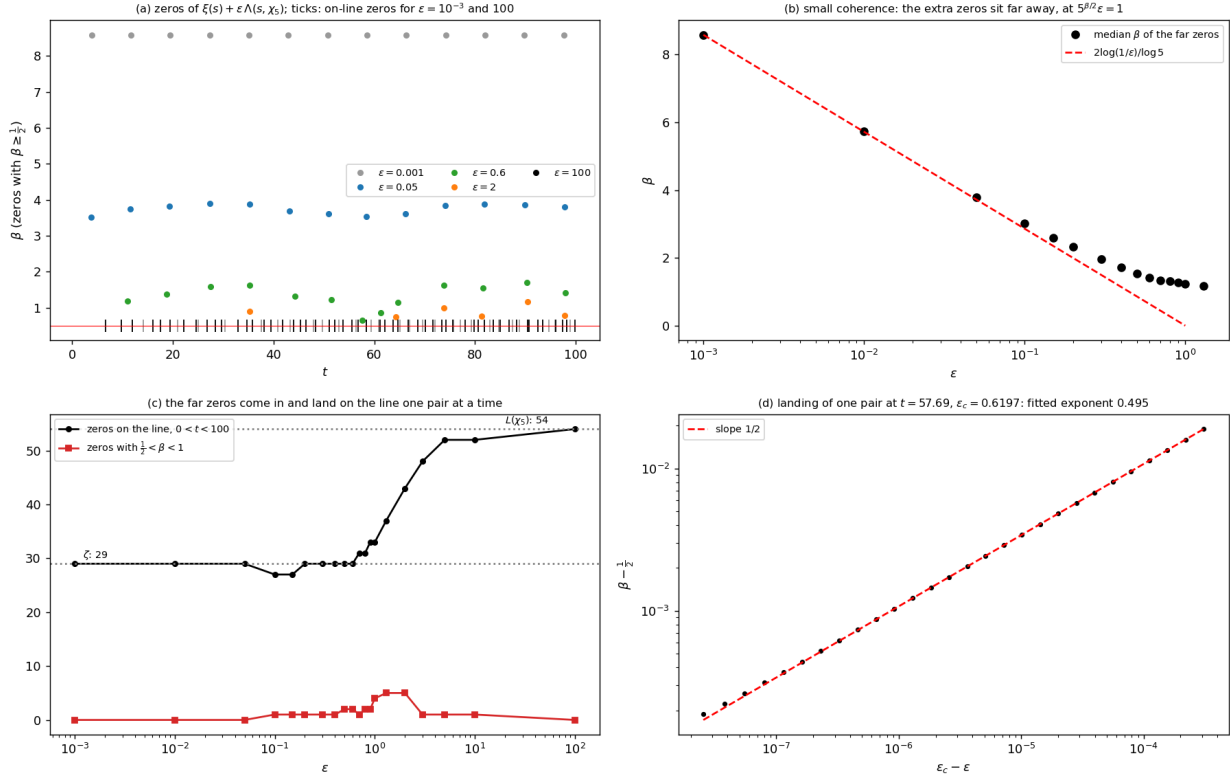


FIGURE 60. Superposition of ξ and $\Lambda(\cdot, \chi_5)$ (Remark 6.16). (a) Zeros of G_ε with $\beta \geq \frac{1}{2}$, $t < 100$, for five values of ε . (b) Median β of the distant zeros against $2 \log(1/\varepsilon)/\log 5$. (c) Number of zeros on the line and in $\frac{1}{2} < \beta < 1$. (d) Landing of one pair: $\beta - \frac{1}{2}$ against $\varepsilon_c - \varepsilon$.

$2.2 \cdot 10^{-3}$, $4.1 \cdot 10^{-5}$), and after subtracting the correction $\frac{4}{\log 5}(2^{-s_k} + 3^{-s_k})$ the differences are $3.1 \cdot 10^{-5}$, $5.6 \cdot 10^{-7}$, $2.0 \cdot 10^{-10}$. For $\varepsilon = 5^{-5/2}$ the argument principle gives exactly one zero in each of $R_{-3}, \dots, R_5$.

Proposition 6.18 (the line as a level set). Let $X(t) = \xi(\frac{1}{2} + it)$ and $Y(t) = \Lambda(\frac{1}{2} + it, \chi_5)$, both real for real t , and $r = -X/Y$, a real meromorphic function of t . For real ε and real t with $Y(t) \neq 0$:

- (i) $\frac{1}{2} + it$ is a zero of G_ε if and only if $r(t) = \varepsilon$, and its multiplicity is the order of vanishing of $r - \varepsilon$ at t .
- (ii) If $r(t_0) = \varepsilon_0$ and $r'(t_0) \neq 0$, then for ε close to ε_0 the function G_ε has exactly one zero near $\frac{1}{2} + it_0$, and it lies on the line.
- (iii) If $r'(t_c) = 0 \neq r''(t_c)$ and $\varepsilon_c = r(t_c)$, then for ε close to ε_c the function G_ε has exactly two zeros near $\frac{1}{2} + it_c$. If $(\varepsilon - \varepsilon_c)/r''(t_c) > 0$ they lie on the line, at $t = t_c \pm \sqrt{2(\varepsilon - \varepsilon_c)/r''(t_c)} + O(|\varepsilon - \varepsilon_c|)$; if $(\varepsilon - \varepsilon_c)/r''(t_c) < 0$ they are

$$\rho_{\pm} = \frac{1}{2} \pm \sqrt{\frac{2|\varepsilon - \varepsilon_c|}{|r''(t_c)|}} + it_c + O(|\varepsilon - \varepsilon_c|).$$

- (iv) Let K be a compact subset of the strip and I an interval such that no G_ε , $\varepsilon \in I$, vanishes on ∂K . Then the number of zeros of G_ε in K off the line is constant on every subinterval

of I that contains no critical value $r(t_c)$ with $\frac{1}{2} + it_c \in K$, and it changes by 2 at each nondegenerate one.

Proof. Near the line, $G_\varepsilon(\frac{1}{2} + iz) = X(z) + \varepsilon Y(z) = Y(z)(\varepsilon - r(z))$, with X, Y entire and real on the real axis, which gives (i). The zeros of G_ε are symmetric under $s \mapsto 1 - \bar{s}$ by the functional equation and $G_\varepsilon(\bar{s}) = \overline{G_\varepsilon(s)}$; in the variable z this is $z \mapsto \bar{z}$. In (ii), $r - \varepsilon$ has a single zero near t_0 by the implicit function theorem; since the zero set is invariant under conjugation, that zero is real. In (iii), write $r(t_c + w) - \varepsilon_c = \frac{1}{2}r''(t_c)w^2u(w)$ with u analytic and $u(0) = 1$, and $\phi(w) = w\sqrt{u(w)}$, which is analytic and invertible near 0 with $\phi'(0) = 1$. Then $r(t_c + w) = \varepsilon$ if and only if $\phi(w) = \pm c$ with $c^2 = 2(\varepsilon - \varepsilon_c)/r''(t_c)$, so $w = \pm c + O(|c|^2)$. If c is real the two zeros are real, that is, on the line. If $c = i|c|$ they are $t_c \pm i|c| + O(|c|^2)$, and $s = \frac{1}{2} + iz$ gives $\rho_\pm$. (iv) By the argument principle the number of zeros in K is constant on I . The number on the line changes only where a zero on the line ceases to be simple, which by (ii) and (iii) happens only at critical values, where it changes by 2. $\square$

The proposition reduces the zeros on the line to the graph of the single real function $r(t) = -\xi(\frac{1}{2} + it)/\Lambda(\frac{1}{2} + it, \chi_5)$: a horizontal line at height ε meets it at the zeros on the line, a local maximum of r above 0 is a value of ε at which a pair leaves the line, and a local minimum is a value at which a pair lands (Figure 61a). For $0 < t < 100$ the counts of solutions of $r(t) = \varepsilon$ coincide with the zeros found directly for all twenty values of ε in Remark 6.16 (Figure 61b). The positive critical values below 1 are 0.0931 (maximum at $t = 60.434$, where 29 zeros on the line become 27), 0.1742 (minimum at $t = 3.777$, back to 29), 0.6197 (minimum at $t = 57.668$) and 0.8747 (minimum at $t = 11.020$). For the landing at $t_c = 57.668$, $r''(t_c) = 1.7278$, and (iii) predicts $\beta - \frac{1}{2} = 1.0759\sqrt{\varepsilon_c - \varepsilon}$; the zero followed by Newton's method gives 1.0768, and the ratio stays at 1.0759 for $10^{-6} < \varepsilon_c - \varepsilon < 10^{-3}$ (Figure 61c). Together with Proposition 6.17, zeros of G_ε reach the strip off the line either from infinity or through a critical point of r ; at no other place can they leave the line.

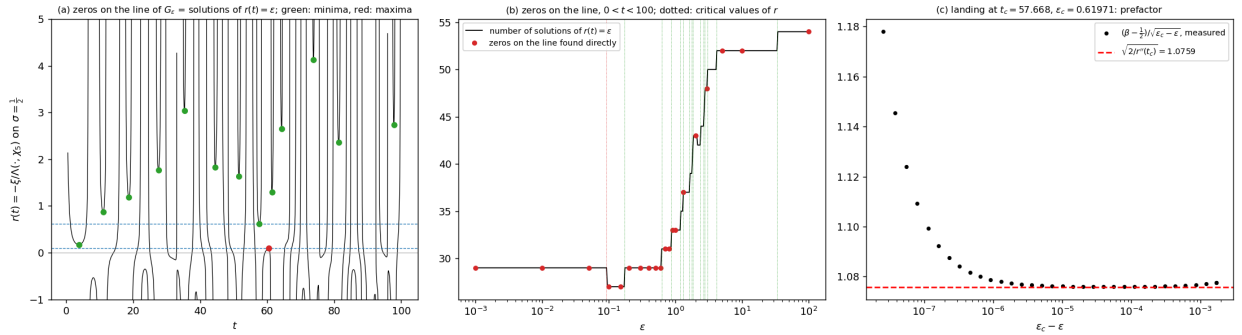


FIGURE 61. The line as a level set (Proposition 6.18). (a) $r(t) = -\xi/\Lambda(\cdot, \chi_5)$ on the critical line, clipped at the poles; dots: positive critical values (green minima, red maxima); dashed: $\varepsilon = 0.0931$ and 0.6197 . (b) Number of solutions of $r(t) = \varepsilon$ with $0 < t < 100$ and the zeros on the line found directly; dotted: critical values. (c) $(\beta - \frac{1}{2})/\sqrt{\varepsilon_c - \varepsilon}$ for the pair landing at $t_c = 57.668$, against $\sqrt{2/r''(t_c)}$; the departure for $\varepsilon_c - \varepsilon < 10^{-6}$ reflects the precision of ε_c .

Proposition 6.19 (the Euler state). Let L be multiplication by $\log n$ on $\ell^2(\mathbb{N})$, P_N the projection onto $\text{span}\{|n\rangle : n \leq N\}$, and for $\sigma > \frac{1}{2}$ (respectively $N < \infty$ and any σ) let $\psi_s = \zeta(2\sigma)^{-1/2} \sum_n n^{-s} |n\rangle$ (respectively $\psi_{s,N} = P_N \psi$ normalized).

(i) $\sum_n n^{-s} |n\rangle \in \ell^2(\mathbb{N})$ if and only if $\sigma > \frac{1}{2}$, and then $\psi_s = e^{-itL} \psi_\sigma$; likewise $\psi_{s,N} = e^{-itL} \psi_{\sigma,N}$.

- (ii) Let $\ell^2(\mathbb{N}) = \bigotimes_p' \ell^2(\mathbb{N}_0)$ by $|n\rangle = \bigotimes_p |v_p(n)\rangle$ (restricted product with respect to $|0\rangle$). Then $e^{-itL} = \bigotimes_p e^{-itL_p}$ with $L_p|k\rangle = k \log p |k\rangle$. Consequently every function of a state that is invariant under diagonal unitaries in the basis $|n\rangle$ (populations, every coherence measure of the resource theory of coherence in this basis, the entropy of the populations) and every function invariant under local unitaries for a bipartition of the primes (entanglement entropies, Schmidt coefficients) takes the same value on ψ_s and ψ_σ , and on $\psi_{s,N}$ and $\psi_{\sigma,N}$, for every t .

- (iii) The relative entropy of coherence of ψ_s is the entropy of the populations $n^{-2\sigma}/\zeta(2\sigma)$:

$$C(\sigma) = \log \zeta(2\sigma) - 2\sigma \frac{\zeta'}{\zeta}(2\sigma) = \frac{1}{2\sigma - 1} - \log(2\sigma - 1) + 1 - \gamma_0 + O(2\sigma - 1) \quad (\sigma \rightarrow \frac{1}{2}^+),$$

and for the truncation on the line $C_N(\frac{1}{2}) = \frac{1}{2} \log N + \log \log N - \frac{\gamma_0}{2} + O(1/\log N)$.

- (iv) For $\sigma > 0$ and the alternating vector $a_N = N^{-1/2} \sum_{n \leq N} (-1)^{n+1} |n\rangle$,

$$\sqrt{N \sum_{n \leq N} n^{-2\sigma}} \langle a_N | e^{-itL} | \psi_{\sigma,N} \rangle = \sum_{n \leq N} (-1)^{n+1} n^{-s} = (1 - 2^{1-s})\zeta(s) + O_s(N^{-\sigma}),$$

uniformly for s in compact subsets of $\sigma > 0$. In particular the zeros of ζ in $0 < \sigma < 1$ are the points at which the rescaled overlap tends to 0.

Proof. (i) $\|\sum n^{-s}|n\rangle\|^2 = \sum n^{-2\sigma}$, and $n^{-s} = e^{-it \log n} n^{-\sigma}$; P_N is diagonal and commutes with L . (ii) By unique factorization $\log n = \sum_p v_p(n) \log p$, so $e^{-it \log n} = \prod_p e^{-it v_p(n) \log p}$, which is the stated product; for a bipartition $S \cup S^c$ of the primes it is $U_S \otimes U_{S^c}$ with diagonal factors, and since P_N commutes with e^{-itL} , also $\psi_{s,N} = (U_S \otimes U_{S^c}) \psi_{\sigma,N}$, although $\psi_{\sigma,N}$ is not a product state. (iii) With $p_n = n^{-2\sigma}/\zeta(2\sigma)$, $-\sum p_n \log p_n = \log \zeta(2\sigma) + 2\sigma \sum p_n \log n$ and $\sum n^{-b} \log n = -\zeta'(b)$. The expansion follows from $\zeta(b) = \frac{1}{b-1} + \gamma_0 + O(b-1)$ and $\frac{\zeta'}{\zeta}(b) = -\frac{1}{b-1} + \gamma_0 + O(b-1)$ with $b = 2\sigma$. For the truncation, $\sum_{n \leq N} n^{-1} = \log N + \gamma_0 + O(N^{-1})$ and $\sum_{n \leq N} n^{-1} \log n = \frac{1}{2} \log^2 N + O(1)$. (iv) The left side equals $\sum_{n \leq N} (-1)^{n+1} n^{-s}$ by definition. Grouping consecutive terms, $|(2k-1)^{-s} - (2k)^{-s}| \leq |s|(2k-1)^{-\sigma-1}$, so the tail beyond N is $O(|s|N^{-\sigma}/\sigma) + O(N^{-\sigma})$, and the full series is $(1 - 2^{1-s})\zeta(s)$ for $\sigma > 0$. $\square$

Numerically, $C(\sigma)$ computed from ζ differs from the expansion by 0.039, 0.019, 0.0038, 0.0004 at $\sigma = 0.6, 0.55, 0.51, 0.501$, and $C_{4096}(\frac{1}{2}) = 6.066$ against 5.988 from the first three terms. Part (ii) says that the arithmetic of the zeros cannot be read from any intrinsic property of the Euler state; only overlaps with vectors that are not diagonal in the multiplicative structure, such as a_N , depend on t .

Proposition 6.20 (convolution identities do not see s). Let f, g be arithmetic functions, $h = f * g$, $G_M(s) = \sum_{m \leq M} g(m) m^{-s}$ and $H_N(s) = \sum_{n \leq N} h(n) n^{-s}$. Then for every N and every $s \in \mathbb{C}$

$$\sum_{d \leq N} f(d) d^{-s} G_{[N/d]}(s) = H_N(s).$$

More generally the identity remains true when n^{-s} is replaced by $x^{v(n)} = \prod_p x_p^{v_p(n)}$ for an arbitrary sequence (x_p) of complex numbers.

Proof. The left side is $\sum_{dm \leq N} f(d) g(m) (dm)^{-s}$, and grouping the terms with $dm = n$ gives $H_N(s)$. Only $(dm)^{-s} = d^{-s} m^{-s}$ was used, which also holds for $x^{v(dm)} = x^{v(d)} x^{v(m)}$. $\square$

With $f = \mu$, $g = 1$ this gives $\sum_{d \leq N} \mu(d) d^{-s} X_{[N/d]}(s) = 1$; with $f = \Lambda$, $g = 1$ it gives $-X'_N(s) = \sum_{d \leq N} \Lambda(d) d^{-s} X_{[N/d]}(s)$; and for $\sqrt{N} < y \leq N$ the decomposition by the largest prime factor gives $X_N(s) = E_{y,N}(s) + \sum_{y < p \leq N} p^{-s} X_{[N/p]}(s)$, where $E_{y,N}$ is the sum over the y -smooth $n \leq N$. All three hold at every s , at a zero of ζ or not, and for every value of β ; they are consequences of the Euler product in the sense of unique factorization, and by the proposition they cannot restrict the position of the zeros. In Bohr's correspondence [Bh] the Euler product is

the statement that X_N is the restriction of a polynomial in the infinitely many variables x_p to the points $x_p = p^{-s}$, and as t varies these points wind densely around the torus $|x_p| = p^{-\sigma}$. Identities that follow from multiplicativity hold on the whole space of the x_p . The real part β enters only through the radii $p^{-\sigma}$ and through the continuation term $N^{1-s}/(1-s)$ of the Euler–Maclaurin expansion, which is not an identity between Dirichlet polynomials.

The computations confirm this. With the primes up to 10^6 , the prime sum

$$P_N(s) = 2 \operatorname{Re} \sum_{d < N} \Lambda(d) N^{-\bar{s}} d^{-s} X_{\lfloor (N-1)/d \rfloor}(s)$$

agrees with $-2 \operatorname{Re}[N^{-\bar{s}} X'_{N-1}(s)]$ to 10^{-15} at ρ_1, ρ_2 and at $0.37 + 20i$, which is not a zero, and the Möbius identity holds to 10^{-11} at the same points for $N = 10^4, 10^5, 10^6$. At a zero, Euler–Maclaurin and $\sum_{d < N} \Lambda(d)/d = \log N - \gamma_0 + o(1)$ give

$$P_N(\rho) - C_2(N, \rho) \sum_{d < N} \frac{\Lambda(d)}{d} = K(\rho) N^{1-2\beta} - 2 \operatorname{Re} [N^{-\bar{\rho}} \zeta'(\rho)] + o(N^{1-2\beta})$$

$$K(\rho) = \gamma_0 A_\rho - 2 \operatorname{Re} \frac{1}{(1-\rho)^2},$$

with $A_\rho = 2 \operatorname{Re} \frac{1}{1-\rho}$ the constant of the centre. At ρ_1 , $K = 0.01286$ and the left side is 0.0133, 0.0084, 0.0116 for $N = 10^4, 10^5, 10^6$, the deviations being of the size $2|\zeta'(\rho_1)|N^{-1/2}$ of the second term. The leading terms $A_\rho N^{1-2\beta} \log N$ cancel, but the term $N^{1-2\beta}$ remains with the coefficient $K(\rho)$, whose only arithmetic ingredient is the constant γ_0 of Mertens' theorem; the primes reproduce the expansion of C_2 and do not constrain it. Likewise, in the decomposition by the largest prime factor with $y = N^\theta$, the main term $N^{1-\rho}/(1-\rho)$ of $X_N(\rho)$ is shared by the large primes and the smooth part in the proportions $-\log \theta$ and $1 + \log \theta$ [Bu]; for $\theta = \frac{1}{2}$ the smooth share at ρ_1 is 0.245, 0.267, 0.270 for $N = 10^4, 10^5, 10^6$, approaching $1 - \log 2 = 0.307$ slowly. Iterating the decomposition distributes $N^{1-2\beta}$ over the scales N^{θ^k} with weights summing to 1 and does not change the exponent.

A quantity that sees β must compare ρ with $1 - \bar{\rho}$, as in Remark 6.9; a product such as $p^{-\rho} p^{-(1-\bar{\rho})} = p^{-1-2i\gamma}$ does not, since its modulus is p^{-1} , whereas the ratio $|p^{-\rho}|/|p^{-(1-\bar{\rho})}| = p^{1-2\beta}$ does. Summed over the zeros with a test function, the comparisons of this kind form Weil's functional $\sum_\rho h(\rho) \overline{h(1-\bar{\rho})}$ of Section 5. Status: the proposition is proved; the rest is numerical.

Proposition 6.21 (entropy forms and Bochner duality). Let T be an even real tempered distribution on $\mathbb{R}$ and $Q(f) = \iint f(x) \overline{f(y)} T(x-y) dx dy$ for $f \in C_c^\infty(\mathbb{R})$.

- (i) $Q(f) \geq 0$ for all f if and only if $\hat{T}$ is a positive measure, and then $Q(f) = \int |\hat{f}(\xi)|^2 d\hat{T}(\xi)$.
- (ii) If there is a positive measure p with $Q(f) = \int |f(x)|^2 dp(x)$ for all f , then p is a constant multiple of Lebesgue measure and $T = c \delta$.

Proof. (i) is the Bochner–Schwartz theorem, with Parseval. (ii) By polarization the bilinear forms $\iint f(x) \overline{g(y)} T(x-y)$ and $\int f \bar{g} dp$ coincide, so $T(x-y) = p(x) \delta(x-y)$ as distributions on $\mathbb{R}^2$. The left side is invariant under $(x, y) \mapsto (x+a, y+a)$, hence so is p , which is therefore $c dx$. $\square$

For ζ , Weil's form of Section 5 is Q with

$$T = (\text{pole}) + (\text{archimedean part}) - \sum_n \frac{\Lambda(n)}{\sqrt{n}} (\delta_{\log n} + \delta_{-\log n}),$$

and $\hat{T} = \sum_\rho \delta_\gamma$ when the zeros are on the line. A second-order expansion of a relative entropy $D(P_\theta \| P_0)$, a variance or a Fisher information, for distributions P_θ on the integers or on $x = \log n$, is a form $\int |h|^2 dp$ with $p \geq 0$. By (ii) such a form cannot be Weil's, since T is not a multiple of δ ; its masses at the prime powers are negative. By (i) Weil's form is a form of this kind on the spectral side, $Q(f) = \sum_\gamma |\hat{f}(\gamma)|^2$, with the zeros as reference measure. A divergence between

positive distributions built from the integers therefore produces positivity on the wrong side of the duality, and on the right side the positive measure is the set of zeros itself. This explains why the entropies of Remarks 6.15 and 6.16, which are positive by construction, carry no information on β .

We checked the spectral representation in the basis of cubic B-splines with mesh $D = 0.03$ used in Section 5, where the form is the Toeplitz matrix $[w(j - k)]$. The values $w(m)$ computed from the primes, the pole and the archimedean term agree with $\sum_{\gamma} 2\hat{K}(\gamma) \cos(\gamma m D)$, $\hat{K}(r) = (2 \sin(rD/2)/r)^8 \geq 0$, summed over the 341 zeros with $\gamma < 600$ plus the tail from the zero density, to 10^{-10} ($w(0) = 21.0981776757$ from both sides), and to $5 \cdot 10^{-7}$ with the 341 zeros alone, for $0 \leq m \leq 180$. In this form the matrix is $\sum_{\gamma} 2\hat{K}(\gamma) v_{\gamma} v_{\gamma}^*$ with $v_{\gamma} = (e^{i\gamma k D})_k$, a sum of positive rank-one terms. For the Davenport–Heilbronn function the corresponding decomposition has complex γ , and the smallest eigenvalue is -2.90 at $L = 4.2$ and -7.47 at $L = 5.4$. Status: the proposition is proved; the comparison is numerical.

7. DISCUSSION

Each of the criteria above is equivalent to RH or implied by it, and each was tested on the Davenport–Heilbronn function f . The results are summarized below.

quantity	ζ	Davenport–Heilbronn f
limit of C_2 (Thm. A, Cor. 2.1)	trichotomy at $\sigma = \frac{1}{2}$	0 for all $\sigma > 0$
helix crossings to $T = 400$ (§2)	202 of 202	287 of 303
spiral of a zero (Prop. 2.3)	rate $n^{-1/2}$	rates $n^{-0.81}$, $n^{-0.19}$
zeros on prime circles (§2)	bias toward π	spikes of both signs
prime-signal energy (Thm. B)	$\approx C_B$ up to 10^8	grows, rate $\approx 2\beta - 1$
local energy (Thm. 3.4)	inside $C_B \pm B(H)$	grows from 0.22 to 2.06
energy slope (Thms. C, D)	$\min_t S \approx 0.39(\sigma - \frac{1}{2})$	one failure interval per zero
W_L (Thm. E)	inside the RH band	negative outside the disc
Weil matrices (§5)	margin $10^{-6.3L^2}$	negative for $L > 3.6$
Euler product (§6)	converges right of zeros	converges right of zeros
Euler helix, pairs $\frac{1}{2} \pm \delta$ (Rem. 6.6)	$\delta_{\min} \approx 4 \cdot 10^{-5}$ at $x = 10^8$	—
heat flow H_t (Rem. 6.8)	first collision at $t_c = -0.47$ (height 49.7)	—
explicit spiral (Prop. 2.3')	rate $N^{-3/2}$	$M^{-2.81}$, $M^{-2.19}$
extrema of Z without crossing	none, $t < 10^5$	8, at the pairs
two-window Weil form (§5)	$ Y /D \leq 0.94$	—

The row of the Euler product is the only one in which f behaves like ζ : convergence of a logarithmic Dirichlet series is controlled by the zeros to its right, whether or not there is an Euler product. In every other row the difference comes from the coefficients. Those of $\log \zeta$ are $\Lambda(n)/\log n \geq 0$ and live on prime powers; those of $\log f$ change sign and live mostly on composite integers. Landau's formula (2.3) displays this difference most directly: summed over the zeros, x^{ρ} returns $-\frac{T}{2\pi}\Lambda(x)$, a one-signed function supported on prime powers.

The results also show where the Euler product is used and where it is not.

1. *In linear form the pole cancels positivity.* Any inequality $\sum a_j \cos j\theta \geq 0$ has $a_0 > 0$, so its use in the strip pays a correction of size $x^{1-\sigma}$ from the point $t = 0$ (Section 6). More generally, no combination of cosines without a constant term is nonnegative at all $\log p$, because the angles $t \log p$ are uniformly distributed (Proposition 6.12). This is why the classical method gives zero-free regions only near $\sigma = 1$, and twisting by characters does not change it (Proposition 6.4).
2. *In quadratic form the primes create positivity.* The pole enters Weil's form as a rank-two term, and the prime terms move the zeros of the minimizer onto the zeros of ζ (Section

- 5). The margin, however, decreases at least like $10^{-6.3L^2}$, and positivity on all windows is equivalent to RH. The size of the margin is fixed by the density of the zeros alone: points with the same density and no arithmetic content give the same values, and for curves over finite fields, whose zeros have constant density, the margin is the constant $2 \log p$ (Proposition 5.2). In a basis of Dirichlet polynomials the collapse of the margin is a counting effect, the same for the Davenport–Heilbronn function.
3. *In energy form positivity is automatic and finiteness is the question.* Averaged over t , the square of the cross-energy keeps only its diagonal; this quadratic positivity without a pole gives the theorem of Bohr and Landau that off-line zeros have density zero (Proposition 4.5) and, with higher moments, the density theorems, but an average cannot exclude a single zero. With a mollifier the bound improves until it meets the gap in Jensen’s inequality, the variance of $\log |\zeta M_y|$, which mollifiers of length at most T cannot reduce below about $T^{1-2\sigma}$ (Remark 4.6). Theorem B expresses RH as the finiteness of $\sum_{\rho} \lim_n C_2(n, \rho)$, that is, of the energy of the prime signal.
4. *Pointwise, the obstruction is local.* By Theorems C and D, a failure of the energy slope at abscissa σ occurs only inside the discs of off-line zeros, and the density theorems, which use the Euler product through mean values of Dirichlet polynomials, make the failure set sparse. For functions of Davenport–Heilbronn type it has positive proportion [KK]. A proof along these lines would require a version of these mean-value arguments valid inside a single disc. By Theorem E this is Weil positivity along the Poisson family, and the truncated functionals W_L show where the difficulty lies: they are exact and computable for each L , but the limit $L \rightarrow \infty$ at fixed $\sigma < 1$ is equivalent to controlling all zeros with $\beta > \sigma$.
5. *From finite prime data the obstruction is uniformity, not resolution.* The Euler helix fixes the real parts of the zeros near a given height to within $4 \cdot 10^{-5}$ with the primes up to 10^8 , although symmetric off-line pairs enter only at second order (Remark 6.6). Combined with the functional equation this can give exact local statements, as in Turing’s method, but it requires a list of the zeros and a number of primes that grows with the height.
6. *Amplification needs an upper bound for the primes.* Over $\mathbb{Q}$ the amplification that proves the analogue of RH for curves becomes a one-sided bound $\psi(x) \leq x + O(x^{1/2} \log^2 x)$. Sieves give upper bounds by positivity, but they stop at the parity barrier, which we measure as a factor close to 2 in Selberg’s sieve on short intervals; removing it needs bilinear information of the kind used by Matomäki and Radziwiłł, which the cross-energy does not provide (Remark 6.7).
7. *The Laguerre hierarchy is the finiteness of the helix energy.* The properties that force real zeros are the Laguerre inequalities $L_n \geq 0$ for all n . On the heat-flow deformation, L_1 and the energy slope change sign exactly at the first collision, while low-degree Jensen polynomials miss it, and wide pairs need higher n . On the helix side, L_1 and L_2 are the mean and half the variance of $|H_s|^2$, and all L_n are moments of a sum of circles: nonnegative when they exist, infinite when a zero leaves the line (Remark 6.8). The crossing law measures the spectral variable $\lambda = s(1-s)$: $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} + \dots$, with rational coefficients, and so do the X – Y energies, $|X_N^C(\rho)|^2 = N/\lambda - \frac{1}{6N} + \dots$ (Section 2).
8. *What the energy slope proves.* The slope gives the zero-free region of de la Vallée Poussin with the constants of the elementary method (Proposition 4.4), and the cross-energy evaluated at its own crossing reduces to $Z(t) \cos \varphi(t)$ on the line, with the zeros of ζ as dislocations of its nodal pattern in the strip (Section 2). Both are proved; neither reaches inside the strip beyond the classical results.
9. *Holomorphic sums are blind, energies are expensive.* Sums over the zeros that are holomorphic in ρ come linearly from the primes and do not see β ; non-holomorphic ones, such as $\sum |\rho|^{-2}$ or the crossing points, see β and are reached from the primes only through energies.

The damped energy has its first singularity at $2\sup\beta - 1$, but detects an off-line zero at height T only with the primes up to about $T^{1/\delta}$ (Remark 6.9).

10. *The wave and the helix are one object.* Sieving the partial sums by the primes up to y multiplies the free amplitude of the wave W by the partial Euler product; after the pole correction is removed, what remains is $\exp(-H_s(\log y)/\log y)$, the Euler helix (Remark 6.10). The two halves of the paper, partial sums and partial products, meet there.
11. *The primes act on all waves by the same real operator, and the reflection pairs convergence with divergence.* On the critical line each prime acts on the wave by $1-p^{-1}T_p$, independently of the height, with spectral form $1/\zeta(2\sigma + \lambda)$; the Davenport–Heilbronn function has no such operator. The functional equation, sieved, keeps modulus 1 on the line by symmetry, but maps the half of the strip where the Euler product converges onto the half where it always diverges (Remark 6.11). This is where the two structures that RH requires meet, and where finite prime data stop.
12. *A single combination of the three structures.* The functional equation, the logarithm of the C_2 energy and the derivative combine into one criterion: RH holds if and only if $\log|\xi(\frac{1}{2} + \sqrt{v} + it)|$ is concave in $v = (\sigma - \frac{1}{2})^2$ for every t (Proposition 4.3). Its slope at the line is the energy of the Euler helix, the Euler product makes it positive for $\sigma > 1$, and one off-line pair breaks the concavity only near its own height, as the Davenport–Heilbronn function shows.
13. *Scale recursions are neutral.* The decompositions of $\Lambda - 1$ into type I and type II pieces (Vaughan’s identity, $\Lambda - 1 = \mu * (\log - \tau)$, the recursion $\sum_{k \leq x} M(x/k) = 1$) are covariant under dilations: by Mellin the scale operator $f \mapsto \sum_{n \leq x} f(x/n)$ is multiplication by $\zeta(s)$, a profile x^a is mapped to x^a , and a bilinear sum inherits the exponent of its factors (for $\sum_{m \sim U} \mu(m)m^{-s} \sum_{x < mn \leq 2x} (\Lambda(n) - 1)n^{-s}$ the ratio to $|A_U||B_V|$ stays between 0.3 and 0.5 with no power trend). Such recursions gain only logarithms, which is why elementary methods stop at savings $\exp(-(\log x)^c)$. Exact periodicity of $N(x) - cx$ and multiplicativity together determine the ordinary primes among Beurling systems up to finitely many primes [Hil]; each alone fails, as the Davenport–Heilbronn function and Beurling systems with large oscillation show [DMV, BDV]. A proof that gains a power must use both, in the part of the sums where the sawtooth $\{x/n\}$ is linear in n ; in $\sum_{n \leq x} \mu(n)\{x/n\} = x \sum_{n \leq x} \mu(n)/n - 1$ the part $n \leq \sqrt{x}$ is trivially $O(\sqrt{x})$ and the whole size comes from $n > \sqrt{x}$.
14. *At large height the window is short, and positivity there is rigid.* By Proposition 5.3 an isolated zero with $\beta = \frac{1}{2} + \delta$ at height γ is detected by the prime powers up to $(C \log \gamma)^{1/\delta}$. The positivity that would exclude it holds for ζ with a margin of 6 to 15% up to $3 \cdot 10^6$, but fails for a random multiplicative model with the same mean: it is not a property of the distribution of the primes, and an argument for it must use the arithmetic that fixes the positions of the zeros.

We list the questions that the computations suggest and that we cannot answer.

- Is there a positive quantity built from the partial Euler products that does not pay the pole correction of item 1, for instance an average over twists in which the $t = 0$ term cancels?
- Can the exceptional set of Theorem D be bounded inside one disc, using only the information in $\sum_{n \leq e^L} \Lambda(n)n^{-s}$ for L of size $C/(\sigma - \frac{1}{2})$?
- The one-signed spectrum of (2.3) is a property of the exact positions of the zeros: displacements of a small fraction of the mean spacing destroy it (Section 2). Is there a statement about $\{\gamma \log p \bmod 2\pi\}$ that is equivalent to the Euler product and implies a constraint on β ?

Two known structures and a missing one. The results can be summarized by the role of three ingredients. The functional equation alone does not force the zeros onto the line: the Davenport–Heilbronn function has one of the same type and off-line zeros, and every criterion of

this paper separates it from ζ only through its coefficients. The Euler product alone does not force them either: it controls the half of the strip to the right of the zeros and diverges on the other half, and products without a functional equation need not even continue into the strip. Together the two are expected to imply RH for every function of the Selberg class, but they do not meet directly: the functional equation maps the half where the partial Euler products converge onto the half where they diverge (Remark 6.11). What is missing is a third ingredient, a positivity that holds for structural reasons. Over function fields it is supplied by the geometry of the curve, through intersection theory and the cohomology used by Weil and Deligne; for an elliptic curve it is the positivity of the degree $\deg(m - nF) = |m - n\alpha|^2$, which the wave of Section 2 reproduces in every other respect but does not see (Remark 6.13), and for graphs and for Dirichlet L -functions over $\mathbb{F}_q[T]$ it is the positivity of a Toeplitz matrix of Weil type coming from a unitary or self-adjoint operator (Remark 6.14). Read as quantum states, the Euler product is a product state whose coherence and entropy do not depend on t , its zeros are orthogonality times of the evolution $e^{-it \log n}$, and zeros leave the line exactly when characters are superposed (Remarks 6.15, 6.16); over $\mathbb{Q}$ it has been sought as a self-adjoint operator (Hilbert–Pólya), as the positivity of Weil’s functional, or as the real-rootedness of Ψ (Pólya). In the language of this paper it appears as the finiteness of the energy of the prime signal and of the Euler helix (Theorem B, Remark 6.5), as the Laguerre inequalities (Remark 6.8), as the energy slope (Theorem 4.1), and as positivity of the Weil matrices (Section 5). Each of these is equivalent to RH, and none is supplied by the partial sums, the partial products or the explicit formula. Every construction made here from the cross-energy either reduces to one of them or detects zeros without constraining them: the crossing law measures the spectral variable, the wave W counts zeros wherever it is applied, the damped energy sees any off-line zero at a cost of primes up to about $T^{1/\delta}$, and sieving by primes stops at the parity barrier. The contribution of the paper is a description of where the missing positivity would have to act, from these several directions, with numerical verification and with the Davenport–Heilbronn function as a control throughout.

Transport without a closed finite space. A proof for all zeros needs either a limit at infinity or a compactification that brings infinity into a bounded region, and the computations above show the same obstruction in both: the limits cannot be reached by computation (Corollary 3.5(c)), and the compactifications of Section 3 and of Propositions 3.5’–3.5’’ concentrate the difficulty at a boundary point. Where RH is proved, a third mechanism is used: an operation that transports the problem from one scale to the next and conserves a norm, so that one verification covers every scale. Over function fields this is Frobenius, unitary on a finite-dimensional cohomology that it maps to itself. Over $\mathbb{Q}$ the primes supply the operations. In the variable $x = \log u$ the dilation by p is the translation $T_p g(x) = g(x - \log p)$, which on the wave is the prime operator $T_p W(n) = W(n/p)$ of Remark 6.11; the translations by $\log p$ commute, and on the zero side T_p multiplies $\hat{g}(\gamma)$ by the phase $p^{i\gamma}$. Weil’s form depends on g only through $h = g * \tilde{g}$, which is unchanged by translation, so $Q(T_p g) = Q(g)$ exactly, for every p and every g : the prime operators are isometries of Weil’s form. If $Q \geq 0$, the test functions completed under Q form a Hilbert space on which the translations act unitarily, and the spectrum of their generator is the set of ordinates γ , which are then real; this is the construction of Weil and Connes, and its positivity is RH. What fails is closure. Weil’s form sees g only through $h = g * \tilde{g}$, whose support is the difference set $S - S$ of the support S of g , and its margin is governed by the length of $S - S$, not by the dimension of the space or by the arithmetic of the translations. With 50 digits, on the prime side and in the full basis of sines and cosines, the smallest eigenvalue is $3.7 \cdot 10^{-10}$ on a window of length 1.25 (24 functions) and $3.0 \cdot 10^{-10}$ on the same window with 48 functions; adding one copy of the window translated by $\log 2$ lowers it to $6.9 \cdot 10^{-21}$, a translation by 0.6, which is not the logarithm of a prime, gives $7.2 \cdot 10^{-21}$, and the full window of length $1.25 + \log 2$ gives $2.5 \cdot 10^{-26}$. A single translate is enough: it adds the lags $c \pm L$ to $S - S$, new prime powers enter the form through them, and positivity has to be established again with no margin inherited from the smaller set (Section 5, “The prime powers one at a time”).

For a curve over $\mathbb{F}_q$ all the translations are multiples of $\log q$, the difference set of a space closed under them stays in the lattice $(\log q)\mathbb{Z}$, the space is closed and of dimension $2g$, and the margin on windows is the constant $2 \log q$ of Proposition 5.2 (there $q = p$). Over $\mathbb{Q}$ the numbers $\log p$ are linearly independent, the translations generate a dense subgroup, and every translate brings new lags. Nor can a common degree be imposed approximately: the best ℓ with $\log p \approx k_p \ell$ for all $p \leq P$ leaves an error $\varepsilon \approx \ell^{1+1/(n-1)}$ with $n = \pi(P)$ (for $p \leq 13$, $\varepsilon = 0.10, 7 \cdot 10^{-3}, 9 \cdot 10^{-5}$ at $\ell \approx 0.33, 0.03, 0.001$), so keeping ε below a margin of 10^{-55} would require $\ell \approx 10^{-46}$, that is $q = e^\ell \rightarrow 1$. The difference between $\mathbb{Q}$ and function fields can therefore be stated in terms of the objects of this paper: the prime operators and Weil's norm exist and are compatible, and what is missing is a finite set of lags, closed under the prime translations, on which the norm can be checked once. Status: the invariance is elementary; the computations are numerical (`anc/transport/`).

We do not prove the Riemann hypothesis. The material of Parts I–V that is not included here (the Fresnel transition law, the monotonicity windows, the Jensen formula for the C_2 wave and the anatomy of individual zeros) is available in [C1]–[C5].

8. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) were computed once and are reused throughout. ζ is evaluated by Euler–Maclaurin summation, and the Davenport–Heilbronn function f through Hurwitz zeta functions as in [C5]; its eight off-line zeros with $\gamma < 400$ are those of [C5]. $\Lambda(n)$ is computed by sieve up to 10^8 , and the coefficients $b(n)$ of $-f'/f$ by Dirichlet convolution up to $2 \cdot 10^7$. Every value in the tables was obtained by two independent routes where one was available, from the primes (or coefficients) and from the zeros, and the agreement is stated in the text.

- *Section 2.* Z_n for $n \leq 2 \cdot 10^4$ against (2.1); the detector $\max_n |D_n(t)|$ on a grid of step 0.002; the sums $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$, their analogues for f , and the displacement and GUE controls (directories `spiral/`, `landau/`).
- *Section 3.* $E(u)$ from the prime powers up to 10^8 ; autocorrelations; $\mathcal{J}(\delta)$ by quadrature of $|\zeta'/\zeta + s/(s-1)|^2 |s|^{-2}$ on $|t| \leq T$ with a tail estimate; the window energies of E_f up to $x = 2 \cdot 10^7$ (`signal/`, `c2law/`).
- *Section 4.* $S(\sigma, t)$ by central differences with step 10^{-5} applied to $\log |\xi|$ and to the completed f , on grids of step 0.01 in t and 0.005 in σ ; GUE spectra of size 400, bulk half, unfolded by the semicircle law (`slope/`).
- *Section 5.* W_L from (5.1) with prime powers or coefficients up to e^{14} and the archimedean integral over $|r| \leq 3000$ with an analytic tail; the Weil matrices in the sine basis with numerical integration over $|r| \leq 800$ (step 0.004), and in the cubic B-spline basis of mesh 0.03 (Toeplitz, archimedean term by Gauss–Legendre quadrature); eigenvalues by `numpy.linalg.eigh`, and on the zero side by singular values, which resolve values below the floor of the explicit side (`weil/`). The minimiser as an approximate spectrum: the even sector of the same sine basis in double precision, and an independent mpmath implementation in a cosine basis with Gauss–Legendre quadrature, at 100 digits for ζ ($L = \log 13$, 44 basis functions) and 50 digits for f ($L = 3.4$, 67 basis functions); zeros of f refined with Hurwitz zeta in mpmath (`connes/`). Möbius energy and Littlewood's criterion: $\mu(n)$ up to 10^8 by a sieve, Dirichlet inverses of the coefficients of $L(s, \chi)$ and f up to 10^6 , $\zeta'(\rho)$ and $L'(\rho)$ by central differences (`mobius_energy/`). Zero counts in rectangles: argument principle with adaptive step ($|f(b) - f(a)| < \frac{1}{2}|f(a)|$ between consecutive points), Littlewood sums with composite Gauss–Legendre quadrature, mpmath with 20 digits (`zerocount/`).

- *Sections 2, 4, 5 (twins and Dirichlet polynomials).* Zeros of $L(s, \chi)$, $\chi \bmod 5$, from sign changes of $W^{-1/2}\Lambda(\frac{1}{2} + it, \chi)$ on a grid of step 0.01, refined by Brent's method; the matrices A_{mn} from (5.1) with prime powers up to 4000 and from the zeros (`twin/`, `joint/`).
- *Section 6.* Partial products over $p \leq 2 \cdot 10^6$; E_1 from `scipy.special.exp1`; mode sums over the 682 zeros with $|\gamma| < 600$ (`euler/`). Proposition 6.1': Gaussian weight in $\log x$, K_g by the trapezoidal rule on 4000 nodes, and the 1517 zeros below height 2000 located from sign changes of Hardy's Z and refined by Brent's method (`euler_local/`); the crossing series by exact rational arithmetic (`crossing/nstar_series.py`).
- *Remarks 6.6–6.8 and the spectral variable (Section 2).* Euler helix with exact bin averages and Gaussian smoothing (`rigidity/`); Selberg's sieve, Liouville's function and the explicit formula on $(10^8, 10^8 + y]$ (`parity/`); moments b_k , the series Ψ , Jensen polynomials, the heat flow H_t and the Laguerre expressions in Python's `decimal` arithmetic with 60 to 500 digits (`spectral/`, `heat/`).
- *Proposition 3.8, the energy matrix and Remark 3.9.* $I(X)$ integrated exactly on each interval $[k, k+1)$ from the primes up to 10^7 ; the constant K from the three sums of Proposition 3.8, with $\zeta'/\zeta(a)$ from $\Lambda(n)$, $n \leq 10^6$, and truncation at $k \leq 32000$ extrapolated; Φ and $M(U)$ from the 682 zeros $|\gamma| < 600$ (`K_constant/`); Cramér's model, independent signs and random multiplicative functions with 100 realizations each up to 10^7 , seed 20261008 (`montecarlo/`).

How this work was produced. The cross-energy C_2 and the research program are the author's, beginning with his preprints of 2017. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures; other AI assistants were used for exploratory discussion. Several claims of the author's earlier preprints were found to be incorrect in this process and are not included as results (Appendix A). All results were checked by the author, who is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package. The preprints [C1]–[C8] on which this paper is based, with their own scripts and data, are deposited on Zenodo under the DOIs listed in the references.

APPENDIX A. COMPENDIUM OF FORMULAS

This appendix collects the formulas used in the paper and some related ones, grouped by structure. Each group states its status: an exact identity, an asymptotic result, a criterion equivalent to RH, or a statement that would imply RH and is not proved. Throughout, $s = \sigma + it$, $\rho = \beta + i\gamma$ is a nontrivial zero, $\rho^* = 1 - \bar{\rho}$ is its reflection, and $X_N(s) = \sum_{n \leq N} n^{-s}$.

A.1. Partial sums. *Status: exact identities and classical asymptotics.*

$$X_N(s) - X_{N-1}(s) = N^{-s}, \quad -X'_N(s) = \sum_{n \leq N} \frac{\log n}{n^s}. \quad (\text{A.1})$$

For fixed $s \neq 1$ with $0 < \sigma < 1$, Euler–Maclaurin summation gives

$$X_N(s) = \frac{N^{1-s}}{1-s} + \zeta(s) + \frac{1}{2}N^{-s} + O_s(N^{-\sigma-1}), \quad (\text{A.2})$$

and at a zero

$$\begin{aligned} X_N(\rho) &= \frac{N^{1-\rho}}{1-\rho} + \frac{1}{2}N^{-\rho} + O_\rho(N^{-\beta-1}), \\ -X'_N(\rho) &= N^{1-\rho} \left[\frac{\log N}{1-\rho} - \frac{1}{(1-\rho)^2} \right] - \zeta'(\rho) + O_\rho(N^{-\beta} \log N). \end{aligned} \quad (\text{A.3})$$

The $X - Y$ decomposition. Put $X_N^C(s) = X_N(s) - \frac{1}{2}N^{-s}$ and $Y_N^C(s) = N^{1-s}/(1-s)$. By (A.2), for $0 < \sigma < 1$,

$$\zeta(s) = \lim_{N \rightarrow \infty} [X_N^C(s) - Y_N^C(s)], \quad D_N(s) := X_N^C(s) - Y_N^C(s) = \zeta(s) + O_s(N^{-\sigma-1}). \quad (\text{A.4})$$

The two components diverge separately for $\sigma < 1$; only their difference converges. The continuous component has

$$|Y_N^C(s)|^2 = \frac{N^{2-2\sigma}}{(1-\sigma)^2 + t^2}, \quad \frac{d}{dN} |Y_N^C(s)|^2 = \frac{2(1-\sigma)N^{1-2\sigma}}{(1-\sigma)^2 + t^2}, \quad (\text{A.5})$$

which is linear in N if and only if $\sigma = \frac{1}{2}$, with slope $1/(t^2 + \frac{1}{4})$. At a zero, $D_N(\rho) = O_\rho(N^{-\beta-1})$, hence

$$|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2 = O_\rho(N^{-2\beta}), \quad \frac{|X_N^C(\rho)|^2}{|Y_N^C(\rho)|^2} = 1 + O_\rho(N^{-2}). \quad (\text{A.6})$$

The two energies agree to leading order at every zero, on or off the critical line; their comparison therefore does not locate the zeros. More precisely, the first Bernoulli term gives $X_N^C(\rho)/Y_N^C(\rho) = 1 - \rho(1-\rho)/(12N^2) + O_\rho(N^{-4})$, so that on the critical line $|X_N^C(\rho)|^2/|Y_N^C(\rho)|^2 = 1 - \lambda/(6N^2) + O(N^{-4})$ with $\lambda = \frac{1}{4} + \gamma^2$; at ρ_1 and $N = 1600$, $N^2(X_N^C/Y_N^C - 1) = -16.6701 + 6 \cdot 10^{-6}i$ against $-\lambda/12 = -16.6700$. The normalised gap reads the abscissa of a zero (Section 2): for every zero ρ , trivial or not,

$$N^{2\beta}(|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2) \rightarrow \kappa(\rho) = -\frac{1}{6} \frac{\operatorname{Re} \lambda(\rho)}{|1-\rho|^2} = -\frac{1}{6} - \frac{(1-\beta)(2\beta-1)}{6|1-\rho|^2}, \quad (\text{A.6}')$$

so $\kappa(\rho) = -B_2$ if and only if $\beta = \frac{1}{2}$, and $\kappa(-2k) = k/(3(2k+1))$. On the critical line, at a zero and to all orders in N ,

$$|X_N^C(\rho)|^2 = \frac{N}{\lambda} - \frac{1}{6N} + \left(\frac{1}{90} + \frac{\lambda}{240}\right) \frac{1}{N^3} + \left(-\frac{1}{210} + \frac{19\lambda}{7560} + \frac{\lambda^2}{6048}\right) \frac{1}{N^5} + \dots, \quad (\text{A.6}'')$$

every coefficient a rational polynomial in λ ; for two zeros $|X_N^C(\rho_1)|^2 - |X_N^C(\rho_2)|^2 = N(\lambda_1^{-1} - \lambda_2^{-1}) + (\lambda_1 - \lambda_2)/(240N^3) + O(N^{-5})$, and $N(|X_N^C(s)|^2 - N/\lambda)$ converges on the line if and only if $\zeta(s) = 0$.

A.2. The cross-energy. *Status: exact identities; asymptotics as in Theorem A and Corollary 2.1.*

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} [N^{-\bar{s}} X_{N-1}(s)],$$

$$C_2(N, s) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos\left(t \log \frac{N}{n}\right). \quad (\text{A.7})$$

From (A.2),

$$C_2(N, s) = A_s N^{1-2\sigma} + 2N^{-\sigma} \operatorname{Re} [\zeta(s) e^{it \log N}] - N^{-2\sigma} + O_s(N^{-2\sigma-1}), \quad (\text{A.8})$$

with $A_s = 2(1-\sigma)/((1-\sigma)^2 + t^2)$. At a zero the oscillating term is absent:

$$C_2(N, \rho) = A_\rho N^{1-2\beta} - N^{-2\beta} + O_\rho(N^{-2\beta-1}), \quad (\text{A.9})$$

and on the critical line Corollary 2.1 refines this to $C_2(N, \rho) = \frac{1}{\gamma^2 + 1/4} - \frac{1}{N} - \frac{1}{12N^2} + \frac{\operatorname{Re}(\rho)_3}{360N^4} + R_N$. Consequently

$$\lim_{N \rightarrow \infty} C_2(N, \rho) = \begin{cases} +\infty, & \beta < \frac{1}{2}, \\ \frac{1}{\gamma^2 + 1/4}, & \beta = \frac{1}{2}, \\ 0, & \beta > \frac{1}{2}, \end{cases} \quad \lim_{N \rightarrow \infty} \frac{\log C_2(N, \rho)}{\log N} = 1 - 2\beta, \quad (\text{A.10})$$

the second for $0 < \beta < 1$, and

$$\lim_{N \rightarrow \infty} N^{2\beta-1} C_2(N, \rho) = A_\rho = N^{2\beta-1} \frac{d}{dN} |Y_N^C(\rho)|^2, \quad (\text{A.11})$$

so that at a zero C_2 is asymptotically the growth rate of the continuous energy (A.5). The leading terms of (A.9) on the critical line vanish at $N \approx \gamma^2 + \frac{1}{4}$; the crossing point of Proposition 2.2 is $n^* = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$.

Reflected zeros. $A_{\rho^*} = 2\beta/(\beta^2 + \gamma^2)$ and $C_2(N, \rho^*) = A_{\rho^*} N^{2\beta-1} - N^{2\beta-2} + O(N^{2\beta-3})$. The product and ratio satisfy

$$C_2(N, \rho)C_2(N, \rho^*) = A_{\rho}A_{\rho^*} - \frac{A_{\rho} + A_{\rho^*}}{N} + O_{\rho}(N^{-2}) \rightarrow \frac{4\beta(1-\beta)}{|\rho|^2|1-\rho|^2}, \quad (\text{A.12})$$

$$\frac{C_2(N, \rho)}{C_2(N, \rho^*)} \sim \frac{A_{\rho}}{A_{\rho^*}} N^{2-4\beta}, \quad \lim_{N \rightarrow \infty} \left[\frac{\log(C_2(N, \rho)/C_2(N, \rho^*))}{\log N} \right]^2 = 4(1-2\beta)^2. \quad (\text{A.13})$$

On the critical line the limit in (A.12) is $(\gamma^2 + \frac{1}{4})^{-2}$ and the limit in (A.13) is 0.

Criterion (Theorem A). RH holds if and only if $0 < \lim_N C_2(N, \rho) < \infty$ for every ρ . *Sufficient condition, not proved:* $C_2(N, \rho) = O_{\rho, \varepsilon}(N^{\varepsilon})$ for every ρ and $\varepsilon > 0$; applied to ρ and ρ^* it forces $\beta = \frac{1}{2}$.

A.3. Functional equation and derivatives. *Status: exact identities; they hold equally for hypothetical off-line zeros.* With $h(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})$ and $\xi(s) = h(s)\zeta(s)$,

$$\xi(s) = \xi(1-s), \quad \xi^{(k)}(s) = (-1)^k \xi^{(k)}(1-s), \quad \xi'(\rho^*) = -\overline{\xi'(\rho)}, \quad \xi''(\rho^*) = \overline{\xi''(\rho)}. \quad (\text{A.14})$$

At a simple zero, $\xi'(\rho) = h(\rho)\zeta'(\rho)$, $\xi''(\rho) = 2h'(\rho)\zeta'(\rho) + h(\rho)\zeta''(\rho)$, and

$$\frac{\xi''(\rho)}{\xi'(\rho)} = 2\frac{h'(\rho)}{h(\rho)} + \frac{\zeta''(\rho)}{\zeta'(\rho)}, \quad \frac{\xi''(\rho^*)}{\xi'(\rho^*)} = -\overline{\left(\frac{\xi''(\rho)}{\xi'(\rho)}\right)}. \quad (\text{A.15})$$

A.4. Primes in the partial sums. *Status: exact identities, except (A.19), which is asymptotic.*

$$\log n = \sum_{d|n} \Lambda(d), \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \quad (\sigma > 1), \quad (\text{A.16})$$

$$-X'_N(s) = \sum_{d \leq N} \frac{\Lambda(d)}{d^s} X_{\lfloor N/d \rfloor}(s), \quad (\text{A.17})$$

$$\frac{\partial C_2(N, s)}{\partial \sigma} = -(\log N) C_2(N, s) - 2 \operatorname{Re} \left[N^{-\bar{s}} \sum_{d < N} \frac{\Lambda(d)}{d^s} X_{\lfloor (N-1)/d \rfloor}(s) \right]. \quad (\text{A.18})$$

At a zero, inserting (A.3) in (A.17) and using $\sum_{d \leq N} \Lambda(d)/d = \log N - \gamma_0 + o(1)$,

$$-X'_N(\rho) - \frac{N^{1-\rho}}{1-\rho} \sum_{d \leq N} \frac{\Lambda(d)}{d} = N^{1-\rho} \left[\frac{\gamma_0}{1-\rho} - \frac{1}{(1-\rho)^2} + o(1) \right]. \quad (\text{A.19})$$

The $o(1)$ comes from the error in Mertens' formula and the replacement of $\lfloor N/d \rfloor$ by N/d ; it dominates the constant $-\zeta'(\rho)$ of (A.3), which is therefore not visible at this precision.

A.5. The prime error and its energy. *Status: classical identities and criteria; (A.24) is exact; (A.25) is Proposition 3.8, conditional on RH, simple zeros and a spacing condition.* With $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $x > 1$ not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1-x^{-2}), \quad -\frac{x^{\rho}}{\rho} - \frac{x^{\bar{\rho}}}{\bar{\rho}} = -\frac{2x^{\beta}}{|\rho|} \cos(\gamma \log x - \arg \rho), \quad (\text{A.20})$$

the sum over zeros taken in symmetric order. In the variable $u = \log x$ the normalized error of the paper is $E(u) = e^{-u/2}(\psi(e^u) - e^u)$, and a pair of zeros contributes an oscillation of amplitude $2e^{(\beta-1/2)u}/|\rho|$ and frequency γ . The Mellin transform of $\psi(x) - x$ is

$$\int_1^\infty (\psi(x) - x)x^{-s-1} dx = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} \quad (\sigma > 1), \quad (\text{A.21})$$

and RH holds if and only if this function continues holomorphically to $\sigma > \frac{1}{2}$, if and only if $\psi(x) - x = O_\varepsilon(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$.

Energy. Put

$$I(X) = \int_1^X \frac{(\psi(x) - x)^2}{x^2} dx = \int_0^{\log X} E(u)^2 du, \quad Q(X) = \frac{I(X)}{\log X}. \quad (\text{A.22})$$

By Theorem 3.2 and Theorem B, RH holds if and only if $\sup_{X \geq 2} Q(X) < \infty$, and then $Q(X) \rightarrow \sum_\rho m_\rho^2 |\rho|^{-2}$, which equals $C_B = 2 + \gamma_0 - \log 4\pi$ when the zeros are simple. If $I(X) = O(\log X)$, then $\int_1^\infty |\psi(x) - x|x^{-\sigma-1} dx < \infty$ for $\sigma > \frac{1}{2}$ by the Cauchy–Schwarz inequality on dyadic intervals, which excludes zeros with $\beta > \frac{1}{2}$. Expanding the square,

$$\int_1^X \frac{\psi(x)^2}{x^2} dx = \sum_{m,n \leq X} \Lambda(m)\Lambda(n) \left(\frac{1}{\max(m,n)} - \frac{1}{X} \right), \quad (\text{A.23})$$

$$I(X) = \sum_{m,n \leq X} \Lambda(m)\Lambda(n) \left(\frac{1}{\max(m,n)} - \frac{1}{X} \right) - 2 \sum_{n \leq X} \Lambda(n) \log \frac{X}{n} + X - 1. \quad (\text{A.24})$$

The double sum splits into the diagonal $D(X) = \sum_{n \leq X} \Lambda(n)^2 (\frac{1}{n} - \frac{1}{X}) \sim \frac{1}{2}(\log X)^2$ and the cross terms $T(X) = 2 \sum_{n \leq X} \Lambda(n)\psi(n^-)(\frac{1}{n} - \frac{1}{X})$. Identity (A.24) expresses the energy through prime powers alone, without reference to ζ or its zeros; the terms on the right are of size X , and the statement $I(X) = O(\log X)$ asks for their cancellation down to $O(\log X)$. It is a classical reformulation of the boundedness in Theorem B, not a new constraint.

The constant term. From the primes up to 10^7 ,

X	10^3	10^4	10^5	10^6	10^7
$Q(X)$	0.40866	0.31875	0.26449	0.22795	0.20200
$(Q(X) - C_B) \log X$	2.504	2.510	2.513	2.511	2.511

in agreement with Proposition 3.8:

$$I(X) = C_B \log X + K + \Phi(\log X) + o(1),$$

$$K = \frac{1}{2} \left(C_B^2 - \sum_\rho \frac{1}{|\rho|^4} \right) - 2 \sum_\rho \frac{R(1-\rho)}{\rho} + \frac{1}{2\pi} \int_{-\infty}^\infty |R(\tfrac{1}{2} + it)|^2 dt = 2.51248 \dots, \quad (\text{A.25})$$

with $R(s) = \frac{1}{s} \left(\frac{1}{2} \frac{\Gamma'}{\Gamma} \left(1 + \frac{s}{2} \right) + \frac{\gamma_0}{2} - \log 2\pi \right)$ and $\Phi(u) = \sum_{\rho \neq \rho'} e^{i(\gamma - \gamma')u} / (i(\gamma - \gamma')\rho\bar{\rho}')$, under RH, simple zeros and $\sum_\rho |\rho|^{-2} \delta_\rho^{-1} < \infty$. The decrease of Q is the approach to C_B at the rate $K/\log X$: subtracting $K/\log X$ from $Q(10^7)$ gives 0.0462. K is almost entirely the trivial part (the interval $1 \leq x < 2$, where $\psi = 0$, contributes exactly 1); the oscillation of $(Q(X) - C_B) \log X$ in the table is Φ . A finite computation does not establish boundedness.

A.6. The two factorizations. *Status: a summary of exact identities of the paper; the last line of the figure is the open step.* In the critical strip ζ has an additive factorization, $\zeta = \lim_N [X_N^C - Y_N^C]$ (A.4), equivalently the C-transform of 1, and a multiplicative one, $\zeta = E_g^* Z_g$ (Proposition 6.1'), the smoothed continuation of the Euler product. The cross-energy links them: $X_{N-1} = \frac{1}{2} N^{\bar{s}} W_N$ with $W_N = C_2 + iS_2$, and the charges of $\Phi = \log |X^C - Y^C|$ give Z_g (Section 6, the zero factor from C_2). Figure 62 collects the formulas and the equivalent forms of RH, together with the two controls: the Davenport–Heilbronn function has the additive side and the symmetry without an Euler product, and the Beurling system of Section 6 has the multiplicative side without the integers $1, 2, 3, \dots$; both have zeros off the line.

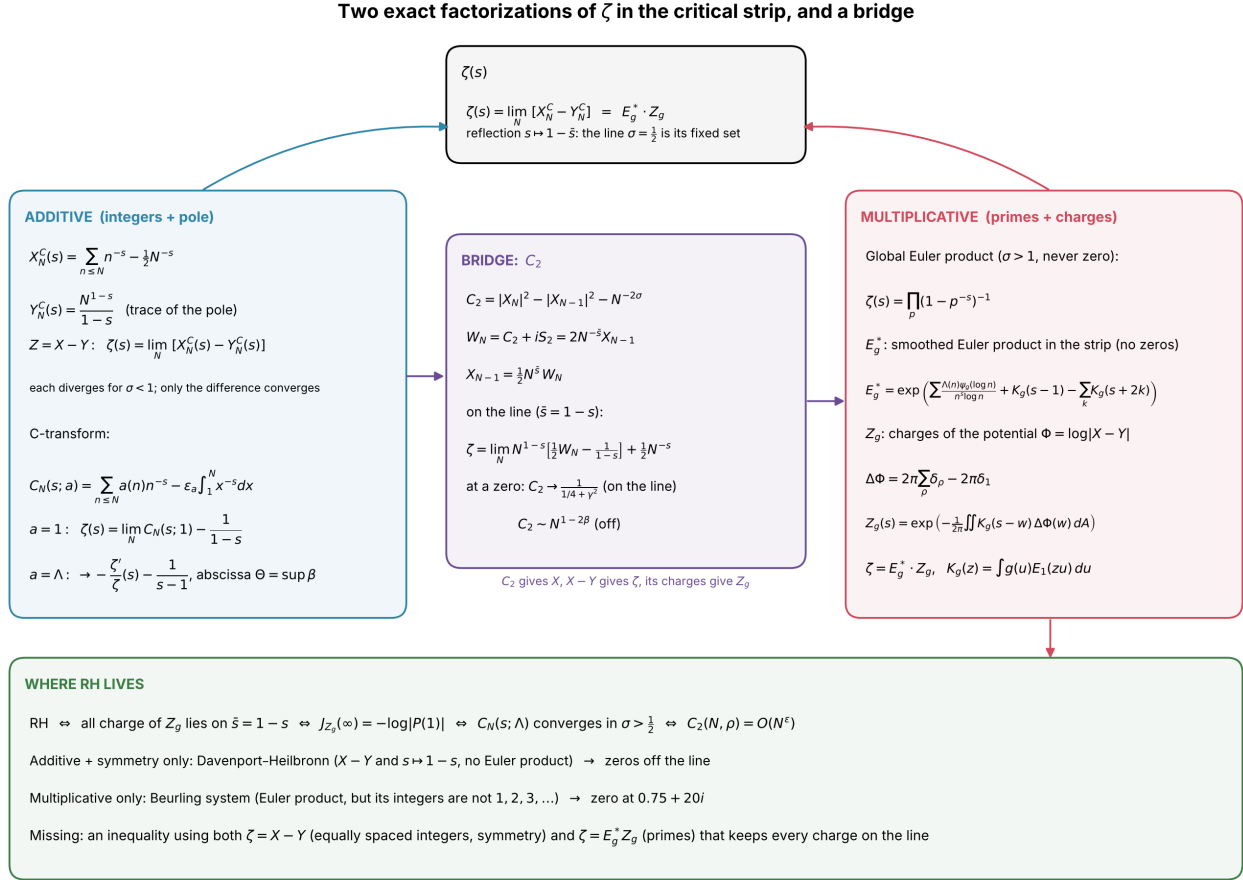


FIGURE 62. The additive and multiplicative factorizations of ζ in the critical strip, the bridge C_2 between them, and the equivalent forms of RH.

Primes minus integers. Put $X_N^\Lambda(s) = \sum_{n \leq N} \Lambda(n) n^{-s}$ and $W_N^\Lambda = C_2^\Lambda + iS_2^\Lambda = 2N^{-\bar{s}} X_{N-1}^\Lambda$. Since Λ has mean 1, the prime sum carries the same wave of the pole $Y_N = N^{1-s}/(1-s)$ as the integers, and the function

$$F(s) = 1 - \zeta(s) - \frac{\zeta'}{\zeta}(s) = \sum_{n \geq 2} \frac{\Lambda(n) - 1}{n^s} \quad (\sigma > 1)$$

has three equivalent forms in which the pole cancels identically:

$$F(s) = 1 + \lim_{N \rightarrow \infty} [C_N(s; \Lambda) - C_N(s; 1)] = 1 + \lim_{N \rightarrow \infty} [X_N^\Lambda - X_N] = 1 + \lim_{N \rightarrow \infty} \frac{1}{2} N^{\bar{s}} (W_N^\Lambda - W_N).$$

The limits exist for $\sigma > \Theta$ and, by Landau's theorem, for no $\sigma < \Theta$, so their existence in $\sigma > \frac{1}{2}$ is RH (the classical form via $\psi(x) - x = O(x^{1/2+\varepsilon})$). The poles of $-\zeta'/\zeta$ and ζ at $s = 1$ cancel, with $F(1) = 1 - 2\gamma_0$. Read backwards, the last form gives $C_2^\Lambda(N, s) - C_2(N, s) = 2 \operatorname{Re} (N^{-\bar{s}}(F(s) - 1))$ plus the waves of the zeros, each of size $N^{\beta-2\sigma}$: the cross-energies of the primes and of the integers agree asymptotically, and F measures their difference. In the multiplicative factorization, $F = 1 - E_g^* Z_g - E_g^{*'} / E_g^* + \sum_\rho K_g'(s - \rho)$, so each charge of Z_g is a pole of F . *Status: classical identities, collected here in the notation of the paper.*

A.7. Finite parts at a zero. *Status: exact identities from the Hadamard product and the functional equation; they hold for hypothetical off-line zeros.* With $F = 1 - \zeta - \zeta'/\zeta$, at a simple zero $F(s) = -1/(s - \rho) + c_0(\rho) + O(s - \rho)$ with $c_0(\rho) = 1 - \zeta''(\rho)/(2\zeta'(\rho))$. With $G(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \psi(\frac{s}{2})$ and $\xi'/\xi(s) = \sum_\rho 1/(s - \rho)$ in symmetric order,

$$c_0(\rho) = 1 + G(\rho) - \sum_{\rho' \neq \rho} \frac{1}{\rho - \rho'}. \quad (\text{A.28})$$

If $\beta = \frac{1}{2}$ the real parts of the sum cancel in pairs, so $\operatorname{Re} c_0(\rho) = 1 - \frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \psi(\frac{1}{4} + \frac{i\gamma}{2})$ whether or not RH holds; for an off-line zero the reflection alone contributes $-1/(2\delta)$ to $\operatorname{Re} c_0(\rho) - 1 - \operatorname{Re} G(\rho)$. Numerically the identity holds to $8 \cdot 10^{-8}$ at the first 100 zeros of ζ and to $3 \cdot 10^{-8}$ at 122 on-line zeros of f (with its own gamma factor), and the four off-line zeros of f below height 200 lie below by 1.88, 3.46, 6.81, 2.51, against $1/(2\delta) = 1.62, 3.32, 6.72, 2.23$. On the line the imaginary part is

$$\operatorname{Im} c_0(\rho) = \sum_{\gamma' \neq \gamma} \frac{1}{\gamma - \gamma'} + \operatorname{Im} G(\frac{1}{2} + i\gamma) = \frac{Z''(\gamma)}{2Z'(\gamma)}, \quad (\text{A.29})$$

the force of the other zeros along the line, which drives the de Bruijn–Newman flow [Pm]; it correlates with $1/g_- - 1/g_+$, $g_\pm$ the gaps to the neighbours, with coefficient 0.98 for ζ and 0.96 for f .

A graded form of Theorem C. For $0 < \eta \leq \frac{1}{2}$, ζ has no zeros in $\sigma > 1 - \eta$ if and only if $S(\sigma, t) = \operatorname{Re} \xi'/\xi(s) > 0$ there, if and only if

$$\operatorname{Re} F(s) < \Phi(s) := 1 - \operatorname{Re} \zeta(s) + \operatorname{Re} G(s) \quad (\sigma > 1 - \eta), \quad (\text{A.30})$$

since $S = \Phi - \operatorname{Re} F$ and $S = \sum_\rho (\sigma - \beta)/|s - \rho|^2$. On the line $\operatorname{Re} F = \Phi$; the right side is explicit, of size $\frac{1}{2} \log \frac{t}{2\pi}$, and the left side is the sum over the primes minus the integers.

Jensen moment at a general centre and the Euler–Cáceres mollifier (Proposition 6.1'''). *Status: (A.31) exact; (A.32) a proved inequality; (A.33) heuristic, checked numerically.* For $\operatorname{Re} s_0 > \frac{1}{2}$, with P_{s_0} the Poisson kernel and $b_\rho(s) = (s - \rho)/(s - 1 + \bar{\rho})$,

$$\int \log |\zeta(\frac{1}{2} + it)| P_{s_0}(t) dt = \log \left| \frac{(s_0 - 1)\zeta(s_0)}{s_0} \right| + \sum_{\beta > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|}, \quad (\text{A.31})$$

$$\sum_{\beta > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|} \leq \frac{1}{2} \log \int \left| \frac{\zeta}{E_g^*}(\frac{1}{2} + it) \right|^2 P_{s_0}(t) dt - \log \left| \frac{\zeta}{E_g^*}(s_0) \right|, \quad (\text{A.32})$$

$$\text{right side of (A.32)} \approx \frac{\kappa \sum_\rho P_{s_0}(\gamma)}{\log X}, \quad \kappa = \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv = 1.828 \dots \quad (\text{A.33})$$

If the real parts of the zeros attain their supremum $\Theta > \frac{1}{2}$, then for $\operatorname{Re} s_0 > \Theta$, $g_X(u) = g_0(u - \log X)$ and some $r > 0$ (Theorem 6.1'''),

$$\text{right side of (A.32)} \geq \frac{1}{2} \operatorname{Ei}(r(\log X - a)) - C \log \log X \geq \frac{X^r}{3re^{ra} \log X}. \quad (\text{A.34})$$

Under RH, with $\mathcal{E}(X) = \int |\zeta/E_{g_X}^*(\frac{1}{2} + it)|^2 dt / (1 + t^2)$ (Theorem 6.1⁽⁵⁾, Corollary 6.1⁽⁶⁾),

$$\log |Z_{g_X}(\frac{1}{2} + it)| = \sum_{\gamma} \int g_X(u) \text{Ci}(|t - \gamma|u) du \leq \varepsilon \log(|t| + 3) + C_{\varepsilon}, \quad \mathcal{E}(X) \rightarrow \pi, \quad (\text{A.35})$$

Consequently (Corollary 6.1⁽⁶⁾), unconditionally RH holds if and only if the right side of (A.32) with $g = g_X$ tends to 0, for any single s_0 with $\text{Re } s_0 > \frac{1}{2}$, $\zeta(s_0) \neq 0$; and if the supremum of the real parts of the zeros is attained, RH holds if and only if $\limsup_X \mathcal{E}(X) < \infty$.

A.8. Statements that would imply RH. *Status: each is equivalent to RH or implies it; none is proved.* For every zero ρ and every $\varepsilon > 0$:

$$C_2(N, \rho) = O_{\rho, \varepsilon}(N^{\varepsilon}), \quad X_N(\rho) = O_{\rho, \varepsilon}(N^{1/2+\varepsilon}), \quad (\text{A.26})$$

and, for the primes,

$$\psi(x) - x = O_{\varepsilon}(x^{1/2+\varepsilon}), \quad I(X) = O(\log X), \quad \sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma_0 - \log 4\pi. \quad (\text{A.27})$$

The last is the vanishing of the non-holomorphic gap $\mathcal{Q}$ of Section 3. The two energy structures of the paper, the cross-energy $C_2(N, \rho)$ of the integers and the energy $I(X)$ of the prime powers, are linked by Theorem B; what is missing is an inequality, independent of the location of the zeros, that excludes the growth $N^{1-2\beta}$ of $C_2(N, \rho)$ for $\beta < \frac{1}{2}$.

A.9. Statements not included as results. Two assertions of the author's earlier preprints are not valid in general. First, $C_2(N, \sigma, t) = 0$ does not imply $\sigma = \frac{1}{2}$: finite crossings of the cross-energy occur off the critical line. Second, the asymptotic vanishing of $X_N^C - Y_N^C$ does not force $|X_N^C|^2$ and $|Y_N^C|^2$ to be linear in N ; by (A.6) the two energies agree at every zero whatever its abscissa. These points do not affect the exact identities above.

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