

The torus with asymptotically the fewest simple closed geodesics

Phuc Thinh Dang, Trong Toan Dao, Nhat Minh Doan

Abstract

We prove that the modular torus uniquely minimizes the area of the stable-norm unit ball among complete finite-area hyperbolic once-punctured tori, resolving the McShane–Rivin area conjecture. It therefore has fewer simple closed geodesics of length at most L than any fixed nonmodular torus for all sufficiently large L . We also bound the area increase away from the modular torus and determine its leading growth and first correction term as the systole tends to zero, with error bounds independent of the twist.

CONTENTS

1. Introduction	1
2. Preliminaries	5
3. Estimates shared by the regional comparisons	12
4. Region A: comparison outside the local disk	23
5. Region B: comparison inside the local disk	27
6. Region C: comparison when the systole is short	31
7. Proofs of the main theorems	40
Appendix	41

1 Introduction

Let $S = S_{1,1}$ be a once-punctured torus. Choose an integral homology basis to identify $H_1(S; \mathbb{Z})$ with $\mathbb{Z}^2$ and $H_1(S; \mathbb{R})$ with $\mathbb{R}^2$. For a complete finite-area hyperbolic structure X on S , let $\|\cdot\|_X$ be its stable norm. For each nonzero primitive class $h \in H_1(S; \mathbb{Z})$, $\|h\|_X$ is the length of the unique oriented simple closed geodesic representing h . The classes h and $-h$ represent the same geodesic with opposite orientations. Set

$$\mathcal{B}(X) = \{w \in \mathbb{R}^2 : \|w\|_X \leq 1\}, \quad \mathcal{A}(X) = \text{Area}(\mathcal{B}(X)),$$

where Area is the usual Euclidean area. Changes of integral homology basis have determinant ± 1 , so $\mathcal{A}(X)$ is independent of this choice. Isometries preserve the stable norm and the integral homology lattice, so $\mathcal{A}$ depends only on the isometry class of X and defines a function on the moduli space.

The stable norm belongs to the theory of geometric norms on homology developed by Federer and Gromov [Fed74, Gro83]. For a hyperbolic once-punctured torus X , McShane and Rivin showed that its unit-ball area $\mathcal{A}(X)$ determines the leading term in simple-geodesic counting [MR95a, MR95b]. If $N_X(L)$ counts unoriented simple closed geodesics of length at most L , then

$$N_X(L) = \frac{3}{\pi^2} \mathcal{A}(X) L^2 + O_X(L \log L). \quad (1)$$

Thus, minimizing $\mathcal{A}$ asks which hyperbolic torus has the fewest simple closed geodesics in the asymptotic sense. For the modular torus X_{mod} , this counting problem is equivalent, after accounting for symmetries and converting traces to lengths, to the counting of Markoff triples studied by Zagier [Zag82]. McShane and Rivin conjectured that $\mathcal{A}$ is minimized at X_{mod} [MR95b, Conjecture 3.4]. In his lecture [Bon18], Bonahon presented numerical evidence for this conjecture, obtained in joint work with Enriquez using shear coordinates.

2020 *Mathematics Subject Classification*. Primary 30F60; Secondary 53C22, 52A10.

Keywords. Hyperbolic surfaces, modular torus, simple closed geodesics, stable norm, McShane–Rivin area conjecture, Farey polygons, Mirzakhani functions.

Mirzakhani extended this counting result to complete finite-area hyperbolic surfaces of genus g with n cusps and $3g - 3 + n > 0$: the growth exponent is $6g - 6 + 2n$, and the leading coefficient is a positive topological constant times the Mirzakhani function [Mir08a]. This function is the Thurston measure of the compactly supported measured laminations of length at most one. With the integral multicurve lattice normalization, it equals $\mathcal{A}(X)/2$ on the once-punctured torus, since each nonzero integral multicurve corresponds to two opposite homology classes [Mir08a]. Continuity and properness ensure that a minimum exists on each moduli space, but do not identify the minimizing surfaces [Mir08a, Proposition 3.2 and Theorem 3.3]. Arana-Herrera and Athreya refined the estimates near degeneration and proved square-integrability for Weil–Petersson measure [AHA20]. The Weil–Petersson integral of this function is also related to Masur–Veech volumes through work of Mirzakhani [Mir08b] and explicit formulas of Delecroix, Goujard, Zograf, and Zorich [DGZZ21].

The function also appears in other counting problems. For closed surfaces, Rafi and Souto give counting formulas for mapping class group orbits of filling geodesic currents and for mapping classes counted using the Thurston metric [RS19]. Aougab and Athreya obtain analogous formulas for harmonic representatives of a fixed weighted triangulation under the mapping class group, counted by weighted length [AA26, Theorem 1.2]. For two closed hyperbolic surfaces of the same genus $g \geq 2$, Lowe, Marques, and Neves showed that the leading coefficient for counting π_1 -injective minimal Lagrangian immersions of genus g into their product by area is proportional to the product of their Mirzakhani functions [LMN26, Corollary 1.5]. These connections motivate the broader question suggested in Parlier’s survey [Par14]: which surfaces minimize the Mirzakhani function for each topological type?

Main results

In this paper, we prove the McShane–Rivin conjecture, establish uniqueness, and give quantitative bounds for the increase away from the modular torus. The symmetry of the modular torus plays an important role in our proof and may also guide the search for minimizers in higher complexity.

Theorem 1.1. *Every complete finite-area hyperbolic once-punctured torus X satisfies*

$$\mathcal{A}(X_{\text{mod}}) \leq \mathcal{A}(X),$$

with equality exactly when $X = X_{\text{mod}}$. Consequently, if $X \neq X_{\text{mod}}$, then by (1)

$$N_{X_{\text{mod}}}(L) < N_X(L)$$

for all sufficiently large L , with the threshold depending on X .

The minimum of $\mathcal{A}$ has consequences for several counting problems. In each case below, the leading coefficient is a positive constant independent of X times $\mathcal{A}(X)$, and hence is uniquely minimized by the modular torus.

- Closed geodesics in any fixed mapping class group orbit of a nontrivial, nonperipheral closed curve, including curves with self-intersections [ES16, Mir16]. This also applies to primitive closed geodesics with at most k self-intersections, for fixed $k \geq 0$, since only finitely many orbits occur. See [ES22] for a detailed account.
- Essential cusp-to-cusp geodesic arcs, or nonempty finite collections of them, in a fixed mapping class group orbit [Bel23, Tri24]. Their total length is measured after truncating the ends at a horocycle of fixed sufficiently small length. Examples include ideal triangulations, counted by the sum of their three truncated edge lengths.
- Conjugacy classes of finitely generated, noncyclic, infinite-index subgroups of $\pi_1(S)$ in a fixed mapping class group orbit [Sas25]. The length assigned to a subgroup is half the total length of the geodesic boundary of its convex core in the corresponding covering surface.

The minimum also transfers to four-punctured spheres. A classical correspondence pairs once-punctured tori with four-punctured spheres, uniquely up to isometry (see [HT14]). Their simple closed geodesics correspond bijectively, with lengths doubled on the sphere, so

$$N_Y(L) = N_X(L/2).$$

The four-punctured sphere Y_* corresponding to X_{mod} therefore uniquely minimizes the Mirzakhani function among four-punctured spheres. Its leading simple-geodesic counting coefficient is one quarter of that of X_{mod} . Together with Mirzakhani’s growth theorem [Mir08a], this gives the following comparison across topological types.

Corollary 1.2. *For every complete finite-area hyperbolic surface Z without boundary, other than Y_* and the thrice-punctured sphere,*

$$N_{Y_*}(L) < N_Z(L)$$

for all sufficiently large L , with the threshold depending on Z .

Write $s_X = \text{sys}(X)$ for the length of a shortest essential simple closed geodesic on a hyperbolic surface X . Corollary 1.2 indicates that Y_* asymptotically has the fewest simple closed geodesics among complete finite-area hyperbolic surfaces without boundary, except for the thrice-punctured sphere, which has none. The boundary restriction is essential: one-holed tori can have $s_X \rightarrow \infty$ [Sch93], forcing $\mathcal{A}(X) \rightarrow 0$ by Minkowski's convex-body theorem [Cas59, Theorem II, p. 71].

Each torus admits a reduced trace triple (x, y, z) , formed from three simple closed geodesics that intersect pairwise once. The trace of a geodesic of length ℓ is $2 \cosh(\ell/2)$. The triple satisfies

$$x^2 + y^2 + z^2 = xyz, \quad 2 < x \leq y \leq z \leq \frac{xy}{2}.$$

These coordinates are recalled in Section 2 and illustrated in Figure 2. For a reduced triple, set

$$p_2 = \frac{1}{3} \left(\log^2 \frac{x}{y} + \log^2 \frac{y}{z} + \log^2 \frac{z}{x} \right). \quad (2)$$

This quantity measures the differences between the three trace logarithms and vanishes exactly at the modular triple $(3, 3, 3)$. Our next theorem gives matching upper and lower bounds for the area gap throughout moduli space and a sharp asymptotic as the systole tends to zero.

Theorem 1.3 (Uniform bounds and cusp growth). *There are universal constants $c, C > 0$ such that every X satisfies*

$$c \frac{p_2 e^{\sqrt{3p_2/2}}}{(1+p_2)^{3/2}} \leq \mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) \leq C \frac{p_2 e^{\sqrt{3p_2/2}}}{(1+p_2)^{3/2}}.$$

As $s_X \rightarrow 0$,

$$\mathcal{A}(X) = \frac{1}{s_X \log(2/s_X)} \left(1 - \frac{\pi^2}{12 \log^2(2/s_X)} + O(\log^{-3}(2/s_X)) \right). \quad (3)$$

The same expansion holds for $\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}})$.

For once-punctured tori, the estimates of Arana-Herrera and Athreya [AHA20, Theorem 1.5] bound $\mathcal{A}(X)$ above and below by positive constant multiples of $1/(s_X |\log s_X|)$ as $s_X \rightarrow 0$. Our expansion (3) determines the leading constant and the first correction term.

Proof strategy

We approximate the stable-norm unit ball by a finite Farey polygon. For each surface X , choose a reduced trace triple (x, y, z) and an ordered integral basis α, β realizing it, so that x, y, z are the traces of $\alpha, \beta, \alpha + \beta$. In each of the three sectors between consecutive classes

$$\alpha, \quad \alpha + \beta, \quad \beta, \quad -\alpha,$$

write $w = pu_0 + qv_0$, where u_0, v_0 are the sector endpoints and $p, q \geq 0$, and define the *sector height* by $h(w) = p + q$. Let $P_6(X)$ be the centrally symmetric inscribed polygon whose vertices are the normalized primitive classes $w/\|w\|_X$ of sector height at most six, together with their negatives (see Figure 1), and set

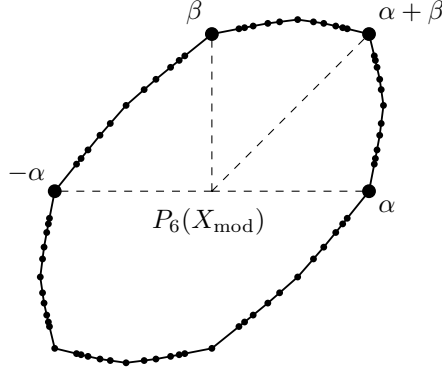
$$\mathcal{A}_6(X) = \text{Area}(P_6(X)).$$

The polygon P_6 is small enough for an explicit comparison and approximates the norm ball closely enough to control the omitted area

$$(\mathcal{A} - \mathcal{A}_6)(X) := \mathcal{A}(X) - \mathcal{A}_6(X).$$

Since $P_6(X) \subsetneq \mathcal{B}(X)$, the omitted area $(\mathcal{A} - \mathcal{A}_6)(X)$ is positive. We use the identity

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) = [\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}})] + [(\mathcal{A} - \mathcal{A}_6)(X) - (\mathcal{A} - \mathcal{A}_6)(X_{\text{mod}})], \quad (4)$$

Figure 1: The Farey polygon $P_6(X_{\text{mod}})$.

where the first bracket is the change in polygon area and the second is the change in omitted area. For nonmodular surfaces with $\text{sys}(X) \geq 7/5$, we show that the polygon area increases by more than any possible decrease in the omitted area. For $\text{sys}(X) < 7/5$, we estimate the full area directly.

Cyclic symmetry is central to the local comparison. The permutation $(x, y, z) \mapsto (y, z, x)$ fixes the modular point. For any twice continuously differentiable function invariant under this permutation, Lemma 2.6 gives, in local coordinates (x, y) ,

$$\nabla F(3, 3) = 0, \quad \text{Hess } F(3, 3) = \kappa \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Thus there is no linear term, and the quadratic term is determined by one number. This applies to both the polygon area $\mathcal{A}_6$ and the omitted area $\mathcal{A} - \mathcal{A}_6$.

For the polygon area, we compute a positive quadratic term and show that it dominates the higher terms throughout an explicit neighborhood. To control the uncomputed Taylor terms, we extend the polygon formula to complex parameter values and bound the average of its squared modulus on a circle. This controls all remaining Taylor terms at once. Proposition 3.10 gives

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > \frac{7}{80}p_2$$

for nonmodular X in that neighborhood. The same average bound, combined with further Taylor coefficients, controls the remainder farther from the modular point.

We combine these polygon estimates with bounds for the omitted area in three regions (Figure 2).

- Region A consists of tori represented by reduced triples with $\text{sys}(X) \geq 7/5$ outside $(x - 3)^2 + (y - 3)^2 \leq 1/1024$. Here the polygon increase exceeds the entire omitted area at the modular torus. Since $p_2 > 1/12800$ by (72), Corollary 4.5 and Lemma 3.4 give

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{19}{1000}p_2 - \frac{1}{700000} > 0.$$

- Region B consists of nonmodular tori represented by reduced triples with $\text{sys}(X) \geq 7/5$ inside that disk. The polygon increase tends to zero at the modular point, so we must control the change in omitted area. Its vanishing first derivatives and a lower bound for its second derivatives limit its possible decrease to less than $p_2/50$. Lemma 3.11 and Corollary 5.4 therefore give

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \left(\frac{7}{80} - \frac{1}{50} \right) p_2 > 0.$$

- In region C , where $\text{sys}(X) < 7/5$, curves winding around a shortest geodesic give a direct lower bound for the full area. Theorem 6.1 and Proposition 3.5 yield

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{1}{1000}.$$

These comparisons give the explicit regional bound and prove Theorem 1.1. For Theorem 1.3, the estimates in region B give the quadratic increase near the modular torus, while the curves used in region

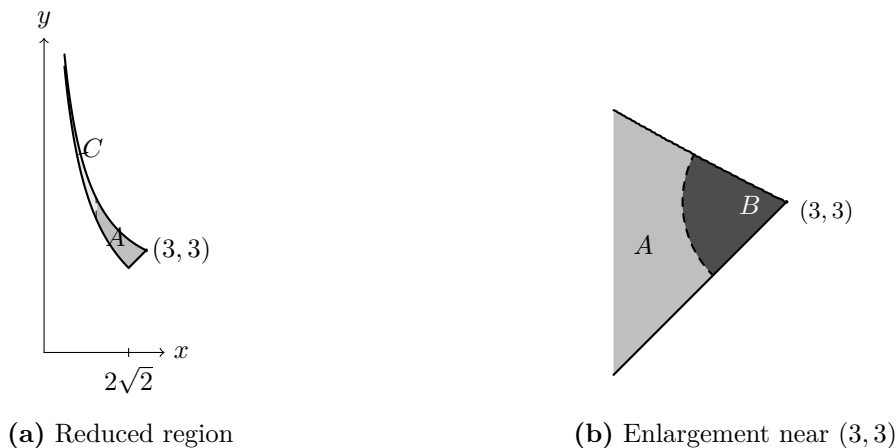


Figure 2: The three comparison regions in reduced trace coordinates. Dashed boundaries mark $\text{sys}(X) = 7/5$ in (a) and $(x - 3)^2 + (y - 3)^2 = 1/1024$ in (b).

C give the expansion as $s_X \rightarrow 0$. In each limit, the ratio of the area gap to $p_2 e^{\sqrt{3p_2/2}} / (1 + p_2)^{3/2}$ tends to a positive constant. After excluding a small neighborhood of the modular point and all sufficiently small systoles, the remaining reduced triples form a compact set. On this set, the ratio is continuous and positive by Theorem 1.1, so it has positive lower and finite upper bounds. Together, these bounds prove the global comparison.

Organization of the paper. Sections 2 and 3 develop the coordinates and estimates used in the three regional comparisons in Sections 4–6. Section 7 completes the proofs of both main theorems.

Acknowledgements. The ideas of subdividing the reduced region for different comparisons and estimating Taylor coefficients along two lines through the modular point (Subsection 3.4) were developed in part from discussions with Ser Peow Tan during the last-named author’s visit to the National University of Singapore. The last-named author thanks him for these helpful discussions and for encouraging us to pursue the conjecture. The background and some estimates were influenced by the recent work [DLN26] with Xiaobin Li and Van Nguyen. The second-named author is supported by NUS President Graduate Fellowship.

Declaration of AI use. ChatGPT (versions 5.4, 5.5, 5.6, and 6.0) was used extensively by the last-named author to assist with technical details, check calculations and arguments, and prepare TikZ figures. The authors developed the proof strategy, directed this work through questions and critical feedback, contributed ideas along the way to improve the results, carefully checked the resulting arguments and computations, and revised the exposition for clarity, simplicity, and accessibility to a broader readership. They take full responsibility for the content.

2 Preliminaries

This section introduces the notations, coordinates, area formulas, and differential tools used in the proof.

2.1 Trace coordinates and reduction

Oriented essential simple closed curves on the once-punctured torus S correspond to nonzero primitive classes in $H_1(S; \mathbb{Z})$. Two primitive classes u, v are Farey neighbors exactly when their algebraic intersection number has absolute value one. Fix a complete finite-area hyperbolic structure X on S and let $\|\cdot\|_X$ be the stable norm on $H_1(S; \mathbb{R})$. For a primitive class u , define its trace by

$$x_u = 2 \cosh \frac{\|u\|_X}{2}.$$

For any Farey neighbors u, v , the traces x_u, x_v, x_{u+v} satisfy the Markoff equation

$$x_u^2 + x_v^2 + x_{u+v}^2 = x_u x_v x_{u+v}, \quad (5)$$

and the two traces across the corresponding Farey edge satisfy

$$x_{u+v} + x_{u-v} = x_u x_v. \quad (6)$$

See [Bow98]. Thus crossing the edge replaces the Farey triangle $(u, v, u-v)$ by $(u, v, u+v)$ and induces the Vieta move

$$x_{u-v} \mapsto x_u x_v - x_{u-v} = x_{u+v}.$$

The positive solutions of the Markoff equation give the usual trace coordinates for marked complete hyperbolic punctured tori, see [Gol09, Theorem 4.1.2 and Section 4.4]. We call a solution $(x, y, z) \in (2, \infty)^3$ a *positive Markoff triple*. For $(x, y, z) = (x_u, x_v, x_{u+v})$, replacing the ordered basis (u, v) by $(v, -u-v)$ gives the three classes $v, -u-v, -u$. Since $x_{-w} = x_w$, this change induces the cyclic permutation

$$\rho(x, y, z) = (y, z, x). \quad (7)$$

For fixed x, y , solving the Markoff equation for z gives

$$z_{\pm}(x, y) = \frac{xy \pm \sqrt{\mathcal{D}(x, y)}}{2}, \quad \mathcal{D}(x, y) := x^2 y^2 - 4x^2 - 4y^2.$$

For any real Markoff triple (x, y, z) ,

$$\mathcal{D}(x, y) = x^2 y^2 - 4xyz + 4z^2 = (xy - 2z)^2 \geq 0.$$

Thus, on the open set

$$\mathcal{U} = \{(x, y) \in (2, \infty)^2 : \mathcal{D}(x, y) > 0\},$$

the two branches $z_{\pm}$ are distinct and real analytic. They coincide exactly on $\mathcal{D} = 0$, where

$$z_+(x, y) = z_-(x, y) = \frac{xy}{2}.$$

A positive Markoff triple (x, y, z) is *reduced* if

$$2 < x \leq y \leq z \leq \frac{xy}{2},$$

see the left panel of Figure 3. Thus, for a reduced triple,

$$z = z_-(x, y).$$

A Farey triangle is called a *sink* if none of its three adjacent Vieta moves decreases the trace it replaces.

Proposition 2.1 (Markoff reduction). *Every positive Markoff triple is equivalent under permutations and Vieta moves to a reduced triple. If (x, y, z) is a reduced triple representing X , then*

$$x = 2 \cosh \frac{\text{sys}(X)}{2}, \quad x \leq 3, \quad y \leq \frac{x}{\sqrt{x-2}}.$$

Proof. Associate X with its Markoff map on the Farey tree. The reduction argument [Bow98, Section 3] gives a sink triangle. Order its traces as $x \leq y \leq z$. Then $z \leq xy - z$, so the triple is reduced.

Conversely, every Vieta move from a reduced triple produces a trace at least as large as z . Continue along any path without reversing a move. At each later step $(U, V, W) \mapsto (U, V, UV - W)$, one of U, V is the trace produced at the previous step. Inductively, it is at least as large as W , so

$$UV - W \geq UV - \max(U, V) > \max(U, V),$$

since $U, V > 2$. Thus x is the smallest trace, so $x = 2 \cosh(\text{sys}(X)/2)$. Finally, (5) and reducedness give

$$x^2 + y^2 = z(xy - z) \geq y(xy - y) = (x - 1)y^2,$$

since $t(xy - t)$ is increasing on $(-\infty, xy/2]$ and $y \leq z \leq xy/2$. Therefore $y \leq x/\sqrt{x-2}$. Since $y \geq x$, it follows that $x \leq 3$. $\square$

For the local estimates near the modular torus, fix an ordered integral basis (α, β) such that the traces of $\alpha, \beta, \alpha + \beta$ at X_{mod} are all 3. As the hyperbolic structure varies, x, y, z denote the respective traces of these fixed classes. In particular, x always denotes the trace of α , even when it is not the smallest trace. These coordinates need not form a reduced triple, and each length derivative follows a fixed curve. We use the branch $z = z_-(x, y)$ on

$$\mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 : (x-3)^2 + (y-3)^2 \leq \frac{1}{1024}, x \leq 3, y \geq x \right\}. \quad (8)$$

The set $\mathcal{R}$ is convex, since it is the intersection of a closed disk and two closed half-planes (see

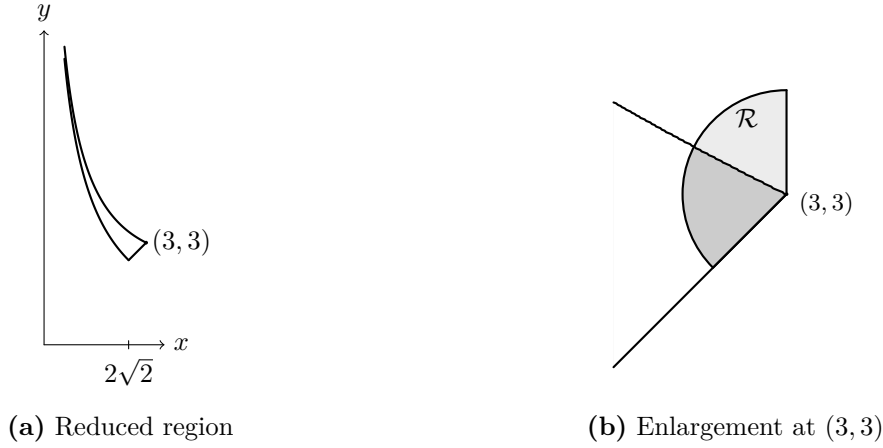


Figure 3: The reduced region in the (x, y) -plane and the convex set $\mathcal{R}$ near $(3, 3)$.

Figure 3(b)). It contains the comparison region B (see Figure 2(b)) and also points whose trace triples become reduced after reordering (Lemma 2.3). Working on $\mathcal{R}$ allows us to apply Taylor's formula along the entire segment from $(3, 3)$ to each point of region B .

Lemma 2.2 (Local discriminant bound). *For every $(x, y) \in \mathcal{R}$,*

$$\mathcal{D}(x, y) = x^2 y^2 - 4x^2 - 4y^2 > \frac{121}{16}.$$

Consequently, $z(x, y) := z_-(x, y)$ is real analytic on a neighborhood of $\mathcal{R}$.

Proof. The region lies in the square $[95/32, 97/32]^2$. Throughout this square,

$$0 < \partial_x \mathcal{D} = 2x(y^2 - 4) \leq 2 \frac{97}{32} \left(\left(\frac{97}{32} \right)^2 - 4 \right) < 32,$$

and the same bound holds for $\partial_y \mathcal{D}$. Since $\mathcal{D}(3, 3) = 9$, the mean value theorem and the disk condition give

$$\mathcal{D}(x, y) \geq 9 - 32(|x-3| + |y-3|) \geq 9 - \sqrt{2} > \frac{121}{16}.$$

The positive lower bound persists on a neighborhood of $\mathcal{R}$, where the formula for z_- is therefore real analytic. $\square$

By Lemma 2.2, the branch $z_-(x, y)$ is real analytic near $\mathcal{R}$. On a sufficiently small open neighborhood, $x, y, z_-(x, y) > 2$, so the trace parametrization defines a marked hyperbolic torus $X_{x,y}$ satisfying

$$x_\alpha(X_{x,y}) = x, \quad x_\beta(X_{x,y}) = y, \quad x_{\alpha+\beta}(X_{x,y}) = z_-(x, y). \quad (9)$$

Set

$$\mathcal{A}(x, y) := \mathcal{A}(X_{x,y}).$$

This function is real analytic on a neighborhood of $\mathcal{R}$ by [MR95b, Section 3, Corollary 1]. See also Bonahon's discussion of this property in [Bon18, slide 30].

We shall repeatedly use the following uniform bounds on $\mathcal{R}$.

Lemma 2.3 (Local trace and length bounds). *For $(x, y) \in \mathcal{R}$, let $z = z_-(x, y)$. Then*

$$\frac{74}{25} < x, y, z < \frac{61}{20},$$

and the corresponding geodesic lengths satisfy

$$2 \operatorname{arccosh} \frac{x}{2}, \quad 2 \operatorname{arccosh} \frac{y}{2}, \quad 2 \operatorname{arccosh} \frac{z}{2} > L_0 := \frac{236}{125}.$$

Moreover, the fixed Farey triangle determined by $\alpha, \beta, \alpha + \beta$ is the unique sink throughout $\mathcal{R}$, even when the ordered triple (x, y, z) is not reduced. Every move along a path away from this triangle produces the larger Vieta root. The three moves adjacent to it produce traces greater than $14279/2500$.

Proof. From the definition of $\mathcal{R}$, $95/32 \leq x, y \leq 97/32$, so only $z = z_-(x, y)$ needs to be bounded. Set

$$f_{x,y}(t) = t^2 - xyt + x^2 + y^2.$$

A direct estimate on $\mathcal{R}$ gives

$$f_{x,y}(74/25) \geq f_{3,97/32}(74/25) > 0, \quad f_{x,y}(61/20) < 0.$$

Since both test points lie to the left of $xy/2$, the lower root satisfies

$$\frac{74}{25} < z < \frac{61}{20}.$$

Hence all three corresponding geodesic lengths are greater than

$$2 \operatorname{arccosh} \frac{37}{25} > \frac{236}{125}.$$

The trace bounds also give

$$2 \max(x, y, z) < \frac{61}{10} < \min(xy, yz, zx).$$

Thus each of the three Vieta moves strictly increases the trace it replaces, so the fixed triangle is a strict sink. The growth argument in the proof of Proposition 2.1 shows that every subsequent move along a path away from this triangle also increases the replaced trace and selects the larger Vieta root. This excludes any other sink. The three adjacent moves produce traces

$$x_{2\alpha+\beta} = xz - y, \quad x_{\alpha+2\beta} = yz - x, \quad x_{\beta-\alpha} = xy - z.$$

Each is greater than

$$\left(\frac{74}{25}\right)^2 - \frac{61}{20} > \frac{61}{20}.$$

□

The next lemma shows that every surface with systole greater than $381/200$ has reduced coordinates in $\mathcal{R}$.

Lemma 2.4 (High systole localization). *If $\operatorname{sys}(X) > 381/200$, then a reduced representative of X belongs to $\mathcal{R}$.*

Proof. For a reduced triple, x is the systolic trace, so

$$x > 2 \cosh \frac{381}{400} > \frac{74}{25}.$$

Since $x \in [74/25, 3]$, a direct squaring gives

$$\frac{x}{\sqrt{x-2}} \leq 6 - x.$$

Together with Proposition 2.1, this yields $x \leq y \leq 6 - x$, and hence

$$(x-3)^2 + (y-3)^2 \leq 2(3-x)^2 < \frac{1}{1024}.$$

Reducedness also gives $z = z_-(x, y)$, so $(x, y) \in \mathcal{R}$. □

2.2 The exact Farey area expansion

To construct the Farey polygons for a surface X , choose a reduced trace triple (x, y, z) and an ordered integral basis (α, β) realizing it:

$$(x_\alpha, x_\beta, x_{\alpha+\beta}) = (x, y, z).$$

The Farey triangle $(\alpha, \beta, \alpha + \beta)$ is a sink by Proposition 2.1. We use it as the initial triangle; this choice may depend on X .

When differentiating in the local family (9), we keep the basis already fixed at X_{mod} . Lemma 2.3 ensures that its triangle remains a sink throughout $\mathcal{R}$, although its traces need not be ordered. Thus the curves and sector heights remain fixed as (x, y) varies. In either setting, a move away from the initial triangle is called an *outward Farey insertion*.

For each of the three initial sectors, let (u_0, v_0) denote its endpoints:

$$(u_0, v_0) \in \{(\alpha, \alpha + \beta), (\alpha + \beta, \beta), (\beta, -\alpha)\}.$$

Every primitive class in this sector has a unique expression $w = pu_0 + qv_0$ with nonnegative integers p, q . Define its *sector height* by

$$h(w) = p + q.$$

Starting from these three intervals, recursively insert $u + v$ between the endpoints of each current interval (u, v) , replacing it by $(u, u + v)$ and $(u + v, v)$. At each insertion, we call u, v the *parents* and $u + v$ their *child*. Let $\mathcal{F}$ be the set of all intervals occurring in this subdivision, including the initial ones. Heights are always measured relative to the fixed endpoints (u_0, v_0) of the initial sector containing the interval. Thus, for every $(u, v) \in \mathcal{F}$,

$$h(u + v) = h(u) + h(v).$$

Every primitive class in the interior of an initial sector is inserted exactly once.

For $K \geq 1$, let $P_K(X)$ be the centrally symmetric inscribed polygon whose vertices are the normalized primitive classes of sector height at most K , together with their negatives, and set

$$\mathcal{A}_K(X) = \text{Area}(P_K(X)).$$

For a reduced triple, this is the polygon used in the introduction (see Figure 1). In the local family, the sectors and heights are fixed; only the geodesic lengths used to normalize the vertices vary. On region B , the fixed traces form a reduced triple and the two constructions agree. The local construction also defines $\mathcal{A}_K$ at the nonreduced triples in $\mathcal{R}$ needed for differentiation.

For a primitive class u , write

$$L_u = \|u\|_X = 2 \operatorname{arccosh} \frac{x_u}{2}.$$

In the marked family (9), we write $L_w(x, y) = \|w\|_{X_{x,y}}$. If $u + v$ is inserted between Farey neighbors u, v , the edge joining u/L_u to v/L_v is replaced by the two edges through $(u + v)/L_{u+v}$ (see Figure 4).

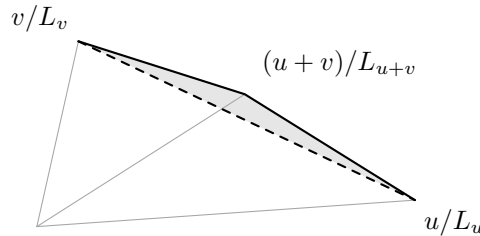


Figure 4: The area added by one Farey insertion on one side of the norm ball.

Since

$$|\det(u, v)| = |\det(u, u + v)| = |\det(u + v, v)| = 1,$$

the shaded triangle has area

$$\frac{1}{2} \left(\frac{1}{L_u L_{u+v}} + \frac{1}{L_{u+v} L_v} - \frac{1}{L_u L_v} \right).$$

Including its antipodal copy, the total area gain is

$$S_X(u, v) := \frac{1}{L_u L_{u+v}} + \frac{1}{L_{u+v} L_v} - \frac{1}{L_u L_v} = \frac{L_u + L_v - L_{u+v}}{L_u L_v L_{u+v}}. \quad (10)$$

This is positive by the strict triangle inequality $L_{u+v} < L_u + L_v$ [MR95b, Section 2]. We obtained the following expansion independently. O'Sullivan [O'S26, Proposition 6.5] generalizes Zagier's counting constant formula for Markoff triples [Zag82, equation (3)] to rising trees of $x^2 + y^2 + z^2 - xyz = D$, with $D \in \mathbb{R}$. His $D = 0$ sector formula gives our expansion after converting half-lengths to lengths and summing over the sectors.

Proposition 2.5 (Farey area expansion). *The polygons $P_K(X)$ are nested and*

$$\mathcal{A}_K(X) \nearrow \mathcal{A}(X).$$

Moreover,

$$\mathcal{A}_K(X) = \mathcal{A}_1(X) + \sum_{\substack{(u,v) \in \mathcal{F} \\ h(u+v) \leq K}} S_X(u, v),$$

and hence

$$\mathcal{A}(X) = \mathcal{A}_1(X) + \sum_{(u,v) \in \mathcal{F}} S_X(u, v).$$

Proof. Increasing K only adds boundary points of $\mathcal{B}(X)$, so the polygons are nested. Since primitive directions are dense and the norm is continuous, $P_K(X)$ converges to $\mathcal{B}(X)$ in the Hausdorff metric. Hence $\mathcal{A}_K(X)$ converges to $\mathcal{A}(X)$ by [Sch14, Theorem 1.8.20]. Summing the corrections of all insertions of height at most K gives the finite formula, and letting $K \rightarrow \infty$ gives the expansion of $\mathcal{A}$. $\square$

We record three facts that will be used repeatedly. First, Proposition 2.5 gives the tail identity

$$\mathcal{A}(X) - \mathcal{A}_K(X) = \sum_{\substack{(u,v) \in \mathcal{F} \\ h(u+v) > K}} S_X(u, v). \quad (11)$$

Second, at sector height $H \geq 2$, each initial sector contains exactly $\varphi(H)$ inserted classes, where $\varphi(H) = \#\{1 \leq p < H : \gcd(p, H) = 1\}$ is Euler's totient function. Indeed, these classes correspond to primitive pairs $p, q > 0$ with $p + q = H$. Hence

$$\#\{(u, v) \in \mathcal{F} : h(u + v) = H\} = 3\varphi(H) \leq 3(H - 1). \quad (12)$$

Third, for an initial trace triple (x, y, z) , write $\mathcal{A}_K(x, y, z)$ for the polygon area $\mathcal{A}_K(X)$ determined by that initial Farey triangle. Permuting its three vertices only relabels the three sectors and preserves sector height and lattice area. Hence, for every permutation σ of the three coordinates,

$$\mathcal{A}_K(\sigma(x, y, z)) = \mathcal{A}_K(x, y, z). \quad (13)$$

As for $\mathcal{A}$, we regard $\mathcal{A}_K$ as a function of the local coordinates (x, y) by writing

$$\mathcal{A}_K(x, y) := \mathcal{A}_K(X_{x,y}) = \mathcal{A}_K(x, y, z_-(x, y)).$$

In particular, $\mathcal{A}_K(3, 3) = \mathcal{A}_K(X_{\text{mod}})$.

We next express the added area in terms of the two parent lengths. For an outward insertion, the Markoff equation gives

$$t^2 - x_u x_v t + x_u^2 + x_v^2 = 0$$

for the two traces across the Farey edge. Whenever the inserted trace is the larger root, substituting $x_w = 2 \cosh(L_w/2)$ gives

$$\cosh \frac{L_{u+v}}{2} = \cosh \frac{L_u}{2} \cosh \frac{L_v}{2} + \sqrt{\sinh^2 \frac{L_u}{2} \sinh^2 \frac{L_v}{2} - 1}. \quad (14)$$

Lemma 2.3 ensures this root choice for every outward insertion in the marked family on $\mathcal{R}$.

The formula also defines L_{u+v} as a function of two independent lengths on

$$\mathcal{E} = \{(L_u, L_v) \in (0, \infty)^2 : \sinh(L_u/2) \sinh(L_v/2) \geq 1\}, \quad (15)$$

the domain where the square root is real (Figure 5). On this domain, define the *length deficit*

$$\delta(L_u, L_v) = L_u + L_v - L_{u+v} \quad (16)$$

and the added-area function

$$\mathcal{D}(L_u, L_v) = \frac{\delta(L_u, L_v)}{L_u L_v L_{u+v}}.$$

For an insertion using the larger root, (10) reads $S_X(u, v) = \mathcal{D}(L_u, L_v)$. This formulation will also be used outside $\mathcal{R}$, after checking the root choice.

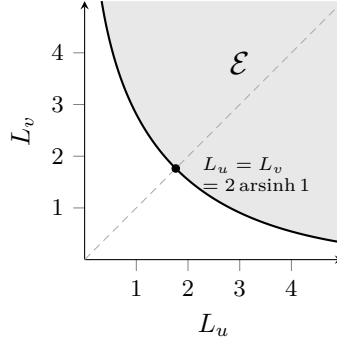


Figure 5: The region $\mathcal{E}$ and its boundary $L_v = 2 \log \coth(L_u/4)$.

2.3 Differential tools

We recall the gradient and Hessian and explain how cyclic symmetry constrains them at the modular point.

Gradient and Hessian. For a twice continuously differentiable function $G(x, y)$, recall that

$$\nabla G = \begin{pmatrix} G_x \\ G_y \end{pmatrix}, \quad \text{Hess } G = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{pmatrix}.$$

For a direction $v \in \mathbb{R}^2$, the directional derivative is

$$D_v G = \nabla G \cdot v.$$

Thus $\nabla G = 0$ means that G has zero first-order slope in every direction. The Hessian describes the quadratic variation:

$$G(q + v) = G(q) + \nabla G(q) \cdot v + \frac{1}{2} v^\top \text{Hess } G(q) v + o(|v|^2).$$

Let

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = xyz\}$$

be the Markoff surface, and let $p = (3, 3, 3)$ be the modular point (see Figure 6). The following lemma is an elementary consequence of cyclic symmetry (see [Pal79]).

Lemma 2.6 (Order-three symmetry at the modular torus). *Suppose $\tilde{F}$ is twice continuously differentiable on Σ near p and invariant under cyclic permutation of the coordinates. On the branch $z = z_-(x, y)$, define*

$$F(x, y) = \tilde{F}(x, y, z_-(x, y)).$$

Then

$$\nabla F(3, 3) = 0, \quad \text{Hess } F(3, 3) = \kappa \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

for some real κ .

Proof. Since $\nabla(x^2 + y^2 + z^2 - xyz)(p) = (-3, -3, -3)$, the tangent plane is $T_p\Sigma = \{a + b + c = 0\}$. Translating it to p gives the affine tangent plane $p + T_p\Sigma$ as shown in Figure 6.

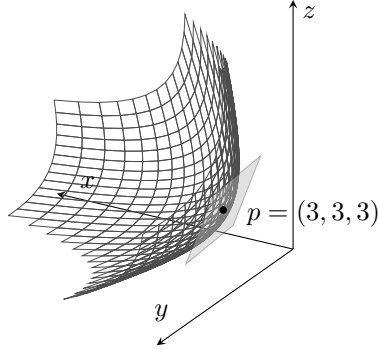


Figure 6: The Markoff surface near $p = (3, 3, 3)$ and its affine tangent plane.

By (7), ρ fixes $p = (3, 3, 3)$ and acts on $T_p\Sigma$ as a rotation through 120° . Hence the three vectors $\xi, \rho\xi, \rho^2\xi$ sum to zero for any $\xi \in T_p\Sigma$. Cyclic invariance makes the directional derivatives along these vectors equal, so linearity gives

$$3D_\xi \tilde{F}(p) = D_{\xi + \rho\xi + \rho^2\xi} \tilde{F}(p) = 0.$$

Thus every tangential derivative vanishes, and the chain rule yields $\nabla F(3, 3) = 0$.

For the second derivatives, let γ be a twice continuously differentiable curve in Σ with $\gamma(0) = p$ and $\gamma'(0) = \xi = (a, b, -a - b)$. Since $\nabla F(3, 3) = 0$, the acceleration terms vanish in the chain rule, giving

$$\left. \frac{d^2}{dt^2} \tilde{F}(\gamma(t)) \right|_{t=0} = F_{xx}(3, 3)a^2 + 2F_{xy}(3, 3)ab + F_{yy}(3, 3)b^2.$$

This depends only on ξ , so it defines a quadratic form Q on $T_p\Sigma$. Cyclic invariance gives $Q(\rho\xi) = Q(\xi)$. If Q is not constant on the Euclidean unit circle in $T_p\Sigma$, its maximum is attained at exactly two opposite points. A 120° rotation cannot preserve this pair, so cyclic invariance forces Q to be constant on the unit circle. By homogeneity,

$$Q(\xi) = \kappa|\xi|^2 = \kappa(a^2 + b^2 + (a + b)^2).$$

Comparing coefficients yields the formula for $\text{Hess } F(3, 3)$ as desired. $\square$

3 Estimates shared by the regional comparisons

To apply (4), we bound individual insertions, give the polygon's area formula, and estimate the modular area and its omitted terms. We then compute the Taylor coefficients, control the remainder, and obtain polygon bounds near the modular point.

Derivative notation. For functions of the independent lengths L_u, L_v , we use $\partial_1 = \partial/\partial L_u$ and $\partial_2 = \partial/\partial L_v$, and write $f_i = \partial_i f$, $f_{ij} = \partial_i \partial_j f$ for $i, j \in \{1, 2\}$. Derivatives in the trace coordinates are indicated by x, y . In $\partial_x^k \partial_y^l$, the nonnegative integers k, l specify the respective derivative orders. These conventions also apply in later sections. Indices on sequences and coefficients are defined where they appear.

3.1 Length and added-area estimates

We estimate the deficit δ and added area $\mathcal{D}$ defined in Subsection 2.2. First, their monotonicity and convexity will control changes in the omitted area in region B and added triangles in region C . We then bound the deficit and its first two length derivatives, and deduce a lower length bound in terms of sector height. Related exponential estimates appear in [DLN26]; here we need explicit constants and derivative bounds. Derivatives with respect to x, y are treated in Subsection 5.1 for the comparison in region B .

Proposition 3.1 (Properties of the deficit and added area). *The set $\mathcal{E}$ defined in (15) is convex. On $\mathcal{E}$, the functions δ and $\mathcal{D}$ are positive, symmetric, strictly decreasing in each length, and have convex logarithms. In particular, $\text{Hess}_{L_u, L_v} \mathcal{D} \geq 0$ in the interior, and*

$$\mathcal{D}(L_u, L_v) > \frac{8e^{-L_u-L_v}}{L_u L_v (L_u + L_v)}. \quad (17)$$

Proof. On $(0, \infty)^2$, the function $\log \sinh(L_u/2) + \log \sinh(L_v/2)$ is concave. Its value is nonnegative exactly on $\mathcal{E}$, so $\mathcal{E}$ is convex. Work first in the interior. Set $q = e^{-L_u}$, $p = e^{-L_v}$, and put

$$a = (1-q)(1-p), b = 2\sqrt{qp} \left(\cosh \frac{L_u + L_v}{2} - \cosh \frac{L_{u+v}}{2} \right).$$

The addition formula and (14) give

$$b = \frac{a - \sqrt{a^2 - 16qp}}{2}, e^{-\delta/2} + qpe^{\delta/2} = 1 + qp - b. \quad (18)$$

The definition of b and monotonicity of $\cosh$ give $\delta > 0$. Write $\psi = 1 - e^{-\delta/2}$, so $0 < \psi < 1$ and $\delta = -2\log(1 - \psi)$. The two relations give

$$b = \frac{4qp + b^2}{a}, \psi = b + \frac{qp\psi}{1 - \psi}. \quad (19)$$

The chosen roots give the positive factors

$$a - 2b = \sqrt{a^2 - 16qp} > 0, 1 - \frac{qp}{(1 - \psi)^2} = 1 - e^{-L_{u+v}} > 0. \quad (20)$$

Since $a_i > 0$ and $(qp)_i = -qp$, implicit differentiation gives, for $i \in \{1, 2\}$,

$$\begin{aligned} (a - 2b)b_i &= -4qp - a_i b < 0, \\ \left(1 - \frac{qp}{(1 - \psi)^2}\right) \psi_i &= b_i - \frac{qp\psi}{1 - \psi} < 0. \end{aligned} \quad (21)$$

Thus b , ψ , and $\delta = -2\log(1 - \psi)$ strictly decrease in each length.

For convexity of the logarithms, differentiate along any straight line in the length variables. Primes denote derivatives along this line. For positive functions f, g , direct differentiation gives

$$(\log(f + g))'' = \frac{f(\log f)'' + g(\log g)''}{f + g} + \frac{fg}{(f + g)^2} ((\log f)' - (\log g'))^2. \quad (22)$$

Here $\log(qp) = -L_u - L_v$ is affine, and $-\log a$ is convex because $(-\log(1 - e^{-L}))'' = e^{-L}/(1 - e^{-L})^2 > 0$. Applying (22) to the first equation in (19) gives

$$\left(1 - \frac{2b}{a}\right) (\log b)'' \geq (-\log a)'' \geq 0.$$

For the equation defining ψ , use

$$\left(\log \frac{\psi}{1 - \psi}\right)'' = \frac{(\log \psi)''}{1 - \psi} + \frac{\psi((\log \psi)')^2}{(1 - \psi)^2}.$$

The same identity, after collecting $(\log \psi)''$ on the left, gives

$$\left(1 - \frac{qp}{(1 - \psi)^2}\right) (\log \psi)'' \geq \frac{b}{\psi} (\log b)'' \geq 0.$$

The positive coefficients in (20) show that $\log b$ and $\log \psi$ are convex. Since $\delta = -2\log(1 - \psi) > 2\psi$, we also have

$$(\log \delta)'' = \frac{2\psi}{(1 - \psi)\delta} \left((\log \psi)'' + \frac{\delta - 2\psi}{(1 - \psi)\delta} ((\log \psi)')^2 \right) \geq 0. \quad (23)$$

Since δ is convex and decreasing, $L_{u+v} = L_u + L_v - \delta$ is positive, concave, and increasing. Consequently every term in

$$\log \mathcal{D} = \log \delta - \log L_u - \log L_v - \log L_{u+v} \quad (24)$$

is convex. Differentiating $\mathcal{D} = e^{\log \mathcal{D}}$ twice in any direction $\xi \in \mathbb{R}^2$ gives

$$\xi^\top (\text{Hess } \mathcal{D}) \xi = \mathcal{D} (\xi^\top (\text{Hess } \log \mathcal{D}) \xi + (\xi^\top \nabla \log \mathcal{D})^2) \geq 0.$$

Here all derivatives are in L_u, L_v ; the first term in parentheses is nonnegative by convexity of $\log \mathcal{D}$, and the second is a square. Hence $\text{Hess}_{L_u, L_v} \mathcal{D} \geq 0$. The numerator of $\mathcal{D}$ strictly decreases and its denominator increases in each length, so $\mathcal{D}$ strictly decreases as well. Symmetry is immediate. Continuity extends convexity and monotonicity to the boundary; strict decrease there follows by comparing through an interior point.

Finally, $b(a - b) = 4qp$ and $0 < a - b < 1$ give $b > 4qp$. The equation for ψ gives $\psi > b$, hence $\delta > 2\psi > 2b > 8qp$, also on the boundary. Together with $L_{u+v} < L_u + L_v$, this proves (17). $\square$

Proposition 3.2 (Bounds for the length deficit). *Suppose $L_u, L_v \geq L_0 = 236/125$ and $\max\{L_u, L_v\} \geq 84/25$. Then*

$$\begin{aligned} 0 < \delta < 11e^{-L_u - L_v}, \\ 0 < -\delta_1, -\delta_2 < 20e^{-L_u - L_v}, \\ 0 < \delta_{11}, \delta_{12}, \delta_{22} < 40e^{-L_u - L_v}. \end{aligned}$$

Proof. We bound the denominators in the implicit equations, then use their first two derivatives. The degree-six Taylor polynomials for the exponential give

$$e^{-L_0} < \frac{19}{125}, e^{-42/25} < \frac{3}{16}. \quad (25)$$

By symmetry, assume $L_v \geq 84/25$. Then $q < 19/125$ and $p < (3/16)^2 < 1/25$. Throughout this rectangle, $a = (1 - q)(1 - p) > 2544/3125$ and $qp < 1/160$, so

$$a^2 - 16qp > \frac{9}{16}, a - 2b > \frac{3}{4}.$$

The formula $b = 8qp/(a + \sqrt{a^2 - 16qp})$, together with $q < 19/125$ and $p < 1/25$, gives $b < 1/32$ and $b < (21/4)qp$. The selected root of $\psi^2 - (1 - qp + b)\psi + b = 0$ is the smaller one by (20). This polynomial is positive at zero and negative at $1/31$, since

$$\frac{1}{31^2} - \frac{159}{160 \cdot 31} + \frac{30}{32 \cdot 31} < 0.$$

Its smaller root thus satisfies

$$0 < \psi < \frac{1}{31}, 1 - \frac{qp}{(1 - \psi)^2} > \frac{99}{100}. \quad (26)$$

The equation $\psi(1 - qp/(1 - \psi)) = b$ now gives $\psi < (100/99)b$. Consequently

$$\delta < \frac{2\psi}{1 - \psi} < \frac{2 \cdot 31}{30} \frac{100}{99} \frac{21}{4} qp < 11qp.$$

For $i, j \in \{1, 2\}$, we have $0 < a_i < 1/6$ and $|a_{ij}| < 1/6$. Equation (21) gives $|b_i| < 7qp$. Differentiating once more gives

$$(a - 2b)b_{ij} = 2b_i b_j - a_i b_j - a_j b_i - a_{ij} b + 4qp,$$

so $0 < b_{ij} < 11qp$. For positivity, use $a_{ii} < 0$ and $a_{12} = qp$ with $b < 1$; for the upper bound, substitute $|b_i| < 7qp$, $b < (21/4)qp$, and $qp < 1/160$. Similarly, the equation for ψ gives

$$\left(1 - \frac{qp}{(1 - \psi)^2}\right) \psi_{ij} = b_{ij} + \frac{qp\psi}{1 - \psi} - \frac{qp(\psi_i + \psi_j)}{(1 - \psi)^2} + \frac{2qp\psi_i\psi_j}{(1 - \psi)^3}. \quad (27)$$

Using (21) first, and then this formula, gives $|\psi_i| < 8qp$ and $0 < \psi_{ij} < 12qp$. Finally,

$$\delta_i = \frac{2\psi_i}{1 - \psi}, \delta_{ij} = \frac{2\psi_{ij}}{1 - \psi} + \frac{2\psi_i\psi_j}{(1 - \psi)^2}$$

give the stated bounds and signs. $\square$

We now deduce a lower length bound in terms of sector height, using the Farey induction of [Bow98, Lemma 2.1.1(2)]. Since both parent lengths are positive, (14) gives $L_{u+v} > \max(L_u, L_v)$ and hence $x_{u+v} > \max(x_u, x_v)$. Thus the first-insertion bound in Lemma 2.3 persists under further insertions: throughout $\mathcal{R}$, every primitive class w of sector height at least two satisfies

$$x_w > \frac{14279}{2500}. \quad (28)$$

Lemma 3.3 (A length lower bound in terms of sector height). *On $\mathcal{R}$, every primitive class in an initial sector satisfies*

$$L_w > \frac{42}{25}h(w) + \frac{3}{50}.$$

Proof. For height one, $L_w > L_0 > 42/25 + 3/50$. For every height at least two, (28) and $2 \cosh(171/100) < 14279/2500$ give $L_w > 171/50$. The latter comparison follows from (99) with $N = 6$, and proves the claimed bound at height two.

In every later insertion, one parent has height at least two, so its length exceeds $171/50 > 84/25$; the other length exceeds L_0 . Proposition 3.2 and (25) therefore give

$$0 < \delta < 11e^{-L_u-L_v} < 11 \frac{19}{125} \left(\frac{3}{16} \right)^2 < \frac{3}{50}. \quad (29)$$

Thus the constant $3/50$ compensates for the loss at each insertion: if the bound holds for both parents, then

$$L_{u+v} = L_u + L_v - \delta > \frac{42}{25}(h(u) + h(v)) + \frac{3}{50}.$$

Induction proves the result. $\square$

3.2 The finite polygon and its area formula

We give an explicit area formula for the polygon P_6 used in regions A and B . A trace $T > 2$ determines the geodesic length

$$L(T) = 2 \log \frac{T + \sqrt{T^2 - 4}}{2}.$$

Write $f(T) = 1/L(T)$. By Subsection 2.2, each consecutive vertex pair and its antipodal pair contribute $f(T)f(T')$. Set

$$T_n = x_{(n+1)\alpha+\beta}.$$

These are the traces obtained by successive Dehn twists of β about α ; we call this sequence a *fan*. Since $x = x_\alpha$, $y = x_\beta$, and $z = x_{\alpha+\beta}$, the Vieta relation gives

$$T_{-1} = y, T_0 = z, T_{n+1} = xT_n - T_{n-1} \text{ for } n \in \mathbb{Z}. \quad (30)$$

For P_6 , take $T_1, \dots, T_5$, of sector heights $2, \dots, 6$, and insert $5\alpha + 2\beta$ between $2\alpha + \beta$ and $3\alpha + \beta$. Its trace is $T_* = T_1T_2 - x$, and its sector height is five since

$$5\alpha + 2\beta = 3\alpha + 2(\alpha + \beta).$$

The boundary order is

$$x, T_5, T_4, T_3, T_2, T_*, T_1.$$

The six permutations of (x, y, z) give the two sides of the fans about the three initial curves, together with these six additional insertions (Figure 7). Thus

$$\mathcal{A}_6(X) = \sum_{\sigma \in \mathfrak{S}_3} \left(f(x)f(T_5) + f(T_*)(f(T_1) + f(T_2)) + \sum_{n=2}^4 f(T_n)f(T_{n+1}) \right) \circ \sigma. \quad (31)$$

At the modular point,

$$(x, T_1, T_2, T_3, T_4, T_5, T_*) = (3, 6, 15, 39, 102, 267, 87).$$

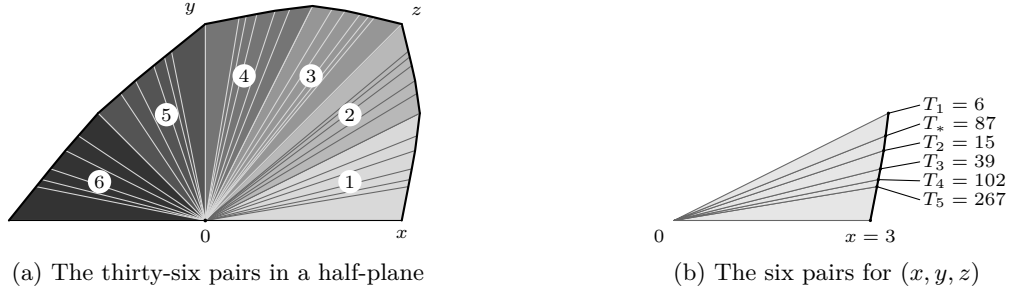


Figure 7: The upper half of $P_6(X_{\text{mod}})$ in the basis (α, β) of Figure 1. Each numbered group contains six consecutive pairs, with permutations (x, y, z) , (z, y, x) , (z, x, y) , (y, x, z) , (y, z, x) , and (x, z, y) for groups 1–6, respectively.

3.3 The modular area and its omitted terms

The estimates below are inspired by the exponential normal-turn bounds in [DLN26]: in both settings, the Farey recurrence yields exponentially small contributions at large heights. Here we bound added areas and compute the required constants separately. We bound the omitted area at the modular torus by $1/700000$ for region A , then obtain the upper bound for $\mathcal{A}(X_{\text{mod}})$ used in region C .

Lemma 3.4. *One has $0 < \mathcal{A}(X_{\text{mod}}) - \mathcal{A}_6(X_{\text{mod}}) < 1/700000$.*

Proof. We compute the insertions of heights seven and eight explicitly, then bound the remaining contributions by a geometric series.

Let u_0, v_0 be the endpoints of an initial sector and write $w = pu_0 + qv_0$, so $h(w) = p + q$. The symmetry exchanging u_0, v_0 gives $\|pu_0 + qv_0\|_{X_{\text{mod}}} = \|qu_0 + pv_0\|_{X_{\text{mod}}} = L_w$. The triangle inequality therefore yields

$$(p + q)\|u_0 + v_0\|_{X_{\text{mod}}} \leq \|pu_0 + qv_0\|_{X_{\text{mod}}} + \|qu_0 + pv_0\|_{X_{\text{mod}}} = 2L_w.$$

Since $u_0 + v_0$ has trace 6, its length is $\|u_0 + v_0\|_{X_{\text{mod}}} = 2 \operatorname{arccosh} 3$. Hence

$$L_w \geq h(w) \operatorname{arccosh} 3 > \frac{7}{4} h(w), \quad e^{-\operatorname{arccosh} 3} = 3 - 2\sqrt{2} < \frac{43}{250}. \quad (32)$$

The Farey recurrence gives five insertion types at heights seven and eight, each occurring six times:

H	parent traces	child trace	$10^9 S_{X_{\text{mod}}}(u, v) <$
7	(3, 267)	699	69
7	(15, 39)	582	48
7	(6, 87)	507	82
8	(3, 699)	1830	8
8	(15, 87)	1299	7

(33)

Appendix A3.1 verifies the bounds of $S_{X_{\text{mod}}}(u, v)$ in the table. For any parent pair (u, v) with total height $H = h(u) + h(v) \geq 9$, Proposition 3.2 and (32) give

$$S_{X_{\text{mod}}}(u, v) = \frac{\delta(L_u, L_v)}{L_u L_v L_{u+v}} < \frac{11(43/250)^H}{(7/4)^3 h(u) h(v) H} \leq \frac{11(43/250)^H}{(7/4)^3 (H-1) H}.$$

Since there are at most $3(H-1)$ insertions at height H (see (12)),

$$\mathcal{A}(X_{\text{mod}}) - \mathcal{A}_8(X_{\text{mod}}) < \frac{33}{(7/4)^3} \sum_{H \geq 9} \frac{(43/250)^H}{H} \leq \frac{33(43/250)^9}{9(7/4)^3 (1 - 43/250)} < \frac{11}{10^8}.$$

Adding the explicitly computed insertions in (33) gives

$$\mathcal{A}(X_{\text{mod}}) - \mathcal{A}_6(X_{\text{mod}}) < \frac{6(69 + 48 + 82 + 8 + 7)}{10^9} + \frac{11}{10^8} < \frac{1}{700000}.$$

□

Proposition 3.5 (An elementary modular upper bound). *One has $\mathcal{A}(X_{\text{mod}}) < 223/250$.*

Proof. Evaluating (31) at the modular torus and bounding the logarithms by (101) gives

$$\mathcal{A}_6(X_{\text{mod}}) < \frac{89183}{100000}.$$

Lemma 3.4 therefore gives

$$\mathcal{A}(X_{\text{mod}}) < \frac{89183}{100000} + \frac{1}{700000} < \frac{223}{250}.$$

□

3.4 Taylor expansion of the polygon area

We now study how $\mathcal{A}_6$ changes near the modular point. Lemma 2.6 determines the form of its Hessian there. To organize the higher-order Taylor terms and estimate the remainder, we introduce centered logarithmic coordinates:

$$(u, v, w) := (\log x, \log y, \log z) - \frac{\log(xyz)}{3}(1, 1, 1).$$

Then $u + v + w = 0$, and the modular triple $(3, 3, 3)$ corresponds to $(u, v, w) = (0, 0, 0)$. These coordinates are linearly related to the shear coordinates discussed in Bonahon's lecture [Bon18]. We retain (u, v, w) because permutations of the traces simply permute these coordinates. Consequently, each homogeneous Taylor term of the polygon area $\mathcal{A}_6$ is symmetric in u, v, w , simplifying the coefficient and remainder estimates below. The Markoff equation gives

$$(x, y, z) = (e^{2u} + e^{2v} + e^{2w})(e^u, e^v, e^w). \quad (34)$$

Conversely, this parametrization satisfies the Markoff equation whenever $u + v + w = 0$. The quantity p_2 introduced in (2) becomes

$$p_2 = \frac{1}{3}((u - v)^2 + (v - w)^2 + (w - u)^2) = u^2 + v^2 + w^2.$$

In these coordinates, we expand $\mathcal{A}_6$ at $u = v = w = 0$. Symmetry in u, v, w makes each homogeneous Taylor term a symmetric polynomial, hence by the fundamental theorem of symmetric polynomials [Mac95, p. 21, Remark following (2.4)], each homogeneous Taylor term is a polynomial in $u + v + w$, $uv + vw + wu$, and uvw . Put $p_3 = u^3 + v^3 + w^3$. Since $u + v + w = 0$, we have $uv + vw + wu = -p_2/2$ and $uvw = p_3/3$. Thus, the Taylor expansion through degree eight has the form

$$\begin{aligned} \mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) &= c_2 p_2 + c_3 p_3 + c_4 p_2^2 + c_5 p_2 p_3 + c_{6'} p_2^3 + c_{6''} p_3^2 \\ &\quad + c_7 p_2^2 p_3 + c_{8'} p_2^4 + c_{8''} p_2 p_3^2 + R_9, \end{aligned} \quad (35)$$

where R_9 contains the terms of degree at least nine. We determine the coefficients by restricting the area formula to two lines through the modular point. On $(u, v, w) = (t, 0, -t)$, we have $p_2 = 2t^2$ and $p_3 = 0$, so the Taylor polynomial through degree eight of (35) becomes

$$2c_2 t^2 + 4c_4 t^4 + 8c_{6'} t^6 + 16c_{8'} t^8.$$

Comparing coefficients on the first line determines $c_2, c_4, c_{6'}, c_{8'}$. On $(u, v, w) = (t, t, -2t)$, we have $p_2 = 6t^2$ and $p_3 = -6t^3$, so the Taylor polynomial through degree eight becomes

$$6c_2 t^2 - 6c_3 t^3 + 36c_4 t^4 - 36c_5 t^5 + (216c_{6'} + 36c_{6''})t^6 - 216c_7 t^7 + (1296c_{8'} + 216c_{8''})t^8.$$

The coefficients of t^3, t^5, t^7 determine c_3, c_5, c_7 . Subtracting the contributions of $c_{6'}$ and $c_{8'}$, determined on the first line, from the coefficients of t^6, t^8 then determines $c_{6''}, c_{8''}$.

For each substitution, we expand the traces using (30), then expand their reciprocal lengths and sum the products in (31). Lemma A2.1 in Appendix A2 evaluates the quadratic term as a sum of six explicitly bounded contributions and proves

$$c_2 > \frac{23}{200}. \quad (36)$$

The remaining coefficient bounds are recorded in Lemma A2.2. The coefficient c_2 identifies the scalar factor in the Hessian formula from Lemma 2.6 and gives the following lower bound in the trace coordinates (x, y) .

Proposition 3.6. *At the modular point,*

$$\text{Hess } \mathcal{A}_6(3, 3) = \frac{2c_2}{9} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} > \frac{23}{900} I,$$

where I is the 2×2 identity matrix. The inequality means that the second derivative in every unit direction is greater than $23/900$.

Proof. By Lemma 2.6,

$$\nabla \mathcal{A}_6(3, 3) = 0, \quad \text{Hess } \mathcal{A}_6(3, 3) = \kappa \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $(x(t), y(t), z(t))$ be the trace curve corresponding to $(u, v, w) = (t, 0, -t)$ under (34). Then $(x'(0), y'(0)) = (3, 0)$, and $\mathcal{A}_6(x(t), y(t))$ has quadratic Taylor coefficient $2c_2$. Since the gradient vanishes, the chain rule gives

$$4c_2 = \frac{d^2}{dt^2} \mathcal{A}_6(x(t), y(t)) \Big|_{t=0} = (3, 0) \text{Hess } \mathcal{A}_6(3, 3) \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 18\kappa.$$

Thus $\kappa = 2c_2/9$. For every nonzero vector (a, b) , (36) gives

$$(a, b) \text{Hess } \mathcal{A}_6(3, 3) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{2c_2}{9} (a^2 + b^2 + (a+b)^2) > \frac{23}{900} (a^2 + b^2).$$

This is precisely the stated matrix inequality. $\square$

3.5 Complex extension and Taylor remainder bounds

To control the uncomputed Taylor terms of $\mathcal{A}_6$, we extend (31) to complex values of t along each line through the modular point in logarithmic coordinates. We first bound the lengths away from zero, then bound the average of $|\mathcal{A}_6|^2$ on the circle $|t| = 3/8$. Proposition 3.9 turns this average bound into a Taylor remainder estimate. We apply it with $N = 9$ in Proposition 3.10, and with $N = 20$, together with explicit bounds for degrees nine through nineteen, in Theorem 4.2 for region A .

Lemma 3.7 (Two elementary analytic estimates). *One has*

$$\text{Re} \log \cosh z \geq \frac{\text{Re}(z^2)}{2} - \frac{|z|^4}{10} \text{ for } |z| \leq 3/4, \quad (37)$$

where the logarithm is zero at the origin. Also

$$\text{Re } e^z > \frac{5}{6} - \frac{|z|}{2} \text{ for } |z| \leq 9/8. \quad (38)$$

Proof. Write $z = x + iy$. For the first estimate, the identity $\cosh z = \cosh x \cos y + i \sinh x \sin y$ gives

$$|\cosh z|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y \geq \cosh^2 x \cos^2 y.$$

Since $|y| \leq |z| \leq 3/4 < \pi/2$, we have $\cos y > 0$. Taking square roots and logarithms therefore yields

$$\text{Re} \log \cosh z = \log |\cosh z| \geq \log \cosh x + \log \cos y. \quad (39)$$

It thus suffices to bound these two real functions. For $s \geq 0$, $0 \leq \tanh s \leq s$ gives $(\tanh s)' = 1 - \tanh^2 s \geq 1 - s^2$. Integrating twice from zero and using evenness gives, for every $t \in \mathbb{R}$,

$$\log \cosh t \geq \frac{t^2}{2} - \frac{t^4}{12} \geq \frac{t^2}{2} - \frac{t^4}{10}. \quad (40)$$

Direct differentiation shows that $(-\log \cos t - t^2/2)/t^4$ is increasing on $(0, \pi/2)$. At $t = 3/4$, this quotient is less than $1/10$. Monotonicity and evenness therefore give

$$\log \cos t \geq -t^2/2 - t^4/10, \text{ for } |t| \leq 3/4. \quad (41)$$

Combining (39), (40), and (41) gives, for $|z| \leq 3/4$,

$$\operatorname{Re} \log \cosh z \geq \frac{x^2 - y^2}{2} - \frac{x^4 + y^4}{10} \geq \frac{\operatorname{Re}(z^2)}{2} - \frac{|z|^4}{10},$$

which proves (37).

For the second estimate, apply $e^u \geq 1 + u$ with $u = x + \log 2$ gives

$$e^x \geq \frac{1 + x + \log 2}{2} > \frac{x + 5/3}{2}.$$

Also,

$$(|z| - x)(x + 5/3) \leq \frac{((|z| - x) + (x + 5/3))^2}{4} < 2.$$

Multiplying by $|z| + x \geq 0$ and using $y^2 = (|z| - x)(|z| + x)$ gives $y^2(x + 5/3) \leq 2(|z| + x)$. Thus, since $\cos y \geq 1 - y^2/2 > 0$,

$$\operatorname{Re} e^z = e^x \cos y > \frac{x + 5/3}{2} - \frac{y^2(x + 5/3)}{4} \geq \frac{x + 5/3}{2} - \frac{|z| + x}{2} = \frac{5}{6} - \frac{|z|}{2}.$$

□

Proposition 3.8 (Complex extension of the polygon area). *Fix $a, b, c \in \mathbb{R}$ satisfy $a + b + c = 0$ and $\max\{|a|, |b|, |c|\} = 1$. For $(u, v, w) = t(a, b, c)$, define (x, y, z) by (34). Then $F(t) = \mathcal{A}_6(x(t), y(t), z(t))$, initially defined for real t near zero, extends analytically to an open disk containing $|t| \leq 3/8$. On $|t| \leq 3/8$, the lengths in the polygon formula satisfy $\operatorname{Re} L_w > l_{h(w)}$, with the bounds in (50).*

Proof. The area formula (31) is a finite sum of terms $1/(L_u L_v)$, so we need lower bounds for $|L_u|$ and $|L_v|$. Throughout, assume $|t| \leq 3/8$ and put

$$p = |\operatorname{Re} t|, \quad q = |\operatorname{Im} t|, \quad p^2 + q^2 \leq 9/64.$$

Step 1: We first show $\operatorname{Re} x, \operatorname{Re} y, \operatorname{Re} z > 17/8$. By (34) and $a + b + c = 0$,

$$x = e^{3at} + e^{(b-c)t} + e^{-(b-c)t} = e^{3at} + 2 \cosh((b-c)t). \quad (42)$$

Since $|b - c| + |a| = 2 \max\{|b|, |c|\} \leq 2$, we have $|(b - c)t| \leq 3(2 - |a|)/8 \leq 3/4$. Using $\cosh s \geq 1$ and $\cos s \geq 1 - s^2/2$ for real s gives

$$\operatorname{Re} \cosh((b - c)t) = \cosh((b - c) \operatorname{Re} t) \cos((b - c) \operatorname{Im} t) \geq 1 - \frac{9}{128}(2 - |a|)^2. \quad (43)$$

Since $|3at| \leq 9/8$, combining (42), (43), and (38) gives

$$\operatorname{Re} x \geq \frac{5}{6} - \frac{9|a|}{16} + 2 - \frac{9}{64}(2 - |a|)^2 = \frac{109}{48} - \frac{9}{64}|a|^2 > \frac{17}{8}.$$

The same argument applies to y and z .

Step 2: We next show that each first child has larger modulus than both parents. For the child $xy - z$, we use

$$\frac{xy - z}{x} = 2e^{(b-c)t} \cosh((b-a)t). \quad (44)$$

It suffices to show that $\log |(xy - z)/x| > 0$. Put $d = |b - a| \leq 2$. Since $2(b - c) = (b - a) - 3c$ and $|c| \leq 1$, we have $|b - c| \leq (3 + d)/2$. Combining this with $|b - c| \leq 2$, using $0 \leq 7p/4 \leq 1$, gives

$$|b - c| \leq \left(1 - \frac{7p}{4}\right) 2 + \frac{7p}{4} \frac{3 + d}{2} = 2 + \frac{7p}{8}(d - 1) \leq 2 + \frac{7p}{16}(d^2 - 1). \quad (45)$$

Taking logarithms of the modulus in (44) and applying (37) gives

$$\log \left| \frac{xy - z}{x} \right| \geq \log 2 - |b - c|p + d^2(p^2 - 2/25),$$

where we used $q^2 \leq 9/64 - p^2$, and $d^4 \leq 4d^2$. Now (45), $d^2 \leq 4$, and $9p^2/16 < 2/25$ yield

$$\log \left| \frac{xy - z}{x} \right| \geq \log 2 - 2p + \frac{7p^2}{16} + d^2 \left(\frac{9p^2}{16} - \frac{2}{25} \right) \geq \log 2 - \frac{8}{25} - 2p + \frac{43p^2}{16} > 0. \quad (46)$$

Hence $|xy - z| > |x|$. Interchanging x and y gives $|xy - z| > |y|$.

Step 3: For the later use, we will show that $|xy - z| > 14/5$. Since $|a|, |b|, |c| \leq 1$,

$$|e^{2at} + e^{2bt} + e^{2ct}| \geq \cos(2q) (e^{2a \operatorname{Re} t} + e^{2b \operatorname{Re} t} + e^{2c \operatorname{Re} t}) \geq 3 \cos(2q).$$

Combining this with (34), (44), $|a - b| \leq 2 - |c|$, $\operatorname{Re} t \leq p$, and $\operatorname{Im} t \leq q$ gives

$$|xy - z| \geq 6 \cos(2q) e^{-2c \operatorname{Re} t} |\cosh((a - b)t)| \geq 6 \cos(2q) e^{-2|c|p} \cos((2 - |c|)q). \quad (47)$$

By (41), $\log \cos s \geq -3s^2/5$ for $|s| \leq 3/4$. Since $(2 - |c|)^2 \leq 4 - 3|c|$, it follows that

$$\log \frac{|xy - z|}{6} \geq -(1 - |c|) \frac{24q^2}{5} - |c|(2p + 3q^2) > -\frac{19}{25},$$

because $p^2 + q^2 \leq 9/64$ gives $24q^2/5 \leq 27/40 < 19/25$ and $2p + 3q^2 \leq 145/192 - 3(p - 1/3)^2 < 19/25$. Therefore

$$|xy - z| > 6e^{-19/25} > \frac{14}{5}.$$

Permuting the coordinates gives the same bound for every first child.

Step 4: By Steps 1–3, the initial traces have modulus $> 17/8$, and each first child exceeds both parents and has modulus $> 14/5$. We propagate these comparisons along each outward branch. At a later insertion $UV - W$, W is a parent of the newer trace among U, V . The comparison at the previous step therefore gives $|W| < \max(|U|, |V|)$. Since $|U|, |V| > 17/8$,

$$|UV - W| \geq |U||V| - \max(|U|, |V|) > \frac{9}{17}|U||V| > \frac{9}{8} \max(|U|, |V|). \quad (48)$$

Thus the new trace again exceeds both parents, completing the induction. The height-one and height-two bounds from Steps 1 and 3, together with the product estimate, now give

$$|x_w| > \begin{cases} 17/8, & h = 1, \\ \frac{14}{5} \left(\frac{9}{8} \right)^{h-2}, & h \geq 2, \end{cases} \quad (49)$$

for every primitive class w in an initial sector, where $h = h(w)$. Indeed, heights add. If one parent has height one, the product estimate preserves the stated bound. If both parent heights are at least two, substituting their stated bounds gives an extra factor $\frac{9}{17} \frac{14}{5} \left(\frac{8}{9} \right)^2 > 1$. Any stronger bounds inherited from earlier insertions only improve this estimate. This follows the induction scheme of [Bow98, Lemma 2.1.1(2)].

Step 5: We now convert the trace bounds into length bounds. For the finitely many traces in P_6 , Step 4 gives $|x_w| > 2$ on a slightly larger open disk containing $|t| \leq 3/8$. Thus we can choose an analytic square root of $x_w^2 - 4$, positive at $t = 0$ [SS03b, Theorem 6.2, p. 100], and define

$$\lambda = \frac{x_w + \sqrt{x_w^2 - 4}}{2}.$$

Here $\lambda(x_w - \lambda) = 1$, so λ never vanishes. Moreover, $|\lambda| = 1$ would give $x_w = \lambda + \lambda^{-1} = \lambda + \bar{\lambda} = 2 \operatorname{Re} \lambda \in [-2, 2]$, contradicting $|x_w| > 2$. At $t = 0$, the modular trace satisfies $x_w(0) > 2$, so $\lambda(0) > 1$. Since $|\lambda| \neq 1$, continuity gives $|\lambda| > 1$ throughout the disk. The logarithm theorem [SS03b, Theorem 6.2, p. 100] gives an analytic length $L_w = 2 \log \lambda$ that agrees with the real length at $t = 0$. Since $\operatorname{Re} L_w = 2 \log |\lambda| > 0$ and

$$|x_w| = |\lambda + \lambda^{-1}| \leq |\lambda| + |\lambda|^{-1} = 2 \cosh \frac{\operatorname{Re} L_w}{2},$$

(49) gives $\operatorname{Re} L_w \geq 2 \operatorname{arccosh}(|x_w|/2) > l_h$, where $h = h(w)$ and

$$\frac{h}{l_h} \left| \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 7/10 & 17/10 & 2 & 23/10 & 13/5 & 14/5 \end{array} \right. \quad (50)$$

Appendix A3.2 verifies these comparisons. The nonvanishing lengths make the finite sum defining F analytic on the larger disk. $\square$

Proposition 3.9 (Taylor remainder bound for the polygon area). *For every function F in Proposition 3.8, every integer $N \geq 1$, and $0 < |t| < 3/8$, one has*

$$\left| F(t) - \sum_{n=0}^{N-1} \frac{F^{(n)}(0)}{n!} t^n \right| < \sqrt{21} \frac{(8|t|/3)^N}{\sqrt{1 - (8|t|/3)^2}}. \quad (51)$$

Proof. The idea is to control all remaining Taylor terms through the circle average of $|F|^2$. Averaging cancels products of different Taylor powers, leaving the sum of their squared sizes on the circle (see Parseval's identity [SS03a, Theorem 1.3(ii), p. 80]).

We first bound this average using the known lengths at the modular torus. By Proposition 3.8, each reciprocal length is analytic on a neighborhood of $|t| \leq 3/8$. Averaging its real part around this circle therefore gives its value at the center (see [SS03b, Corollary 7.3, p. 103]),

$$\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{1}{L_v((3/8)e^{i\theta})} d\theta = \operatorname{Re} \frac{1}{L_v(0)} = \frac{1}{L_v(0)}. \quad (52)$$

For each of the thirty-six terms $1/(L_u L_v)$ in (31), order u, v so that $L_v(0) \geq L_u(0)$. The bounds $|L_u| \geq \operatorname{Re} L_u > l_{h(u)}$ and $\operatorname{Re} L_v > l_{h(v)}$ (see (50)) imply, for every such pair,

$$\frac{l_{h(u)} l_{h(v)}}{|L_u L_v|^2} = \frac{l_{h(u)}^2}{|L_u|^2} \frac{l_{h(v)}}{\operatorname{Re} L_v} \frac{\operatorname{Re}(1/L_v)}{l_{h(u)}} < \frac{\operatorname{Re}(1/L_v)}{l_{h(u)}}.$$

Thus Cauchy–Schwarz gives

$$|F(t)|^2 \leq \left(\sum \frac{1}{l_{h(u)} l_{h(v)}} \right) \left(\sum \frac{l_{h(u)} l_{h(v)}}{|L_u L_v|^2} \right) < \left(\sum \frac{1}{l_{h(u)} l_{h(v)}} \right) \sum \frac{\operatorname{Re}(1/L_v)}{l_{h(u)}} = \frac{94260}{10829} \sum \frac{\operatorname{Re}(1/L_v)}{l_{h(u)}}, \quad (53)$$

where each sum runs over the thirty-six polygon summands.

By (52), using (31) at the modular point and the bounds in (50), the circle average of $\sum \operatorname{Re}(1/L_v)/l_{h(u)}$ is

$$\sum \frac{1}{l_{h(u)} L_v(0)} = 6 \left(\frac{165}{91L(267)} + \frac{37}{34L(87)} + \frac{1}{2L(39)} + \frac{10}{23L(102)} \right) < \frac{12}{5}.$$

Averaging the preceding bound (53) for $|F|^2$ therefore yields

$$\frac{1}{2\pi} \int_0^{2\pi} \left| F\left(\frac{3}{8}e^{i\theta}\right) \right|^2 d\theta < \frac{94260}{10829} \frac{12}{5} < 21.$$

Since F is analytic beyond this circle, its Taylor series converges uniformly there. Averaging its squared modulus cancels the mixed terms (Parseval's identity) and gives

$$\sum_{n=0}^{\infty} \left| \frac{F^{(n)}(0)}{n!} \right|^2 \left(\frac{3}{8} \right)^{2n} < 21.$$

Cauchy–Schwarz and the geometric series now give

$$\left| F(t) - \sum_{n=0}^{N-1} \frac{F^{(n)}(0)}{n!} t^n \right| \leq \sum_{n=N}^{\infty} \left| \frac{F^{(n)}(0)}{n!} \right| |t|^n < \sqrt{21} \left(\sum_{n=N}^{\infty} (8|t|/3)^{2n} \right)^{1/2} = \sqrt{21} \frac{(8|t|/3)^N}{\sqrt{1 - (8|t|/3)^2}}.$$

□

3.6 Polygon area increase near the modular point

To prove that the full area increases near the modular point, we need the polygon area $\mathcal{A}_6$ to increase by more than the omitted area can decrease. Here we establish the polygon bound $\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > 7p_2/80$ and relate the logarithmic coordinates to the trace coordinates used in region B . For a real triple (u, v, w) , put

$$t = \max\{|u|, |v|, |w|\}.$$

Here t is real and nonnegative. For $0 < t < 3/8$, write $(u, v, w) = t(a, b, c)$ with $\max\{|a|, |b|, |c|\} = 1$. Applying (51) with $N = 9$ to the corresponding function in Proposition 3.8 gives

$$|R_9| < \sqrt{21} \frac{(8t/3)^9}{\sqrt{1 - (8t/3)^2}}. \quad (54)$$

Proposition 3.10 (A local polygon bound). *In the logarithmic coordinates (34),*

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > \frac{7}{80}p_2 \text{ whenever } 0 < t \leq \frac{1}{20}.$$

Proof. Assume $0 < t \leq 1/20$. Since $u + v + w = 0$, using elementary inequalities, one can show that

$$\frac{3}{2}t^2 \leq p_2 \leq 2t^2, \quad |p_3| = 3|uvw| \leq \frac{3}{4}t^3 \leq \frac{t}{2}p_2, \quad (55)$$

where p_2 and p_3 are denoted before (35). Combining (54) with $p_2 \geq 3t^2/2$ and $t \leq 1/20$, we obtain

$$\frac{|R_9|}{p_2} < \frac{2\sqrt{21}}{3t^2} \frac{(8t/3)^9}{\sqrt{1 - (8t/3)^2}} < \frac{1}{25000}. \quad (56)$$

Using the coefficient bounds (105) in Appendix A2, discard the nonnegative terms $c_{6''}p_3^2$ and $c_{8'}p_2^4$ in (35) and apply (55) to obtain

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > p_2 \left(c_2 - \frac{347}{1000}t - \frac{81}{250}t^2 - \frac{163}{200}t^3 - \frac{157}{125}t^4 - \frac{277}{500}t^5 - \frac{15077}{1000}t^6 \right) + R_9. \quad (57)$$

Using $0 < t \leq 1/20$ with (36) and (56), this gives

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > p_2 \left(\frac{23}{200} - \frac{1169288557}{64000000000} \right) - \frac{1}{25000}p_2 > \frac{7}{80}p_2. \quad \square$$

The trace bounds in Lemma 2.3 allow us to apply the following lemma on the domain $\mathcal{R}$ defined in (8). It places $\mathcal{R}$ within the range of Proposition 3.10 and compares the two coordinate distances.

Lemma 3.11. *Let (x, y, z) be a positive Markoff triple with $74/25 < x, y, z < 61/20$, and let (u, v, w) be its coordinates in (34). Then $t = \max\{|u|, |v|, |w|\} < 1/60$. If the triple is nonmodular, then*

$$(x-3)^2 + (y-3)^2 + (z-3)^2 < 10p_2.$$

Proof. Since $u+v+w=0$, we have $e^{2u}+e^{2v}+e^{2w} \geq 3$. From the hypothesis, $|x-3|, |y-3|, |z-3| < 1/20$. The Markoff equation and (34) give $x^2+y^2+z^2 = (e^{2u}+e^{2v}+e^{2w})^3$ and

$$x^2 + y^2 + z^2 = 27 + 6((x-3)^2 + (y-3)^2 + (z-3)^2) - 3(x+y+z-9)^2 - 2(x-3)(y-3)(z-3),$$

where the right-hand side is smaller than $27 + 6 \times 3/20^2 - 0 + 2 \times 1/20^3 < (1501/500)^3$. Consequently,

$$3 \leq e^{2u} + e^{2v} + e^{2w} < 1501/500.$$

Combining with the hypothesis $74/25 < x, y, z < 61/20$, we have $60/61 < (74/25)/(1501/500) < e^u = x/(e^{2u}+e^{2v}+e^{2w}) < (61/20)/3 = 61/60$. The same holds for e^v , and e^w , so

$$t = \max\{|u|, |v|, |w|\} < \log \frac{61}{60} < \frac{1}{60}.$$

For the distance comparison, the exponential series and $u+v+w=0$ give

$$|e^u - 1| \leq \frac{|u|}{1-t}, \quad 0 \leq e^{2u} + e^{2v} + e^{2w} - 3 \leq \frac{2p_2}{1-2t},$$

for $0 \leq t < 1/2$. The first bound also holds for v, w . By (34),

$$(x-3, y-3, z-3) = 3(e^u - 1, e^v - 1, e^w - 1) + (e^{2u} + e^{2v} + e^{2w} - 3)(e^u, e^v, e^w).$$

Applying the Euclidean triangle inequality to the two terms on the right, using the preceding bounds and $e^u, e^v, e^w \leq e^t \leq 1/(1-t)$ ($0 \leq t < 1$), gives

$$\sqrt{(x-3)^2 + (y-3)^2 + (z-3)^2} \leq \frac{3\sqrt{p_2}}{1-t} + \frac{2\sqrt{3}p_2}{(1-t)(1-2t)} \leq \frac{3\sqrt{p_2}}{(1-t)(1-2t)}, \quad (58)$$

where the last inequality uses $\sqrt{p_2} \leq \sqrt{3}t$. For $t < 1/60$, the coefficient is less than $\sqrt{10}$, proving the required bound when $p_2 > 0$. $\square$

4 Region A: comparison outside the local disk

Region A consists of tori X represented by reduced triples (x, y, z) with $\text{sys}(X) = 2 \operatorname{arccosh}(x/2) \geq 7/5$ and $(x-3)^2 + (y-3)^2 > 1/1024$. We will show that upper bounds for $|R_9|$ and lower bounds for the polynomial part of (35) yield $\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > 19p_2/1000$ in Corollary 4.5. Combining this with $p_2 > 1/12800$ in (72), $\mathcal{A}(X_{\text{mod}}) - \mathcal{A}_6(X_{\text{mod}}) < 1/700000$ in Lemma 3.4, and (4) implies the desired comparison for all X in region A :

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > 19p_2/1000 - 1/700000 > p_2/1400.$$

4.1 Bounding the Taylor remainder R_9 on a larger neighborhood

The direct remainder estimate (56) suffices for $t \leq 1/20$, but is too large at the larger radii allowed in region A by (66). To control R_9 throughout region A , we estimate the terms of degrees nine through nineteen explicitly and apply (51) with $N = 20$ to the remaining tail. The full Taylor expansion extending (35) is

$$\mathcal{A}_6(u, v, w) = \mathcal{A}_6(X_{\text{mod}}) + \sum_{n=2}^{\infty} H_n(u, v, w), \quad (59)$$

where H_n is the homogeneous Taylor term of degree n .

Lemma 4.1 (Four directions determine the required coefficients). *For $9 \leq n \leq 19$, let ϵ be zero for even n and one for odd n , and put $m = (n - 3\epsilon)/2$. There are unique real constants $c_{n,k}$ such that*

$$H_n = p_3^\epsilon \sum_{k=0}^{\lfloor m/3 \rfloor} c_{n,k} p_2^{m-3k} p_3^{2k}. \quad (60)$$

All these constants are determined by the Taylor expansions through degree nineteen on the four rays

$$(u, v, w) = t(1, -s, s-1), \quad t \geq 0, \quad s \in \{1/10, 2/10, 3/10, 4/10\}.$$

Proof. As in (35), permutation symmetry expresses each H_n as a polynomial in p_2, p_3 . Homogeneity gives (60). We determine its coefficients by evaluating a one-variable polynomial.

Along the ray $(u, v, w) = t(1, -s, s-1)$, we have $p_2 = t^2 p_2(s)$, and $p_3 = t^3 p_3(s)$, where

$$p_2(s) := 2(1-s+s^2), \quad p_3(s) := 3s(1-s).$$

Since H_n is homogeneous of degree n , $H_n(1, -s, s-1)$ is the coefficient of t^n in $\mathcal{A}_6(t, -st, (s-1)t)$. Factoring out $p_2(s)^m p_3(s)^\epsilon$, we put $q(s) = p_3(s)^2/p_2(s)^3$ and obtain

$$\frac{H_n(1, -s, s-1)}{p_2(s)^m p_3(s)^\epsilon} = \sum_{k=0}^d c_{n,k} q(s)^k, \quad d = \lfloor m/3 \rfloor. \quad (61)$$

Note that $q(s) > 0$ on $(0, 1/2)$ and its logarithmic derivative is

$$\frac{q'(s)}{q(s)} = (1-2s) \left(\frac{2}{s(1-s)} + \frac{3}{1-s+s^2} \right) > 0.$$

Since $n \leq 19$, one has $d \leq 3$. Use $s_j = (j+1)/10$ for $0 \leq j \leq d$. The values $q(s_j)$ are distinct, and $p_3(s_j) > 0$. Note that (61) is linear with the Vandermonde determinant $\det(V) = \prod_{0 \leq i < j \leq d} (q(s_j) - q(s_i)) \neq 0$, hence it determines every coefficient in (60). $\square$

Applying these four substitutions to the polygon formula gives the coefficient bounds in Lemma A2.3 (Appendix A2). These yield the following bounds for the remainder R_9 .

Theorem 4.2 (Two uniform Taylor remainder bounds). *For every real zero-sum triple, the remainder beyond degree eight satisfies*

$$|R_9| < \frac{3}{250} p_2 \text{ whenever } 0 < t \leq \frac{6}{25},$$

and

$$|R_9| < \frac{11}{200} p_2 \text{ whenever } 0 < t \leq \frac{21}{80}.$$

Proof. We estimate degrees nine through nineteen using Lemma A2.3. For the remaining terms, apply (51) with $N = 20$ along $(a, b, c) = (u, v, w)/t$ and divide by $p_2 \geq 3t^2/2$ to obtain

$$\frac{1}{p_2} \left| \sum_{n=20}^{\infty} H_n \right| < \frac{2\sqrt{21}}{3t^2} \frac{(8t/3)^{20}}{\sqrt{1 - (8t/3)^2}}. \quad (62)$$

For $9 \leq n \leq 19$, put $\epsilon = n \bmod 2$ and $m = (n - 3\epsilon)/2$. The formula (60) gives

$$\frac{|H_n|}{p_2} \leq \frac{2}{3} t^{n-2} \sum_{k=0}^{\lfloor m/3 \rfloor} |c_{n,k}| 2^{m-3k} \left(\frac{3}{4} \right)^{\epsilon+2k}. \quad (63)$$

This uses $p_2 \leq 2t^2$ and $|p_3| \leq 3t^3/4$ from (55). Bound each $|c_{n,k}|$ by the larger absolute value of its two endpoints in Lemma A2.3. Both bounds increase with t . Substituting $6/25$ and $21/80$ gives

$$\begin{array}{c|cc} t \leq & \frac{1}{p_2} \sum_{n=9}^{19} |H_n| & \frac{1}{p_2} \left| \sum_{n=20}^{\infty} H_n \right| \\ \hline 6/25 & < 1/400 & < 19/2000 \\ 21/80 & < 1/200 & < 1/20 \end{array} \quad (64)$$

For completeness, let B_n denote the right-hand coefficient of t^{n-2} in (63), with each $|c_{n,k}|$ replaced by the larger absolute endpoint from Lemma A2.3. Thus

$$B_n = \frac{2}{3} \sum_{k=0}^{\lfloor m/3 \rfloor} \frac{\max(|a_{n,k}|, |a_{n,k} + 1|)}{1000} 2^{m-3k} \left(\frac{3}{4} \right)^{\epsilon+2k},$$

where $a_{n,k}$ is the corresponding integer in that table. The two finite sums are

$$\begin{aligned} \sum_{n=9}^{19} B_n (6/25)^{n-2} &= \frac{335916951272113891872267}{145519152283668518066406250} < \frac{1}{400}, \\ \sum_{n=9}^{19} B_n (21/80)^{n-2} &= \frac{141381022750318299872952246014482329}{28823037615171174400000000000000000000} < \frac{1}{200}. \end{aligned}$$

For the final column of (64), put

$$V(t)^2 = \frac{28}{3t^4} \frac{(8t/3)^{40}}{1 - (8t/3)^2}.$$

This function increases for $0 < t < 3/8$; substituting $t = 6/25$ and $t = 21/80$ gives the stated bounds. $\square$

4.2 The degree-eight comparison on the reduced region

We now prove the polygon bound needed in region A . It holds for every nonmodular reduced triple with $\text{sys}(X) \geq 7/5$. The inequalities defining a reduced triple bound p_2 , p_3/p_2 , and t . These bounds show that the degree-eight polynomial is larger than the possible error $|R_9|$ from Theorem 4.2.

Lemma 4.3 (Bounds for p_2 , p_3/p_2 , and t). *For a nonmodular reduced triple, one has*

$$\frac{p_3}{p_2} \leq \frac{\log 2}{6} < \frac{7}{60}. \quad (65)$$

If in addition $\text{sys}(X) \geq 7/5$, then

$$t = \max(|u|, |v|, |w|) < \frac{21}{80}, p_2 < \frac{441}{3200}. \quad (66)$$

If also $t \geq 6/25$, then

$$\frac{p_3}{p_2} < \frac{1}{120}. \quad (67)$$

Proof. For a reduced triple, the largest of e^{2u}, e^{2v}, e^{2w} is at most the sum of the other two. We use this inequality to bound p_3/p_2 , then use the systole bound to bound t . The first assertion is immediate when $p_3 \leq 0$. Otherwise the coordinates are, up to permutation, $(t, -t/2 + d, -t/2 - d)$ with $0 \leq d < t/2$. The preceding inequality becomes

$$e^{3t} \leq 2 \cosh(2d), 3t \leq \log 2 + 2d^2.$$

The second inequality follows from $\log \cosh(2d) \leq 2d^2$. Since $2d \leq t$, we also have $e^{3t} \leq 2 \cosh(2d) \leq 2e^t$. Thus $e^{2t} \leq 2$ and $t < 6$. Direct algebra gives

$$3t - 6\frac{p_3}{p_2} = \frac{24td^2}{3t^2/2 + 2d^2} \geq \frac{12d^2}{t} \geq 2d^2.$$

Thus $6p_3/p_2 \leq 3t - 2d^2 \leq \log 2 < 7/10$.

To bound t , fix the systolic trace $x = 2 \cosh a$ and vary y among reduced triples. Wherever $2z < xy$, the lower root $z = z_-(x, y)$ decreases with y . Implicit differentiation also gives

$$\frac{d}{dy} \log(yz) = \frac{2(z^2 - y^2)}{yz(2z - xy)} \leq 0. \quad (68)$$

Thus the smallest logarithmic coordinate $(2 \log x - \log y - \log z)/3$ increases with y , while the largest $(2 \log z - \log x - \log y)/3$ decreases. The maximum of t is therefore at the smallest y for which the triple is reduced. The conclusion extends to an endpoint with $2z = xy$ by continuity.

For $x \geq \sqrt{8}$, that endpoint is $y = x$. Put

$$q(x) = \frac{x - \sqrt{x^2 - 8}}{2}. \quad (69)$$

Here $1 \leq q \leq \sqrt{2}$ and the logarithmic coordinates are $(-\log q/3, -\log q/3, 2 \log q/3)$. Hence $t \leq (\log 2)/3 < 21/80$. For $2 < x \leq \sqrt{8}$ it is

$$y = \frac{2x}{\sqrt{x^2 - 4}}, z = \frac{xy}{2}.$$

In terms of a the endpoint is $(x, y, z) = (2 \cosh a, 2 \coth a, 2 \cosh a \coth a)$. The absolute values of its smallest and largest logarithmic coordinates are respectively

$$\frac{1}{3} \log \frac{\cosh a}{\sinh^2 a}, \frac{1}{3} \log \frac{\cosh^2 a}{\sinh a}.$$

Both functions decrease for $7/10 \leq a \leq \operatorname{arsinh} 1$. It therefore suffices to bound them at $a = 7/10$. The Taylor sums for $\cosh$ and $\sinh$, with the remainder bounds in Appendix A1, give

$$\cosh(7/10) < \frac{157}{125}, \sinh(7/10) > \frac{379}{500}.$$

The resulting two quotients are less than $219/100$, which is less than $e^{63/80}$ by its Taylor sum. Thus $t < 21/80$. Finally $p_2 \leq 2t^2$ gives the stated bound for p_2 .

Now we prove (67), suppose $t \geq 6/25$ and $p_3 > 0$. Using the representation $(t, -t/2 + d, -t/2 - d)$, since $p_2 = 3t^2/2 + 2d^2 \leq 2t^2$ and $2d^2 \geq 3t - \log 2$, we have

$$\frac{p_3}{p_2} = \frac{t}{2} - \frac{4td^2}{p_2} \leq \frac{t}{2} - 3 + \frac{\log 2}{t}.$$

The last expression decreases for $6/25 \leq t \leq 21/80$. The Taylor sum of degree nine for the exponential gives $e^{1733/2500} > 2$, so $\log 2 < 1733/2500$. Consequently

$$\frac{p_3}{p_2} < \frac{3}{25} - 3 + \frac{1733}{600} = \frac{1}{120}.$$

□

Let T_8 be the sum of the Taylor terms of degrees two through eight in (35). Thus

$$\mathcal{A}_6 - \mathcal{A}_6(X_{\text{mod}}) = T_8 + R_9. \quad (70)$$

Proposition 4.4 (A lower bound for the degree-eight polynomial). *Suppose (x, y, z) is a reduced triple for X and $\text{sys}(X) \geq 7/5$. If $0 < t \leq 6/25$, then*

$$T_8(u, v, w) > \frac{31}{1000}p_2.$$

If instead $t \geq 6/25$, then

$$T_8(u, v, w) > \frac{2}{25}p_2.$$

Proof. We reduce the comparison to one variable: first show that T_8/p_2 decreases with p_3/p_2 , then replace that ratio by its upper bound. The resulting polynomial in p_2 can be bounded at its endpoints. Regard T_8/p_2 as a quadratic polynomial in p_3/p_2 , with p_2 fixed. Its quadratic coefficient is $c_6''p_2 + c_8''p_2^2 > 0$ for $0 < p_2 \leq 441/3200$. On the range $p_3/p_2 \leq 7/60$ from (65), its derivative with respect to this ratio is at most

$$-\frac{692}{1000} + \frac{815}{1000}p_2 + \frac{277}{1000}p_2^2 + 2\frac{6367}{1000}p_2\frac{7}{60} < 0.$$

The last expression increases with p_2 ; at $441/3200$ it equals $-3785555003/1024000000 < 0$. Replacing p_3/p_2 by $7/60$ therefore gives a lower bound. Apply (36) and the lower bounds in (105). With $a = 7/60$, the constant, linear, quadratic, and cubic coefficients are, respectively,

$$\begin{aligned} \frac{23}{200} - \frac{694}{1000}a &= \frac{1021}{30000}, \\ -\frac{162}{1000} + \frac{812}{1000}a + \frac{6354}{1000}a^2 &= \frac{11531}{600000}, \\ -\frac{314}{1000} + \frac{270}{1000}a - \frac{15077}{1000}a^2 &= -\frac{1755773}{3600000}, \\ \frac{1118}{1000} &= \frac{559}{500}. \end{aligned}$$

Consequently,

$$\frac{T_8}{p_2} > \frac{1021}{30000} + \frac{11531}{600000}p_2 - \frac{1755773}{3600000}p_2^2 + \frac{559}{500}p_2^3. \quad (71)$$

For $t \leq 6/25$, we have $p_2 \leq 72/625$. The cubic on the right is concave on this interval: its second derivative is increasing and is negative at $72/625$. Its endpoint values exceed $31/1000$. At the nonzero endpoint its value is

$$\frac{46119182759}{1464843750000} > \frac{31}{1000}.$$

Concavity proves the first bound throughout the interval.

If $t \geq 6/25$, use (67) instead. Since T_8/p_2 decreases as p_3/p_2 increases, substituting $p_3/p_2 = 1/120$ again gives a lower bound. At this value the terms with coefficients c_5, c_7, c_8' are positive, as is the combined quadratic term $(c_6''p_2 + c_8''p_2^2)(p_3/p_2)^2$. Discarding them gives

$$\frac{T_8}{p_2} > \frac{23}{200} - \frac{347}{60000} - \frac{81}{500}p_2 - \frac{157}{500}p_2^2 > \frac{2}{25}.$$

The last expression decreases with p_2 . At $p_2 = 441/3200$ it equals

$$\frac{1243045849}{15360000000} > \frac{2}{25},$$

which proves the final comparison. $\square$

Corollary 4.5 (Uniform polygon comparison). *For every nonmodular reduced triple with $\text{sys}(X) \geq 7/5$,*

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > \frac{19}{1000}p_2.$$

Proof. Use (70), Proposition 4.4, and Theorem 4.2. For $0 < t \leq 6/25$, they give

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > \left(\frac{31}{1000} - \frac{3}{250} \right) p_2 = \frac{19}{1000} p_2.$$

For $6/25 < t < 21/80$, they give

$$\mathcal{A}_6(X) - \mathcal{A}_6(X_{\text{mod}}) > \left(\frac{2}{25} - \frac{11}{200} \right) p_2 = \frac{1}{40} p_2 > \frac{19}{1000} p_2.$$

The radius bound (66) covers the whole reduced region under consideration. $\square$

We now compare the full areas on region A . If $t \leq 1/20$, (58), $e^t < 20/19$, and $e^{2t} < 10/9$ give

$$\frac{1}{1024} < (x-3)^2 + (y-3)^2 < \left(\frac{200}{57} \right)^2 p_2 < \frac{25}{2} p_2.$$

If $t > 1/20$, then (55) gives $p_2 \geq 3t^2/2 > 3/800$. Thus throughout region A ,

$$p_2 > \frac{1}{12800}. \quad (72)$$

Corollary 4.5, Lemma 3.4, and (4) therefore imply

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{19}{1000} p_2 - \frac{1}{700000} > \left(\frac{19}{1000} - \frac{12800}{700000} \right) p_2 = \frac{1}{1400} p_2 > 0. \quad (73)$$

Section 7 uses the sharper polygon bound near the disk to obtain the coefficient $1/80$.

5 Region B: comparison inside the local disk

Region B consists of nonmodular tori represented by reduced triples with $\text{sys}(X) \geq 7/5$ and $(x-3)^2 + (y-3)^2 \leq 1/1024$. Reducedness gives $x \leq 3$ and $y \geq x$, so these points lie in $\mathcal{R}$. Here we compare the change in omitted area with the polygon gain. Its zero gradient and a lower Hessian bound give

$$(\mathcal{A} - \mathcal{A}_6)(X) - (\mathcal{A} - \mathcal{A}_6)(X_{\text{mod}}) > -\frac{(x-3)^2 + (y-3)^2}{500}$$

away from the modular point. We prove this by bounding length derivatives, differentiating the area series, and summing the possible negative contributions. Combining it with the polygon bound will yield $\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > 27p_2/400$ throughout $\mathcal{R}$, which is stronger than needed on region B .

5.1 Derivatives of all simple lengths

To control the change of the omitted area in region B , we need bounds for length derivatives in x, y . Differentiating the Farey recursion expresses them as the parent derivatives plus exponentially small corrections, bounded in Subsection 3.1.

Proposition 5.1 (Length derivatives bounded by sector height). *On $\mathcal{R}$, every simple curve w in an initial Farey sector satisfies*

$$\begin{aligned} \max\{|\partial_x L_w|, |\partial_y L_w|\} &< \frac{3}{2} h(w), \\ \max\{|\partial_{xx} L_w|, |\partial_{xy} L_w|, |\partial_{yy} L_w|\} &< 12h(w). \end{aligned}$$

Proof. We first check heights one and two, then sum the corrections introduced by later insertions.

By Lemmas 2.2 and 2.3, $\sqrt{\mathcal{D}} > 11/4$, $95/32 \leq x, y \leq 97/32$, and $74/25 < x, y, z < 61/20$. Implicit differentiation gives

$$z_x = \frac{2x - yz}{\sqrt{\mathcal{D}}}, z_y = \frac{2y - xz}{\sqrt{\mathcal{D}}},$$

so $-5/4 < z_x, z_y < 0$. Differentiating once more gives

$$z_{xx} = \frac{2 - 2yz_x + 2z_x^2}{\sqrt{\mathcal{D}}}, z_{xy} = \frac{-z - yz_y - xz_x + 2z_xz_y}{\sqrt{\mathcal{D}}},$$

and the analogous formula for z_{yy} . The preceding bounds imply $|z_{xx}|, |z_{xy}|, |z_{yy}| < 6$. For $L(t) = 2 \operatorname{arccosh}(t/2)$,

$$L'(t) = \frac{2}{\sqrt{t^2 - 4}}, L''(t) = -\frac{2t}{(t^2 - 4)^{3/2}}.$$

On $[74/25, 61/20]$, these satisfy $0 < L' < 1$ and $|L''| < 3/5$. Thus the three initial lengths have first derivatives bounded by $5/4$ and second derivatives bounded by $6 + (3/5)(5/4)^2 < 7$.

The first insertion traces $xy - z, yz - x, zx - y$ exceed $14279/2500$. Their first derivatives have absolute value less than 5 and second derivatives less than 21, by the bounds for z above. For the first derivatives use $z_x, z_y < 0$; for example, $|z + yz_y| \leq \max\{z, -yz_y\}$. On $t \geq 14279/2500$, $L'(t) < 3/8$ and $|L''(t)| < 2/25$. The chain rule therefore bounds the first derivatives by $15/8 < 5/2$ and the second by $(3/8)21 + (2/25)5^2 < 14$. We have proved, at heights one and two,

$$\max\{|\partial_x L_w|, |\partial_y L_w|\} < \frac{5}{4}h(w), \max\{|\partial_{xx} L_w|, |\partial_{xy} L_w|, |\partial_{yy} L_w|\} < 7h(w). \quad (74)$$

All comparisons for L' and L'' follow by squaring positive quantities.

For a later insertion $w = u + v$, put $H = h(u) + h(v) \geq 3$. For example,

$$\partial_x L_w = (1 - \delta_1)\partial_x L_u + (1 - \delta_2)\partial_x L_v,$$

with the same formula for y . Proposition 3.2 and (25) give

$$1 < 1 - \delta_1, 1 - \delta_2 < 1 + 20 \left(\frac{3}{16}\right)^H, 0 < \delta_{11}, \delta_{12}, \delta_{22} < 40 \left(\frac{3}{16}\right)^H. \quad (75)$$

After division by H , the parent bounds are averaged with weights $h(u)/H$ and $h(v)/H$. Since both parent heights are smaller than H , induction bounds each first derivative divided by $h(w)$ by $5/4$ times the following product:

$$\prod_{H=3}^{\infty} \left(1 + 20 \left(\frac{3}{16}\right)^H\right) < e^{135/832} < \frac{6}{5}. \quad (76)$$

Here $\log(1+t) < t$ bounds the product, and $\log(6/5) > 1/5 - (1/5)^2/2 > 135/832$. This proves the first-derivative bound $3h(w)/2$.

A second differentiation has the same inherited parent terms, plus products of two first derivatives. For example,

$$\partial_{xx} L_w = (1 - \delta_1)\partial_{xx} L_u + (1 - \delta_2)\partial_{xx} L_v - \delta_{11}(\partial_x L_u)^2 - 2\delta_{12}\partial_x L_u \partial_x L_v - \delta_{22}(\partial_x L_v)^2.$$

For each of $\partial_{xx}, \partial_{xy}, \partial_{yy}$, the extra terms have total absolute value less than $40(3/2)^2 H^2 (3/16)^H$. Divide by H , iterate from the base bound 7, and use (76). Each second derivative in absolute value, divided by $h(w)$, is less than

$$\frac{6}{5} \left(7 + 90 \sum_{H=3}^{\infty} H \left(\frac{3}{16}\right)^H\right) = \frac{6}{5} \left(7 + 90 \frac{567}{21632}\right) < 12.$$

□

5.2 Differentiating the area and bounding the omitted terms

Proposition 2.5 gives

$$\mathcal{A} - \mathcal{A}_6 = \sum_{\substack{(u,v) \in \mathcal{F} \\ h(u+v) > 6}} S_X(u, v).$$

We show that this sum can be differentiated twice term by term, then bound its Hessian to control the possible decrease in omitted area.

Proposition 5.2 (Differentiating the area formula twice). *For all nonnegative integers k, l with $k + l \leq 2$, the series*

$$\sum_{(u,v) \in \mathcal{F}} \partial_x^k \partial_y^l S_X(u, v) \quad (77)$$

converges uniformly on $\mathcal{R}$ and locally uniformly on an open neighborhood of $\mathcal{R}$. Thus the area series and its first and second derivative series converge, and the area formula may be differentiated twice term by term. In particular, on $\mathcal{R}$,

$$\text{Hess}(\mathcal{A} - \mathcal{A}_K) = \sum_{\substack{(u,v) \in \mathcal{F} \\ h(u+v) > K}} \text{Hess} S_X(u, v). \quad (78)$$

Proof. We group the insertions by height $H = h(u) + h(v)$. For $H \geq 3$, Proposition 3.2 bounds δ and its first two derivatives in L_u, L_v by constants times $e^{-L_u - L_v}$. Since $L_{u+v} = L_u + L_v - \delta$, its first derivatives are $1 - \delta_1, 1 - \delta_2$, and its second derivatives are $-\delta_{11}, -\delta_{12}, -\delta_{22}$, all uniformly bounded. Together with $L_u, L_v, L_{u+v} \geq L_0$, this implies that $(L_u L_v L_{u+v})^{-1}$ and its first two derivatives are uniformly bounded. The product rule therefore bounds $\mathcal{D} = \delta / (L_u L_v L_{u+v})$ and its first and second partial derivatives in L_u, L_v by uniform constants times $e^{-L_u - L_v}$.

Proposition 5.1 bounds every first and second partial derivative of L_u, L_v in x, y by a constant times H . For the second derivative in x , the chain rule gives

$$\partial_{xx} S_X(u, v) = \mathcal{D}_{11} (\partial_x L_u)^2 + 2\mathcal{D}_{12} (\partial_x L_u) (\partial_x L_v) + \mathcal{D}_{22} (\partial_x L_v)^2 + \mathcal{D}_1 \partial_{xx} L_u + \mathcal{D}_2 \partial_{xx} L_v.$$

The first three terms are bounded by a constant times $H^2 e^{-L_u - L_v}$, and the last two by a constant times $H e^{-L_u - L_v}$. The same argument applies to $\partial_{xy} S_X(u, v)$ and $\partial_{yy} S_X(u, v)$, so all second derivatives are bounded by a constant times $H^2 e^{-L_u - L_v}$. First derivatives involve only one parent-length derivative and are bounded by a constant times $H e^{-L_u - L_v}$.

Together with the bound for $S_X(u, v)$ obtained above and $L_u + L_v > (42/25)H$ from Lemma 3.3, this gives a constant C , independent of the insertion and the point in $\mathcal{R}$, such that, for $H \geq 3$,

$$|\partial_x^k \partial_y^l S_X(u, v)| \leq C H^{k+l} e^{-42H/25}, \quad k, l \geq 0, \quad k + l \leq 2. \quad (79)$$

The finitely many height-two terms and their derivatives are continuous on the compact set $\mathcal{R}$, hence bounded. Increasing C if necessary extends this estimate to every $H \geq 2$. There are at most $3H$ insertions of height H by (12). Since $\sum_{H \geq 2} H^3 e^{-42H/25} < \infty$, the Weierstrass test gives uniform convergence of all six series in (77).

To differentiate the sum at boundary points of $\mathcal{R}$, we also need convergence on an open neighborhood. The bounds used to start the height and derivative inductions concern only the finitely many curves of heights one and two. These bounds, including the inequalities selecting the larger child root, are strict and depend continuously on (x, y) . By compactness, they remain valid on a common open neighborhood of $\mathcal{R}$. Repeating the inductions there gives (79), after increasing C if necessary, and hence uniform convergence of all six series.

Applying the termwise differentiation theorem successively in x and y [Rud76, Theorem 7.17], we may differentiate the area formula twice term by term. Applied to (11), this gives (78) for every integer $K \geq 1$. $\square$

Proposition 5.3 (A lower Hessian bound for the omitted area). *Throughout $\mathcal{R}$,*

$$\text{Hess}(\mathcal{A} - \mathcal{A}_6) > -\frac{1}{250} I,$$

where I is the 2×2 identity matrix.

Proof. We bound each omitted insertion, then sum over their heights. Fix $m = h(u)$, $n = h(v)$, and $H = m + n \geq 7$. Since $S_X(u, v) = \mathcal{D}(L_u, L_v)$, the chain rule gives

$$\text{Hess} S_X(u, v) = J^\top (\text{Hess}_{L_u, L_v} \mathcal{D}) J + \mathcal{D}_1 \text{Hess} L_u + \mathcal{D}_2 \text{Hess} L_v, \quad (80)$$

where J is the derivative matrix of $(x, y) \mapsto (L_u, L_v)$. The first term is nonnegative in every direction by Proposition 3.1. Proposition 5.1 bounds each entry of $\text{Hess} L_u$ by $12m$ in absolute value. Thus

$$|\xi^\top (\text{Hess} L_u) \xi| \leq 12m(|\xi_1| + |\xi_2|)^2 \leq 24m|\xi|^2$$

for every $\xi \in \mathbb{R}^2$, and similarly for L_v . Multiplying by $\mathcal{D}_1, \mathcal{D}_2$ and using absolute values to bound each possible negative contribution gives

$$\xi^\top (\text{Hess } S_X(u, v)) \xi \geq -24(m|\mathcal{D}_1| + n|\mathcal{D}_2|)|\xi|^2.$$

Since this holds in every direction,

$$\text{Hess } S_X(u, v) \geq -24(m|\mathcal{D}_1| + n|\mathcal{D}_2|)I.$$

We now bound these two derivatives. Proposition 3.2, Lemma 3.3, and (25) give $\delta < 11e^{-L_u-L_v}$, $|\delta_1|, |\delta_2| < 20e^{-L_u-L_v}$, and

$$0 < 1 - \delta_1, 1 - \delta_2 < 4/3.$$

Differentiating $\mathcal{D} = \delta/(L_u L_v L_{u+v})$, with $\partial_1 L_{u+v} = 1 - \delta_1$, gives

$$|\mathcal{D}_1| < \frac{e^{-L_u-L_v}}{L_u L_v L_{u+v}} \left(20 + \frac{11}{L_u} + \frac{44}{3L_{u+v}} \right).$$

The bound for $|\mathcal{D}_2|$ has L_v in place of L_u . Multiplying by m, n and adding, we obtain

$$m|\mathcal{D}_1| + n|\mathcal{D}_2| < \frac{e^{-L_u-L_v}}{L_u L_v L_{u+v}} \left(20H + 11 \left(\frac{m}{L_u} + \frac{n}{L_v} + \frac{4H}{3L_{u+v}} \right) \right).$$

The height bounds $L_u > (42/25)m$, $L_v > (42/25)n$, and $L_{u+v} > (42/25)H$ bound the second term in parentheses by $11(25/42)(2 + 4/3) < 22$. Therefore

$$m|\mathcal{D}_1| + n|\mathcal{D}_2| < \left(\frac{25}{42} \right)^3 \frac{20H + 22}{mnH} e^{-L_u-L_v}. \quad (81)$$

Since $mn \geq H - 1$ and $H \geq 7$,

$$24 \left(\frac{25}{42} \right)^3 \frac{20H + 22}{mnH} < 20.$$

Thus each omitted insertion satisfies

$$\text{Hess } S_X(u, v) > -20e^{-L_u-L_v} I > -20 \left(\frac{3}{16} \right)^H I.$$

There are at most $3(H - 1)$ insertions of height H by (12). Thus (78) gives

$$\text{Hess}(\mathcal{A} - \mathcal{A}_6) \geq -60 \sum_{H=7}^{\infty} (H - 1) \left(\frac{3}{16} \right)^H I > -\frac{1}{250} I.$$

Here the sum is evaluated by differentiating $\sum_{H=7}^{\infty} t^{H-1} = t^6/(1 - t)$ and multiplying by t^2 , at $t = 3/16$. $\square$

5.3 Completing the comparison in region B

Combining the polygon gain with the omitted-area estimate gives the following inequality throughout $\mathcal{R}$, and hence on region B .

Corollary 5.4 (Uniform local area inequality). *For $(x, y) \in \mathcal{R} \setminus \{(3, 3)\}$,*

$$\mathcal{A}(x, y) - \mathcal{A}(3, 3) > \frac{7}{80} p_2 - \frac{1}{500} ((x - 3)^2 + (y - 3)^2).$$

Proof. By Proposition 5.2, $\mathcal{A} - \mathcal{A}_6$ is C^2 near $\mathcal{R}$. Its permutation symmetry and Lemma 2.6 give $\nabla(\mathcal{A} - \mathcal{A}_6)(3, 3) = 0$, while Proposition 5.3 gives $\text{Hess}(\mathcal{A} - \mathcal{A}_6) > -I/250$ on $\mathcal{R}$. For $(x, y) \neq (3, 3)$, put $d = (x - 3, y - 3)$. Taylor's formula along the segment in $\mathcal{R}$ gives

$$(\mathcal{A} - \mathcal{A}_6)(x, y) - (\mathcal{A} - \mathcal{A}_6)(3, 3) > -\frac{1}{500} |d|^2.$$

By Lemma 3.11, $t < 1/60$ and $|d|^2 < 10p_2$. Adding the polygon bound from Proposition 3.10 in (4) gives

$$\mathcal{A}(x, y) - \mathcal{A}(3, 3) > \frac{7}{80} p_2 - \frac{1}{500} |d|^2.$$

$\square$

5.3.1 Local convexity and the area inequality

Theorem 5.5 (Local estimate near the modular torus). *Every X with $\text{sys}(X) > 381/200$ admits reduced coordinates in $\mathcal{R}$. Moreover,*

$$\mathcal{A}(x, y) - \mathcal{A}(3, 3) \geq \frac{27}{4000}((x - 3)^2 + (y - 3)^2), (x, y) \in \mathcal{R},$$

and

$$\nabla \mathcal{A}(3, 3) = 0, \text{Hess } \mathcal{A}(3, 3) > \frac{97}{4500}I.$$

In the centered logarithmic coordinates, as $p_2 \rightarrow 0$,

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) = \kappa p_2 + o(p_2), \kappa > \frac{97}{1000}.$$

In particular, $\mathcal{A}$ is strictly convex on a sufficiently small disk centered at $(3, 3)$.

Proof. Corollary 5.4 and Lemma 3.11 give the area inequality and its equality case. Since $\mathcal{A}$ is symmetric in the three traces, Lemma 2.6 gives

$$\nabla \mathcal{A}(3, 3) = 0.$$

Propositions 3.6 and 5.3 give

$$\text{Hess } \mathcal{A}(3, 3) > \left(\frac{23}{900} - \frac{1}{250} \right) I = \frac{97}{4500} I.$$

By Proposition 5.2, the Hessian is continuous on an open neighborhood of $(3, 3)$. Hence there exists $\varepsilon > 0$ such that

$$\text{Hess } \mathcal{A}(x, y) > \frac{1}{100} I \quad \text{for } (x - 3)^2 + (y - 3)^2 < \varepsilon^2.$$

This proves strict convexity on that disk.

In centered logarithmic coordinates, cyclic symmetry makes the quadratic term a multiple κp_2 . At the modular point, $d(u, v, w) = (dx, dy, -dx - dy)/3$, so this term has Hessian $\frac{2\kappa}{9} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in (x, y) . Its smaller eigenvalue is $2\kappa/9$; the preceding Hessian bound therefore gives $\kappa > 97/1000$. Finally, Lemma 2.4 places every torus with $\text{sys}(X) > 381/200$ in $\mathcal{R}$. $\square$

Remark 5.6. Figure 8 numerically illustrates the area above its modular value on the disk of radius $1/32$ centered at $(3, 3)$, with vertical scale enlarged by 10^5 . The graph suggests strict convexity throughout this disk. Our analytic proof establishes it on a sufficiently small neighborhood of the modular point.

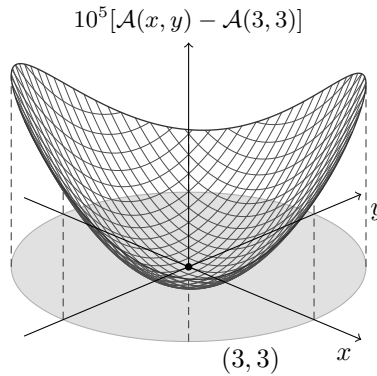


Figure 8: Area above its modular value.

6 Region C: comparison when the systole is short

We base the fan construction from Subsection 3.2 on a systole. After establishing its area bounds and cusp growth, we prove the comparison for region C successively for systole at most $2/3$, $6/5$, and $7/5$.

6.1 Area bounds from curves around a systole

Mirzakhani's cusp estimate implies $\mathcal{A}(X) \rightarrow \infty$ as $\text{sys}(X) \rightarrow 0$ [Mir08a, Proposition 3.6]. To join this region to our polygon comparison, we need an explicit cutoff. Twisting a curve about a systole gives vertices of an inscribed polygon and hence a lower area bound. We refine this bound in Subsections 6.2 and 6.3 to prove the following theorem.

Theorem 6.1 (Short-systole estimate). *If $\text{sys}(X) \leq 7/5$, then*

$$\mathcal{A}(X) > \frac{893}{1000} > \mathcal{A}(X_{\text{mod}}),$$

where $\mathcal{A}(X_{\text{mod}}) < 223/250$.

The recurrence from Subsection 3.2 can be solved explicitly, expressing every fan trace in terms of the base length and one twist parameter.

Lemma 6.2 (Trace formula for a fan). *Let γ be a simple class of length $2a$ and let γ' be a Farey neighbor. After replacing γ' by $\gamma' + k\gamma$, there is $|\theta| \leq a/2$ such that, for every $n \in \mathbb{Z}$,*

$$x_{\gamma' + n\gamma} = x_n(a, \theta) := 2 \coth a \cosh \left(\left(n + \frac{1}{2} \right) a + \theta \right).$$

Proof. The recurrence (30), with $x = 2 \cosh a$, has general solution $x_n = Ae^{na} + Be^{-na}$. Positivity for all $n \in \mathbb{Z}$ gives $A, B \geq 0$, and the Markoff identity for $(2 \cosh a, x_n, x_{n+1})$ gives $AB = x^2/(x^2 - 4) = \coth^2 a$. Thus $x_n = 2 \coth a \cosh(na + a/2 + \theta)$ for some real θ . Replacing γ' by $\gamma' + k\gamma$ changes θ by ka , so we may arrange $|\theta| \leq a/2$. $\square$

From now on, take γ to be a systole. Consecutive fan classes, and each fan class paired with γ , have absolute determinant one. Including the opposite vertices and closing at $\pm\gamma/(2a)$ gives a centrally symmetric inscribed polygon. Its area is a sum of reciprocal length products, as in (31). Upper bounds for its lengths give a lower area bound; further Farey insertions increase the area by (10).

Proposition 6.3 (A lower bound independent of the twist). *For $n \geq 0$, put*

$$l_a(t) = 2 \operatorname{arccosh}(\coth a \cosh t), c_n = l_a(na), m_n = \frac{c_n + c_{n+1}}{2}. \quad (82)$$

For every nonnegative integer M ,

$$\mathcal{A}(X) \geq \frac{1}{m_0^2} + 2 \sum_{n=0}^{M-1} \frac{1}{m_n m_{n+1}} + \frac{1}{am_M}. \quad (83)$$

The sum is empty when $M = 0$. These lower bounds increase strictly with M and converge to

$$\frac{1}{m_0^2} + 2 \sum_{n=0}^{\infty} \frac{1}{m_n m_{n+1}}.$$

Proof. Convexity bounds each length above by the line joining two consecutive endpoint values. After this substitution, we pair the area terms with twist parameters θ and $-\theta$. Each pair is smallest at $\theta = 0$, giving a bound independent of the twist. Set $k = \coth a$. Direct differentiation gives

$$l_a''(t) = \frac{2k(k^2 - 1) \cosh t}{(k^2 \cosh^2 t - 1)^{3/2}} > 0. \quad (84)$$

Write $s = 1/2 + \theta/a \in [0, 1]$, and put

$$p_n(s) = (1 - s)c_n + sc_{n+1}.$$

Convexity gives

$$l_a((n + s)a) \leq p_n(s).$$

Consider the finite fan with endpoint parameters $-(M+1-s)a$ and $(M+s)a$, closed by the two points $\pm\gamma/(2a)$. Its area is

$$\begin{aligned} & \frac{1}{l_a(sa)l_a((1-s)a)} \\ & + \sum_{n=0}^{M-1} \left(\frac{1}{l_a((n+s)a)l_a((n+1+s)a)} + \frac{1}{l_a((n+1-s)a)l_a((n+2-s)a)} \right) \\ & + \frac{1}{2a} \left(\frac{1}{l_a((M+s)a)} + \frac{1}{l_a((M+1-s)a)} \right). \end{aligned}$$

Replace $l_a((n+s)a)$ by its upper bound $p_n(s)$, and likewise for $1-s$. This gives a lower bound for the area. Since $p_0(s) + p_0(1-s) = 2m_0$, its central term is at least m_0^{-2} . For positive affine functions $u(s), v(s)$, direct differentiation gives

$$\frac{d^2}{ds^2} \frac{1}{uv} = \frac{2}{uv} \left(\left(\frac{u'}{u} \right)^2 + \left(\frac{v'}{v} \right)^2 + \frac{u'v'}{u v} \right) \geq 0. \quad (85)$$

By (85), the middle terms at s and $1-s$ have sum at least twice their value at $s = 1/2$, namely $2/(m_n m_{n+1})$. Convexity of the reciprocal gives at least $1/(am_M)$ for the two terms from edges ending at $\pm\gamma/(2a)$. This proves the bound.

For $t \geq 0$ we have $0 \leq l'_a(t) < 2$, so $0 < m_{M+1} - m_M < 2a$. The increment of the lower bound is exactly

$$\frac{2a - (m_{M+1} - m_M)}{am_M m_{M+1}} > 0.$$

Finally $m_n = 2na + O_a(1)$, which proves convergence. $\square$

6.1.1 Two-sided bounds and growth at the cusp

The same twisting curves give an upper bound as well: their limiting directions determine a diamond containing the norm ball. The full fan then approximates its area closely enough to determine the first correction in Theorem 1.3.

Proposition 6.4. *Every X satisfies*

$$0 \leq 1 - s_X \log \coth(s_X/2) \mathcal{A}(X) \leq \frac{\pi^2}{12 \log^2 \coth(s_X/2)} + \frac{s_X^2 \cosh(s_X/2)}{32 \log \coth(s_X/2)}.$$

The cusp expansion in Theorem 1.3 holds uniformly in the twist.

Proof. Put $a = s_X/2$ and $k = \coth a$, and use the classes and parameter θ from Lemma 6.2. Their lengths are $L_n = l_a((n+1/2)a + \theta)$. As $n \rightarrow \pm\infty$,

$$L_n - 2a|n| \rightarrow 2 \log k \pm (a + 2\theta).$$

For $q > 0$ and $n \in \mathbb{Z}$, the triangle inequality gives $\|p\gamma + q\gamma'\|_X \geq qL_n - 2a|qn - p|$. Taking the two limits, and using central symmetry for $q < 0$, gives

$$\|p\gamma + q\gamma'\|_X \geq |2ap + (a + 2\theta)q| + 2 \log k |q|.$$

The diamond defined by the right-hand side has area $1/(2a \log k)$, which proves the upper bound for $\mathcal{A}(X)$.

Let $\mathcal{A}_{\text{fan}}$ denote the area of the full inscribed fan, and let $\tilde{l}_a$ interpolate l_a affinely on the grid $a\mathbb{Z} + a/2 + \theta$. The area formula and integration on each grid interval give

$$\mathcal{A}_{\text{fan}} = \sum_{n \in \mathbb{Z}} \frac{1}{L_n L_{n+1}} = \frac{1}{a} \int_{\mathbb{R}} \frac{dt}{\tilde{l}_a(t)^2} \leq \mathcal{A}(X).$$

The closing edges at $\pm\gamma/(2a)$ tend to zero as the finite fans exhaust this polygon. Differentiating (82) gives $0 < l_a''(t) \leq 2 \cosh a$, and therefore

$$0 \leq \tilde{l}_a(t) - l_a(t) \leq \frac{a^2 \cosh a}{4}.$$

For $g(t) = \log(1 + e^{-2t})$, $t \geq 0$, elementary bounds for arccosh give

$$2(\log k + |t|) \leq l_a(t) \leq 2(\log k + |t| + g(|t|)).$$

Using $(1+z)^{-2} \geq 1-2z$ for $z \geq 0$ in the integral above, together with

$$\int_0^\infty g(t) dt = \frac{1}{2} \sum_{j \geq 1} \frac{(-1)^{j+1}}{j^2} = \frac{\pi^2}{24},$$

gives

$$\mathcal{A}_{\text{fan}} \geq \frac{1}{2a \log k} \left(1 - \frac{\pi^2}{12 \log^2 k} - \frac{a^2 \cosh a}{8 \log k} \right).$$

This proves the stated two-sided bound.

For the correction term we must also control the area missed by the fan. For sufficiently small a , all fan lengths are at least $l_a(0) > 4$. The opposite curve is the short systole, so every outward insertion uses the larger Markoff root; its trace exceeds both parent traces, and this persists under insertion. By Proposition 3.2, every length deficit is at most $11e^{-2l_a(0)}$. Farey induction shows that subtracting this amount from each endpoint length gives a lower linear bound for the norm in that sector: at a class $pu + qv$, the accumulated loss is at most $(p+q-1)11e^{-2l_a(0)}$. Homogeneity and continuity extend this bound to the whole sector. Comparing sector areas therefore gives

$$\mathcal{A}_{\text{fan}} \leq \mathcal{A}(X) \leq \left(1 - \frac{11e^{-2l_a(0)}}{l_a(0)} \right)^{-2} \mathcal{A}_{\text{fan}}.$$

Since $e^{-2l_a(0)} = O(a^4)$, the omitted area is $O(a^3/\log^2 k)$. The interpolation estimate also gives

$$0 \leq \frac{1}{a} \int_{\mathbb{R}} \frac{dt}{l_a(t)^2} - \mathcal{A}_{\text{fan}} \leq \frac{a \cosh a}{16 \log^2 k}.$$

Finally, $2 \log(2k \cosh t) - l_a(t) = O(a^2 \operatorname{sech}^2 t)$, uniformly in t . Hence

$$\mathcal{A}(X) = \frac{1}{2a} \int_0^\infty \frac{dt}{(\log k + t + g(t))^2} + O(a/\log^2 k).$$

Expanding in $g(t)$ first, whose first moment and square are integrable, yields

$$\begin{aligned} \int_0^\infty \frac{dt}{(\log k + t + g(t))^2} &= \frac{1}{\log k} - 2 \int_0^\infty \frac{g(t)}{(\log k + t)^3} dt + O(\log^{-4} k) \\ &= \frac{1}{\log k} - \frac{\pi^2}{12 \log^3 k} + O(\log^{-4} k). \end{aligned}$$

Since $\log k = \log(1/a) + O(a^2)$, this is the claimed expansion. Subtracting the fixed value $\mathcal{A}(X_{\text{mod}})$ does not change it. All error bounds are independent of θ . $\square$

6.1.2 Area bound for systole at most $2/3$

We first bound the lengths of all fan curves when the fan is based on a systole. This will also let us apply Proposition 3.1 to the area added by an insertion between two fan classes.

Lemma 6.5 (A lower bound for fan lengths). *Suppose the fan is based on a systole of length $2a$. Every length of a curve in this fan, written as*

$$l_a(t) = 2 \operatorname{arccosh}(\coth a \cosh t)$$

is at least $2 \operatorname{arsinh} 1$.

Proof. If $a \leq \operatorname{arsinh} 1$, then $\coth a \geq \sqrt{2}$, so $l_a(t) \geq 2 \operatorname{arccosh} \sqrt{2} = 2 \operatorname{arsinh} 1$. If $a \geq \operatorname{arsinh} 1$, the systolic assumption gives $l_a(t) \geq 2a \geq 2 \operatorname{arsinh} 1$ for every curve in the fan. $\square$

Proposition 6.6 (Area bound for systole at most $2/3$). *If $\operatorname{sys}(X) \leq 2/3$, then*

$$\mathcal{A}(X) > \frac{893}{1000}.$$

Proof. Here the smallest fan polygon already suffices. We bound its mean length m_0 above, then show that the resulting area bound is smallest at $a = 1/3$. Put $a = \operatorname{sys}(X)/2 \leq 1/3$. Taking $M = 0$ in Proposition 6.3 gives

$$\mathcal{A}(X) \geq \frac{1}{am_0} + \frac{1}{m_0^2}.$$

To bound $m_0 = (c_0 + c_1)/2$ above, we use

$$\coth x \leq \frac{1}{x} + \frac{x}{3}, \log \cosh x \leq \frac{x^2}{2} (x > 0).$$

For the first, the function $(1 + x^2/3) \sinh x - x \cosh x$ vanishes at zero and has derivative $x(x \cosh x - \sinh x)/3 \geq 0$. The second follows by integrating $\tanh x \leq x$. We also use

$$2 \operatorname{arccosh} z \leq 2 \log(2z) - \frac{1}{2z^2} (z > 1).$$

To see this, apply $\sqrt{1-u} \leq 1 - u/2$ and $\log(1 - u/4) \leq -u/4$ with $u = z^{-2}$.

Since $c_0 = 2 \log \coth(a/2)$, these estimates imply

$$c_0 \leq 2 \log(2/a) + \frac{a^2}{6}.$$

For $z = \coth a \cosh a$ we have

$$z \leq a^{-1} \exp(5a^2/6).$$

Therefore, using $e^{-x} \geq 1 - x$,

$$c_1 \leq 2 \log(2/a) + \frac{5a^2}{3} - \frac{a^2}{2} e^{-5a^2/3} \leq 2 \log(2/a) + \frac{7a^2}{6} + \frac{5a^4}{6}.$$

Consequently

$$m_0 \leq b(a) := 2 \log(2/a) + \frac{2a^2}{3} + \frac{5a^4}{12}.$$

The function

$$f(a) = \frac{1}{ab(a)} + \frac{1}{b(a)^2}$$

is decreasing on $(0, 1/3]$. Indeed $b > 3$, $ab' \geq -2$, and $2a^2b' \geq -4a$, while

$$f'(a) = -\frac{b(b + ab') + 2a^2b'}{a^2b^3}.$$

Its numerator satisfies $b(b + ab') + 2a^2b' \geq b(b - 2) - 4a > 3 - \frac{4}{3} > 0$, where $b > 3$ follows from $b(a) > 2 \log(2/a) \geq 2 \log(6) > 3$, thus $\mathcal{A}(X) \geq f(1/3) > \frac{893}{1000}$. $\square$

6.2 Area bounds for systole at most $6/5$

We extend the short-systole estimate from $2/3$ to $6/5$ in one step. Convexity in a bounds the fan lengths above by the lines joining their values at $a = 1/3$ and $a = 3/5$. Inserting one vertex between the two central fan classes adds enough area to complete the comparison.

Lemma 6.7 (Convexity in the systole and twist parameters). *For every real k , the function*

$$(a, \theta) \mapsto l_a(ka + \theta) = 2 \operatorname{arccosh}(\coth a \cosh(ka + \theta))$$

is convex on $(0, \infty) \times \mathbb{R}$. In particular, for fixed k ,

$$\frac{d^2}{da^2} l_a(ka) \geq \frac{2}{\sinh^2 a}.$$

For $k \geq 1$, the function $a \mapsto l_a''(ka)$ is decreasing, where the primes in l_a'' refer to the second argument.

Proof. Write $b(a) = \log \coth(a/2)$ and $h(b, t) = \operatorname{arccosh}(\cosh b \cosh t)$, so $l_a(t) = 2h(b(a), t)$. We prove that h is convex and increasing in b ; convexity of $b(a)$ will then give the required convexity by the chain rule. If $J = \sinh h$, direct differentiation gives

$$\operatorname{Hess}_{b,t} h = \frac{1}{J^3} \begin{pmatrix} \cosh b \cosh t \sinh^2 t & -\sinh b \sinh t \\ -\sinh b \sinh t & \cosh b \cosh t \sinh^2 b \end{pmatrix}.$$

Its diagonal entries are nonnegative and its determinant is $\sinh^2 b \sinh^2 t / J^4 \geq 0$. Also $h_b > 0$, $b' = -1/\sinh a$ and $b'' = \cosh a / \sinh^2 a > 0$. The second-order chain rule therefore proves convexity. Since

$$h_b = \frac{\sinh b \cosh t}{J} \geq \tanh b = \frac{1}{\cosh a},$$

the second derivative of $l_a(ka) = 2h(b(a), ka)$ is at least $2h_b b'' \geq 2/\sinh^2 a$.

For the last assertion, write

$$g_a(t) = l_a''(t) = \frac{2 \cosh a \cosh t}{(1 + \cosh^2 a \sinh^2 t)^{3/2}}. \quad (86)$$

For $u = \tanh a$, $v = \tanh(ka)$ and $d = 1 - u^2 + u^2 v^2 \leq 1$, logarithmic differentiation gives

$$\frac{d}{da} \log g_a(ka) = \frac{u(d - 3v^2) + kv(d - 3)}{d} \leq \frac{u - 2kv}{d} < 0,$$

because $k \geq 1$ and $v \geq u > 0$. $\square$

One central insertion improves the finite fan-polygon bound. Its two parent lengths $L_{\pm} = l_a(a/2 \pm \theta)$ have mean at most $m_0(a)$, by convexity of l_a . The opposite trace is systolic, so the inserted trace is the larger root in (14). Lemma 6.5 puts these lengths, and the pair obtained by replacing both by their mean, in $\mathcal{E}$. By Proposition 3.1, symmetry and convexity show that this replacement cannot increase $\mathcal{D}$. Increasing the common length to m_0 cannot increase it either. Thus (17) gives

$$\mathcal{D}(L_-, L_+) \geq \mathcal{D}(m_0, m_0) > \frac{4e^{-2m_0}}{m_0^3}. \quad (87)$$

This holds for any fan based on a systole and will also be used in Subsection 6.3.

Proposition 6.8 (Area bound for systole at most $6/5$). *If $\operatorname{sys}(X) \leq 6/5$, then*

$$\mathcal{A}(X) > \frac{893}{1000}.$$

Proof. Proposition 6.6 covers systole at most $2/3$. For the remaining range, put $L = 1/3$, $U = 3/5$ and $a = \operatorname{sys}(X)/2 \in [L, U]$. Let $b_n(a)$ be affine with the following endpoint values:

n	$b_n(L)$	$b_n(U)$
0	73189/20000	13313/5000
1	38763/10000	66299/20000
2	8537/2000	86537/20000
3	95591/20000	109373/20000
4	53619/10000	6651/1000

(88)

These values bound m_n at the endpoints, as verified in Appendix A3.3. Convexity of m_n (Lemma 6.7) therefore gives $m_n(a) < b_n(a)$ throughout $[L, U]$.

Since $m_0 < b_0$, (87) bounds the added area below by $4e^{-2b_0}/b_0^3$. Adding this to the finite fan-polygon bound in the proof of Proposition 6.3, with $M = 4$, gives

$$\mathcal{A}(X) > Q(a) := \frac{1}{b_0(a)^2} + 2 \sum_{n=0}^3 \frac{1}{b_n(a)b_{n+1}(a)} + \frac{1}{ab_4(a)} + \frac{4e^{-2b_0(a)}}{b_0(a)^3}.$$

We need only one tangent line to bound Q on the whole interval. Each term involving reciprocal lengths is convex by (85); its last term has convex logarithm $\log 4 - 2b_0 - 3 \log b_0$, hence is convex as well. At $a_* = 279/500$, exact rational bounds give

$$Q(a_*) > \frac{8931}{10000}, -\frac{1}{1000} < Q'(a_*) < 0. \quad (89)$$

The evaluation is given in Appendix A3.3. Since a convex function lies above its tangent, for every $a \in [L, U]$ we have

$$Q(a) \geq Q(a_*) + Q'(a_*)(a - a_*) > \frac{8931}{10000} - \frac{1}{1000} \left(\frac{3}{5} - \frac{279}{500} \right) > \frac{893}{1000}.$$

This proves the assertion. $\square$

6.3 Area bounds for systole at most $7/5$

To reach systole $7/5$, we improve the polygon bound by considering pairs of lengths with twist parameters θ and $-\theta$. We estimate both their mean and their difference, then add one Farey vertex between the two central fan classes.

For consecutive endpoint lengths, write

$$c_n(a) = l_a(na), m_n(a) = \frac{c_n(a) + c_{n+1}(a)}{2}, \Delta_n(a) = \frac{c_{n+1}(a) - c_n(a)}{2}.$$

Thus m_n is their mean and Δ_n is half their difference. We compare these values with the lengths at twist parameters θ and $-\theta$.

Lemma 6.9 (Lengths at opposite twist parameters). *Let $0 \leq \theta \leq a/2$, set $\tau = 2\theta/a$, and put*

$$\begin{aligned} \mu_n &= \frac{l_a((n + \frac{1}{2})a + \theta) + l_a((n + \frac{1}{2})a - \theta)}{2}, \\ d_n &= \frac{l_a((n + \frac{1}{2})a + \theta) - l_a((n + \frac{1}{2})a - \theta)}{2}. \end{aligned}$$

Then $d_n \geq \Delta_n(a)\tau$. Moreover $\Delta_n(a)/a$ is increasing in a . If $\kappa_n \leq g_a(t)$ on $na \leq t \leq (n+1)a$, then

$$\mu_n \leq m_n(a) - \frac{\kappa_n a^2}{8}(1 - \tau^2). \quad (90)$$

The same mean bound holds with

$$\kappa_0 = \frac{33}{20}, \kappa_n = g_a((n + \frac{1}{2})a) \text{ for } n \geq 1$$

whenever $2/3 \leq a \leq 7/10$.

Proof. Dividing the half-difference by θ gives the average of l'_a on the interval between the two arguments. We first bound this average to estimate d_n , then use l''_a to estimate how far μ_n lies below the line joining the endpoint lengths. The function l'_a is increasing and concave on $[0, \infty)$, since $g_a > 0$ and $g'_a(t) < 0$ for $t > 0$. For a concave function, the average on an interval is at least the average of its endpoint values. Differentiating the interval average with respect to its half-length therefore shows that the average decreases as the interval expands about its center. Apply this to l'_a on the intervals centered at $(n + 1/2)a$, with half-lengths θ and $a/2$. It gives $d_n/\theta \geq 2\Delta_n/a$, including the limit at $\theta = 0$. Furthermore,

$$\frac{\Delta_n(a)}{a} = \int_n^{n+1} \frac{\sinh(av)}{\sqrt{\sinh^2(av) + \operatorname{sech}^2 a}} dv$$

is increasing, because the integrand is increasing in a for $v \geq 0$.

If $l''_a \geq \kappa_n$, subtract the quadratic $\kappa_n t^2/2$ and use the fact that the resulting convex function lies below the line joining its endpoint values. Averaging at the two arguments equally spaced about $(n + 1/2)a$ proves the bound for μ_n .

For the stronger mean bounds, we improve the lower bound for $g_a = l''_a$ before integrating twice. Put $w = \sinh^2 a$ and $\xi = (1 + \cosh^2 a \sinh^2 t)^{-1}$. Direct differentiation gives

$$\frac{g''_a(t)}{g_a(t)} = 4 + (12w - 6)\xi - 15w\xi^2. \quad (91)$$

For $n \geq 1$, we use the tangent to g_a at the midpoint of $[na, (n+1)a]$. It lies below g_a because g_a is convex for $t \geq a$. Indeed $w > 1/2$, and $0 < \xi \leq (1+w+w^2)^{-1}$ on this range. The quadratic in (91) is concave in ξ , so it suffices to check its two endpoint values. At zero its value is 4; at the other endpoint, its sign is that of

$$4w^4 + 20w^3 + 18w^2 - w - 2 > 18w^2 - w - 2 \geq 2.$$

Integrating the tangent twice, with the endpoint lengths fixed, gives a cubic upper bound for l_a . Averaging its values at $(n+1/2)a \pm \theta$ gives (90) with $\kappa_n = g_a((n+1/2)a)$.

For the central interval $[0, a]$, a quadratic lower bound for g_a is enough. On $0 \leq \xi \leq 1$, the same concave quadratic is at least $-2 - 3w$. Since $0 < g_a(t) \leq 2 \cosh a$, we obtain

$$g_a''(t) \geq -2 \cosh a (3 \cosh^2 a - 1).$$

Using $g_a'(0) = 0$ and integrating twice gives

$$g_a(t) \geq 2 \cosh a - \cosh a (3 \cosh^2 a - 1) t^2.$$

Integrate this quadratic twice, again matching the endpoint lengths. The resulting quartic lies above l_a . Averaging at $a/2 - \theta$ and $a/2 + \theta$ gives

$$\mu_0 \leq m_0 - \frac{a^2(1-\tau^2)}{8} \left(2 \cosh a - \frac{a^2(7+\tau^2)}{24} \cosh a (3 \cosh^2 a - 1) \right).$$

Here $0 \leq \tau \leq 1$, so the expression in parentheses is at least

$$2 \cosh a - \frac{a^2}{3} \cosh a (3 \cosh^2 a - 1) > \frac{246}{100} - \frac{49}{300} \frac{63}{50} \left(3 \left(\frac{63}{50} \right)^2 - 1 \right) > \frac{33}{20}.$$

We used $123/100 < \cosh a < 63/50$ on $[2/3, 7/10]$; the endpoint comparisons are recorded in (109). This proves the bound for μ_0 . $\square$

Substituting these bounds for the means and differences into the polygon area gives a lower bound that is quadratic in the twist parameter θ . The next lemma records this bound.

Lemma 6.10 (An area bound using pairs of fan lengths). *Suppose the mean bounds in Lemma 6.9 hold with nonnegative constants $\kappa_0, \dots, \kappa_M$. Set*

$$\tilde{m}_n = m_n - \frac{\kappa_n a^2}{8},$$

and assume $\tilde{m}_n > 0$. Define

$$F(a) = \frac{1}{\tilde{m}_0^2} + 2 \sum_{n=0}^{M-1} \frac{1}{\tilde{m}_n \tilde{m}_{n+1}} + \frac{1}{a \tilde{m}_M}, \quad (92)$$

$$\begin{aligned} N(a) &= \frac{\kappa_0}{4 \tilde{m}_0^3} + \frac{1}{4} \sum_{n=0}^{M-1} \frac{\kappa_n / \tilde{m}_n + \kappa_{n+1} / \tilde{m}_{n+1}}{\tilde{m}_n \tilde{m}_{n+1}} + \frac{\kappa_M}{8 a \tilde{m}_M^2}, \\ P(a) &= \frac{(\Delta_0/a)^2}{m_0^4} \\ &\quad + 2 \sum_{n=0}^{M-1} \left(\frac{(\Delta_n/a)^2}{m_n^3 m_{n+1}} + \frac{(\Delta_n/a)(\Delta_{n+1}/a)}{m_n^2 m_{n+1}^2} + \frac{(\Delta_{n+1}/a)^2}{m_n m_{n+1}^3} \right) \\ &\quad + \frac{(\Delta_M/a)^2}{a m_M^3}. \end{aligned} \quad (93)$$

Then the finite fan polygon, before further insertions and with end vertices joined to $\pm\gamma/(2a)$, has area at least

$$F(a) + a^2 \tau^2 (P(a) - N(a)).$$

If additionally $m_0 < 8/3$, one insertion in the central sector gives

$$\mathcal{A}(X) > F(a) + a^2 \tau^2 (P(a) - N(a)) + \frac{1}{1000}.$$

Proof. For the terms paired at θ and $-\theta$, increasing the mean length lowers the area, while increasing the half-difference raises it. The possible loss gives $N(a)$; the guaranteed increase gives $P(a)$. By (90), the means satisfy $\mu_n \leq \bar{m}_n := \tilde{m}_n + \kappa_n a^2 \tau^2 / 8$. For the central term, this bound and $d_0 \geq \Delta_0 \tau$ give

$$\frac{1}{\mu_0^2 - d_0^2} \geq \frac{1}{\bar{m}_0^2 - \Delta_0^2 \tau^2} \geq \frac{1}{\tilde{m}_0^2} - \frac{\kappa_0 a^2}{4\tilde{m}_0^3} \tau^2 + \frac{\Delta_0^2}{\tilde{m}_0^4} \tau^2.$$

The last inequality uses the tangent bound for $(\tilde{m}_0 + \kappa_0 a^2 \tau^2 / 8)^{-2}$ and the elementary inequality $(1-t)^{-1} \geq 1+t$ for $0 \leq t < 1$.

For the two side terms corresponding to θ and $-\theta$, their sum decreases with either mean and increases with either nonnegative half-difference. These facts follow by differentiating the two reciprocal products. Replace the means by $\bar{m}_n, \bar{m}_{n+1}$ and the differences by $\Delta_n \tau, \Delta_{n+1} \tau$. The substituted lengths remain positive: in each pair, the smaller one is at least the smaller actual length. For $x, y \in [0, 1)$,

$$\frac{1}{2} \left(\frac{1}{(1-x)(1-y)} + \frac{1}{(1+x)(1+y)} \right) \geq 1 + x^2 + xy + y^2.$$

Combining the fractions gives $(1+xy)/((1-x^2)(1-y^2))$; apply $(1-t)^{-1} \geq 1+t$ to both reciprocal factors. Since $\bar{m}_n \leq m_n$, this gives the three positive terms in P . The tangent inequality

$$\frac{1}{\bar{m}_n \bar{m}_{n+1}} \geq \frac{1}{\tilde{m}_n \tilde{m}_{n+1}} - \frac{a^2 \kappa_n / \tilde{m}_n + \kappa_{n+1} / \tilde{m}_{n+1}}{8 \tilde{m}_n \tilde{m}_{n+1}} \tau^2$$

gives the corresponding negative terms. For the two edges ending at $\pm \gamma / (2a)$, use

$$\frac{\bar{m}_M}{a(\bar{m}_M^2 - \Delta_M^2 \tau^2)} \geq \frac{1}{a\tilde{m}_M} - \frac{\kappa_M a}{8\tilde{m}_M^2} \tau^2 + \frac{\Delta_M^2}{a\tilde{m}_M^3} \tau^2.$$

Adding these inequalities gives (92) and the coefficients in (93), proving the first assertion.

For the central insertion, (87) and $m_0 < 8/3$ give an added area greater than

$$\frac{27}{128} e^{-16/3} > \frac{1}{1000}.$$

The last elementary inequality follows from $e^{2/3} < 39/20$ and $(39/20)^8 < 3375/16$. $\square$

6.3.1 Applying the bound on two intervals

It remains to apply Lemma 6.10 for $3/5 \leq a \leq 7/10$. We use $M = 4$ and divide this range at $a = 2/3$, where the stronger mean bound in Lemma 6.9 applies. We will prove $P(a) > N(a)$, so the twist term is nonnegative, and $F(a) + 1/1000 > 893/1000$. We first choose the constants κ_n for the bounds on the means, then prove these two inequalities on each interval. The following rows give $[L, U]$ and the constants $1000\kappa_n$:

row	L	U	0	1	2	3	4
1	3/5	2/3	1276	365	96	25	6
2	2/3	7/10	1650	621	153	37	9

(94)

For row 1, $\kappa_n < g_U((n+1)U)$, so the first mean bound applies throughout the interval by monotonicity of $a \mapsto g_a(ka)$. For row 2 and $n \geq 1$, use $\kappa_n < g_U((n+1/2)U)$ and the stronger mean bound. Its central bound is $\kappa_0 = 33/20$ from Lemma 6.9.

To bound N above, we bound its length denominators below. To bound P below, we bound its lengths above and Δ_n/a below. The endpoint calculations in Appendix A3.4 give

row	$N(a) <$	$P(a) >$
1	9/200	59/1000
2	131/2000	667/10000

(95)

Thus $P(a) > N(a)$ on both intervals. The same length bounds give $m_0 < 8/3$, so the central insertion adds more than $1/1000$.

We now bound the remaining term $F(a)$ using affine upper bounds for its lengths. By Lemma 6.7, every $\tilde{m}_n(a) = m_n(a) - \kappa_n a^2/8$ is convex on these intervals, since

$$\tilde{m}_n''(a) \geq \frac{2}{\sinh^2 a} - \frac{\kappa_n}{4} > 0.$$

Let $\hat{m}_n$ be the affine function joining the rational upper bounds for $\tilde{m}_n(L)$ and $\tilde{m}_n(U)$ specified in Appendix A3.4. Convexity of $\tilde{m}_n$ gives $\tilde{m}_n \leq \hat{m}_n$ throughout $[L, U]$. Hence

$$F(a) \geq Q(a) := \frac{1}{\hat{m}_0(a)^2} + 2 \sum_{n=0}^3 \frac{1}{\hat{m}_n(a)\hat{m}_{n+1}(a)} + \frac{1}{a\hat{m}_4(a)}. \quad (96)$$

The function Q is convex by (85). Its endpoint value and derivative satisfy

row	$Q(U) >$	$Q'(U) <$
1	2231/2500	0
2	8929/10000	0

(97)

The rational evaluations are given in Appendix A3.4. Convexity gives $Q(a) \geq Q(U) + Q'(U)(a - U) \geq Q(U)$ for $a \in [L, U]$, since both $Q'(U)$ and $a - U$ are nonpositive.

Proof of Theorem 6.1. For $a \leq 3/5$, Proposition 6.8 gives $\mathcal{A}(X) > 893/1000$. For $3/5 \leq a \leq 2/3$, (95) gives $P - N > 0$, and we have $m_0 < 8/3$ as shown above. Combining Lemma 6.10 with the first row of (97) gives

$$\mathcal{A}(X) > \frac{2231}{2500} + \frac{1}{1000} > \frac{893}{1000}.$$

For $2/3 \leq a \leq 7/10$, the second row of (95) also gives $P - N > 0$. The second row of (97) and Lemma 6.10 therefore give

$$\mathcal{A}(X) > \frac{8929}{10000} + \frac{1}{1000} > \frac{893}{1000}.$$

This proves the claimed strict inequality for every $\text{sys}(X) = 2a \leq 7/5$. The modular upper bound is Proposition 3.5. $\square$

7 Proofs of the main theorems

We first combine the regional comparisons into an explicit lower bound, which proves the minimum theorem. We then establish the global comparison using the local and cusp estimates.

Proposition 7.1 (Explicit regional estimate). *Every nonmodular torus X satisfies*

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \begin{cases} p_2/80, & \text{if } s_X \geq 7/5, \\ 1/1000, & \text{if } s_X < 7/5. \end{cases}$$

Proof. In region C , Theorem 6.1 and Proposition 3.5 give

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{893}{1000} - \frac{223}{250} = \frac{1}{1000}.$$

In region B , Lemma 3.11 and Corollary 5.4 give $\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > 27p_2/400 > p_2/80$.

In region A , (72) gives $p_2 > 1/12800$. If $t \leq 1/20$, the local polygon bound in Proposition 3.10 gives

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{7}{80}p_2 - \frac{1}{700000} > \left(\frac{7}{80} - \frac{12800}{700000}\right)p_2 > \frac{1}{80}p_2.$$

If $t > 1/20$, then $p_2 > 3/800$, and (73) gives

$$\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) > \frac{19}{1000}p_2 - \frac{1}{700000} > \left(\frac{19}{1000} - \frac{800}{3 \cdot 700000}\right)p_2 > \frac{1}{80}p_2.$$

These cases cover every nonmodular torus. $\square$

Proof of Theorem 1.1. For a nonmodular torus, $p_2 > 0$, so Proposition 7.1 gives $\mathcal{A}(X) > \mathcal{A}(X_{\text{mod}})$. Equality holds at the modular torus. $\square$

Proof of Theorem 1.3. Proposition 6.4 proves the cusp expansion. Near the modular point, Theorem 5.5 gives $\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}}) = \kappa p_2 + o(p_2)$ with $\kappa > 0$, so the required quotient tends to κ . At the other end, the discriminant and Proposition 2.1 give

$$\frac{2x}{\sqrt{x^2 - 4}} \leq y \leq \frac{x}{\sqrt{x - 2}}, 1 \leq \frac{z}{y} \leq \frac{x}{2}.$$

With $x = 2 \cosh(s_X/2)$, these imply $y/x = (2/s_X)(1 + O(s_X^2))$ and $z/y = 1 + O(s_X^2)$. Since

$$\log(y/x) \leq \sqrt{3p_2/2} \leq \log(z/x),$$

we obtain $\sqrt{3p_2/2} = \log(2/s_X) + O(s_X^2)$. The cusp expansion consequently gives

$$\frac{\mathcal{A}(X) - \mathcal{A}(X_{\text{mod}})}{p_2 e^{\sqrt{3p_2/2}/(1+p_2)^{3/2}}} \rightarrow \frac{1}{\sqrt{6}} \text{ as } s_X \rightarrow 0.$$

Finally, the displayed trace bounds make the reduced region with $s_X \geq \varepsilon$ compact for each $\varepsilon > 0$. Outside a fixed small neighborhood of the modular point, continuity and Theorem 1.1 bound this quotient above and below by positive constants. Together with the two limits, this proves the uniform bounds. $\square$

Appendix

A1 Elementary bounds for series remainders

The numerical comparisons use rational arithmetic and explicit bounds for the remaining series terms. We collect those bounds here. Appendix A2 applies them to lengths and Taylor coefficients; Appendix A3 records the remaining evaluations used in the main text.

For a nonnegative integer N and $0 \leq t < N + 2$,

$$0 \leq e^t - \sum_{j=0}^N \frac{t^j}{j!} \leq \frac{t^{N+1}}{(N+1)!} \frac{1}{1 - t/(N+2)}. \quad (98)$$

Indeed, after the first omitted term, the ratio of successive terms is at most $t/(N+2)$, so their sum is bounded by a geometric series.

The same argument applied to the even terms gives, for real t with $t^2 < (2N+3)(2N+4)$,

$$0 \leq \cosh t - \sum_{j=0}^N \frac{t^{2j}}{(2j)!} \leq \frac{t^{2N+2}}{(2N+2)!} \frac{1}{1 - t^2/((2N+3)(2N+4))}. \quad (99)$$

For example, $N = 8$ gives $\cosh(118/125) < 37/25$, the inequality used to bound L_0 in Lemma 2.3.

For the fan endpoint calculations, it is convenient to use fixed polynomials. Put

$$\begin{aligned} C(x) &= \sum_{j=0}^{12} \frac{x^{2j}}{(2j)!}, & S(x) &= \sum_{j=0}^{12} \frac{x^{2j+1}}{(2j+1)!}, \\ C^+(x) &= C(x) + \frac{2x^{26}}{26!}, & S^+(x) &= S(x) + \frac{2x^{27}}{27!}. \end{aligned}$$

Then, for $0 \leq x \leq 5$,

$$C(x) \leq \cosh x \leq C^+(x), S(x) \leq \sinh x \leq S^+(x). \quad (100)$$

The ratios after the first omitted terms are at most $25/(27 \cdot 28)$ and $25/(28 \cdot 29)$, respectively. Both are less than $1/2$, so twice the first omitted term bounds each remainder.

For logarithms, let $v \geq 1$ and put $u = (v - 1)/(v + 1)$. Then $0 \leq u < 1$, and

$$0 \leq \log v - 2 \sum_{j=0}^N \frac{u^{2j+1}}{2j+1} \leq \frac{2u^{2N+3}}{(2N+3)(1-u^2)}. \quad (101)$$

This follows from the positive series $\log v = 2 \sum_{j \geq 0} u^{2j+1}/(2j+1)$ by bounding the omitted denominators below by $2N+3$ and summing the remaining geometric series. For an arbitrary positive argument, first write it as $2^k v$ with $1 \leq v < 2$ and use the same bound at v and at 2. All finite sums and remainder bounds are rational when their arguments are rational.

A2 Length bounds and Taylor coefficients of the polygon area

The polygon calculations use $f(T) = 1/L(T)$, where $L(T) = 2 \operatorname{arccosh}(T/2)$. We give the modular length bounds and their refinements, an explicit evaluation of the quadratic term, the recursions and two-line calculation through degree eight, and the coefficient table used in the degree-nineteen remainder estimate.

The modular length bounds. For $t \in \{3, 6, 15, 39, 87, 102, 267\}$, enclose $\sqrt{t^2 - 4}$ between consecutive multiples of 10^{-6} by squaring. Substitute these bounds into $L(t) = 2 \log((t + \sqrt{t^2 - 4})/2)$ and apply (101) with $N = 11$, after reducing its arguments to $[1, 2]$ by powers of two. The resulting rational bounds give the numerical comparisons in Proposition 3.5 and Proposition 3.9.

Rational input bounds. Put $S_T = \sqrt{T^2 - 4}$. The following integers satisfy

$$\frac{R_T}{10^{20}} < S_T < \frac{R_T + 1}{10^{20}}, \quad \frac{J_T}{10^{20}} < L(T) < \frac{J_T + 1}{10^{20}}.$$

T	R_T	J_T
3	223606797749978969640	192484730023841378999
6	565685424949238019520	352549434807817210093
15	1486606874731850552261	540715166186280463466
39	3894868418830089310004	732580706919993661986
87	8697700845625814003528	893155194923016687644
102	10198039027185569660056	924975336509101048101
267	26699250925821870330699	1117446926139105944726

The square-root bounds follow by checking $R_T^2 < 10^{40}(T^2 - 4) < (R_T + 1)^2$. For the length bounds, refine the square-root intervals as in (102) below and apply (101) with $N = 99$ to $(T + S_T)/2$, first dividing by a power of two to reduce the argument to $[1, 2]$. Bound $\log 2$ by the same formula.

For the Taylor calculations through degree nineteen, use the same procedure with $D = 10^{100}$ and

$$K_T = \max\{k \in \mathbb{Z}_{\geq 0} : k^2 \leq D^2(T^2 - 4)\}.$$

Then

$$K_T/D \leq S_T \leq (K_T + 1)/D. \quad (102)$$

Apply (101), again with $N = 99$, to these endpoints to bound $L(T)$. Every subsequent operation uses interval arithmetic: addition adds endpoints, multiplication takes the minimum and maximum of the four endpoint products, and $[a, b]^{-1} = [1/b, 1/a]$ when $0 \notin [a, b]$. After an operation, an interval $[a, b]$ may be replaced by

$$\left[\frac{\lfloor Da \rfloor}{D}, \frac{\lfloor Db \rfloor}{D} \right].$$

Thus all endpoints are rational and every rounding is outward.

The quadratic coefficient as six contributions.

Lemma A2.1. *The coefficient in (35) satisfies*

$$c_2 > \frac{23}{200}.$$

Proof. Along $(u, v, w) = (t, 0, -t)$ the initial traces are

$$x(t) = e^{3t} + e^t + e^{-t}, \quad y(t) = e^{2t} + 1 + e^{-2t}, \quad z(t) = e^t + e^{-t} + e^{-3t}.$$

Their values, first derivatives, and second derivatives at zero are $(3, 3, 3)$, $(3, 0, -3)$, and $(11, 8, 11)$. For each of the six permutations, differentiate the trace recurrence:

$$\begin{aligned} T'_{n+1} &= x'T_n + xT'_n - T'_{n-1}, \\ T''_{n+1} &= x''T_n + 2x'T'_n + xT''_n - T''_{n-1}, \\ T'_* &= T'_1T_2 + T_1T'_2 - x', \\ T''_* &= T''_1T_2 + 2T'_1T'_2 + T_1T''_2 - x''. \end{aligned}$$

All values in these formulas are taken at $t = 0$. For a modular trace T , write

$$A_T = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_3} T'_\sigma(0)^2, \quad B_T = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_3} T''_\sigma(0),$$

where T_σ denotes the corresponding trace after permutation. For a consecutive pair put $D_{T,U} = \sum_{\sigma \in \mathfrak{S}_3} T'_\sigma(0)U'_\sigma(0)$. The derivative recurrences give

T	3	6	15	39	87	102	267
A_T	18	288	4446	57258	409014	648288	6705198
B_T	30	132	618	2622	7746	10236	37554

and the six values of $D_{T,U}$ displayed below. Differentiating $f(T) = 1/L(T)$ gives

$$f'(T) = -\frac{2}{S_T L(T)^2}, \quad f''(T) = \frac{2T}{S_T^3 L(T)^2} + \frac{8}{S_T^2 L(T)^3}.$$

The contribution of the pair (T, U) , summed over all permutations, to the coefficient of t^2 is therefore

$$\begin{aligned} Q_{T,U} &= A_T f''(T) f(U) + A_U f(T) f''(U) + D_{T,U} f'(T) f'(U) \\ &\quad + B_T f'(T) f(U) + B_U f(T) f'(U). \end{aligned}$$

Substituting the rational input intervals above gives the following strict bounds. The final two columns bound $10^9 Q_{T,U}$.

(T, U)	$D_{T,U}$	lower	upper
$(3, 267)$	21078	144338437	144338438
$(6, 87)$	21096	31985371	31985372
$(15, 39)$	31338	16852085	16852086
$(15, 87)$	84222	12863676	12863677
$(39, 102)$	383112	12782608	12782609
$(102, 267)$	4160232	11679178	11679179

If r_2 is the coefficient of t^2 in $\mathcal{A}_6(t, 0, -t) - \mathcal{A}_6(0, 0, 0)$, then $r_2 = 2c_2$ and

$$r_2 = \sum_{(T,U)} Q_{T,U} > \frac{230501355}{10^9} > \frac{23}{100}.$$

Dividing by two proves the claim. $\square$

Taylor coefficient recursions. Suppose the trace series is real analytic near $t = 0$ and has the form

$$x(t) = \sum_{j \geq 0} x_j t^j, \quad x_0 > 2.$$

Set $s(x) = \sqrt{x^2 - 4}$. We compute the square root, its reciprocal, the length, and finally the reciprocal length in that order.

First expand the square root and its reciprocal:

$$s(x(t)) = \sum_{j \geq 0} s_j t^j, \frac{1}{s(x(t))} = \sum_{j \geq 0} v_j t^j.$$

Their constant terms are $s_0 = \sqrt{x_0^2 - 4}$ and $v_0 = 1/s_0$. Squaring the first series and multiplying the two series give, for $j \geq 1$,

$$s_j = \frac{\sum_{i=0}^j x_i x_{j-i} - \sum_{i=1}^{j-1} s_i s_{j-i}}{2s_0}, v_j = -\frac{1}{s_0} \sum_{i=1}^j s_i v_{j-i}. \quad (103)$$

Each right-hand side uses only coefficients already computed.

Next write

$$L(x(t)) = \sum_{j \geq 0} l_j t^j, f(x(t)) = \sum_{j \geq 0} f_j t^j,$$

with $l_0 = L(x_0)$ and $f_0 = 1/l_0$. The chain rule

$$\frac{d}{dt} L(x(t)) = \frac{2x'(t)}{s(x(t))}$$

and the identity $L(x(t))f(x(t)) = 1$ give

$$l_j = \frac{2}{j} \sum_{i=1}^j i x_i v_{j-i}, f_j = -\frac{1}{l_0} \sum_{i=1}^j l_i f_{j-i}, j \geq 1. \quad (104)$$

Thus the trace coefficients determine the reciprocal-length coefficients successively, using only finite sums and division by the positive constants s_0, l_0 .

The coefficients through degree eight.

Lemma A2.2. *The quadratic coefficient satisfies*

$$\frac{115250679305}{10^{12}} < c_2 < \frac{115250679306}{10^{12}}.$$

The remaining coefficients through degree eight satisfy

j	3	4	5	6'	6''	7	8'	8''
$1000c_j >$	-694	-162	812	-314	6354	270	1118	-15077
$1000c_j <$	-692	-161	815	-312	6367	277	1128	-14964.

(105)

Proof. Let r_j and q_j denote the coefficients of t^j in

$$\mathcal{A}_6(t, 0, -t) - \mathcal{A}_6(0, 0, 0), \quad \mathcal{A}_6(t, t, -2t) - \mathcal{A}_6(0, 0, 0),$$

respectively. For the first line, use the initial trace formulas in Lemma A2.1. For the second, they are

$$x(t) = y(t) = 2e^{3t} + e^{-3t}, \quad z(t) = 2 + e^{-6t}.$$

Their Taylor coefficients are rational: replace each e^{at} by $\sum_{j=0}^8 a^j t^j / j!$, discarding terms of degree greater than eight after each multiplication. For every permutation, apply the trace recurrence, then (103) and (104), and sum the six products in (31). This gives the following explicit intervals. Each entry is a strict lower bound for $10^{12}r_j$ or $10^{12}q_j$; adding one gives a strict upper bound. The odd r_j vanish because the first restriction is even.

j	$10^{12}r_j$	$10^{12}q_j$
2	230501358611	691504075834
3	—	4158193745227
4	-646178134908	-5815603214170
5	—	-29291413002270
6	-2503246530490	161369799006231
7	—	-59139571788055
8	17964933816381	-1789267475604971

The conversion to the invariant coefficients is

$$\begin{aligned} c_2 &= r_2/2, & c_3 &= -q_3/6, & c_4 &= r_4/4, \\ c_5 &= -q_5/36, & c_{6'} &= r_6/8, & c_{6''} &= (q_6 - 27r_6)/36, \\ c_7 &= -q_7/216, & c_{8'} &= r_8/16, & c_{8''} &= (q_8 - 81r_8)/216. \end{aligned}$$

In particular, the first row gives

$$\frac{230501358611}{2 \cdot 10^{12}} < c_2 < \frac{230501358612}{2 \cdot 10^{12}}.$$

For the two coefficients obtained by subtraction, the same table gives

$$\begin{aligned} \frac{228957455329434}{36 \cdot 10^{12}} < c_{6''} < \frac{228957455329462}{36 \cdot 10^{12}}, \\ -\frac{3244427114731913}{216 \cdot 10^{12}} < c_{8''} < -\frac{3244427114731831}{216 \cdot 10^{12}}. \end{aligned}$$

These bounds and the remaining divisions give every entry of (105). $\square$

Coefficients of degrees nine through nineteen. The following table supplies the bounds used in Theorem 4.2. Its computation uses the recursions above and the four substitutions of Lemma 4.1.

Lemma A2.3 (Explicit higher-coefficient bounds). *For $9 \leq n \leq 19$, each table entry is a strict lower bound for $1000c_{n,k}$; adding one gives a strict upper bound. A dash means that the coefficient does not occur.*

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$
9	2407	-37499	—	—
10	-1373	14063	—	—
11	-13233	135121	—	—
12	985	-39829	286516	—
13	22984	-201453	—	—
14	-1948	185062	-1374518	—
15	-20400	405076	-1909443	—
16	6289	-450043	3030012	—
17	31287	-1786116	11542319	—
18	-12548	721576	-6517347	14598450
19	-111754	5301317	-32106332	—

Proof. Use the finer square-root bounds (102) and the length bounds derived from them. For each ordered permutation of the initial traces, generate $T_0, \dots, T_5$ by (30), set $T_* = T_1T_2 - x$, and sum the six contributions in (31).

On a ray $t(1, -s, s-1)$ the initial traces are

$$\begin{aligned} x(t) &= e^{3t} + e^{(1-2s)t} + e^{(2s-1)t}, \\ y(t) &= e^{(2-s)t} + e^{-3st} + e^{(s-2)t}, \\ z(t) &= e^{(1+s)t} + e^{-(1+s)t} + e^{3(s-1)t}. \end{aligned}$$

At the four rational values of s in Lemma 4.1, their Taylor coefficients are rational. Products are evaluated by

$$(ab)_j = \sum_{i=0}^j a_i b_{j-i}, 0 \leq j \leq 19.$$

The trace recurrence, followed by the reciprocal-length recursions (103)–(104), therefore bounds each $h_n(s)$ above and below. With m, ϵ, d as in Lemma 4.1 and $s_j = (j+1)/10$, Lagrange interpolation gives $c_{n,k}$ as the coefficient of q^k in

$$\sum_{j=0}^d \frac{h_n(s_j)}{p_2(s_j)^m p_3(s_j)^\epsilon} \prod_{\substack{0 \leq l \leq d \\ l \neq j}} \frac{q - q(s_l)}{q(s_j) - q(s_l)}. \quad (106)$$

Substituting the bounds for $h_n(s_j)$ in this formula gives the table. All divisions are by the positive constant terms of the square-root and length series or by the nonzero rational interpolation differences. Thus the table follows from the stated rational bounds by keeping lower and upper bounds at every arithmetic step. $\square$

A3 Verification of the numerical bounds

We collect the endpoint calculations used in the modular-area, complex-length, and short-systole estimates. Each calculation uses the rational series bounds in Appendix A1.

A3.1 The omitted area at the modular torus

In (33), the area added by parent traces a, b and child trace c is

$$\frac{1}{L(a)L(c)} + \frac{1}{L(b)L(c)} - \frac{1}{L(a)L(b)}.$$

To check the stated upper bounds, bracket each square root in $L(T)$ between consecutive multiples of 10^{-20} and apply (101) with $N = 24$, after rescaling its argument into $[1, 2]$. In the added-area formula, use lower length bounds in the two positive terms and upper length bounds in the subtracted term. These are rational comparisons.

A3.2 The complex length bounds

To check (50), compare the corresponding trace bound in (49) with $2 \cosh(l_h/2)$. Each comparison follows from (99) with $N = 8$:

$$\cosh s < \sum_{j=0}^8 \frac{s^{2j}}{(2j)!} + \frac{s^{18}}{18!} \frac{1}{1 - s^2/(19 \cdot 20)} (0 < s \leq 3/2).$$

A3.3 The fan bound for systole at most 6/5

We verify the endpoint bounds and tangent-line values used in Proposition 6.8. Here $L = 1/3$, $U = 3/5$, and $c_n(a) = 2 \operatorname{arccosh}(\coth a \cosh(na))$.

The following integer entries are strict upper bounds for $10000c_n(a)$ at the two endpoints:

a	0	1	2	3	4	5
1/3	36020	37169	40357	45013	50578	56660
3/5	24668	28584	37715	48822	60551	72469

To check an entry r in column n , use the polynomial bounds (100). In every case, rational arithmetic gives

$$S(a)C(r/20000) - C^+(a)C^+(na) > \frac{1}{100000} > 0.$$

Thus $\sinh a \cosh(r/20000) > \cosh a \cosh(na)$, which is equivalent to the stated upper bound for $c_n(a)$.

Averaging consecutive entries and dividing by 10000 gives the endpoint bounds for $m_n = (c_n + c_{n+1})/2$ in (88).

For the function Q in that proof, take $a_* = 279/500$. All $b_n(a_*)$ and b'_n are rational, and $2b_0(a_*) = 22556831/4000000$. Use (98) with this argument and $N = 20$, then take reciprocals to bound $e^{-2b_0(a_*)}$ above and below. Substitution in Q and its derivative proves (89).

A3.4 The fan bounds on the two remaining intervals

As in (94), rows 1 and 2 refer to $[L, U] = [3/5, 2/3]$ and $[2/3, 7/10]$, respectively. We verify the curvature constants in (94), the length and difference bounds in (107), and the tangent-line values in (97).

To bound N above we need lower bounds for its length denominators. To bound P below we need upper bounds for the lengths and lower bounds for Δ_n/a . The following table gives all three bounds throughout each interval:

$$\tilde{m}_n > \frac{\lambda_n}{1000}, m_n < \frac{\eta_n}{1000}, \frac{\Delta_n}{a} > \frac{\pi_n}{1000}.$$

row	array	0	1	2	3	4
1	λ	2440	3270	4300	5440	6630
1	η	2663	3315	4472	5762	7084
1	π	326	760	925	977	993
2	λ	2347	3259	4458	5756	7080
2	η	2514	3303	4556	5920	7311
2	π	366	810	949	986	996

(107)

We verify these bounds and the curvature constants using the polynomials C, S, C^+, S^+ from (100) on $[0, 5]$. The integers in this table mean $v_n/10000 < c_n(a) < (v_n + 1)/10000$:

a	v_0	v_1	v_2	v_3	v_4	v_5
3/5	24667	28583	37714	48821	60550	72468
2/3	22694	27581	38382	51040	64195	77482
7/10	21790	27204	38845	52261	66118	80083

(108)

For every entry the two required rational inequalities are

$$C^+(v_n/20000)S^+(a) < C(a)C(na),$$

$$C((v_n + 1)/20000)S(a) > C^+(a)C^+(na).$$

Since $\cosh$ is increasing on $[0, \infty)$, these inequalities prove the lower and upper bounds for $c_n(a)$. At argument zero use its exact values $\cosh 0 = 1$, $\sinh 0 = 0$. The bounds for $g_a = l_a''$ follow by substituting C, S, C^+, S^+ in (86) and squaring positive quantities. The same polynomials give the remaining comparisons used above:

$$\begin{aligned} S(2/3)^2 &> \frac{1}{2}, C(2/3) > \frac{123}{100}, C^+(7/10) < \frac{63}{50}, \\ C^+(2/3) + S^+(2/3) &< \frac{39}{20}. \end{aligned} \quad (109)$$

Since $\tilde{m}_n$ is convex, its tangent at U gives a lower bound on $[L, U]$. Explicit upper bounds for $\tilde{m}'_n(U)$ are

row	0	1	2	3	4
1	0	0	12/5	47/10	67/10
2	0	14/100	261/100	481/100	688/100

They follow from

$$c'_n(a) = \frac{2(-\cosh(na)/\sinh a + n \cosh a \sinh(na))}{\sqrt{1 + \cosh^2 a \sinh^2(na)}}, \tilde{m}'_n = \frac{c'_n + c'_{n+1}}{2} - \frac{\kappa_n a}{4}.$$

If the corresponding slope bound is $s_n \geq 0$, the length bounds give

$$\tilde{m}_n(a) > \frac{v_n(U) + v_{n+1}(U)}{20000} - \frac{\kappa_n U^2}{8} - s_n(U - L) > \frac{\lambda_n}{1000}.$$

Convexity of m_n bounds it above by the larger endpoint value, giving η_n . The monotonicity of Δ_n/a gives π_n from the two strict length bounds at L . For example, in row 2,

$$\frac{\Delta_1(2/3)}{2/3} > \frac{(38382 - 27582)/20000}{2/3} = \frac{81}{100}.$$

Substitute (94) and (107) into (93). Specifically use $\lambda_n/1000$ and $a \geq L$ to bound N above, and $\eta_n/1000$, $\pi_n/1000$ and $a \leq U$ to bound P below. Evaluating these rational expressions gives (95). The bounds $m_0 < 2663/1000$ in row 1 and $m_0 < 2514/1000$ in row 2 also give $m_0 < 8/3$ on both intervals.

Finally, we specify the affine upper bounds used in (96). Using (108), define $\hat{m}_n$ by

$$\hat{m}_n(L) = \frac{v_n(L) + v_{n+1}(L) + 2}{20000} - \frac{\kappa_n L^2}{8},$$

$$\widehat{m}_n(U) = \frac{v_n(U) + v_{n+1}(U) + 2}{20000} - \frac{\kappa_n U^2}{8}.$$

These bound $\widetilde{m}_n$ at both endpoints. The resulting values at U and slopes are

n	row 1		row 2	
	$\widehat{m}_n(U)$	$\widehat{m}'_n$	$\widehat{m}_n(U)$	$\widehat{m}'_n$
0	439733/180000	−145997/60000	187899/80000	−17627/8000
1	118007/36000	−7399/24000	2611611/800000	1833/80000
2	16747/3750	43001/20000	3636823/800000	199989/80000
3	1036883/180000	105457/24000	4733427/800000	1130323/240000
4	425017/60000	64933/10000	5847679/800000	542757/80000

Substituting these rational numbers in (96) and its derivative gives both rows of (97).

References

- [ABC+21] J. E. Andersen, G. Borot, S. Charbonnier, A. Giacchetto, D. Lewański, and C. Wheeler, On the Kontsevich geometry of the combinatorial Teichmüller space, arXiv:2010.11806, version 3, 2021, <https://arxiv.org/abs/2010.11806v3>.
- [AA26] T. Aougab and J. S. Athreya, Currents with corners and counting weighted triangulations, *J. Lond. Math. Soc.* **114** (2026), e70675, <https://doi.org/10.1112/jlms.70675>.
- [AHA20] F. Arana-Herrera and J. S. Athreya, Square-integrability of the Mirzakhani function and statistics of simple closed geodesics on hyperbolic surfaces, *Forum Math. Sigma* **8** (2020), e9, 33 pp., <https://doi.org/10.1017/fms.2019.49>.
- [Bel23] N. Bell, Counting arcs on hyperbolic surfaces, *Groups Geom. Dyn.* **17** (2023), no. 2, 459–478, <https://doi.org/10.4171/GGD/705>.
- [Bon18] F. Bonahon, *The Mirzakhani function on the moduli space of a surface*, recorded lecture at the 2018 Fields Medal Symposium: Maryam Mirzakhani, Fields Institute, Toronto, November 7, 2018, reporting joint work with Sabrina Enriquez, <https://www.fields.utoronto.ca/talks/Mirzakhani-i-function-moduli-space-surface>.
- [Bow98] B. H. Bowditch, Markoff triples and quasifuchsian groups, *Proc. Lond. Math. Soc.* (3) **77** (1998), no. 3, 697–736, <https://doi.org/10.1112/S0024611598000604>.
- [Cas59] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Grundlehren der mathematischen Wissenschaften, vol. 99, Springer-Verlag, Berlin, 1959.
- [DGZZ21] V. Delecroix, É. Goujard, P. Zograf, and A. Zorich, Masur–Veech volumes, frequencies of simple closed geodesics, and intersection numbers of moduli spaces of curves, *Duke Math. J.* **170** (2021), no. 12, 2633–2718, <https://doi.org/10.1215/00127094-2021-0054>.
- [DLN26] N. M. Doan, X. Li, and V. Nguyen, McShane–Rivin norm balls and simple-length multiplicities, arXiv:2605.14574, version 2, 2026, <https://arxiv.org/abs/2605.14574v2>.
- [ES16] V. Erlandsson and J. Souto, Counting curves in hyperbolic surfaces, *Geom. Funct. Anal.* **26** (2016), no. 3, 729–777, <https://doi.org/10.1007/s00039-016-0374-7>.
- [ES22] V. Erlandsson and J. Souto, *Mirzakhani’s Curve Counting and Geodesic Currents*, Progress in Mathematics **345**, Birkhäuser, Cham, 2022, <https://doi.org/10.1007/978-3-031-08705-9>.
- [Fed74] H. Federer, Real flat chains, cochains and variational problems, *Indiana Univ. Math. J.* **24** (1974/75), 351–407.
- [FM12] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series **49**, Princeton University Press, Princeton, 2012.
- [HT14] H. Hu and S. P. Tan, *New identities for small hyperbolic surfaces*, Bull. Lond. Math. Soc. **46** (2014), no. 5, 1021–1031.
- [Gol03] W. M. Goldman, The modular group action on real $SL(2)$ -characters of a one-holed torus, *Geom. Topol.* **7** (2003), 443–486, <https://doi.org/10.2140/gt.2003.7.443>.
- [Gol09] W. M. Goldman, Trace coordinates on Fricke spaces of some simple hyperbolic surfaces, in *Handbook of Teichmüller Theory*, Vol. II, IRMA Lect. Math. Theor. Phys. **13**, European Mathematical Society, Zürich, 2009, 611–684, <https://doi.org/10.4171/055-1/16>.
- [Gro83] M. Gromov, Filling Riemannian manifolds, *J. Differential Geom.* **18** (1983), no. 1, 1–147, <https://doi.org/10.4310/jdg/1214509283>.

- [LMN26] B. Lowe, F. C. Marques, and A. Neves, Counting minimal Lagrangians via Mirzakhani functions, arXiv:2605.04614, version 3, 2026, <https://arxiv.org/abs/2605.04614v3>.
- [Ma09] T.-W. Ma, Higher chain formula proved by combinatorics, *Electron. J. Combin.* **16** (2009), no. 1, N21, 7 pp., <https://doi.org/10.37236/259>.
- [Mac95] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.
- [Mir07] M. Mirzakhani, Random hyperbolic surfaces and measured laminations, in *In the Tradition of Ahlfors–Bers, IV*, Contemp. Math. **432**, American Mathematical Society, Providence, RI, 2007, 179–198.
- [Mir08a] M. Mirzakhani, Growth of the number of simple closed geodesics on hyperbolic surfaces, *Ann. of Math.* (2) **168** (2008), no. 1, 97–125, <https://doi.org/10.4007/annals.2008.168.97>.
- [Mir08b] M. Mirzakhani, Ergodic theory of the earthquake flow, *Int. Math. Res. Not. IMRN* **2008** (2008), Art. ID rnm116, 39 pp., <https://doi.org/10.1093/imrn/rnm116>.
- [Mir16] M. Mirzakhani, Counting mapping class group orbits on hyperbolic surfaces, arXiv:1601.03342, 2016, <https://arxiv.org/abs/1601.03342>.
- [MR95a] G. McShane and I. Rivin, A norm on homology of surfaces and counting simple geodesics, *Internat. Math. Res. Notices* **1995** (1995), no. 2, 61–69, <https://doi.org/10.1155/S1073792895000055>.
- [MR95b] G. McShane and I. Rivin, Simple curves on hyperbolic tori, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), no. 12, 1523–1528; electronic text corresponding to the published article: arXiv:math/0005220.
- [O’S26] C. O’Sullivan, Patterns in the Markov numbers and their generalizations, arXiv:2609.14149, 2026, <https://arxiv.org/abs/2609.14149>.
- [Pal79] R. S. Palais, The principle of symmetric criticality, *Communications in Mathematical Physics* **69** (1979), no. 1, 19–30.
- [Par14] H. Parlier, Simple closed geodesics and the study of Teichmüller spaces, in *Handbook of Teichmüller Theory*, Vol. IV, IRMA Lect. Math. Theor. Phys. **19**, European Mathematical Society, Zürich, 2014, 113–134, <https://doi.org/10.4171/117-1/3>.
- [RS19] K. Rafi and J. Souto, Geodesic currents and counting problems, *Geom. Funct. Anal.* **29** (2019), 871–889, <https://doi.org/10.1007/s00039-019-00502-7>.
- [Roc70] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series **28**, Princeton University Press, Princeton, 1970.
- [Rud76] W. Rudin, *Principles of Mathematical Analysis*, third edition, McGraw–Hill, New York, 1976.
- [Sas25] D. Sasaki, Counting subgroups via Mirzakhani’s curve counting, arXiv:2409.08109, version 3, 2025, <https://arxiv.org/abs/2409.08109v3>.
- [Sch93] P. Schmutz, *Riemann surfaces with shortest geodesic of maximal length*, Geometric and Functional Analysis **3** (1993), 564–631.
- [Sch14] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, second expanded edition, Encyclopedia of Mathematics and its Applications **151**, Cambridge University Press, Cambridge, 2014.
- [Ser85] C. Series, The geometry of Markoff numbers, *Math. Intelligencer* **7** (1985), no. 3, 20–29.
- [SS03a] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lectures in Analysis, vol. I, Princeton University Press, Princeton, 2003.
- [SS03b] E. M. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis, vol. II, Princeton University Press, Princeton, 2003.
- [Thu88] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.* (N.S.) **19** (1988), no. 2, 417–431.
- [Tri24] M. Trin, Counting arcs of the same type, *Int. Math. Res. Not. IMRN* **2024** (2024), no. 16, 11749–11765, <https://doi.org/10.1093/imrn/rnae143>.
- [Zag82] D. Zagier, On the number of Markoff numbers below a given bound, *Math. Comp.* **39** (1982), no. 160, 709–723, <https://doi.org/10.1090/S0025-5718-1982-0669663-7>.

PHUC THINH DANG

Vietnam National University, Ho Chi Minh City, Vietnam

Email address: dpthinhmaths@gmail.com

TRONG TOAN DAO

National University of Singapore, Singapore

Email address: daotrongtoan.dtt4@gmail.com and daotrongtoan@u.nus.edu

NHAT MINH DOAN

Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam

Email address: dnminh@math.ac.vn

URL: <https://sites.google.com/view/dnminh>