

Uniform Kazhdan Monsters and Schmidt's Property M

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Abstract

Let G be an infinite finitely generated group whose proper subgroups are finite. We prove that a positive Kazhdan constant uniform over all finite generating sets forces every unit vector in every representation without invariant vectors to have a cofinite displacement gap. For ergodic probability-measure-preserving actions this gives

$$\liminf_{g \rightarrow \infty} \mu(gA \triangle A) \geq \kappa_u(G)^2 \mu(A)(1 - \mu(A)),$$

and hence Schmidt's Property M . A tensor-square argument also gives a uniform gap from modulus one for diagonal matrix coefficients. Conditional on the graded-diagram package underlying the Osin–Sonkin construction, the theorem shows that their infinite finitely generated simple ICC group of bounded exponent has Property M . We include a corrected, constant-tracked proof of the Osin–Sonkin deduction from that package and conclude with questions about Ozawa's quasifinite Kazhdan quotients.

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1 Introduction and context

Furstenberg and Weiss introduced mild mixing through the multiplier problem, and Schmidt and Walters developed it for locally compact group actions [10, 39]. Related foundations and later formulations appear in [9, 1, 44, 5, 15, 12]. Schmidt associated to a noncompact locally compact second countable group the properties W , M , and S : every ergodic probability-measure-preserving action is, respectively, weakly, mildly, or strongly mixing, with $S \Rightarrow M \Rightarrow W$ [38]. He also proved that an infinite discrete M -group is finitely generated and that every proper subgroup is finite; it is perfect and simple modulo a finite center [38, Lemma 6.2 and Theorem 6.3].¹ Schmidt knew no discrete example and pointed to newly constructed simple groups without infinite proper subgroups as candidates; the absence of a known countably infinite example was still noted in [21, pp. 83–84].

Property M is distinct from two neighboring rigidity notions. The Connes–Weiss characterization of Property (T) concerns strong ergodicity: it rules out nontrivial sequences of sets that are

¹The conclusion of Theorem 6.3 is printed with *every subgroup* where the assumptions and proof require *every proper subgroup*.

asymptotically invariant under every fixed group element [8, 37, 19]. Failure of mild mixing instead fixes one nontrivial set and lets the group element escape to infinity. At the stronger end, Property S is the Howe–Moore property: all coefficients of representations without invariant vectors vanish at infinity [16, 38, 6]. Mayer’s representation-theoretic condition also called mild mixing is stronger than the Schmidt–Walters notion and concerns a different classification problem [25]. Cluckers–Cornulier–Louvè–Tessera–Valette showed, more specifically, that an infinite discrete Howe–Moore group is finitely generated and finite-by-(quasifinite simple), while emphasizing how little was known about representations of quasifinite groups [6, Lemma 3.4 and Remark 3.5].

The relevant groups originate in small-cancellation constructions. Ol’shanskii produced finitely generated groups with finite cyclic proper subgroups, and later work produced periodic and quasifinite quotients of hyperbolic groups [26, 13, 28, 29, 18]. Ozawa used this machinery to obtain uncountably many quasifinite ICC quotients, which are Kazhdan when the hyperbolic source is Kazhdan [34]. Osin and Sonkin introduced Property U and used Semenov’s graded small-cancellation construction to produce the first infinite groups whose Kazhdan constants are bounded away from zero over all finite generating sets [41, 31]. Further periodic groups with Property U were constructed in [3, 36], but those papers do not assert Property (T) .

For context, Lubotzky asked whether every Kazhdan group has such a uniform lower bound [23]. The answer is negative: examples and broad zero-constant results occur in [11, 32, 17]; a recent finite-quotient obstruction appears in [46]. Lubotzky and Yao subsequently proved that every infinite finitely generated linear group has zero uniform Kazhdan constant [24, Theorem 1.1]. Uniformity for prescribed families of generating sets or representations, and uniform nonamenability, have different quantifiers [14, 35, 2]. The elementary observation behind this note is that quasifiniteness converts every infinite subset into a generating set, so a generating-set-uniform spectral gap becomes a gap at infinity. The contribution of this note is the resulting displacement, action, and coefficient estimates; all construction-theoretic inputs are quoted from their sources.

2 Preliminaries

Throughout, G is a finitely generated discrete group, $g \rightarrow \infty$ means that g leaves every finite subset of G , and complex Hilbert space inner products are linear in the first variable.

Definition 2.1. An infinite group is *quasifinite* if every proper subgroup is finite.

For a finite generating set S , put

$$\kappa(G, S) = \inf_{\substack{\pi: G \rightarrow \mathcal{U}(\mathcal{H}) \\ \mathcal{H}^G = \{0\}}} \inf_{\|\xi\|=1} \max_{s \in S} \|\pi(s)\xi - \xi\|, \quad (2.1)$$

$$\kappa_u(G) = \inf_S \kappa(G, S), \quad (2.2)$$

where the last infimum ranges over all finite generating sets, without a cardinality bound.

The exclusion of invariant vectors in (2.1) is the standard norm-valued convention [20, 4].²

For an ergodic p.m.p. action $G \curvearrowright (X, \mu)$, let $U_g f = f \circ g^{-1}$ be the Koopman representation on $L_0^2(X) = L^2(X) \ominus \mathbb{C}\mathbf{1}$. It has no invariant vector. Schmidt’s rigid-set criterion says that the action

²It is omitted from the displayed prose in [31], although the subsequent assertions there require it.

is mildly mixing precisely when

$$\liminf_{g \rightarrow \infty} \mu(gA \triangle A) > 0 \quad (0 < \mu(A) < 1), \quad (2.3)$$

[38, Proposition 2.3]. The group has Property M when every ergodic p.m.p. action on a standard nonatomic probability space is mildly mixing [38, Definition 3.1].

3 The uniform displacement theorem

Theorem 3.1. *Let G be infinite, finitely generated, and quasifinite, and suppose that $\kappa = \kappa_{\mathfrak{u}}(G) > 0$.*

1. *If $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ has no invariant vector and $\xi \in \mathcal{H}$ is a unit vector, then*

$$\{g \in G : \|\pi(g)\xi - \xi\| < \kappa\}$$

is finite.

2. *Every ergodic p.m.p. action $G \curvearrowright (X, \mu)$ and every measurable A with $0 < \mu(A) < 1$ satisfy, for all but finitely many g ,*

$$\mu(gA \triangle A) \geq \kappa^2 \mu(A)(1 - \mu(A)). \quad (3.1)$$

In particular, G has Property M .

3. *For every such representation π , every $v \neq 0$, and all but finitely many g ,*

$$|\langle \pi(g)v, v \rangle| \leq \sqrt{1 - \frac{\kappa^2}{2}} \|v\|^2. \quad (3.2)$$

The passage from displacement to the absolute value in part (3) uses two standard facts, included to make the dependence explicit.

Lemma 3.2. *Every infinite finitely generated quasifinite group is minimally almost periodic: every finite-dimensional unitary representation is trivial.*

Proof. The group is torsion: an infinite cyclic subgroup would be the whole group, which is impossible because an infinite cyclic group has an infinite proper subgroup. The image of a finite-dimensional unitary representation is a finitely generated periodic complex linear group, hence finite by Schur's theorem [40]. Its kernel is infinite and therefore, by quasifiniteness, is the whole group. $\square$

Lemma 3.3. *If G is minimally almost periodic and π has no invariant vector, then $\pi \otimes \bar{\pi}$ has no invariant vector.*

Proof. The representation π has no nonzero finite-dimensional invariant subspace. Under $\mathcal{H} \otimes \bar{\mathcal{H}} \cong \text{HS}(\mathcal{H})$, an invariant vector would give a nonzero Hilbert–Schmidt operator T commuting with $\pi(G)$. A nonzero eigenspace of the compact positive operator T^*T would then be a nonzero finite-dimensional invariant subspace, a contradiction. $\square$

Proof of Theorem 3.1. For part (1), call the displayed set E and suppose that it is infinite. Then $\langle E \rangle = G$ by quasifiniteness. Since G is finitely generated, finitely many elements $F \subset E$ generate G . Hence

$$\kappa(G, F) \leq \max_{s \in F} \|\pi(s)\xi - \xi\| < \kappa \leq \kappa(G, F),$$

a contradiction.

For part (2), set $p = \mu(A)$ and $\xi = (\mathbf{1}_A - p\mathbf{1})/\sqrt{p(1-p)} \in L_0^2(X)$. Then

$$\|U_g \xi - \xi\|^2 = \frac{\mu(gA \triangle A)}{p(1-p)}.$$

Part (1) gives (3.1) outside a finite set. Its positive lower bound and (2.3) prove Property M .

For part (3), put $\eta = v/\|v\|$, $\rho = \pi \otimes \bar{\pi}$, and $\zeta = \eta \otimes \bar{\eta}$. Lemmas 3.2 and 3.3 allow part (1) to be applied to ρ, ζ ; thus, off a finite set,

$$\kappa^2 \leq \|\rho(g)\zeta - \zeta\|^2 = 2(1 - |\langle \pi(g)\eta, \eta \rangle|^2).$$

This is (3.2). Finally $\kappa \leq \sqrt{2}$, since the left regular representation and δ_e give $\kappa(G, S) \leq \sqrt{2}$ for every finite generating set S . The radicand is therefore nonnegative and, because $\kappa > 0$, strictly less than one. $\square$

4 The Osin–Sonkin construction

We give a corrected proof of the Osin–Sonkin deduction, with all constants and noncommutation hypotheses made explicit. There is an important boundary to the claimed completeness. The argument below is complete once the graded-diagram package (GD0)–(GD4) is granted. Proving that package from first principles would require reproducing the least-parameter machinery and the diagram theory of Chapters 8–9 of [27], together with Semenov’s hyperbolic-source modifications [42, 43]. The latter is an unpublished 31-page author manuscript; no single published self-contained proof of the combined quantitative package appears to be available. Semenov’s short published announcement states the basic quotient theorem [41]. Kulikova later invokes thesis Theorem 2.6, identified there with Semenov’s 1993 Theorem 1 [22, Theorem 4, Section 4]; this is later expert use, not an independent proof of the input. Neither published paper supplies the quantitative pair estimates needed for Property U .

For a subset S of a group, $|g|_S$ denotes word length with respect to $S \cup S^{-1}$. We use $[x, y] = xyx^{-1}y^{-1}$.

Definition 4.1. Let Q be generated by a finite set X . The group Q has *Property U* if there is $C_U < \infty$ such that, for every generating set Y of Q , there is $t \in Q$ satisfying

$$\max_{x \in X} |t^{-1}xt|_Y \leq C_U.$$

4.1 The graded-diagram input

Let

$$\Gamma = \langle a_1, \dots, a_m \mid \mathcal{R}_0 \rangle$$

be a non-elementary torsion-free hyperbolic group. We take $\mathcal{R}_0$ to be the complete set of cyclically reduced X -words trivial in Γ , after deleting trivial generators and inverse or duplicate generators, and put $X = \{a_1, \dots, a_m\}$. The Ol'shanskii–Semenov least-parameter construction supplies constants

$$d > 0, \quad h, n, N \in \mathbb{N}, \quad \sigma > 0, \quad N \text{ odd}, \quad \sigma < N/4,$$

a graded quotient $Q = G(\infty)$ of Γ , and sets of periods. For an X -word W , write $\|W\|_X$ for its combinatorial length. Capital letters denoting periods or auxiliary words always denote chosen X -words as well as the elements they represent; equations between them are equations in the indicated group. The exact input used below is:

- (GD0) If A is a period of rank i , the rank- i relations contain $A^N = 1$. For each A , a maximal set $\mathcal{G}_A$ is chosen of words T satisfying

$$0 < \|T\|_X < d \|A\|_X,$$

with at most one minimum-length representative from each double coset $\langle A \rangle T \langle A \rangle$ in $G(i-1)$. If

$$a_j \notin \langle A \rangle, \quad a_j \notin \langle A \rangle a_k \langle A \rangle \quad (k < j),$$

then for every $T \in \mathcal{G}_A \setminus \langle A \rangle a_j \langle A \rangle$ the rank- i relations contain

$$a_j A^{n+j-1} T A^{n+m+j-1} \dots T A^{n+m(h-1)+j-1} = 1. \quad (\text{GD0.1})$$

There are h displayed powers of A and $h-1$ occurrences of T .

- (GD1) The group Q is infinite, has exponent dividing N , and every proper subgroup of Q is cyclic of order dividing N .
- (GD2) Every period has order exactly N in Q . Every nonidentity element is conjugate to a nonzero power of a period, and $C_Q(x)$ is cyclic of order exactly N for every $x \neq 1$.
- (GD3) Let C be a period, let $\sigma < k < N/2$, and let W have minimum $\|\cdot\|_X$ -length in $\langle C^k \rangle W \langle C^k \rangle$. Suppose that $[C^k, W] \neq 1$ and that $[C^k, W]$ is conjugate in Q to A^ℓ , where A is a period and $1 \leq |\ell| \leq (N-1)/2$. Then $|\ell| \leq \sigma$, and there exist $g \in Q$ and an X -word B such that

$$[C^k, W] = g^{-1} A^\ell g, \quad B = g C^k g^{-1} \text{ in } Q, \quad \|B\|_X < d \|A\|_X.$$

- (GD4) Let A be a period, let B be a chosen X -word, and suppose

$$0 < |\ell| \leq \sigma, \quad [A^\ell, B] \neq 1, \quad \|B\|_X < d \|A\|_X.$$

Then there are $s \in \mathbb{Z}$, $z \in Q$, $\varepsilon \in \{-1, 1\}$, a period F , and a word T such that

$$F^\varepsilon = z^{-1} B A^{\ell s} z, \quad T = z^{-1} B z, \quad [F^\varepsilon, T] \neq 1, \quad \|T\|_X < d \|F\|_X.$$

The exact source map is as follows. Item (GD0) is Semenov's Section 11, and condition R is his Lemma 11.2 (27.2). Item (GD1) is Theorem A, using Theorems 10.1, 10.2, 10.5 and Lemma 11.4 (27.4); item (GD2) uses Theorems 10.2, 10.4 and 10.5 [43, pp. 29–31]. Item (GD3) is Semenov's Lemma 9.23 (25.21) [43, p. 27], modifying Ol'shanskii's Lemma 25.21 [27, pp. 281–282]. Item (GD4) is the intermediate pair-straightening step in the last five paragraphs of Ol'shanskii's Lemma 27.3

[27, pp. 301–303], with the hyperbolic-source changes specified in Semenov’s Lemma 11.3 [43, pp. 30–31]. Semenov also uses the published hyperbolic estimates [28, Lemmas 26, 27, 34, 42, pp. 527–528, 531–532, 538]; that paper refers its Sections 6–9 back to the book’s definitions and proofs [28, p. 520].

The noncommutation needed to invoke (GD4) is asserted immediately after Lemma 25.21 is applied inside the source proof of Lemma 27.3. Rather than importing it, we derive it from (GD1)–(GD2) in Lemma 4.4. For comparison, [33, Lemmas 2.4–2.5, pp. 406–407] gives the parallel free-Burnside-group argument, not the hyperbolic-source modification; its displayed Lemma 2.5 has the same local zero-exponent omission. The literal formulation of [31, Lemma 2.5] allows $\ell = 0$, in which case it is false; (GD4) records the corrected hypothesis used in the application.

Put

$$q = \frac{N-1}{2}, \quad I = \lfloor \sigma \rfloor + 1, \quad E = 2N + 2I, \quad M = (q+1)E,$$

and

$$L = \max\{N, hq + (h-1)N + 2q\}, \quad C_U = LM. \quad (4.1)$$

These constants depend only on the fixed quotient and its fixed generating set.

Lemma 4.2 (Normalizer rigidity). *If $1 \neq x \in Q$ and xyx^{-1} commutes with x , then y commutes with x .*

Proof. Let $H = C_Q(x)$. By (GD2), H is cyclic of order N . If $a = yxy^{-1}$ commutes with x , then $a \in H$. Since H is abelian, $H \subseteq C_Q(a)$; both groups have order N , so

$$yHy^{-1} = C_Q(yxy^{-1}) = C_Q(a) = H.$$

If $\langle H, y \rangle$ is cyclic, then y commutes with x . Otherwise (GD1) forces $\langle H, y \rangle = Q$, and H is a nontrivial finite normal subgroup. The conjugation homomorphism $Q \rightarrow \text{Aut}(H)$ has an infinite kernel, which by (GD1) must be all of Q . Then $H \subseteq Z(Q)$, contradicting $C_Q(x) = H \neq Q$. $\square$

Lemma 4.3 (A short period pair). *Let F be a period and let T be a word satisfying*

$$[F, T] \neq 1, \quad \|T\|_X < d\|F\|_X.$$

Then

$$\max_{1 \leq j \leq m} |a_j|_{\{F, T\}} \leq L.$$

Proof. Let i be the rank of F , and make the following double-coset comparisons in $G(i-1)$. Since $F^N = 1$ in Q , every power of F has $\{F, T\}$ -length at most q .

Fix j . If $a_j \in \langle F \rangle$, then $|a_j|_{\{F, T\}} \leq q$. If $T \in \langle F \rangle a_j \langle F \rangle$, reducing the two flanking powers modulo N gives

$$|a_j|_{\{F, T\}} \leq 2q + 1 = N. \quad (4.2)$$

In the remaining case, in $G(i-1)$,

$$a_j \notin \langle F \rangle, \quad T \notin \langle F \rangle a_j \langle F \rangle.$$

Let j_0 be the least index such that $a_{j_0} \in \langle F \rangle a_j \langle F \rangle$. Thus $a_{j_0} \notin \langle F \rangle$, and by minimality it lies in no double coset $\langle F \rangle a_k \langle F \rangle$ with $k < j_0$. The double coset $\langle F \rangle T \langle F \rangle$ has a representative $T_1 \in \mathcal{G}_F$: the

word T itself satisfies the required length bound, this double coset is nontrivial because $[F, T] \neq 1$, and maximality supplies a minimum-length representative. Moreover, $T_1 \notin \langle F \rangle a_{j_0} \langle F \rangle$. Thus (GD0) contains the relation

$$a_{j_0} F^{n+j_0-1} T_1 F^{n+m+j_0-1} \dots T_1 F^{n+m(h-1)+j_0-1} = 1. \quad (4.3)$$

The index in (4.3) is j_0 , rather than the j printed at the corresponding point of [31].

Because $T_1 \in \langle F \rangle T \langle F \rangle$,

$$|T_1|_{\{F, T\}} \leq 2q + 1 = N. \quad (4.4)$$

Solving (4.3) for a_{j_0} gives

$$|a_{j_0}|_{\{F, T\}} \leq hq + (h-1)N.$$

Finally $a_j \in \langle F \rangle a_{j_0} \langle F \rangle$, whence

$$|a_j|_{\{F, T\}} \leq hq + (h-1)N + 2q \leq L.$$

Together with (4.2), this proves the assertion. In particular, the safe constant depends on h , and the bound in (4.4) need not be strict. $\square$

Lemma 4.4 (Uniform pair reduction). *For every $u, v \in Q$ with $[u, v] \neq 1$, there are a period F , a word T , and $t \in Q$ such that*

$$[F, T] \neq 1, \quad \|T\|_X < d \|F\|_X,$$

and

$$\max\{|t^{-1}Ft|_{\{u, v\}}, |t^{-1}Tt|_{\{u, v\}}\} \leq M. \quad (4.5)$$

Proof. By (GD2),

$$u = P^{-1}C^{k'}P$$

for a period C . Replace the pair (u, v) by (u^{-1}, v) and relabel it, if necessary; the symmetric word metric is unchanged and noncommutation is preserved. We may then assume $1 \leq k' \leq q$. If $k' > \sigma$, put $i = 1$; otherwise put

$$i = \left\lfloor \frac{\sigma}{k'} \right\rfloor + 1.$$

Then $i \leq I$ and

$$\sigma < k := ik' < N/2. \quad (4.6)$$

Indeed, if $k' > \sigma$, then $k = k' \leq q < N/2$; otherwise $k \leq \sigma + k' \leq 2\sigma < N/2$. The exact order assertion in (GD2) gives $C^k \neq 1$. If u^i commuted with v , then $u, v \in C_Q(u^i)$, a cyclic group, contrary to $[u, v] \neq 1$. Therefore

$$[C^k, P v P^{-1}] = P[u^i, v]P^{-1} \neq 1. \quad (4.7)$$

Put $x = C^k$, $W_0 = P v P^{-1}$, and choose W of minimum $\|\cdot\|_X$ -length in $\langle x \rangle W_0 \langle x \rangle$. There are $\alpha, \beta \in \mathbb{Z}$, with $|\alpha|, |\beta| \leq q$, such that

$$W = x^\alpha W_0 x^\beta.$$

The element W does not commute with x , since

$$[x, x^\alpha W_0 x^\beta] = 1 \iff [x, W_0] = 1.$$

Moreover

$$P^{-1}WP = u^{i\alpha}v u^{i\beta}.$$

Reducing both u -exponents modulo N , rather than using the incorrect signed estimate printed in [31], gives

$$\left| P^{-1}WP \right|_{\{u,v\}} \leq 2q + 1 = N. \quad (4.8)$$

Let $D = [x, W]$. From (4.6)–(4.8),

$$\left| P^{-1}xP \right|_{\{u,v\}} \leq I, \quad \left| P^{-1}DP \right|_{\{u,v\}} \leq 2I + 2N = E. \quad (4.9)$$

By (GD2), choose a period A and an integer ℓ with $1 \leq |\ell| \leq q$ such that D is conjugate to A^ℓ . By (GD3), $|\ell| \leq \sigma$, and there exist $g \in Q$ and an X -word B such that

$$D = g^{-1}A^\ell g, \quad B = gxg^{-1} \text{ in } Q, \quad \|B\|_X < d \|A\|_X. \quad (4.10)$$

For clarity, we verify the noncommutation that is tacit in the preprint. If D commuted with x , put $a = WxW^{-1}$. Since $D = xa^{-1}$, the elements a and x would commute. Lemma 4.2, applied with $y = W$, would make W commute with x , contradicting the equivalence above and (4.7). Thus $[D, x] \neq 1$, and simultaneous conjugation by g gives

$$[A^\ell, B] \neq 1. \quad (4.11)$$

Apply (GD4), obtaining s, z, ε, F, T . Since $[F^\varepsilon, T] \neq 1$, also $[F, T] \neq 1$. Replace s by its residue in $[-q, q]$; this does not change $BA^{\ell s}$. Since

$$g^{-1}Bg = x, \quad g^{-1}A^\ell g = D,$$

we have

$$g^{-1}BA^{\ell s}g = xD^s, \quad g^{-1}Bg = x.$$

Set $t = z^{-1}gP$. Equations (4.9) and the preceding identities give

$$\left| t^{-1}Ft \right|_{\{u,v\}} = \left| t^{-1}F^\varepsilon t \right|_{\{u,v\}} \leq I + qE \leq (q+1)E = M, \quad \left| t^{-1}Tt \right|_{\{u,v\}} \leq I \leq M.$$

□

Theorem 4.5 (Corrected Osin–Sonkin deduction). *Assume (GD0)–(GD4). Then Q has Property U with respect to X . More precisely, every generating set Y of Q admits $t \in Q$ such that*

$$\max_{1 \leq j \leq m} \left| t^{-1}a_j t \right|_Y \leq C_U.$$

Proof. The group Q is nonabelian by (GD2), so Y contains noncommuting elements u, v . The subgroup $\langle u, v \rangle$ is noncyclic and therefore equals Q by (GD1). In particular, $|g|_Y \leq |g|_{\{u,v\}}$.

Apply Lemma 4.4 to obtain F, T, t . By Lemma 4.3, each a_j is represented by a word of length at most L in $F^{\pm 1}, T^{\pm 1}$. Conjugating that word by t and using (4.5) gives

$$\left| t^{-1}a_j t \right|_Y \leq \left| t^{-1}a_j t \right|_{\{u,v\}} \leq LM = C_U.$$

□

Lemma 4.6 (Kazhdan comparison). *Let S, T be finite generating sets of a group G , and let $g \in G$. Then*

$$\kappa(G, g^{-1}Sg) = \kappa(G, S).$$

If $D = \max_{s \in S} |s|_T$, then

$$\kappa(G, T) \geq D^{-1} \kappa(G, S).$$

Proof. For a unitary representation π and a unit vector ξ ,

$$\left\| \pi(g^{-1}sg)\xi - \xi \right\| = \left\| \pi(s)\eta - \eta \right\|, \quad \eta = \pi(g)\xi.$$

The map $\xi \mapsto \eta$ permutes the unit sphere, proving conjugacy invariance after taking the two infima in (2.1). This corrects the false pointwise equality in [31, Lemma 3.1].

If $s = t_1 \cdots t_r$, where $r \leq D$ and $t_i \in T \cup T^{-1}$, telescoping gives

$$\left\| \pi(s)\xi - \xi \right\| \leq r \max_{t \in T} \left\| \pi(t)\xi - \xi \right\|.$$

Here $\left\| \pi(t^{-1})\xi - \xi \right\| = \left\| \pi(t)\xi - \xi \right\|$, so inverse letters are controlled by the same maximum. Taking maxima and infima gives the second assertion. $\square$

Corollary 4.7. *If the initial group Γ has Property (T), then*

$$\kappa_u(Q) \geq C_U^{-1} \kappa(Q, X) > 0.$$

Proof. Property (T) passes to quotients, so $\kappa(Q, X) > 0$. For any finite generating set Y , Theorem 4.5 supplies t with $\max_{x \in X} |t^{-1}xt|_Y \leq C_U$. Lemma 4.6 yields

$$\kappa(Q, Y) \geq C_U^{-1} \kappa(Q, t^{-1}Xt) = C_U^{-1} \kappa(Q, X).$$

Taking the infimum over Y proves the result. $\square$

Corollary 4.8 (Conditional on (GD0)–(GD4)). *The quotient $Q = G(\infty)$ constructed from a torsion-free non-elementary hyperbolic Kazhdan group has Property M and satisfies (3.1)–(3.2). It is an infinite finitely generated simple ICC group of bounded odd exponent, and every proper subgroup is finite cyclic.*

Proof. The group is finitely generated because it is a quotient of Γ . By (GD1) and Corollary 4.7, it is infinite, quasifinite, and has positive uniform Kazhdan constant, so Theorem 3.1 applies.

By (GD2), every nontrivial centralizer is finite, so every nontrivial conjugacy class is infinite and Q is ICC. A nontrivial proper normal subgroup would be finite by quasifiniteness, contradicting ICC; hence Q is simple. Finally, (GD1) gives the stated cyclic proper subgroups and bounded odd exponent. $\square$

The algebraic and uniform Kazhdan properties are the Osin–Sonkin claims reconstructed above. The new consequences are Property M and the quantitative estimates. The conditional label records exactly the remaining source-verification issue: the elementary reduction is now complete, while the hyperbolic-source proof of (GD0)–(GD4) remains the composite Ol’shanskii–Semenov input described at the start of this section.

5 Concluding remarks and questions

Fix a Kazhdan hyperbolic source in Ozawa’s construction. The resulting groups Γ_α are finitely generated Kazhdan quotients, quasifinite and ICC, and therefore simple [34]. Ozawa proves quasifiniteness, but does not assert bounded exponent, prescribed cyclic proper subgroups, Property U , or a positive uniform Kazhdan constant. Theorem 3.1 therefore suggests: *Do Ozawa’s groups Γ_α have Schmidt’s Property M ? Does $\kappa_u(\Gamma_\alpha) > 0$ hold for any, or for every, member of the family?* The second question gives a sufficient route to the first, not a necessary condition. Ordinary Property (T) supplies a positive constant for each finite generating set separately; it gives no common lower bound for the tail-dependent generating sets arising from a sequence escaping to infinity.

Ozawa’s motivation was operator algebraic. His Theorem 2 says that, for a fixed separable II_1 -factor M , the unitary group $\mathcal{U}(M)$ contains an isomorphic copy of Γ_α for at most countably many α . His Corollary 3 says that M contains a subfactor stably equivalent to $L(\Gamma_\alpha)$ for at most countably many α [34]. Neither statement asserts that the factors $L(\Gamma_\alpha)$ are pairwise nonisomorphic.

The broader landscape changed in August 2026. Connes had conjectured that an ICC Kazhdan group is determined by its group von Neumann algebra [7, p. 45]. An OpenAI manuscript constructs a countably infinite family of pairwise nonisomorphic, mutually commensurable, finitely generated ICC Kazhdan groups with mutually isomorphic group factors [30, Chapter 4]. A contemporaneous preprint of Zhou constructs an explicit pair of nonisomorphic ICC Kazhdan groups with isomorphic group factors [47]. These very recent sources refute the original conjecture for arbitrary ICC Kazhdan groups, not the narrower higher-rank-lattice variant discussed in [47, Section 1].

Neither construction enters the quasifinite regime. The OpenAI groups and Zhou’s groups are semidirect products with infinite proper abelian normal kernels. Schmidt’s subgroup obstruction [38, Lemma 6.2] therefore shows that none has Property M . These examples do not determine whether Property M is invariant under isomorphism of group factors, and they do not resolve the questions above. Finally, our coefficient bound is a gap from modulus one, not decay to zero, so it proves M rather than S . Since an infinite amenable Kazhdan group cannot exist, the method does not address the amenable quasifinite case.

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