

Moment defects and sharp iid Berry–Esseen bounds

Structure, lattice asymptotics, and effective estimates

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Abstract

Shevtsova’s sharp third-moment inequality supplies an exact defect that identifies the standardized two-point family and controls transport and characteristic functions. We develop the corresponding Fourier reduction and give the qualitative eventual upper-bound argument. A penalized extremal problem yields eventual two-point optimality and a uniform linear deficit, with sharp cubic Wasserstein stability. A fifth-order lattice saddle gives the full oscillatory n^{-2} correction for the optimal constants. A separate effective argument proves $C_n < C_E$ for every $n \geq 31$, by finite-index certificates and a first-failure argument. A completed finite continuation at indices $20, \dots, 24$ lowers the cutoff to 20 under the explicitly retained analytic and historical certificate premises stated in the paper. Exact polynomial certificates give bounds for two and three summands. For four summands, a common-measure Fourier comparison and rational certificates give the unrestricted bound $C_4 \leq 0.4097 < C_E$; a sharper estimate $C_{4, \leq 3} < 0.409$ holds on at most three points. An elementary historical bound $C \leq 0.45$ explains the passage from a symmetric reference to the matching two-point law; its finite arithmetic is recorded in an appendix. ChatGPT and Codex assisted the development of the proofs and computational certificates and the preparation of this manuscript, using the GPT-5.6 Sol, GPT-6.1 Sol and GPT-6 Astra models.

Introduction

The common normalization is mean zero, variance one and finite third absolute moment. Put

$$R_n(P) = \frac{\sqrt{n} \Delta_n(P)}{\rho(P)}, \quad C_n = \sup_P R_n(P), \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

In Part I we use

$$\beta(P) = \rho(P), \quad f(\gamma) = \rho_2(\gamma), \quad B_n = C_Q^*(n), \quad d(P) = \rho(P) - \rho_2(\gamma(P)).$$

The normalized defect $q(P) = d(P)/\rho(P)$ is distinct from the Bernoulli probability $q = 1 - p$. Lattice spans and conditional-noise variances are introduced locally.

The principal qualitative conclusion is that, for all sufficiently large n , every standardized law satisfies

$$R_n(P) + 10^{-4} \frac{d(P)}{\rho(P)} \leq C_Q^*(n) = C_n < C_E.$$

Theorem 8.1 proves this statement. The asymptotic refinement, including its arithmetic oscillation, is given in Corollary 9.3. The effective theorem $C_n < C_E$ for $n \geq 31$ is Theorem 12.1. The finite continuation in Theorem 11.2 extends the cutoff to 20 under Assumption 11.1. Neither effective statement asserts two-point optimality from its stated cutoff onward.

Part I develops the defect, Fourier stability, coherent response and two-point saddle, followed by the quantitative and second-order refinements. Part II proves the effective estimates by analytic reductions and complete arithmetic certificates. The unrestricted fourth-index bound, Theorem 26.1, localizes a hypothetical extremizer and retains one defect measure throughout the Fourier comparison. Literature inputs are stated where used. The unrestricted indices $5, \dots, 19$ remain open in this work; the extension through $20, \dots, 30$ has the retained-premise status specified in Section 11.

He–Cheng [9] prove the eventual upper bound $C_n \leq C_E$, with the explicit threshold $2[\exp(10^{17})]$. The equality $C_n = C_Q^*(n)$, the stability statement and the lattice correction are separate structural assertions; an effective strict upper bound does not by itself imply them.

The manuscript and reproducibility material are maintained in the **moment-defects** directory of the accompanying repository. The original 0.45 certificate is preserved separately there. The finite extension to 20 and its retained-premise ledger are supplied in the separate continuation [11].

Principal results

Let P have mean zero, variance one and finite $\rho = \mathbb{E}|X|^3$; write $\gamma = \mathbb{E}X^3$, $R_n(P) = \sqrt{n} \Delta_n(P)/\rho$, $C_n = \sup_P R_n(P)$, and $C_Q^*(n) = \sup_{Q \text{ two-point}} R_n(Q)$. Put $C_E = (3 + \sqrt{10})/(6\sqrt{2\pi})$.

1. **Exact two-point defect.** With Q_γ the matching standardized two-point law,

$$d(P) = \rho - \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}} \geq 0, \quad d(P) = 0 \iff P = Q_\gamma.$$

Known sharp moment inequality [3]; Theorem 2.1 records its defect form.

2. **Linear Fourier stability.** For every $t \in \mathbb{R}$,

$$|\varphi_P(t) - \varphi_{Q_\gamma}(t)| \leq \frac{|t|^3 d(P)}{4(1 - |c_\gamma|)}, \quad c_\gamma = \frac{\gamma(\gamma^2 + 6)}{(\gamma^2 + 4)^{3/2}}.$$

Theorem 2.2.

3. **Eventual two-point optimality.** For all sufficiently large n ,

$$C_n = C_Q^*(n) < C_E.$$

Every maximizing law is two-point. Theorem 8.1.

4. **The full oscillatory second-order asymptotic.** With the explicit constants and phase of Section 9,

$$C_n = C_E - \frac{\kappa_\infty}{n} - \frac{\kappa_2 + A_2 \omega_n^2}{n^2} + o(n^{-2}).$$

The limit set of $n^2(C_n - C_E + \kappa_\infty/n)$ is exactly $[-\kappa_2 - A_2/4, -\kappa_2]$. Corollary 9.3.

5. **An effective unrestricted threshold.** $C_n < C_E$ for every integer $n \geq 31$. Theorem 12.1. The completed finite continuation gives the cutoff 20 under Assumption 11.1; see Theorem 11.2. These thresholds concern the strict upper bound, not two-point optimality.

6. **Finite-index certificates.**

$$C_2 \leq \frac{2\sqrt{2}}{7}, \quad C_3 \leq \frac{59\sqrt{3}}{250}, \quad C_4 \leq \frac{4097}{10000} < C_E.$$

All three bounds are unrestricted. On at most three points the sharper bound is $C_{4,\leq 3} < 409/1000$. Theorems 24.1, 25.1, 26.1 and 27.1.

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1 From a symmetric comparison to the moment defect

We begin with a simpler bound. It isolates the comparison principle without the extremizer or lattice arguments used later. Throughout this section, $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\rho = \mathbb{E}|X|^3$, $\gamma = \mathbb{E}X^3$, and $f_n(t) = \varphi_X(t/\sqrt{n})^n$.

1.1 The Rademacher reference

The least possible third absolute moment is one. Equality holds only for the symmetric law on $\{-1, 1\}$: Jensen gives $\mathbb{E}|X|^3 \geq (\mathbb{E}X^2)^{3/2}$, and equality forces $|X| = 1$. Thus $d_R = \rho - 1$ is a defect from one fixed reference law. Mattner–Shevtsova [2, Theorem 1.2 and Example 1.4] give the distribution-free comparison

$$|\varphi_X(u) - \cos u| \leq \frac{B(\rho)|u|^3}{6}, \quad B(\rho)^2 = \frac{\rho^2 + \rho\sqrt{\rho^2 + 8}}{2} - 2. \quad (1.1)$$

This uses the sharp third-moment geometry of Shevtsova [3, Theorem 6]. In particular,

$$B(\rho) = \sqrt{\frac{8}{3}(\rho - 1)} + o(\sqrt{\rho - 1}) \quad (\rho \downarrow 1).$$

The small parameter in this comparison is therefore a square root of the symmetric defect.

Put $u = t/\sqrt{n}$, $c = |\cos u|$, $\delta = B(\rho)|u|^3/6$, and $a = \min(1, c + \delta)$. Factoring a difference of powers gives

$$|f_n(t) - \cos^n u| \leq D_{n,\rho}(t) := \min \left\{ 2, n\delta, \delta \sum_{j=0}^{n-1} a^j c^{n-1-j} \right\}. \quad (1.2)$$

Consequently the two quantities needed in smoothing have majorants

$$\begin{aligned} |f_n(t) - e^{-t^2/2}| &\leq \min\{2, |\cos^n u - e^{-t^2/2}| + D_{n,\rho}(t)\}, \\ |f_n(t)| &\leq \min\{1, c^n + D_{n,\rho}(t)\}. \end{aligned}$$

For the second quantity one also takes the minimum with the Prawitz–Shevtsova modulus bound described in Appendix C. Each bound applies to the original law; the Rademacher law supplies a tractable reference, not a restriction on the marginal.

1.2 Smoothing and the elementary bound

Let

$$K(s) = \frac{1-s}{2} + \frac{i}{2} \left((1-s) \cot(\pi s) + \frac{1}{\pi} \right), \quad L(s) = K(s) - \frac{i}{2\pi s}.$$

If $R(t) \geq |f_n(t) - e^{-t^2/2}|$ and $M(t) \geq |f_n(t)|$, Prawitz smoothing [5] gives, for $T > 0$, $0 < t_0 < 1$,

$$\begin{aligned} \Delta_n &\leq \int_0^{t_0} 2|K(s)|R(Ts) \, ds + \int_{t_0}^1 2|K(s)|M(Ts) \, ds \\ &\quad + \int_0^{t_0} 2|L(s)|e^{-T^2 s^2/2} \, ds + \frac{1}{2\pi} E_1(T^2 t_0^2/2), \end{aligned} \quad (1.3)$$

where $E_1(x) = \int_x^\infty e^{-y} dy/y$. The reference comparison controls the low frequencies. The modulus estimate controls the remaining frequencies. These are distinct roles and both are needed.

For completeness, the finite reduction is short. The one-summand estimate of Bentkus–Kirša [1] and Shevtsova’s bounds [7, 4] imply

$$R_1(P) \leq .3704, \quad R_n(P) \leq .3322(1 + .429/\rho), \quad R_n(P) \leq C_E + \frac{3\rho}{\sqrt{n}}.$$

Put $\rho_0 = 712569/589000$. The second bound is at most $9/20$ for $\rho \geq \rho_0$; the third is strictly below $9/20$ for $n \geq 8124$ and $\rho \leq \rho_0$. It remains to bound (1.3) on

$$2 \leq n \leq 8123, \quad 1 \leq \rho \leq \rho_0.$$

The closed-interval certificate in Appendix C does precisely this and proves $C \leq 9/20$. The reduction is analytic; its last finite inequality is computer-assisted. The supplementary arithmetic removes endpoint singularities before taking upper sums and encloses differences of nearby positive quantities without subtractive cancellation.

1.3 Why the matching two-point law is the right reference

An asymmetric two-point law has $\rho > 1$, so its Rademacher error allowance in (1.1) does not vanish. The reference should instead match the signed third moment. Shevtsova’s sharp moment inequality [3, Theorem 6] is equivalently

$$\rho \geq \rho_2(\gamma) := \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}}. \quad (1.4)$$

Equality identifies the standardized two-point law Q_γ . Thus the appropriate defect is

$$d(P) = \rho(P) - \rho_2(\gamma(P)), \quad d(P) = 0 \iff P = Q_\gamma.$$

Theorem 2.1 gives its explicit nonnegative density. Theorem 2.2 then proves the linear comparison

$$|\varphi_P(t) - \varphi_{Q_\gamma}(t)| \leq \frac{|t|^3 d(P)}{4(1 - |c_\gamma|)}, \quad c_\gamma = \frac{\gamma(\gamma^2 + 6)}{(\gamma^2 + 4)^{3/2}}. \quad (1.5)$$

The allowance now vanishes on the entire two-point family and is linear in the nonbinary excess. This is the structural improvement: replace a fixed symmetric reference by the binary law matching the first three moments. It is not a pointwise domination of distribution functions, nor a claim that (1.5) is smaller than (1.1) for every law and frequency.

The same matching also fixes the smoothing scale. If $m_1 = \mathbb{E}|X|$ and $s = \sqrt{\gamma^2 + 4}$, then

$$\rho_2(\gamma) + \mathbb{E}|Q_\gamma| = s, \quad 0 \leq \rho + m_1 - s \leq K_1(\gamma)d(P)$$

by Proposition 3.2. Here s is the binary lattice span. Retaining m_1 , as in Shevtsova’s modulus estimate [4], therefore aligns the Fourier cutoff with the same reference. This comparison, its coherent defect measure and the signed lattice smoothing form the starting point of Part I.

Part I

Structure, stability, and lattice asymptotics

2 Normalization and main results

Let Φ and $\phi = \Phi'$ denote respectively the standard normal cdf and density, and for any real random variable Z write

$$\varphi_Z(t) := \mathbb{E}e^{itZ}.$$

Let $Y, Y_1, Y_2, \dots$ be iid with

$$\mathbb{E}Y = 0, \quad \mathbb{E}Y^2 = 1, \quad \rho = \mathbb{E}|Y|^3 < \infty,$$

and put

$$S_n = \frac{Y_1 + \dots + Y_n}{\sqrt{n}}, \quad \Delta_n(Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}\{S_n \leq x\} - \Phi(x)|,$$

$$C_n = \sup_Y \frac{\sqrt{n} \Delta_n(Y)}{\rho}, \quad C_E = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}}.$$

The supremum in C_n is over centered variance-one laws with finite third absolute moment. For later local arguments, let X be centered and standardized and put

$$\gamma = \mathbb{E}X^3, \quad \rho = \mathbb{E}|X|^3, \quad s = \sqrt{\gamma^2 + 4}, \quad p = \frac{s - \gamma}{2s}, \quad q = 1 - p.$$

Then $\gamma = (q - p)/\sqrt{pq}$ and $s = (pq)^{-1/2}$. If $x = s(\eta + \theta)/\sqrt{n}$, $np + \eta \in \mathbb{Z}$, and $|\theta| \leq 1/2$, write

$$\mathcal{C}_n^+(X; \eta, \theta) := \frac{\sqrt{n}}{\mathbb{E}|X|^3} \left\{ \mathbb{P}\left(\frac{X_1 + \dots + X_n}{\sqrt{n}} \leq x\right) - \Phi(x) \right\},$$

$$\mathcal{C}_n^-(X; \eta, \theta) := \frac{\sqrt{n}}{\mathbb{E}|X|^3} \left\{ \Phi(x) - \mathbb{P}\left(\frac{X_1 + \dots + X_n}{\sqrt{n}} < x\right) \right\}.$$

Thus

$$C_n(X) := \frac{\sqrt{n} \Delta_n(X)}{\rho} = \sup_{\eta, \theta} \max\{\mathcal{C}_n^+, \mathcal{C}_n^-\}.$$

The following sharp moment inequality is due to Shevtsova [3, Theorem 6]; see also Mattner–Shevtsova [2, Example 1.3]. Their bound $|\gamma| \leq B(\rho)$ is equivalent to $\rho \geq \rho_2(\gamma)$, since

$$B(\rho) = \sqrt{\frac{\rho^2 + \rho\sqrt{\rho^2 + 8}}{2}} - 2$$

is the inverse of ρ_2 restricted to $[0, \infty)$. For $\gamma \neq 0$, the equality case is also contained in the osculatory interpolation of [2, Lemma 2.4(b)]; for $\gamma = 0$, it follows from Jensen's equality case. We include a self-contained factorized proof, recording the nonnegative defect density needed below.

Theorem 2.1 (exact two-point defect). *Let $\gamma = \mathbb{E}Y^3$ and*

$$s = \sqrt{\gamma^2 + 4}, \quad a = \frac{s - \gamma}{2}, \quad b = \frac{s + \gamma}{2}.$$

Let Z_γ have values $-a, b$ with probabilities $b/s, a/s$. Then

$$d(Y) := \rho - \rho_2(\gamma) \geq 0, \quad \rho_2(\gamma) = \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}},$$

and equality holds if and only if $Y \stackrel{d}{=} Z_\gamma$.

For any centered variance-one random variable Z , we use the functional notation

$$\gamma(Z) := \mathbb{E}Z^3, \quad \rho(Z) := \mathbb{E}|Z|^3, \quad d(Z) := \rho(Z) - \rho_2(\gamma(Z)), \quad (\text{N0})$$

and write

$$s_\gamma(\gamma) := \sqrt{\gamma^2 + 4}, \quad p(Z) := \frac{s_\gamma(\gamma(Z)) - \gamma(Z)}{2s_\gamma(\gamma(Z))}. \quad (\text{N1})$$

When the random variable is fixed, the corresponding unsubscripted symbols γ, ρ, d, s, p retain the shorter meanings introduced above.

Theorem 2.2 (linear Fourier stability). *With $c_\gamma = \gamma(\gamma^2 + 6)/(\gamma^2 + 4)^{3/2}$ and $\chi_\gamma(t) = \mathbb{E}e^{itZ_\gamma}$, one has, for all $t \in \mathbb{R}$,*

$$|\mathbb{E}e^{itY} - \chi_\gamma(t)| \leq \frac{|t|^3}{4(1 - |c_\gamma|)}d(Y).$$

We henceforth write Q_γ for the standardized two-point law Z_γ in Theorem 2.1. Thus, with $a = (\sqrt{\gamma^2 + 4} - \gamma)/2$ and $b = (\sqrt{\gamma^2 + 4} + \gamma)/2$,

$$\mathbb{P}\{Q_\gamma = -a\} = \frac{b}{a+b}, \quad \mathbb{P}\{Q_\gamma = b\} = \frac{a}{a+b}. \quad (\text{Q0})$$

This notation always denotes the law (or a random variable having that law), never an auxiliary polynomial.

Write

$$\overline{\mathbb{R}} := [-\infty, +\infty]$$

for the two-ended order compactification of $\mathbb{R}$.

Corollary 2.3 (coherent defect representation). *Let X be centered with variance one and third moment γ . There is a unique $p \in (0, 1)$ such that, with $q = 1 - p$,*

$$\gamma = \gamma_p := \frac{q - p}{\sqrt{pq}}.$$

Put $s = (pq)^{-1/2}$ and $Y = p + X/s$. Then

$$\mathbb{E}Y = \mathbb{E}Y^2 = \mathbb{E}Y^3 = p.$$

There are an explicit nonnegative defect density W_p , an explicit continuous response curve $r_{p,t} : \overline{\mathbb{R}} \rightarrow \mathbb{C}$, and a probability measure ν such that, with $b_p(t) = q + pe^{2\pi it}$,

$$\boxed{\varphi_Y(2\pi t) = b_p(t) + d r_\nu(t), \quad r_\nu(t) = \int r_{p,t}(y) d\nu(y),} \quad (\text{RF})$$

where

$$d = \mathbb{E}|X|^3 - \rho_2(\gamma_p), \quad \rho_2(\gamma_p) = \frac{p^2 + q^2}{\sqrt{pq}},$$

The probability measure ν is independent of t . The centered Bernoulli transform is

$$c_p(t) = e^{-2\pi i p t} b_p(t).$$

When $d = 0$, the law of Y is Bernoulli and the perturbation vanishes.

For clarity we fix here, before its first use, the signed smoothing notation. For a centered variance-one summand X , let

$$F_{n,X}(x) = \mathbb{P}\{(X_1 + \cdots + X_n)/\sqrt{n} \leq x\}, \quad F_{n,X}(x-) = \lim_{u \uparrow x} F_{n,X}(u),$$

and put

$$D_X^+(x) = F_{n,X}(x) - \Phi(x), \quad D_X^-(x) = \Phi(x) - F_{n,X}(x-).$$

For $\sigma \in \{+, -\}$, $D_X^\sigma(x)$ denotes the corresponding one of these two signed errors. The same sign convention is used for $\mathcal{C}_n^\sigma$ and for the one-sided operators below. Thus

$$\Delta_n(X) = \sup_x \max\{D_X^+(x), D_X^-(x)\}.$$

The Prawitz kernel used throughout is

$$K(t) = \frac{1-t}{2} + \frac{i}{2} \left\{ (1-t) \cot(\pi t) + \frac{1}{\pi} \right\}, \quad 0 < t < 1,$$

and

$$K(t) = \frac{i}{2\pi t} + \frac{1}{2} + O(t), \quad \overline{K(t)} = -\frac{i}{2\pi t} + \frac{1}{2} + O(t) \quad (t \downarrow 0).$$

The corresponding one-sided complex improper integral need not exist. For $\kappa \in \{K, \overline{K}\}$, $0 < b \leq 1$, and $g(t) = g(0) + O(t)$ with $g(0) \in \mathbb{R}$, define instead

$$\mathfrak{I}_{\kappa,b}(g) := 2 \lim_{\varepsilon \downarrow 0} \operatorname{Re} \int_\varepsilon^b \kappa(t) g(t) dt, \quad \mathfrak{I}_\kappa := \mathfrak{I}_{\kappa,1}. \quad (\text{P0})$$

Thus every later expression of the form $2 \operatorname{Re} \int_0^b \kappa(t) g(t) dt$ is understood as the limit in (P0).

Lemma 2.4 (Prawitz–Beurling half-line kernel). *Define, in the real-part improper sense of (P0),*

$$M(u) := \frac{1}{2} + 2\Re \int_0^1 K(t) e^{2\pi i t u} dt, \quad e_B(u) := M(u) - 1_{(-\infty, 0]}(u). \quad (\text{P1})$$

Then

$$M(u) \geq 1_{(-\infty, 0]}(u), \quad 1 - M(-u) \leq 1_{(-\infty, 0)}(u), \quad e_B(u) \geq 0 \quad (\text{P2})$$

for every $u \in \mathbb{R}$. At the integer nodes the identities are exact:

$$M(j) = 1_{\{j \leq 0\}}, \quad 1 - M(-j) = 1_{\{j < 0\}} \quad (j \in \mathbb{Z}). \quad (\text{P3})$$

If U is integrable and $\psi_U(t) = \mathbb{E} e^{2\pi i t U}$, then all expectations below are finite and

$$\mathbb{E} M(U) = \frac{1}{2} + \mathfrak{I}_K(\psi_U), \quad \mathbb{E}\{1 - M(-U)\} = \frac{1}{2} - \mathfrak{I}_{\overline{K}}(\psi_U). \quad (\text{P4})$$

Thus taking expectations in (P2) involves no formal interchange with the t^{-1} part of the kernel.

Proof. The one-sided smoothing inequality of [5], in the normalization whose Fourier kernel is the displayed K , applied to the point mass at u , gives $1_{(-\infty, 0]}(u) \leq M(u)$. Thus the only imported assertion here is exactly Prawitz's standard half-line inequality; the second inequality in (P2) follows from the first at $-u$. We verify the cardinal and integrability facts used later. The independent partial-fraction calculation (4.52a)–(4.52b) below also checks the sign of the resulting Beurling error away from the integer nodes.

For $j = 0$, the real part of K gives $M(0) = 1$. If $j \in \mathbb{Z} \setminus \{0\}$, the elementary cosine term and the bare sine term integrate to zero, and hence

$$M(j) = \frac{1}{2} - \int_0^1 (1-t) \cot(\pi t) \sin(2\pi j t) dt. \quad (\text{P5})$$

The product following $(1 - t)$ is invariant under $t \mapsto 1 - t$. Moreover, for $m \geq 1$,

$$\cot(\pi t) \sin(2\pi m t) = 1 + \cos(2\pi m t) + 2 \sum_{\ell=1}^{m-1} \cos(2\pi \ell t), \quad (\text{P6})$$

and the formula for $-m$ is its negative. Thus the integral in (P5) is $\frac{1}{2} \operatorname{sgn} j$, proving (P3).

Finally $2 \int_0^1 \Re K(t) dt = 1/2$, so for each real u

$$M(u) = 1 + 2\Re \int_0^1 K(t) \{e^{2\pi i t u} - 1\} dt. \quad (\text{P7})$$

This integral is absolutely convergent near zero because $K(t) = i/(2\pi t) + O(1)$ and $|e^{2\pi i t u} - 1| \leq 2\pi t|u|$. For integrable U , the same bound with $\mathbb{E}|U|$ permits Fubini in (P7). It gives the first identity in (P4), and conjugation gives the second. The same estimate supplies an integrable affine bound for $M(U)$. $\square$

We also fix the associated order operators here, before any asymptotic use. If X is real-valued and integrable, $n \geq 1$, $h > 0$, and $x \in \mathbb{R}$, put

$$g_{X,n,h,x}(t) = e^{-2\pi i x \sqrt{n} t/h} \varphi_X(2\pi t/h)^n, \\ \mathcal{U}_{X,n,h}^+(x) := \frac{1}{2} + \Im K(g_{X,n,h,x}), \quad \mathcal{U}_{X,n,h}^-(x) := \frac{1}{2} - \Im \overline{K}(g_{X,n,h,x}). \quad (\text{P8})$$

For $Z = (X_1 + \dots + X_n - x\sqrt{n})/h$, the lemma gives

$$\mathcal{U}_{X,n,h}^+(x) = \mathbb{E}M(Z), \quad \mathcal{U}_{X,n,h}^-(x) = \mathbb{E}\{1 - M(-Z)\}, \quad (\text{P9})$$

and therefore

$$F_{n,X}(x) = \mathcal{U}_{X,n,h}^+(x) - \mathbb{E}e_B(Z), \\ F_{n,X}(x-) = \mathcal{U}_{X,n,h}^-(x) + \mathbb{E}e_B(-Z). \quad (\text{P10})$$

Thus $\mathcal{U}^+$ is an upper operator for the closed half-line and $\mathcal{U}^-$ a lower operator for the open half-line. In particular,

$$D_X^+(x) \leq \mathcal{U}_{X,n,h}^+(x) - \Phi(x), \quad D_X^-(x) \leq \Phi(x) - \mathcal{U}_{X,n,h}^-(x). \quad (\text{P11})$$

Theorem 2.5 (two-point asymptotics). *If $C_Q^*(n)$ is the supremum defining C_n restricted to standardized two-point laws, then*

$$C_Q^*(n) = C_E - \frac{\kappa_\infty}{n} + o(n^{-1}), \quad \kappa_\infty = \frac{867 + 287\sqrt{10}}{25920\sqrt{2\pi}}.$$

Theorem 2.6 (qualitative eventual upper bound). *There exists an integer N_0 such that*

$$C_n \leq C_E \quad (n \geq N_0).$$

3 The exact moment defect

The elementary identities

$$ab = 1, \quad a + b = s, \quad b - a = \gamma$$

show that Z_γ has moments $0, 1, \gamma$ of orders one, two and three. Its third absolute moment is

$$\rho_2(\gamma) = \frac{b}{s}a^3 + \frac{a}{s}b^3 = s - \frac{2}{s} = \frac{\gamma^2 + 2}{s}.$$

Proof of Theorem 2.1. For $\gamma \neq 0$, the density w_γ below is the normalized osculatory error of Mattner–Shevtsova [2, Lemma 2.4(b)], applied to x^3 at the nodes $-a, b$. Its formula extends continuously to $\gamma = 0$. We record its explicit factorization. Define

$$c_\gamma = \frac{\gamma(\gamma^2 + 6)}{s^3}$$

and

$$w_\gamma(x) = |x|^3 - c_\gamma x^3 - \frac{12}{s^3} x^2 + \frac{6\gamma}{s^3} x + \frac{4}{s^3}.$$

Notice that the denominator occurring below is strictly positive, since

$$1 - c_\gamma^2 = \frac{(\gamma^2 + 4)^3 - \gamma^2(\gamma^2 + 6)^2}{(\gamma^2 + 4)^3} = \frac{4(3\gamma^2 + 16)}{(\gamma^2 + 4)^3} > 0.$$

Using $ab = 1$, $a + b = s$ and $b - a = \gamma$, direct multiplication gives

$$w_\gamma(x) = \frac{2a^2}{s^3}(x - b)^2\{2 + (a + 3b)x\}, \quad x \geq 0, \quad (3.1)$$

$$w_\gamma(-t) = \frac{2b^2}{s^3}(t - a)^2\{2 + (3a + b)t\}, \quad t \geq 0. \quad (3.2)$$

Both right-hand sides are nonnegative, and they vanish only at b and a , respectively. Taking expectations and using the first three moments,

$$\mathbb{E}w_\gamma(Y) = \rho - c_\gamma\gamma - \frac{8}{s^3}.$$

The identity

$$c_\gamma\gamma + \frac{8}{s^3} = \frac{\gamma^4 + 6\gamma^2 + 8}{s^3} = \frac{(\gamma^2 + 2)(\gamma^2 + 4)}{s^3} = \rho_2(\gamma)$$

proves $d(Y) = \mathbb{E}w_\gamma(Y) \geq 0$. Equality forces Y to be supported on $\{-a, b\}$. If $r = \mathbb{P}\{Y = b\}$, the equation $\mathbb{E}Y = 0$ gives $r = a/(a + b) = a/s$; hence $Y \stackrel{d}{=} Z_\gamma$. The converse follows from the displayed factorization. $\square$

Corollary 3.1 (quadratic concentration). *Put $a_* = \frac{1}{2}(s - |\gamma|)$. Then*

$$\mathbb{E} \operatorname{dist}(Y, \{-a, b\})^2 \leq \frac{s^3}{4a_*^2} d(Y).$$

Proof. In (3.1) and (3.2) the final linear factor is at least 2, while $\min(a, b) = a_*$. Therefore

$$w_\gamma(x) \geq \frac{4a_*^2}{s^3} \operatorname{dist}(x, \{-a, b\})^2.$$

Integrate and use $\mathbb{E}w_\gamma(Y) = d(Y)$. $\square$

Proposition 3.2 (stability of the smoothing scale). *With*

$$b_* = \frac{s + |\gamma|}{2}, \quad K_1(\gamma) = \frac{s^2}{a_*(a_* + 3b_*)},$$

one has

$$0 \leq \rho + \mathbb{E}|Y| - s \leq K_1(\gamma)d(Y).$$

Proof. By reflection assume $\gamma \geq 0$ and define

$$v_\gamma(x) = |x|^3 + |x| - \frac{\gamma}{s}x^3 - \frac{4}{s}x^2 + \frac{\gamma}{s}x.$$

Direct factorization gives

$$v_\gamma(x) = \frac{2a}{s}x(x-b)^2 \quad (x \geq 0), \quad v_\gamma(-t) = \frac{2b}{s}t(t-a)^2 \quad (t \geq 0).$$

Thus $v_\gamma \geq 0$ and $\mathbb{E}v_\gamma(Y) = \rho + \mathbb{E}|Y| - s$. Dividing these factorizations by (3.1)–(3.2), and using monotonicity of the resulting fractional-linear functions, yields

$$\frac{v_\gamma(x)}{w_\gamma(x)} \leq \frac{s^2}{a(a+3b)}, \quad \frac{v_\gamma(-t)}{w_\gamma(-t)} \leq \frac{s^2}{b(3a+b)}.$$

The first constant is larger when $b \geq a$. Take expectations and then reflect to cover $\gamma < 0$. $\square$

4 Proof of the characteristic-function stability theorem

Let $H_{\gamma,t}$ be the cubic Hermite interpolant of $x \mapsto e^{itx}$ at $-a, b$, matching values and first derivatives. Since Y and Z_γ have identical moments through order three,

$$\mathbb{E}e^{itY} - \chi_\gamma(t) = \mathbb{E}\{e^{itY} - H_{\gamma,t}(Y)\}. \quad (4.1)$$

By reflection we may suppose $\gamma, t \geq 0$. Put

$$p = \frac{a}{s}, \quad q = \frac{b}{s} = 1 - p, \quad z = ts,$$

and let P_z be the cubic Hermite interpolant of e^{izy} at 0, 1. Write $E_z(y) = e^{izy} - P_z(y)$.

For $x \geq 0$, $y = (x+a)/s$ and a direct change of variables gives

$$\frac{w_\gamma(x)}{1 - c_\gamma} = s^3(y-1)^2(y - \alpha_p), \quad \alpha_p = \frac{p}{3-2p}.$$

The function $p \mapsto \alpha_p$ is increasing. Hence it suffices to prove

$$|E_z(y)| \leq \frac{|z|^3}{4}(y-1)^2 \frac{2y(1-y)}{3-2y}, \quad 0 < y < \frac{1}{2}, \quad (4.2)$$

$$|E_z(y)| \leq \frac{|z|^3}{4}(y-1)^2(y - \frac{1}{4}), \quad y \geq \frac{1}{2}. \quad (4.3)$$

Indeed, when $p \leq y \leq 1/2$,

$$y - \alpha_p \geq y - \alpha_y = \frac{2y(1-y)}{3-2y},$$

and when $y \geq 1/2$, one has $y - \alpha_p \geq y - \alpha_{1/2} = y - 1/4$. On the negative half-axis the identity

$$E_z(1-y) = e^{iz} \overline{E_z(y)}$$

follows from uniqueness of Hermite interpolation. To make the reduction explicit, write $t = -x \geq 0$ and $y' = (t+b)/s = 1-y \geq q$, where $q = 1-p \geq 1/2$. Direct substitution in (3.2) gives

$$\frac{w_\gamma(-t)}{1 - c_\gamma} = s^3(y'-1)^2 A_q(y' - \beta_q), \quad A_q = \frac{q^2(3-2q)}{p^2(1+2q)}, \quad \beta_q = \frac{q}{3-2q}.$$

Here $A_q \geq 1$, so the difference $A_q(y' - \beta_q) - (y' - 1/4)$ is increasing in y' . After multiplication by the positive denominator $4p^2(1 + 2q)$, its value at $y' = q$ is

$$p(16q^3 - 6q^2 - 3q + 1) = p(2q - 1)(8q^2 + q - 1) \geq 0,$$

because $q \geq 1/2$. Thus $A_q(y' - \beta_q) \geq y' - 1/4$, and (4.2)–(4.3) cover both half-axes.

The Hermite–Genocchi formula gives

$$E_z(y) = y^2(y - 1)^2 f_z[0, 0, y, 1, 1], \quad f_z(u) = e^{izu}.$$

Let M_y denote the probability B-spline with knots $0, 0, y, 1, 1$. The normalized formula, including the simplex-volume factor, is

$$f_z[0, 0, y, 1, 1] = \frac{1}{4!} \int_0^1 f_z^{(4)}(u) M_y(u) du. \quad (\text{HG})$$

For $0 < y < 1$ its density is

$$M_y(u) = \begin{cases} \frac{4u^2}{y^2} \{3y - (1 + 2y)u\}, & 0 \leq u \leq y, \\ \frac{4(1 - u)^2}{(1 - y)^2} \{(3 - 2y)u - y\}, & y \leq u \leq 1, \end{cases}$$

with the limiting interpretation at $y = 0, 1$. Direct differentiation shows that M_y is unimodal, vanishes at the endpoints, and has maximum at most 2; hence its total variation is at most 4. Stieltjes integration by parts in (HG) yields

$$|E_z(y)| \leq \frac{|z|^3}{6} y^2 (y - 1)^2, \quad 0 \leq y \leq 1. \quad (4.4)$$

For $0 < y \leq 1/2$, (4.2) follows from $2y^2 - 6y + 3 \geq 0$. For $1/2 \leq y \leq 1$, (4.3) follows from $8y^2 - 12y + 3 \leq 0$. It remains to treat $y \geq 1$.

Put $v = z(y - 1)$, $w = zy = v + z$, and

$$A(r) = \int_0^1 e^{i\theta r} d\theta.$$

Define the normalized error

$$F = \frac{e^{-iz} E_z(y)}{z^2 (y - 1)^2},$$

with its continuous value at $z = 0$ or $y = 1$. Cancelling the double zero at $y = 1$ gives the exact identity

$$F = iw \int_0^1 u(1 - u) e^{-izu} \{1 - A(wu)\} du.$$

Consequently, by Cauchy–Schwarz,

$$|F| \leq w \sqrt{J_2(w)/6}, \quad J_2(w) = \int_0^1 u(1 - u) |1 - A(wu)|^2 du. \quad (4.5)$$

Lemma 4.1. *For every $w \geq 0$,*

$$J_2(w) \leq \frac{243}{1250} = 6 \left(\frac{9}{50} \right)^2.$$

Proof. For $r \geq 0$,

$$|1 - A(r)|^2 = 1 - 2\frac{\sin r}{r} + 2\frac{1 - \cos r}{r^2}.$$

On $0 \leq r \leq 10$ insert the alternating Taylor bounds

$$\begin{aligned} \frac{\sin r}{r} &\geq \sum_{k=0}^8 \frac{(-1)^k r^{2k}}{(2k+1)!} - \frac{r^{18}}{19!}, \\ \frac{1 - \cos r}{r^2} &\leq \sum_{k=1}^8 \frac{(-1)^{k+1} r^{2k-2}}{(2k)!} + \frac{r^{16}}{18!}. \end{aligned}$$

These inequalities are not an appeal to a numerical alternating-series test: the first is the global lower sine polynomial through degree 19 and the second is the global upper polynomial for $1 - \cos r$ through degree 18. Repeated differentiation reduces both remainders to $1 - \cos r \geq 0$, proving the stated signs for every $r \geq 0$. Let $P(r)$ be the resulting upper polynomial and $Q(w) = \int_0^1 u(1-u)P(wu) du$. Each monomial is integrated by

$$\int_0^1 u^{j+1}(1-u) du = \frac{1}{(j+2)(j+3)}.$$

After $x = w^2/100$, the degree-nine polynomial $243/1250 - Q(10\sqrt{x})$ has the positive Bernstein coefficients printed in Appendix A.1, separately on $[0, 1/2]$ and $[1/2, 1]$. This proves the estimate for $w \leq 10$.

For $w \geq 10$, direct integration gives

$$J_2(w) = \frac{1}{6} + \frac{2 \operatorname{Cin}(w)}{w^2} - \frac{4}{w^2} + \frac{4 \sin w}{w^3}, \quad \operatorname{Cin}(w) := \int_0^w \frac{1 - \cos t}{t} dt. \quad (\text{J1})$$

We bound the tail without invoking a special-function identity. The global lower Taylor polynomial of degree 22 for cosine gives, after termwise integration on $[0, 10]$,

$$\operatorname{Cin}(10) \leq C_{22} := \sum_{k=1}^{11} \frac{(-1)^{k+1} 10^{2k}}{2k(2k)!} = \frac{2702150949482225}{905566992594264}.$$

The direction of the Taylor bound follows, as above, by repeated differentiation down to $1 - \cos t \geq 0$. Moreover, integration by parts gives for every $w \geq 10$

$$\left| \int_{10}^w \frac{\cos t}{t} dt \right| \leq \frac{1}{w} + \frac{1}{10} + \int_{10}^w \frac{dt}{t^2} = \frac{1}{5}.$$

Consequently, with

$$D := C_{22} + \frac{1}{5} = \frac{14416321740005389}{4527834962971320},$$

we have

$$\operatorname{Cin}(w) \leq D + \log(w/10)$$

and hence, by (J1),

$$J_2(w) \leq K(w) := \frac{1}{6} + \frac{2D + 2\log(w/10) - 4}{w^2} + \frac{4}{w^3}.$$

Since $D > 5/2$,

$$K'(w) = \frac{10 - 4D - 4\log(w/10)}{w^3} - \frac{12}{w^4} < 0 \quad (w \geq 10).$$

Finally, exact rational arithmetic yields

$$K(10) = \frac{43998176831418013}{226391748148566000}, \quad \frac{243}{1250} - K(10) = \frac{61895043316087}{1131958740742830000} > 0.$$

□

Lemma 4.1 and (4.5) give $|F| \leq 9w/50$. For $y \geq 1$,

$$\frac{z(y - 1/4)}{4} = \frac{w}{4} \left(1 - \frac{1}{4y}\right) \geq \frac{3w}{16} > \frac{9w}{50}.$$

Thus (4.3) holds for all $y \geq 1$. Substituting the resulting pointwise Hermite estimate into (4.1) and using $\mathbb{E}w_\gamma(Y) = d(Y)$ proves Theorem 2.2.

5 The coherent defect representation

The common defect measure retains the signed Hermite error at every frequency. Its explicit density and moment cancellation enter the outer, shape and low-frequency estimates below.

Proof of Corollary 2.3. The function $p \mapsto (1-2p)/\sqrt{p(1-p)}$ is continuous and strictly decreasing from $+\infty$ to $-\infty$, so the required p exists and is unique. From $Y = p + \sqrt{pq} X$, $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\mathbb{E}X^3 = \gamma_p$, direct expansion gives $\mathbb{E}Y = \mathbb{E}Y^2 = \mathbb{E}Y^3 = p$.

The density is

$$W_p(y) = \begin{cases} \frac{2q}{p\sqrt{pq}} y^2 \{3p - (1+2p)y\}, & y \leq p, \\ \frac{2p}{q\sqrt{pq}} (1-y)^2 \{(3-2p)y - p\}, & y \geq p. \end{cases} \quad (3.2)$$

Both bracketed factors are positive on their stated half-lines: at the common endpoint they equal $2pq$, and they improve away from it. Hence $W_p \geq 0$, with zero set $\{0, 1\}$. To make the moment cancellation explicit, put $r = \sqrt{pq}$. On both half-lines the same polynomial identity holds:

$$r^3 \{W_p(y) - |r^{-1}(y-p)|^3\} = \mathcal{Q}_p(y),$$

where

$$\mathcal{Q}_p(y) = -p^3 + 3p^2y + (6p^3 - 12p^2 + 3p)y^2 + (-4p^3 + 6p^2 - 1)y^3.$$

Since $\mathbb{E}Y = \mathbb{E}Y^2 = \mathbb{E}Y^3 = p$,

$$\mathbb{E}\mathcal{Q}_p(Y) = -pq(p^2 + q^2).$$

Consequently

$$\mathbb{E}W_p(Y) = \mathbb{E}|s(Y-p)|^3 - \rho_2(\gamma_p) = d. \quad (3.3)$$

Let H_t be the cubic Hermite interpolant of $y \mapsto e^{2\pi ity}$ at 0 and 1, matching values and first derivatives, and define

$$r_{p,t}(y) = \frac{e^{2\pi ity} - H_t(y)}{W_p(y)}. \quad (3.4)$$

The double zeros in numerator and denominator make the singularities at 0, 1 removable. As $y \rightarrow +\infty$ and $y \rightarrow -\infty$, separately, the exponential term divided by $W_p(y)$ tends to zero, while $H_t(y)/W_p(y)$ has a finite limit, since both polynomials have degree at most three and W_p has a nonzero cubic leading coefficient on each tail. Thus (3.4) has finite, possibly different, limits at the two ends and extends continuously to $\mathbb{R}$ and is bounded.

If $d > 0$, set

$$d\nu(y) = d^{-1}W_p(y) d\mathbb{P}_Y(y).$$

Equation (3.3) says that ν is a probability measure. Since H_t is cubic and Y has the first three moments of a Bernoulli variable B of parameter p ,

$$\mathbb{E}H_t(Y) = \mathbb{E}H_t(B) = \mathbb{E}e^{2\pi itB} = b_p(t).$$

Therefore

$$\varphi_Y(2\pi t) - b_p(t) = \mathbb{E}\{e^{2\pi itY} - H_t(Y)\} = d \int r_{p,t}(y) d\nu(y),$$

which proves (RF). If $d = 0$, take, for example, $\nu = \delta_0$; its choice is immaterial because it is multiplied by d . Equation (3.3) and the zero set of W_p force $Y \in \{0, 1\}$ almost surely; its mean fixes the Bernoulli weights and the same identity holds with zero right side. $\square$

6 The standardized two-point branch

Let $0 < p \leq 1/2$, $q = 1 - p$, and let Q_p denote the standardized two-point law which takes the values $\sqrt{q/p}$ and $-\sqrt{p/q}$ with probabilities p and q , respectively. Equivalently, $Q_p = Q_{\gamma_p}$ for $\gamma_p = (q - p)/\sqrt{pq}$ in the notation (Q0). Reflection sends p to q , preserves $\rho(p)$ and the Kolmogorov distance, and therefore makes the restriction $p \leq 1/2$ harmless. If $K \sim \text{Bin}(n, p)$, then

$$\frac{Q_{p,1} + \cdots + Q_{p,n}}{\sqrt{n}} = \frac{K - np}{\sqrt{npq}}, \quad \rho(p) = \frac{p^2 + q^2}{\sqrt{pq}}.$$

Writing $z_k = (k - np)/\sqrt{npq}$ and $B_{n,p}(k) = \mathbb{P}\{K \leq k\}$, with $B_{n,p}(-1) := 0$, constancy of the distribution function between neighbouring atoms gives the exact identity

$$\Delta_n(Q_p) = \max_{0 \leq k \leq n} \max(|B_{n,p}(k) - \Phi(z_k)|, |B_{n,p}(k-1) - \Phi(z_k)|).$$

Put $s_p = \sqrt{pq}$, $D = p^2 + q^2$, and $\phi_0 = (2\pi)^{-1/2}$. We use the following uniform saddle expansion. Its purpose is not merely to identify formal coefficients: the displayed remainder is uniform in every parameter subsequently optimized.

Lemma 6.1 (uniform central binomial saddle). *Let $\epsilon = n^{-1/2} \leq 10^{-24}$, $p \in [2/5, 11/25]$, and let $k = np + \eta$ be an integer with $|\eta| \leq 1$. Then*

$$\frac{B_{n,p}(k) - \Phi(\eta\epsilon/s_p)}{\rho(p)\epsilon} = G(p) + \epsilon^2 F(p, \eta) + \mathcal{R}(p, \eta, \epsilon), \quad (6.1)$$

where

$$G(p) = \phi_0 \frac{2-p}{3D},$$

$$F(p, \eta) = -\frac{\phi_0}{540pqD} \{270q\eta^2 + 45(p^2 - 7p + 5)\eta + p^3 + 21p^2 - 69p + 46\},$$

and

$$|\mathcal{R}(p, \eta, \epsilon)| \leq 10^{10} \epsilon^{11/4}.$$

On the symmetric wider compact range $p \in [1/6, 5/6]$ the same formula holds with the bound

$$|\mathcal{R}(p, \eta, \epsilon)| \leq 10^{20} \epsilon^{11/4}.$$

Under $K \mapsto n - K$, $p \leftrightarrow 1 - p$ and $\eta \leftrightarrow -\eta$, this gives the signed expansion at the left edge of a jump. Its leading coefficient is

$$G_-(p) = \phi_0 \frac{1+p}{3D} \leq \phi_0 \quad (0 < p \leq 1/2).$$

Proof. The exact incomplete-beta identity is

$$B_{n,p}(k) = I_q(n - k, k + 1) = \frac{\int_0^q t^{n-k-1} (1-t)^k dt}{\int_0^1 t^{n-k-1} (1-t)^k dt}.$$

Set $t = q + \epsilon s_p y$, $A = \sqrt{p/q}$ and $B = \sqrt{q/p}$. The common constant cancels, and the logarithm of the remaining density is

$$L(y) = (nq - \eta - 1) \log(1 + \epsilon A y) + (np + \eta) \log(1 - \epsilon B y). \quad (\text{S1})$$

The denominator has limits $-B/\epsilon, A/\epsilon$, while the numerator ends at zero. Expanding the two logarithms algebraically gives

$$L(y) = -\frac{y^2}{2} + \sum_{j \geq 1} \epsilon^j \ell_j(y), \quad \ell_j(y) = c_j y^{j+2} + d_j y^j, \quad (\text{S2})$$

where

$$c_j = \frac{q(-1)^{j+3} A^{j+2} - p B^{j+2}}{j+2}, \quad d_j = \frac{-(\eta+1)(-1)^{j+1} A^j - \eta B^j}{j}. \quad (\text{S3})$$

Define $E_0 = 1$ and

$$E_m = \frac{1}{m} \sum_{j=1}^m j \ell_j E_{m-j}.$$

Thus $\exp(\sum_{j \geq 1} \epsilon^j \ell_j) = \sum_{m \geq 0} \epsilon^m E_m$. Equations (S2)–(S3) imply $E_m(-y) = (-1)^m E_m(y)$. This parity is retained throughout the quotient; in particular, no spurious order ϵ^4 term is created.

We give the complete remainder ledger. Put $R_\epsilon = \epsilon^{-1/12}$ and restrict first to $|y| \leq R_\epsilon$. Here $\max(A, B) < 5/4$ and $\max(A, B)\epsilon R_\epsilon < 1/2$. The coefficient norm in y obeys

$$\|\ell_j\|_1 \leq b_j := \frac{(5/4)^{j+2}}{j+2} + \frac{3(5/4)^j}{j} \quad (1 \leq j \leq 4),$$

and hence, by the displayed recursion,

$$\|E_1\|_1 < 4.402, \quad \|E_2\|_1 < 12.65, \quad \|E_3\|_1 < 29.9, \quad \|E_4\|_1 < 62.5. \quad (\text{S4})$$

The geometric remainder of the logarithms after ℓ_4 is at most

$$\left(\frac{(5/4)^7}{7} + \frac{3(5/4)^5}{5} \right) \frac{\epsilon^5 R_\epsilon^7}{1 - (5/4)\epsilon R_\epsilon} < 6\epsilon^5 R_\epsilon^7. \quad (\text{S5})$$

If $P_4 = \sum_{m=0}^4 \epsilon^m E_m$, coefficient majorization by $\exp(z + z^2 + z^3 + z^4)$ gives

$$\left| \exp \left(\sum_{j=1}^4 \epsilon^j \ell_j \right) - P_4 \right| \leq 88(12\epsilon R_\epsilon^3)^5.$$

Together with (S5), this yields

$$|e^{L(y)+y^2/2} - P_4(y)| \leq 2.2 \cdot 10^7 \epsilon^{15/4}. \quad (\text{S6})$$

No tail of the beta integral is hidden in (S6). Differentiating (S1), using $pq \geq 6/25$ and $n \geq 10^{48}$, gives

$$L''(y) \leq -0.4, \quad L(0) = 0, \quad |L'(0)| < 5\epsilon.$$

Consequently

$$L(y) \leq 32\epsilon^2 - 0.2(y - y_0)^2, \quad |y_0| < 12.5\epsilon,$$

and both omitted tails are smaller than $8R_\epsilon^{-1}e^{-0.19R_\epsilon^2} < 8\epsilon^5$. The tails of $e^{-y^2/2}P_4(y)$ are smaller than $10^{40}e^{-R_\epsilon^2/3} < \epsilon^5$. On $[-1, 1]$ one has $L(y) \geq -1/2 - 11\epsilon$, so the denominator in the beta quotient is larger than one.

For $0 \leq m \leq 4$ define

$$D_m = \int_{\mathbb{R}} e^{-y^2/2} E_m(y) dy, \quad A_m = \int_{-\infty}^0 e^{-y^2/2} E_m(y) dy, \quad N_4 = \sum_{m=0}^4 \epsilon^m A_m.$$

The parity noted above then gives

$$N_4 = \frac{1}{2}(D_0 + \epsilon^2 D_2 + \epsilon^4 D_4) + \epsilon A_1 + \epsilon^3 A_3.$$

Here is the coefficient calculation. Write $r = s_p = \sqrt{pq}$, so that $A = p/r$, $B = q/r$, and put $D = p^2 + q^2$. From (S3),

$$\begin{aligned} c_1 &= \frac{p-q}{3r}, & d_1 &= -\frac{\eta+p}{r}, & E_1 &= c_1 y^3 + d_1 y, \\ c_2 &= -\frac{1-3r^2}{4r^2}, & d_2 &= \frac{\eta(p-q)+p^2}{2r^2}, & E_2 &= c_2 y^4 + d_2 y^2 + \frac{1}{2}(c_1 y^3 + d_1 y)^2, \\ c_3 &= \frac{(p-q)D}{5r^3}, & d_3 &= -\frac{\eta(1-3r^2)+p^3}{3r^3}, & E_3 &= (c_3 y^5 + d_3 y^3) + E_1(c_2 y^4 + d_2 y^2) + \frac{1}{6}E_1^3. \end{aligned} \quad (\text{S6a})$$

Using

$$D_0 = \sqrt{2\pi}, \quad \frac{1}{D_0} \int_{\mathbb{R}} y^{2j} e^{-y^2/2} dy = (2j-1)!!, \quad \int_{-\infty}^0 y^{2j+1} e^{-y^2/2} dy = -2^j j!,$$

one obtains, without any numerical integration,

$$A_1 = -2c_1 - d_1 = \frac{1}{r} \left\{ \eta + \frac{2-p}{3} \right\}, \quad (\text{S6b})$$

$$\frac{D_2}{D_0} = 3c_2 + d_2 + \frac{15}{2}c_1^2 + 3c_1 d_1 + \frac{d_1^2}{2} =: \delta_2, \quad (\text{S6c})$$

$$\begin{aligned} A_3 &= -8c_3 - 2d_3 - 48c_1 c_2 - 8(c_1 d_2 + d_1 c_2) - 2d_1 d_2 \\ &\quad - 64c_1^3 - 24c_1^2 d_1 - 4c_1 d_1^2 - \frac{d_1^3}{3}. \end{aligned} \quad (\text{S6d})$$

In particular, direct collection over the common denominator $540r^3$ gives the checkable identity

$$A_3 - A_1 \delta_2 + \frac{\eta^3}{6r^3} = -\frac{270q\eta^2 + 45(p^2 - 7p + 5)\eta + p^3 + 21p^2 - 69p + 46}{540r^3}. \quad (\text{S6e})$$

For clarity, we also spell out the quotient bookkeeping. Equations (S4)–(S6), including the two truncated tails, give for the exact denominator and numerator of the beta quotient

$$\begin{aligned} \mathcal{D} &= D_0 + \epsilon^2 D_2 + \epsilon^4 D_4 + e_D, \\ \mathcal{N} &= \frac{1}{2}(D_0 + \epsilon^2 D_2 + \epsilon^4 D_4) + \epsilon A_1 + \epsilon^3 A_3 + e_N, \quad |e_D| + |e_N| < 1.2 \cdot 10^8 \epsilon^{15/4}. \end{aligned}$$

The Gaussian moments and (S4) give

$$|A_1| < 12, \quad |D_2| < 500, \quad |A_3| < 10^5, \quad |D_4| < 10^8.$$

Thus, with $x = (\epsilon^2 D_2 + \epsilon^4 D_4)/D_0$, one has $|x| < 201\epsilon^2$. The exact identity $(1+x)^{-1} = 1 - x + x^2/(1+x)$ yields

$$\frac{\mathcal{N}}{\mathcal{D}} = \frac{1}{2} + \epsilon \frac{A_1}{D_0} + \epsilon^3 \left\{ \frac{A_3}{D_0} - \frac{A_1 D_2}{D_0^2} \right\} + O(1.2 \cdot 10^8 \epsilon^{15/4}) + O(2.1 \cdot 10^8 \epsilon^5). \quad (\text{S6f})$$

Finally,

$$\Phi(\eta\epsilon/r) = \frac{1}{2} + \phi_0 \frac{\eta\epsilon}{r} - \phi_0 \frac{\eta^3 \epsilon^3}{6r^3} + O(10\epsilon^5).$$

Since $D_0^{-1} = \phi_0$ and $\rho(p) = D/r$, (S6b) gives the coefficient $G(p) = \phi_0(2-p)/(3D)$, while (S6e) gives exactly the displayed F . After division by $\rho(p)\epsilon \geq \epsilon$, (S6f) gives $|\mathcal{R}| < 10^{10} \epsilon^{11/4}$.

For the wider interval replace $5/4$ by $9/4$. Then

$$(b_1, b_2, b_3, b_4) < (10.55, 14.01, 22.93, 40.85)$$

and the same recursion gives

$$(\|E_1\|_1, \|E_2\|_1, \|E_3\|_1, \|E_4\|_1) < (10.55, 69.7, 367, 1676).$$

The logarithmic remainder constant in (S5) is then less than 160, and the analogue of (S6) is

$$|e^{L(y)+y^2/2} - P_4(y)| \leq 9.8 \cdot 10^{13} \epsilon^{15/4}.$$

Since $pq \geq 5/36$, the same concavity argument applies (already for $n \geq 120$); the raw quotient error is below $5 \cdot 10^{14} \epsilon^{15/4}$. The deliberately coarse coefficient bounds $(10^4, 10^8, 10^{12}, 10^{16})$ for (A_1, D_2, A_3, D_4) leave, after $|x| < 40000001 \epsilon^2$, and the same exact reciprocal identity leaves a polynomial residual below $5 \cdot 10^{19} \epsilon^5$. Together with the $10 \epsilon^5$ normal remainder and division by $\rho \epsilon$, this gives the asserted $10^{20} \epsilon^{11/4}$ bound. Reflection $K \mapsto n - K$, $p \leftrightarrow q$ gives the last assertion. $\square$

Lemma 6.2 (bounded-node consequence). *Put*

$$p_* = \frac{4 - \sqrt{10}}{2}, \quad q_* = 1 - p_*, \quad s_* = (p_* q_*)^{-1/2}, \quad \rho_* = \frac{p_*^2 + q_*^2}{\sqrt{p_* q_*}}, \quad \eta_* = -\frac{20 + \sqrt{10}}{72},$$

and $a_{\text{tan},*} = s_*^3 q_* \phi_0 / 2$. For every fixed $M < \infty$ and every fixed interval U with $p_* \in U \subseteq (2/5, 11/25)$, uniformly for $p \in U$ and $|\eta| \leq M$ with $np + \eta \in \mathbb{Z}$,

$$\frac{\sqrt{n}}{\rho(p)} \{B_{n,p}(np + \eta) - \Phi(\eta/\sqrt{npq})\} = G(p) + \frac{F(p, \eta)}{n} + o_M(n^{-1}). \quad (\text{S7})$$

Consequently, if $p \rightarrow p_*$, then

$$\frac{\sqrt{n}}{\rho(p)} \{B_{n,p}(np + \eta) - \Phi(\eta/\sqrt{npq})\} \leq C_Q^*(n) - \frac{a_{\text{tan},*}}{\rho_* n} (\eta - \eta_*)^2 + o_M(n^{-1}), \quad (\text{S8})$$

Proof. In the proof of Lemma 6.1, η occurs only in the coefficients d_j in (S3). Replacing $|\eta| \leq 1$ by $|\eta| \leq M$ merely replaces the fixed coefficient majorants by constants depending on M ; the concavity and tail arguments are unchanged. This proves (S7). To compare with the optimized two-point value without using the theorem being proved, write

$$\eta_0(p) = -\frac{p^2 - 7p + 5}{12(1 - p)}, \quad a(p) = \frac{\phi_0}{2p^{3/2}(1 - p)^{1/2}}.$$

For the integer $k = np + \eta$, set $p' = p + (\eta - \eta_0(p))/n$. Then $k = np' + \eta_0(p)$, so the right-edge score of the two-point law with parameter p' at this same integer is an admissible competitor in $C_Q^*(n)$. Applying (S7) to p and p' and using the exact quadratic identity

$$F(p, \eta) = F(p, \eta_0(p)) - \frac{a(p)}{\rho(p)} (\eta - \eta_0(p))^2$$

gives, as $p \rightarrow p_*$,

$$G(p) - G(p') = o_M(n^{-1}), \quad F(p', \eta_0(p)) = F(p, \eta_0(p)) + o_M(1).$$

Consequently the current score is at most that competitor minus $(a_{\text{tan},*}/\rho_*)(\eta - \eta_*)^2/n + o_M(n^{-1})$, which is (S8). $\square$

The equation $G'(p) = 0$ is $2p^2 - 8p + 3 = 0$, so the unique maximizer is

$$p_* = \frac{4 - \sqrt{10}}{2}, \quad G(p_*) = C_E.$$

More explicitly,

$$G''(p) = \frac{\phi_0}{3D^3}(-8p^3 + 48p^2 - 36p + 4).$$

On $[2/5, 11/25]$ the negative of the displayed cubic is minimized at an endpoint and is at least $50448/15625 > 3.2286$. Since $D \leq 13/25$ and $\phi_0 > 199/500$, it follows that $-G'' > 3$ throughout that interval. The quadratic $\eta \mapsto F(p_*, \eta)$ has its vertex at

$$r_* = -\frac{20 + \sqrt{10}}{72},$$

and exact substitution gives

$$\max_{\eta \in \mathbb{R}} F(p_*, \eta) = -\frac{867 + 287\sqrt{10}}{25920\sqrt{2\pi}} = -\kappa_\infty.$$

Proof of Theorem 2.5. For the lower bound choose an integer k_n nearest to $np_* + r_*$ and put $p_n = (k_n - r_*)/n$. Then $p_n = p_* + O(n^{-1})$ and $k_n - np_n = r_*$. Equation (6.1) gives

$$C_Q^*(n) \geq C_E - \frac{\kappa_\infty}{n} + o(n^{-1}).$$

For completeness we spell out the global localization used for the upper bound. Proposition A.1 states the imported theorem with all four edge terms and the normalization used here. In particular, for $p \in [1/6, 5/6]$ it bounds the normalized distance by those four terms and $c_0 \widehat{G}(p)$, where $c_0 < 0.948$, and on $p \in [1/3, 2/3]$ it gives the strict bound by $\widehat{G}(p)$. On the half-range $p \leq 1/2$ one has $\widehat{G}(p) = G(p)$. For $p \leq 1/6$ the universal estimate [7]

$$\frac{\sqrt{n}\Delta_n(Q_p)}{\rho(p)} \leq 0.3322 \left(1 + \frac{0.429}{\rho(p)}\right) \leq 0.405740$$

has a fixed gap from C_E ; reflection covers $p \geq 5/6$. The term $c_0 \widehat{G}(p)$ likewise has a fixed gap. The imported strict bound by G , together with continuity and the unique maximum at p_* , first eliminates the compact set $[1/3, 1/2] \setminus (2/5, 11/25)$.

It remains to inspect the four central edges on $[1/6, 1/3] \cup [2/5, 11/25]$, all of which lie in the wider parameter range of Lemma 6.1. On the left edges the reflected leading coefficient satisfies $G_-(p) \leq \phi_0 < C_E$; on right edges with $p \in [1/6, 1/3]$ one has $G(p) \leq G(1/3) < C_E$. These positive leading coefficients also fix the signs of the corresponding signed expansions for all sufficiently large n . Thus every sequence attaining the supremum up to $o(n^{-1})$ eventually consists of right central endpoints with $p \in [2/5, 11/25]$. On that interval one has $-G''(p) > 3$. Comparison with the explicit lower-bound sequence therefore gives $p - p_* = O(\epsilon)$. Write $p = p_* + u\epsilon$; the numbers u remain bounded, and the central-point theorem gives $|\eta| \leq 1$. Lemma 6.1, continuity of F , and Taylor's theorem now give, uniformly on bounded u and $|\eta| \leq 1$,

$$\frac{\sqrt{n}\Delta_n(Q_p)}{\rho(p)} \leq C_E + \epsilon^2 \left\{ \frac{1}{2}G''(p_*)u^2 + F(p_*, \eta) \right\} + o(\epsilon^2).$$

The first term in braces is nonpositive, while the concave quadratic $F(p_*, \eta)$ has maximum $-\kappa_\infty$ at $\eta = r_*$. Taking upper limits proves

$$\limsup_{n \rightarrow \infty} n(C_Q^*(n) - C_E) \leq -\kappa_\infty.$$

Together with the lower bound this proves the theorem. $\square$

7 The original eventual upper bound

We now prove Theorem 2.6 with its natural qualitative quantifiers. Put

$$p_* = \frac{4 - \sqrt{10}}{2}, \quad q_* = 1 - p_*, \quad s_* = (p_* q_*)^{-1/2},$$

$$r_E = 1 - 2p_*, \quad \gamma_* = s_* r_E, \quad \eta_* := r_* = -\frac{20 + \sqrt{10}}{72},$$

and let $\rho_* := \rho(p_*)$. We write

$$Q_* := Q_{p_*} = Q_{\gamma_*}. \quad (4.0)$$

For a centered variance-one law X write

$$\mathcal{C}_n(X) = \frac{\sqrt{n} \Delta_n(X)}{\rho}, \quad \gamma = \mathbb{E}X^3, \quad d = \rho - \rho_2(\gamma).$$

After reflection we use the unique $p \leq 1/2$ for which $\gamma = (1 - 2p)/\sqrt{p(1 - p)}$. Thus d is precisely the defect in Corollary 2.3. All little-oh statements below are sequential: they are uniform along every triangular array satisfying the hypotheses in which they occur. This is enough for the final $\exists N_0 \forall n$ statement and avoids introducing an artificial cutoff.

We use the following terminology throughout this section. A one-sided cdf point (x, σ) in row (n, X) is *active* if its normalized signed score $\sqrt{n}D_X^\sigma(x)/\rho$ is larger than C_E . A positive cell is the set of points, as $|\theta| \leq 1/2$ varies, written as

$$x = \frac{s(\eta + \theta)}{\sqrt{n}}, \quad np + \eta \in \mathbb{Z}, \quad |\theta| \leq \frac{1}{2}. \quad (4.0a)$$

(At a cell boundary either adjacent representation may be chosen.) The cell is *active* if it contains an active point; when (η, θ) is displayed, “active” refers to that particular point. A triangular array of points is said to lie on the *localized positive branch* when $p_n \rightarrow p_*$, $d_n \rightarrow 0$, its sign is positive, and $x_n \rightarrow 0$. Thus “active” records the inequality being contradicted, whereas “localized” records precisely the four sequential conclusions just listed; neither term hides an additional estimate.

Outer stability

Proposition 7.1 (outer stability). *Let $n_j \rightarrow \infty$ and let X_j be centered and standardized. If*

$$\mathcal{C}_{n_j}(X_j) > C_E, \quad (\text{OS0})$$

then, after reflection and passage to a subsequence,

$$p_j \longrightarrow p_*, \quad d_j \longrightarrow 0. \quad (\text{OS})$$

Moreover every sequence of cdf points whose normalized one-sided error tends to C_E lies in the positive central branch and tends to zero in the standardized cdf coordinate.

Proof. The published structural estimate [7]

$$\frac{\sqrt{n} \Delta_n(X)}{\rho} \leq \frac{1661}{5000} \left(1 + \frac{429}{1000\rho} \right) \quad (4.1)$$

shows, by direct comparison at $\rho = 37/20$, that every row in (OS0) has $\rho < 37/20$. Hence its moment parameters are precompact.

Put

$$r = \frac{\gamma}{\sqrt{4 + \gamma^2}}, \quad u = \sqrt{1 - r^2}, \quad s = \frac{2}{u}, \quad \varrho_2(r) := \rho_2(2r/u) = \frac{1 + r^2}{u}, \quad (4.2)$$

and set $\lambda = \rho + \mathbb{E}|X|$. Proposition 3.2 gives

$$s \leq \lambda \leq s + \frac{2d}{(1 - r)(2 + r)}. \quad (4.3)$$

We record why the bandwidth λ removes every nonzero leading Fourier cell. If X' is an independent copy and $D = X - X'$, then

$$z^2 \leq |z|^3 + \frac{|\sin(\pi z)|}{\pi} \quad (z \in \mathbb{R}). \quad (4.4)$$

For $|z| \geq 1$ this is immediate; on $[0, 1]$ it follows from $\sin(\pi z) \geq \pi z(1 - z)$, and symmetry finishes the proof. For $H > 0$ apply (4.4) to D/H . Independence and centering give

$$\mathbb{E}|D|^3 \leq 2(\rho + \mathbb{E}|X|) = 2\lambda, \quad \mathbb{E}\sin^2(\pi D/H) = \frac{1 - |\varphi_X(2\pi/H)|^2}{2}.$$

Here the first inequality is elementary but worth recording: pointwise,

$$|x - y|^3 \leq (x - y)^2(|x| + |y|).$$

After expansion with $(x, y) = (X, X')$, the two mixed terms containing $\mathbb{E}X$ vanish, while the remaining terms are $2\mathbb{E}|X|^3 + 2\mathbb{E}X^2\mathbb{E}|X| = 2(\rho + \mathbb{E}|X|)$. Cauchy–Schwarz therefore yields

$$H \leq \lambda + \frac{H^3}{2\pi\sqrt{2}}\sqrt{1 - |\varphi_X(2\pi/H)|^2}. \quad (4.5)$$

With $H = \lambda/t$ this becomes

$$\sqrt{1 - |\varphi_X(2\pi t/\lambda)|^2} \geq \frac{2\pi\sqrt{2}}{\lambda^2}t^2(1 - t). \quad (4.6)$$

Indeed $2 \leq \lambda \leq 37/20 + 1 = 57/20$. Fix temporarily $0 < \tau < 1/4$. Squaring (4.6) and using $1 - y \leq e^{-y}$ gives constants $q_\tau < 1$ and $c_\tau > 0$, independent of the row, such that

$$\begin{aligned} |\varphi_X(2\pi t/\lambda)|^n &\leq q_\tau^n, & \tau \leq t \leq 1 - \tau, \\ |\varphi_X(2\pi t/\lambda)|^n &\leq e^{-c_\tau n(1-t)^2}, & 1 - \tau \leq t \leq 1. \end{aligned} \quad (4.6a)$$

This is the common Cramér-gap partition used below. Since K is bounded on the middle interval, that integral is $O(q_\tau^n)$. Since $|K(1 - h)| \leq C_\tau h$, the terminal integral is bounded by

$$C_\tau \int_0^\tau h e^{-c_\tau n h^2} dh = O(n^{-1}). \quad (4.6b)$$

After multiplication by $\sqrt{n}$ both pieces vanish. We subsequently shrink τ , uniformly in $\lambda \in [2, 57/20]$, so that the remaining zero cell $[0, \tau]$ lies in the small-frequency region of (4.7).

We next evaluate the zero cell uniformly for triangular arrays. This is the point at which uniformity in the cdf coordinate and in a possibly nonvanishing defect is needed. We give the replacement argument explicitly. Apply the two one-sided Prawitz operators with physical span parameter $h = \lambda$, equivalently with Fourier cutoff $T_{n,\lambda} = 2\pi\sqrt{n}/\lambda$. At a standardized cdf coordinate x their Fourier factors are $e^{-iT_{n,\lambda}tx}\varphi_X(2\pi t/\lambda)^n$; hence every estimate below is unchanged when x varies. Suppress the row index and write

$$\varphi(v) = \mathbb{E}e^{ivX}, \quad \chi(v) = \mathbb{E}e^{ivQ_\gamma}.$$

The moment cap above bounds, uniformly in the row, the third absolute moments of both X and Q_γ . Symmetrization and $1 - \cos z \geq z^2/2 - |z|^3/6$ therefore give constants $c, \delta > 0$ such that

$$\max\{|\varphi(v)|, |\chi(v)|\} \leq e^{-cv^2} \quad (|v| \leq \delta). \quad (4.7)$$

Indeed, if Z' is an independent copy, then $\mathbb{E}(Z - Z')^2 = 2$ and

$$1 - |\mathbb{E}e^{ivZ}|^2 = \mathbb{E}\{1 - \cos(v(Z - Z'))\} \geq v^2 - \frac{|v|^3}{6} \mathbb{E}|Z - Z'|^3. \quad (4.7a)$$

The just-proved difference inequality and the common third-moment bound make the last expectation uniformly bounded for $Z = X, Q_\gamma$. Choose δ so that the right side of (4.7a) is at least $v^2/2$. Then $|\mathbb{E}e^{ivZ}|^2 \leq 1 - v^2/2 \leq e^{-v^2/2}$, which proves (4.7), for instance with $c = 1/4$.

Use the exact divided power

$$\mathcal{P}_n(v) = \frac{\varphi(v)^n - \chi(v)^n}{n\{\varphi(v) - \chi(v)\}} = \int_0^1 \{(1-a)\chi(v) + a\varphi(v)\}^{n-1} da, \quad (4.8)$$

with its continuous value if the denominator vanishes. Since both transforms equal $1 - v^2/2 + O(|v|^3)$, with a row-independent constant, the difference-of-powers identity and (4.7) imply, for $0 \leq w \leq V_n := n^{1/21}$,

$$\left| \mathcal{P}_n(w/\sqrt{n}) - e^{-w^2/2} \right| \leq C \left\{ \frac{w^3}{\sqrt{n}} + \frac{w^2 + w^4}{n} \right\} e^{-cw^2}. \quad (4.9)$$

On $V_n < w \leq \delta\sqrt{n}$ the left side is instead bounded by $2e^{-cw^2}$. These bounds are independent of the cdf phase. The Prawitz substitution $t = \lambda w/(2\pi\sqrt{n})$ also gives, uniformly on the zero cell,

$$2|K(t)| dt \leq \left\{ \frac{1}{\pi w} + \frac{C}{\sqrt{n}} \right\} dw. \quad (4.10)$$

Consequently every error in replacing the exact divided power by its Gaussian limit is dominated by an integrable Gaussian polynomial. In particular, this replacement is simultaneous in the cdf coordinate x , because the omitted factor e^{-iwx} has modulus one.

It remains to identify the contribution of $\varphi - \chi$. In the Bernoulli coordinate $Y = p + \sqrt{p(1-p)}X$ of Corollary 2.3, the two cubic tails of W_p have leading coefficients

$$C_+(p) = \frac{2p(3-2p)}{(1-p)\sqrt{p(1-p)}}, \quad C_-(p) = \frac{2(1-p)(1+2p)}{p\sqrt{p(1-p)}}. \quad (4.10a)$$

The precompact moment range keeps p away from 0 and 1. Let P_z be the cubic Hermite interpolant of e^{izy} at 0, 1 and put $E_z(y) = e^{izy} - P_z(y)$. Moment matching and the common centering phase give

$$\varphi(v) - \chi(v) = e^{-ispv} \mathbb{E}E_{sv}(Y).$$

Hermite–Genocchi and the explicit branches of W_p give, uniformly in this compact p -range,

$$|E_z(y)| \leq \frac{|z|^4}{24} y^2(1-y)^2, \quad \frac{y^2(1-y)^2}{(1+|y|)W_p(y)} \leq C. \quad (4.10b)$$

For the tails write

$$g(v) = e^{iv} - 1 - iv + \frac{v^2}{2} + \frac{iv^3}{6}.$$

The phase-uniform bound

$$|g(v)| \leq \frac{|v|^3}{6} \quad (4.10c)$$

follows by setting $A = \cos v - 1 + v^2/2$, $B = \sin v - v + v^3/6$ and observing

$$\frac{d}{dv} \{v^6/36 - A^2 - B^2\} = \frac{v^3}{3} (1 - \cos v) \geq 0 \quad (v \geq 0);$$

conjugation covers $v < 0$. Moreover the Hermite basis gives

$$|E_z(y) - g(zy)| \leq C|z|^4(1 + |y|^3) \quad (4.10d)$$

outside a fixed compact interval.

Split the defect probability $d^{-1}W_p(y) d\mathbb{P}_Y(y)$ at $H_n = n^{1/8}$ (with the assertion trivial when $d = 0$). Equations (4.10b) and (4.10d), followed by (4.9)–(4.10), make the part $|y| \leq H_n$ and the interpolant correction on $|y| > H_n$ at most $CdH_n/\sqrt{n}$ and $Cd/\sqrt{n}$, respectively, in normalized score. The divided-power replacement itself costs $O(d/\sqrt{n})$, by the characteristic-function stability theorem and the Gaussian moments in (4.9). These quantities are $o(1)$ uniformly in the row and in x . On the remaining tail, (4.10c) and the exact cubic leading terms in (4.10a) give

$$\frac{n^{3/2}}{d} \left| \mathbb{E}\{E_{sw/\sqrt{n}}(Y); |Y| > H_n\} \right| \leq \frac{s^3|w|^3}{6} [\max\{C_+(p)^{-1}, C_-(p)^{-1}\} + o(1)] + C \frac{w^4}{\sqrt{n}}. \quad (4.10e)$$

Here the $o(1)$ is independent of the row and of w on $0 \leq w \leq V_n$. Beyond V_n the characteristic-function stability theorem and the second bound following (4.9) give a vanishing Gaussian tail. Thus dominated convergence applies to the full zero-cell integral, without first assuming $d \rightarrow 0$. Since $p \leq 1/2$, one has

$$\frac{C_-(p)}{C_+(p)} - 1 = \frac{(1-2p)(1+2p-2p^2)}{p^2(3-2p)} \geq 0;$$

hence the maximum in (4.10e) is $C_+(p)^{-1}$, and

$$\frac{1}{\pi} \int_0^\infty w^2 e^{-w^2/2} dw = \phi_0, \quad \frac{s^3}{6C_+(p)} = \frac{1}{3(1-r)^2(2+r)}. \quad (4.10f)$$

To make the replacement just proved directly reusable, its conclusion is the following uniform inequality. With the fixed zero-cell width τ chosen after (4.6b), there is $\varepsilon_n \rightarrow 0$, independent of the row, the cdf coordinate, and the choice $\kappa \in \{K, \bar{K}\}$, such that

$$\begin{aligned} \sqrt{n} \sup_{x \in \mathbb{R}} \left| 2\Re \int_0^\tau \kappa(t) e^{-iT_n \lambda t x} \{ \varphi(2\pi t/\lambda)^n - \chi(2\pi t/\lambda)^n \} dt \right| \\ \leq d \left\{ \frac{\phi_0 s^3}{6C_+(p)} + \varepsilon_n \right\}. \end{aligned} \quad (4.10h)$$

When $d = 0$ both sides vanish. Indeed the part $w \leq V_n$ is exactly the product of (4.8), (4.9), and (4.10e); (4.10) supplies the factor $1/(\pi w)$, and (4.10f) evaluates its Gaussian majorant. The terms $CdH_n/\sqrt{n}$, $Cd/\sqrt{n}$, and the tail $w > V_n$ listed above are $\varepsilon_n d$. This proves (4.10h), rather than merely a pointwise limit for a fixed x .

The complementary frequencies have already been partitioned in (4.6a)–(4.6b): their raw integrals are $O(q_\tau^n) + O(n^{-1})$ for either kernel. The estimates discard only a spatial factor of modulus one, so after score normalization they are $o(1)$ uniformly in x .

The matching-law zero cell follows from the same expansion, now one order further. Uniformly for $0 \leq w \leq V_n$,

$$\chi(w/\sqrt{n})^n = e^{-w^2/2} \left(1 - \frac{i\gamma w^3}{6\sqrt{n}} \right) + O \left(\frac{(w^4 + w^6)e^{-cw^2}}{n} \right).$$

For $V_n < w \leq \delta\sqrt{n}$ both the exact power and the displayed Gaussian polynomial have a vanishing Gaussian tail by (4.7). In signed form the kernel expansion used here is

$$2K(t) dt = \left\{ \frac{i}{\pi w} + \frac{\lambda}{2\pi\sqrt{n}} \right\} dw + O(w/n) dw,$$

uniformly throughout the fixed zero cell, with the conjugate expansion for the other one-sided operator. Together with normal inversion, this gives, uniformly for all real x , the two profiles

$$\frac{\phi(x)}{\rho} \left\{ \frac{\lambda}{2} \pm \frac{\gamma}{6}(1-x^2) \right\} + o(1). \quad (4.10g)$$

Because $\lambda > |\gamma|$, the positive maximum of either profile is at $x = 0$, and the plus profile is the larger one. Hence (P11), (4.10h), and the vanishing complementary cells give the explicit uniform assembly

$$\mathcal{C}_n(X) \leq \frac{\phi_0}{\rho} \left\{ \frac{\lambda}{2} + \frac{\gamma}{6} + \frac{s^3}{6C_+(p)} d \right\} + o(1). \quad (4.10i)$$

Now use (4.3), $\rho = \varrho_2(r) + d$, $s = 2/u$, and $\gamma = 2r/u$:

$$\begin{aligned} \frac{\lambda}{2} + \frac{\gamma}{6} + \frac{s^3 d}{6C_+(p)} &\leq \frac{s}{2} + \frac{\gamma}{6} + \left\{ \frac{1}{(1-r)(2+r)} + \frac{s^3}{6C_+(p)} \right\} d \\ &= \frac{3+r}{3u} + \left\{ \frac{1}{(1-r)(2+r)} + \frac{1}{3(1-r)^2(2+r)} \right\} d. \end{aligned}$$

Passing to any subsequence on which $(r_j, d_j) \rightarrow (r, d)$ therefore proves

$$\limsup_j \mathcal{C}_{n_j}(X_j) \leq \phi_0 \frac{N(r) + L(r)d}{\varrho_2(r) + d}, \quad (4.11)$$

where

$$N(r) = \frac{3+r}{3u}, \quad L(r) = \frac{1}{(1-r)(2+r)} + \frac{1}{3(1-r)^2(2+r)}. \quad (4.12)$$

It remains to prove the strict scalar assertion behind (4.11). For $d = 0$ it is the elementary maximization

$$\phi_0 \frac{3+r}{3(1+r^2)} \leq C_E, \quad (4.13)$$

with equality only at $r = r_E$. Indeed the derivative of $(3+r)/(3(1+r^2))$ has the sign of $1-6r-r^2$, whose only nonnegative zero is $\sqrt{10}-3 = r_E$; substitution there gives $(3+\sqrt{10})/6$.

For fixed r the quotient in (4.11) is fractional linear in d , hence monotone or constant, so it suffices to inspect the two endpoints of the allowed interval. Enlarge the moment cap to $R = 37/20$. Since $\varrho'_2(r) = r(3-r^2)/(1-r^2)^{3/2} > 0$ and $\varrho_2(2/3) > R$, every admissible row has $r < 2/3$ and $0 \leq d < R - \varrho_2(r)$. It remains to prove, for $0 \leq r \leq 2/3$,

$$\frac{N(r) + L(r)\{R - \varrho_2(r)\}}{R} < \frac{41}{40} < \frac{3 + \sqrt{10}}{6}. \quad (4.14)$$

The second inequality is equivalent to $\sqrt{10} > 63/20$, and follows on squaring because $10 > 3969/400$. For the first put

$$P = r^4 + 6r^3 - 7r^2 - 4r + 2, \quad B = \frac{123r^3 - 249r + 86}{40}.$$

The algebraic equivalence is checkable without expansion:

$$\frac{41R}{40} - N - L\{R - \varrho_2(r)\} = \frac{RuB - P}{3u(1-r)^2(2+r)}.$$

Thus the first inequality in (4.14) is exactly

$$RuB - P > 0. \quad (4.15)$$

On $0 \leq r \leq 37/100$, $B > 0$ and $u \geq 1 - r^2$, so the left side of (4.15) is at least $f = R(1 - r^2)B - P$. Writing $800f = F$ gives

$$F = -4551r^5 - 800r^4 + 8964r^3 + 2418r^2 - 6013r + 1582.$$

Here is an exact derivative ledger. Since

$$B'(r) = \frac{369r^2 - 249}{40} < 0, \quad B(37/100) = \frac{100319}{40000000} > 0,$$

the asserted sign of B holds. Furthermore

$$F''' = 53784 - 19200r - 273060r^2, \quad F'''' = -19200 - 546120r < 0,$$

and

$$\begin{aligned} F'''(37/100) &= \frac{4649043}{500} > 0, \quad F''(0) = 4836, \\ F'(37/100) &= -\frac{22614402711}{20000000} < -1000, \quad F(37/100) = \frac{957159691693}{10^{10}} > 80. \end{aligned}$$

Thus $F''' > 0$, then $F'' > 0$, on the entire interval; consequently F' is increasing but remains negative, and F is decreasing with a positive right-end value.

For $r \geq 37/100$ one has $P < 0$. Indeed $P''(37/100) = 2407/2500 > 0$, $P''' = 24r + 36 > 0$, and $P'(2/3) = -112/27 < 0$; hence $P' < 0$ on this interval, while $P(37/100) = -11564039/10^8 < 0$. If $B \geq 0$, (4.15) is immediate. If $B < 0$, use

$$u \leq U := 1 - \frac{r^2}{2} - \frac{r^4}{8}, \quad U^2 - (1 - r^2) = \frac{r^6(r^2 + 8)}{64},$$

so that $RuB - P \geq h := RUB - P$. Since B is decreasing,

$$B(37/100) = \frac{100319}{40000000}, \quad B(3/8) = -\frac{91}{4096},$$

and the preceding monotonicity of P gives, on $[37/100, 3/8]$, $-P > 1/10$ and $|B| < 1/40$. As $0 < U \leq 1$, this yields $h > 1/10 - 37/800 = 43/800 > 1/20$ whenever $B < 0$.

For the remaining derivative checks, direct multiplication gives

$$\begin{aligned} h(r) &= \frac{791}{400} - \frac{6013}{800}r + \frac{4009}{800}r^2 + \frac{1743}{320}r^3 - \frac{4791}{3200}r^4 \\ &\quad - \frac{8991}{6400}r^5 - \frac{4551}{6400}r^7. \end{aligned} \quad (4.15a)$$

Thus

$$h'''(r) = -\frac{14373}{400} - \frac{26973}{160}r - \frac{95571}{160}r^3 < 0. \quad (4.15b)$$

The function h'' is concave, so its minimum on $[3/8, 1/2]$ is at an endpoint; exact substitution gives

$$h''(3/8) = \frac{1892502967}{104857600} > 17, \quad h''(1/2) = \frac{1784437}{102400} > 17. \quad (4.15c)$$

At $r_0 = 12/25$,

$$h(r_0) = \frac{17639835737}{2441406250000} > \frac{7}{1000}, \quad h'(r_0) = -\frac{7154739661}{195312500000}, \quad |h'(r_0)| < \frac{1}{25}. \quad (4.15d)$$

Taylor's formula and $h'' \geq 17$ yield

$$h(r) \geq h(r_0) - \frac{|h'(r_0)|^2}{34} > \frac{7}{1000} - \frac{1}{34 \cdot 25^2} > 0. \quad (4.15e)$$

Finally (4.15b) and $h'''(1/2) = -803367/51200 < 0$ show that h' is concave on $[1/2, 2/3]$, while

$$h'(1/2) = \frac{128931}{409600} > \frac{1}{4}, \quad h'(2/3) = \frac{549953}{194400} > 2.$$

A concave function lies above its endpoint chord, so $h' > 0$ throughout this interval; hence h is increasing there. This proves (4.14), and consequently the second inequality in (4.11), with equality only at $(r, d) = (r_E, 0)$.

Every score in (OS0) is larger than C_E . Thus (4.11) first forces the scores to converge to C_E , and its equality statement forces $r_j \rightarrow r_E$ and $d_j \rightarrow 0$, equivalently $p_j \rightarrow p_*$. At this point let

$$P_{\pm}(x) = \frac{\phi(x)}{\rho_*} \left\{ \frac{s_*}{2} \pm \frac{\gamma_*}{6}(1 - x^2) \right\}.$$

The derivative calculation used after (4.10g) shows that P_+ has its unique global maximum $P_+(0) = C_E$ at zero, whereas

$$\sup_x P_-(x) = P_-(0) = C_E - \frac{\phi_0 \gamma_*}{3\rho_*} < C_E.$$

Consequently, for every fixed $\epsilon > 0$, compactness together with the Gaussian tails supplies $c_\epsilon > 0$ such that

$$\sup_{|x| \geq \epsilon} P_+(x) \leq C_E - c_\epsilon, \quad \sup_x P_-(x) \leq C_E - \frac{\phi_0 \gamma_*}{3\rho_*}.$$

The convergence in (4.10g) and the replacement bound (4.10h) are uniform in x . Hence an above- C_E point must have the plus sign and $x_j \rightarrow 0$. This proves every assertion of Proposition 7.1. $\square$

The certified shape input and the low-frequency defect

The only computer-assisted premise in the eventual argument is the next one-dimensional inequality. It is stated separately so that its trust boundary is not mixed with the analytic proof.

Lemma 7.2 (shape-contraction). *For $0 < p < 1$ put $q = 1 - p$, and let H_p be the cubic Hermite interpolant at $0, 1$ of*

$$g_p(y) = (y - p)e^{2\pi i y}.$$

Put $E_p^{\text{sh}} = g_p - H_p$ and

$$W_p(y) = \begin{cases} \frac{2q}{p\sqrt{pq}} y^2 \{3p - (1 + 2p)y\}, & y \leq p, \\ \frac{2p}{q\sqrt{pq}} (1 - y)^2 \{(3 - 2p)y - p\}, & y \geq p. \end{cases} \quad (\text{SC0})$$

At the certified parameter $p = p_*$, for every real y ,

$$\boxed{|E_p^{\text{sh}}(y)| \leq \frac{19}{16} W_p(y) + |e^{2\pi i y} - 1|^2.} \quad (\text{SC})$$

Only the compact inequality at the single parameter p_* is computer-certified: the checker uses exact outward-rounded dyadic interval arithmetic on 187 accepted intervals. The tail proof and the passage from p_* to nearby p given below are analytic. No floating-point number enters an acceptance decision.

Proof. The exact real and imaginary parts are

$$\begin{aligned}\Re E_p^{\text{sh}}(y) &= (y - p)(\cos 2\pi y - 1), \\ \Im E_p^{\text{sh}}(y) &= (y - p) \sin 2\pi y - 2\pi\{(1 - 2p)y^3 + (3p - 1)y^2 - py\}.\end{aligned}\tag{SC1}$$

The tails require no interval arithmetic. Direct substitution in the radical defining p_* , followed only by squaring positive quantities, gives

$$.418 < p_* < .419, \quad .581 < q_* < .582, \quad 0 < 1 - 2p_* < .164, \quad \sqrt{p_* q_*} < 1/2, \quad 2\pi < 44/7,$$

and

$$\frac{2q_*}{p_* \sqrt{p_* q_*}} > \frac{2324}{419}, \quad \frac{2p_*}{q_* \sqrt{p_* q_*}} > \frac{836}{291}.$$

Consequently, with $y = -x$ and $x \geq 1$, (SC0)–(SC1) give

$$|E_p^{\text{sh}}(-x)| < \frac{11}{10}x^3 + 9x^2, \quad W_p(-x) > 10x^3 + 6x^2.$$

Their difference is positive at $x = 1$ and has positive derivative thereafter. With $y = 1 + x$ and $x \geq 2$, the same expansion gives

$$|E_p^{\text{sh}}(1 + x)| < \frac{11}{10}x^3 + \frac{47}{10}x^2 + \frac{67}{10}x + \frac{9}{5},$$

$$W_p(1 + x) > \frac{31}{5}x^3 + 5x^2.$$

The difference is positive at $x = 2$ and has positive derivative thereafter. Thus the stronger inequality $|E_p^{\text{sh}}| < W_p$ holds outside $[-1, 3]$.

For completeness we describe the exact compact certificate. Put

$$c(t) = \frac{1 - \cos(2\pi t)}{t^2}, \quad s_0(t) = \frac{\sin(2\pi t)}{2\pi t}, \quad b(t) = \frac{1 - s_0(t)}{t}.$$

On the left branch the checker divides the triangle gap $(19/16)W_p + |e^{2\pi i y} - 1|^2 - |\Re E_p^{\text{sh}}| - |\Im E_p^{\text{sh}}|$ by y^2 and evaluates the regularized identities

$$\begin{aligned}\frac{\Re E_p^{\text{sh}}(y)}{y^2} &= -(y - p)c(y), \\ \frac{\Im E_p^{\text{sh}}(y)}{y^2} &= 2\pi\{s_0(y) - (1 - 2p)y - (3p - 1) + pb(y)\}, \\ \frac{W_p(y)}{y^2} &= \frac{2q}{p\sqrt{pq}}\{3p - (1 + 2p)y\}, \quad \frac{|e^{2\pi i y} - 1|^2}{y^2} = 2c(y).\end{aligned}\tag{SC2}$$

The right branch is the same calculation after $t = y - 1$, divided by t^2 . The entire alternating series of s_0, b, c remove the apparent singularities. The checker encloses $\sqrt{10}$ by integer square roots, encloses π through Machin's identity and alternating rational series, and rounds every rational operation outward on a fixed dyadic grid. It subdivides only when the resulting lower endpoint of the normalized triangle gap is not yet positive. All 187 terminal intervals have a strictly positive rational lower endpoint, cover $[-1, 3]$, and overlap across the irrational branch point $y = p_*$. Since both terms on the right of (SC) are nonnegative and $|E| \leq |\Re E| + |\Im E|$, positivity of the triangle gap proves (SC). The endpoints $y = 0, 1$ follow directly from (SC1). $\square$

We next make precise the continuity step which transports this single-parameter certificate. This continuity step is analytic. For p near p_* and $y \neq 0, 1$ define

$$\mathcal{R}(p, y) := \frac{(|E_p^{\text{sh}}(y)| - |e^{2\pi i y} - 1|^2)_+}{W_p(y)}. \quad (\text{SC3})$$

Here $W_p > 0$ off $\{0, 1\}$. The following endpoint expansions are uniform when p ranges over a sufficiently small closed neighbourhood of p_* :

$$\begin{aligned} E_p^{\text{sh}}(y) &= \{2\pi^2 p + 2\pi i(2 - 3p)\}y^2 + O(|y|^3), & y \rightarrow 0, \\ |e^{2\pi i y} - 1|^2 &= 4\pi^2 y^2 + O(y^4), & W_p(y) = \frac{6(1-p)}{\sqrt{p(1-p)}}y^2 + O(|y|^3), \end{aligned} \quad (\text{SC4})$$

$$\begin{aligned} E_p^{\text{sh}}(1+t) &= \{-2\pi^2(1-p) + 2\pi i(3p-1)\}t^2 + O(|t|^3), & t \rightarrow 0, \\ |e^{2\pi i(1+t)} - 1|^2 &= 4\pi^2 t^2 + O(t^4), & W_p(1+t) = \frac{6p}{\sqrt{p(1-p)}}t^2 + O(|t|^3). \end{aligned} \quad (\text{SC5})$$

At $p = p_*$ the moduli of the two complex coefficients in (SC4)–(SC5) are strictly smaller than $4\pi^2$ (squaring and using $.418 < p_* < .419$ verifies both inequalities). They therefore remain strictly smaller on a fixed neighbourhood of p_* . Thus $\mathcal{R}$ is identically zero in fixed punctured neighbourhoods of 0 and 1, and extends there jointly continuously by the value zero. On the two tails, uniformly in the same p -neighbourhood,

$$\begin{aligned} E_p^{\text{sh}}(y) &= -2\pi i(1-2p)y^3 + O(y^2), \\ W_p(y) &= C_+(p)y^3 + O(y^2) & (y \rightarrow +\infty), \\ W_p(y) &= C_-(p)|y|^3 + O(y^2) & (y \rightarrow -\infty), \end{aligned} \quad (\text{SC6})$$

where $C_{\pm}$ are defined in (4.10a). Hence the two remaining removable limits (in the notation of (SC3)) are

$$\mathcal{R}(p, +\infty) = \frac{2\pi(1-2p)}{C_+(p)}, \quad \mathcal{R}(p, -\infty) = \frac{2\pi(1-2p)}{C_-(p)}. \quad (\text{SC7})$$

Equations (SC1) and (SC4)–(SC7), including the agreement of the two branches of W_p at $y = p$, show that $\mathcal{R}$ is jointly continuous on $U_0 \times \mathbb{R}$ for some compact $U_0 \subseteq (0, 1/2)$ containing p_* . Since the second factor is the two-point compactification, uniform continuity and (SC) give a modulus $\omega(\delta) \downarrow 0$ such that

$$|E_p^{\text{sh}}(y)| \leq \{19/16 + \omega(|p - p_*|)\}W_p(y) + |e^{2\pi i y} - 1|^2 \quad (\text{SC8})$$

for all $p \in U_0$ and $y \in \mathbb{R}$.

For a Bernoulli-coordinate law with the first three moments equal to p , write

$$\zeta = \mathbb{E}e^{2\pi i Y} = re^{i\alpha}, \quad m = \mathbb{E}\{(Y - p)e^{2\pi i Y}\}.$$

Moment matching annihilates H_p (its Bernoulli expectation is $qg_p(0) + pg_p(1) = 0$), while (3.3) gives $\mathbb{E}W_p(Y) = d$. Therefore (SC8) gives, whenever $p \rightarrow p_*$ and $d \rightarrow 0$,

$$|m| \leq \{19/16 + o(1)\}d + (1 - r^2) + O(d^2). \quad (4.16)$$

Indeed $2(1 - \Re \zeta) = 1 - r^2 + |1 - \zeta|^2$, while the linear Fourier stability theorem gives $|1 - \zeta| = O(d)$ uniformly for $p \in U_0$. Thus the $o(1)$ in (4.16) is uniform along every array with $p \rightarrow p_*$; there is no unproved pointwise-to-uniform passage. We shall repeatedly use

$$r^{n-1}(1 - r^2) \leq \frac{2}{n}(1 - r^n), \quad (4.17)$$

which follows after cancelling $1 - r$ from $1 - r^n = (1 - r) \sum_{j < n} r^j$.

At the matching physical bandwidth s , define the zero-frequency piece at a cdf point x by

$$L_{X,n}(x) := \frac{1}{2} - \Phi(x) + 2\Re \int_0^{1/4} K(t) e^{-2\pi i x \sqrt{n} t/s} \varphi_X(2\pi t/s)^n dt, \quad (4.17a)$$

and let $L_{Q,n}$ be the same expression for the matching two-point law.

For $p \in (0, 1)$ and $d \geq 0$ define, before its first use,

$$\begin{aligned} q &= 1 - p, & s(p) &:= \{p(1 - p)\}^{-1/2}, & \gamma(p) &:= \frac{1 - 2p}{\sqrt{p(1 - p)}}, \\ \rho_2(p) &:= \rho_2(\gamma(p)) = \frac{p^2 + (1 - p)^2}{\sqrt{p(1 - p)}}, & \rho(p, d) &:= \rho_2(p) + d, \\ \Lambda_{\text{abs}}(p, d) &:= \frac{s(p)^3 \phi_0}{6\rho(p, d)} \max\{C_+(p)^{-1}, C_-(p)^{-1}\}. \end{aligned} \quad (4.18a)$$

Here and below, $\rho_2(p)$ is only the explicitly declared Bernoulli-parameter shorthand $\rho_2(\gamma(p))$; the function $\rho_2(\gamma)$ itself retains the meaning fixed in Theorem 2.1. Also put

$$\Lambda_{\text{abs},*} := \Lambda_{\text{abs}}(p_*, 0), \quad c_{\text{sh}}(p, d) := \frac{s(p)^3 \phi_0}{4\pi\rho(p, d)}, \quad c_{\text{sh},*} := c_{\text{sh}}(p_*, 0). \quad (4.18c)$$

Lemma 7.3 (phase-free low-frequency bound). *Fix a compact interval $U \subseteq (0, 1)$ containing p_* . Let $n_j \rightarrow \infty$, let X_j be centered and variance one, and suppose that their matching parameters satisfy*

$$p_j \in U, \quad d_j := \mathbb{E}|X_j|^3 - \rho_2(p_j) \rightarrow 0. \quad (4.18b)$$

Let $Q_j := Q_{p_j}$ denote the matching standardized two-point law. Then the difference of the zero-frequency parts of the calibrated one-sided operators satisfies

$$\sup_{x \in \mathbb{R}} \frac{\sqrt{n_j}}{\rho(p_j, d_j)} |L_{X_j, n_j}(x) - L_{Q_j, n_j}(x)| \leq \{\Lambda_{\text{abs}}(p_j, d_j) + o_U(1)\} d_j. \quad (4.18)$$

Here $o_U(1) \rightarrow 0$ along every array satisfying (4.18b), independently of the cdf coordinate. In particular, no assertion is made for arrays on which d_j fails to tend to zero.

Proof. The case $d_j = 0$ is separate and immediate from Theorem 2.1; below pass to the rows with $d_j > 0$ and suppress the index j . Write $\varphi = \varphi_X$ and let χ_p be the characteristic function of the standardized matching two-point law. If E_z denotes the cubic Hermite error for $y \mapsto e^{izy}$, the defect representation from Corollary 2.3 is

$$\varphi(u) - \chi_p(u) = e^{-is(p)pu} d \int \frac{E_{s(p)u}(y)}{W_p(y)} d\nu(y), \quad d\nu = \frac{W_p}{d} d\mathbb{P}_Y. \quad (4.20)$$

Direct expansion of the four interpolation data, uniformly for $p \in U$ and small $|z|$, gives

$$|E_z(y) - g(zy)| \leq C_U |z|^4 (1 + |y|^3), \quad g(w) = e^{iw} - 1 - iw + w^2/2 + iw^3/6. \quad (4.21)$$

Choose $H = n^{1/4}$. Hermite–Genocchi and the explicit branches of W_p give

$$|E_z(y)| \leq \frac{|z|^4}{24} y^2 (1 - y)^2, \quad \frac{y^2 (1 - y)^2}{W_p(y)} \leq C_U (1 + |y|), \quad (4.21a)$$

while $|g(w)| \leq |w|^3/6$ by (4.10c). Consequently, for each fixed $v \geq 0$, with $z = s(p)v/\sqrt{n}$,

$$\begin{aligned} n^{3/2} \left| \int \frac{E_z(y)}{W_p(y)} d\nu(y) \right| &\leq \frac{s(p)^3 v^3}{6} [\max\{C_+(p)^{-1}, C_-(p)^{-1}\} + O_U(H^{-1})] \\ &\quad + C_U \frac{v^4 (H + 1)}{\sqrt{n}}. \end{aligned} \quad (4.21c)$$

Indeed, (4.21a) gives the last bound on $|y| \leq H$; on $|y| > H$, (4.21) handles the interpolant correction and the exact cubic leading terms give

$$\frac{|y|^3}{W_p(y)} \leq \max\{C_+(p)^{-1}, C_-(p)^{-1}\} \{1 + O_U(H^{-1})\}.$$

We now give an explicit frequency partition. The exact divided power is

$$\mathcal{P}_n(u) = \frac{\varphi(u)^n - \chi_p(u)^n}{n\{\varphi(u) - \chi_p(u)\}} = \int_0^1 \{(1-a)\chi_p(u) + a\varphi(u)\}^{n-1} da, \quad (4.21d)$$

with its continuous value when the denominator vanishes. The third absolute moments are uniformly bounded. From

$$1 - |\mathbb{E}e^{iuZ}|^2 = \mathbb{E}\{1 - \cos(u(Z - Z'))\}, \quad 1 - \cos w \geq w^2/2 - |w|^3/6,$$

and $\mathbb{E}(Z - Z')^2 = 2$, there are $c > 0$ and $\delta > 0$, depending only on U , such that

$$\max\{|\varphi(u)|, |\chi_p(u)|\} \leq e^{-cu^2} \quad (|u| \leq \delta). \quad (4.21e)$$

The same estimate, with a smaller c , holds for every convex combination in (4.21d). Hence

$$\mathcal{P}_n(v/\sqrt{n}) \longrightarrow e^{-v^2/2}, \quad |\mathcal{P}_n(v/\sqrt{n})| \leq e^{-cv^2} \quad (0 \leq v \leq \delta\sqrt{n}), \quad (4.21f)$$

where the convergence is uniform for v in a fixed compact interval.

For clarity, the rest of the zero cell is not covered by the words ‘‘Cramér gap.’’ Its standardized frequency is $u = 2\pi t/s(p)$ and its right endpoint is $b(p) = \pi/(2s(p))$. Decrease δ so that $0 < \delta < \min_{p \in U} b(p)$. Since

$$|\chi_p(u)|^2 = 1 - 2p(1-p)\{1 - \cos(s(p)u)\},$$

compactness gives

$$q_U := \sup_{p \in U, \delta \leq |u| \leq b(p)} |\chi_p(u)| < 1. \quad (4.21g)$$

Theorem 2.2 gives $|\varphi(u) - \chi_p(u)| \leq C_U d|u|^3$. Thus, for all sufficiently large rows, both endpoints and every convex combination in (4.21d) have modulus at most $q'_U := (1 + q_U)/2 < 1$ throughout the interval in (4.21g). Its normalized operator contribution is therefore bounded by

$$C_U n^{3/2} d (q'_U)^{n-1} = o_U(d), \quad (4.21h)$$

uniformly in the cdf phase. This proves the required common gap without any restriction on how slowly $d \downarrow 0$.

It remains only to justify a single dominated-convergence passage on the root-scale interval. Put

$$B_n(v) := n^{3/2} \int \frac{E_{s(p)v/\sqrt{n}}(y)}{W_p(y)} d\nu(y).$$

Theorem 2.2 applied in (4.20) supplies the global local-cell majorant $|B_n(v)| \leq C_U v^3$. Under the substitution

$$t = \frac{s(p)v}{2\pi\sqrt{n}}, \quad 2K(t) dt = \left\{ \frac{i}{\pi v} + \frac{s(p)}{2\pi\sqrt{n}} + O_U(v/n) \right\} dv, \quad (4.21i)$$

the absolute value of the complete normalized integrand, after the spatial phase is discarded, is at most

$$C_U d \{v^2 + v^3/\sqrt{n} + v^4/n\} e^{-cv^2}. \quad (4.21j)$$

This is integrable on $[0, \infty)$ and independent of the cdf coordinate. Equations (4.21c), (4.21f), and (4.21j), first on a fixed v -interval and then with its Gaussian tail removed, give

$$\frac{d s(p)^3}{6\pi\rho(p, d)} \max\{C_+(p)^{-1}, C_-(p)^{-1}\} \int_0^\infty v^2 e^{-v^2/2} dv + o_U(d).$$

Because $\pi^{-1} \int_0^\infty v^2 e^{-v^2/2} dv = \phi_0$, this is the right side of (4.18). The bounded and $O(v/n)$ kernel terms vanish under the same majorant. The conjugate one-sided kernel has identical absolute bounds. No linearization of the divided power, and hence no unrecorded d^2 remainder, has been used. This proves (4.18). $\square$

We isolate the elementary arithmetic needed later. Direct substitution of $p_* = (4 - \sqrt{10})/2$, followed only by squaring positive radicals, gives

$$\frac{C_E}{\rho_*} > \frac{39}{100}, \quad c_{\text{sh},*} < \frac{51}{200}, \quad \Lambda_{\text{abs},*} < \frac{17}{200}, \quad (4.22)$$

$$G_{\text{span},*} := \frac{C_E - \phi_0 K_1(\gamma_*)/2}{\rho_*} > \frac{9}{50}. \quad (4.23)$$

In particular the two margins which occur below are strictly positive:

$$\frac{C_E}{\rho_*} - \frac{19}{16} c_{\text{sh},*} - \Lambda_{\text{abs},*} > \frac{7}{3200}, \quad G_{\text{span},*} - \Lambda_{\text{abs},*} > \frac{19}{200}. \quad (4.24)$$

A direct outward-rational evaluation of the same radical formulas gives the following exact integer enclosures (the common denominator is 10^{15}):

$$\begin{aligned} 10^{15} c_{\text{sh},*} &\in [254150792637701, 254150792637702], \\ 10^{15} \Lambda_{\text{abs},*} &\in [84254560311404, 84254560311405], \\ 10^{15} \frac{C_E}{\rho_*} &\in [393927535745323, 393927535745324], \\ 10^{15} \frac{19c_{\text{sh},*}}{16} &\in [301804066257270, 301804066257271], \\ 10^{15} G_{\text{span},*} &\in [182181753528660, 182181753528661], \\ 10^{15} \left(\frac{C_E}{\rho_*} - \frac{19c_{\text{sh},*}}{16} - \Lambda_{\text{abs},*} \right) &\in [7868909176648, 7868909176649], \\ 10^{15} (G_{\text{span},*} - \Lambda_{\text{abs},*}) &\in [97927193217255, 97927193217256]. \end{aligned} \quad (4.24e)$$

All endpoints in (4.24e) are integers, not rounded floating-point output; cross multiplication proves the two rational inequalities in (4.24).

Uniform endpoint and tangent expansions

For later bookkeeping we now define the bandwidth-dependent operators explicitly. For any real-valued law X , integer $n \geq 1$, $h > 0$, and $x \in \mathbb{R}$, put

$$g_{X,n,h,x}(t) := e^{-2\pi i x \sqrt{n} t/h} \varphi_X(2\pi t/h)^n, \quad 0 \leq t \leq 1.$$

With

$$\epsilon_+ = 1, \quad \epsilon_- = -1, \quad \tilde{K}_+ = K, \quad \tilde{K}_- = \overline{K},$$

define

$$\boxed{\mathcal{U}_{X,n,h}^\sigma(x) := \frac{1}{2} + \epsilon_\sigma \mathfrak{J}_{\tilde{K}_\sigma}(g_{X,n,h,x}), \quad \sigma \in \{+, -\}. \quad (4.24u)}$$

Thus, written without the notation from (P0),

$$\mathcal{U}_{X,n,h}^+(x) = \frac{1}{2} + 2\Re \int_0^1 K(t)g_{X,n,h,x}(t) dt, \quad \mathcal{U}_{X,n,h}^-(x) = \frac{1}{2} - 2\Re \int_0^1 \overline{K(t)}g_{X,n,h,x}(t) dt.$$

All integrals meeting zero are understood in the real-part improper sense of (P0). For an interval $I \subset [0, 1]$, denote its integral contribution (without the common $1/2$) by

$$\mathcal{J}_{X,n,h;I}^\sigma(x) := 2\epsilon_\sigma \Re \int_I \tilde{K}_\sigma(t)g_{X,n,h,x}(t) dt. \quad (4.24v)$$

To record the half-line conventions, let

$$Z = \frac{X_1 + \dots + X_n - x\sqrt{n}}{h}, \quad M(u) = \frac{1}{2} + 2\Re \int_0^1 K(t)e^{2\pi i t u} dt,$$

and let $e_B(u) := M(u) - 1_{(-\infty, 0]}(u)$ be the upper-majorant excess over the closed negative half-line. Then

$$\mathcal{U}_{X,n,h}^+(x) = \mathbb{E}M(Z), \quad \mathcal{U}_{X,n,h}^-(x) = \mathbb{E}\{1 - M(-Z)\},$$

and hence the exact identities are

$$\begin{aligned} F_{n,X}(x) &= \mathcal{U}_{X,n,h}^+(x) - \mathbb{E}e_B(Z), \\ F_{n,X}(x-) &= \mathcal{U}_{X,n,h}^-(x) + \mathbb{E}e_B(-Z). \end{aligned} \quad (4.24w)$$

In particular,

$$F_{n,X}(x) \leq \mathcal{U}_{X,n,h}^+(x), \quad F_{n,X}(x-) \geq \mathcal{U}_{X,n,h}^-(x),$$

so

$$D_X^+(x) \leq \mathcal{U}_{X,n,h}^+(x) - \Phi(x), \quad D_X^-(x) \leq \Phi(x) - \mathcal{U}_{X,n,h}^-(x).$$

Fix $\delta := 1/4$, put $x_{\eta,\theta} = s(\eta + \theta)/\sqrt{n}$ and $k = np + \eta \in \mathbb{Z}$. All pieces in the next display are in raw cdf units. Indeed, for the Bernoulli coordinate $Y = p + X/s$ and matching bandwidth $h = s$,

$$g_{X,n,s,x_{\eta,\theta}}(t) = e^{-2\pi i(k+\theta)t} \varphi_Y(2\pi t)^n, \quad (4.24x)$$

which is the common phase used below:

$$\begin{aligned} D_{X,n}^{(0)}(\eta, \theta) &= \frac{1}{2} - \Phi(x_{\eta,\theta}) + 2\Re \int_0^\delta K(t)e^{-2\pi i(k+\theta)t} \varphi_Y(2\pi t)^n dt, \\ D_{X,n}^{(1)}(\eta, \theta) &= 2\Re \int_{1-\delta}^1 K(t)e^{-2\pi i(k+\theta)t} \varphi_Y(2\pi t)^n dt, \\ M_{X,n}(\eta, \theta) &= 2\Re \int_\delta^{1-\delta} K(t)e^{-2\pi i(k+\theta)t} \varphi_Y(2\pi t)^n dt. \end{aligned} \quad (4.24a0)$$

Set $\Psi_Q(z) := q + pe^{iz}$. The pieces $D_{Q,n}^{(j)}$ and $M_{Q,n}$ are obtained by replacing φ_Y by Ψ_Q with the same cuts, phase and normal increment. Let $\mathcal{P}_n^+(X; \eta, \theta)$ be the normalized Prawitz upper operator. The actual positive cdf score $\mathcal{C}_n^+(X; \eta, \theta)$ satisfies $\mathcal{C}_n^+(X; \eta, \theta) \leq \mathcal{P}_n^+(X; \eta, \theta)$, and the exact operator decomposition is

$$\begin{aligned} \mathcal{P}_n^+(X; \eta, \theta) &= \frac{\sqrt{n}}{\rho} [D_{Q,n}^{(0)}(\eta, \theta) + D_{Q,n}^{(1)}(\eta, \theta) + \{D_{X,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, \theta)\} \\ &\quad + \{D_{X,n}^{(1)}(\eta, \theta) - D_{Q,n}^{(1)}(\eta, \theta)\} + M_{X,n}(\eta, \theta)]. \end{aligned} \quad (4.24a)$$

For the matching law write, without changing any cdf convention,

$$N_{Q,n}(\eta, \theta) := \frac{1}{2} - \Phi(x_{\eta,\theta}), \quad Z_{Q,n}(\eta, \theta) := D_{Q,n}^{(0)}(\eta, \theta) - N_{Q,n}(\eta, \theta), \quad (4.24b)$$

and define the complete integer-node value

$$E_{Q,n}(\eta) := N_{Q,n}(\eta, 0) + Z_{Q,n}(\eta, 0) + D_{Q,n}^{(1)}(\eta, 0) + M_{Q,n}(\eta, 0). \quad (4.24c)$$

By the cardinal identity (P3), $\sqrt{n}E_{Q,n}(\eta)/\rho_2(p)$ is the exact right-node Bernoulli score. These definitions will make explicit which phase increment owns each later term. The matching endpoint is (4.29), the zero-cell phase increment is (4.31), the joint saddle is (4.40), and the actual endpoint is first factorized by (4.26); its complete remainder is (4.43). This identity is the owner of the later assembly, so no zero or endpoint row is silently duplicated. For reference, the ownership ledger is

quantity	owner	used in
$N_{Q,n}(\eta, 0) + Z_{Q,n}(\eta, 0)$	$E_{Q,n}(\eta)$	(4.40)
$D_{Q,n}^{(1)}(\eta, 0)$	$E_{Q,n}(\eta)$	(4.40)
$M_{Q,n}(\eta, 0)$	$E_{Q,n}(\eta)$	(4.40)
$D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)$	(4.31)	(4.34)
$\Re\{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\}$	(4.43b)	(4.43e), (4.43g)
$M_{Q,n}(\eta, \theta) - M_{Q,n}(\eta, 0)$	middle gap	exponentially small.

(4.24d)

For the rest of the proof use Bernoulli coordinates

$$Y = p + \frac{X}{s}, \quad q = 1 - p, \quad s = (pq)^{-1/2}, \quad \rho_2(p) = \frac{p^2 + q^2}{\sqrt{pq}},$$

$$\Psi(z) = \mathbb{E}e^{izY}, \quad \Psi_Q(z) = q + pe^{iz} \quad (\text{as above}).$$

For an integer k and a circular phase $|\theta| \leq 1/2$, put

$$\eta = k - np, \quad x = \frac{s(\eta + \theta)}{\sqrt{n}}. \quad (4.25)$$

The terminology in this paragraph repeats (4.0a). For a fixed row (n, X) , the positive cell indexed by k is the set of physical cdf points in (4.25) as θ ranges over $[-1/2, 1/2]$. It is *active* if at least one of those points has $\mathcal{C}_n^+(X; \eta, \theta) > C_E$; when a particular point is under discussion, “active” means that this strict inequality holds at that point. Every sequence of active points on the *localized positive branch* after the reduction in Proposition 7.1 satisfies, explicitly,

$$p \rightarrow p_*, \quad d \rightarrow 0, \quad x \rightarrow 0, \quad \eta/\sqrt{n} \rightarrow 0, \quad (4.25a)$$

and the positive, rather than reflected negative, one-sided score is being used. The last condition in (4.25a) follows from the preceding three and $|\theta| \leq 1/2$. Thus later uses of either phrase carry no additional unstated localization assumption.

Lemma 7.4 (endpoint factorization). *If p remains in a fixed neighbourhood of p_* , then, uniformly for $0 \leq h \leq 1/4$,*

$$\Psi(2\pi(1 - h)) = \zeta \Psi_Q(2\pi(1 - h)) - 2\pi i h m + O(h^2 d), \quad (4.26)$$

where $\zeta = \Psi(2\pi)$ and $m = \mathbb{E}\{(Y - p)e^{2\pi i Y}\}$.

Proof. Let $A(h) := \Psi(2\pi(1 - h))$ and $B(h) := \zeta \Psi_Q(2\pi(1 - h))$. Then $A(0) = B(0) = \zeta$ and $A'(0) - B'(0) = -2\pi i m$. Twice differentiating the remaining error reduces, apart from a fixed factor, to the expectation of

$$e^{2\pi i y} \{y^2 e^{-2\pi i h y} - p e^{-2\pi i h}\}. \quad (4.27)$$

Subtract its cubic Hermite interpolant at 0, 1. Its expectation is zero by the three moment identities. The quotient of the Hermite remainder by $W_p(y)$ extends through the two double zeros and stays bounded at infinity, because numerator and denominator have cubic growth. Compactness in (p, h) therefore bounds (4.27) by Cd . Integrating twice proves (4.26). $\square$

Lemma 7.5 (uniform first endpoint shape mode). *Fix $U \Subset (0, 1)$. In the notation of Lemma 7.4, let $p \in U$, let $d \rightarrow 0$, and let $n \rightarrow \infty$. The contribution of the term*

$$-2\pi i n h m \zeta^{n-1} \Psi_Q(2\pi(1-h))^{n-1} \quad (4.27a)$$

to the normalized returned-endpoint operator is, uniformly in the integer node and in the full circular phase $|\theta| \leq 1/2$, at most

$$\{c_{\text{sh}}(p, d) + o_U(1)\} r^{n-1} |m|. \quad (4.27b)$$

The $o_U(1)$ is independent of the rates of p, d and of the node and phase. After one derivative in the cdf phase the same term is bounded by $C_U r^{n-1} |m|$. Neither assertion uses a fourth moment or a previously localized η -window.

Proof. The spatial and circular phase factors have modulus one. The exact Bernoulli identity

$$|\Psi_Q(2\pi(1-h))|^2 = 1 - 2p(1-p)\{1 - \cos(2\pi h)\} \quad (4.27c)$$

gives, uniformly for $p \in U$, a Gaussian bound $e^{-c_U(n-1)h^2}$ on $0 \leq h \leq \delta$ and a fixed Cramér gap on $\delta \leq h \leq 1/4$. Thus the latter part of the endpoint cell is $O_U(n^{3/2}e^{-c_U n})r^{n-1}|m|$ after normalization.

On the root-scale part first restrict to $0 \leq v = \sqrt{n}h \leq n^{1/12}$. Directly from the displayed kernel definition,

$$2K(1-h) = h + O(h^2), \quad (4.27d)$$

uniformly on this subcell. The absolute value of the integral generated by (4.27a), including the factor $\sqrt{n}/\rho(p, d)$, is consequently

$$\frac{2\pi n \sqrt{n}}{\rho(p, d)} r^{n-1} |m| \int_0^{n^{-5/12}} h^2 e^{-2\pi^2 p(1-p)(n-1)h^2} \{1 + O_U(h + nh^4)\} dh. \quad (4.27e)$$

The nh^4 term follows by expanding the logarithm in (4.27c); it is not asserted as a relative expansion on the rest of the endpoint cell. After the substitution it is dominated on this subcell by a fixed Gaussian polynomial. On $n^{1/12} < v \leq \delta\sqrt{n}$, the exact bound (4.27c) makes the integrated contribution $O_U(e^{-c_U n^{1/6}})$, and the fixed remainder was treated above. Dominated convergence is therefore uniform for $p \in U$. Since

$$\int_0^\infty v^2 e^{-2\pi^2 v^2 / s(p)^2} dv = \frac{s(p)^3 \phi_0}{8\pi^2},$$

the limiting coefficient in (4.27e) is

$$\frac{2\pi}{\rho(p, d)} \frac{s(p)^3 \phi_0}{8\pi^2} = \frac{s(p)^3 \phi_0}{4\pi \rho(p, d)} = c_{\text{sh}}(p, d).$$

One phase derivative only inserts the bounded factor $-2\pi i(1-h)$ before this argument, proving the last assertion. $\square$

Lemma 7.6 (uniform signed tangent expansion). *Let $p \rightarrow p_*$ and $d \rightarrow 0$. Define*

$$A(p) = \frac{\phi_0}{\rho_2(p)} \left(\frac{s}{2} + \frac{\gamma}{6} \right), \quad a(p) = \frac{s^3 q \phi_0}{2}, \quad \eta_0(p) = -\frac{p^2 - 7p + 5}{12q}. \quad (4.28)$$

Write $a_{\text{tan},*} := a(p_*)$. Then $A(p) \leq C_E$, with equality only at p_* , and $\eta_0(p_*) = \eta_*$. The following assertions hold uniformly.

(a) If $|\eta| = o(\sqrt{n})$, then the matching two-point high endpoint is

$$2 \int_{3/4}^1 K(t) e^{-2\pi i(k+\theta)t} (q + p e^{2\pi i t})^n dt = e^{-2\pi i \theta} \left\{ \frac{C_0(p)}{n} + \frac{i C_1(p, \eta + \theta)}{n^{3/2}} \right\} + O\left(\frac{1 + \eta^2}{n^2}\right), \quad (4.29)$$

with one uniform θ derivative, where

$$C_0(p) = \frac{s^2}{4\pi^2}, \quad C_1(p, z) = \frac{s^3 \phi_0}{24\pi} \{6z + 3(1 - 2p) + 1\}. \quad (4.30)$$

(b) The zero-end increment, after subtracting the normal increment, is

$$D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0) = \frac{-a(p)\theta^2 + \{\ell(p) - 2a(p)\eta\}\theta}{n^{3/2}} + R_n^{(0)}, \quad (4.31)$$

where

$$\ell(p) = -\frac{s^3 \phi_0}{48} \{(1 - 2p)^2 + 12(1 - 2p) + 7\} = 2a(p)\eta_0(p), \quad (4.32)$$

and, whenever $|\eta| \leq C\sqrt{nd + 1}$,

$$\sqrt{n} |R_n^{(0)}| \leq C \left\{ \frac{1 + \eta^4}{n^{3/2}} + \frac{1 + |\eta|^5}{n^2} \right\}. \quad (4.33)$$

(c) If $d = o(n^{-1/2})$, $nd \rightarrow \infty$, and $|\eta| \leq C\sqrt{nd + 1}$, the integer-node matching score together with the zero-cell phase increment, namely

$$\frac{\sqrt{n}}{\rho_2(p)} \{E_{Q,n}(\eta) + D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)\},$$

is at most

$$A(p) - \frac{a(p)}{\rho_2(p)n} \{\eta + \theta - \eta_0(p)\}^2 + o(d). \quad (4.34)$$

The returned-endpoint phase increment (4.29) is deliberately not included in (4.34); it is inserted exactly once in Lemma 7.8.

Proof. For (a), put $t = 1 - v/\sqrt{n}$, $z = \eta + \theta$ and $\xi = z/\sqrt{n}$. After extracting the Bernoulli quadratic term the integrand equals

$$e^{-2\pi i \theta} e^{-2\pi^2 v^2/s^2} e^{2\pi i \xi v} \left[1 + \frac{iB(p)v^3}{\sqrt{n}} + \frac{P_{p,n}(v)}{n} \right], \quad B(p) = \frac{4\pi^3}{3} pq(1 - 2p), \quad (4.35)$$

where $P_{p,n}$ is bounded by a fixed polynomial under the Gaussian. Keep $e^{2\pi i \xi v}$ exact. Its Gaussian integrals are entire in ξ ; Taylor's formula at zero gives an error $O(z^2/n^2)$ in the leading v/n integral and $O((1 + |z|)/n^2)$ in the next integrals. Using

$$\int_0^\infty v e^{-2\pi^2 v^2/s^2} dv = \frac{s^2}{4\pi^2},$$

$$\int_0^\infty v^2 e^{-2\pi^2 v^2/s^2} dv = \frac{s^3 \phi_0}{8\pi^2}, \quad \int_0^\infty v^4 e^{-2\pi^2 v^2/s^2} dv = \frac{3s^5 \phi_0}{32\pi^4}$$

gives (4.29)–(4.30). Splitting at $v = n^{1/12}$ and using the exact Bernoulli Gaussian envelope on the complement proves the stated uniform remainder and its phase derivative.

At the zero endpoint put again $z = \eta + \theta$ and $C_z(v) = 2\pi zv + B(p)v^3$. The centered Bernoulli cumulants through order five give

$$e^{-2\pi i(k+\theta)t}(q + pe^{2\pi it})^n = e^{-2\pi^2 v^2/s^2} [1 - in^{-1/2}C_z(v) + n^{-1}\{E_4v^4 - C_z(v)^2/2\} + in^{-3/2}\{J_5v^5 - C_z(v)E_4v^4 + C_z(v)^3/6\} + O(n^{-2}P_{12}(v, z))], \quad (4.36)$$

where

$$E_4 = \frac{2\pi^4}{3}pq(1 - 6pq), \quad J_5 = \frac{4\pi^5}{15}pq(1 - 2p)(1 - 12pq).$$

Also

$$2K(v/\sqrt{n})n^{-1/2} = \frac{i}{\pi v} + n^{-1/2} - n^{-1} \left(v + \frac{i\pi v}{3} \right) + \frac{i\pi v^2}{3n^{3/2}} + O(n^{-2}P_3(v)). \quad (4.37)$$

Here is the uniform remainder statement implicit in these two displays. There are a neighbourhood U_1 of p_* , constants $c, C > 0$, and fixed positive polynomials P_j such that, for $p \in U_1$, $0 \leq v \leq n^{1/20}$, and $|z| \leq \varepsilon_0\sqrt{n}$ for a fixed small $\varepsilon_0 > 0$, the remainder inside the brackets in (4.36), together with one z derivative, is bounded by

$$\frac{C(1 + |z|^4)v^4P_8(v)}{n^2}. \quad (4.37a)$$

After multiplication by (4.37), its integrable majorant is the same bound times $(1 + v^{-1})e^{-cv^2}$. To verify (4.37a), expand $n \log\{e^{-2\pi i p t}(q + pe^{2\pi it})\}$ through the fifth derivative; on $|t| \leq n^{-9/20}$ its sixth derivative is uniformly bounded for $p \in U_1$. Taylor's integral remainder for the exponential is then bounded by $Cn^{-2}v^4P_8(v)(1 + |z|^4)e^{-cv^2}$. Here the spatial factor is expanded separately, using the global integral-remainder inequality for e^{-iw} ; hence no assumption $z = O(n^{1/4})$ is hidden. Differentiating in z merely removes one factor of z and inserts $v/\sqrt{n}$. On $n^{1/20} < v \leq \sqrt{n}/4$, the exact identity

$$|q + pe^{2\pi i v/\sqrt{n}}|^2 = 1 - 2pq\{1 - \cos(2\pi v/\sqrt{n})\}$$

gives Ce^{-cv^2} up to a fixed small cell and Ce^{-cn} on its complement. The same estimates hold after one z derivative. Thus the root-scale expansion may be integrated and differentiated term by term; this is a quantitative C^1 statement, not a formal Edgeworth calculation.

For transparency, the parity ledger for the real part of the product of (4.36) and (4.37) is

order	real coefficient before Gaussian integration	z -dependence after the normal term
1	0	0
$n^{-1/2}$	$1 + C_z(v)/(\pi v)$	linear term cancels
n^{-1}	$-v$	none
$n^{-3/2}$	$R_p(z, v)$	degrees 0, 1, 2, 3,

(4.37b)

because the omitted coefficients in the first three rows are purely imaginary. In the second row, $2z \int_0^\infty e^{-2\pi^2 v^2/s^2} dv = s\phi_0 z$, which cancels the $-s\phi_0 z$ from $1/2 - \Phi(sz/\sqrt{n})$. The order- n^{-1} row is independent of z and therefore cancels when two phases in the same cell are subtracted. The last-row polynomial is

$$R_p(z, v) = E_4v^4 - \frac{C_z(v)^2}{2} - \frac{J_5v^5 - C_z(v)E_4v^4 + C_z(v)^3/6}{\pi v} - \frac{\pi v C_z(v)}{3}. \quad (4.38)$$

Using

$$I_2 = \frac{s^3\phi_0}{8\pi^2}, \quad I_4 = \frac{3s^5\phi_0}{32\pi^4}, \quad I_6 = \frac{15s^7\phi_0}{128\pi^6}, \quad I_8 = \frac{105s^9\phi_0}{512\pi^8}, \quad (4.38a)$$

where $I_j = \int_0^\infty v^j e^{-2\pi^2 v^2/s^2} dv$, direct collection of the four powers of z gives

$$\begin{array}{l|l} z^0 & \beta_0(p) = \phi_0 s^3 \left(-\frac{1}{32} - \frac{7\tau}{160} - \frac{\tau^2}{96} + \frac{\tau^3}{4320} \right) \\ z^1 & -\frac{s^3 \phi_0}{48} (\tau^2 + 12\tau + 7) = \ell(p) \\ z^2 & -\frac{s^3 q \phi_0}{2} = -a(p) \\ z^3 & -\frac{s^3 \phi_0}{6}, \end{array} \quad \tau = 1 - 2p. \quad (4.38b)$$

For example, the linear row before simplification is

$$s^3 \phi_0 \left[-\frac{1}{12} + \frac{1 - 6pq - 2\tau}{8} - \frac{5\tau^2}{24} \right],$$

which is the displayed value of $\ell(p)$; this also independently checks the sign. Finally

$$\Phi(sz/\sqrt{n}) = \frac{1}{2} + \phi_0 \left(\frac{sz}{\sqrt{n}} - \frac{s^3 z^3}{6n^{3/2}} \right) + O(|z|^5 n^{-5/2}) \quad (4.39)$$

shows that the z^3 row of (4.38b) is cancelled exactly by the normal term. The value $\beta_0(p)$ and every earlier z -independent row cancel on subtracting two phases. To state this without overloading the two-argument cell notation, define

$$\mathcal{D}_{Q,n}^{(0)}[z] := \frac{1}{2} - \Phi(sz/\sqrt{n}) + 2\Re \int_0^{1/4} K(t) e^{-2\pi i z t} \{e^{-2\pi i p t} \Psi_Q(2\pi t)\}^n dt. \quad (4.38b1)$$

Thus $D_{Q,n}^{(0)}(\eta, \theta) = \mathcal{D}_{Q,n}^{(0)}[\eta + \theta]$. More precisely, the estimates just proved and Taylor's remainder in (4.39) give a function $\mathcal{E}_{n,p}$ such that

$$\begin{aligned} \mathcal{D}_{Q,n}^{(0)}[z] - \mathcal{D}_{Q,n}^{(0)}[0] &= n^{-3/2} \{ \ell(p)z - a(p)z^2 \} + \mathcal{E}_{n,p}(z), \\ |\partial_z \mathcal{E}_{n,p}(z)| &\leq C \left\{ \frac{1 + |z|^4}{n^2} + \frac{1 + |z|^5}{n^{5/2}} \right\} \end{aligned} \quad (4.38c)$$

uniformly for $p \in U_1$ and $|z| \leq \varepsilon_0 \sqrt{n}$.

For completeness, the replacement of the finite root cell by the positive half-line in the leading row is uniform at the sharper spatial scale needed below. Write $G_{0,p}(v)$ for the unphased leading coefficient in the score-normalized root integral corresponding to the first two rows of (4.37b), including the bounded Gaussian smoothing term and the uniformly bounded factor $1/\rho$. For two phases in one cell put $x = s(p)\eta/\sqrt{n}$ and $y = s(p)(\eta + \theta)/\sqrt{n}$. On $p \in U_1$ there are a fixed positive polynomial P and $c, C > 0$ such that $|G_{0,p}(v)| \leq CP(v)e^{-cv^2}$. Hence, for $A > 0$,

$$\left| \int_A^\infty \Re \{ (e^{-iyv} - e^{-ixv}) G_{0,p}(v) \} dv \right| \leq \frac{C|y-x|}{A^2} \int_0^\infty v^3 P(v) e^{-cv^2} dv. \quad (4.38c1)$$

Indeed $|e^{-iyv} - e^{-ixv}| \leq |y-x|v$, and on $v > A$ one inserts $1 \leq v^2/A^2$. The real cosine smoothing summand satisfies the same estimate directly from $|\cos(yv) - \cos(xv)| \leq |y-x|v$. Taking $A = \delta\sqrt{n}$ and using $|y-x| = s(p)|\theta|/\sqrt{n} \leq Cn^{-1/2}$ makes the omitted spatial tail $O(n^{-3/2})$. Thus passing to the half-line in the coefficient calculation neither loses an n^{-1} term nor hides a merely $O(n^{-1})$ remainder.

Apply the mean-value theorem between $z = \eta$ and $z = \eta + \theta$. Since $|\theta| \leq 1/2$ and the window in (b) has $|\eta|/\sqrt{n} = O(\sqrt{d} + n^{-1/2}) = o(1)$, (4.38c) proves (4.31) and exactly the stated bound (4.33).

It remains to retain the node loss rather than maximize it away. The exact beta-integral representation used in Lemma 6.1, with $\epsilon = n^{-1/2}$ and $\xi = \epsilon\eta$ kept as a parameter, gives the following explicit joint saddle. On a fixed rectangle about $(p_*, 0)$ put

$$\begin{aligned} P_1(p, \xi; y) &= \frac{s(p-q)}{3}y^3 + \frac{s^2\xi(p-q)}{2}y^2 - spy, \\ P_2(p, \xi; y) &= -\frac{s^2(1-3pq)}{4}y^4 + \frac{s^2p^2}{2}y^2 - \frac{s^3\xi(1-3pq)}{3}y^3, \end{aligned} \quad (4.39a)$$

and $W_\xi(y) = \exp(-y^2/2 - s\xi y)$. For $j = 0, 1, 2$ define

$$\begin{aligned} \mathcal{Q}_0 &= 1, \quad \mathcal{Q}_1 = P_1, \quad \mathcal{Q}_2 = P_2 + \tfrac{1}{2}P_1^2, \\ \mathcal{D}_j(p, \xi) &= \int_{\mathbb{R}} W_\xi(y) \mathcal{Q}_j(p, \xi; y) dy, \quad \mathcal{N}_j(p, \xi) = \int_{-\infty}^0 W_\xi(y) \mathcal{Q}_j(p, \xi; y) dy. \end{aligned} \quad (4.39b)$$

Taylor expansion of the two logarithms in the beta density gives

$$L(y) = -y^2/2 - s\xi y + \epsilon P_1(p, \xi; y) + \epsilon^2 P_2(p, \xi; y) + \epsilon^3 R_\epsilon(p, \xi; y). \quad (4.39c)$$

Consequently, if R_1, R_2 denote the two quotient coefficients, then

$$\begin{aligned} R_1 &= \frac{\mathcal{N}_1}{\mathcal{D}_0} - \frac{\mathcal{N}_0 \mathcal{D}_1}{\mathcal{D}_0^2}, \\ R_2 &= \frac{\mathcal{N}_2}{\mathcal{D}_0} - \frac{\mathcal{N}_1 \mathcal{D}_1}{\mathcal{D}_0^2} + \mathcal{N}_0 \left\{ \frac{\mathcal{D}_1^2}{\mathcal{D}_0^3} - \frac{\mathcal{D}_2}{\mathcal{D}_0^2} \right\}. \end{aligned} \quad (4.39d)$$

The leading quotient is $\mathcal{N}_0/\mathcal{D}_0 = \Phi(s\xi)$. Thus the required analytic functions are, explicitly,

$$H_0 = \frac{R_1}{\rho_2(p)}, \quad H_1 = \frac{R_2}{\rho_2(p)}. \quad (4.39e)$$

They satisfy, uniformly on that rectangle,

$$\frac{\sqrt{n}}{\rho_2(p)} E_{Q,n}(\eta) = H_0(p, \xi) + \epsilon H_1(p, \xi) + O(\epsilon^2), \quad (4.40)$$

with the same estimate after one ξ derivative. Here is a quantitative version of the remainder assertion. On $|y| \leq \epsilon^{-1/10}$, both R_ϵ and $\partial_\xi R_\epsilon$ are bounded by $C(1 + |y|^M)$ for one fixed M , and Taylor's integral formula gives

$$\begin{aligned} W_\xi(y) |e^{\epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 R_\epsilon} - \{1 + \epsilon \mathcal{Q}_1 + \epsilon^2 \mathcal{Q}_2\}| &\leq C\epsilon^3(1 + |y|^{3M})e^{-y^2/3}, \\ W_\xi(y) |\partial_\xi [e^{\epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 R_\epsilon} - \{1 + \epsilon \mathcal{Q}_1 + \epsilon^2 \mathcal{Q}_2\}]| &\leq C\epsilon^3(1 + |y|^{3M+1})e^{-y^2/3}. \end{aligned} \quad (4.39f)$$

Strict concavity makes the two complementary tails, with the same one derivative, $O(e^{-c\epsilon^{-1/5}})$. Moreover,

$$\partial_\xi L(y) = \epsilon^{-1} \log \frac{1 - \epsilon s q y}{1 + \epsilon s p y};$$

on the central interval this is polynomially bounded, and at either endpoint its logarithmic factor is absorbed by the beta power there. Thus the same tail estimate holds after one ξ derivative. In particular, if $\mathcal{N}_\epsilon, \mathcal{D}_\epsilon$ denote the exact truncated and full beta integrals, (4.39f) implies

$$\left\| \frac{\mathcal{N}_\epsilon}{\mathcal{D}_\epsilon} - \frac{\mathcal{N}_0 + \epsilon \mathcal{N}_1 + \epsilon^2 \mathcal{N}_2}{\mathcal{D}_0 + \epsilon \mathcal{D}_1 + \epsilon^2 \mathcal{D}_2} \right\|_{C^1(\xi)} \leq C\epsilon^3. \quad (4.39g)$$

The denominator is bounded away from zero on the fixed rectangle. Expanding the quotient in (4.39g), subtracting its leading normal term, and multiplying by $\epsilon^{-1} = \sqrt{n}$ proves the asserted genuine $O_{C^1}(\epsilon^2)$ remainder in (4.40).

For an independently checkable evaluation of the coefficients, put $z = s\xi$ and

$$I_m(z) = \int_{-\infty}^0 y^m e^{-y^2/2 - zy} dy.$$

Integration by parts gives

$$I_0(z) = \sqrt{2\pi} e^{z^2/2} \Phi(z), \quad I_1(z) = -1 - zI_0(z), \quad I_m(z) = (m-1)I_{m-2}(z) - zI_{m-1}(z). \quad (4.40a)$$

The corresponding full-line moments are obtained by differentiating $\sqrt{2\pi} e^{z^2/2}$. Substitution of (4.39a) in (4.39d), followed by (4.40a), gives

$$H_0(p, \xi) = \frac{\phi(s\xi)}{\rho_2(p)} \left\{ \frac{s}{2} + \frac{\gamma}{6} (1 - s^2 \xi^2) \right\}. \quad (4.40b)$$

At $\xi = 0$, Q_2 is even, so $\mathcal{N}_2 = \mathcal{D}_2/2$, while $\mathcal{D}_1 = 0$ and $\mathcal{N}_0 = \mathcal{D}_0/2$; hence $H_1(p, 0) = 0$. Here and below a prime means $\partial_\xi|_{\xi=0}$. The moment recursion (4.40a) gives the scalar reductions

$$\begin{aligned} \mathcal{N}_1 &= \frac{s(2-p)}{3}, & \mathcal{D}'_1 &= \frac{s^2}{2} \mathcal{D}_0, & \mathcal{D}'_0 &= 0, & \mathcal{N}'_0 &= s, \\ \mathcal{D}_2 &= \frac{s^2(1-pq)}{12} \mathcal{D}_0, & \mathcal{D}'_2 &= 0, & \mathcal{N}'_2 &= \frac{s^3 p}{3}. \end{aligned} \quad (4.40c)$$

Differentiating the second line of (4.39d) at zero therefore gives

$$\begin{aligned} R'_2(p, 0) &= \phi_0 s^3 \left\{ \frac{p}{3} - \frac{2-p}{6} - \frac{1-pq}{12} \right\} \\ &= -\frac{\phi_0(p^2 - 7p + 5)}{12(pq)^{3/2}} \\ &= -\frac{s^3 \phi_0}{48} \{(1-2p)^2 + 12(1-2p) + 7\} = \ell(p). \end{aligned} \quad (4.40d)$$

Thus

$$H_1(p, 0) = 0, \quad \partial_\xi H_1(p, 0) = \frac{\ell(p)}{\rho_2(p)}. \quad (4.41)$$

As a direct cross-check, (S6e) says exactly

$$F(p, \eta) - F(p, 0) = -\frac{a(p)}{\rho_2(p)} \eta^2 + \frac{\ell(p)}{\rho_2(p)} \eta;$$

this is also obtained by putting $\xi = \epsilon\eta$ in the C^1 expansion (4.40).

Since $\phi(s\xi) = \phi_0(1 - s^2 \xi^2/2 + O(\xi^4))$ and $s + \gamma = 2sq$, the quadratic coefficient in (4.40b) is $-\phi_0 s^2(s + \gamma)/(4\rho_2) = -a/\rho_2$. Together with (4.41), this gives

$$H_0 = A(p) - \frac{a(p)}{\rho_2(p)} \xi^2 + O(\xi^4), \quad H_1 = \frac{\ell(p)}{\rho_2(p)} \xi + O(\xi^2). \quad (4.42)$$

On the scale in (c),

$$\xi^2 = O(d + n^{-1}) = O(d), \quad n^{-1} = o(d).$$

Thus the three node remainders in (4.42) and (4.40) are, respectively, $O(\eta^4/n^2) = O(d^2)$, $O(\eta^2/n^{3/2}) = O(d/\sqrt{n})$, and $O(n^{-1})$, all $o(d)$. The zero-cell remainder is also $o(d)$: (4.33) and $\eta^2 = O(nd)$ give

$$\sqrt{n}|R_n^{(0)}| \leq O(n^{-3/2}) + O(\sqrt{n} d^2) + O(n^{-2}) + O(\sqrt{n} d^{5/2}) = o(d), \quad (4.42a0)$$

because $nd \rightarrow \infty$, $\sqrt{n}d \rightarrow 0$, and $d \rightarrow 0$. Equations (4.40)–(4.41) are the expansion of the complete node $E_{Q,n}(\eta)$, so its high-end node value is already present. Adding only the zero-cell increment (4.31), and using $\ell = 2a\eta_0$, completes the exact square in (4.34). The high-end phase increment from (4.29) has not been added here and is therefore not double counted. Finally $A(p)$ is exactly the function in (4.13) under the change $r = 1 - 2p$, proving its last assertion. $\square$

Lemma 7.7 (coarse phase localization). *Assume $p \rightarrow p_*$, $d \rightarrow 0$, $\sqrt{n}d \rightarrow 0$, and let an active positive cell have $\xi = \eta/\sqrt{n} = o(1)$ and $|\theta| \leq 1/2$. Before using the windowed remainder (4.33), there exist fixed $c_{\text{loc}}, C_{\text{loc}} > 0$ such that, uniformly,*

$$C_n^+(X; \eta, \theta) \leq A(p) - c_{\text{loc}} \frac{\eta^2}{n} + C_{\text{loc}} d + C_{\text{loc}} n^{-1} + o(d + \eta^2/n). \quad (4.42a)$$

Consequently every cell whose score exceeds C_E satisfies $\eta^2 = O(nd + 1)$.

Proof. We give the complete assembly because this estimate is used to obtain the η -window and therefore cannot itself use that window. All constants below are uniform for p in a sufficiently small fixed neighbourhood of p_* , and we continue to write $\rho = \rho_2(p) + d$. Let

$$\begin{aligned} \mathcal{E}_{0,n} &= \frac{\sqrt{n}}{\rho} \{D_{X,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, \theta)\}, \\ \mathcal{E}_{M,n} &= \frac{\sqrt{n}}{\rho} \{M_{X,n}(\eta, \theta) - M_{Q,n}(\eta, \theta)\}. \end{aligned} \quad (4.42b)$$

Put $\zeta = re^{i\alpha}$ and choose $\vartheta \in [-1/2, 1/2]$ so that $n\alpha = 2\pi(j + \vartheta)$; this is possible for all sufficiently large rows because $|1 - \zeta| = O(d)$. When $d = 0$ take $r = 1$, $\alpha = \vartheta = 0$. With $u = \theta - \vartheta$, the exact decomposition (4.24a), followed only by the endpoint factorization (4.26), has the following ledger:

$$\begin{aligned} J_{Q,n}(\eta, \theta) &:= 2 \int_{3/4}^1 K(t) e^{-2\pi i(k+\theta)t} \Psi_Q(2\pi t)^n dt, \quad D_{Q,n}^{(1)} = \Re J_{Q,n}. \\ \mathcal{P}_n^+(X; \eta, \theta) &= \frac{\sqrt{n}}{\rho} E_{Q,n}(\eta) \\ &\quad + \frac{\sqrt{n}}{\rho} \{D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)\} \\ &\quad + \frac{\sqrt{n}}{\rho} \Re \{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\} \\ &\quad + \frac{\sqrt{n}}{\rho} \{M_{Q,n}(\eta, \theta) - M_{Q,n}(\eta, 0)\} \\ &\quad + \mathcal{E}_{0,n} + \mathcal{E}_{M,n} + \mathcal{S}_n + \mathcal{R}_n. \end{aligned} \quad (4.42c)$$

Here $\mathcal{S}_n$ is the normalized returned-endpoint integral generated by

$$-2\pi i n h m \zeta^{n-1} \Psi_Q(2\pi(1-h))^{n-1}, \quad (4.42d)$$

and, writing $R_n^{\text{end}}(h)$ for the power remainder defined below,

$$\begin{aligned} \mathcal{S}_n &= \frac{\sqrt{n}}{\rho} 2\Re \int_0^{1/4} K(1-h) e^{-2\pi i(k+\theta)(1-h)} \{-2\pi i n h m \zeta^{n-1} \Psi_Q(2\pi(1-h))^{n-1}\} dh, \\ \mathcal{R}_n &= \frac{\sqrt{n}}{\rho} 2\Re \int_0^{1/4} K(1-h) e^{-2\pi i(k+\theta)(1-h)} R_n^{\text{end}}(h) dh. \end{aligned} \quad (4.42d1)$$

Thus (4.42c) is an identity, not a comparison of cdf expansions at two sides of a lattice jump.

We first bound the four matching-law terms. Equations (4.40)–(4.42), with $\xi = \eta/\sqrt{n}$, give

$$\frac{\sqrt{n}}{\rho_2(p)} E_{Q,n}(\eta) = A(p) - \frac{a(p)}{\rho_2(p)} \xi^2 + \frac{\ell(p)}{\rho_2(p)} \frac{\xi}{\sqrt{n}} + O\left(\xi^4 + \frac{\xi^2}{\sqrt{n}} + \frac{1}{n}\right). \quad (4.42e)$$

Since $\xi = o(1)$ and $a(p)/\rho_2(p)$ stays bounded away from zero, Young's inequality yields fixed $c_1, C_1 > 0$ such that

$$\frac{\sqrt{n}}{\rho} E_{Q,n}(\eta) \leq A(p) - c_1 \xi^2 + \frac{C_1}{n}. \quad (4.42f)$$

Indeed, the left side of (4.42e) is $A(p) + o(1) > 0$, uniformly along the rows under consideration, and

$$\frac{\sqrt{n}}{\rho} E_{Q,n}(\eta) = \frac{\rho_2(p)}{\rho} \frac{\sqrt{n}}{\rho_2(p)} E_{Q,n}(\eta) \leq \frac{\sqrt{n}}{\rho_2(p)} E_{Q,n}(\eta).$$

For the zero-cell phase increment define, at real root-scale coordinate x ,

$$Z_{0,n}(p, x) := \frac{\sqrt{n}}{\rho_2(p)} \left\{ \frac{1}{2} - \Phi(sx) + 2\Re \int_0^{1/4} K(t) e^{-2\pi i \sqrt{n} x t} c_p(t)^n dt \right\}. \quad (4.42f0)$$

Thus the normal term is included. Since both the normal argument and the Fourier phase depend on $\eta + \theta$, one has the exact identity

$$\frac{\sqrt{n}}{\rho} \{D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)\} = \frac{\rho_2(p)}{\rho} \{Z_{0,n}(p, \xi + \theta/\sqrt{n}) - Z_{0,n}(p, \xi)\}. \quad (4.42f1)$$

On a fixed compact set of (p, x) , the substitution $t = v/\sqrt{n}$ gives

$$Z_{0,n}(p, x) = H_{0,0}(p, x) + n^{-1/2} H_{1,0}(p, x) + O_{C^1}(n^{-1}), \quad (4.42g)$$

where $H_{1,0}$ has bounded C^1 norm and

$$H_{0,0}(p, x) = \frac{\phi(sx)}{\rho_2(p)} \left\{ \frac{s}{2} + \frac{\gamma}{6} (1 - s^2 x^2) \right\} \quad (4.42g1)$$

is even. Indeed, Taylor expansion one order beyond the leading profile gives, uniformly for $v \leq n^{1/10}$,

$$\begin{aligned} c_p(v/\sqrt{n})^n &= e^{-2\pi^2 p q v^2} \left\{ 1 + n^{-1/2} i \beta_3(p) v^3 + n^{-1} P_6(p, v) + O(n^{-3/2} P_9(v)) \right\}, \\ 2n^{-1/2} K(v/\sqrt{n}) &= \frac{i}{\pi v} + n^{-1/2} + n^{-1} \kappa_K v + O(n^{-3/2} v^2), \end{aligned} \quad (4.42g2)$$

where $\beta_3(p) := -(4\pi^3/3)p(1-p)(1-2p)$, $P_6(p, \cdot)$ has degree at most six with coefficients uniformly bounded near p_* , P_9 is a fixed positive polynomial majorant, and κ_K is fixed. The order-one singular integral cancels the normal term, by

$$\frac{1}{2} - \Phi(sx) + \frac{1}{\pi} \int_0^\infty e^{-2\pi^2 p q v^2} \frac{\sin(2\pi x v)}{v} dv = 0. \quad (4.42g3)$$

The coefficients of $n^{-1/2}$ and n^{-1} in the raw zero-cell integral give respectively $\rho_2(p)H_{0,0}$ and $\rho_2(p)H_{1,0}$. Gaussian integration gives (4.42g1). After one x derivative, the unnormalized remainder in (4.42g2), including all products, is bounded by $Cn^{-3/2}P_{12}(v)e^{-cv^2}$. The complement of this root-scale part has the exact Bernoulli Gaussian bound, also after that derivative. Keeping $e^{-2\pi i x v}$ exact and multiplying by $\sqrt{n}/\rho_2(p)$ therefore proves the $O_{C^1}(n^{-1})$ remainder in (4.42g). Since $\partial_x H_{0,0}(p, 0) = 0$, (4.42f1) and the mean-value theorem give

$$\frac{\sqrt{n}}{\rho} \{D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)\} = O\left(\frac{|\xi|}{\sqrt{n}} + \frac{1}{n}\right). \quad (4.42h)$$

The matching middle-cell phase increment is $O(\sqrt{n}q_0^n) = o(n^{-1})$, for a fixed $q_0 < 1$.

It remains to keep the returned endpoint, including its value and shape modes. The expansion (4.29), which is uniform for $|\eta| = o(\sqrt{n})$ and has one uniform θ derivative, gives

$$\begin{aligned} \frac{\sqrt{n}}{\rho} \Re\{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\} &= \frac{C_0(p)}{\rho\sqrt{n}} \{r^n \cos(2\pi u) - 1\} \\ &+ \frac{r^n C_1(p, \eta + \theta)}{\rho n} \sin(2\pi u) + O\left(\frac{1 + \eta^2}{n^{3/2}}\right). \end{aligned} \quad (4.42i)$$

In particular, its value part contains

$$-\frac{C_0(p)}{\rho\sqrt{n}}(1 - r^n) + \frac{C_0(p)r^n}{\rho\sqrt{n}}\{\cos(2\pi u) - 1\}, \quad (4.42j)$$

and the second term is nonpositive. Lemma 7.5, used here before any localization of η or of the circular phase, and (4.16) give

$$\begin{aligned} \mathcal{S}_n &\leq \{c_{\text{sh}}(p, d) + o(1)\}r^{n-1}|m| \\ &\leq Cd + \frac{2c_{\text{sh}}(p, d) + o(1)}{n}(1 - r^n). \end{aligned} \quad (4.42k)$$

Here we used $|1 - \zeta| = O(d)$, the $O(d^2)$ term was included in Cd , and (4.17) was used for the displayed radial term. Since $C_0(p)/\rho$ stays bounded away from zero while $c_{\text{sh}}(p, d)$ stays bounded,

$$-\left\{\frac{C_0(p)}{\rho\sqrt{n}} - \frac{2c_{\text{sh}}(p, d) + o(1)}{n}\right\}(1 - r^n) \leq 0 \quad (4.42l)$$

for all sufficiently large n . This explicitly absorbs the radial shape mode into the returned value mode.

For completeness, we also bound the endpoint power remainder rather than hiding it in the word ‘‘Taylor.’’ Write

$$A_h = \Psi(2\pi(1 - h)), \quad B_h = \zeta\Psi_Q(2\pi(1 - h)), \quad l_h = -2\pi i h m, \quad e_h = A_h - B_h - l_h.$$

Lemma 7.4 gives $|e_h| \leq Cdh^2$. The Bernoulli gap and $|1 - \zeta| + |m| \leq Cd$ imply, uniformly for $0 \leq \tau \leq 1$,

$$\max\{|A_h|, |B_h|, |B_h + \tau(A_h - B_h)|\} \leq \exp\{-ch^2 + Cd^2\}. \quad (4.42m)$$

Indeed, for some fixed $c_0 > 0$, $|B_h| \leq e^{-c_0 h^2}$ and $|A_h - B_h| \leq |l_h| + |e_h| \leq Cdh$. On the root-scale part this gives $|B_h + \tau(A_h - B_h)| \leq 1 - c_1 h^2 + Cdh$, and $Cdh \leq c_1 h^2/2 + Cd^2$ proves (4.42m); the fixed part of the endpoint cell is covered by the strict Bernoulli gap. Taylor’s exact integral identity

$$(a + b)^n - a^n - na^{n-1}l = na^{n-1}e + n(n-1)b^2 \int_0^1 (1 - \tau)(a + \tau b)^{n-2} d\tau,$$

with $a = B_h$, $b = l_h + e_h$, and $e = e_h$, now gives

$$|R_n^{\text{end}}(h)| \leq C\{ndh^2 + n^2|m|^2h^2 + n^2d^2h^4\}e^{-cnh^2 + Cnd^2}. \quad (4.42n)$$

Here $R_n^{\text{end}}(h) = A_h^n - B_h^n - nB_h^{n-1}l_h$, and $\mathcal{R}_n$ in (4.42c) is its normalized returned-endpoint integral with the common spatial phase and kernel. Because $nd^2 \rightarrow 0$, $|K(1 - h)| \leq Ch$, $\int_0^\infty h^3 e^{-cnh^2} dh = O(n^{-2})$, and $\int_0^\infty h^5 e^{-cnh^2} dh = O(n^{-3})$, its normalized integral satisfies

$$|\mathcal{R}_n| \leq C\left\{\frac{d}{\sqrt{n}} + \sqrt{n}|m|^2 + \frac{d^2}{\sqrt{n}}\right\} = o(d + n^{-1}). \quad (4.42o)$$

Finally Lemma 7.3 gives $|\mathcal{E}_{0,n}| \leq \{\Lambda_{\text{abs}}(p, d) + o(1)\}d = O(d)$. On the middle cell the matching transform has a fixed Cramér gap and Theorem 2.2 gives a uniform $O(d)$ transform difference; hence

$$|\mathcal{E}_{M,n}| \leq Cn^{3/2}dq_1^{n-1} = o(d)$$

for some $q_1 < 1$ (and it is exactly zero when $d = 0$).

Insert (4.42f), (4.42h)–(4.42o) into the exact ledger (4.42c). Since $C_1(p, z)$ is affine in z , all still unsigned mixed terms are bounded by

$$C \frac{|\xi|}{\sqrt{n}} + \frac{C}{n} + O\left(\frac{1 + \eta^2}{n^{3/2}}\right).$$

For any fixed $\varepsilon > 0$, Young's inequality gives $C|\xi|/\sqrt{n} \leq \varepsilon\xi^2 + C_\varepsilon/n$; moreover $(1 + \eta^2)n^{-3/2} = o(n^{-1} + \xi^2)$. Choose ε smaller than half the loss in (4.42f), absorb every pure $o(n^{-1})$ term into C_{loc}/n , and use $\mathcal{C}_n^+ \leq \mathcal{P}_n^+$. This proves (4.42a) with fixed $c_{\text{loc}}, C_{\text{loc}} > 0$.

Since $A(p) \leq C_E$, a score larger than C_E forces, after absorbing the sequential $o(d + \xi^2)$ term, $\xi^2 \leq C(d + n^{-1})$. Equivalently, $\eta^2 = O(nd + 1)$, as asserted. $\square$

We now record the signed endpoint consequence of the factorization. This is the common input for all three defect scales below.

Lemma 7.8 (perturbed endpoint assembly). *Assume $p \rightarrow p_*$, $d \rightarrow 0$, and $\sqrt{n}d \rightarrow 0$. Since $|1 - \zeta| = O(d)$, one has $\zeta \neq 0$ for all sufficiently large n . Write $\zeta = re^{i\alpha}$ and choose $\vartheta \in [-1/2, 1/2]$ so that $n\alpha = 2\pi(j + \vartheta)$ for an integer j . For each physical $\theta \in [-1/2, 1/2]$ put $u = \theta - \vartheta$; only the trigonometric class of u modulo $\mathbb{Z}$ will be used. Define the complex matching endpoint integral*

$$J_{Q,n}(\eta, \theta) := 2 \int_{3/4}^1 K(t) e^{-2\pi i(k+\theta)t} \Psi_Q(2\pi t)^n dt; \quad D_{Q,n}^{(1)} = \Re J_{Q,n}.$$

The phase-modulated analytic baseline is

$$\mathcal{B}_{n,p,\zeta}^+(\eta, \theta) := \frac{\sqrt{n}}{\rho} \left\{ D_{Q,n}^{(0)}(\eta, \theta) + \Re(\zeta^n J_{Q,n}(\eta, \theta)) + M_{Q,n}(\eta, \theta) \right\}. \quad (4.42P)$$

Every term here uses the same integer k , the same physical phase θ , and hence the same cdf point $x_{\eta,\theta}$. This is only an analytic comparison object; unless $\zeta = 1$ it is not asserted to be the Prawitz operator of a probability law. Uniformly under these hypotheses,

$$\begin{aligned} \Psi(2\pi(1-h))^n &= \zeta^n \Psi_Q(2\pi(1-h))^n \\ &\quad - 2\pi i n h m \zeta^{n-1} \Psi_Q(2\pi(1-h))^{n-1} + \mathcal{R}_n(h), \end{aligned} \quad (4.43)$$

where

$$\frac{\sqrt{n}}{\rho} 2 \int_0^{1/4} |K(1-h)| |\mathcal{R}_n(h)| dh \leq C\{d/\sqrt{n} + \sqrt{n}|m|^2\}. \quad (4.43a)$$

The complete value mode relative to the calibrated two-point node is

$$\begin{aligned} &\frac{\sqrt{n}}{\rho} \Re\{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\} \\ &= \frac{C_0(p)}{\rho\sqrt{n}} \{r^n \cos(2\pi u) - 1\} + \frac{r^n C_1(p, \eta + \theta)}{\rho n} \sin(2\pi u) + O\left(\frac{1 + \eta^2}{n^{3/2}}\right). \end{aligned} \quad (4.43b)$$

The first shape mode is at most

$$\{c_{\text{sh}}(p, d) + o(1)\}r^{n-1}|m|, \quad c_{\text{sh}}(p, d) \longrightarrow c_{\text{sh},*}. \quad (4.43c)$$

Consequently its radial part is absorbed by the value mode:

$$-\left\{\frac{C_0(p)}{\rho\sqrt{n}} - \frac{2c_{\text{sh}}(p, d) + o(1)}{n}\right\}(1 - r^n) \leq 0. \quad (4.43d)$$

For every fixed $D, M < \infty$, under $nd \leq D$ and $|\eta| \leq M$, the global circular-phase comparison, followed by the zero-cell increment, gives uniformly in the admissible η and $|\theta| \leq 1/2$

$$\mathcal{B}_{n,p,\zeta}^+(\eta, \theta) \leq C_Q^*(n) - \frac{a_{\text{tan},*}}{\rho_* n}(\eta + \theta - \eta_*)^2 + o_{D,M}(n^{-1}). \quad (4.43e)$$

On the localized positive branch the phase-free low-frequency bound gives

$$\mathcal{C}_n^+(X; \eta, \theta) \leq C_Q^*(n) - \{C_E/\rho_* - \frac{19}{16}c_{\text{sh},*} - \Lambda_{\text{abs},*} - o(1)\}d - \frac{a_{\text{tan},*}}{\rho_* n}(\eta + \theta - \eta_*)^2 + o_{D,M}(n^{-1}). \quad (4.43f)$$

If instead $d = o(n^{-1/2})$, $nd \rightarrow \infty$, and $\eta^2 = O(nd + 1)$, then

$$\max\{\mathcal{C}_n^+(X; \eta, \theta), \mathcal{C}_n^-(X; \eta, \theta)\} \leq C_Q^*(n) - \left\{\frac{C_Q^*(n)}{\rho} - \frac{19}{16}c_{\text{sh},*} - \Lambda_{\text{abs},*} - o(1)\right\}d + o(d). \quad (4.43g)$$

In particular, the coefficient in braces tends to the first positive margin in (4.24).

Proof. Write $A_h = \Psi(2\pi(1 - h))$, $B_h = \zeta\Psi_Q(2\pi(1 - h))$, $l_h = -2\pi i h m$, and $e_h = h^2 d R_{p,h}$, where $|R_{p,h}| \leq C$ by (4.26). The Bernoulli gap and $|1 - \zeta| + |m| \leq Cd$ imply, uniformly for $0 \leq t \leq 1$,

$$\max\{|A_h|, |B_h|, |B_h + t(A_h - B_h)|\} \leq \exp\{-c_1 h^2 + C_1 d^2\}. \quad (4.43a0)$$

Near zero this follows from $e^{-ch^2} + Cdh \leq 1 - c'h^2 + Cdh$ and completion of the square; outside a smaller fixed cell it follows from the strict Bernoulli gap. Taylor's literal integral identity is

$$(a + b)^n - a^n - na^{n-1}l = na^{n-1}e + n(n-1)b^2 \int_0^1 (1-t)(a+tb)^{n-2} dt, \quad (4.43a1)$$

where $a = B_h$, $b = l_h + e_h$, and $e = e_h$. Equations (4.43a0)–(4.43a1) give

$$|\mathcal{R}_n(h)| \leq C\{ndh^2 + n^2|m|^2h^2 + n^2d^2h^4\}e^{-cnh^2 + Cnd^2}.$$

Since $nd^2 \rightarrow 0$, $|K(1 - h)| \leq Ch$, $\int h^3 e^{-cnh^2} dh = O(n^{-2})$, and $\int h^5 e^{-cnh^2} dh = O(n^{-3})$, this proves (4.43)–(4.43a). Differentiating the endpoint operator in the cdf phase only multiplies the integrand by the bounded factor $-2\pi i(1 - h)$, so the same bound holds for one phase derivative. No fourth moment is used. Multiplying (4.29) by ζ^n , using $e^{-2\pi i \theta} \zeta^n = r^n e^{-2\pi i u}$, and taking real parts gives (4.43b). Lemma 7.5, which is independent of the present phase localization, gives (4.43c). Equations (4.16)–(4.17) and $r^n \cos(2\pi u) - 1 \leq -(1 - r^n)$ give (4.43d). Indeed, separately from the case $r = 1$, where the radial factor vanishes, $p \rightarrow p_*$ and $d \rightarrow 0$ imply

$$\frac{C_0(p)}{\rho\sqrt{n}} - \frac{2c_{\text{sh}}(p, d) + o(1)}{n} \geq \frac{c}{\sqrt{n}} - \frac{C}{n} > 0 \quad (4.43d1)$$

for all sufficiently large n . Thus no division by $1 - r$, $1 - r^n$, or d is hidden in the absorption.

Fix $D, M < \infty$, assume $nd \leq D$ and $|\eta| \leq M$, and split off the radial part in (4.43b) and put

$$a_n = \frac{C_0(p)r^n}{\rho\sqrt{n}}, \quad b_n(\theta) = \frac{r^n C_1(p, \eta + \theta)}{\rho n}, \quad B_n = \sup_{|\theta| \leq 1/2} |b_n(\theta)|.$$

Characteristic-function stability gives $1 - r \leq |1 - \zeta| \leq Cd \leq CD/n$, so $r^n \geq c_D > 0$ for all sufficiently large n . Hence $a_n \asymp_D n^{-1/2}$ and $B_n = O_{D,M}(n^{-1})$. For every physical θ , not merely at a maximizing phase,

$$a_n\{\cos(2\pi u) - 1\} + b_n(\theta)\sin(2\pi u) \leq a_n\{\cos(2\pi u) - 1\} + B_n|\sin(2\pi u)| \leq \sqrt{a_n^2 + B_n^2} - a_n \leq \frac{B_n^2}{2a_n}.$$

Thus the possible positive phase residue is $O_{D,M}(n^{-3/2})$ uniformly in ϑ and θ . Retaining the radial part, (4.43b) gives

$$\frac{\sqrt{n}}{\rho} \Re\{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\} \leq -\frac{C_0(p)}{\rho\sqrt{n}}(1 - r^n) + O_{D,M}(n^{-3/2}). \quad (4.43b1)$$

To identify the reference object, note that the already defined quantity (4.24c) is equivalently

$$E_{Q,n}(\eta) = D_{Q,n}^{(0)}(\eta, 0) + \Re J_{Q,n}(\eta, 0) + M_{Q,n}(\eta, 0).$$

At $\theta = 0$ the cardinal excess vanishes on the integer-valued Bernoulli sum by (P3), so

$$\frac{\sqrt{n}E_{Q,n}(\eta)}{\rho_2(p)}$$

is exactly the right-node Bernoulli score to which Lemma 6.2 applies. Direct subtraction, with no change of the physical point inside the baseline, gives

$$\begin{aligned} \mathcal{B}_{n,p,\zeta}^+(\eta, \theta) &= \frac{\sqrt{n}}{\rho} E_{Q,n}(\eta) + \frac{\sqrt{n}}{\rho} \{D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0)\} \\ &\quad + \frac{\sqrt{n}}{\rho} \Re\{\zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0)\} \\ &\quad + \frac{\sqrt{n}}{\rho} \{M_{Q,n}(\eta, \theta) - M_{Q,n}(\eta, 0)\}. \end{aligned}$$

The last line is exponentially small uniformly on the bounded node/phase set. Put

$$\lambda_n := \frac{\rho_2(p)}{\rho}, \quad \tau_n := \frac{a_{\text{tan},*}}{\rho_* n}(\eta + \theta - \eta_*)^2.$$

Lemma 6.2, (4.31), and (4.43b1) first give the naturally normalized inequality

$$\mathcal{B}_{n,p,\zeta}^+(\eta, \theta) \leq \lambda_n \{C_Q^*(n) - \tau_n\} - \frac{C_0(p)}{\rho\sqrt{n}}(1 - r^n) + \tilde{R}_n, \quad \tilde{R}_n = o_{D,M}(n^{-1}). \quad (4.43e0)$$

Here the factor λ_n multiplies the complete tangent square; it cannot be removed merely from $\lambda_n \leq 1$, because the tangent enters with a negative sign. Instead, $\rho = \rho_2(p) + d$ gives

$$1 - \lambda_n = \frac{d}{\rho}, \quad \lambda_n \{C_Q^*(n) - \tau_n\} = \lambda_n C_Q^*(n) - \tau_n + R_n^{\text{norm}},$$

where

$$R_n^{\text{norm}} := (1 - \lambda_n)\tau_n = \frac{d}{\rho} \frac{a_{\text{tan},*}}{\rho_* n}(\eta + \theta - \eta_*)^2.$$

This is uniformly $o_{D,M}(n^{-1})$. Indeed, under the quantifiers in (S.3),

$$|\eta + \theta - \eta_*| \leq M + \frac{1}{2} + |\eta_*|, \quad d \leq \frac{D}{n}, \quad \rho \longrightarrow \rho_* > 0,$$

and hence, for all sufficiently large n ,

$$0 \leq R_n^{\text{norm}} \leq \frac{2a_{\text{tan},*}D}{\rho_*^2} \left(M + \frac{1}{2} + |\eta_*|\right)^2 n^{-2} = o_{D,M}(n^{-1}).$$

After absorbing R_n^{norm} into $\tilde{R}_n$, use $\ell = 2a\eta_0$ to obtain algebraically

$$-\frac{a_{\text{tan},*}}{\rho_* n} \{(\eta - \eta_*)^2 + \theta^2 - 2(\eta_* - \eta)\theta\} = -\frac{a_{\text{tan},*}}{\rho_* n} (\eta + \theta - \eta_*)^2.$$

Consequently,

$$\mathcal{B}_{n,p,\zeta}^+(\eta, \theta) \leq \frac{\rho_2(p)}{\rho} C_Q^*(n) - \frac{a_{\tan,*}}{\rho_* n} (\eta + \theta - \eta_*)^2 - \frac{C_0(p)}{\rho \sqrt{n}} (1 - r^n) + o_{D,M}(n^{-1}). \quad (4.43e1)$$

Only at this point do $C_Q^*(n) > 0$ and $\rho_2(p)/\rho \leq 1$ imply the displayed weaker form (4.43e). In particular, the retained phase ζ^n has not been replaced by one.

It remains to compare the actual operator with this baseline. The exact decomposition (4.24a) counts the zero cell, returned endpoint, and middle once each. The zero-cell difference is at most $\{\Lambda_{\text{abs},*} + o(1)\}d$. The endpoint linear term is at most

$$\{c_{\text{sh}}(p, d) + o(1)\}r^{n-1}|m| \leq \{\frac{19}{16}c_{\text{sh},*} + o(1)\}d + \{c_{\text{sh}}(p, d) + o(1)\}r^{n-1}(1 - r^2) + O(d^2).$$

The second term is at most $\{2c_{\text{sh}}(p, d) + o(1)\}(1 - r^n)/n$ and is absorbed by the negative radial term in (4.43b1). The endpoint remainder is $o_D(n^{-1})$, and $M_{X,n} - M_{Q,n}$ is exponentially small. Finally, since $\rho = \rho_2(p) + d$ and the reference right-node score tends to C_E ,

$$\frac{\rho_2(p)}{\rho} C_Q^*(n) = C_Q^*(n) - \{C_E/\rho_* + o(1)\}d.$$

There is no rate conversion hidden here. Indeed (4.16) and $|1 - \zeta| = O(d)$ give $|m| = O(d)$. Under $nd \leq D$ and $|\eta| \leq M$, the terms not displayed with a sign satisfy the explicit ledger

term	bound
endpoint power remainder (4.43a)	$O(d/\sqrt{n} + \sqrt{n}d^2) = O_D(n^{-3/2})$
C_1 phase residue after the sine-cosine completion	$O_{D,M}(n^{-3/2})$
shape-continuity and quadratic errors	$o(1)d + O(d^2) = o_D(n^{-1})$
low-frequency remainder	$o(1)d = o_D(n^{-1})$
middle-cell difference	$O(n^{3/2}dq_1^n) = o_D(n^{-1})$

(4.43f0)

The second row uses the global completion preceding (4.43b1), not an assertion that the affine C_1 term has a sign by itself. On the localized branch Lemma 7.7 supplies $\eta^2 = O(nd + 1)$, so boundedness of η in this regime is a conclusion, not an additional hypothesis. Combining these signed terms proves (4.43f); its coefficient is positive by (4.24). Notice that the loss $C_E d/\rho_*$ and the radial absorption belong to the combined baseline comparison, not to a false standalone inequality asserting that the actual operator is already below the baseline by a fixed multiple of d .

In the growing window use the two signed rows at the same active point, rather than extracting a tangent. The complete upper and lower two-point cells at the calibrated integer node are at most $\{\rho_2(p)/\rho\}C_Q^*(n)$. The corresponding zero-cell spatial increments have a common error $O((1 + |\eta|)/n) = o(d)$. Indeed $\eta^2 = O(nd)$ and $nd \rightarrow \infty$ give

$$\frac{1 + |\eta|}{nd} = O((nd)^{-1} + (nd)^{-1/2}) = o(1), \quad \frac{1 + \eta^2}{n^{3/2}d} = o(1) + O(n^{-1/2}) = o(1). \quad (4.43g0)$$

The signed middle-cell error is $o(n^{-1}) = o(d)$ because $nd \rightarrow \infty$. The physical returned-endpoint owner contributes $\{19c_{\text{sh},*}/16 + o(1)\}d + o(d)$ simultaneously for both signs, and the actual-minus-matching zero/middle owner contributes $\{\Lambda_{\text{abs},*} + o(1)\}d + o(d)$. These are the same owners as in (4.24d), each inserted once. Moreover (4.16), (4.43a), and $\sqrt{n}d \rightarrow 0$ give

$$|m| = O(d), \quad \frac{d}{\sqrt{n}} + \sqrt{n}|m|^2 \leq d\{n^{-1/2} + O(\sqrt{n}d)\} = o(d). \quad (4.43g1)$$

The remaining low-frequency and exponential errors are $o(d)$. Finally the exact denominator identity

$$\frac{\rho_2(p)}{\rho} C_Q^*(n) = C_Q^*(n) - \frac{C_Q^*(n)}{\rho} d, \quad \frac{C_Q^*(n)}{\rho} \rightarrow \frac{C_E}{\rho_*}, \quad (4.43g2)$$

turns the paired owner ledger into (4.43g). Its coefficient tends to $C_E/\rho_* - 19c_{\text{sh},*}/16 - \Lambda_{\text{abs},*} > 0$ by (4.24). This proof is uniform in the active point and does not require the more detailed $A(p)$ tangent expansion. $\square$

The active signed subcritical estimates

This supplement makes explicit two points which are easy to lose in the compressed proof of Lemma 7.8. First, all pieces of a row are evaluated at one and the same integer, phase, and physical cdf point. Second, in the growing window the Gaussian boundary/quadrature and the signed binomial stop-loss prefix are one quantity. Neither of them is estimated in absolute value before they have cancelled.

Signed orientation and uniform quantifiers. The upper and lower signed estimates use the same orientation. Their node-zero, middle, and returned contributions are treated together. We display the upper notation,

$$k = np + \eta \in \mathbb{Z}, \quad |\theta| \leq \frac{1}{2}, \quad x_{\eta,\theta} = \frac{s(p)(\eta + \theta)}{\sqrt{n}}. \quad (\text{S.1})$$

The lower row uses the conjugate kernel and its own lower node-zero identity at this same physical point. Thus no formula combines an upper integer-node score with a lower returned endpoint, and every occurrence of k , θ , and $x_{\eta,\theta}$ refers to the same physical point. Reflection gives an equivalent proof, but is unnecessary for these paired estimates.

We use the following precise meaning of uniform little-oh. Fix $D, M < \infty$. An assertion

$$R_n(\eta, \theta) = o_{D,M}(n^{-1}) \quad (\text{S.2})$$

means that for every sequence of admissible rows with

$$p_n \rightarrow p_*, \quad d_n \rightarrow 0, \quad \sqrt{n} d_n \rightarrow 0, \quad nd_n \leq D,$$

one has

$$\sup_{\substack{|\eta| \leq M, |\theta| \leq 1/2, \\ np_n + \eta \in \mathbb{Z}}} n |R_n(\eta, \theta)| \rightarrow 0. \quad (\text{S.3})$$

Only cells occurring in the row enter the supremum. Similarly, $R_n = o_{C_\eta}(d_n)$ in the growing window means that the supremum of $|R_n|/d_n$ tends to zero along every admissible sequence satisfying

$$d_n = o(n^{-1/2}), \quad nd_n \rightarrow \infty, \quad \eta^2 \leq C_\eta(nd_n + 1). \quad (\text{S.4})$$

Thus the quantifier over the active cell and physical phase is inside the limit. The coarse localization lemma is used only to supply a finite $M = M(D)$ in the bounded branch and a finite C_η in the growing branch. The assembly itself applies to every cell satisfying the displayed bounds. An *active point* has exactly the meaning from (4.0a), namely $\mathcal{C}_n^+(X; \eta, \theta) > C_E$; it carries no extra analytic assumption.

One exact finite-row ledger. Let $J_{Q,n}$ be as in Lemma 7.8. In oriented upper notation put

$$\begin{aligned} T_n &= \frac{\sqrt{n}}{\rho} \left\{ E_{Q,n}(\eta) + D_{Q,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, 0) \right\}, \\ V_n &= \frac{\sqrt{n}}{\rho} \Re \{ \zeta^n J_{Q,n}(\eta, \theta) - J_{Q,n}(\eta, 0) \}, \\ H_n &= \frac{\sqrt{n}}{\rho} \{ M_{Q,n}(\eta, \theta) - M_{Q,n}(\eta, 0) \}, \\ L_n &= \frac{\sqrt{n}}{\rho} \{ D_{X,n}^{(0)}(\eta, \theta) - D_{Q,n}^{(0)}(\eta, \theta) \}, \\ E_n &= \frac{\sqrt{n}}{\rho} \{ M_{X,n}(\eta, \theta) - M_{Q,n}(\eta, \theta) \}. \end{aligned} \quad (\text{S.5})$$

Let S_n be the normalized real part of the first-order endpoint term

$$-2\pi i n h m \zeta^{n-1} \Psi_Q(2\pi(1-h))^{n-1} \quad (\text{S.6})$$

integrated on $0 \leq h \leq 1/4$ with the returned Prawitz kernel, and let R_n^{pow} be the integral of the literal remainder $\mathcal{R}_n(h)$ in (4.43). These are definitions, not estimates. The exact decomposition (4.24a), with (4.24c) added and subtracted once, gives

$$\boxed{\mathcal{C}_n^+(X; \eta, \theta) \leq \mathcal{P}_n^+(X; \eta, \theta) = T_n + V_n + H_n + L_n + S_n + R_n^{\text{pow}} + E_n.} \quad (\text{S.7})$$

No term uses an independently chosen node or phase. In particular, $E_{Q,n}$ owns all four upper integer-node pieces in (4.24c), and the returned endpoint occurs only in $V_n + S_n + R_n^{\text{pow}}$.

There are constants $C, c > 0$, $0 < q_1 < 1$, and uniform moduli $\omega_{\text{lo}}, \omega_{\text{sh}} \rightarrow 0$ such that

$$|L_n| \leq \{\Lambda_{\text{abs},*} + \omega_{\text{lo}}(n)\}d, \quad (\text{S.8})$$

$$|E_n| \leq Cn^{3/2}dq_1^{n-1}, \quad (\text{S.9})$$

$$|H_n| \leq C\sqrt{n}q_1^n, \quad (\text{S.10})$$

$$|R_n^{\text{pow}}| \leq C\{d/\sqrt{n} + \sqrt{n}|m|^2 + d^2/\sqrt{n}\}, \quad (\text{S.11})$$

$$|m| \leq Cd. \quad (\text{S.12})$$

Here (S.8) is (4.18), (S.9) is the middle-cell estimate, (S.10) is the matching middle gap, and (S.11) follows from (4.43a). The harmless last summand in (S.11) merely displays all quadratic endpoint remainders. Equation (S.12) follows from (4.16) and $|1 - \zeta| = O(d)$.

The bounded active-localized branch

Fix $D, M < \infty$, assume $nd \leq D$, and restrict to active cells with $|\eta| \leq M$. The bounded-node expansion, zero-cell increment, returned-endpoint sine-cosine completion, and (S.10) give the single signed inequality

$$\begin{aligned} T_n + V_n + H_n &\leq \frac{\rho_2(p)}{\rho} C_Q^*(n) - \frac{a_{\text{tan},*}}{\rho_* n} (\eta + \theta - \eta_*)^2 \\ &\quad - \frac{C_0(p)}{\rho\sqrt{n}} (1 - r^n) + R_n^{\text{base}}, \quad R_n^{\text{base}} = o_{D,M}(n^{-1}). \end{aligned} \quad (\text{S.13})$$

The possible positive returned-endpoint residue is bounded before it is placed in R_n^{base} . Indeed, with

$$a_n = \frac{C_0(p)r^n}{\rho\sqrt{n}}, \quad B_n = \sup_{|\theta| \leq 1/2} \frac{r^n |C_1(p, \eta + \theta)|}{\rho n},$$

one has $a_n \asymp_D n^{-1/2}$, $B_n = O_{D,M}(n^{-1})$, and

$$a_n \{\cos(2\pi u) - 1\} + B_n |\sin(2\pi u)| \leq \frac{B_n^2}{2a_n} = O_{D,M}(n^{-3/2}). \quad (\text{S.14})$$

Thus (S.13) is uniform in both the retained phase ϑ and the physical phase θ . It does not assign a sign to the affine C_1 -term by itself.

More explicitly, one may decompose

$$R_n^{\text{base}} = R_n^{\text{node}} + R_n^{0,\text{loc}} + R_n^{0,\text{tail}} + R_n^{\text{circ}} + R_n^{\text{coef}} + R_n^{\text{matchmid}}, \quad (\text{S.14a})$$

where, uniformly under the quantifiers in (S.3),

$$\begin{aligned} R_n^{\text{node}} &= o_M(n^{-1}), & |R_n^{0,\text{loc}}| &\leq C_M n^{-3/2}, \\ |R_n^{0,\text{tail}}| &\leq C_M n^{-3/2}, & |R_n^{\text{circ}}| &\leq C_{D,M} n^{-3/2}, \\ R_n^{\text{coef}} &= o_{D,M}(n^{-1}), & |R_n^{\text{matchmid}}| &\leq C\sqrt{n}q_1^n. \end{aligned} \quad (\text{S.14b})$$

The first row is the uniform remainder in the bounded-node lemma; the second is (4.33) with bounded η ; the third is the finite-cell to half-line remainder (4.38c1); the fourth contains (S.14), the remainder in (4.43b); the fifth contains the $o(1)/n$ coefficient replacements; and the last is (S.10). Thus every remainder in (S.13) has a named owner. In particular, the fifth owner contains the normalization correction

$$R_n^{\text{norm}} = \left(1 - \frac{\rho_2(p)}{\rho}\right) \frac{a_{\text{tan},*}}{\rho_* n} (\eta + \theta - \eta_*)^2, \quad 0 \leq R_n^{\text{norm}} \leq C_{D,M} n^{-2}.$$

Thus moving $\rho_2(p)/\rho$ off the negative tangent square is an explicit $o_{D,M}(n^{-1})$ coefficient replacement, rather than an application of $\rho_2(p)/\rho \leq 1$.

The shape row is retained until its radial part is paired with (S.13). Equations (4.16), (4.17), and (4.43c) give

$$S_n \leq \left\{ \frac{19}{16} c_{\text{sh},*} + \omega_{\text{sh}}(n) \right\} d + \frac{2c_{\text{sh}}(p, d) + \omega_{\text{sh}}(n)}{n} (1 - r^n) + C d^2. \quad (\text{S.15})$$

The radial rows in (S.13) and (S.15) obey, uniformly for all sufficiently large n ,

$$\left[-\frac{C_0(p)}{\rho\sqrt{n}} + \frac{2c_{\text{sh}}(p, d) + \omega_{\text{sh}}(n)}{n} \right] (1 - r^n) \leq 0. \quad (\text{S.16})$$

No division by d , $1 - r$, or $1 - r^n$ occurs in (S.16).

The denominator loss is the exact identity

$$\frac{\rho_2(p)}{\rho} C_Q^*(n) = C_Q^*(n) - b_n d, \quad b_n = \frac{C_Q^*(n)}{\rho} \longrightarrow \frac{C_E}{\rho_*}. \quad (\text{S.17})$$

Substituting (S.8)–(S.17) into (S.7) gives

$$\begin{aligned} C_n^+(X; \eta, \theta) &\leq C_Q^*(n) - \left\{ \frac{C_E}{\rho_*} - \frac{19}{16} c_{\text{sh},*} - \Lambda_{\text{abs},*} - \varepsilon_{n,D,M} \right\} d \\ &\quad - \frac{a_{\text{tan},*}}{\rho_* n} (\eta + \theta - \eta_*)^2 + R_{n,D,M}, \end{aligned} \quad (\text{S.18})$$

where $\varepsilon_{n,D,M} \rightarrow 0$ and

$$\begin{aligned} |R_{n,D,M}| &\leq |R_n^{\text{base}}| + C \{ d/\sqrt{n} + \sqrt{n} d^2 + d^2 \} + \{ \omega_{\text{lo}}(n) + \omega_{\text{sh}}(n) \} d \\ &\quad + C n^{3/2} d q_1^{n-1}. \end{aligned} \quad (\text{S.19})$$

Every term on the right of (S.19) is $o_{D,M}(n^{-1})$: since $d \leq D/n$, the first two endpoint terms are $O_D(n^{-3/2})$, $d^2 = O_D(n^{-2})$, and exponential decay dominates every power of n . This is (4.43f) with its finite-row, active-localized, and signed quantifiers exposed.

The growing branch: the direct paired active ledger

Assume (S.4) with $nd \rightarrow \infty$. For each sign, let $T_n^\pm$ be the complete matching two-point cell at the calibrated integer node plus the signed matching zero-cell spatial increment from that node to the active point. Let $R_{n,\pm}^{\text{mid}}$, $R_{n,\pm}^{\text{ret}}$, and $R_{n,\pm}^{\text{act}}$ denote respectively the matching middle, physical returned-endpoint, and actual-minus-matching zero/middle owners. The exact finite-row grouping is

$$C_n^\pm(X; \eta, \theta) \leq T_n^\pm + R_{n,\pm}^{\text{mid}} + R_{n,\pm}^{\text{ret}} + R_{n,\pm}^{\text{act}}. \quad (\text{S.G0})$$

The calibrated upper and lower node-zero blocks admit one signed error r_n^{node} such that

$$T_n^\pm \leq \frac{\rho_2(p)}{\rho} C_Q^*(n) + r_n^{\text{node}}, \quad \frac{r_n^{\text{node}}}{d} \longrightarrow 0. \quad (\text{S.G1})$$

The estimate is obtained before taking the maximum of the two signs. The integer-node terms use the exact two-point bound, while the two zero-cell increments are bounded by $C(1 + |\eta|)/n$; (4.43g0) makes the latter $o(d)$. No binomial prefix is separated from a Gaussian boundary term.

With the same cutoff and the same physical point, the remaining concrete owners give errors satisfying

$$\begin{aligned} |R_{n,\pm}^{\text{mid}}| &\leq r_n^{\text{mid}}, & r_n^{\text{mid}}/d &\longrightarrow 0, \\ R_{n,\pm}^{\text{ret}} &\leq \{\frac{19}{16}c_{\text{sh},*} + o(1)\}d + r_n^{\text{ret}}, & r_n^{\text{ret}}/d &\longrightarrow 0, \\ R_{n,\pm}^{\text{act}} &\leq \{\Lambda_{\text{abs},*} + o(1)\}d + r_n^{\text{act}}, & r_n^{\text{act}}/d &\longrightarrow 0. \end{aligned} \quad (\text{S.G2})$$

Here $0 \leq r_n^{\text{mid}} = o(n^{-1}) = o(d)$, the returned owner includes the power error (4.43a), and the actual owner is the full zero/middle actual-minus-matching difference. Hence the exact finite-row assembly (S.G0) gives

$$C_n^\pm(X; \eta, \theta) \leq \frac{\rho_2(p)}{\rho} C_Q^*(n) + \{\frac{19}{16}c_{\text{sh},*} + \Lambda_{\text{abs},*} + o(1)\}d + r_n, \quad r_n/d \longrightarrow 0. \quad (\text{S.G3})$$

Since $\rho = \rho_2(p) + d$,

$$\frac{\rho_2(p)}{\rho} C_Q^*(n) = C_Q^*(n) - \frac{C_Q^*(n)}{\rho} d, \quad \frac{C_Q^*(n)}{\rho} \longrightarrow \frac{C_E}{\rho_*}. \quad (\text{S.G4})$$

Taking the maximum in (S.G3) and using (S.G4) proves (4.43g).

The transition to one adaptive lattice

The scale $\sqrt{n}d \asymp 1$ cannot be treated by expanding around a fixed two-point resonance: the resonance itself may move. The next lemma supplies the required signed compactness without approximating the discontinuous sawtooth.

In this subsection an *active near-extremal positive-branch sequence* means a sequence of points x_n which are active in the sense of (4.0a), are chosen on a putative extremizing sequence, and satisfy, with $\rho_n = \mathbb{E}|X_n|^3$,

$$\frac{\sqrt{n}}{\rho_n} \{F_{n,X_n}(x_n) - \Phi(x_n)\} \longrightarrow C_E$$

and the positive error is the competitive one. The outer localization proved above then gives $p_n \rightarrow p_*$, $d_n \rightarrow 0$, $x_n \rightarrow 0$, while the reflected branch has a fixed gap. We shall always state explicitly when only the corresponding Prawitz operator, rather than the cdf error, is being used; thus “active” carries no unmentioned Fourier estimate.

Lemma 7.9 (spectral selection and the vanishing-power branch). *Let X_n be centered, variance-one laws, set $\gamma_n = \mathbb{E}X_n^3$, let p_n be the matching Bernoulli parameter, and put $d_n = \mathbb{E}|X_n|^3 - \rho_2(\gamma_n)$. Suppose $p_n \rightarrow p_*$ and $d_n \rightarrow 0$. Put*

$$R_n(u) = |\varphi_{X_n}(u)|^2, \quad u_n^* = \frac{2\pi}{s(p_n)}.$$

There are fixed $\delta, c > 0$ such that, for all sufficiently large n , R_n has a unique interior maximizer u_n on $I_n = [u_n^* - \delta, u_n^* + \delta]$ and $R_n'' \leq -c$ there. Choose once and for all

$$\max\{s_*/2, \gamma_*\} < h_0 < s_*.$$

If $|\varphi_{X_n}(u_n)|^n \rightarrow 0$, then

$$\limsup_n \mathcal{C}_n(X_n) \leq \frac{\phi_0}{\rho_*} \left(\frac{h_0}{2} + \frac{\gamma_*}{6} \right) < C_E. \quad (4.43h)$$

The reflected one-sided score has a strictly smaller limit.

Proof. The defect identity and $d_n \rightarrow 0$ concentrate the law at the two zeros of the defect density. The first two moments determine their limiting weights, so X_n converges to its matching two-point law. Since the second and third absolute moments also converge to those of that law, $|X_n|^2$ and $|X_n|^3$ are uniformly integrable. For an independent copy X'_n set $D_n = X_n - X'_n$. Then

$$R''_n(u) = -\mathbb{E}\{(X_n - X'_n)^2 \cos(u(X_n - X'_n))\}.$$

Uniform integrability of D_n^2 , followed by a fixed tail truncation, shows that both R'_n and R''_n converge uniformly on every fixed compact u -interval. In particular, uniformly on a fixed neighbourhood of u_n^* ,

$$-R''_n(u) \rightarrow 2 \cos\{s_*(u - 2\pi/s_*)\}.$$

Choose $\delta > 0$ and $c_0 > 0$ so that the last cosine is at least $4c_0$ when $|u - 2\pi/s_*| \leq 2\delta$. For all large n ,

$$R''_n(u) \leq -2c_0 \quad (u \in I_n), \quad R'_n(u_n^* - \delta) > c_0\delta, \quad R'_n(u_n^* + \delta) < -c_0\delta. \quad (4.43h1)$$

The derivative signs follow by integrating the limiting curvature from the limiting critical point. Thus strict concavity gives the asserted unique $u_n \in \text{int } I_n$. Uniform convergence of R'_n and uniqueness of the zero of its limit on this interval also give $u_n \rightarrow 2\pi/s_*$, and hence $R_n(u_n) \rightarrow 1$. Taylor's theorem at u_n then gives, after decreasing δ if necessary,

$$|\varphi_{X_n}(u)|^n \leq |\varphi_{X_n}(u_n)|^n e^{-c_1 n(u-u_n)^2}, \quad |u - u_n| \leq \delta, \quad (4.43h2)$$

with fixed $c_1 > 0$. Indeed (4.43h1) gives $R_n(u) \leq R_n(u_n) - c_0(u - u_n)^2$, while $R_n(u_n) \rightarrow 1$; divide by $R_n(u_n)$ and use $(1 - y)^{n/2} \leq e^{-ny/2}$. This proves (4.43h2) after a uniform decrease of c_1 .

We make the frequency split used here explicit. At physical bandwidth h_0 , put $U_0 = 2\pi/h_0$. The assumptions $s_*/2 < h_0 < s_*$ imply $2\pi/s_* < U_0 < 4\pi/s_*$. Hence the limiting two-point modulus is one on $[0, U_0]$ precisely at 0 and $2\pi/s_*$. Choose fixed disjoint neighbourhoods

$$Z = [0, \delta_0], \quad E_n = [u_n - \delta_1, u_n + \delta_1] \subset (0, U_0),$$

with $\delta_0, \delta_1 > 0$ small enough for the Gaussian zero-frequency bound and (4.43h2), respectively. On the compact remainder $M_n = [0, U_0] \setminus (Z \cup E_n)$, uniform convergence of characteristic functions and the displayed description of the limiting equality set give constants $q < 1$ and n_0 such that

$$\sup_{u \in M_n} |\varphi_{X_n}(u)| \leq q \quad (n \geq n_0). \quad (4.43h3)$$

Thus the normalized middle contribution is $O(\sqrt{n}q^n)$. On E_n the kernel and the change of physical frequency are bounded, and (4.43h2) bounds the normalized contribution by

$$C\sqrt{n} |\varphi_{X_n}(u_n)|^n \int_{\mathbb{R}} e^{-c_1 n v^2} dv = o(1). \quad (4.43h4)$$

This proves, with uniform constants, that only the zero cell remains. Uniform integrability of the third moment gives the uniform Peano expansion

$$\varphi_{X_n}(w) = 1 - \frac{w^2}{2} - \frac{i\gamma_n w^3}{6} + |w|^3 \eta_n(w),$$

where

$$\lim_{\delta \downarrow 0} \limsup_n \sup_{|w| \leq \delta} |\eta_n(w)| = 0,$$

together with a uniform Gaussian power envelope $|\varphi_{X_n}(w)|^n \leq C e^{-cnw^2}$ for small w . In the Prawitz integral put

$$t = \frac{h_0 v}{2\pi\sqrt{n}}, \quad \text{equivalently} \quad w = \frac{2\pi t}{h_0} = \frac{v}{\sqrt{n}}.$$

The kernel expansion is then

$$2K(t) \, dt = \left\{ \frac{i}{\pi v} + \frac{h_0}{2\pi\sqrt{n}} + O(v/n) \right\} dv,$$

with the conjugate expansion for the reflected operator. After multiplication by $\sqrt{n}$, every remainder is bounded, independently of the cdf coordinate, by a constant times

$$(v^2 + v^3 + v^4)e^{-cv^2}.$$

First pass to the limit on $0 \leq v \leq M$ and then let $M \rightarrow \infty$. This two-stage domination yields, uniformly in the cdf coordinate, the two signed limits

$$\frac{\phi(x)}{\rho_*} \left\{ \frac{h_0}{2} \pm \frac{\gamma_*}{6}(1-x^2) \right\}. \quad (4.43i)$$

For the plus sign the positive maximum is at zero. For the minus sign it is also at zero because $h_0 > \gamma_*$, and it is smaller. Finally

$$\frac{\phi_0}{\rho_*} \left(\frac{h_0}{2} + \frac{\gamma_*}{6} \right) < \frac{\phi_0}{\rho_*} \left(\frac{s_*}{2} + \frac{\gamma_*}{6} \right) = C_E.$$

This proves the lemma. $\square$

Lemma 7.10 (non-circular adaptive majorant). *Let X_n be centered variance-one laws with matching parameters $p_n \rightarrow p_*$, defects $d_n \rightarrow 0$, third moments γ_n , and third absolute moments ρ_n . If $h_n \rightarrow s_*$, define the exact Prawitz operators at physical bandwidth h_n . Then, uniformly in $x \in \mathbb{R}$,*

$$\frac{\sqrt{n}}{\rho_n} \{ \mathcal{U}_{X_n, n, h_n}^+(x) - \Phi(x) \} = \frac{\phi(x)}{\rho_n} \left\{ \frac{h_n}{2} + \frac{\gamma_n}{6}(1-x^2) \right\} + o(1), \quad (4.43j)$$

and

$$\frac{\sqrt{n}}{\rho_n} \{ \Phi(x) - \mathcal{U}_{X_n, n, h_n}^-(x) \} = \frac{\phi(x)}{\rho_n} \left\{ \frac{h_n}{2} - \frac{\gamma_n}{6}(1-x^2) \right\} + o(1). \quad (4.43k)$$

Consequently the positive adaptive majorant is at most $C_E + o(1)$ and the reflected one has a fixed limiting gap. No residual-power conclusion and no residual-variance estimate of order $o(n^{-1})$ is used.

Proof. Write $\varphi_n := \varphi_{X_n}$. Uniformly for γ near γ_* , the exact defect density satisfies

$$w_\gamma(x) \geq c|x|^3 \quad (|x| \geq R_0), \quad \mathbb{E}w_\gamma(X_n) = d_n. \quad (4.43l)$$

Thus $\{|X_n|^3\}$ is uniformly integrable: for all sufficiently large n its tail is at most Cd_n , and the finitely many earlier rows are handled by increasing the cutoff. Quadratic defect localization and the moment equations also give $X_n \Rightarrow Q_*$. Hence

$$\omega(r) := \sup_n \mathbb{E}\{|X_n|^3 \min(r|X_n|, 1)\} \longrightarrow 0 \quad (r \downarrow 0), \quad (4.43m)$$

and

$$\varphi_n(w) = 1 - \frac{w^2}{2} - \frac{i\gamma_n w^3}{6} + |w|^3 \epsilon_n(w), \quad |\epsilon_n(w)| \leq C\omega(|w|). \quad (4.43n)$$

Symmetrization and $1 - \cos y \geq y^2/2 - |y|^3/6$ give fixed $c, \delta_0 > 0$ such that $|\varphi_n(w)| \leq e^{-cw^2}$ for $|w| \leq \delta_0$.

Put

$$f_n(v) = \varphi_n(v/\sqrt{n})^n, \quad r_n(v) = f_n(v) - e^{-v^2/2} \left(1 - \frac{i\gamma_n v^3}{6\sqrt{n}} \right), \quad B_n = n^{1/21}.$$

The exact difference of powers and (4.43n) give, for $v \leq B_n$,

$$|r_n(v)| \leq C e^{-cv^2} \left\{ \frac{v^3}{\sqrt{n}} \omega(B_n/\sqrt{n}) + \frac{v^4 + v^6}{n} \right\},$$

whereas for $B_n \leq v \leq \delta_0 \sqrt{n}$ the Gaussian envelope gives $|r_n(v)| \leq C e^{-cv^2} (1 + v^3/\sqrt{n})$. Therefore

$$\sqrt{n} \int_0^{\delta_0 \sqrt{n}} \frac{|r_n(v)|}{v} dv = o(1), \quad \int_0^{\delta_0 \sqrt{n}} |f_n(v) - e^{-v^2/2}| dv = o(1), \quad (4.43o)$$

and $\int v |f_n(v)| dv = O(1)$ on that interval.

On the zero cell put $t = h_n v / (2\pi \sqrt{n})$. The signed kernel expansion is

$$2K(t) dt = \left\{ \frac{i}{\pi v} + \frac{h_n}{2\pi \sqrt{n}} + O(v/n) \right\} dv, \quad (4.43p)$$

with the singular sign reversed for the conjugate kernel. Equation (4.43o) makes the replacement by the Gaussian-skewness expression an $o(n^{-1/2})$ error uniformly in x , because the omitted spatial phase has modulus one. Normal inversion and

$$\frac{1}{\pi} \int_0^\infty e^{-v^2/2} \cos(vx) dv = \phi(x), \quad \frac{1}{\pi} \int_0^\infty v^2 e^{-v^2/2} \cos(vx) dv = (1 - x^2) \phi(x)$$

give (4.43j)–(4.43k) on the zero cell.

Here is the promised uniform partition of the rest of the physical t -interval. Fix $0 < \delta < 1/4$ so small that the zero-frequency estimate used above holds on $I_0 = [0, \delta]$ and the curvature interval from Lemma 7.9 contains $2\pi I_1/h_n$, where $I_1 = [1 - \delta, 1]$, for all sufficiently large n . On $I_m = [\delta, 1 - \delta]$ one has, uniformly in n ,

$$\sup_{t \in I_m} |\varphi_n(2\pi t/h_n)| \leq q < 1. \quad (4.43q0)$$

Indeed $X_n \Rightarrow Q_*$ and $h_n \rightarrow s_*$ give uniform convergence on this compact interval, whereas

$$|\varphi_{Q_*}(2\pi t/s_*)|^2 = 1 - 2p_*q_*\{1 - \cos(2\pi t)\} < 1 \quad (\delta \leq t \leq 1 - \delta).$$

The same q works for both kernels, so the normalized middle contribution is $O(\sqrt{n}q^n)$, uniformly in the spatial phase.

For I_1 , let u_n be the unique critical spectral maximum from Lemma 7.9 and put $\tau_n = h_n u_n / (2\pi) \rightarrow 1$. The curvature estimate (4.43h1), the affine change $u = 2\pi t/h_n$, and $R_n(u_n) \leq 1$ give fixed $c_1 > 0$ such that

$$|\varphi_n(2\pi t/h_n)|^n \leq e^{-c_1 n(t - \tau_n)^2}, \quad t \in I_1. \quad (4.43q1)$$

Because $|K(1 - y)| + |\bar{K}(1 - y)| \leq Cy$, either normalized endpoint is at most

$$C\sqrt{n} \int_0^\delta y e^{-c_1 n(y - e_n)^2} dy \leq C\{n^{-1/2} + |e_n|\} = o(1), \quad e_n = 1 - \tau_n. \quad (4.43q)$$

For the inequality use $y \leq |y - e_n| + |e_n|$, extend the two nonnegative Gaussian integrals to the real line, and evaluate them. Notice that e_n may have either sign. This argument uses no residual-power conclusion and no estimate on the selected height other than $|\varphi_n(u_n)| \leq 1$.

The three cells prove the two uniform formulas. Their limiting plus profile is maximized at zero and equals $\phi_0(s_*/2 + \gamma_*/6)/\rho_* = C_E$. The reflected profile is also maximized at zero because $s_* > \gamma_*$ and is smaller by $\phi_0\gamma_*/(3\rho_*)$. $\square$

Lemma 7.11 (uniform marked lattice local limit). *For every row let $(J_{n,r}, \eta_{n,r})$, $1 \leq r \leq n$, be independent copies of (J_n, η_n) , where J_n is integer valued. Put*

$$m_n = \mathbb{E}J_n, \quad Y_n = J_n - m_n, \quad V_n = \mathbb{E}Y_n^2, \quad K_n = \sum_{r=1}^n J_{n,r}, \quad T_n = \sum_{r=1}^n \eta_{n,r}.$$

Assume, with constants independent of the row, that

$$\mathbb{E}\eta_n = \mathbb{E}(Y_n \eta_n) = 0, \quad B_n := \mathbb{E}\eta_n^2 \leq \frac{C_B}{n}, \quad 0 < v_- \leq V_n \leq v_+ < \infty, \quad (4.48u0)$$

that two adjacent values of J_n each have probability at least $c_0 > 0$, and that, for some $\bar{d}, C_0 < \infty$ and some $d_n \in [0, \bar{d}]$,

$$\mathbb{E}\{|J_n|^r; |J_n| > H\} \leq \frac{C_0 d_n}{H^{3-r}}, \quad r = 0, 1, 2, \quad H \geq 2. \quad (4.48u1)$$

For fixed $T, M, X < \infty$, let

$$\mathcal{G}(T, M) = \left\{ g \in C_c^2(\mathbb{R}) : \text{supp } g \subset [-T, T], \quad \max_{0 \leq r \leq 2} \|g^{(r)}\|_\infty \leq M \right\}.$$

Then, uniformly over all row laws satisfying the displayed bounds, all $g_n \in \mathcal{G}(T, M)$, and all $k_n \in \mathbb{Z}$ for which

$$x_n = \frac{k_n - nm_n}{\sqrt{n}}, \quad |x_n| \leq X,$$

one has

$$\sqrt{n} \mathbb{E}\{g_n(T_n) 1_{\{K_n=k_n\}}\} = \frac{e^{-x_n^2/(2V_n)}}{\sqrt{2\pi V_n}} \mathbb{E}g_n(T_n) + o(1). \quad (4.48)$$

In particular, the error in (4.48) is one common error for the whole C^2 class and for every bounded centering x_n ; it does not depend on a limiting law for T_n .

Proof. We give all estimates uniformly over the class in the statement. First the tail assumption controls the translation from J_n to Y_n . Indeed, taking $H = 2$ in (4.48u1) gives

$$\mathbb{E}|J_n| \leq 2 + \mathbb{E}\{|J_n|; |J_n| > 2\} \leq 2 + C_0 \bar{d}/4,$$

so $|m_n| \leq M_0$ for a fixed M_0 . Consequently there are fixed H_0, C_1 such that, for $r = 0, 1, 2$ and $H \geq H_0$,

$$\mathbb{E}\{|Y_n|^r; |Y_n| > H\} \leq \frac{C_1 d_n}{H^{3-r}}. \quad (4.48u3)$$

For if $H \geq 2M_0$, the event $|Y_n| > H$ implies $|J_n| > H/2$, and $|Y_n|^r \leq C_r(|J_n|^r + M_0^r)$. Applying (4.48u1) at $H/2$ proves (4.48u3). In particular, the family Y_n^2 is uniformly integrable:

$$\sup_n \mathbb{E}\{Y_n^2; |Y_n| > H\} \leq \frac{C_1 \bar{d}}{H} \longrightarrow 0. \quad (4.48u4)$$

Fix temporarily $U < \infty$ and, for $|u| \leq U$, define

$$f_{n,u} = \mathbb{E}e^{iu\eta_n}, \quad c_{1,n}(u) = \mathbb{E}\{Y_n e^{iu\eta_n}\}, \quad c_{2,n}(u) = \mathbb{E}\{(Y_n^2 - V_n) e^{iu\eta_n}\}.$$

We first prove the two uniform decorrelation estimates

$$\sup_{|u| \leq U} |c_{1,n}(u)| = o(n^{-1/2}), \quad \sup_{|u| \leq U} |c_{2,n}(u)| = o(1). \quad (4.48u5)$$

Let $C_H = \{|Y_n| \leq H\}$. The two normal equations in (4.48u0) give the exact identity

$$\begin{aligned} c_{1,n}(u) &= \mathbb{E}\{Y_n(e^{iu\eta_n} - 1 - iu\eta_n); C_H\} \\ &\quad + \mathbb{E}\{Y_n(e^{iu\eta_n} - 1); C_H^c\} - iu\mathbb{E}\{Y_n\eta_n; C_H^c\}. \end{aligned} \quad (4.48u6)$$

The inequalities $|e^{iy} - 1 - iy| \leq y^2/2$ and $|e^{iy} - 1| \leq 2$, (4.48u3), and Cauchy–Schwarz bound the three terms by

$$C_U H B_n, \quad \frac{C_U d_n}{H^2}, \quad C_U \left(\frac{B_n d_n}{H} \right)^{1/2}.$$

Since

$$(B_n d_n / H)^{1/2} \leq \frac{1}{2}(H B_n + d_n / H^2),$$

we have

$$\sup_{|u| \leq U} |c_{1,n}(u)| \leq C_U \{H B_n + d_n / H^2\}. \quad (4.48u7)$$

If $nd_n \leq H_0^3$, use $H = H_0$; then the right side is $O(n^{-1})$. If $nd_n > H_0^3$, use $H = (nd_n)^{1/3}$; after multiplication by $\sqrt{n}$ both terms are bounded by $C_U d_n^{1/3} n^{-1/6} \leq C_U d^{1/3} n^{-1/6}$. This proves the first part of (4.48u5).

For the second part, use $\mathbb{E}(Y_n^2 - V_n) = 0$ to write

$$c_{2,n}(u) = \mathbb{E}\{(Y_n^2 - V_n)(e^{iu\eta_n} - 1)\}.$$

On C_H , Cauchy–Schwarz and $Y_n^4 1_{C_H} \leq H^2 Y_n^2$ give

$$\mathbb{E}\{|Y_n^2 - V_n| |\eta_n|; C_H\} \leq CH \sqrt{B_n}.$$

On C_H^c , (4.48u3) and $V_n \leq v_+$ give

$$\mathbb{E}\{|Y_n^2 - V_n|; C_H^c\} \leq C d_n / H.$$

Thus

$$\sup_{|u| \leq U} |c_{2,n}(u)| \leq C_U \{H \sqrt{B_n} + d_n / H\}. \quad (4.48u8)$$

Taking $H = \max(H_0, n^{1/8})$ proves the second part of (4.48u5). Finally, because $\mathbb{E}\eta_n = 0$,

$$f_{n,u} - 1 = \mathbb{E}(e^{iu\eta_n} - 1 - iu\eta_n) = O_U(B_n) = O_U(n^{-1}), \quad (4.48u9)$$

so $f_{n,u}$ is bounded away from zero, uniformly on $|u| \leq U$.

Put

$$\psi_n(t, u) = \mathbb{E}e^{itY_n + iu\eta_n}.$$

For $q = v/\sqrt{n}$, Taylor's formula in the first coordinate gives the exact expansion

$$\psi_n(q, u) = f_{n,u} + iq c_{1,n}(u) - \frac{q^2}{2} \{V_n f_{n,u} + c_{2,n}(u)\} + R_n(q, u), \quad (4.48u10)$$

where

$$|R_n(q, u)| \leq C q^2 \mathbb{E}\{Y_n^2 \min(|qY_n|, 1)\}. \quad (4.48u11)$$

The last expectation tends to zero uniformly for bounded v : for fixed H its part on $|Y_n| \leq H$ is at most $|v|H\mathbb{E}Y_n^2/\sqrt{n}$, while its complement tends uniformly to zero by (4.48u4). Divide (4.48u10) by $f_{n,u}$ and use (4.48u5), (4.48u9), and (4.48u11). Uniformly on every compact $|u| \leq U$, $|v| \leq V_0$,

$$n \log \frac{\psi_n(v/\sqrt{n}, u)}{f_{n,u}} = -\frac{V_n v^2}{2} + o(1).$$

Since $|f_{n,u}|^n \leq 1$, exponentiation yields

$$\psi_n(v/\sqrt{n}, u)^n = f_{n,u}^n e^{-V_n v^2/2} + o(1) \quad (4.48u12)$$

with one uniform error on each such compact set.

We next prove an integrable bound valid outside compact v -sets. This is the step for which convergence (4.48u12) alone is insufficient. Let (Y'_n, η'_n) be an independent copy. Then

$$|\psi_n(t, u)|^2 = \mathbb{E} e^{it(Y_n - Y'_n) + iu(\eta_n - \eta'_n)}.$$

Taylor expansion in t gives

$$\begin{aligned} |\psi_n(t, u)|^2 &= |f_{n,u}|^2 + it \mathbb{E}\{(Y_n - Y'_n) e^{iu(\eta_n - \eta'_n)}\} \\ &\quad - \frac{t^2}{2} \mathbb{E}\{(Y_n - Y'_n)^2 e^{iu(\eta_n - \eta'_n)}\} + \mathcal{R}_n(t, u), \end{aligned} \quad (4.48u13)$$

where

$$|\mathcal{R}_n(t, u)| \leq Ct^2 \mathbb{E}\{(Y_n - Y'_n)^2 \min(|t(Y_n - Y'_n)|, 1)\}. \quad (4.48u14)$$

Put $r_{1,n}(U) = \sup_{|u| \leq U} |c_{1,n}(u)|$. The coefficient of t in (4.48u13) has absolute value at most $2r_{1,n}(U)$. The coefficient inside the quadratic expectation is

$$2\Re\{(V_n f_{n,u} + c_{2,n}(u)) \overline{f_{n,u}}\} - 2|c_{1,n}(u)|^2 = 2V_n + o(1), \quad (4.48u15)$$

uniformly for $|u| \leq U$. The uniform integrability needed for the difference is a direct consequence of (4.48u3)–(4.48u4): with $D_n = Y_n - Y'_n$,

$$\mathbb{E}\{D_n^2; |D_n| > H\} \leq 4\mathbb{E}\{Y_n^2; |Y_n| > H/2\} + 4V_n \mathbb{P}\{|Y_n| > H/2\} \longrightarrow 0 \quad (4.48u15b)$$

uniformly in the row. Hence one can first choose a fixed $\delta > 0$ such that the right side of (4.48u14) is at most $v_- t^2/8$ whenever $|t| \leq \delta$, uniformly in the row, and then choose n_U so large that the quadratic expectation in (4.48u15) is at least $3v_-/2$ for $n \geq n_U$ and $|u| \leq U$. Therefore (4.48u13) gives

$$|\psi_n(t, u)|^2 \leq 1 + 2|t|r_{1,n}(U) - \frac{3v_-}{4}t^2 + \frac{v_-}{8}t^2. \quad (4.48u15a)$$

Young's inequality, in the explicit form

$$2|t|r_{1,n}(U) \leq \frac{v_-}{8}t^2 + \frac{8}{v_-}r_{1,n}(U)^2,$$

now gives

$$|\psi_n(t, u)|^2 \leq 1 - \frac{v_-}{2}t^2 + \frac{8}{v_-}r_{1,n}(U)^2, \quad |t| \leq \delta, \quad |u| \leq U, \quad n \geq n_U. \quad (4.48u16)$$

The right side of (4.48u16) is nonnegative because it bounds the nonnegative left side. Thus $\log y \leq y - 1$, applied after raising the squared modulus to the power $n/2$, yields

$$|\psi_n(t, u)|^n \leq \exp \left\{ -\frac{v_-}{4}nt^2 + \frac{4}{v_-}nr_{1,n}(U)^2 \right\}. \quad (4.48u16a)$$

By (4.48u5), $nr_{1,n}(U)^2 = o(1)$. For the indices $n < n_U$, the trivial bound $|\psi_n(t, u)| \leq 1$ is dominated on $|t| \leq \delta$ by the same Gaussian after choosing, for example, $C_U \geq \exp\{v_- n_U \delta^2/4\}$. Thus, for every row,

$$|\psi_n(t, u)|^n \leq C_U e^{-v_- nt^2/4}, \quad |t| \leq \delta, \quad |u| \leq U. \quad (4.48u17)$$

It remains to obtain a common gap away from the origin. If $j_n, j_n + 1$ have masses at least c_0 , the independent-copy identity gives, for every $t \in [-\pi, \pi]$,

$$1 - |\mathbb{E} e^{itY_n}|^2 = \mathbb{E}\{1 - \cos(t(Y_n - Y'_n))\} \geq 2c_0^2(1 - \cos t). \quad (4.48u18)$$

Moreover

$$|\psi_n(t, u) - \mathbb{E} e^{itY_n}| \leq |u| \mathbb{E} |\eta_n| \leq U \sqrt{B_n}. \quad (4.48u19)$$

For the fixed δ in (4.48u17), (4.48u18)–(4.48u19) imply that there are $q_U < 1$ and n_U such that

$$|\psi_n(t, u)| \leq q_U, \quad \delta \leq |t| \leq \pi, \quad |u| \leq U, \quad n \geq n_U. \quad (4.48u20)$$

Thus (4.48u12), (4.48u17), and (4.48u20) give respectively compact convergence and domination in the three ranges

$$|v| \leq V_0, \quad V_0 < |v| \leq \delta\sqrt{n}, \quad \delta\sqrt{n} < |v| \leq \pi\sqrt{n}. \quad (4.48u21)$$

Fourier inversion on the unit lattice, followed by $v = t\sqrt{n}$, is the exact identity

$$\sqrt{n} \mathbb{E}\{e^{iuT_n} 1_{\{K_n=k_n\}}\} = \frac{1}{2\pi} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} e^{-ivx_n} \psi_n(v/\sqrt{n}, u)^n dv. \quad (4.48u22)$$

On the first range in (4.48u21), use (4.48u12). On the second use the integrable Gaussian in (4.48u17); on the third, the integral is at most $2\pi\sqrt{n}q_U^n$. Let $n \rightarrow \infty$, then $V_0 \rightarrow \infty$. The bounds are independent of $|x_n| \leq X$, and

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivx_n} e^{-V_n v^2/2} dv = \frac{e^{-x_n^2/(2V_n)}}{\sqrt{2\pi V_n}}.$$

We have therefore proved, uniformly for $|u| \leq U$ and $|x_n| \leq X$,

$$\sqrt{n} \mathbb{E}\{e^{iuT_n} 1_{\{K_n=k_n\}}\} = \frac{e^{-x_n^2/(2V_n)}}{\sqrt{2\pi V_n}} f_{n,u}^n + o(1). \quad (4.48u23)$$

We finally pass from exponential marks to the whole class $\mathcal{G}(T, M)$. Put

$$\widehat{g}(u) = \int_{\mathbb{R}} e^{-iut} g(t) dt.$$

Two integrations by parts give the uniform estimate

$$|\widehat{g}(u)| \leq C_{T,M}(1+u^2)^{-1}, \quad g \in \mathcal{G}(T, M). \quad (4.48u24)$$

The unmarked case $u = 0$ of (4.48u17)–(4.48u20), inserted into ordinary lattice inversion, also gives

$$\sup_{n,k} \sqrt{n} \mathbb{P}\{K_n = k\} \leq C. \quad (4.48u25)$$

Since

$$|\mathbb{E}\{e^{iuT_n} 1_{\{K_n=k_n\}}\}| \leq \mathbb{P}\{K_n = k_n\}, \quad |f_{n,u}|^n \leq 1,$$

the contribution of $|u| > U$ to the Fourier inversion of either side of (4.48) is bounded by

$$C \int_{|u|>U} (1+u^2)^{-1} du, \quad (4.48u26)$$

uniformly in n, k_n, g_n . On $|u| \leq U$ use (4.48u23), integrate the uniform error against (4.48u24), and note the exact identity $f_{n,u}^n = \mathbb{E} e^{iuT_n}$. First let $n \rightarrow \infty$ and then $U \rightarrow \infty$ in (4.48u26). This proves (4.48). $\square$

Lemma 7.12 (smooth cardinal gap). *Suppose that, for a triangular array,*

$$X = a + hJ + \varepsilon, \quad J \in \mathbb{Z}, \quad \mathbb{E}\varepsilon = \mathbb{E}(J\varepsilon) = 0, \quad A := \mathbb{E}\varepsilon^2 = O(n^{-1}), \quad d = d_n \rightarrow 0, \quad (4.44)$$

where h stays in a compact subinterval of $(0, \infty)$, two adjacent values of J have probabilities bounded below, and

$$\mathbb{E}\{|J|^k; |J| > H\} \leq \frac{Cd}{H^{3-k}} \quad (k = 0, 1, 2; H \geq 2). \quad (4.45)$$

Put

$$K_n = \sum_{j=1}^n J_j, \quad T_n = \frac{1}{h} \sum_{j=1}^n \varepsilon_j, \quad z_n = \mathbb{E}e^{2\pi i \varepsilon/h}.$$

If $V_n = \text{Var } J \rightarrow V \in (0, \infty)$, $|k_n - n\mathbb{E}J| \leq X_0\sqrt{n}$, and $|z_n|^n \leq 1 - \kappa$ for some fixed $\kappa > 0$, then, uniformly for $|\theta_n| \leq 1/2$,

$$\liminf_n \sqrt{n} \mathbb{E}e_B(K_n - k_n + T_n - \theta_n) > 0, \quad (4.46)$$

where e_B is the excess of the one-sided Beurling majorant over the indicator of the closed negative half-line.

Proof. Put

$$Y = J - \mathbb{E}J, \quad \eta = \frac{\varepsilon}{h}, \quad B = \mathbb{E}\eta^2.$$

The normal equations in (4.44) give

$$\mathbb{E}\eta = \mathbb{E}(Y\eta) = 0, \quad B = \frac{A}{h^2} = O(n^{-1}).$$

After discarding finitely many rows, $V/2 \leq V_n \leq 2V$, while d_n is bounded. Thus (4.45), the two adjacent masses, and the compact range of h verify every hypothesis of Lemma 7.11. In particular, for fixed $T, M < \infty$, let

$$\mathcal{G}(T, M) = \left\{ g \in C_c^2(\mathbb{R}) : \text{supp } g \subset [-T, T], \quad \max_{0 \leq j \leq 2} \|g^{(j)}\|_\infty \leq M \right\}. \quad (4.47b)$$

For $x_n = (k_n - n\mathbb{E}J)/\sqrt{n}$, $|x_n| \leq X_0$, equation (4.48) therefore holds with one $o(1)$, uniformly over $g_n \in \mathcal{G}(T, M)$, the phase, the bounded center, and all row parameters occurring here. No limiting distribution for T_n is assumed.

The elementary cardinal subkernel is

$$e_B(x) \geq \frac{\sin^2(\pi x)}{4\pi^2(1+x^2)}. \quad (4.52)$$

Here are the partial-fraction details. Inserting the paired expansion of $\pi \cot \pi x$ in the Beurling formula gives, for $x > 0$,

$$e_B(x) = \left(\frac{\sin \pi x}{\pi} \right)^2 \left\{ \sum_{j=0}^{\infty} \frac{1}{(x+j)^2} - \frac{1}{x} \right\}, \quad (4.52a)$$

$$e_B(-x) = \left(\frac{\sin \pi x}{\pi} \right)^2 \left\{ \frac{1}{x} - \sum_{j=1}^{\infty} \frac{1}{(x+j)^2} \right\}. \quad (4.52b)$$

Let $q(t) = (x+t)^{-2}$. Since q is decreasing and convex, the trapezoidal inequality and telescoping give

$$\sum_{j=0}^{\infty} q(j) - \int_0^{\infty} q(t) dt \geq \frac{1}{2} \sum_{j=0}^{\infty} \{q(j) - q(j+1)\} = \frac{1}{2x^2}.$$

For the other sign, the tangent inequality at the right endpoint of each unit interval gives

$$\begin{aligned} \int_{j-1}^j q(t) dt - q(j) &\geq -\frac{q'(j)}{2} = \frac{1}{(x+j)^3}, \\ \frac{1}{x} - \sum_{j=1}^{\infty} \frac{1}{(x+j)^2} &\geq \sum_{j=1}^{\infty} \frac{1}{(x+j)^3} \geq \int_1^{\infty} \frac{dt}{(x+t)^3} = \frac{1}{2(x+1)^2}. \end{aligned}$$

Now $1/(2x^2) \geq 1/\{4(1+x^2)\}$ and $1/\{2(x+1)^2\} \geq 1/\{4(1+x^2)\}$. Multiplication by $(\sin \pi x/\pi)^2$ proves (4.52) for both signs. At nonzero integers it follows by continuity. At $x = 0$ the right side is zero and $e_B(0) = M(0) - 1 = 0$ by (P3), so no continuity of the positive-side auxiliary expression at zero is being assumed. Now

$$\mathbb{E} \sin^2\{\pi(T_n - \theta_n)\} = \frac{1 - \Re(e^{-2\pi i \theta_n} z_n^n)}{2} \geq \frac{\kappa}{2}. \quad (4.53)$$

Moreover $\mathbb{E} T_n^2 = nA/h^2 \leq C_T$ for a fixed C_T . Choose R so large that $C_T/R^2 \leq \kappa/4$, and choose one fixed $\chi \in C_c^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, equal to one on $[-R, R]$, and supported in $[-R-1, R+1]$. Put

$$g_n(t) = \chi(t) \sin^2\{\pi(t - \theta_n)\}.$$

Equations (4.53) and Chebyshev give $\mathbb{E} g_n(T_n) \geq \kappa/4$, uniformly in the phase. This family belongs to one class $\mathcal{G}(R+1, M)$. On its support, (4.52) gives the explicit phase-uniform bound

$$e_B(t - \theta_n) \geq c_R g_n(t), \quad c_R = \frac{1}{4\pi^2\{1 + (R + 3/2)^2\}}. \quad (4.54)$$

Since $e_B \geq 0$, retain only the event $K_n = k_n$ and apply the uniform version of (4.48) to g_n . If $x_n = (k_n - n\mathbb{E}J)/\sqrt{n}$, then

$$\begin{aligned} \sqrt{n} \mathbb{E} e_B(K_n - k_n + T_n - \theta_n) &\geq c_R \sqrt{n} \mathbb{E}\{g_n(T_n) 1_{\{K_n = k_n\}}\} \\ &= c_R \frac{e^{-x_n^2/(2V_n)}}{\sqrt{2\pi V_n}} \mathbb{E} g_n(T_n) + o(1). \end{aligned} \quad (4.54a)$$

The Gaussian factor has a positive lower bound because $|x_n| \leq X_0$ and $V_n \rightarrow V \in (0, \infty)$; the expectation is at least $\kappa/4$. This proves (4.46), uniformly in the phase, without an infinite-cell interchange. $\square$

Corollary 7.13 (residual-power selection). *In addition to the hypotheses of Lemma 7.12, suppose (x_n) is an active near-extremal positive-branch sequence in the sense fixed at the start of this subsection, and coordinate its thresholds by*

$$x_n \sqrt{n} = na_n + h_n(k_n + \theta_n), \quad |\theta_n| \leq \frac{1}{2}.$$

Suppose also that $\sup_n \rho_n < \infty$ and that the adaptive majorant $\mathcal{U}_{X_n, n, h_n}^+$ satisfies

$$\frac{\sqrt{n}}{\rho_n} \sup_x \{\mathcal{U}_{X_n, n, h_n}^+(x) - \Phi(x)\} \leq C_E + o(1), \quad (4.47)$$

while the reflected branch has a fixed gap. Then $|z_n|^n \rightarrow 1$.

Proof. Write

$$S_n = \frac{\sqrt{n}}{\rho_n} \{F_{n,X_n}(x_n) - \Phi(x_n)\}, \quad M_n = \frac{\sqrt{n}}{\rho_n} \{\mathcal{U}_{X_n,n,h_n}^+(x_n) - \Phi(x_n)\}.$$

The upper identity in (4.24w) and the exact threshold coordinate give, with no inequality,

$$S_n = M_n - \frac{\sqrt{n}}{\rho_n} \mathbb{E}e_B(K_n - k_n + T_n - \theta_n). \quad (4.55)$$

By near extremality, $S_n \rightarrow C_E$, whereas (4.47) gives $M_n \leq C_E + o(1)$. If $|z_n|^n \not\rightarrow 1$, then $0 \leq |z_n|^n \leq 1$ supplies a subsequence and a fixed $\kappa > 0$ on which $|z_n|^n \leq 1 - \kappa$. Lemma 7.12 gives a fixed $c_\kappa > 0$ such that

$$\liminf_n \sqrt{n} \mathbb{E}e_B(K_n - k_n + T_n - \theta_n) \geq c_\kappa.$$

Choose $R_\rho < \infty$ with $\rho_n \leq R_\rho$. Then (4.55) would imply $\limsup S_n \leq C_E - c_\kappa/R_\rho$, a contradiction. Hence $|z_n|^n \rightarrow 1$. The reflected fixed gap is used only one step upstream, to ensure that the near-extremal absolute score has the positive orientation; no lower cardinal estimate is hidden in (4.55). $\square$

Lemma 7.14 (active-point exact-lattice envelope). *Let V_n be centered variance-one laws supported on maximal lattices $a_n + b_n\mathbb{Z}$, where b_n stays in a compact subset of $(0, \infty)$. Put $\gamma_n = \mathbb{E}V_n^3$, $\rho_n = \mathbb{E}|V_n|^3$, let p_n be the matching Bernoulli parameter, and set $d_n = \rho_n - \rho_2(\gamma_n)$. Assume that two adjacent sites have probabilities at least $c_0 > 0$,*

$$p_n \rightarrow p_*, \quad d_n \rightarrow 0, \quad 0 < c \leq \sqrt{n} d_n \leq C.$$

If $x_n \rightarrow 0$, then, uniformly in the lattice phase of x_n ,

$$\frac{\sqrt{n}}{\rho_n} \{\mathcal{U}_{V_n,n,b_n}^+(x_n) - \Phi(x_n)\} \leq \frac{\phi_0}{\rho_n} \left(\frac{b_n}{2} + \frac{\gamma_n}{6} \right) + \{\Lambda_{\text{abs},*} + o(1)\}d_n + O(n^{-1}). \quad (4.58)$$

Consequently

$$\frac{\sqrt{n}}{\rho_n} \{\mathcal{U}_{V_n,n,b_n}^+(x_n) - \Phi(x_n)\} \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_n + O(n^{-1}) < C_E. \quad (4.58a)$$

The reflected operator has a fixed leading gap.

Proof. All constants in this proof are uniform after the row index is sufficiently large. We suppress that index and write $\gamma = \gamma_n$, $\rho = \rho_n$, $d = d_n$, and $b = b_n$. The assumptions put γ in a fixed compact neighbourhood Γ of γ_* , put b in a compact subset of $(0, \infty)$, and give

$$d \asymp n^{-1/2}, \quad \rho = \rho_2(\gamma) + d \asymp 1. \quad (4.58b0)$$

The actual span. Let w_γ be the nonnegative defect density from the proof of Theorem 2.1. If an atom v of V_n has mass at least c_0 , then

$$0 \leq w_\gamma(v) \leq c_0^{-1} \mathbb{E}w_\gamma(V_n) = c_0^{-1}d. \quad (4.58b1)$$

Uniformly for $\gamma \in \Gamma$, the density w_γ is coercive and its only zeros are $-a_\gamma$ and b_γ , whose distance is $s_\gamma(\gamma)$. Thus each of the two adjacent atoms in the hypothesis has distance $o(1)$ from this two-point zero set. The distance between the two atoms is the lattice span b_n , which is bounded away from zero; consequently the two atoms cannot converge to the same zero. Their order then forces the lower atom to converge to $-a_{\gamma_n}$ and the upper atom to b_{γ_n} . Hence

$$b_n - s_\gamma(\gamma_n) \rightarrow 0, \quad b_n \rightarrow s_*. \quad (4.58b)$$

Exact folding. Choose the unique $k \in \mathbb{Z}$ and $\theta \in [-1/2, 1/2)$ such that

$$x = \frac{na_n + b_n(k + \theta)}{\sqrt{n}},$$

put, for $0 \leq q \leq 1/2$,

$$f(q) = \mathbb{E}e^{2\pi i q V_n / b_n}, \quad g(q) = e^{-2\pi i q x \sqrt{n} / b_n} f(q)^n.$$

Since $V_n \in a_n + b_n \mathbb{Z}$, $f(1 - q) = e^{2\pi i a_n / b_n} \overline{f(q)}$. Substitution of the displayed coordinate for x therefore gives, with no approximation,

$$g(1 - q) = e^{-2\pi i \theta} \overline{g(q)}.$$

It follows that

$$\mathcal{U}_{V_n, n, b_n}^+(x) = \frac{1}{2} + 2\Re \int_0^{1/2} \{K(q)g(q) + K(1 - q)e^{-2\pi i \theta} \overline{g(q)}\} dq. \quad (4.58c)$$

In (4.58c) replace $f(q)$ by $\chi_\gamma(2\pi q/b)$, and in the returned term replace $\overline{f(q)}$ by the corresponding conjugate. Denote the resulting analytic expression by $\mathcal{S}_n^{\text{lat}}(x, \theta)$. This definition uses the actual interval and the actual folding phase; it does not assert that the matching two-point law has span b .

A uniform second-order expansion of the surrogate. The support of Q_γ is uniformly bounded for $\gamma \in \Gamma$. Hence there are $\delta, C > 0$ such that the principal logarithm of $\chi_\gamma(t)$ is defined for $|t| \leq \delta$ and, uniformly in $\gamma \in \Gamma$,

$$\log \chi_\gamma(t) = -\frac{t^2}{2} - \frac{i\gamma t^3}{6} + \frac{\gamma^2 - 2}{24}t^4 + r_\gamma(t), \quad |r_\gamma(t)| \leq C|t|^5. \quad (4.58c0)$$

Indeed, Taylor's formula for χ_γ , followed by Taylor's formula for $\log(1 + z)$, is uniform on this compact family, and $\mathbb{E}Q_\gamma^4 = \gamma^2 + 1$. For $t = w/\sqrt{n}$, put

$$A = -\frac{i\gamma w^3}{6}, \quad B = \frac{\gamma^2 - 2}{24}w^4, \quad E = nr_\gamma(w/\sqrt{n}).$$

Then (4.58c0) says

$$n \log \chi_\gamma(w/\sqrt{n}) = -\frac{w^2}{2} + \frac{A}{\sqrt{n}} + \frac{B}{n} + E, \quad |A| \leq Cw^3, \quad |B| \leq Cw^4, \quad |E| \leq \frac{Cw^5}{n^{3/2}}. \quad (4.58c0a)$$

For all sufficiently large n , on $0 \leq w \leq n^{1/16}$, the sum $|A|/\sqrt{n} + |B|/n + |E|$ tends to zero uniformly. Taylor's formula for the exponential, with its integral remainder, therefore gives

$$\left| e^{A/\sqrt{n} + B/n + E} - 1 - \frac{A}{\sqrt{n}} - \frac{B + A^2/2}{n} \right| \leq C \left\{ \frac{w^5 + w^7 + w^9}{n^{3/2}} + \frac{w^8}{n^2} \right\}. \quad (4.58c0b)$$

Here the four displayed monomials respectively dominate E , the product $AB/n^{3/2}$, the cubic term $A^3/n^{3/2}$, and B^2/n^2 ; every product containing E , and every higher power, is smaller on the stated window. Since $B + A^2/2 = R_{\text{even}, \gamma}(w)$, (4.58c0a)–(4.58c0b) give

$$\chi_\gamma(w/\sqrt{n})^n = e^{-w^2/2} \left[1 - \frac{i\gamma w^3}{6\sqrt{n}} + \frac{R_{\text{even}, \gamma}(w)}{n} + \mathcal{R}_{n, \gamma}(w) \right], \quad (4.58c1)$$

where

$$R_{\text{even}, \gamma}(w) = \frac{\gamma^2 - 2}{24}w^4 - \frac{\gamma^2}{72}w^6, \quad (4.58c2)$$

and, for example,

$$|\mathcal{R}_{n,\gamma}(w)| \leq \frac{C}{n^{3/2}} w^5 (1+w)^{10}. \quad (4.58c2a)$$

Indeed (4.58c2a) follows term by term from (4.58c0b), using $n^{-2}w^8 \leq n^{-3/2}w^5(1+w)^{10}$. The factor w^5 in (4.58c2a) is recorded because the direct kernel is singular at zero. Notice also that the conjugate power has the same real even term and the opposite cubic term.

We next justify extending the root-scale integrals to infinity. By symmetrisation there are $c, \delta_1 > 0$, uniform in $\gamma \in \Gamma$, such that $|\chi_\gamma(u)| \leq e^{-cu^2}$ for $|u| \leq \delta_1$. Shrink δ_1 , if necessary, so that $s_- \delta_1 < \pi/4$, where $s_- := \inf_{\gamma \in \Gamma} s_\gamma(\gamma) > 0$. By (4.58b), after removing finitely many rows, $3/4 \leq b/s_\gamma(\gamma) \leq 5/4$. The explicit identity

$$|\chi_\gamma(u)|^2 = 1 - 2p(1-p)\{1 - \cos(s_\gamma(\gamma)u)\}$$

has $p(1-p)$ bounded away from zero on Γ . On the rest of the moving interval,

$$\delta_1 \leq u \leq \frac{\pi}{b} \implies s_\gamma(\gamma)u \in [s_- \delta_1, 4\pi/3]. \quad (4.58c2d)$$

The compact interval on the right is disjoint from $2\pi\mathbb{Z}$. The explicit identity therefore gives one common modulus gap on $\delta_1 \leq u \leq \pi/b$. Consequently, under $u = w/\sqrt{n}$,

$$|\chi_\gamma(w/\sqrt{n})|^n \leq C e^{-cw^2} \quad (0 \leq w \leq \delta_1 \sqrt{n}),$$

and it is at most $C e^{-cn}$ beyond that interval. In particular the part $w \geq n^{1/16}$, including the factor w^{-1} from the singular kernel, is $O(e^{-cn^{1/8}})$. This proves the required tail claim on the moving actual-span interval rather than on a fixed surrogate interval.

Set $q = bw/(2\pi\sqrt{n})$. Taylor expansion of the displayed Prawitz kernel, with uniform remainders on $w \leq n^{1/16}$, gives

$$\begin{aligned} 2K(q) \, dq &= \left\{ \frac{i}{\pi w} + \frac{b}{2\pi\sqrt{n}} - \frac{b^2 w}{4\pi^2 n} - \frac{ib^2 w}{12\pi n} + O\left(\frac{w^2(1+w)}{n^{3/2}}\right) \right\} dw, \\ 2K(1-q) \, dq &= \left\{ \frac{b^2 w}{4\pi^2 n} + O\left(\frac{w^2(1+w)}{n^{3/2}}\right) \right\} dw. \end{aligned} \quad (4.58c2b)$$

The following identities fix all signs in the calculation:

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin(xw)}{w} e^{-w^2/2} dw &= \Phi(x), \\ \frac{1}{\pi} \int_0^\infty e^{-w^2/2} \cos(xw) dw &= \phi(x), \quad \frac{1}{\pi} \int_0^\infty w^2 e^{-w^2/2} \cos(xw) dw = (1-x^2)\phi(x). \end{aligned} \quad (4.58c2c)$$

The first identity follows by differentiating in x and checking $x = 0$; the other two are the Gaussian Fourier transform and its second derivative.

Insert (4.58c1)–(4.58c2b), retain every product through raw order n^{-1} , and keep the spatial factors e^{-ixw} and e^{ixw} until real parts are taken. The complete direct ledger is

kernel factor	power factor	direct real contribution
$i/(\pi w)$	1	$\Phi(x) - 1/2$
$b/(2\pi\sqrt{n})$	1	$b\phi(x)/(2\sqrt{n})$
$i/(\pi w)$	$-i\gamma w^3/(6\sqrt{n})$	$\gamma(1-x^2)\phi(x)/(6\sqrt{n})$
$i/(\pi w)$	$R_{\text{even},\gamma}(w)/n$	$O(x /n)$
$b/(2\pi\sqrt{n})$	$-i\gamma w^3/(6\sqrt{n})$	$O(x /n)$
$-ib^2 w/(12\pi n)$	1	$O(x /n)$
$-b^2 w/(4\pi^2 n)$	1	$-b^2/(4\pi^2 n) + O(x^2/n)$

(4.58c3)

For each of the three $O(|x|/n)$ entries the coefficient before e^{-ixw} is purely imaginary and odd; its real part therefore contains $\sin(xw)$, and $|\sin(xw)| \leq |x|w$ proves the stated bound by Gaussian integration. For the last entry use $|1 - \cos(xw)| \leq x^2 w^2/2$ and $\int_0^\infty w e^{-w^2/2} dw = 1$.

The returned kernel has neither a singular term nor a term of order $n^{-1/2}$. Its only contribution of raw order n^{-1} is

$$\frac{b^2}{4\pi^2 n} \Re \left\{ e^{-2\pi i \theta} \int_0^\infty w e^{-w^2/2} e^{ixw} dw \right\} = \frac{b^2}{4\pi^2 n} \cos(2\pi \theta) + O(|x|/n), \quad (4.58c4)$$

uniformly for $|\theta| \leq 1/2$. Products not displayed in (4.58c3)–(4.58c4) are integrable, by (4.58c2a), against the singular kernel and are bounded by $Cn^{-3/2}$ after integration. The already estimated tail contributes exponentially less. We have therefore proved, with a constant independent of $|x| \leq 1$ and $|\theta| \leq 1/2$,

$$\frac{\sqrt{n}}{\rho} \{ \mathcal{S}_n^{\text{lat}}(x, \theta) - \Phi(x) \} = \frac{\phi(x)}{\rho} \left\{ \frac{b}{2} + \frac{\gamma}{6}(1 - x^2) \right\} + \frac{b^2}{4\pi^2 \rho \sqrt{n}} \{ \cos(2\pi \theta) - 1 \} + O\left(\frac{|x|}{\sqrt{n}} + \frac{1}{n} \right). \quad (4.58d)$$

Uniform comparison of the actual and surrogate powers. Let $p = p(V_n)$, $s = s_\gamma(\gamma) = s(p)$, $\varphi = \varphi_{V_n}$, $\chi = \chi_\gamma$, and $\Delta(u) = \varphi(u) - \chi(u)$. In the Bernoulli coordinate $Y = p + V_n/s$, cubic Hermite interpolation at 0, 1, together with the matching moments of Y and the Bernoulli law, gives the exact identity

$$\Delta(u) = e^{-isp u} d \int \frac{E_{su}(y)}{W_p(y)} d\nu(y), \quad d\nu(y) = \frac{W_p(y) d\mathbb{P}_Y(y)}{d}. \quad (4.58e)$$

Here ν is a probability measure because $\mathbb{E}W_p(Y) = d$.

We record all uniformity needed below. Every convex combination

$$\psi_a(u) = (1 - a)\chi(u) + a\varphi(u), \quad 0 \leq a \leq 1,$$

is the characteristic function of a mixture of centered variance-one laws. Their third absolute moments are uniformly bounded, since $p \rightarrow p_*$ and $d \rightarrow 0$. Applying

$$1 - |\mathbb{E}e^{iuZ}|^2 = \mathbb{E}\{1 - \cos(u(Z - Z'))\}, \quad 1 - \cos v \geq v^2/2 - |v|^3/6,$$

to these mixtures gives fixed $c, \delta_0 > 0$ such that

$$|\psi_a(u)| \leq e^{-cu^2} \quad (|u| \leq \delta_0, \ 0 \leq a \leq 1). \quad (4.58e1)$$

Shrink δ_0 , if necessary, so that $b\delta_0/(2\pi) < 1/2$ for every b in the fixed span compactum. Thus the exact divided power

$$\mathcal{P}_n(u) := \frac{\varphi(u)^n - \chi(u)^n}{n\{\varphi(u) - \chi(u)\}} = \int_0^1 \psi_a(u)^{n-1} da \quad (4.58e2)$$

(with its continuous value if the denominator vanishes) satisfies

$$|\mathcal{P}_n(w/\sqrt{n})| \leq C e^{-cw^2} \quad (0 \leq w \leq \delta_0 \sqrt{n}). \quad (4.58e3)$$

Moreover, uniformly for w in each fixed compact interval and $a \in [0, 1]$,

$$\psi_a(w/\sqrt{n}) = 1 - \frac{w^2}{2n} + O(w^3 n^{-3/2}),$$

so (4.58e2) converges there uniformly to $e^{-w^2/2}$. No linearisation in d has been made.

Put $H = n^{1/8}$. Equations (4.21)–(4.21a), already proved uniformly for p in a compact subset of $(0, 1)$, say

$$|E_z(y) - g(zy)| \leq C|z|^4(1 + |y|^3), \quad g(v) = e^{iv} - 1 - iv + v^2/2 + iv^3/6,$$

$$|E_z(y)| \leq \frac{|z|^4}{24}y^2(1 - y)^2, \quad \frac{y^2(1 - y)^2}{W_p(y)} \leq C(1 + |y|). \quad (4.58e4)$$

On $|y| \leq H$, the second line of (4.58e4) bounds the integral by $C|z|^4(H + 1)$. On $|y| > H$, use the first line, $|g(v)| \leq |v|^3/6$, and the explicit cubic tails of W_p :

$$\frac{|y|^3}{W_p(y)} \leq \max\{C_+(p)^{-1}, C_-(p)^{-1}\}\{1 + O(H^{-1})\}. \quad (4.58e5)$$

Since ν is a probability measure, these two estimates yield, uniformly for $0 \leq w \leq \delta_0\sqrt{n}$,

$$\frac{n^{3/2}|\Delta(w/\sqrt{n})|}{d} \leq \frac{s^3}{6} \max\{C_+(p)^{-1}, C_-(p)^{-1}\}\{1 + O(H^{-1})\}w^3 + C(H + 1)\frac{w^4}{\sqrt{n}}. \quad (4.58f)$$

We now insert (4.58f) into the folded operator, rather than merely invoke a pointwise $o(1)$. On the direct root-scale cell, the singular part of (4.58c2b), (4.58e2), and (4.58e3) give

$$\begin{aligned} \frac{\sqrt{n}}{\rho} |\text{direct singular defect}| &\leq \frac{d}{\pi\rho} \int_0^{\delta_0\sqrt{n}} \frac{|\mathcal{P}_n(w/\sqrt{n})|}{w} \frac{n^{3/2}|\Delta(w/\sqrt{n})|}{d} dw \\ &\leq \frac{ds^3}{6\pi\rho} \max\{C_+(p)^{-1}, C_-(p)^{-1}\} \int_0^\infty w^2 e^{-w^2/2} dw + o(d) \quad (4.58g) \\ &= \{\Lambda_{\text{abs}}(p, d) + o(1)\}d = \{\Lambda_{\text{abs},*} + o(1)\}d. \end{aligned}$$

To justify the second line with its sharp Gaussian coefficient, first restrict to $0 \leq w \leq M$. There (4.58f) and the uniform compact convergence following (4.58e3) give the displayed limsup with $e^{-w^2/2}$. On $M \leq w \leq \delta_0\sqrt{n}$, (4.58e3) and (4.58f) give the row-independent majorant $C(w^2 + w^3)e^{-cw^2}$; let $M \rightarrow \infty$. The coefficient errors in (4.58f) are $O(H^{-1})$ and $O((H + 1)/\sqrt{n})$, both tending to zero. The identity $\pi^{-1} \int_0^\infty w^2 e^{-w^2/2} dw = \phi_0$ then gives exactly the definition (4.18a) of Λ_{abs} .

For completeness, all nonsingular pieces are smaller on the asserted scale. Since $(H + 1)/\sqrt{n} \leq 1$ eventually, (4.58f) and (4.58e3) imply

$$|\varphi(w/\sqrt{n})^n - \chi(w/\sqrt{n})^n| \leq \frac{Cd}{\sqrt{n}}(w^3 + w^4)e^{-cw^2}. \quad (4.58g1)$$

The bounded part of the direct kernel is $O(n^{-1/2})dw$, so its normalized contribution is $O(d/\sqrt{n})$. On the returned copy, $|2K(1 - q)dq| \leq Cw dw/n$, and hence

$$\frac{\sqrt{n}}{\rho} |\text{returned defect}| \leq \frac{Cd}{n} \int_0^\infty w(w^3 + w^4)e^{-cw^2} dw = O(d/n). \quad (4.58g2)$$

It remains to cover, uniformly, the complement of the root-scale cell. Choose a fixed $0 < u_0 < \min\{\delta_0, \pi/(2s_*)\}$. For the actual law, the two adjacent atoms of mass at least c_0 give

$$1 - |\varphi(u)|^2 = \mathbb{E}\{1 - \cos(u(V_n - V'_n))\} \geq 2c_0^2\{1 - \cos(bu)\}. \quad (4.58g3)$$

For $u_0 \leq u \leq \pi/b$, (4.58b) implies, for all sufficiently large rows,

$$bu \in [s_*u_0/2, \pi], \quad s(p)u \in [s_*u_0/2, 3\pi/2]. \quad (4.58g4)$$

Both intervals in (4.58g4) are compact and avoid $2\pi\mathbb{Z}$. Hence (4.58g3) gives a fixed positive lower bound for $1 - |\varphi(u)|^2$. The explicit identity

$$1 - |\chi(u)|^2 = 2p(1-p)\{1 - \cos(s(p)u)\},$$

together with $p \rightarrow p_* \in (0, 1)$, gives another fixed positive lower bound for $1 - |\chi(u)|^2$. Thus both powers are at most r^n , for one row-independent $r < 1$, on the remainder of $0 \leq q \leq 1/2$. The kernel is bounded there, so the normalized contribution is $O(\sqrt{n}r^n) = o(d)$ by (4.58b0). Equations (4.58g)–(4.58g4) therefore compare the actual folded operator with the surrogate on the whole actual fundamental interval.

Now take the sequence $x = x_n \rightarrow 0$. The cosine term in (4.58d) is nonpositive. For all sufficiently large rows the positive leading profile is at most its value at zero. Furthermore $|x_n|/\sqrt{n} = o(d)$, and the nonsingular comparison error $d/\sqrt{n}$ is $O(n^{-1})$, by (4.58b0). Combining (4.58d)–(4.58g4) gives

$$\frac{\sqrt{n}}{\rho_n} \{\mathcal{U}_{V_n, n, b_n}^+(x_n) - \Phi(x_n)\} \leq \frac{\phi_0}{\rho_n} \left(\frac{b_n}{2} + \frac{\gamma_n}{6} \right) + \{\Lambda_{\text{abs},*} + o(1)\}d_n + O(n^{-1}),$$

uniformly in θ .

The signed denominator loss, with the current p frozen. If V' is an independent copy of V , then $V - V' \in b\mathbb{Z}$, and hence

$$2b = b\mathbb{E}(V - V')^2 \leq \mathbb{E}|V - V'|^3 \leq 2\{\rho + \mathbb{E}|V|\}.$$

The last inequality follows by expanding $|x - y|^3 \leq (x - y)^2(|x| + |y|)$ and using centering. Proposition 3.2 now gives

$$b \leq s(p) + K_1(\gamma(p))d. \quad (4.58h)$$

Define, at the current value of p ,

$$\begin{aligned} N_0(p) &= \phi_0\{s(p)/2 + \gamma(p)/6\}, & N_1(p) &= \phi_0 K_1(\gamma(p))/2, \\ D_0(p) &= \rho_2(p), & A(p) &= N_0(p)/D_0(p). \end{aligned}$$

Then (4.58h) and the exact algebraic identity

$$\frac{N_0(p) + N_1(p)d}{D_0(p) + d} = A(p) - \frac{A(p) - N_1(p)}{D_0(p) + d}d \quad (4.58i)$$

show, without any joint expansion in (p, d) , that

$$\frac{\phi_0}{\rho} \left(\frac{b}{2} + \frac{\gamma}{6} \right) \leq A(p) - \{G_{\text{span},*} + o(1)\}d \leq C_E - \{G_{\text{span},*} + o(1)\}d. \quad (4.58j)$$

Indeed $A(p) \leq C_E$ by Lemma 7.6, while continuity and $p \rightarrow p_*$, $d \rightarrow 0$, give

$$\frac{A(p) - N_1(p)}{D_0(p) + d} \longrightarrow \frac{C_E - \phi_0 K_1(\gamma_*)/2}{\rho_*} = G_{\text{span},*}.$$

Thus (4.58i) freezes p and expands only in d ; in particular no unproved relation such as $|p - p_*| = o(d)$ is used. Combining (4.58) and (4.58j), and then using (4.24), gives

$$\frac{\sqrt{n}}{\rho_n} \{\mathcal{U}_{V_n, n, b_n}^+(x_n) - \Phi(x_n)\} \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_n + O(n^{-1}) < C_E,$$

because $n^{-1} = o(d_n)$.

Finally, the reflected assertion is not merely a change of notation. Directly from (4.24u), conjugation gives

$$\mathcal{U}_{-V,n,b}^+(-x) = 1 - \mathcal{U}_{V,n,b}^-(x), \quad \Phi(x) - \mathcal{U}_{V,n,b}^-(x) = \mathcal{U}_{-V,n,b}^+(-x) - \Phi(-x). \quad (4.58k)$$

The reflected law has the same b, ρ, d , skewness $-\gamma$, and the same adjacent-atom and comparison bounds. Repeating the already proved surrogate calculation on the reflected compact parameter set (the defect term is $O(d) = o(1)$) yields, uniformly in phase,

$$\frac{\sqrt{n}}{\rho_n} \{\Phi(x_n) - \mathcal{U}_{V,n,b_n}^-(x_n)\} = \frac{\phi_0}{\rho_*} \left(\frac{s_*}{2} - \frac{\gamma_*}{6} \right) + o(1) = C_E - \frac{\phi_0 \gamma_*}{3\rho_*} + o(1). \quad (4.58l)$$

Since $\gamma_* > 0$, this is bounded above by C_E minus a fixed positive constant for all sufficiently large n , which proves the claimed reflected gap. $\square$

Lemma 7.15 (near-lattice transfer). *Let X_n be centered, variance-one laws, let p_n be their matching Bernoulli parameters, and put $d_n = \mathbb{E}|X_n|^3 - \rho_2(\mathbb{E}X_n^3)$. Assume $p_n \rightarrow p_*$, $d_n \rightarrow 0$, and*

$$0 < \liminf \sqrt{n} d_n \leq \limsup \sqrt{n} d_n < \infty. \quad (4.56)$$

Let (x_n) be an active near-extremal positive-branch sequence. If $\liminf |\varphi_{X_n}(u_n)|^n > 0$ at the selected critical resonance, then there is a centered, variance-one maximal-span exact lattice projection L_n° , physical spans h_n before and b_n after standardization, and points $y_n \rightarrow 0$ such that, for both one-sided Prawitz operators,

$$\sqrt{n} |\mathcal{U}_{X_n,n,h_n}^\pm(x_n) - \mathcal{U}_{L_n^\circ,n,b_n}^\pm(y_n)| = o(d_n), \quad \rho(L_n^\circ) = \rho(X_n) + o(d_n), \quad d(L_n^\circ) = d_n + o(d_n). \quad (4.57)$$

Consequently the positive active score satisfies

$$\frac{\sqrt{n}}{\rho(X_n)} \{\mathcal{U}_{X_n,n,h_n}^+(x_n) - \Phi(x_n)\} \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_n + O(n^{-1}). \quad (4.57a)$$

Proof. We suppress row subscripts until the last step. Let u_0 be the unique critical spectral maximum from Lemma 7.9, write

$$\varphi_X(u_0) = r e^{i\alpha}, \quad h_0 = \frac{2\pi}{u_0},$$

and choose $a_0 = \alpha/u_0$ modulo h_0 . Let J be the nearest integer to $(X - a_0)/h_0$ and put

$$X = a_0 + h_0 J + \varepsilon_0, \quad |\varepsilon_0| \leq h_0/2.$$

This phase convention gives

$$\mathbb{E} e^{iu_0 \varepsilon_0} = r.$$

Since $1 - \cos(2\pi v) \geq 8v^2$ for $|v| \leq 1/2$, the assumed lower bound for r^n implies

$$A_0 := \mathbb{E} \varepsilon_0^2 \leq \frac{h_0^2}{8} (1 - r) = O(n^{-1}). \quad (4.59)$$

We now derive, rather than assume, the adjacency and tail assertions used below. Uniform convergence of R'_n in Lemma 7.9 puts its unique critical point at $u_0 - 2\pi/s_\gamma(\gamma) = o(1)$; hence

$$h_0 = \frac{2\pi}{u_0} \longrightarrow s_*. \quad (4.59a0)$$

Let $-a_\gamma, b_\gamma$ be the two zeros of w_γ , so that $a_\gamma + b_\gamma = s_\gamma(\gamma)$, and set

$$\delta_n = (d + A_0)^{1/4}.$$

Corollary 3.1 and Chebyshev's inequality give

$$\mathbb{P}\{\text{dist}(X, \{-a_\gamma, b_\gamma\}) > \delta_n\} = o(1), \quad \mathbb{P}\{|\varepsilon_0| > \delta_n\} = o(1). \quad (4.59a1)$$

If Z is a nearest one of these two roots and $\pi_n = \mathbb{P}\{Z = b_\gamma\}$, then $\mathbb{E}|X - Z| = O(\sqrt{d})$ and centering gives

$$0 = \mathbb{E}X = s_\gamma(\gamma)\pi_n - a_\gamma + O(\sqrt{d}), \quad \pi_n = \frac{a_\gamma}{s_\gamma(\gamma)} + O(\sqrt{d}). \quad (4.59a2)$$

Thus both root clusters have masses bounded below.

Intersect each cluster with the two good events in (4.59a1). On either intersection every lattice site $a_0 + h_0J = X - \varepsilon_0$ lies within $2\delta_n$ of the corresponding root. Since h_0 is bounded below, two different labels cannot occur in one such cluster once $4\delta_n < h_0$. Call the resulting labels j_- and j_+ . They have uniformly positive masses, and

$$h_0(j_+ - j_-) = a_\gamma + b_\gamma + O(\delta_n) = s_\gamma(\gamma) + o(1). \quad (4.59a3)$$

The integer $j_+ - j_-$ therefore tends to one and equals one for all sufficiently large rows. Translate J so that these labels are 0, 1; this only changes a_0 by an integral multiple of h_0 and preserves the phase convention.

Finally h_0 is in a fixed compact span range, $|\varepsilon_0| \leq h_0/2$, and the labels 0, 1 are the only cells containing the two defect roots. The explicit cubic tails of w_γ consequently give, uniformly in the row,

$$|J|^3 1_{\{|J| > H\}} \leq Cw_\gamma(X) 1_{\{|J| > H\}} \quad (H \geq 2).$$

Since $\mathbb{E}w_\gamma(X) = d$, multiplication by $|J|^{k-3} \leq H^{k-3}$ proves, simultaneously for every $H \geq 2$,

$$\mathbb{E}\{|J|^k; |J| > H\} \leq \frac{Cd}{H^{3-k}}, \quad k = 0, 1, 2. \quad (4.59a)$$

We give the spectral-to-least-squares bridge explicitly. Since the phase choice gives $\mathbb{E}e^{iu_0\varepsilon_0} = r \in \mathbb{R}$, Taylor's theorem gives

$$|\mathbb{E}\varepsilon_0| \leq CA_0. \quad (4.59b)$$

Stationarity of $|\varphi_X|^2$ at u_0 gives $\Im\mathbb{E}\{Xe^{iu_0\varepsilon_0}\} = 0$. We record the quantitative argument rather than hide it in this stationarity statement. Since u_0 is bounded above and below,

$$u_0\mathbb{E}(X\varepsilon_0) = \mathbb{E}X\{u_0\varepsilon_0 - \sin(u_0\varepsilon_0)\}. \quad (4.59b1)$$

On $C_H = \{|J| \leq H\}$, one has $|X| \leq C(1 + H)$ and $|y - \sin y| \leq Cy^2$, whence the right side of (4.59b1), restricted to C_H , is at most CHA_0 . On C_H^c , use $|y - \sin y| \leq C|y|$, Cauchy-Schwarz, and (4.59a):

$$\begin{aligned} \mathbb{E}\{|X\varepsilon_0|; C_H^c\} &\leq C\{\mathbb{E}[(1 + J^2); C_H^c]\}^{1/2} A_0^{1/2} \\ &\leq C(A_0d/H)^{1/2} \leq C\{HA_0 + d/H^2\}. \end{aligned} \quad (4.59b2)$$

All constants here are uniform for $H \geq 2$ and for every row under consideration. Equations (4.59b1)–(4.59b2) prove

$$|\mathbb{E}(X\varepsilon_0)| \leq C\{HA_0 + d/H^2 + A_0\}. \quad (4.59c)$$

If $A_0 = 0$ the bridge is immediate. Otherwise take $H = A_0^{-1/3}$, replaced by 2 for the finitely many rows for which it is smaller. Then $HA_0 = A_0^{2/3}$ and $d/H^2 = dA_0^{2/3}$. Because d is bounded, (4.59b)–(4.59c) and $X = a_0 + h_0J + \varepsilon_0$ imply

$$|\mathbb{E}(J\varepsilon_0)| + |\mathbb{E}\varepsilon_0| = O(A_0^{2/3}). \quad (4.59d)$$

The two core labels give $\text{Var } J \geq c > 0$. Hence the affine least-squares projection

$$L = a + hJ, \quad \varepsilon = X - L$$

satisfies

$$\mathbb{E}\varepsilon = \mathbb{E}(J\varepsilon) = 0, \quad A := \mathbb{E}\varepsilon^2 \leq A_0, \quad |a - a_0| + |h - h_0| = O(A_0^{2/3}). \quad (4.60a0)$$

Put $u_L = 2\pi/h$ and $z = \mathbb{E}e^{iu_L\varepsilon}$. Exact lattice phase cancellation gives $|z| = |\varphi_X(u_L)|$. On the selected window $|\varphi_X|$ has a uniformly bounded second derivative, while its first derivative vanishes at u_0 . Thus

$$n||z| - r| \leq Cn|u_L - u_0|^2 = O(nA_0^{4/3}) = o(1). \quad (4.60)$$

This is the required bridge: the selected spectral power and the least-squares residual power have the same subsequential limit.

Since $u_0 \rightarrow 2\pi/s_*$ and (4.60a0) gives $h - h_0 = o(1)$, one has $h \rightarrow s_*$. Lemma 7.10 now gives exactly the upper-majorant hypothesis (4.47) in Corollary 7.13, without using residual power or $A = o(n^{-1})$; it also gives the reflected fixed gap. The middle has the adjacent-mass Cramér gap. Moreover $A = O(n^{-1})$, so $\text{Var } J = (1 - A)/h^2 \rightarrow s_*^{-2} \in (0, \infty)$, while outer localization gives $x_n = o(1)$. Define the integer k_n and $\theta_n \in [-1/2, 1/2)$ by the exact threshold identity

$$x_n\sqrt{n} = na + h(k_n + \theta_n). \quad (4.60a00)$$

The first normal equation gives $a = -h\mathbb{E}J$, and hence

$$\frac{k_n - n\mathbb{E}J}{\sqrt{n}} = \frac{x_n}{h} - \frac{\theta_n}{\sqrt{n}} \rightarrow 0. \quad (4.60a01)$$

Moreover, with $K_n = \sum J_i$ and $T_n = h^{-1} \sum \varepsilon_i$, the closed-half-line convention is exactly

$$\sum_{i=1}^n X_i \leq x_n\sqrt{n} \iff K_n - k_n + T_n - \theta_n \leq 0. \quad (4.60a02)$$

Thus the cardinal-gap hypotheses are now all explicit: the normal equations are (4.60a0), $A = O(n^{-1})$, the variance has the displayed positive limit, (4.59a) supplies adjacent masses and uniform tails, (4.60a01) supplies the centering, (4.60a02) fixes the sign and index, and $\rho_n = \rho_2(p_n) + d_n = O(1)$ follows from $p_n \rightarrow p_*$ and $d_n \rightarrow 0$. Consequently every hypothesis of Corollary 7.13 is satisfied. Since the actual score tends to C_E , that corollary yields

$$|z|^n \rightarrow 1. \quad (4.60a1)$$

Only $A = O(n^{-1})$ was used to obtain (4.60a1). The stronger residual estimate $A = o(n^{-1})$ is deduced below from (4.60a1), so the argument is not circular. Choose $\beta \in [-1/2, 1/2)$ so that $e^{-2\pi i\beta}z = |z|$, let N be a nearest integer to $\varepsilon/h - \beta$, and write exactly

$$X = (a + h\beta) + h(J + N) + \tilde{\varepsilon}, \quad \tilde{\varepsilon} = \varepsilon - h(\beta + N). \quad (4.60a2)$$

The chord-to-coset inequality and (4.60a1) give

$$\mathbb{E}\tilde{\varepsilon}^2 \leq \frac{h^2}{8} \{1 - |z|\} = o(n^{-1}). \quad (4.60a)$$

Here there is no hidden compactness step. By the definition of N , $v := \varepsilon/h - \beta - N$ lies in $[-1/2, 1/2]$, and therefore

$$\begin{aligned} 1 - |z| &= \mathbb{E}\{1 - \cos(2\pi(\varepsilon/h - \beta))\} \\ &= \mathbb{E}\{1 - \cos(2\pi v)\} \geq 8\mathbb{E}v^2 = \frac{8}{h^2} \mathbb{E}\tilde{\varepsilon}^2. \end{aligned} \quad (4.60a3)$$

Moreover $|z|^n \rightarrow 1$ implies $n(1 - |z|) \rightarrow 0$: otherwise a subsequence with $n(1 - |z|) \geq c > 0$ would satisfy $|z|^n \leq e^{-n(1-|z|)} \leq e^{-c} < 1$. Thus (4.60a) follows from (4.60a3), with the explicit constant $1/8$. Take the affine least-squares projection once more onto $J + N$ and rename it $L = a + hJ$, with residual ε . Least squares cannot increase the last squared distance, so

$$\mathbb{E}\varepsilon = \mathbb{E}(J\varepsilon) = \mathbb{E}(L\varepsilon) = 0, \quad A = \mathbb{E}\varepsilon^2 = o(n^{-1}). \quad (4.61)$$

We verify here that the lattice hypotheses survive both the reindexing and this second projection. Before the renaming, let $j_0, j_0 + 1$ be the two core labels constructed immediately before (4.59a), put $J' = J + N$, and let $\varepsilon^{(1)}$ denote the first least-squares residual in (4.60a0). Since $\mathbb{E}\varepsilon^{(1)} = 0$, $u_L = 2\pi/h$ stays bounded, and $A_1 := \mathbb{E}(\varepsilon^{(1)})^2 = O(n^{-1})$,

$$1 - \Re z \leq \frac{u_L^2 A_1}{2}, \quad |\Im z| = |\mathbb{E}(\sin(u_L \varepsilon^{(1)}) - u_L \varepsilon^{(1)})| \leq C u_L^2 A_1. \quad (4.61a)$$

Thus $\arg z = O(A_1)$ and the representative chosen in (4.60a2) satisfies $\beta = O(A_1)$. If $N \neq 0$, the nearest-integer definition implies $|\varepsilon^{(1)}/h - \beta| \geq 1/2$. Since h is bounded away from zero, for all sufficiently large n this entails $|\varepsilon^{(1)}| \geq c > 0$; hence

$$\mathbb{P}\{N \neq 0\} \leq C \mathbb{E}(\varepsilon^{(1)})^2 = O(n^{-1}). \quad (4.61b)$$

If the two old core masses are at least c_0 , then

$$\mathbb{P}\{J' = j_0 + r\} \geq \mathbb{P}\{J = j_0 + r, N = 0\} \geq c_0 - \mathbb{P}\{N \neq 0\} \geq \frac{c_0}{2}, \quad r = 0, 1. \quad (4.61c)$$

In particular $\text{Var } J'$ is bounded below. Write the second projection, before renaming, as $L = a' + h'J'$. From the preliminary representation (4.60a2) and the normal equations,

$$|h' - h| \leq \frac{\|\tilde{\varepsilon}\|_2}{\sqrt{\text{Var } J'}} = o(n^{-1/2}). \quad (4.61d)$$

Consequently $h' > 0$ and remains in the same compact span range. Projection changes neither J' nor its probabilities, so (4.61c) still holds for the renamed lattice law. Its support is contained in $a' + h'\mathbb{Z}$ and contains both $a' + h'j_0$ and $a' + h'(j_0 + 1)$. Any lattice containing the whole support must therefore have span at most h' , whereas h' itself is a span. Hence h' is the maximal span. Centering and standardizing later in (4.68a) merely replace it by h'/σ and preserve both adjacent masses and maximality.

We next prove the full operator transfer. Put $\alpha_0 = (\mathbb{E}\tilde{\varepsilon}^2)^{1/2} = o(n^{-1/2})$ for the preliminary reindexed residual in (4.60a2). The normal equations, the two core labels, and (4.59a) give the pointwise least-squares geometry as follows. Before the second projection write $X = c + \bar{h}J + \tilde{\varepsilon}$, choosing c modulo $\bar{h}$ so $|c| \leq C$. The span $\bar{h}$ is bounded above and below and $|\tilde{\varepsilon}| \leq \bar{h}/2$. Centering gives $|\mathbb{E}J| \leq C$, while the adjacent core labels give $\text{Var } J \geq c_0$. The least-squares normal equations therefore imply

$$|h - \bar{h}| = \frac{|\text{Cov}(J, \tilde{\varepsilon})|}{\text{Var } J} \leq C\alpha_0, \quad |a - c| \leq |\mathbb{E}\tilde{\varepsilon}| + |h - \bar{h}| |\mathbb{E}J| \leq C\alpha_0.$$

Moreover $|J| \leq C(1 + |X|)$. Hence

$$|X - L| \leq C\{1 + \alpha_0(1 + |X|)\}. \quad (4.62)$$

For the exact defect density, uniformly on the critical compactum,

$$w_\gamma(x) \geq c|x|^3 \quad (|x| \geq R_0), \quad \mathbb{E}w_\gamma(X) = d. \quad (4.63)$$

Consequently, for $R \geq R_0$, $\mathbb{E}\{|X|^k; |X| > R\} \leq Cd/R^{3-k}$, $k = 0, 1, 2$. From

$$||x|^3 - |l|^3| + |x^3 - l^3| \leq C(x^2 + l^2)|x - l|$$

and (4.62)–(4.63), we now record the split which also fixes the dependence on the preliminary residual α_0 . Write $C_R = \{|X| \leq R\}$. On C_R , $L = X - (X - L)$ and (4.62) give

$$|L| \leq C(1 + R), \quad |X - L| \leq C(1 + \alpha_0 R),$$

and hence, using $\mathbb{E}|X - L| \leq \sqrt{A}$,

$$\mathbb{E}\{(X^2 + L^2)|X - L|; C_R\} \leq C\{R^2\sqrt{A} + (1 + \alpha_0 R)A\}. \quad (4.63a)$$

Indeed, with $e = X - L$, the elementary bound $X^2 + L^2 \leq C(X^2 + e^2)$ makes the two terms on the left at most $CR^2\mathbb{E}|e|$ and $\mathbb{E}|e|^3$. The first is at most $CR^2\sqrt{A}$, while (4.62) gives $\mathbb{E}\{|e|^3; C_R\} \leq \sup_{C_R} |e| \mathbb{E}e^2 \leq C(1 + \alpha_0 R)A$. Thus (4.63a) uses no fourth moment. On C_R^c , after increasing R_0 and taking n large, (4.62) also gives $|L| \leq C(1 + |X|)$ and

$$(X^2 + L^2)|X - L| \leq C\{1 + X^2 + \alpha_0(1 + |X|^3)\}.$$

The three tail estimates following (4.63), together with $\mathbb{E}\{|X|^3; |X| > R\} \leq Cd$, therefore yield

$$\mathbb{E}\{(X^2 + L^2)|X - L|; C_R^c\} \leq C\{d/R + \alpha_0 d\}. \quad (4.63b)$$

More explicitly, the part without α_0 is controlled by $\mathbb{P}(C_R^c) + \mathbb{E}\{X^2; C_R^c\} \leq C(d/R^3 + d/R)$, and the α_0 part by $\alpha_0\mathbb{E}\{1 + |X|^3; C_R^c\} \leq C\alpha_0 d$. All constants in (4.62)–(4.63b) are independent of $R \geq R_0$ and of the row once p lies in the fixed critical compactum. Combining (4.63a)–(4.63b) proves

$$|\mathbb{E}X^3 - \mathbb{E}L^3| + |\mathbb{E}|X|^3 - \mathbb{E}|L|^3| \leq C\{R^2\sqrt{A} + (1 + \alpha_0 R)A + d/R + \alpha_0 d\}. \quad (4.64)$$

The relation needed here is

$$\sqrt{A} = o(n^{-1/2}), \quad n^{-1/2} = O(d), \quad \sqrt{A} = o(d); \quad (4.64a)$$

the previously suggested relation $n^{-1/2} = o(d)$ would be false on the scale (4.56). We choose all later free cutoffs simultaneously. With a vanishing denominator interpreted as $+\infty$, put

$$H_n^{\text{cut}} = \min \left\{ \left(\frac{d}{\sqrt{A}} \right)^{1/4}, \left(\frac{d}{A} \right)^{1/2}, (nd)^{1/2}, \alpha_0^{-1/2} \right\}, \quad \text{and below we take } R = H = H_n^{\text{cut}}. \quad (4.64b)$$

Every entry in the minimum tends to infinity. Moreover

$$\frac{(H_n^{\text{cut}})^2 \sqrt{A}}{d} \leq \sqrt{\frac{\sqrt{A}}{d}} \rightarrow 0, \quad \frac{H_n^{\text{cut}} A}{d} \leq \sqrt{\frac{A}{d}} \rightarrow 0, \quad \frac{H_n^{\text{cut}}}{nd} \leq \frac{1}{\sqrt{nd}} \rightarrow 0, \quad \alpha_0 H_n^{\text{cut}} \leq \sqrt{\alpha_0} \rightarrow 0. \quad (4.64c)$$

It follows at once that

$$R^2 \sqrt{A} = o(d), \quad (1 + \alpha_0 R)A = o(d), \quad d/R = o(d), \quad HA = o(d), \quad H = o(nd), \quad (4.64d)$$

as well as $d/H^2 = o(d)$ and $\alpha_0 d/H = o(d)$. Thus the right side of (4.64) is $o(d)$ and the same single cutoff will be used in (4.67b)–(4.67d). For later reference, the normalized assertions just used are

$$\frac{R^2 \sqrt{A}}{d} \rightarrow 0, \quad \frac{A}{d} \rightarrow 0, \quad \alpha_0 R \rightarrow 0, \quad \frac{1}{R} \rightarrow 0, \quad \frac{HA}{d} \rightarrow 0, \quad \frac{H}{nd} \rightarrow 0. \quad (4.64e)$$

They follow respectively from (4.64c), (4.64a), (4.64c), $R \rightarrow \infty$, (4.64c), and (4.64c). This list is the complete scale input used below; in particular no step assumes the false relation $n^{-1/2} = o(d)$.

Here is the zero-cell comparison, including its full frequency range. Set

$$G(y) = e^{iy} - 1 - iy + \frac{y^2}{2} + \frac{iy^3}{6}, \quad |G'(y)| \leq C \min\{|y|^3, 1 + y^2\}. \quad (4.65)$$

Put $\kappa = 2\pi/h$; it ranges in the compact set of scaling factors generated by the admissible spans. Put $u = \kappa v/\sqrt{n}$, and initially restrict

$$0 \leq v \leq \delta\sqrt{n}. \quad (4.65a)$$

For $|X| \geq R_0$, write $q_X = |X - L|/|X|$ and $s_X = |X|/\sqrt{n}$. Increase R_0 once so that $|L| \leq C|X|$ on this set. The mean-value theorem and (4.65) then give

$$|G(uX) - G(uL)| \leq Cq_X \min\{|u|^4|X|^4, |u||X| + |u|^3|X|^3\}.$$

Dividing by the cubic lower bound in (4.63), and using $u = \kappa v/\sqrt{n}$, proves the first inequality in

$$\frac{n^{3/2}|G(uX) - G(uL)|}{w_\gamma(X)} \leq Cq_X \min\{v^4 s_X, v/s_X^2 + v^3\} \leq C(\alpha_0 + n^{-1/2})(v^3 + v^4). \quad (4.65b)$$

Indeed (4.62) implies $q_X \leq C(\alpha_0 + |X|^{-1})$. If $s_X \leq 1$, use the first member of the minimum and $q_X s_X \leq C(\alpha_0 + n^{-1/2})$. If $s_X \geq 1$, then $q_X \leq C(\alpha_0 + n^{-1/2})$, while

$$\min\{v^4 s_X, v/s_X^2 + v^3\} \leq \min\{v^4 s_X, v/s_X^2\} + v^3 \leq 2v^3;$$

the last inequality follows by separating $vs_X \leq 1$ and $vs_X \geq 1$. This proves the second inequality in (4.65b). On $|X| < R_0$, no division by w_γ is made (so both zeros of the defect are included), and

$$\mathbb{E}\{|G(uX) - G(uL)|; |X| < R_0\} \leq C \frac{v^4}{n^2} \sqrt{A}. \quad (4.65c)$$

After integrating (4.65b) against $w_\gamma(X) d\mathbb{P}/d$ and adding (4.65c),

$$\frac{n^{3/2}}{d} \mathbb{E}|G(uX) - G(uL)| \leq C\vartheta_n(v^3 + v^4), \quad \vartheta_n = \alpha_0 + n^{-1/2} + \frac{\sqrt{A}}{\sqrt{n}d} = o(1). \quad (4.66)$$

This single estimate covers compact, root-scale, mesoscopic, and super-root atoms.

Let $R(u) = \varphi_X(u) - \varphi_L(u)$ and $B_3 = \mathbb{E}X^3 - \mathbb{E}L^3 = o(d)$. Since both variables are centered and $\mathbb{E}X^2 - \mathbb{E}L^2 = A$, the identity defining G gives

$$R(u) = -\frac{Au^2}{2} - \frac{iB_3u^3}{6} + \mathbb{E}\{G(uX) - G(uL)\}. \quad (4.66a)$$

Thus, with $\eta_n = |B_3|/d + \vartheta_n \rightarrow 0$ and $P(v) = v^3 + v^4$,

$$|R(u)| \leq C \left\{ \frac{Av^2}{n} + \frac{d\eta_n P(v)}{n^{3/2}} \right\}. \quad (4.66b)$$

The variances of X, L are bounded below and their third absolute moments are bounded above. Applying $1 - \cos y \geq y^2/2 - |y|^3/6$ to an independent difference gives fixed $c, \delta > 0$ such that

$$\max\{|\varphi_X(\kappa t)|, |\varphi_L(\kappa t)|\}^{n-j} \leq Ce^{-cnt^2}, \quad 0 \leq t \leq \delta, \quad j = 0, 1, 2. \quad (4.66c)$$

For complex a, b ,

$$|a^n - b^n - nb^{n-1}(a - b)| \leq \frac{n(n-1)}{2} |a - b|^2 \max(|a|, |b|)^{n-2}. \quad (4.66d)$$

Equations (4.66b)–(4.66d), only on the range (4.65a), imply

$$|\varphi_X(u)^n - \varphi_L(u)^n| \leq Ce^{-cv^2} \{n|R(u)| + n^2|R(u)|^2\}. \quad (4.66e)$$

Finally

$$\frac{2}{\sqrt{n}} |K(v/\sqrt{n})| \leq \frac{C_\delta}{v}, \quad 0 < v \leq \delta\sqrt{n}. \quad (4.66f)$$

Write $I_0 = [0, \delta]$ and $I_1 = [1 - \delta, 1]$ for the zero and return cells. After $t = v/\sqrt{n}$, the zero-cell integral is bounded by

$$\begin{aligned} & \frac{\sqrt{n}}{d} \max_{\sigma \in \{+, -\}} \sup_x |\mathcal{J}_{X,n,h;I_0}^\sigma(x) - \mathcal{J}_{L,n,h;I_0}^\sigma(x)| \\ & \leq C \int_0^{\delta\sqrt{n}} \frac{e^{-cv^2}}{v} \left\{ \frac{\sqrt{n}A}{d} v^2 + \eta_n P(v) + \frac{\sqrt{n}A^2}{d} v^4 \right. \\ & \quad \left. + A\eta_n v^2 P(v) + \frac{d\eta_n^2}{\sqrt{n}} P(v)^2 \right\} dv = o(1). \end{aligned} \quad (4.67)$$

Every coefficient in the last integral tends to zero:

$$\frac{\sqrt{n}A}{d} = o(1), \quad \eta_n = o(1), \quad \frac{\sqrt{n}A^2}{d} = A \frac{\sqrt{n}A}{d} = o(1), \quad A\eta_n = o(1), \quad \frac{d\eta_n^2}{\sqrt{n}} = o(1). \quad (4.67a0)$$

Here the first relation is exactly $A = o(n^{-1})$ together with $d \asymp n^{-1/2}$; all Gaussian polynomial integrals are finite. Thus (4.67) says precisely that the raw zero-cell difference is $o(d/\sqrt{n})$, uniformly in the cdf coordinate and for both signs. Only the nonnegative majorant in (4.67) is extended to $[0, \infty)$. The estimate (4.66c) is not asserted beyond the fixed zero cell. On the fixed physical middle interval, the adjacent masses give $|\varphi_L(u)| \leq 1 - c$, while

$$|\varphi_X(u) - \varphi_L(u)| \leq |u| \mathbb{E}|X - L| \leq C\sqrt{A} = o(1).$$

Thus both powers are exponentially small there. The remaining endpoint is handled below.

For that endpoint put $t_0 = 2\pi/h$,

$$z = \mathbb{E}e^{it_0\varepsilon} = r_0\omega, \quad \lambda = e^{it_0a}, \quad M_1 = \mathbb{E}\{Xe^{it_0\varepsilon}\}.$$

Since $e^{it_0L} = \lambda$, the marked first moment below satisfies

$$m = \mathbb{E}\{Xe^{it_0X}\} = \lambda M_1, \quad |m| = |M_1|. \quad (4.67a00)$$

Also put

$$q_2 = \mathbb{E}\{(X^2 - 1)e^{it_0X}\}, \quad q_3 = \mathbb{E}\{(X^3 - \gamma)e^{it_0X}\}. \quad (4.67a)$$

Since $\mathbb{E}\varepsilon = 0$,

$$1 - r_0 \leq 1 - \Re z \leq \frac{t_0^2 A}{2}, \quad |\arg z| \leq C|\Im z| \leq C\mathbb{E}|\sin(t_0\varepsilon) - t_0\varepsilon| \leq CA.$$

Consequently

$$\frac{1 - r_0^n + |\omega^n - 1|}{\sqrt{n}} \leq C\sqrt{n}A = o(d). \quad (4.67a1)$$

The common unit phase relates m and M_1 . Since $\mathbb{E}\varepsilon = \mathbb{E}(L\varepsilon) = 0$, one has $\mathbb{E}(X\varepsilon) = A$. Because also $\mathbb{E}X = 0$, the exact centered Taylor identity is

$$M_1 = it_0A + \mathbb{E}\{X(e^{it_0\varepsilon} - 1 - it_0\varepsilon)\}. \quad (4.67a2)$$

On $|X| \leq H$, the remainder bound $|e^{iy} - 1 - iy| \leq Cy^2$ makes the last expectation at most CHA . On $|X| > H$, use instead $|e^{iy} - 1 - iy| \leq C|y|$, followed by (4.62), to get

$$\mathbb{E}\{|X||\varepsilon|; |X| > H\} \leq C\{d/H^2 + \alpha_0 d/H\}.$$

Thus, for every $H \geq \max\{2, R_0\}$,

$$|M_1| \leq C\{A + HA + d/H^2 + \alpha_0 d/H\}. \quad (4.67b0)$$

Taking the single common choice $H = H_n^{\text{cut}}$ from (4.64b)–(4.64d) gives

$$|M_1| = o(d), \quad \sqrt{n}|M_1|^2 = o(d), \quad (4.67b)$$

the second conclusion using $\sqrt{n}d = O(1)$. For completeness we prove the two marked estimates used next without any moment beyond the third. Put

$$P_2(x) = x^2 - \gamma x - 1, \quad R_3(x) = (x + \gamma)P_2(x).$$

Uniformly for γ in the critical compactum, the factorizations (3.1)–(3.2) imply, for $H \geq 2$,

$$\begin{aligned} P_2(x)^2 1_{\{|x| \leq H\}} &\leq CH w_\gamma(x), \\ |P_2(x)| 1_{\{|x| > H\}} &\leq \frac{C}{H} w_\gamma(x), \\ |R_3(x)| &\leq C\{w_\gamma(x)^{1/2} + w_\gamma(x)\}. \end{aligned} \quad (4.67b1)$$

Indeed the quotients extend through the two zeros of w_γ , and their limits at infinity give the displayed powers of H . Since $z = \mathbb{E}e^{it_0\varepsilon} = r_0\omega$, $|\omega| = 1$,

$$\mathbb{E}|e^{it_0\varepsilon} - \omega|^2 = 2(1 - r_0). \quad (4.67b2)$$

Moreover $\mathbb{E}P_2(X) = 0$, while e^{it_0L} is a constant unit phase. Subtracting ω inside the expectation, applying Cauchy–Schwarz on $|X| \leq H$, and using the second bound in (4.67b1) on its complement gives

$$|q_2 - \gamma m| \leq C\{\sqrt{Hd(1 - r_0)} + d/H\}. \quad (4.67c)$$

Finally

$$x^3 - \gamma = (\gamma^2 + 1)x + R_3(x), \quad \mathbb{E}R_3(X) = 0.$$

One more subtraction of ω , followed by (4.67b1)–(4.67b2), yields

$$|q_3 - (\gamma^2 + 1)m| \leq C\{\sqrt{d(1 - r_0)} + d\}.$$

Since $|m| = |M_1| = o(d)$ by (4.67b), this proves

$$|q_3| \leq C\{\sqrt{1 - r_0} + d\}. \quad (4.67c1)$$

Use the common $H = H_n^{\text{cut}}$ from (4.64b)–(4.64d). Then (4.67a1)–(4.67c) give

$$q_2 = o(d), \quad q_3/\sqrt{n} = o(d), \quad \frac{1 - r_0^n}{\sqrt{n}} = o(d). \quad (4.67d)$$

Indeed the three normalized estimates, including the terms suppressed in the preceding sentence, are

$$\begin{aligned} \frac{|q_2|}{d} &\leq C\left\{\sqrt{HA/d} + \frac{1}{H} + \frac{|M_1|}{d}\right\} = o(1), \\ \frac{|q_3|}{\sqrt{n}d} &\leq C\left\{\frac{\sqrt{A}}{\sqrt{n}d} + \frac{1}{\sqrt{n}} + \frac{|M_1|}{\sqrt{n}d}\right\} = o(1), \\ \frac{1 - r_0^n + |\omega^n - 1|}{\sqrt{n}d} &\leq C\frac{\sqrt{n}A}{d} = o(1). \end{aligned} \quad (4.67d1)$$

The first line uses $1 - r_0 \leq CA$ and (4.64e); in the second, $\sqrt{n}d$ is bounded below and $\sqrt{A} = o(n^{-1/2})$; the last is (4.67a1). This verifies all three $o(d)$ statements on the full scale $d \asymp n^{-1/2}$. The same tail comparison makes $|X|^3$ uniformly integrable as $d \rightarrow 0$. Taylor’s Peano

remainder is therefore uniform along the triangular array, and the marked factorization at $t_0 + s$ is

$$\varphi_X(t_0 + s) - \varphi_X(t_0)\varphi_X(s) = ism - \frac{s^2}{2}q_2 - \frac{is^3}{6}q_3 + o(|s|^3). \quad (4.67e)$$

Recall that $e^{it_0L} = \lambda$, $\varphi_X(t_0) = \lambda z$, and $m = \lambda M_1$. If

$$D_n(s) = \varphi_X(t_0 + s) - \varphi_X(t_0)\varphi_X(s),$$

uniform third-moment Peano expansion gives

$$D_n(s) = ism - \frac{s^2}{2}q_2 - \frac{is^3}{6}q_3 + s^3\epsilon_n(s), \quad \limsup_{\delta \downarrow 0} \sup_n \sup_{|s| \leq \delta} |\epsilon_n(s)| = 0. \quad (4.67e1)$$

For $\mathcal{F}_n(s) = \varphi_X(t_0 + s)$,

$$\mathcal{F}_n(0) = \lambda z, \quad \mathcal{F}'_n(0) = im, \quad \mathcal{F}''_n(0) = -\mathcal{F}_n(0) - q_2.$$

Consequently

$$(|\mathcal{F}_n|^2)''(0) = -2r_0^2 - 2\Re\{q_2\overline{\lambda z}\} + 2|m|^2 = -2 + o(1). \quad (4.67e2)$$

Uniform integrability of X_n^2 makes these second derivatives equicontinuous. Thus $(|\mathcal{F}_n|^2)'' \leq -c$ on a fixed endpoint cell, and Taylor's theorem plus completion of the square gives

$$|\mathcal{F}_n(s)|^n \leq C \exp\{-cns^2 + Cn|m|^2\} \leq C'e^{-c'ns^2}. \quad (4.67e3)$$

We now verify all bases which occur in the divided powers. Write

$$\chi(s) = \varphi_X(s), \quad \psi(s) = \varphi_L(s), \quad F(s) = \varphi_X(t_0 + s), \quad B(s) = \lambda z\chi(s), \quad C(s) = \lambda\psi(s),$$

and put $D = F - B$, $R = \chi - \psi$. For $\alpha, \beta, \kappa \geq 0$, $\alpha + \beta + \kappa = 1$, set $G_{\alpha\beta\kappa} = \alpha F + \beta B + \kappa C$, and abbreviate this function by G in the next two displays. At the endpoint,

$$\begin{aligned} G(0) &= \lambda\{(\alpha + \beta)z + \kappa\}, & G'(0) &= i\alpha m, \\ G''(0) &= -\lambda\{(\alpha + \beta)z + \kappa(1 - A)\} - \alpha q_2. \end{aligned} \quad (4.67e4)$$

It follows, uniformly on this compact simplex, that

$$(|G|^2)'(0) = O(|m|), \quad (|G|^2)''(0) = -2 + o(1). \quad (4.67e5)$$

Uniform integrability of X^2 and L^2 makes these second derivatives equicontinuous. Taylor's theorem and completion of the square therefore give, for $j = 0, 1, 2$,

$$|G_{\alpha\beta\kappa}(s)|^{n-j} \leq C \exp\{-cns^2 + Cn|m|^2\} \leq C'e^{-c'ns^2}. \quad (4.67e6)$$

The last inequality uses $n|m|^2 = o(1)$, which follows from (4.67b) and $\sqrt{n}d = O(1)$. The zero-cell divided power uses $Z_\tau = (1 - \tau)\psi + \tau\chi$; here

$$Z_\tau(0) = 1, \quad Z'_\tau(0) = 0, \quad Z''_\tau(0) = -\{1 - (1 - \tau)A\},$$

so the same Gaussian bound holds uniformly in $0 \leq \tau \leq 1$.

For completeness, these local bounds have a common continuation over the whole endpoint cell. If two adjacent lattice sites have probabilities at least c_0 , then

$$1 - |\psi(s)|^2 \geq 2c_0^2\{1 - \cos(hs)\}. \quad (4.67e7)$$

Away from $s = 0$ this is a fixed gap. Exact lattice periodicity and the residual bound give

$$|\chi(s) - \psi(s)| \leq |s|\sqrt{A}, \quad |F(s) - C(s)| \leq |t_0 + s|\sqrt{A}, \quad |z - 1| = O(A).$$

The triangle inequality therefore transfers the gap to all three vertices F, B, C , and hence to their convex hull. This proves the claimed power domination without interpolating unproved endpoint estimates.

The algebraic ledger is also exact. With $E_\tau = (1 - \tau)B + \tau F$,

$$\begin{aligned} F^n - C^n &= \lambda^n z^n n R \int_0^1 Z_\tau^{n-1} d\tau + \lambda^n (z^n - 1) \psi^n + n B^{n-1} D \\ &\quad + n(n-1) D^2 \int_0^1 (1 - \tau) E_\tau^{n-2} d\tau. \end{aligned} \quad (4.67e8)$$

For the first term in the first row, put $s = -t_0 v / \sqrt{n}$, so that $t = 1 - v / \sqrt{n}$, $dt = -dv / \sqrt{n}$, and $|K(1 - v / \sqrt{n})| \leq C v / \sqrt{n}$. The zero-cell divided-power bound for Z_τ following (4.67e6) gives

$$\left| \int_0^1 Z_\tau(s)^{n-1} d\tau \right| \leq C e^{-cv^2}.$$

Consequently, since the phases and z^n have modulus at most one, (4.66b) gives

$$\begin{aligned} \sqrt{n} \left| 2 \int_{1-\delta}^1 K(t) \lambda^n z^n n R(s) \int_0^1 Z_\tau(s)^{n-1} d\tau dt \right| \\ \leq C \sqrt{n} \int_0^{\delta \sqrt{n}} v e^{-cv^2} |R(-t_0 v / \sqrt{n})| dv \\ \leq C \left\{ \frac{A}{\sqrt{n}} + \frac{d\eta_n}{n} \right\}. \end{aligned} \quad (4.67e8a)$$

Thus the first term has the explicit normalized-score bound

$$C \left\{ \frac{A}{\sqrt{n}} + \frac{d\eta_n}{n} \right\} = o(d). \quad (4.67e9)$$

The second term in that row costs $C\{1 - r_0^n + |\omega^n - 1|\} / \sqrt{n}$. Insert (4.67e1) in the linear term and use (4.67e6) in the quadratic term. With $s = -t_0 v / \sqrt{n}$ and $|K(1 - v / \sqrt{n})| \leq C v / \sqrt{n}$, their costs are respectively

$$O\left(|m| + \frac{|q_2|}{\sqrt{n}} + \frac{|q_3|}{n}\right) + o(d), \quad O(\sqrt{n}|m|^2) + o(d).$$

Here is the full content of the second $o(d)$. Squaring the four terms in (4.67e1), absorbing cross products by $2ab \leq a^2 + b^2$, and using the returned-kernel factor v/n gives the normalized bound

$$C \left\{ \sqrt{n}|m|^2 + \frac{|q_2|^2}{\sqrt{n}} + \frac{|q_3|^2}{n^{3/2}} \right\} + o(n^{-1}). \quad (4.67e10)$$

For the Peano terms choose $B_n = C_0 \sqrt{\log(n+2)}$, with C_0 large enough that $n^3 e^{-cB_n^2} \rightarrow 0$. Then $B_n / \sqrt{n} \rightarrow 0$, so the uniform modulus in (4.67e1) tends to zero on $v \leq B_n$; on $v > B_n$, (4.67e6) and the choice of C_0 dominate every polynomial prefactor. This proves the $o(n^{-1})$ in (4.67e10), rather than merely asserting a diagonal cutoff. Moreover

$$\begin{aligned} |m| &= o(d), \quad |q_2| / \sqrt{n} = o(d), \quad |q_3| / n = o(d), \\ \sqrt{n}|m|^2 &= o(d), \quad |q_2|^2 / \sqrt{n} = o(d), \quad |q_3|^2 / n^{3/2} = o(d), \quad n^{-1} = o(d), \end{aligned} \quad (4.67e11)$$

by (4.56), (4.67b), and (4.67d); boundedness of the third absolute moments suffices for the penultimate relation. The fixed physical middle is $O(\text{poly}(n)e^{-cn}) = o(d)$ by (4.56). Notice in particular that the cubic marked term costs $|q_3|/n$ after the outer $\sqrt{n}$ normalization: s^3 supplies $n^{-3/2}$, the linear divided power supplies n , and the returned kernel together with dt supplies n^{-1} .

The stronger estimate $q_3/\sqrt{n} = o(d)$ in (4.67d) is therefore more than is needed. Consequently, uniformly in the cdf coordinate and in $\sigma \in \{+, -\}$,

$$\begin{aligned} & \sqrt{n} \max_{\sigma \in \{+, -\}} \sup_x |\mathcal{J}_{X,n,h;I_1}^\sigma(x) - \mathcal{J}_{L,n,h;I_1}^\sigma(x)| \\ & \leq C \left\{ \frac{1 - r_0^n + |\omega^n - 1|}{\sqrt{n}} + |M_1| + \frac{|q_2|}{\sqrt{n}} \right. \\ & \quad \left. + \sqrt{n}|M_1|^2 + \frac{|q_3|}{n} \right\} = o(d). \end{aligned} \quad (4.67f)$$

Thus (4.67), (4.67e4)–(4.67e8), (4.67f), and the adjacent-mass middle Cramér gap transfer both one-sided operators with $o(d)$ cost. The defining outer minus in the reflected contribution (4.24v) is already included and is immaterial in these absolute-value estimates.

It remains to center and standardize the lattice comparison. Least squares gives

$$\mathbb{E}L = 0, \quad \sigma_L^2 := \mathbb{E}L^2 = 1 - A, \quad L^\circ = \frac{L}{\sigma_L}, \quad b = \frac{h}{\sigma_L}. \quad (4.68a)$$

The moment estimate (4.64), followed by $\sigma_L^{-3} = 1 + O(A)$, yields

$$\gamma(L^\circ) = \gamma(X) + o(d), \quad \rho(L^\circ) = \rho(X) + o(d), \quad d(L^\circ) = d(X) + o(d). \quad (4.68)$$

For $\Phi_{\sigma_L}(x) = \Phi(x/\sigma_L)$,

$$\sup_x |\Phi_{\sigma_L}(x) - \Phi(x)| \leq CA, \quad \sqrt{n}A = o(d). \quad (4.68b)$$

Exact scaling gives

$$\mathcal{U}_{L,n,h}^\pm(x) = \mathcal{U}_{L^\circ,n,b}^\pm(x/\sigma_L). \quad (4.68c)$$

At the active point put $y_n = x_n/\sigma_L \rightarrow 0$, with the row index on σ_L suppressed. Equations (4.67), (4.67f), and (4.68b)–(4.68c) prove (4.57), including the true third-absolute-moment denominators. Before invoking the exact-lattice lemma, its hypotheses are as follows: (4.61) gives centering and variance (1-A); (4.61c) preserves two adjacent masses; (4.61d) keeps the span in a fixed compact subset of $(0, \infty)$; the support argument following (4.61d) proves maximality; and standardization preserves both adjacency and maximality. Finally (4.64) and (4.68) give

$$p(L^\circ) \rightarrow p_*, \quad d(L^\circ) = d + o(d), \quad \rho(L^\circ) = \rho(X) + o(d), \quad 0 < c' \leq \sqrt{n} d(L^\circ) \leq C', \quad y_n \rightarrow 0. \quad (4.68d)$$

Thus every hypothesis is checked before Lemma 7.14 is applied directly at y_n ; that lemma gives

$$\frac{\sqrt{n}}{\rho(X_n)} \{\mathcal{U}_{X_n,n,h_n}^+(x_n) - \Phi(x_n)\} \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_n + O(n^{-1}) + o(d_n).$$

Since $n^{-1} = o(d_n)$ in (4.56), this proves (4.57a). The dependency order is therefore

$$\begin{aligned} \text{spectral selection} & \longrightarrow \text{adaptive majorant} \longrightarrow \text{cardinal gap} \\ & \longrightarrow A = o(n^{-1}) \longrightarrow \text{operator transfer} \longrightarrow \text{exact-lattice envelope.} \end{aligned}$$

In particular, neither the adaptive majorant nor the residual-power selection uses the near-lattice transfer that it is later used to prove. $\square$

Lemma 7.16 (moving-bandwidth low-frequency expansion). *Let $n_j \rightarrow \infty$, and for each j let X_j be centered and variance one. Denote its third moment and third absolute moment by γ_j and ρ_j , let p_j be the matching Bernoulli parameter, and put*

$$d_j = \rho_j - \rho_2(p_j), \quad p_j \rightarrow p_*, \quad d_j \rightarrow 0,$$

and

$$\lambda_j = \rho_j + \mathbb{E}|X_j|.$$

For the pointwise Prawitz operator at physical bandwidth λ_j one has, uniformly for $x \in \mathbb{R}$,

$$\begin{aligned} \frac{\sqrt{n_j}}{\rho_j} \{ \mathcal{U}_{X_j, n_j, \lambda_j}^+(x) - \Phi(x) \} &\leq \frac{\phi(x)}{\rho_j} \left\{ \frac{\lambda_j}{2} + \frac{\gamma_j}{6} (1 - x^2) \right\} \\ &+ \{ \Lambda_{\text{abs}}(p_j, d_j) + o(1) \} d_j + O(n_j^{-1/2}). \end{aligned} \quad (4.69)$$

The $o(1)$ and $O(n_j^{-1/2})$ are uniform in x . For the conjugate kernel the sign of the skewness term is reversed. The assertion is pointwise and uniform in the cdf coordinate; it is not notation for a supremum over that coordinate. More precisely, along every array in the hypotheses there are numbers $\varepsilon_j \downarrow 0$ and a constant $C < \infty$, independent of x , such that the two remainders on the right of (4.69) are bounded by $\varepsilon_j d_j$ and $C n_j^{-1/2}$, respectively.

Proof. The rows with $d = 0$ are separate: Theorem 2.1 gives $X \stackrel{d}{=} Q_p$, so the defect contribution (4.69i) is zero. For the defect calculation below pass to rows with $d > 0$. The matching-law expansion and the three frequency-region estimates below apply unchanged to the rows $d = 0$, where $\lambda = s$. Suppress the row index; in either case Proposition 3.2 gives

$$0 \leq \lambda - s \leq K_1(\gamma)d, \quad (4.69a)$$

so λ stays in a fixed compact subinterval of $(0, \infty)$. Choose a fixed zero cell on which the physical argument is within the range of (4.7). The substitution

$$t = \frac{\lambda v}{2\pi\sqrt{n}} \quad (4.69b)$$

has two exact consequences which are important here: the characteristic function is evaluated at $v/\sqrt{n}$, and

$$2K(t) \, dt = \frac{i}{\pi v} \, dv + \frac{\lambda}{2\pi\sqrt{n}} \, dv + O(v/n) \, dv. \quad (4.69c)$$

Thus neither the singular coefficient nor the physical Fourier scale changes when the bandwidth moves.

We first expand the matching two-point transform. Uniformly on $0 \leq v \leq n^{1/21}$,

$$\chi_p(v/\sqrt{n})^n = e^{-v^2/2} \left(1 - \frac{i\gamma v^3}{6\sqrt{n}} \right) + O \left(\frac{(v^4 + v^6)e^{-cv^2}}{n} \right), \quad (4.69d)$$

and both sides have a common Gaussian bound up to the end of the fixed zero cell. Insert (4.69c)–(4.69d), retain e^{-ivx} exactly, and use normal inversion. The singular skewness term and the bounded kernel term give respectively

$$\frac{\gamma}{6\rho}(1 - x^2)\phi(x), \quad \frac{\lambda}{2\rho}\phi(x), \quad (4.69e)$$

with an $O(n^{-1/2})$ normalized remainder. Its dominator is a fixed Gaussian polynomial and is independent of x .

We next calculate the defect term at this bandwidth, rather than invoking Lemma 7.3 by analogy. In Bernoulli coordinates,

$$\varphi_X(v/\sqrt{n}) - \chi_p(v/\sqrt{n}) = e^{-ispv/\sqrt{n}} d \int \frac{E_{sv/\sqrt{n}}(y)}{W_p(y)} \, d\nu(y), \quad d \, d\nu = W_p \, d\mathbb{P}_Y. \quad (4.69f)$$

With $H = n^{1/4}$, Hermite–Genocchi gives on $|y| \leq H$

$$\frac{|E_{sv/\sqrt{n}}(y)|}{W_p(y)} \leq \frac{C(1+|y|)v^4}{n^2}, \quad (4.69g)$$

whose normalized singular-kernel contribution is $O(dH/\sqrt{n}) = o(d)$. On $|y| > H$, subtract $g(svy/\sqrt{n})$ and use (4.21); that correction is $O(d/\sqrt{n})$. The remaining term satisfies

$$\frac{n^{3/2}}{d} |\mathbb{E}\{g(svY/\sqrt{n}); |Y| > H\}| \leq \frac{s^3 v^3}{6} \max\{C_+(p)^{-1}, C_-(p)^{-1}\} \{1 + O(H^{-1})\}. \quad (4.69h)$$

For reference, the exact divided power used at this point is

$$\mathcal{P}_n(v) := \frac{\varphi_X(v/\sqrt{n})^n - \chi_p(v/\sqrt{n})^n}{n\{\varphi_X(v/\sqrt{n}) - \chi_p(v/\sqrt{n})\}} = \int_0^1 \{(1-a)\chi_p(v/\sqrt{n}) + a\varphi_X(v/\sqrt{n})\}^{n-1} da, \quad (4.69h1)$$

with the continuous value when the denominator vanishes. Every transform inside the braces is that of a centered variance-one mixture whose third absolute moment is bounded uniformly along the array. Hence (4.7), first on $|v| \leq \delta\sqrt{n}$ and then the fixed Cramér gap on the rest of the zero cell, gives the required common power bound. Explicitly, on every fixed annulus $\delta \leq |u| \leq u_0$ contained in that cell,

$$\sup_{p \text{ near } p_*} |\chi_p(u)| \leq q < 1, \quad |\varphi_X(u) - \chi_p(u)| \leq Cd,$$

so, for all sufficiently large rows, both endpoints and every convex combination in (4.69h1) have modulus at most $(1+q)/2 < 1$. Consequently

$$|\mathcal{P}_n(v)| \leq Ce^{-cv^2}, \quad \mathcal{P}_n(v) \longrightarrow e^{-v^2/2} \quad (4.69h2)$$

locally uniformly in v . The same statements hold after multiplication by any spatial phase, since that phase has modulus one. In particular, dominated convergence applied to the nonnegative majorant $C(v + v^2 + v^3 + v^4)e^{-cv^2}$ supplies a number $\varepsilon_n \downarrow 0$, independent of x , which bounds by $\varepsilon_n d$ the sum of: the compact part in (4.69g), the interpolant correction in (4.21), the $O(H^{-1})$ term in (4.69h), and the replacement of $\mathcal{P}_n$ by its Gaussian limit. Equations (4.69c), (4.69f)–(4.69h2), and $\pi^{-1} \int_0^\infty v^2 e^{-v^2/2} dv = \phi_0$ therefore bound the singular defect by

$$\{\Lambda_{\text{abs}}(p, d) + o(1)\}d. \quad (4.69i)$$

The bounded-kernel part of the defect is $O(d/\sqrt{n})$ in normalized score. Its change caused by $\lambda - s = O(d)$ is no larger; the matching smoothing change itself is not discarded, but is retained exactly as $\lambda/2$ in (4.69e). Every estimate above is unchanged after taking a supremum over x , because the only omitted spatial factor has modulus one.

It remains to justify that no other frequency cell contributes at the displayed scale. Choose $0 < \delta_1 < 1/4$ so small that (4.7) applies whenever $0 \leq t \leq 2\delta_1$, uniformly for all large rows. Near zero the resulting Gaussian envelope comes from (4.7), after $u = 2\pi t/\lambda$; it does not come from the quartic lower bound below. For $\delta_1 \leq t \leq 1 - \delta_1$, (4.6) and compactness of the bandwidth range give constants $q < 1$ and $C < \infty$ such that the whole integral is at most Cq^n in raw units. Finally, at $t = 1 - h$, $0 \leq h \leq \delta_1$, the same inequality gives $|\varphi_X(2\pi t/\lambda)|^n \leq e^{-cnh^2}$, while $|K(1-h)| \leq Ch$. Hence this endpoint has raw size $O(n^{-1})$ and normalized size $O(n^{-1/2})$. These bounds also ignore only a unit spatial phase. Combining the three frequency regions proves (4.69). Conjugation reverses the singular skewness sign and leaves all absolute defect estimates unchanged. $\square$

Lemma 7.17 (moment-band signed gap). *Let $X = X_n$ be a triangular array of centered variance-one laws, let $p = p_n$ be their matching Bernoulli parameters, and put $d = d_n = \mathbb{E}|X_n|^3 - \rho_2(p_n)$.*

Suppose $n \rightarrow \infty$, $p \rightarrow p_*$ and $d \rightarrow 0$, and let $\rho = \rho_2(p) + d$ and $\lambda = \rho + \mathbb{E}|X|$. Uniformly at every cdf coordinate $x \in \mathbb{R}$, the pointwise positive one-sided operator at physical bandwidth λ satisfies

$$\frac{\sqrt{n}}{\rho} \{\mathcal{U}_{X,n,\lambda}^+(x) - \Phi(x)\} \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d + O(n^{-1/2}). \quad (4.70a)$$

For the conjugate operator the same estimate holds with a strictly smaller leading constant: there is a fixed $c_- > 0$ such that its right side may be replaced by $C_E - c_- + o(1)$.

Proof. The three separate envelope sources are, explicitly,

frequency region	owner of the bound
$t \simeq 0$	(4.7), Gaussian in $\sqrt{n}t$,
$\delta \leq t \leq 1 - \delta$	(4.6), a fixed Cramér gap,
$t = 1 - h$	(4.6), Gaussian in $\sqrt{n}h$.

(4.70b)

In particular, the quartic behavior of the right side of (4.6) at zero is not used as a Gaussian estimate. Lemma 7.16, whose proof carries out the moving-bandwidth calculation, gives pointwise

$$\frac{\sqrt{n}}{\rho} \{\mathcal{U}_{X,n,\lambda}^+(x) - \Phi(x)\} \leq \frac{\phi_0}{\rho} \left(\frac{\lambda}{2} + \frac{\gamma}{6} \right) + \{\Lambda_{\text{abs},*} + o(1)\}d + O(n^{-1/2}). \quad (4.70d)$$

Here $\lambda > |\gamma|$ and the positive profile in (4.69) is maximized at $x = 0$, which justifies the replacement of $\phi(x)$ by ϕ_0 in (4.70d). Proposition 3.2 gives $\lambda \leq s + K_1(\gamma)d$. Thus the actual profile in (4.70d) is first bounded above by the numerator displayed below. Expanding that upper envelope with the true denominator, before replacing any parameter by its limit, gives

$$\frac{\phi_0 \{s/2 + \gamma/6 + (K_1(\gamma)/2)d\}}{\rho_2(p) + d} = A(p) - \{G_{\text{span},*} + o(1)\}d \leq C_E - \{G_{\text{span},*} + o(1)\}d.$$

The equality is simply the first-order identity

$$\frac{N_0(p) + N_1(p)d}{D_0(p) + d} = \frac{N_0(p)}{D_0(p)} - \frac{N_0(p)/D_0(p) - N_1(p)}{D_0(p)}d + O(d^2), \quad (4.70e)$$

used with

$$N_0(p) = \phi_0(s/2 + \gamma/6), \quad N_1(p) = \phi_0 K_1(\gamma)/2, \quad D_0(p) = \rho_2(p).$$

At $p = p_*$, $N_0/D_0 = C_E$ and the coefficient of d in (4.70e) is exactly $G_{\text{span},*}$; continuity makes the discrepancy $o(1)$ along the array. Thus no denominator has been replaced by its limit before the signed loss is extracted. The elementary bounds (4.22)–(4.23) give $G_{\text{span},*} - \Lambda_{\text{abs},*} > 19/200 > 0$, proving (4.70a). Reflection reverses the skewness term at the equality parameters and so leaves a strict leading gap. $\square$

Uniform signed local stability and completion

Proposition 7.18 (uniform signed local stability). *Let $n_j \rightarrow \infty$. Suppose that laws X_j satisfy*

$$p_j \rightarrow p_*, \quad d_j \rightarrow 0, \quad (LS0)$$

and that $\mathcal{C}_{n_j}(X_j) > C_E$ for every j . Choose a one-sided cdf point whose normalized score is still larger than C_E . After reflection and passage to a subsequence it lies in the positive central branch. Choose its unique half-open Bernoulli-cell representation

$$x_j = \frac{s(p_j)(\eta_j + \theta_j)}{\sqrt{n_j}}, \quad n_j p_j + \eta_j \in \mathbb{Z}, \quad -\frac{1}{2} \leq \theta_j < \frac{1}{2}. \quad (LS1)$$

It is impossible that the corresponding one-sided scores satisfy

$$\mathcal{C}_{n_j}^+(X_j; \eta_j, \theta_j) > C_E \quad (LS)$$

for every j .

Proof. If $d_j = 0$ on an infinite subsequence, Theorem 2.1 makes X_j exactly its matching two-point law there, and $C_Q^*(n_j) = C_E - \kappa_\infty/n_j + o(n_j^{-1}) < C_E$, a contradiction. After discarding finitely many rows we may therefore take $d_j > 0$; every later normalization by d_j is confined to these rows. Fix $c_{\text{gap}} > 0$ smaller than both positive margins in (4.24). Put $\tau_j = \sqrt{n_j}d_j$. The compactification $[0, +\infty]$ is sequentially compact, so every subsequence has a further subsequence on which either $\tau_j \rightarrow 0$, $\tau_j \rightarrow +\infty$, or $0 < c \leq \tau_j \leq C < \infty$. It is therefore enough to contradict (LS) on each of these three alternatives. In the first alternative the same argument, applied to the positive sequence $n_j d_j$, permits exactly the three further cases used below: convergence to zero, confinement to a compact subinterval of $(0, \infty)$, or convergence to infinity.

Suppose first that $\tau_j \rightarrow \infty$. Lemma 7.17 gives

$$C_{n_j}^+(X_j; \eta_j, \theta_j) \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_j + O(n_j^{-1/2}). \quad (4.71)$$

The coefficient is positive by (4.22)–(4.23), as recorded in the proof of (4.70a), and $n^{-1/2} = o(d_j)$, so (4.71) contradicts (LS).

Next suppose $\tau_j \rightarrow 0$. Pass again to a subsequence according to $n_j d_j$. Outer stability gives $\xi_j = \eta_j/\sqrt{n_j} \rightarrow 0$. Lemma 7.7, proved without (4.31)–(4.33), therefore gives

$$\eta_j^2 = O(n_j d_j + 1). \quad (4.72)$$

This is the non-circular owner of the tangent window.

If $n_j d_j \rightarrow 0$, (4.72) makes η_j bounded. Use Lemma 7.8 with any fixed D . The value/radial mode is nonpositive, the shape and low-frequency costs are $O(d_j) = o(n_j^{-1})$, and (4.43a) is $o(n_j^{-1})$. Therefore

$$C_{n_j}^+(X_j; \eta_j, \theta_j) \leq C_Q^*(n_j) + o(n_j^{-1}) = C_E - \frac{\kappa_\infty}{n_j} + o(n_j^{-1}) < C_E. \quad (4.73)$$

If $n_j d_j$ stays bounded away from both zero and infinity, choose a fixed D containing the subsequence. The complete node and phase have already been joined in the nonpositive square (4.43e), and (4.43f) gives

$$C_{n_j}^+(X_j; \eta_j, \theta_j) \leq C_Q^*(n_j) - \{c_{\text{gap}} + o(1)\}d_j + o(n_j^{-1}) < C_E \quad (4.74)$$

because its linear coefficient is positive and $d_j \asymp n_j^{-1}$.

It remains within $\tau_j \rightarrow 0$ that $n_j d_j \rightarrow \infty$. The window (4.72) is already available, so the paired growing-window conclusion (4.43g) applies. For reference, its normalized scale ledger is

$$\begin{aligned} \frac{\eta^4}{n^{3/2}} &= O(\sqrt{n} d^2) = o(d), & \frac{|\eta|^5}{n^2} &= O(\sqrt{n} d^{5/2}) = o(d), \\ \frac{1 + \eta^2}{n^{3/2}} &= o(d), & \frac{1 + |\eta|}{n} &= o(d), & \frac{1}{n} &= o(d). \end{aligned} \quad (4.77)$$

Indeed division of the first two bounds by d leaves respectively $\sqrt{n} d$ and $\sqrt{n} d^{3/2}$, both tending to zero. Equation (4.43g), with its coefficient converging to the positive first margin in (4.24), gives

$$\max\{C_{n_j}^+(X_j; \eta_j, \theta_j), C_{n_j}^-(X_j; \eta_j, \theta_j)\} \leq C_Q^*(n_j) - \{c_{\text{gap}} + o(1)\}d_j + o(d_j) < C_E. \quad (4.78)$$

This contradicts (LS) throughout the scale $\tau_j \rightarrow 0$.

Finally suppose that τ_j stays in a compact subinterval of $(0, \infty)$. Strict resonance curvature selects the critical maximum u_j used in Lemma 7.15. The selected powers belong to $[0, 1]$; pass once more so that they converge. If their limit is zero, Lemma 7.9, specifically the strict bound (4.43h), contradicts (LS). Pass therefore to a subsequence on which the power is bounded below. Proposition 7.1 implies that the scores tend to C_E and its active points tend to zero. Lemma 7.15

now performs the preliminary regression, invokes the upstream adaptive Lemma 7.10 and the cardinal gap, and only then reindexes to the $o(n_j^{-1})$ lattice tube. Its active-point transfer and Lemma 7.14 give

$$\mathcal{C}_{n_j}^+(X_j; \eta_j, \theta_j) \leq C_E - \{G_{\text{span},*} - \Lambda_{\text{abs},*} - o(1)\}d_j + O(n_j^{-1}) + o(d_j). \quad (4.79)$$

Here $n_j^{-1} = o(d_j)$ by (4.56), and the absolute span margin in (4.24) is positive. Thus (4.79) is strictly below C_E , the final contradiction. Since every putative counterexample sequence has a further subsequence in one of the alternatives just excluded, no such sequence exists. This proves the proposition. Reflection supplies the identical conclusion when the other one-sided error is competitive. $\square$

Proof of Theorem 2.6. If the assertion were false, for every j one could choose an integer $n_j \geq j$ and a centered standardized law X_j such that

$$\mathcal{C}_{n_j}(X_j) > C_E.$$

Proposition 7.1, after reflection and passage to a subsequence, supplies (LS0) and the positive central branch. Proposition 7.18 then gives a contradiction. This proves exactly

$$\boxed{\exists N_0 \forall n \geq N_0 : C_n \leq C_E.}$$

No numerical value of N_0 is asserted or needed. $\square$

8 Eventual two-point optimality and quantitative stability

8.1 The quantitative theorem

For a law P of mean zero, variance one and finite third absolute moment, put

$$\beta = \mathbb{E}|X|^3, \quad \gamma = \mathbb{E}X^3, \quad f(\gamma) = \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}}, \quad d(P) = \beta - f(\gamma), \quad q(P) = \frac{d(P)}{\beta}.$$

The moment-defect theorem of Part I gives $d \geq 0$, with equality exactly for the standardized two-point law Q_γ having third moment γ . In particular $0 \leq q < 1$. Let Φ and ϕ be the standard normal distribution function and density, and let $F_{n,P}(t) = P^{*n}((-\infty, t])$ be the distribution function of the unnormalized sum of n independent variables with law P . Write

$$R_n(P) = \frac{\sqrt{n}}{\beta(P)} \sup_t |F_{n,P}(t) - \Phi(t/\sqrt{n})|, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

$$C_n = \sup_P R_n(P), \quad B_n = \sup_{0 < p < 1} R_n(\text{Bernoulli}(p)).$$

The supremum defining C_n runs over standardized laws with finite third absolute moment. All moment parameters and R_n for an unstandardized nondegenerate law refer to its standardization.

Theorem 8.1. *There are constants $\theta > 0$ and $N < \infty$ such that*

$$R_n(P) + \theta \frac{d(P)}{\beta(P)} \leq B_n \quad (n \geq N) \quad (8.1)$$

for every standardized law P with finite third absolute moment. Consequently, for some $a > 0$ and all sufficiently large n ,

$$C_n - R_n(P) \geq a \min\{d(P), 1\}, \quad C_n = B_n < C_E. \quad (8.2)$$

One may take $\theta = 1/10000$ and $a = 1/18500$. Every maximizing law is two-point. Moreover,

$$C_n = C_E - \frac{\kappa_\infty}{n} + o(n^{-1}), \quad \kappa_\infty = \frac{867 + 287\sqrt{10}}{25920\sqrt{2\pi}}.$$

The inputs from Part I are the exact moment defect, the optimized two-point expansion $B_n = C_E - \kappa_\infty/n + o(n^{-1})$, and the uniform outer bound (4.10i)–(4.14). From He–Cheng [8] we use the sequential smoothing Lemma 2.2 and its Corollary 2.4, the local comparison Lemmas 3.3 and 3.5, and Gaussian cancellation Lemma 4.4. The contact argument is derived below for the penalized functional. Neither earlier eventual exactness theorem is assumed.

8.2 The outer estimate with a penalty

Let $D = 37/20$ and $\phi_0 = \phi(0)$. The structural estimate used in Part I gives, whenever $\beta \geq D$,

$$R_n(P) \leq h_* := \frac{1661}{5000} \left(1 + \frac{429}{1850} \right) < C_E. \quad (8.3)$$

For $\beta \leq D$, reflect the law to have $\gamma \geq 0$ and put

$$r = \frac{\gamma}{\sqrt{4 + \gamma^2}}, \quad u = \sqrt{1 - r^2}, \quad \rho_2(r) = \frac{1 + r^2}{u}.$$

$$H(r) = \frac{3 + r}{3u}, \quad L(r) = \frac{1}{(1 - r)(2 + r)} + \frac{1}{3(1 - r)^2(2 + r)}.$$

The compact parameter region is

$$\mathcal{K} = \{(r, d) : 0 \leq r \leq 2/3, \quad 0 \leq d \leq D - \rho_2(r)\}.$$

For any sequence with $n_j \rightarrow \infty$, $\beta_j \leq D$ and $(r_j, d_j) \rightarrow (r, d)$, the uniform Fourier estimate of Part I is

$$\limsup_j R_{n_j}(P_j) \leq \frac{\phi_0 H(r) + \phi_0 L(r) d}{\rho_2(r) + d}. \quad (8.4)$$

Its proof requires only the moment cap, not $R_n(P) > C_E$. The estimates of the nonzero frequency cells and of the divided-power remainder are uniform on this cap. Thus no uniform integrability of third moments is being assumed in (8.4).

At $d = 0$ the right side has unique maximum C_E at $r_E = \sqrt{10} - 3$. At $d = D - \rho_2(r)$ the scalar estimate (4.14) of Part I bounds it strictly by $41\phi_0/40$. Define

$$\mu = C_E - 41\phi_0/40 > 0.$$

For $0 < \theta < \mu$, the penalized scalar expression

$$\frac{\phi_0 H(r) + (\phi_0 L(r) + \theta) d}{\rho_2(r) + d} \quad (8.5)$$

is still at most C_E , with equality only at $(r_E, 0)$. Indeed it is fractional linear in d ; its lower endpoint is unchanged, and its upper endpoint is at most $41\phi_0/40 + \theta < C_E$.

Set

$$J_n(P) = R_n(P) + \theta q(P), \quad A_n = \sup_P J_n(P).$$

Choose also $\theta < C_E - h_*$. Since $J_n(P) \leq h_* + \theta < C_E$ when $\beta \geq D$, any asymptotically maximizing sequence for A_n has $\beta < D$: here $A_n \geq B_n \rightarrow C_E$. Applying (8.4) and (8.5) along convergent moment parameters proves

$$A_n \rightarrow C_E. \quad (8.6)$$

The same argument shows that if $n_j \rightarrow \infty$ and $A_{n_j} - J_{n_j}(P_j) \rightarrow 0$, then

$$d(P_j) \rightarrow 0, \quad |\gamma(P_j)| \rightarrow \gamma_E, \quad \beta(P_j) \rightarrow \beta_E, \quad R_{n_j}(P_j) \rightarrow C_E. \quad (8.7)$$

Here γ_E, β_E are the moments of the positive-skew Esseen law P_E .

8.3 Penalized extremizers

Lemma 8.2 (Attainment). *For all sufficiently large n , the value A_n is attained by a law P and a positive closed-endpoint discrepancy at some finite t .*

Proof. The penalty q is invariant under reflection, so the supremum also equals the supremum of the penalized positive signed discrepancy. Fix a large n with $A_n > h_* + \theta$ and $A_n > 2\theta$. Choose signed maximizing pairs (P_k, t_k) , with penalized signed values $j_k \rightarrow A_n$. After discarding finitely many terms, $\beta_k < D$ by (8.3). Their unpenalized signed values are at least $j_k - \theta$ and hence stay bounded away from zero. The same normal-tail and Chebyshev argument as in an ordinary extremal problem bounds t_k for this fixed n .

Pass to a subsequence with $P_k \Rightarrow P$, $t_k \rightarrow t$, $\beta_k \rightarrow b$ and $\gamma_k \rightarrow g$. Uniform integrability of squares preserves mean zero and variance one; let $\beta = \beta(P) \leq b$ and $\gamma = \gamma(P)$. Lower semicontinuity of $|x|^3 + x^3$ and $|x|^3 - x^3$ gives

$$|g - \gamma| \leq b - \beta. \quad (8.8)$$

The derivative

$$f'(v) = \frac{v(v^2 + 6)}{(v^2 + 4)^{3/2}}, \quad 1 - f'(v)^2 = \frac{4(3v^2 + 16)}{(v^2 + 4)^3} > 0$$

shows that f is 1-Lipschitz. Portmanteau at the moving closed threshold, followed by continuity of Φ , gives

$$\sqrt{n}\{F_{n,P}(t) - \Phi(t/\sqrt{n})\} \geq (A_n - \theta)b + \theta f(g).$$

The resulting signed penalized value at P, t is at least

$$\theta + \frac{(A_n - \theta)b + \theta f(g) - \theta f(\gamma)}{\beta} \geq A_n + (A_n - 2\theta)\frac{b - \beta}{\beta} \geq A_n.$$

Maximality forces $b = \beta$ and equality of the value with A_n . The limiting signed discrepancy is positive, since $A_n - \theta q(P) > 0$. This proves the claim. $\square$

At such a pair, write $S = A_n$, $R = R_n(P) = S - \theta q(P)$, $z = t/\sqrt{n}$, $M = \mathbb{E}X|X|$ and $G = F_{n-1,P}$. Under raw contamination by a point mass at y , followed by standardization, the moment derivatives are

$$U_\beta(y) = |y|^3 - \beta - 3My - \frac{3}{2}\beta(y^2 - 1), \quad U_\gamma(y) = y^3 - \gamma - 3y - \frac{3}{2}\gamma(y^2 - 1).$$

Differentiating the penalized signed discrepancy at the same raw threshold gives

$$\begin{aligned} \beta I(y) &= n^{3/2}\{G(t - y) - F_{n,P}(t)\} + ny\phi(z) + \frac{\sqrt{n}}{2}z\phi(z)(y^2 - 1) \\ &\quad - (S - \theta)U_\beta(y) - \theta f'(\gamma)U_\gamma(y). \end{aligned} \quad (8.9)$$

For clarity, the penalty derivative is $\beta\theta q' = \theta\{f(\gamma)U_\beta/\beta - f'(\gamma)U_\gamma\}$; combining it with $-RU_\beta$ yields (8.9). Maximality gives $I(y) \leq 0$ for every y . The integral of its right side under P is zero, since $\mathbb{E}U_\beta = \mathbb{E}U_\gamma = 0$ and the convolution term also integrates to zero. Upper semicontinuity then gives

$$I(y) = 0 \quad (y \in \text{supp } P). \quad (8.10)$$

No point mass at an individual support point is required.

8.4 Selection and support control

Suppose $A_n > B_n$ for arbitrarily large n . Apply harmonic selection to $a_n = A_n - B_n \geq 0$, which tends to zero. It gives $n_j \uparrow \infty$ with $a_{n_j} > 0$ and $n_j(a_{n_j-1} - a_{n_j})_+ \rightarrow 0$. This elementary selection follows by summing $a_{n-1} - a_n \geq \epsilon/n$ backwards along a hypothetical arbitrarily late run of positive terms. Writing $B_n = C_E - \kappa_\infty/n + \epsilon_n/n$, $\epsilon_n \rightarrow 0$, we have the exact identity

$$n(B_{n-1} - B_n) = -\frac{\kappa_\infty}{n-1} + \frac{n}{n-1}\epsilon_{n-1} - \epsilon_n \rightarrow 0.$$

Hence

$$A_{n_j} > B_{n_j}, \quad n_j(A_{n_j-1} - A_{n_j})_+ \rightarrow 0. \quad (8.11)$$

Choose positive penalized maximizing pairs by Lemma 8.2.

Suppress the index j . Put $\alpha_n = \sqrt{n/(n-1)}$ and let $H_n(z, y)$ denote the Gaussian cancellation expression

$$n^{3/2} \left\{ \Phi \left(\frac{z - y/\sqrt{n}}{\sqrt{1 - 1/n}} \right) - \Phi(z) \right\} + ny\phi(z) + \frac{\sqrt{n}}{2}z\phi(z)(y^2 - 1).$$

The definition of A_{n-1} , for the same law P , gives

$$G(t - y) - \Phi \left(\frac{z - y/\sqrt{n}}{\sqrt{1 - 1/n}} \right) \leq \frac{\beta(A_{n-1} - \theta q(P))}{\sqrt{n-1}}.$$

Together with the exact active value at n , this implies

$$\begin{aligned} \beta I(y) &\leq H_n(z, y) + \beta n(A_{n-1} - S) \\ &\quad + \beta n(\alpha_n - 1)(A_{n-1} - \theta q(P)) \\ &\quad - (S - \theta)U_\beta(y) - \theta f'(\gamma)U_\gamma(y). \end{aligned} \quad (8.12)$$

All moments in this expression are bounded on the selected sequence. The recursive positive part tends to zero and $n(\alpha_n - 1)$ is bounded. The Gaussian bound from [8, Lemma 4.4] gives $|H_n(z, y)| \leq \phi_0|y|^3/6 + C(1 + |y|)$. Choose $\theta < C_E/8$. Since $S \rightarrow C_E$ and $|f'| < 1$, the cubic coefficient in the upper bound is at most $-S + 2\theta + \phi_0/6$, bounded away from zero on the negative side. Thus (8.10) forces one common compact support.

By (8.7), $d_j \rightarrow 0$ and $R_j \rightarrow C_E$. Extract a W_3 -convergent subsequence using the common compact support. The moment-defect equality identifies its limit as the Esseen law or its reflection. The positive envelope in [8, Corollary 2.4] excludes the reflected law and forces $z_j \rightarrow 0$: the reflected envelope has strictly smaller supremum, while the positive-skew envelope has unique maximum at zero. Thus

$$P_j \rightarrow P_E \text{ in } W_3, \quad z_j \rightarrow 0.$$

Write $p_E = (4 - \sqrt{10})/2$, $q_E = 1 - p_E$, $\sigma_E = \sqrt{p_E q_E}$, $a_E = p_E/\sigma_E$, $b_E = q_E/\sigma_E$, $h_E = a_E + b_E$, and $M_E = q_E - p_E$. For any convergent support sequence $y_j \rightarrow y$, take upper limits in (8.12), using contact. The recursive positive part vanishes, $q_j \rightarrow 0$, and $n(\alpha_n - 1) \rightarrow 1/2$. We obtain

$$0 \leq V(y) + \theta w_E(y), \quad (8.13)$$

where

$$V(y) = \frac{\phi_0}{6}(y^3 - 3y) - C_E|y|^3 + \frac{3}{2}C_E\beta_E y^2 + 3C_E M_E y$$

and w_E is the exact defect polynomial at γ_E . Indeed the additional term is $U_{\beta,E}(y) - f'(\gamma_E)U_{\gamma,E}(y) = w_E(y)$; this follows either by the defect factorization or by direct expansion.

This perturbation preserves a sufficient support interval for small fixed θ . With $c_* = 3 + \sqrt{10}$, the identities $c_* M_E = 1$ and $c_* \beta_E = \sqrt{10}/\sigma_E$ give

$$V(y) = -\frac{\phi_0}{6}y^2 \left\{ (c_* - \operatorname{sgn} y)|y| - \frac{3}{2}c_*\beta_E \right\}.$$

Its nonzero roots are $-\sqrt{10}a_E/2$ and $\sqrt{10}b_E/2$. Under $v = p_E + \sigma_E y$, they lie strictly inside $[-1/2, 3/2]$. Therefore $V < 0$ on the closed set

$$\mathcal{O} = \{y : p_E + \sigma_E y \notin (-1/2, 3/2)\}.$$

Also $w_E > 0$ on this set. The ratio $-V/w_E$ is continuous there and has positive limits at both infinities:

$$\frac{C_E - \phi_0/6}{1 - f'(\gamma_E)}, \quad \frac{C_E + \phi_0/6}{1 + f'(\gamma_E)}.$$

Thus

$$\theta_* := \inf_{y \in \mathcal{O}} \frac{-V(y)}{w_E(y)} > 0. \quad (8.14)$$

Choosing $\theta < \theta_*$ makes (8.13) impossible on $\mathcal{O}$.

It remains to separate distinct support limits. The bounded polynomial terms in (8.9) give, uniformly on the support,

$$G_j(t_j - v) = F_{n_j, P_j}(t_j) - \phi(z_j)v/\sqrt{n_j} + O(n_j^{-1}).$$

Because the unpenalized active values R_j tend to C_E , each such residual threshold has the first-order discrepancy $C_E\beta_E$ at sample size $m_j = n_j - 1$. Apply [8, Lemma 2.2] to that convolution, with uniform jitter of width h_E . If J_j is its jittered CDF and $u_j = t_j - v_j$, this gives

$$\sqrt{m_j}\{J_j(u_j + h_E/2) - G_j(u_j)\} \rightarrow 0, \quad J_j(u_j + h_E/2) - G_j(u_j) = \frac{1}{h_E} \int_0^{h_E} [G_j(u_j + s) - G_j(u_j)] ds.$$

Consequently, for $\ell_j \rightarrow \ell \in (0, h_E)$, monotonicity gives $\sqrt{m_j}[G_j(u_j + \ell_j) - G_j(u_j)] \rightarrow 0$. Two support limits $x < y$ with $y - x < h_E$ would contradict this: subtracting their contact expansions makes the same increment tend to $\phi_0(y - x) > 0$. Distinct support limits are therefore separated by at least h_E .

In the coordinate $v = p_E + \sigma_E y$, all support limits lie in $(-1/2, 3/2)$ and are separated by at least one. Weak convergence supplies the limits zero and one. No other support limit is possible. The common compact support now gives

$$\sup_{x \in \text{supp } P_j} \text{dist}(x, \{-a_E, b_E\}) \rightarrow 0. \quad (8.15)$$

8.5 Absorption of the penalty

We record the local estimate that connects the moment defect to the conditional noise. Suppose $Y = B_p + U$, $\mathbb{E}(U \mid B_p) = 0$, $|U| \leq \varepsilon$, with p in a fixed sufficiently small compact interval about p_E . Put $s = \mathbb{E}U^2$, $v = p(1 - p)$ and $\tau = p^2 + (1 - p)^2$. For sufficiently small fixed ε , the signs of $Y - p$ on the two clusters are fixed, so

$$\mathbb{E}Y = p, \quad \text{Var } Y = v + s, \quad |\mathbb{E}|Y - p|^3 - v\tau| \leq Cs, \quad |\mathbb{E}(Y - p)^3 - v(1 - 2p)| \leq Cs.$$

The two last bounds follow by cubic expansion, conditional centering, and $\mathbb{E}|U|^3 \leq \varepsilon s$. Since v is bounded away from zero, standardization and $|f'| < 1$ give a constant $K < \infty$ with

$$0 \leq d(Y) \leq Ks, \quad q(Y) \leq Ks. \quad (8.16)$$

The second inequality uses the standardized bound $\beta(Y) \geq 1$.

From (8.15), recenter and rescale the conditional means of the two clusters. This writes an affine image of P_j as $Y_j = B_{p_j} + U_j$, where $p_j \rightarrow p_E$, $\mathbb{E}(U_j \mid B_{p_j}) = 0$ and $\|U_j\|_\infty \rightarrow 0$. Explicitly, if $Z_j = p_E + \sigma_E X_j$ and the two cluster radii are $\delta_j \rightarrow 0$, their conditional-mean difference lies in

$[1 - 2\delta_j, 1 + 2\delta_j]$ and the resulting noise is bounded by $2\delta_j/(1 - 2\delta_j)$. Affine invariance and (8.7) give $R_{n_j}(Y_j) \rightarrow C_E$. Lemma 3.5 of [8] therefore gives $\lambda_j = n_j \mathbb{E}U_j^2 \rightarrow 0$.

The proof of [8, Lemma 3.3, equation (38)] supplies fixed $k > 0$, $\lambda_0 > 0$, $\varepsilon_0 > 0$ and a sample-size threshold such that

$$R_n(B_p + U) \leq R_n(B_p) - kn\mathbb{E}U^2 \quad (8.17)$$

whenever its local hypotheses hold and $n\mathbb{E}U^2 \leq \lambda_0$. Indeed its loss is $\min\{c_0\lambda/\sqrt{C_0\lambda + \varepsilon_0^2}, \eta_0\}$, which is at least $k\lambda$ on $[0, \lambda_0]$ for $k = \min\{c_0/\sqrt{C_0\lambda_0 + \varepsilon_0^2}, \eta_0/\lambda_0\} > 0$. Shrink ε_0 and the parameter interval as necessary for (8.16). All these hypotheses hold eventually along the selected sequence. Combining the two estimates gives

$$A_{n_j} = J_{n_j}(Y_j) \leq R_{n_j}(B_{p_j}) - (kn_j - \theta K)\mathbb{E}U_j^2 \leq B_{n_j}$$

for all sufficiently large j , contradicting (8.11).

Choose once and for all

$$0 < \theta < \min\{\mu, C_E - h_*, C_E/8, \theta_*\}.$$

Every restriction used above is then satisfied. We have proved $A_n = B_n$ for all sufficiently large n , which is (8.1). In particular $C_n = B_n$, and $d = 0$ for any maximizing law. A binary maximizer exists: its score is continuous in p , tends to zero at $p = 0, 1$, and has a positive interior maximum. Schulz's strict inequality [6] gives $B_n < C_E$ for each fixed n . The two-point expansion proves the last assertion of Theorem 8.1.

Finally increase N so that $B_n \geq (C_E + h_*)/2$ for $n \geq N$. When $\beta \leq D$, (8.1) gives a loss at least $\theta d/D$. When $\beta > D$, (8.3) gives a loss at least $(C_E - h_*)/2$. Thus (8.2) holds with

$$a = \min\{\theta/D, (C_E - h_*)/2\} > 0.$$

This completes the proof.

8.6 An explicit choice of constants

We verify the constants stated in Theorem 8.1. In this section write $s = \sqrt{10}$. The elementary bounds $79/25 < s < 16/5$ and $\pi < 22/7$ give

$$\phi_0 > 997/2500 > 1/3, \quad C_E > \frac{76769}{187500} > \frac{2}{5}.$$

Indeed $44 \cdot 997^2 < 7 \cdot 2500^2$ and $(3 + s)/6 > 77/75$. Since $h_* = 3785419/9250000$, subtraction yields

$$C_E - h_* > \frac{1111}{5550000} > \frac{1}{5000}, \quad \mu = \phi_0 \frac{20s - 63}{120} > \frac{1}{1800}, \quad C_E/8 > \frac{1}{20}.$$

It remains to bound (8.14). Put $p = p_E$, $q = q_E$ and $\sigma = \sigma_E$. The moduli of the nonzero roots of V are $\ell_- = sa_E/2$ and $\ell_+ = sb_E/2$. On the negative component of $\mathcal{O}$, $|y| \geq (p + 1/2)/\sigma > 1$ and

$$\frac{\ell_-}{|y|} \leq \frac{s}{2} \frac{p}{p + 1/2} < \frac{4}{5} < \frac{7}{8}.$$

On the positive component, $y \geq (q + 1/2)/\sigma > 1$ and, since $q < 3/5$,

$$\frac{\ell_+}{y} \leq \frac{s}{2} \frac{q}{q + 1/2} < \frac{48}{55} < \frac{7}{8}.$$

The factorization of V , together with $c_* - \operatorname{sgn} y > 5$, now gives

$$-V(y) > \frac{5}{144}|y|^3 \quad (y \in \mathcal{O}).$$

For $S_E = \sqrt{\gamma_E^2 + 4} \geq 2$, the exact defect polynomial is

$$w_E(y) = |y|^3 - f'(\gamma_E)y^3 - \frac{12y^2}{S_E^3} + \frac{6\gamma_E y + 4}{S_E^3}.$$

Its two zeros are outside $\mathcal{O}$. Using $|f'| < 1$, $|\gamma_E| \leq S_E$ and $|y| > 1$ gives there

$$0 < w_E(y) \leq 2|y|^3 + \frac{3}{2}|y| + \frac{1}{2} \leq 4|y|^3.$$

Consequently $\theta_* \geq 5/576$. All four restrictions on θ hold for $\theta = 1/10000$, and

$$\min\{\theta/D, (C_E - h_*)/2\} = \theta/D = 1/18500.$$

This makes the coefficients explicit, but not the sample-size threshold.

8.7 Cubic transport stability

Lemma 8.3. *If $\beta(P) \leq 37/20$, then*

$$W_3(P, Q_\gamma)^3 \leq 1400 d(P). \quad (8.18)$$

Proof. Let $S = \sqrt{\gamma^2 + 4}$, $a = (S - \gamma)/2$, $b = (S + \gamma)/2$, $m = \min(a, b)$, $M = \max(a, b)$, and $A = S^3/(4m^2)$. Round X by sign to $Y \in \{-a, b\}$. The defect factorization in Part I is

$$w_\gamma(x) = \frac{2a^2}{S^3}(x - b)^2\{2 + (a + 3b)x\} \quad (x \geq 0),$$

with a, b interchanged on the negative half-line; $\mathbb{E}w_\gamma(X) = d$. Since $ab = 1$, $M \geq 1$ and $a + 3b \geq 2\sqrt{3}$,

$$\frac{x + b}{2 + (a + 3b)x} \leq M/2 \quad (x \geq 0).$$

The same bound holds on the other half-line. Consequently

$$\mathbb{E}|X - Y|^3 \leq AMd.$$

To correct the mass of Y , use the cubic $H(x) = 3t^2 - 2t^3$, $t = (x + a)/S$. Moment matching gives $\mathbb{E}H(X) = a/S$. Direct factorization yields

$$|1_{\{x \geq 0\}} - H(x)| \leq \frac{3S}{4m^2} w_\gamma(x).$$

For example, on $x \geq 0$ the ratio is $(2x + 3a + b)/(2a^2\{(a + 3b)x + 2\})$, bounded by $3S/(4m^2)$; the negative half-line is symmetric. Thus the mass correction needed to turn Y into Q_γ is at most $3Sd/(4m^2)$. Move exactly that mass between the two nodes and call the corrected variable Y_0 . Gluing the coupling gives

$$\mathbb{E}|Y - Y_0|^3 \leq 3SAd, \quad \mathbb{E}|X - Y_0|^3 \leq 4A(M + 3S)d.$$

Here $|\gamma| \leq \beta \leq 37/20$ implies $S < 11/4$, $M < 5/2$, $m > 2/5$, and $A < 33275/1024$. Therefore

$$4A(M + 3S) < \frac{1430825}{1024} < 1400.$$

Taking the infimum over couplings proves (8.18). $\square$

Corollary 8.4. *There is $N < \infty$ such that, for every $n \geq N$ and every standardized law P with finite third absolute moment, if $\delta = C_n - R_n(P) < 1/10000$, then*

$$\beta(P) < 37/20, \quad d(P) \leq 18500\delta, \quad W_3(P, Q_{\gamma(P)})^3 \leq 259000000\delta.$$

In particular $W_3(P, Q_{\gamma(P)}) \leq 300\delta^{1/3}$. No bounded-support or fourth-moment hypothesis is imposed.

Proof. Increase N so that $B_n \geq (C_E + h_*)/2$. Equation (8.3) excludes $\beta \geq 37/20$ when $\delta < 1/10000$, since $(C_E - h_*)/2 > 1/10000$. Then (8.1) gives $d \leq 18500\delta$; apply Lemma 8.3 and $1400 \cdot 18500 = 25900000 < 300^3$. $\square$

The comparison law in this statement is the matching two-point law, not necessarily the optimizing Bernoulli parameter at that n . The conclusion controls moment-weighted transport, not the Hausdorff distance of supports. An arbitrarily small remote atom prevents such a support-distance statement. Only the threshold N remains qualitative; the inequality for every sample size remains open.

8.8 Sharpness of the stability exponents

Let $\mathcal{Q}_2$ be the class of probability laws supported on at most two points. The exponent $1/3$ cannot be increased, even if the comparison law is freely chosen in $\mathcal{Q}_2$.

Theorem 8.5. *For each fixed sufficiently large n , there are standardized three-point laws P_ε , with uniformly bounded supports, such that as $\varepsilon \downarrow 0$,*

$$d(P_\varepsilon) \asymp \varepsilon, \quad C_n - R_n(P_\varepsilon) \asymp \varepsilon, \\ \inf_{Q' \in \mathcal{Q}_2} W_3(P_\varepsilon, Q') \asymp \varepsilon^{1/3}.$$

The comparison constants may depend on n . Thus neither the linear defect loss nor the cubic transport exponent admits a stronger power uniformly near the maximizing class, even at one such fixed index.

Proof. Fix a standardized binary maximizer Q at n and a point y outside its two-point support. Let P_ε be the standardization of

$$Z_\varepsilon \sim (1 - \varepsilon)Q + \varepsilon\delta_y.$$

After reflecting Q and y together if necessary, Q has an active positive closed threshold t_0 for its unnormalized sum. A negative active left limit becomes precisely such a threshold under reflection. At the same raw threshold t_0 , the distribution functions of the n -fold sums of Z_ε and Q differ by at most $n\varepsilon$. Writing $m_\varepsilon = \varepsilon y$ and $\sigma_\varepsilon^2 = 1 + \varepsilon(y^2 - 1) - \varepsilon^2 y^2$, the corresponding standardized threshold is $(t_0 - nm_\varepsilon)/\sigma_\varepsilon$. It changes the normal term by $O_n(\varepsilon)$, and $\beta(P_\varepsilon) = \beta(Q) + O(\varepsilon)$. Evaluating the signed score there gives

$$R_n(P_\varepsilon) \geq C_n - O_n(\varepsilon).$$

The moment derivatives used in (8.9), at a two-point equality law, give

$$d(P_\varepsilon) = \varepsilon w_{\gamma(Q)}(y) + O(\varepsilon^2), \quad w_{\gamma(Q)}(y) > 0.$$

Equation (8.1) supplies the matching positive lower bound for $C_n - R_n(P_\varepsilon)$.

The three support points of P_ε have a common positive separation, and each has mass at least ε for small ε . Choose disjoint balls of a fixed radius $r > 0$ around these points. Any law in $\mathcal{Q}_2$ omits at least one of the balls; every coupling therefore has cubic cost at least $r^3\varepsilon$. Conversely, the cap estimate gives $\beta(Q) < 37/20$ for large n , and hence $\beta(P_\varepsilon) < 37/20$ for small ε . Lemma 8.3, with the matching law $Q_{\gamma(P_\varepsilon)}$, gives a cubic cost $O(\varepsilon)$. This proves all three comparisons. $\square$

9 The oscillatory second-order correction

9.1 Set-up and statement

Let $0 < p \leq 1/2$, $q = 1 - p$, and let Q_p be the standardized two-point law which takes the values $\sqrt{q/p}$ and $-\sqrt{p/q}$ with probabilities p and q . If $K \sim \text{Bin}(n, p)$, then

$$\frac{Q_{p,1} + \cdots + Q_{p,n}}{\sqrt{n}} = \frac{K - np}{\sqrt{npq}}, \quad \rho(p) = \frac{p^2 + q^2}{\sqrt{pq}}.$$

For an integer k , put

$$\eta = k - np, \quad B_{n,p}(k) = \mathbb{P}\{K \leq k\},$$

and define the normalized positive right-edge score

$$S_n(p, \eta) := \frac{\sqrt{n}}{\rho(p)} \left\{ B_{n,p}(k) - \Phi \left(\frac{\eta}{\sqrt{npq}} \right) \right\}. \quad (9.1)$$

The notation presupposes that $k = np + \eta$ is an integer.

Set

$$\begin{aligned} \phi_0 &= (2\pi)^{-1/2}, \quad D = p^2 + q^2, \\ G(p) &= \phi_0 \frac{2-p}{3D}, \end{aligned} \quad (9.2)$$

and

$$F(p, \eta) = -\frac{\phi_0}{540pqD} \{270q\eta^2 + 45(p^2 - 7p + 5)\eta + p^3 + 21p^2 - 69p + 46\}. \quad (9.3)$$

The first-order constants from Section 6 are

$$p_* = \frac{4 - \sqrt{10}}{2}, \quad \eta_* = -\frac{20 + \sqrt{10}}{72}, \quad C_E = G(p_*) = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}}, \quad (9.4)$$

$$\kappa_\infty = \frac{867 + 287\sqrt{10}}{25920\sqrt{2\pi}}, \quad F(p_*, \eta_*) = -\kappa_\infty. \quad (9.5)$$

The new constants are

$$H_* = -\phi_0 \frac{653866157 + 211759711\sqrt{10}}{25395793920}, \quad (9.6)$$

$$F_{p,*} := \partial_p F(p_*, \eta_*) = -\phi_0 \frac{1483 + 463\sqrt{10}}{19440}, \quad (9.7)$$

$$G_*'' := G''(p_*) = -\phi_0 \frac{60 + 19\sqrt{10}}{15}, \quad (9.8)$$

$$\delta_0 := -\frac{F_{p,*}}{G_*''} = \frac{1010 - 397\sqrt{10}}{12960}, \quad (9.9)$$

$$A_2 := -\frac{1}{2}G_*'' = \phi_0 \frac{60 + 19\sqrt{10}}{30}, \quad (9.10)$$

$$\kappa_2 := -H_* + \frac{F_{p,*}^2}{2G_*''} = \phi_0 \frac{3183580225 + 1028276063\sqrt{10}}{126978969600}. \quad (9.11)$$

Numerically,

$$\begin{aligned} \delta_0 &= -0.0189370548678122\dots, \\ \kappa_2 &= 0.0202183334057192\dots, \\ A_2 &= 1.59687652610925\dots, \\ G_*'' &= -3.19375305221850\dots \end{aligned}$$

Theorem 9.1 (second-order lattice oscillation). *Let $C_Q^*(n)$ be the optimal fixed- n Berry–Esseen constant restricted to standardized two-point laws, with the normalization of Section 6. Define*

$$\omega_n := \text{dist}(np_* + \eta_* + \delta_0, \mathbb{Z}) \in [0, 1/2]. \quad (9.12)$$

Then

$$C_Q^*(n) = C_E - \frac{\kappa_\infty}{n} - \frac{1}{n^2} \{ \kappa_2 + A_2 \omega_n^2 \} + o(n^{-2}). \quad (9.13)$$

The $o(n^{-2})$ term is uniform over the compact local parameter family associated with the finitely many central-edge types retained by Schulz’s localization theorem.

Because $A_2 > 0$ and $\kappa_2 > 0$, the second correction is always strictly negative. It varies between approximately $-0.0202/n^2$ and $-0.4194/n^2$.

9.2 The fifth-order central saddle

The main calculation is a one-order extension of the incomplete-beta saddle in the second note.

Lemma 9.2 (uniform fifth-order saddle). *Fix a compact interval $U \Subset (0, 1)$ and $M < \infty$. Uniformly for $p \in U$, $|\eta| \leq M$, and integers $k = np + \eta$,*

$$S_n(p, \eta) = G(p) + \frac{F(p, \eta)}{n} + \frac{H(p, \eta)}{n^2} + o_{U,M}(n^{-2}), \quad (9.14)$$

where

$$H(p, \eta) = \frac{\phi_0}{30240p^2q^2D} P(p, \eta) \quad (9.15)$$

and

$$\begin{aligned} P(p, \eta) = & 1260(4 - 5p)\eta^4 + 420(25p^2 - 55p + 28)\eta^3 \\ & + 1260(-p^3 + 14p^2 - 21p + 9)\eta^2 \\ & + 21(5p^4 - 70p^3 + 355p^2 - 470p + 189)\eta \\ & + 5(25p^5 - 52p^4 - 17p^3 + 88p^2 - 115p + 46). \end{aligned} \quad (9.16)$$

One may replace the little-oh in (9.14) by $O_{U,M}(n^{-181/80})$.

Proof. Write $\varepsilon = n^{-1/2}$ and $r = \sqrt{pq}$. The incomplete-beta identity is

$$B_{n,p}(k) = \frac{\int_0^q t^{n-k-1}(1-t)^k dt}{\int_0^1 t^{n-k-1}(1-t)^k dt}.$$

With $t = q + \varepsilon ry$, $A = p/r$ and $B = q/r$, the common multiplicative constant cancels and the logarithm of the remaining density is

$$L(y) = (nq - \eta - 1) \log(1 + \varepsilon Ay) + (np + \eta) \log(1 - \varepsilon By). \quad (9.17)$$

Expanding the logarithms gives

$$L(y) = -\frac{y^2}{2} + \sum_{j \geq 1} \varepsilon^j \ell_j(y), \quad \ell_j(y) = c_j y^{j+2} + d_j y^j, \quad (9.18)$$

where

$$c_j = \frac{q(-1)^{j+3}A^{j+2} - pB^{j+2}}{j+2}, \quad d_j = \frac{-(\eta+1)(-1)^{j+1}A^j - \eta B^j}{j}.$$

Define $E_0 = 1$ and

$$E_m = \frac{1}{m} \sum_{j=1}^m j \ell_j E_{m-j}. \quad (9.19)$$

Then

$$\exp \left(\sum_{j \geq 1} \varepsilon^j \ell_j \right) \sim \sum_{m \geq 0} \varepsilon^m E_m.$$

Since $\ell_j(-y) = (-1)^j \ell_j(y)$, recursion (9.19) gives $E_m(-y) = (-1)^m E_m(y)$.

For a uniform remainder, restrict first to $|y| \leq R_\varepsilon := \varepsilon^{-1/40}$. On U and for bounded η , Taylor's formula for the two logarithms gives the uniform remainder

$$O_{U,M}(\varepsilon^6 R_\varepsilon^8)$$

after ℓ_5 . The part omitted after total ε -degree five in the exponential is

$$O_{U,M}(\varepsilon^6 R_\varepsilon^{18}).$$

After integration over the central interval this is, conservatively,

$$O_{U,M}(\varepsilon^6 R_\varepsilon^{19}) = O_{U,M}(\varepsilon^{221/40}) = o(\varepsilon^5).$$

Here is a uniform tail justification. Put

$$\alpha = \min\{\inf U, 1 - \sup U\} > 0, \quad t = q + \varepsilon r y.$$

On the beta integration interval $0 < t < 1$, exact differentiation of (9.17) gives

$$L''(y) = -pq \left\{ \frac{q - \varepsilon^2(\eta + 1)}{t^2} + \frac{p + \varepsilon^2 \eta}{(1-t)^2} \right\}. \quad (9.20)$$

For all sufficiently small ε , the two coefficients in braces are at least $\alpha/2$. Since $t^{-2} + (1-t)^{-2} \geq 8$ and $pq \geq \alpha^2$, it follows that

$$L''(y) \leq -4\alpha^3 =: -c_U$$

throughout the whole beta interval. Moreover, cancellation of the ε^{-1} terms gives

$$L'(0) = -\varepsilon\{(\eta + 1)A + \eta B\} = O_{U,M}(\varepsilon).$$

If y_0 is the unique maximizer of L , then $|y_0| = O_{U,M}(\varepsilon)$ and

$$L(y) \leq O_{U,M}(\varepsilon^2) - \frac{c_U}{2}(y - y_0)^2.$$

Consequently the beta tails outside $[-R_\varepsilon, R_\varepsilon]$ are $O_{U,M}(\exp(-c_U R_\varepsilon^2/8))$, hence smaller than every power of ε . The polynomials $E_0, \dots, E_5$ have uniformly bounded coefficients and degree at most 15, so their Gaussian tails satisfy the same conclusion. Thus the exact denominator and numerator have the expansions

$$\mathcal{D} = D_0 + \varepsilon^2 D_2 + \varepsilon^4 D_4 + o_{U,M}(\varepsilon^5), \quad (9.21)$$

$$\mathcal{N} = \frac{1}{2}(D_0 + \varepsilon^2 D_2 + \varepsilon^4 D_4) + \varepsilon A_1 + \varepsilon^3 A_3 + \varepsilon^5 A_5 + o_{U,M}(\varepsilon^5), \quad (9.22)$$

where

$$D_m = \int_{\mathbb{R}} e^{-y^2/2} E_m(y) dy, \quad A_m = \int_{-\infty}^0 e^{-y^2/2} E_m(y) dy, \quad D_0 = \sqrt{2\pi}.$$

Put $\delta_j = D_j/D_0$. Division of (9.22) by (9.21) yields

$$B_{n,p}(k) = \frac{1}{2} + \phi_0 \varepsilon A_1 + \phi_0 \varepsilon^3 (A_3 - A_1 \delta_2) + \phi_0 \varepsilon^5 \{A_5 - A_3 \delta_2 + A_1 (\delta_2^2 - \delta_4)\} + o_{U,M}(\varepsilon^5). \quad (9.23)$$

Here A_1, A_3, A_5 denote the raw half-Gaussian integrals above; the factor $D_0^{-1} = \phi_0$ is displayed explicitly.

The normal cdf contributes

$$\Phi(\eta \varepsilon / r) = \frac{1}{2} + \phi_0 \frac{\eta \varepsilon}{r} - \phi_0 \frac{\eta^3 \varepsilon^3}{6r^3} + \phi_0 \frac{\eta^5 \varepsilon^5}{40r^5} + O_{U,M}(\varepsilon^7). \quad (9.24)$$

The coefficients of ε and ε^3 reproduce (9.2) and (9.3). For the next coefficient, set $\tilde{\ell}_j = r^j \ell_j$ and $\tilde{E}_m = r^m E_m$. If

$$\tilde{A}_m = \int_{-\infty}^0 e^{-y^2/2} \tilde{E}_m(y) dy, \quad \tilde{\delta}_m = \frac{1}{D_0} \int_{\mathbb{R}} e^{-y^2/2} \tilde{E}_m(y) dy,$$

then exact Gaussian integration gives the coefficient ledger in Appendix B. In particular,

$$\frac{H(p, \eta)}{\phi_0} = \frac{\tilde{A}_5 - \tilde{A}_3 \tilde{\delta}_2 + \tilde{A}_1 (\tilde{\delta}_2^2 - \tilde{\delta}_4) - \eta^5 / 40}{D r^4}. \quad (9.25)$$

Substitution simplifies exactly to (9.15)–(9.16). Notice that the apparent fifth power of η cancels identically.

The raw quotient error is $O(\varepsilon^{221/40})$. Dividing by $\rho(p)\varepsilon$, with ρ uniformly bounded away from zero on U , gives $O(\varepsilon^{181/40}) = O(n^{-181/80})$. This proves the lemma. $\square$

9.3 Optimization with the integer node retained

For later use define

$$\eta_0(p) = -\frac{p^2 - 7p + 5}{12(1-p)}.$$

Completing the square in (9.3) gives the exact identity

$$F(p, \eta) = F(p, \eta_0(p)) - \frac{\phi_0}{2p^{3/2}q^{1/2}\rho(p)} \{\eta - \eta_0(p)\}^2. \quad (9.26)$$

In particular, $\eta_0(p_*) = \eta_*$.

Proof of Theorem 9.1. The binomial localization theorem used in the second note reduces the global two-point supremum, up to alternatives separated by a fixed gap from C_E , to the four central jump edges. Reflection and the strict gaps in their leading coefficients leave only positive right edges with p in a fixed neighborhood of p_* . Their lattice coordinate satisfies $|\eta| \leq 1$. Lemma 9.2 is therefore uniform over every candidate that can be asymptotically optimal.

For the lower bound, let k_n be an integer nearest to

$$np_* + \eta_* + \delta_0$$

and put

$$p_n = \frac{k_n - \eta_*}{n}.$$

Then $k_n = np_n + \eta_*$ and

$$u_n := n(p_n - p_*) = k_n - (np_* + \eta_*).$$

Taylor expansion in Lemma 9.2 gives

$$S_n(p_n, \eta_*) = C_E - \frac{\kappa_\infty}{n} + \frac{1}{n^2} \left\{ H_* + F_{p,*} u_n + \frac{1}{2} G''_* u_n^2 \right\} + o(n^{-2}). \quad (9.27)$$

By (9.9)–(9.11),

$$H_* + F_{p,*} u + \frac{1}{2} G''_* u^2 = -\kappa_2 - A_2(u - \delta_0)^2. \quad (9.28)$$

The choice of k_n makes $|u_n - \delta_0| = \omega_n$, proving the lower bound in (9.13).

For the upper bound, choose a central right-edge candidate within $o(n^{-2})$ of $C_Q^*(n)$ and denote its parameters by p_n, η_n, k_n . The preceding lower bound and the strict quadratic maximum of G first give

$$p_n - p_* = O(n^{-1/2}).$$

Using the F/n term and the exact concavity in (9.26) improves this to

$$p_n - p_* = O(n^{-1}), \quad \eta_n - \eta_* = O(n^{-1/2}). \quad (9.29)$$

Indeed, comparison with the lower bound gives, for fixed positive c, b and some C ,

$$c(p_n - p_*)^2 \leq \frac{C|p_n - p_*|}{n} + \frac{C}{n^2},$$

and then

$$\frac{b}{n} \{\eta_n - \eta_0(p_n)\}^2 \leq \frac{C}{n^2}.$$

Put

$$u_n = n(p_n - p_*), \quad d_n = k_n - (np_* + \eta_*).$$

By (9.29), u_n and d_n are bounded and $d_n - u_n = \eta_n - \eta_* = o(1)$. Applying Lemma 9.2, using (9.26) to discard its nonpositive square, and expanding at p_* gives

$$\begin{aligned} C_Q^*(n) &\leq C_E - \frac{\kappa_\infty}{n} + \frac{1}{n^2} \left\{ H_* + F_{p,*} u_n + \frac{1}{2} G''_* u_n^2 \right\} + o(n^{-2}) \\ &= C_E - \frac{\kappa_\infty}{n} - \frac{1}{n^2} \{ \kappa_2 + A_2(d_n - \delta_0)^2 \} + o(n^{-2}). \end{aligned}$$

Since $d_n \in \mathbb{Z} - (np_* + \eta_*)$ and $G''_* < 0$,

$$-\kappa_2 - A_2(d_n - \delta_0)^2 \leq -\kappa_2 - A_2\omega_n^2.$$

This proves the matching upper bound. □

9.4 The full optimal constants

Corollary 9.3. *With the constants in (9.5), (9.9), (9.10) and (9.11), put*

$$\omega_n = \text{dist}(np_* + \eta_* + \delta_0, \mathbb{Z}).$$

Then

$$C_n = C_E - \frac{\kappa_\infty}{n} - \frac{\kappa_2 + A_2\omega_n^2}{n^2} + o(n^{-2}).$$

The complete set of subsequential limits of $n^2(C_n - C_E + \kappa_\infty/n)$ is

$$\left[-\kappa_2 - \frac{A_2}{4}, -\kappa_2 \right].$$

In particular this sequence has no limit, and eventually

$$C_n \leq C_E - \frac{\kappa_\infty}{n} - \frac{\kappa_2}{2n^2}.$$

Proof. Theorem 8.1 gives $C_n = C_Q^*(n)$ for all sufficiently large n . Substitute this equality into Theorem 9.1. Irrationality of p_* makes $np_* + \eta_* + \delta_0$ dense modulo one; consequently the limit set of ω_n^2 is $[0, 1/4]$. Finally $A_2\omega_n^2 \geq 0$ and $\kappa_2 > 0$ absorb the $o(n^{-2})$ remainder into $\kappa_2/(2n^2)$. □

10 Proper convolution roots at every index

For a probability law P with mean zero, variance one and finite third absolute moment, set

$$\beta(P) = \int |x|^3 P(dx), \quad R_n(P) = \frac{\sqrt{n}}{\beta(P)} \sup_x |P^{*n}((-\infty, \sqrt{n}x]) - \Phi(x)|.$$

Write $T_k(Q)$ for the law of $k^{-1/2}(X_1 + \cdots + X_k)$, where the X_i are independent with standardized common law Q . Recall

$$C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

Theorem 10.1. *Suppose $P = T_k(Q)$ for an integer $k \geq 2$. Then, for every $n \geq 1$,*

$$R_n(P) \leq 0.3031 \frac{\beta(Q) + 0.646}{\beta(Q) + \frac{3(k-1)}{2k\beta(Q)^3}} < \frac{39}{100} < C_E. \quad (10.1)$$

The conclusion applies whenever P is a standardized nonzero affine image of a sum of k iid nondegenerate random variables. Finiteness of the third absolute moment of P suffices; no moment assumption on the root is needed separately.

The only Berry–Esseen estimate used below is Shevtsova’s structural bound [7], in the normalization recorded in [8, Eq. (6)]:

$$R_j(Q) \leq 0.3031 \left(1 + \frac{0.646}{\beta(Q)} \right), \quad j \geq 1. \quad (10.2)$$

All terminating decimals denote exact rational constants.

Lemma 10.2. *Let A, B be independent centered real random variables with finite third absolute moments. Put $u = \mathbb{E}|A|$ and $v = \mathbb{E}|B|$. If $u + v > 0$, then*

$$\mathbb{E}|A + B|^3 - \mathbb{E}|A|^3 - \mathbb{E}|B|^3 \geq \frac{3u^2v^2}{u + v}. \quad (10.3)$$

Proof. Define

$$g(x, y) = |x + y|^3 - |x|^3 - |y|^3 - 3x|x|y - 3y|y|x.$$

This is zero for $xy \geq 0$. For opposite signs, put $a = \max\{|x|, |y|\}$ and $b = \min\{|x|, |y|\}$. Then

$$g(x, y) = 2b^2(3a - b) \geq \frac{6a^2b^2}{a + b},$$

since the difference multiplied by $a + b$ is $2b^3(2a - b) \geq 0$. The mixed expectations in $g(A, B)$ vanish. With $D = \{AB < 0\}$, centering and independence give

$$\mathbb{E}[|AB|1_D] = \frac{uv}{2}, \quad \mathbb{E}[(|A| + |B|)1_D] = \frac{u}{2}\mathbb{P}(B \neq 0) + \frac{v}{2}\mathbb{P}(A \neq 0) \leq \frac{u + v}{2}.$$

For $uv > 0$, Cauchy–Schwarz yields

$$\mathbb{E}g(A, B) \geq 6 \frac{(\mathbb{E}[|AB|1_D])^2}{\mathbb{E}[(|A| + |B|)1_D]} \geq \frac{3u^2v^2}{u + v}.$$

The case $uv = 0$ is immediate. □

Lemma 10.3. *For iid standardized X_i with $\beta = \mathbb{E}|X_1|^3 < \infty$, put $M_k = \mathbb{E}|X_1 + \cdots + X_k|^3$. Then*

$$M_k \geq k\beta + \frac{3(k-1)}{2\beta^3}, \quad k \geq 1. \quad (10.4)$$

Proof. Let $m = \mathbb{E}|X_1|$. Cauchy–Schwarz gives $m\beta \geq 1$. Conditional Jensen gives $\mathbb{E}|X_1 + \cdots + X_j| \geq m$. For $u \geq m$,

$$\frac{3u^2m^2}{u+m} \geq \frac{3}{2}m^3.$$

Apply Lemma 10.2 to the partial sum and the next independent summand. Every increment $M_{j+1} - M_j - \beta$ is at least $3m^3/2 \geq 3/(2\beta^3)$. Summation proves the claim. $\square$

Proof of Theorem 10.1. First consider a possibly unnormalized iid root $Y_1, \dots, Y_k$ whose sum S has a finite third absolute moment. Put $T = Y_2 + \cdots + Y_k$ and choose $L < \infty$ with $r = \mathbb{P}(|T| \leq L) > 0$. Independence and Tonelli give

$$r\mathbb{E}|Y_1|^3 = \mathbb{E}[|Y_1|^3 1_{\{|T| \leq L\}}] \leq 4\mathbb{E}|S|^3 + 4L^3r < \infty.$$

Thus the root has finite mean and variance. Its variance is positive, and centering, rescaling and, if needed, reflection give $P = T_k(Q)$.

Write $\beta = \beta(Q)$. Grouping nk iid root summands into n blocks gives the exact identities

$$\beta(P) = \frac{M_k}{k^{3/2}}, \quad R_n(P) = \frac{k\beta}{M_k} R_{nk}(Q). \quad (10.5)$$

Equations (10.2) and (10.4) give the first inequality in (10.1). Since $3(k-1)/(2k) \geq 3/4$, it remains to bound

$$0.3031 \frac{\beta + 0.646}{\beta + 3/(4\beta^3)}.$$

This is less than 0.39 for every $\beta > 0$: after multiplying by the denominator, the assertion is

$$0.0869\beta + 0.2925\beta^{-3} > 0.1958026.$$

Indeed,

$$0.0869\beta + 0.2925\beta^{-3} > \frac{3\beta}{35} + \frac{2}{7\beta^3} \geq 4 \left(\frac{2}{7 \cdot 35^3} \right)^{1/4} > \frac{1}{5} > 0.1958026.$$

The penultimate comparison is equivalent to $320000 > 300125$. Finally $\sqrt{10} > 79/25$ and $\pi < 22/7$ give $C_E > 76769/187500 > 2/5 > 39/100$. $\square$

Corollary 10.4. *For $P = T_k(Q)$ and every $n \geq 1$,*

$$R_n(P) \leq 0.3031 + \frac{0.1958026}{\sqrt{k}\beta(P)}. \quad (10.6)$$

If P is infinitely divisible, then $R_n(P) \leq 0.3031$ for every $n \geq 1$.

Proof. The weaker inequality $M_k \geq k\beta(Q)$ implies $\beta(Q) \leq \sqrt{k}\beta(P)$. Use (10.2) in (10.5) to obtain (10.6). Infinite divisibility provides a root for every k . The preceding moment argument applies to each root; P and n remain fixed. Let $k \rightarrow \infty$. $\square$

Part II

Effective bounds and finite certificates

11 The conditional refinement to twenty

The new finite continuation [11] treats the actual indices 20, 21, 22, 23, 24. It is joined to the retained estimates at 25, $\dots$, 30 and the theorem from 31 onward proved in Section 12. We state its inherited premises explicitly. The completed finite calculation does not by itself reconstruct every earlier analytic or computational dependency.

Assumption 11.1 (Retained inputs for the finite continuation). Let $\mathcal{A}_{20}$ denote the conjunction of the following inputs, at the precise domains recorded in [11].

- (i) At each actual index $20 \leq n \leq 24$, a contrary inequality $C_n \geq C_E$ supplies an attained unrestricted global maximizing law and a positive closed threshold, after reflection if necessary. The same-law contact, support, Fourier and conditional count implications used in its resource acquisition are valid.
- (ii) The inherited support and resource calculations for the low-moment bands and the Fourier exclusions for $\rho \geq 9/8$ are valid, with their original contact allowances, moment domains and normalizations. For $21 \leq n \leq 24$, the independent auxiliary ceilings are $C_n < 419/1000$. The actual-index count inequalities convert the finite coefficient bounds into upper bounds for that same witness.
- (iii) The retained continuation gives $C_n < C_E$ at each of $n = 25, \dots, 30$.

This assumption retains the analytic and historical certificate premises of the supplied continuation; it is not an assertion that all of them have been independently reconstructed here. The result for $n \geq 31$ below is used separately.

Theorem 11.2 (Conditional cutoff twenty). *Under Assumption 11.1,*

$$C_n < C_E \quad \text{for every integer } n \geq 20.$$

The finite continuation is complete at all five actual indices 20, $\dots$, 24. This conclusion does not assert two-point optimality from index 20 onward.

11.1 The moment price and the coupled noise budget

Here are the elementary inequalities used to contract the finite domains. Write

$$X = \frac{B + U - p}{\sqrt{v + s}}, \quad B \sim \text{Bernoulli}(p), \quad q = 1 - p, \quad v = pq, \quad \mathbb{E}(U \mid B) = 0.$$

The sign classes are fixed by the original law. Put

$$b = q\mathbb{E}(U^2 \mid B = 0), \quad c = p\mathbb{E}(U^2 \mid B = 1), \quad s = b + c.$$

Suppose the inward endpoints satisfy $U \leq a_0 < p$ in class 0 and $U \geq -a_1 > -q$ in class 1, where $a_0, a_1 \geq 0$ and $a_0 + a_1 \leq d$. The letter d in this subsection denotes this inward budget, not the third-moment defect of Part I.

Lemma 11.3 (Inward moment price). *If $0 < p \leq 1/2$ and $d > 0$, then*

$$\rho \geq I(p, s, d) := \frac{v(1 - 2v) + (3p - d)s + s^2/(qd)}{(v + s)^{3/2}}. \quad (11.1)$$

Moreover,

$$(v+s)^{5/2}I_s = v(6p-3p^2-d-3/2) + (2p/d - (3p-d)/2)s + s^2/(2qd). \quad (11.2)$$

In particular, if $d \leq p$ and $d \leq 6p-3p^2-3/2$, then $\rho \geq \rho_B(p) = (1-2v)/\sqrt{v}$. At $d=0$, both conditional variances vanish.

Proof. The sign separation gives the exact identity

$$\rho(v+s)^{3/2} = v(1-2v) + 3pb + 3qc - q\mathbb{E}(U^3 | B=0) + p\mathbb{E}(U^3 | B=1).$$

The two-sided cubic bounds of Lemma 12.2 imply

$$\rho(v+s)^{3/2} \geq v(1-2v) + 3pb + 3qc - a_0b - a_1c + \frac{b^2}{qa_0} + \frac{c^2}{pa_1}.$$

A zero inward endpoint forces its centered noise to vanish; its term is omitted. If $a_0 + a_1 = 0$, then $s=0$ and the claim is immediate. Otherwise Cauchy–Schwarz gives

$$\frac{b^2}{qa_0} + \frac{c^2}{pa_1} \geq \frac{(b/\sqrt{q} + c/\sqrt{p})^2}{a_0 + a_1} \geq \frac{s^2}{qd}.$$

Also $3pb + 3qc \geq 3ps$ and $a_0b + a_1c \leq ds$. This proves (11.1); differentiation gives (11.2). Under the stated gates every term in that derivative is nonnegative. Thus $I(p, s, d) \geq I(p, 0, d)$. If $d=0$, each centered one-sided conditional variable is zero. $\square$

The same price gives a finite necessary upper bound on the noise even when its linear coefficient has the unfavorable sign.

Lemma 11.4 (A quadratic noise cap). *Suppose $p \in [l, u] \subset (0, 1/2]$, $0 < d \leq l$, $\rho \leq R$, and $s \leq S$. Put*

$$v_- = l(1-l), \quad v_+ = u(1-u), \quad \varepsilon = \max\{0, R - \rho_B(u)\},$$

$$f = 6l - 3l^2 - d - 3/2, \quad G = \frac{1}{(1-l)d} - \frac{3(1-2v_-)}{8v_-}.$$

Then $G \geq 5/(8v_-) + 3/4 > 0$ and

$$Gs^2 + fs \leq \varepsilon(v_+ + S)^{3/2}. \quad (11.3)$$

Consequently the larger root of this quadratic gives a necessary upper bound for s , without assuming $f \geq 0$.

Proof. For $s \geq 0$, Taylor's theorem gives

$$(v+s)^{3/2} \leq v^{3/2} + \frac{3}{2}\sqrt{v}s + \frac{3}{8}s^2/\sqrt{v}.$$

Subtract $\rho_B(p)(v+s)^{3/2}$ from the numerator lower bound in (11.1). The linear coefficient is $6p-3p^2-d-3/2$, and the quadratic coefficient is $1/(qd) - 3(1-2v)/(8v)$. They are bounded below by f, G . The other side is at most $\max\{0, R - \rho_B(p)\}(v+s)^{3/2} \leq \varepsilon(v_+ + S)^{3/2}$. Finally $d \leq l$ proves the asserted positive lower bound on G . $\square$

An acquired joint resource has the physical form

$$d_{\text{actual}} \leq d_0(1 - s/S_0), \quad S_0 = \Lambda/n.$$

To test $s \geq a$, $0 < a < S_0$, use $d_{\text{actual}} \leq d_0(1 - a/S_0)$ in Lemma 11.4, whenever its positive-budget gates hold. A zero inward budget instead forces $s=0$, without division; this also treats $a=S_0$. If the necessary upper bound is strictly smaller than a , this part of the noise interval is impossible. Every accepted contraction is a finite implication. The original S_0 , rather than the contracted endpoint, remains in the affine price. No convergence or monotonicity of a numerical resource algorithm is assumed. Reflection may fold $p > 1/2$ for these symmetric scalar inequalities; the original orientation remains in the conditional count event and its signed phase.

11.2 The three moment regions

For each actual index the closed regions are

$$B_0 = [1, 11/10], \quad B_1 = [11/10, 9/8], \quad B_\infty = [9/8, \infty). \quad (11.4)$$

These are restrictions on one unrestricted global contrary witness, not three moment-constrained optimization problems. Each low-moment band uses all 100 closed probability panels $[j/100, (j+1)/100]$, $0 \leq j \leq 99$. A panel is either necessarily impossible, excluded by a strict scalar inequality, or covered by the full conditional count comparison. The latter retains all $n+1$ original counts and the common phase of Lemma 12.5.

The high-moment support enclosure uses another elementary consequence of signwise Cauchy–Schwarz. With $\beta_s = \mathbb{E}[X|X|]$ and $\mu = \mathbb{E}X_+ = \mathbb{E}X_- > 0$,

$$\mu^2 \leq \frac{1 - \beta_s^2}{4}, \quad \rho \geq \frac{1 + \beta_s^2}{2\mu}, \quad \rho^2(1 - \beta_s^2) \geq (1 + \beta_s^2)^2. \quad (11.5)$$

Indeed, putting $m_\pm = \mathbb{E}X_\pm^2 = (1 \pm \beta_s)/2$, the probability bounds $\mu^2/m_\pm \leq \mathbb{P}(\pm X > 0)$ prove the first inequality, including a possible atom at zero. The cubic bounds $\mathbb{E}X_\pm^3 \geq m_\pm^2/\mu$ prove the second. For a moment ceiling R , the polynomial $g(t) = (1+t)^2 - R^2(1-t)$ increases for $t \geq 0$. Thus a rational $0 < b < 1$ with $g(b^2) \geq 0$ gives $|\beta_s| \leq b$ on the entire band.

On $[9/8, 6/5]$ the retained support calculation and the supported Fourier enclosure exclude the contrary witness. For each of 21, ..., 24, support in $[-3, 3]$ is acquired at that actual index with zero predecessor allowance and both rays. For $\rho \geq 6/5$, the retained support-free Fourier bounds apply. For 21, ..., 24, these high-moment bounds are at most $4097/10000 < C_E$. No support restriction is imposed on an arbitrary admissible law.

11.3 Finite coverage and composition

The selected low-moment count covers have the following sizes. These are count leaves; the separate support, resource and Fourier certificates remain necessary.

n	B_0	B_1	total
20	37 842	50 144	87 986
21	9 402	12 555	21 957
22	6 748	7 694	14 442
23	5 751	5 322	11 073
24	4 714	3 848	8 562
total	144 020		

The supplied continuation records complete arithmetic acceptance of each selected count forest, with the exact original contract and a rational target $\underline{C}_E < C_E$. The remaining panels are covered by their necessary resource or scalar exclusions. In particular all eight low-moment bands at 21, ..., 24 are complete. Closed cells retain every common endpoint, including both seams of (11.4).

Proof of Theorem 11.2. Work under $\mathcal{A}_{20}$. If $C_{20} \geq C_E$, take its attained unrestricted positive closed maximizing witness. Jensen gives $\rho \geq 1$. The two complete low-moment covers and the high-moment exclusion exhaust (11.4) and contradict that witness. Hence $C_{20} < C_E$. This base uses its own retained contact allowance; no sharp result at index 19 is assumed.

Now let $n \in \{21, 22, 23, 24\}$ and suppose the preceding global conclusion $C_{n-1} < C_E$ has been proved. A contrary $C_n \geq C_E$ has

$$n(C_{n-1} - C_n)_+ = 0.$$

Together with the independent own-score ceiling 419/1000, this licenses the actual-index support and resource calculations. The completed covers at this same n again exhaust all three moment regions and exclude its unrestricted contrary witness. This proves $C_n < C_E$ successively at 21, 22, 23 and 24. Forests at index 20 are not relabelled for these steps.

The retained estimates at 25 through 30 and Theorem 12.1 supply all remaining indices. This is a finite induction followed by an established infinite range, not an assumption of monotonicity of C_n . $\square$

The successful finite refinement combines the inward moment price, the original joint noise budget and the common-phase count bound. Subdivision in the conditional variance allocation completes the remaining arithmetic cases without changing their inequalities. The continuation does not independently reconstruct all inherited analytic and historical resource/Fourier calculations; these remain the explicit premises of Assumption 11.1.

12 The refinement to thirty-one

The original effective argument below gives a self-contained threshold of 49. The refinement uses the same contact, support and Fourier principles, but retains the dependence between inward displacement, conditional variance and threshold phase in the count comparison. Its finite certificates and the full effective continuation are supplied in [10]. The conclusions concern unrestricted laws, not optimizers in a moment-restricted class.

Theorem 12.1. *For every integer $n \geq 31$,*

$$C_n < C_E.$$

This is a computer-assisted statement, using the stated analytic inputs and the finite-certificate implications below.

Proof inputs. The continuation uses unrestricted attainment and reflection, same-law contact and support bounds, the charged Fourier enclosure, nonnegativity of the gap defect, and the conditional count inequalities. Their analytic foundation is given below and in [10]. The finite inputs are the positive-budget certificates at 27–30, the complete sharp covers at 31–34, and the finite supported exclusions at 35–48. The independent proof from 49 onward is included below. The positive-budget enclosure is used in its charged form, including at index 27. The theorem incorporates these previously established implications at their stated analytic and finite-arithmetic scope; the supplement records the complete premises.

12.1 A positive budget before the sharp budget

Put $a_n = C_n - C_E$. The finite continuation first proves

$$a_{27} < \frac{2}{45}, \quad a_{28} < \frac{9}{280}, \quad a_{29} < \frac{3}{145}, \quad a_{30} < \frac{1}{100}. \quad (12.1)$$

These are positive ceilings, not sharp estimates at indices 27–30. Their role is to improve the available contact allowance before attempting the zero budget at 31. For a contrary attained maximum at index n , with $na_n \geq K$ and predecessor bound $a_{n-1} < A$,

$$n(C_{n-1} - C_n)_+ \leq \max\{0, nA - K\}.$$

The precise allowances in the finite certificates are

n	27	28	29	30	31
K	6/5	9/10	3/5	3/10	0
q	813/2000	31/90	93/280	93/290	31/100.

The first row uses $C_{26} \leq 469/1000$ and $C_E > 819/2000$. Every subsequent allowance follows from the preceding strict ceiling. Thus the argument is a finite descent in the permissible excess; it assumes no monotonicity of C_n .

The charge used in a positive-budget contact estimate is conservative: if $n(C - C_E) \geq K$, then

$$n(C - C_E) \geq \max\{K, n(C - 40974/100000)\}.$$

Splitting at $C = 40974/100000 + K/n$ makes this charge affine on each side. This permits the same corner-enclosure argument as at the zero budget, without discarding the score dependence.

The scalar seed at 27 uses $R = 13/5$, $C \leq 469/1000$, $K = 6/5$, and the direct support certificate of Lemma 16.2. Its two complete signed passes each use the $24 \times 24 \times 8 \times 4$ closed moment cover. The first exports

$$\mathcal{R} \leq 36139/12500, \quad D \leq 20807/25000.$$

The inequalities $\mathcal{R} < 2n/5$, $D < 4$ license the density and phase refinement before the second pass. On its complete, nonempty surviving cover the charged enclosure gives $D \leq -1/1000$, contradicting $D \geq 0$. The finite certificate in [10] checks every necessary exclusion, score-moment corner and threshold envelope; the cell count alone is not this assertion. This proves the first ceiling in (12.1); the next three use their separately certified positive-budget enclosures.

12.2 One contrary law and a closed moment cover

Suppose $C_{31} \geq C_E$. Lemma 14.1 supplies one unrestricted global maximizing law and a positive closed threshold; reflection is performed before acquiring any resources. In particular $\rho \geq 1$. The preceding ceiling at 30 gives contact allowance $31/100$. Lemma 15.1 also gives a uniform score cap. Indeed $C_E < 37/90$, $\sqrt{31/30} < 61/60$ and $\sqrt{30} > 5$ imply

$$C_{31} < \frac{61}{60} \left(\frac{37}{90} + \frac{1}{100} + \frac{1}{450} + \frac{2}{3750} \right) = \frac{193919}{450000} < \frac{432313}{10^6}.$$

No separate barrier theorem is required for this cap.

The sharp 31 certificate excludes this same law on each of the eight closed intervals

$$\begin{aligned} &[1, 11/10], [11/10, 6/5], [6/5, 13/10], [13/10, 7/5], \\ &[7/5, 3/2], [3/2, 33/20], [33/20, 7/4], [7/4, 7361/4000]. \end{aligned} \tag{12.2}$$

For every interval, all 100 closed probability panels are either necessarily impossible or covered by an acquired resource and count comparison. The support, moments, endpoint bounds and noise budget in that comparison all belong to the fixed contrary law. Shared endpoints remain included; there is no optimizer chosen separately for each interval.

For the remaining moment range, Shevtsova's bound (13.1) gives the decreasing envelope

$$T(\rho) = \min\{.469, .3322(1 + .429/\rho), .3031(1 + .646/\rho)\}.$$

At $\rho_* = 7361/4000$,

$$T(\rho_*) = \frac{819}{2000} < C_E.$$

Hence the closed cover and this tail exclude every possible $\rho \geq 1$. This proves $C_{31} < C_E$.

12.3 The conditional moment prices

The stronger count comparisons use the following pointwise moment inequalities, not additional hypotheses on an optimizer.

Lemma 12.2 (Two-sided cubic moment bounds). *If $\mathbb{E}Y = 0$, $\mathbb{E}Y^2 = w$, and $-L \leq Y \leq R$, with $L, R > 0$, then*

$$w \leq LR, \quad w^2/L - Lw \leq \mathbb{E}Y^3 \leq Rw - w^2/R.$$

If one endpoint is zero, $Y = 0$ almost surely.

Proof. Integrate $(Y + L)(R - Y) \geq 0$ to obtain the variance bound. Integrating $(R - Y)(Y + w/R)^2 \geq 0$ gives the upper cubic bound; apply it to $-Y$ for the lower one. The assertion at a zero endpoint follows from centering and the sign. $\square$

Let the actual central gap be $(-A, B)$, $A, B > 0$. Write $X = \ell(B_0 + U - p)$, $B_0 = \mathbf{1}_{\{X > 0\}}$, $p = \mathbb{P}(B_0 = 1)$, $q = 1 - p$, and $\mathbb{E}(U \mid B_0) = 0$. Put

$$v = pq, \quad a_0 = p - A/\ell, \quad a_1 = q - B/\ell, \quad s_i = \mathbb{E}(U^2 \mid B_0 = i), \quad t_i = \mathbb{E}(U^3 \mid B_0 = i).$$

Thus $U \leq a_0$ in class 0, $U \geq -a_1$ in class 1, and $a_0, a_1 \geq 0$. Bounded support supplies the opposite endpoints, denoted $-O_0$ and O_1 . The exact normalization and absolute third moment are

$$\ell^{-2} = v + qs_0 + ps_1, \quad \frac{\rho}{\ell^3} = v(p^2 + q^2) + 3v(s_0 + s_1) + pt_1 - qt_0.$$

Lemma 12.2 consequently supplies both lower and upper bounds for ρ and the necessary clamps $s_0 \leq O_0 a_0$, $s_1 \leq a_1 O_1$.

Lemma 12.3 (The priced central-gap defect). *Let $\sigma = \sqrt{10}$, and let D be the nonnegative gap defect of Section 17. With $d = a_0 + a_1$ and $b_0 = v[(\sigma + 1)p^2 + (\sigma - 1)q^2 - 6v]$,*

$$\begin{aligned} \frac{D}{\ell^3} &\geq b_0 + 6v(d - d^2 + a_0 a_1) \\ &\quad + ps_1 [3(\sigma + 1)q - (\sigma + 5)a_1 + (\sigma - 1)s_1/a_1] \\ &\quad + qs_0 [3(\sigma + 3)p - (\sigma + 7)a_0 + (\sigma + 1)s_0/a_0]. \end{aligned}$$

At $a_i = 0$, its variance is zero and its term is defined as zero.

Proof. The Tonelli gap decomposition and its nonnegative outer-gap remainder give

$$D \geq m\{(\sigma + 1)A^2 + (\sigma - 1)B^2 - 6AB\} + (\sigma - 1)T_+ + (\sigma + 1)T_- + 6BR_+ + 6AR_-.$$

Here $m = \ell v$, $A = \ell(p - a_0)$, $B = \ell(q - a_1)$, and direct expansion gives

$$\begin{aligned} T_+/\ell^3 &= p(q^3 + 3qs_1 + t_1) - v(q - a_1)^2, \\ T_-/\ell^3 &= q(p^3 + 3ps_0 - t_0) - v(p - a_0)^2, \\ R_+/\ell^2 &= ps_1 + va_1, \quad R_-/\ell^2 = qs_0 + va_0. \end{aligned}$$

Use $t_1 \geq s_1^2/a_1 - a_1 s_1$ and $t_0 \leq a_0 s_0 - s_0^2/a_0$ from Lemma 12.2. Expansion, with $\sqrt{\sigma^2 - 1} = 3$, gives the assertion. At a zero inward endpoint the conditional noise vanishes, so the convention follows directly, without division by zero. $\square$

These inequalities justify exclusions by the actual defect or absolute third moment. They are evaluated on the whole acquired parameter cell. For example $a_0 + a_1 - a_0^2 - a_1^2 - a_0 a_1$ increases in each coordinate when $a_0 + a_1 < 1/4$; positive reciprocal costs use the upper endpoint of their denominator. Stronger endpoint inversions require their derivative-sign gates. Without them the complete interval enclosure is retained, rather than excluding a cell.

More explicitly, for the right side F of Lemma 12.3,

$$\begin{aligned} \partial_{a_0} F &= 6v(1 - 2a_0 - a_1) - q(\sigma + 7)s_0 - q(\sigma + 1)(s_0/a_0)^2, \\ \partial_{a_1} F &= 6v(1 - 2a_1 - a_0) - p(\sigma + 5)s_1 - p(\sigma - 1)(s_1/a_1)^2. \end{aligned}$$

On $p \in [p_-, p_+] \subset (0, 1/2]$, $a_0 + a_1 \leq d_* < 1/2$, $s_i \leq S_i$, $s_i/a_i \leq O_i$, sufficient lower derivative gates are

$$\begin{aligned} 6p_-(1-p_-)(1-2d_*) - (1-p_-)[(\sigma+7)S_0 + (\sigma+1)O_0^2] &> 0, \\ 6p_-(1-p_-)(1-2d_*) - p_+[(\sigma+5)S_1 + (\sigma-1)O_1^2] &> 0. \end{aligned}$$

When they hold, the actual lower inward endpoints, including s_i/O_i , may be substituted. Zero endpoints are treated by the vanishing-noise case. In the actual central-gap domain $a_0 < p$, $a_1 < q$, all terms of F are nonnegative: $b_0 = v(\sqrt{\sigma+1}p - \sqrt{\sigma-1}q)^2$, $d - d^2 + a_0a_1 \geq 0$, and the two linear variance prices are positive. Thus a negative interval lower bound for F may be raised to zero before division by a positive upper normalization bound; this follows from $F \geq 0$, not merely from $D \geq 0$.

12.4 The count inequality preserves one phase

The decisive refinement of the finite comparison can be seen before any arithmetic. Write the localized law as $X = \ell(B + U - p)$, with $B \sim \text{Bernoulli}(p)$ and $\mathbb{E}(U | B) = 0$. Let W_j be the conditional noise sum at count j , $\pi_j = \binom{n}{j}p^j(1-p)^{n-j}$, and $F_k = \sum_{j \leq k} \pi_j$. At phase u , conditioning gives

$$\begin{aligned} &\mathbb{P}\{\sum_i (B_i + U_i) \leq k + u\} - \Phi(Z(u)) \\ &\leq F_k - \Phi(Z(u)) - \pi_k \mathbb{P}\{W_k > u\} + \sum_{j > k} \pi_j \mathbb{P}\{W_j \leq k + u - j\}, \end{aligned} \quad (12.3)$$

where $Z(u) = (k + u - np)/\sqrt{np(1-p) + \lambda}$ and $\lambda = n\mathbb{E}U^2$. The Gaussian term, the outgoing own-count mass and the incoming count tails must be evaluated at the same phase. Maximizing them independently loses precisely the compensation needed near the sharp level. Polynomial minorants for the outgoing mass and Cantelli or exponential bounds for the incoming tails reduce (12.3) to an explicit polynomial on a closed phase interval. The resource inequalities constrain its coefficients before its maximum is enclosed. The strict outgoing event is the complement of the closed own-count event, including atoms. At $\lambda = 0$, the law is Bernoulli and Schulz's strict theorem applies directly.

For completeness the incoming exponential envelopes retain the conditional variance allocation. If T is centered, $T \leq A$, $A > 0$, and $\mathbb{E}T^2 = w$, then, for $z > 0$,

$$\mathbb{E}e^{zT} \leq 1 + \frac{w}{A^2}(e^{zA} - 1 - zA).$$

Indeed $(e^{zx} - 1 - zx)/x^2 = z^2 \int_0^1 (1-t)e^{ztx} dt$ is increasing in x , including its value at zero. Multiply the resulting pointwise inequality by T^2 and integrate. At $A = 0$, centering forces $T = 0$.

The allocation retained across all counts belongs to the marginal law:

$$\lambda = n(qs_0 + ps_1), \quad \alpha = \frac{ps_1}{qs_0 + ps_1} \quad (\lambda > 0), \quad s_0 = \frac{\lambda(1-\alpha)}{nq}, \quad s_1 = \frac{\lambda\alpha}{np}.$$

For $m_0 = n - j$, $m_1 = j$, write W_j for the corresponding class sum. Independence and $\lambda \leq \Lambda$ give

$$\mathbb{E}e^{zW_j} \leq [1 + r_0(1-\alpha)]^{m_0} [1 + r_1\alpha]^{m_1}, \quad r_0 = \frac{\Lambda(e^{zA_0} - 1 - zA_0)}{nqA_0^2}, \quad r_1 = \frac{\Lambda(e^{zA_1} - 1 - zA_1)}{npA_1^2}.$$

Apply this also to reflected noises for incoming left tails. An absent class or zero endpoint contributes the factor one. On a probability cell, use its positive lower bounds for p, q in these adverse denominators. The case $\lambda = 0$ is deterministic noise. No different count-specific allocation

is substituted for the shared α . The logarithm of the product is concave in α . When $r_0 r_1 > 0$, its only possible interior maximum is

$$\alpha_* = \frac{jr_1(1+r_0) - (n-j)r_0}{nr_0r_1}.$$

Thus its maximum on an acquired closed allocation interval is bounded at the two endpoints and, if contained there, at α_* . Degenerate cases use the endpoints. Chernoff's inequality gives a valid incoming tail bound for each chosen z ; its minimum with a separately proved Cantelli bound is again valid. This establishes the incoming-envelope implication, not merely the evaluation of its coefficients.

Here is the common-phase estimate in a form independent of the chosen minorants. Fix a closed interval $[l, h]$, and nonnegative smooth incoming envelopes $B_j(u) \geq \mathbb{P}\{W_j \leq u - (j - k)\}$. Suppose, on the whole parameter cell,

$$0 \leq \pi_k \leq \pi_k, \quad \pi_j \leq m_j, \quad B'_j \leq L_j, \quad (\Phi \circ Z)' \geq g,$$

and set

$$B_0 \geq F_k - \Phi(Z(l)), \quad I_0 \geq \sum_{j>k} m_j B_j(l), \quad L = \sum_{j>k} m_j L_j.$$

For each shift a in a chosen nonempty finite subset of $(0, \infty)$, let a complete finite list of quartic cases satisfy, for every actual law and phase,

$$\mathbb{P}\{W_k > u\} \geq Q_{a,c}(u)/(M_a a^4), \quad M_a > 0,$$

for at least one applicable case c . The shifted-quartic inequality of Section 22 supplies elementary minorants. The following refinement retains the separate conditional variances and the signed support costs.

Lemma 12.4 (A variance-priced quartic minorant). *Let Z_i be independent centered variables supported in $[l_i, r_i]$, put $v_i = \mathbb{E}Z_i^2$, $V = \sum_i v_i$, $J = \sum_i v_i^2$, and $W = \sum_i Z_i$. For $a > 0$ and $K \geq (9 + 6\sqrt{3})/4$,*

$$\begin{aligned} Ka^4 \mathbb{P}\{W > u\} &\geq 3a^2 V - 3V^2 + 3J + \sum_i v_i \min_{x \in [l_i, r_i]} (4ux - x^2) \\ &\quad + (3a^2 - 6V)u^2 - 2a^3 u - u^4. \end{aligned}$$

Proof. The polynomial $h_a(z) = (-z^4 + 3a^2 z^2 + 2a^3 z)/(Ka^4)$ satisfies $h_a(z) \leq \mathbf{1}_{\{z>0\}}$. Indeed its numerator, in the variable z/a , is $x(2-x)(x+1)^2$: it is nonpositive for $x \leq 0$, and its maximum for $x \geq 0$ is $(9 + 6\sqrt{3})/4$. Apply this inequality to $W - u$, using

$$\mathbb{E}W^2 = V, \quad \mathbb{E}W^3 = \sum_i \mathbb{E}Z_i^3, \quad \mathbb{E}W^4 = \sum_i \mathbb{E}Z_i^4 + 3V^2 - 3J.$$

Concavity of $x \mapsto 4ux - x^2$ on $[l_i, r_i]$ gives

$$4u\mathbb{E}Z_i^3 - \mathbb{E}Z_i^4 = \mathbb{E}[Z_i^2(4uZ_i - Z_i^2)] \geq v_i \min_{x \in [l_i, r_i]} (4ux - x^2).$$

Expansion of $\mathbb{E}h_a(W - u)$ gives the assertion. The strict outgoing event is preserved, since $h_a(0) = 0$. $\square$

For two noise classes, with support intervals $[-O_0, a_0]$ and $[-a_1, O_1]$ and total variances V_0, V_1 , the signed cost is

$$V_0 \min_{x \in [-O_0, a_0]} (4ux - x^2) + V_1 \min_{x \in [-a_1, O_1]} (4ux - x^2).$$

The switches occur at $(a_0 - O_0)/4$ and $(O_1 - a_1)/4$. A closed phase interval crossing a switch retains both cases. For a class containing $m > 0$ summands, Cauchy–Schwarz gives $\sum v_i^2 \geq V_{\text{class}}^2/m$; a class with no summands contributes zero. Thus the minorants use only acquired support and variance bounds. One may take the rational $K = 97/20$, since $(26/15)^2 > 3$.

Lemma 12.5 (Common phase). *The signed error in (12.3) is at most*

$$\min_a \max_c \sup_{u \in [l, h]} \left\{ B_0 + I_0 + (L - g)(u - l) - \frac{\pi_k Q_{a,c}(u)}{M_a a^4} \right\}.$$

Proof. Integrate the incoming and Gaussian derivative bounds from the same l to u . For the applicable quartic case,

$$\pi_k \mathbb{P}\{W_k > u\} \geq \pi_k \max\{0, Q_{a,c}(u)/(M_a a^4)\} \geq \pi_k Q_{a,c}(u)/(M_a a^4).$$

This is valid even when the polynomial is negative. The case may depend on the law and phase, so take the maximum over cases and the supremum over the whole interval. Each shift gives a valid bound; take their minimum. Only smooth envelopes were differentiated, never the actual distribution function. $\square$

The remaining supremum is polynomial. For $P(u) = \sum_{r=0}^4 q_r u^r$, put $w = h - l$ and

$$c_s = w^s \sum_{r=s}^4 \binom{r}{s} q_r l^{r-s}, \quad b_j = \sum_{s=0}^j \frac{\binom{j}{s}}{\binom{4}{s}} c_s.$$

Then

$$P(l + wt) = \sum_{j=0}^4 b_j \binom{4}{j} t^j (1-t)^{4-j}, \quad \sup_{[l, h]} P \leq \max_{0 \leq j \leq 4} b_j.$$

The basis is nonnegative and sums to one. Rational upper enclosures of its coefficients therefore prove the inequality on a whole closed interval; a subdivision retains every shared endpoint. This is the finite algebraic step, not a numerical search for a maximizing phase.

12.5 Signed comparisons on a finite index band

For a Bernoulli law write $q = 1 - p$, $\rho_B = (p^2 + q^2)/\sqrt{pq}$, $F_{-1} = 0$, $F_j = \mathbb{P}\{\text{Bin}(n, p) \leq j\}$, and $z_j = (j - np)/\sqrt{npq}$. Its full normalized discrepancy is

$$B_n(p) = \frac{\sqrt{npq}}{1 - 2pq} \max_{0 \leq j \leq n} \{F_j - \Phi(z_j), \Phi(z_j) - F_{j-1}\}.$$

Both faces of every atom are retained; between atoms the signed discrepancy is monotone. On $[l, r] \subset (0, 1/2]$, coupling shows that F_j decreases with p , while direct differentiation shows that z_j decreases and the positive normalization factor increases. Consequently

$$B_n(p) \leq \frac{\sqrt{nr(1-r)}}{1 - 2r(1-r)} \max_{0 \leq j \leq n} \{0, F_j(l) - \Phi(z_j(r)), \Phi(z_j(l)) - F_{j-1}(r)\}.$$

The zero in the maximum licenses the upward normalization bound. Exact binomial probabilities and outward Gaussian enclosures on subdivisions of the 100 closed panels covering $[\cdot 28, \cdot 48]$ prove the conservative margins

n	40	41	42
$C_E - B_n(p)$	$370197/10^9$	$202018/10^9$	$204942/10^9$

where each entry is a lower bound; the larger cell-specific margins are retained in the finite witnesses.

Lemma 12.6 (Signed finite margin). *Let $0 < N \leq n \leq L < \infty$, $0 < l \leq \lambda \leq U$, $0 < m \leq M \leq 1$, and $\rho_B \geq r_B > 0$. Suppose the same-law comparison gives*

$$\rho_B(C - B_n(p))/\lambda \leq MB(\lambda) + I(\lambda)/n,$$

where B, I are convex on $[l, U]$, and $B_n(p) \leq C_* - \varepsilon$, $\varepsilon > 0$, on the whole admitted index and probability cell. Put

$$F(\lambda) = \max\{mB(\lambda), B(\lambda)\} + \max\{I(\lambda)/N, I(\lambda)/L\}.$$

If $F(l), F(U) < r_B\varepsilon/U$, then $C < C_*$.

Proof. For real b, i , the linear endpoint bounds give $Mb + i/n \leq \max\{mb, b\} + \max\{i/N, i/L\}$. The maxima of positive multiples of convex functions are convex, so $F(\lambda) \leq \max\{F(l), F(U)\}$. Multiply the strict bound by $\lambda \leq U$ to obtain $\rho_B(C - B_n(p)) < r_B\varepsilon \leq \rho_B\varepsilon$. Division by ρ_B proves the assertion. No sign of B or I was assumed. $\square$

For a lower reserve one instead uses $\min\{mb, b\} \leq Mb$. If $\rho_B(C - B_n(p)) \leq -R$, the sufficient test is $R + r_B\varepsilon > 0$. If $\rho_B(C - B_n(p)) \leq -\lambda R_T$, it is $R_T + r_B\varepsilon/U > 0$, uniformly for $0 < \lambda \leq U$. At index 39 use $C_* = C_E + K/39$, $\varepsilon = K/39$, and Schulz's bound; at 40–42 use the finite margins above. Replacing a signed binomial discrepancy by the larger $B_n(p)$ preserves the comparison. The actual lower-noise kernels are a constant plus a nonnegative reciprocal of a positive affine function and a nonnegative linear term; their inverse-index terms are a constant plus a nonnegative multiple of $1/\lambda$. These forms prove the required convexity.

Lemma 12.7 (Signed cubic repair). *Let $0 < p \leq 1/2$, $q = 1 - p$, $v = pq$, $Y = B_0 + U$, $B_0 \sim \text{Bernoulli}(p)$, $\mathbb{E}(U \mid B_0) = 0$, and $s_i = \mathbb{E}(U^2 \mid B_0 = i)$. Put $s = qs_0 + ps_1$, $\lambda = ns$, and $X = (Y - p)/\sqrt{v + s}$. Suppose $U_0 \leq a_0 < p$, $U_1 \geq -a_1 > -q$, $a_i \geq 0$, and $d = a_0 + a_1$. Then*

$$(v + s)^{3/2}(\rho - \rho_B) \geq sQ, \quad Q = 6p - 3p^2 - 3/2 - d - \frac{3(1 - 2v)s}{8v}.$$

On $p \in [p_-, p_+] \subset (0, 1/2]$, $n \geq N > 0$, $\lambda \leq \Lambda$, $d \leq d_*$, set

$$v_- = p_-(1 - p_-), \quad Q_* = 6p_- - 3p_-^2 - 3/2 - d_* - \frac{3(1 - 2v_-)\Lambda}{8v_-N}.$$

If $0 < m_- \leq C/(v + s)^{3/2} \leq m_+$, put $g = \min\{m_-Q_*, m_+Q_*\}$. Thus $C(\rho - \rho_B) \geq sg$, even when $Q_* < 0$. For $P \geq (2/5)/(2v_-^{3/2})$, $D_+ \geq \max\{\delta, 0\}$,

$$\begin{aligned} & \sqrt{n} \left[\Phi \left(\frac{\delta}{\sqrt{nv}} \right) - \Phi \left(\frac{\delta}{\sqrt{n(v + s)}} \right) \right] - C(\rho - \rho_B) \\ & \leq (PD_+ - g)\lambda/n \leq \max\{0, PD_+ - g\}\lambda/N. \end{aligned}$$

Proof. The sign restrictions fix the signs of $Y - p$ in each class. Expansion gives

$$\mathbb{E}|Y - p|^3 = v(1 - 2v) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3.$$

Since $\mathbb{E}U_1^3 \geq -a_1s_1$, $\mathbb{E}U_0^3 \leq a_0s_0$, and $p \leq q$, this is at least $v(1 - 2v) + (3p - d)s$. The bound $(1 + x)^{3/2} \leq 1 + 3x/2 + 3x^2/8$, $x \geq 0$, follows from Taylor's theorem with negative third derivative. It proves the displayed formula for Q . Its adverse terms are enlarged by $d_*, \Lambda/N, v_-$, whereas $6p - 3p^2$ increases, so $Q \geq Q_*$. Multiply by the actual positive multiplier and minimize over $[m_-, m_+]$, retaining either sign of Q_* . The normal correction is nonpositive for $\delta \leq 0$. For $\delta > 0$, its derivative in the added variance is at most $(2/5)\delta/(2v_-^{3/2})$. Integrating from zero to s and subtracting the cubic gain proves the first repair bound. Only its nonnegative part can be enlarged by replacing n by N . $\square$

For $2/5 \leq C \leq 4247/10000$, admissible multipliers satisfy

$$m_- \leq \frac{2/5}{[p_+(1-p_+) + \Lambda/N]^{3/2}}, \quad m_+ \geq \frac{4247/10000}{v_-^{3/2}}.$$

For $d_0, \Lambda > 0$, if the acquired joint resource gives $d/d_0 + \lambda/\Lambda \leq 1$, a noise band $\lambda \geq l$ licenses $d_* = d_0(1 - l/\Lambda)$. The tiny- and middle-noise tests use the clamped repair; the lower-noise signed test retains its coefficient inside $I(\lambda)/n$ and uses Lemma 12.6. At zero noise the law is Bernoulli directly.

Lemma 12.8 (Signed Gaussian phase derivatives). *Let $v = p(1-p) > 0$, $s = \lambda/n \geq 0$, $\ell = (v+s)^{-1/2}$, and $np + H/\ell = k + u$, $|u| \leq 1/2$. Put $h = H/\ell$, $\delta = k - np = h - u$, and $\pi_k = \mathbb{P}\{\text{Bin}(n, p) = k\} > 0$. Suppose the finite Stirling enclosure gives*

$$\sqrt{n}\pi_k = \frac{\phi(0)}{\sqrt{v}} \exp\{-\delta^2/(2nv) + \zeta\}, \quad |\zeta| \leq Z/n.$$

Assume $n \geq N$, $v \geq v_- > 0$, $|\delta| \leq D$, $D^2 < Nv_-$, $\lambda \leq \Lambda$, and $h \leq h_+$, $h_+ \geq 0$. With k, δ fixed, set

$$f(t) = \frac{\Phi((\delta+t)/\sqrt{n(v+s)}) - \Phi(\delta/\sqrt{n(v+s)})}{\pi_k}.$$

Put $b = 1/2$, $t_* = \min\{b, h_+\}$, $G_+ = 2h_+t_* - t_*^2$, $G_- = b^2 + 2h_+b$, and $\eta = G_-/(2v_-) + Z$. If $\eta < N$, then

$$f'(t) \geq 1 - \frac{(\Lambda + G_+)/(2v_-) + Z}{N} \quad (0 \leq t \leq u, \quad u \geq 0),$$

$$f'(-t) \leq (1 - \eta/N)^{-1} \quad (0 \leq t \leq -u, \quad u \leq 0).$$

Proof. Direct differentiation gives

$$\log f'(t) = -\frac{1}{2} \log(1 + s/v) + \frac{\delta^2 s}{2nv(v+s)} - \frac{2\delta t + t^2}{2n(v+s)} - \zeta.$$

For $u \geq 0$, convexity on $[0, u]$ bounds $2(h-u)t + t^2$ by $\max\{0, 2hu - u^2\} \leq G_+$. The first logarithmic term is at least $-s/(2v)$ and the second is nonnegative. Apply $e^{-x} \geq 1 - x$ to obtain the first bound. For $u = -r$, put $t = -\tau$. Since $\delta^2 \leq nv$, the first two terms are at most $-\log(1+x)/2 + x/[2(1+x)] \leq 0$, $x = s/v$. Also $2(h+r)\tau - \tau^2 \leq r^2 + 2h_+r \leq G_-$ on $0 \leq \tau \leq r$. Hence $\log f'(-\tau) \leq \eta/n$; use $e^x \leq (1-x)^{-1}$, $0 \leq x < 1$. The closed seams $u = 0$ and $s = 0$ are included. $\square$

Only the positive threshold cap enters $h_+ = H_+/\ell_-$. The negative cap still enters D and the Stirling enclosure. The symmetric entropy bounds remain valid on $[a, b] \subset (0, 1)$ with $a < 1/2$, $a + b \leq 1$, even when $b > 1/2$. Indeed the farthest endpoint from one half is a , and $r(1-r)$ decreases, whereas $|1-2r|/[r(1-r)]^j$, $j = 1, 2$, increases with this distance. With $y = |1-2r|$, the latter functions are $4^j y/(1-y^2)^j$, with positive derivatives. This extends only these symmetric bounds, not arbitrary signed coefficients. Each entropy segment checks $a + b \leq 1$, $b < 1$.

12.6 Actual-index bridges and the infinite range

The continuation supplies three separate certificate implications

$$C_{m-1} < C_E \implies C_m < C_E, \quad m = 32, 33, 34. \quad (12.4)$$

They have the same closed moment cover (12.2). In the resource transfer, the physical defect and noise prices scale by $31/m$. A uniform pass at floor $N = 31$ retains the two invariants

$$\mathcal{R}_m \leq r_{\text{stored}}, \quad mD_m \leq 31d_{\text{stored}},$$

not merely $D_m \leq d_{\text{stored}}$. Equations (17.3), (17.4), and (17.12) give this interpretation by induction over the enclosure schedule. The floor Fourier constants bound all actual indices above the floor under their monotonicity gates. The stored raw bound licenses the smaller density floor $.3988(1 - 5r_{\text{stored}}/(2N))$. The floor excess subtraction is at most the actual $m(C_m - C_E)\rho$. Dividing its remaining nonnegative budget by $997N/15000$ therefore exports $mD_m \leq Nd_{\text{stored}}$. Intersections take minima and a complete cell union takes maxima; both preserve the weighted invariant. Actual-index support, score caps and every later resource callback are acquired separately, before using these exports. The binomial probabilities, threshold normalization and count covers use the actual index m . Thus (12.4) consists of three actual-index proofs, not relabeled 31 certificates. Discharge their predecessor assumptions in the order 31, 32, 33, 34.

Only finitely many further certificate implications are required, since Theorem 13.1 below treats every index from 49 onward. Put $a_n = C_n - C_E$ and $\delta_n = n(C_{n-1} - C_n)_+$. The supplied finite witnesses exclude an unrestricted attained contrary law under the following contracts, with actual support and resource acquisition included:

1. At 35: $a_{35} \geq 0$, $\delta_{35} \leq 63/340$, $C_{35} < 423493/10^6$.
2. At 36, 37, 38: $a_n \geq 0$, $\delta_n = 0$, with respective own-score caps $1699/4000$, $84867/200000$, $211449/500000$.
3. The positive 39 barrier: $39a_{39} \geq 741/4000$, $\delta_{39} \leq 33/100$, $C_{39} < 2113/5000$, $C_{38}, C_{37} < 4253/10000$, $39a_{39} < 1/2$.
4. The sharp 39 case: $a_{39} \geq 0$, $\delta_{39} = 0$, $C_{39} < 41449/100000$.
5. At 40 and 41: $0 \leq na_n < 27/50$, $C_n < 2113/5000$, $C_{n-1}, C_{n-2} < 4253/10000$, with $\delta_{40} \leq 19/100$ or $\delta_{41} = 0$.
6. At 42–48: $0 \leq na_n \leq 27/50$, $\delta_n \leq 19/100$, $C_n < 2113/5000$, $C_{n-1}, C_{n-2} < 4253/10000$.

These are finite supported exclusions, not an assumed infinite-tail theorem. The resource enclosure, cubic prices, common-phase minorants and signed finite-margin rules above give their analytic meaning; the exact coefficient and cover witnesses are in [10]. In the floor-40 comparison, the support scopes $(N, L, K, q) = (40, 40, 0, .19)$ and $(40, 42, 0, 0)$ are acquired separately, using the upper band endpoint in the favorable fifth-order coefficient. The resource keeps $nD \leq 40d$. On a noise band it simultaneously uses $d \leq d_0(1 - \lambda_-/\Lambda)$ and $\theta \leq \lambda_+/(40v_- + \lambda_+)$. Both threshold signs and every admitted probability cell remain covered, including the left signed phase and arbitrarily small positive noise.

Here is the scalar composition. Whenever $n \geq 35$ and $C_{n-1} < C_E$, the support-free smoothing estimate gives

$$na_n < n(s_n - 1)C_E + s_n^3 \left(\frac{1}{15} + \frac{2}{25\sqrt{n-1}} \right).$$

For $n \geq 35$, $s_n < 203/200$, $\sqrt{n-1} > 29/5$, and $n(s_n - 1) < 64/125$. These follow by squaring at 35 and using $n(s_n - 1) = s_n^2/(s_n + 1)$. Therefore

$$na_n < \frac{20487}{50000} \frac{64}{125} + \left(\frac{203}{200} \right)^3 \left(\frac{1}{15} + \frac{2/25}{29/5} \right) = \frac{5114241437}{17400000000} < \frac{3}{10}.$$

Starting with $C_{34} < C_E$, any first failure at 35–38 has $\delta_n = 0$ and $C_n < .40974 + .30/n$, strictly below its listed own-score cap. The first two finite contracts exclude it. Thus 35–38 are sharp.

Next, if $a_{39} \geq 19/4000$, the same estimate gives $39a_{39} < .30 < .50$, $C_{39} < .4226$, and $\delta_{39} = 0$; both deleted predecessors are sharp. The positive 39 contract contradicts this, proving $a_{39} < 19/4000$. A failure $C_{39} \geq C_E$ then has $C_{39} < C_E + 19/4000 < .41449$, so the sharp 39 contract excludes it. A first failure at 40 or 41 now has $\delta_n = 0$, $na_n < .30 < .54$, $C_n < .4226$,

and two sharp deleted predecessors; its actual-index contract excludes it. A first failure in 42–48 satisfies the identical caps and the last finite contract. Finally the already proved 49 theorem supplies the entire infinite range. This proves Theorem 12.1 without monotonicity or an imported infinite-tail conclusion.

13 The result and the organizing ideas

Let $\mathcal{P}$ be the probability laws on $\mathbb{R}$ with mean zero, variance one and finite third absolute moment $\rho(P) = \mathbb{E}_P|X|^3$. For iid variables with law P , set $S_n = X_1 + \cdots + X_n$ and

$$C_n(P) = \frac{\sqrt{n}}{\rho(P)} \sup_{H \in \mathbb{R}} \left| \Pr_P(S_n \leq H) - \Phi(H/\sqrt{n}) \right|, \quad C_n = \sup_{P \in \mathcal{P}} C_n(P).$$

Here H is a raw sum threshold, Φ is the standard normal CDF, and $\phi = \Phi'$. All terminating decimals in the proof are exact rationals.

Theorem 13.1. *For every integer $n \geq 49$,*

$$C_n < C_E, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

The argument is easiest to read as three ideas.

1. *A first failure is easier than an arbitrary large index.* Once the case $n = 49$ is proved, the first later failure has a good predecessor. Smoothing that predecessor automatically bounds its excess by $.29/n$. This supplies the hypotheses of a local exclusion, so the same argument proves the entire infinite tail.
2. *Spend one budget only once.* At an extremizer, excess score, threshold displacement, nonbinary defect and probability leaking into a support gap are nonnegative terms of one exact identity. Moment, support and concentration estimates refine that common budget. They do not choose a different law for each term. The many numerical passes are instances of one enclosure theorem.
3. *Compare each noisy law with its own Bernoulli law.* Writing $Y = B + U$ with $\mathbb{E}(U | B) = 0$ gives a count and a centered error. One exact score identity and two tail estimates cover every noise and phase regime. Positive noise lowers the signed score; zero noise is already Bernoulli. Schulz's theorem treats every $0 < p < 1$, so no finite binomial subproblem is needed.

Section 15 proves the global implication from explicit finite and local propositions. The remaining sections prove those propositions and specify their exact arithmetic. This order separates the short mathematical mechanism from the estimates needed to certify its numerical hypotheses. The proof does not invoke any earlier project cutoff, an eventual-tail theorem, or monotonicity of C_n .

13.1 External inputs

Shevtsova's finite iid estimates [7] give, in this normalization,

$$C_n(P) \leq \min\{.469, .3322(1 + .429/\rho), .3031(1 + .646/\rho)\}. \quad (13.1)$$

The elementary numerical enclosures used below are

$$.3988 < \phi(0) < .4, \quad .4095 < C_E < .40974, \quad 3.162277660 < \sqrt{10} < 3.162277661.$$

Their rational checks are part of the arithmetic kernel. In particular, a law attaining $C_n \geq C_E$ has $1 \leq \rho < 1.85$ before any higher-moment or bounded-support assumption is made.

Schulz's Theorem 1 [6], for every $n \geq 1$ and $0 < p < 1$, states

$$\sup_{x \in \mathbb{R}} \left| \Pr(K \leq x) - \Phi\left(\frac{x - np}{\sqrt{np(1-p)}}\right) \right| < \frac{C_E\{p^2 + (1-p)^2\}}{\sqrt{np(1-p)}}, \quad K \sim \text{Bin}(n, p). \quad (13.2)$$

Thus every standardized Bernoulli law has score strictly below C_E . The restriction $p \in [1/3, 2/3]$ belongs to a sharper envelope in that work; it is not a restriction of (13.2).

From He–Cheng [8] we use the finite Gaussian bound (Appendix A.1, (70)) in the effective large-index argument. The fixed-index attainment and contact arguments are proved next; the contact identity agrees with their Lemma 4.3. For the unrestricted fourth index we prove a separate Gaussian estimate in Section 26. All remaining analytic reductions in this manuscript are proved here.

14 Fixed-index extremizers and the contact interface

14.1 Attainment, including the critical value

Lemma 14.1. *For every fixed $n \geq 1$, $0 < C_n \leq \sqrt{n}$, and a law $P \in \mathcal{P}$ and threshold H attain C_n as a positive closed discrepancy.*

Proof. A negative closed discrepancy reflects into a positive strict discrepancy. Approach that threshold from below; continuity of Φ gives the same supremum using positive closed discrepancies. Since $\rho \geq 1$, $C_n \leq \sqrt{n}$; the symmetric binary law at $H = n$ gives positivity. Choose positive scores $r_j \rightarrow C_n$ with $r_j \geq c > 0$. Then $\rho_j \leq \sqrt{n}/c =: M$. Tightness gives a subsequence $P_j \Rightarrow P$, and we may also assume $\rho_j \rightarrow b$. The estimates

$$\int_{|x|>A} x^2 dP_j \leq M/A, \quad \int_{|x|>A} |x| dP_j \leq M/A^2$$

preserve the first two moments under weak convergence. Lower semicontinuity gives $\rho(P) \leq b$, so $P \in \mathcal{P}$. Chebyshev’s inequality on a negative threshold, and the Gaussian upper tail on a positive threshold, give $|H_j| \leq n^{3/4}/\sqrt{c}$. Pass to $H_j \rightarrow H$. Since n is fixed, weak convergence of products and the continuous sum map give $P_j^{*n} \Rightarrow P^{*n}$. For every $\varepsilon > 0$, eventually $H_j \leq H + \varepsilon$; closed-set Portmanteau followed by $\varepsilon \downarrow 0$ gives

$$F_{n,P}(H) - \Phi(H/\sqrt{n}) \geq C_n b / \sqrt{n}.$$

Consequently $C_n \geq C_n b / \rho(P) \geq C_n$. Equality proves attainment and $\rho(P) = b$. Thus third-moment convergence is a conclusion of extremality, not an unjustified uniform-integrability assumption. This proof includes $C_n = C_E$; the strict-violation version is Lemma 4.1 of [8]. $\square$

14.2 Contact at every support point

For $n \geq 2$, fix an attaining pair (P, H) , and write

$$C = C_n, \quad F = F_{n,P}(H), \quad z = H/\sqrt{n}, \quad \rho = \mathbb{E}|X|^3, \quad \gamma = \mathbb{E}X^3, \quad \beta = \mathbb{E}[X|X|].$$

Here β is a signed second moment; it is not the third absolute moment denoted by β in [8].

Lemma 14.2 (Contact). *Define*

$$\begin{aligned} \rho I(y) &= n^{3/2} \{F_{n-1,P}(H-y) - F\} + ny\phi(z) + \frac{1}{2}H\phi(z)(y^2 - 1) \\ &\quad + \frac{3}{2}C\rho(y^2 - 1) - C\{|y|^3 - \rho - 3y\beta\}. \end{aligned} \tag{14.1}$$

Then $I \leq 0$ on $\mathbb{R}$ and $I = 0$ on the full closed support of P . The support is compact. Equivalently, for $\nu = \mathcal{L}(H - S_{n-1})$, $\nu([y, \infty)) \leq \mathfrak{q}(y)$, with equality on $\text{supp } P$, where

$$\begin{aligned} n\mathfrak{q}(y) &= nF + \frac{1}{2}\{z\phi(z) + C\rho/\sqrt{n}\} - (\sqrt{n}\phi(z) + 3C\beta/\sqrt{n})y \\ &\quad - \frac{1}{2}\{z\phi(z) + 3C\rho/\sqrt{n}\}y^2 + C|y|^3/\sqrt{n}. \end{aligned} \tag{14.2}$$

Proof. For fixed y , contaminate P by $P_\varepsilon = (1 - \varepsilon)P + \varepsilon\delta_y$, holding H fixed. Its mean and variance are

$$m_\varepsilon = \varepsilon y, \quad v_\varepsilon = 1 + \varepsilon(y^2 - 1) - \varepsilon^2 y^2.$$

Write $r_\varepsilon = \mathbb{E}_{P_\varepsilon}|X - m_\varepsilon|^3$. After standardization, the signed score

$$\frac{\sqrt{n} v_\varepsilon^{3/2}}{r_\varepsilon} \left[F_{n, P_\varepsilon}(H) - \Phi\left(\frac{H - nm_\varepsilon}{\sqrt{nv_\varepsilon}}\right) \right]$$

is at most C and equals C at zero. The raw CDF is a polynomial in ε , whose derivative there is $n\{F_{n-1, P}(H - y) - F\}$. Dominated differentiation, using the finite second moment, gives $r'_0 = |y|^3 - \rho - 3\beta y$. The derivative of the displayed score is precisely $I(y)$, so $I(y) \leq 0$. Integration against P gives $\int I dP = 0$: in particular $\int F_{n-1, P}(H - y) dP(y) = F$. The closed CDF is upper semicontinuous. Hence I is upper semicontinuous, and the closed set $\{I = 0\}$ has full P measure; it contains all of $\text{supp } P$. Rearrangement gives (14.2). Finally, $\rho I(y) \leq -C|y|^3 + O(y^2)$ at this fixed pair, since the CDF term is bounded and $C > 0$. Equality on the support excludes both sufficiently distant rays. No uniform radius is claimed yet. $\square$

The pointwise equality controls even a support point carrying arbitrarily little mass. This is why weak closeness to a binary law will not be the last step of the proof.

15 A finite base and a first-failure induction

Put $a_n = C_n - C_E$ and $\delta_n = n(C_{n-1} - C_n)_+$. Different indices may have different extremizers. This section uses only their scalar suprema.

15.1 A support-free smoothing estimate

Lemma 15.1. *For every integer $n \geq 2$,*

$$C_n \leq \sqrt{\frac{n}{n-1}} C_{n-1} + \frac{\sqrt{n}}{15(n-1)^{3/2}} + \frac{.08\sqrt{n}}{(n-1)^2}.$$

No bounded-support or maximizing-law assumption is required.

Proof. Fix any mean-zero, variance-one iid law with finite $\rho = \mathbb{E}|X|^3 \geq 1$, and let $v = n - 1$. Conditional on the last summand, the defining bound at index v gives

$$|\mathbb{P}(S_n \leq H) - \mathbb{E}\Phi((H - X)/\sqrt{v})| \leq C_{n-1}\rho/\sqrt{v}.$$

Write $F_v(t) = \Phi(t/\sqrt{v})$. Taylor's theorem through degree two, using the actual mean and variance, gives

$$|\mathbb{E}F_v(H - X) - F_v(H) - \frac{1}{2}F_v''(H)| \leq \frac{\phi(0)\rho}{6v^{3/2}} < \frac{\rho}{15v^{3/2}}.$$

Here $\|\phi''\|_\infty = \phi(0) < .4$. Indeed $\phi''(z) = (z^2 - 1)\phi(z)$; on $z^2 \leq 1$ the assertion is immediate, and on $z^2 \geq 1$ the maximum relative factor is $2e^{-3/2} < 1$. This globally bounded remainder needs only the third absolute moment.

The heat identity $\partial_v F_v = F_v''/2$ gives $\partial_v^2 F_v = F_v^{(4)}/4$. Taylor's theorem in the variance variable therefore implies

$$|F_{v+1}(H) - F_v(H) - \frac{1}{2}F_v''(H)| \leq \frac{\|\phi'''\|_\infty}{8v^2} < \frac{.08}{v^2}.$$

For the last bound, Gaussian Fourier inversion gives $\|\phi'''\|_\infty \leq \pi^{-1} \int_0^\infty t^3 e^{-t^2/2} dt = 2/\pi < .64$. Combine these estimates, multiply by $\sqrt{n}/\rho$, and use $\rho \geq 1$. The result is uniform in H and in the law, so taking both suprema proves the lemma. The bound covers closed events and their left limits by the same uniform comparison. $\square$

All three terms obtained by substituting $C_{n-1} < .40974 + .7/(n-1)$ decrease with n . At $n = 46$, rational square-root enclosures therefore give $C_n < .432$. This is the current-score cap at a first failure of the $.7/n$ ceiling.

15.2 The two inputs from the detailed calculation

The resource enclosure and support estimates prove these five *supported exclusions*. Each row says that no attained, positively oriented maximizing pair can have all the listed properties:

	n	$ X \leq$	$C \leq$	$na_n \geq$	passes
A40	40	3.3	.469	1.5	5
P41	41:49	2.9	.469	1.5	6
A45	45	3.5	.4431	.7	12
P46	46:98	3.5	.432	.7	13
A46	46	3.4	.42496	.54	4 + 3

(15.1)

Every pass preserves the same-law constraints and gives a new upper bound for the nonnegative defect D . Its final bound is negative. The first two rows use 64 closed cells per pass. A45 and P46 use 18432; A46 uses four signed passes with 18432 cells, followed by three probability passes with 96 cells. Section 21 gives its complete additional calculation. The upper endpoints 49 and 98 describe those individual certificates; we only apply them through 45 and 50. These arithmetic exclusions do not assert an unrestricted ceiling on their own.

The all-index local input is the following precise proposition.

Proposition 15.2 (Local exclusion). *For any integer $n \geq 47$, the conditions*

$$0 \leq a_n \leq .7/n, \quad \delta_n \leq .19, \quad C_n < .4247, \quad C_{n-1}, C_{n-2} < .4253$$

cannot hold simultaneously.

Proof from the estimates established below. Take the attained positive closed maximum. The all-threshold support estimate bounds its entire support by 3.84. The resource schedule then acquires 83 probability panels covering $p \in [.304, .47]$, together with their moments, signed thresholds, conditional variances and inward-gap resources. Support and concentration refinements are applied before the resulting stronger moments are used.

Write the same law as $X = \ell(B + U - p)$ with $\mathbb{E}(U | B) = 0$. If $\lambda = n\mathbb{E}U^2 > 0$, the complete local cover of Section 22 gives $\rho_B(C - C_B^+(n, p, k)) < -.001\lambda$ at an actual nearest count. Schulz's all-parameter theorem gives $C_B^+ < C_E$, a contradiction. If $\lambda = 0$, then $U = 0$ almost surely and the law is itself Bernoulli; the same published theorem applies directly to its whole Kolmogorov score. This uses no division by noise or limiting argument.

All entry, support, concentration and comparison estimates are uniform for $n \geq 47$. In the initial arithmetic, the apparent upper endpoint 98 occurs only as $.4095 + K/98$, and $K = 0$ for this application. Its other occurrence checks the calculation floor, not the actual index. Weighted resources such as nD and $n\mathbb{E}U^2$ are retained throughout; no unweighted estimate is multiplied by 47 to cover larger n . The excess assumption $.7/n$ is used in the exterior contact calculation and is therefore included explicitly. $\square$

15.3 Only six sharp-ceiling indices are needed

We first prove $a_j < 1.5/j$ for $40 \leq j \leq 45$. If it failed at 40, $C_{39} \leq .469$ and $C_E > .4095$ would give $\delta_{40} < .88$. The support row $(n, K, R, q) = (40, 1.5, 3.3, .88)$ and A40 exclude this. At a first failure $41 \leq j \leq 45$, $\delta_j < 1.5/(j-1) \leq 1.5/40$. The support row on $[41, 49]$ and P41 exclude it.

We now prove

$$a_j < .7/j \quad (45 \leq j \leq 50). \quad (15.2)$$

A failure at 45 would give

$$\delta_{45} < 45(1.5/44) - .7 < .8341, \quad C_{45} < .40974 + 1.5/45 < .4431.$$

The appropriate support row gives $|X| < 3.5$ and A45 excludes the law. At a first failure $46 \leq j \leq 50$, $\delta_j < .7/(j-1) \leq .7/45$, and smoothing gives $C_j < .432$. The support row on [46, 98] and P46 again exclude it. This proves (15.2) using no later index.

For $47 \leq j \leq 50$, it supplies the local score caps because

$$.40974 + .7/47 < .4247, \quad .40974 + .7/45 < .4253.$$

We strengthen just one auxiliary index:

$$a_{46} < .54/46. \tag{15.3}$$

Indeed, if this failed, the preceding finite ceiling gives

$$\delta_{46} < 46(.7/45) - .54 = 79/450, \quad C_{46} < .40974 + .7/46 < .42496.$$

The support row (46, 46, .54, 3.4, 79/450) gives $|X| < 3.4$. The supported exclusion A46 is now applicable, a contradiction. This argument uses only the finite ceiling already established.

If $a_{49} \geq 0$, Proposition 15.2 forces $a_{48} > a_{49} + .19/49 > 0$. Repeat at 48 and 47 to obtain

$$a_{46} > .19 \sum_{j=47}^{49} \frac{1}{j} > \frac{.54}{46}, \quad .19 \sum_{j=47}^{49} \frac{1}{j} - \frac{.54}{46} = \frac{35419}{254251200} > 0.$$

This contradicts (15.3). Thus $C_{49} < C_E$.

15.4 The infinite tail follows from a least failure

Suppose $n \geq 50$ and $C_{n-1} < C_E$. With $s_n = \sqrt{n/(n-1)}$, smoothing gives

$$na_n < C_E n(s_n - 1) + s_n^3/15 + .08s_n^3/\sqrt{n-1}.$$

Every term on the right decreases with n : in particular $n(s_n - 1) = s_n^2/(s_n + 1)$, an increasing function of s_n . Since $s_n < 1.011$, $\sqrt{n-1} \geq 7$ and $n(s_n - 1) < .51$,

$$na_n < .40974(.51) + (1.011)^3(1/15 + .08/7) = \frac{50691940857}{175000000000} < .29. \tag{15.4}$$

If a failure $C_n \geq C_E$ occurs after 49, choose its least index. Then $\delta_n = 0$, and (15.4) gives $0 \leq a_n < .29/n < .7/n$ and $C_n < .40974 + .29/50 < .4247$. Its predecessor is below $C_E < .4253$. If $n = 50$, the second predecessor is 48 and (15.2) bounds it by $.40974 + .7/48 < .4253$. For $n \geq 51$, that predecessor is also below C_E by minimality. Proposition 15.2 now contradicts the failure. This proves Theorem 13.1.

16 One support estimate, uniform in the threshold

16.1 The scale correction

Write $\phi_0 = \phi(0)$. For $n > 1$ put $s_n = \sqrt{n/(n-1)}$ and $e_n = n(s_n - 1) - 1/2$.

Lemma 16.1. *For every integer $n \geq 36$, one has $0 < e_n < .384/n$.*

Proof. Put $x = 1/n$ and $c = .384 = 48/125$. Direct expansion gives

$$(1-x)(1+x/2+cx^2)^2 - 1 = x^2\{(2c-3/4) - (c+1/4)x + (c^2-c)x^2 - c^2x^3\}.$$

The bracket is decreasing for $x \geq 0$, since $0 < c < 1$. At $x = 1/36$ its exact rational value is $823/4050000 > 0$. Squaring positive quantities gives $s_n < 1 + x/2 + cx^2$ for $0 < x \leq 1/36$. Similarly $(1-x)(1+x/2)^2 - 1 = -3x^2/4 - x^3/4 < 0$, which gives the lower bound. $\square$

16.2 A signed Gaussian envelope

Define, for every real H, t and $n > 1$,

$$\mathcal{H}_n(H, t) = n^{3/2} \{ \Phi((H - t)/\sqrt{n-1}) - \Phi(H/\sqrt{n}) \} + nt\phi(H/\sqrt{n}) + \frac{H}{2}(t^2 - 1)\phi(H/\sqrt{n}).$$

Lemma 16.2 (Direct support at every index). *Let $n \geq 2$, and let (P, H) be an unrestricted attained positive closed maximizing pair, with $C = C_n > 0$. Put $s = \sqrt{n/(n-1)}$, $e = n(s-1) - 1/2$, $A = \sqrt{n-1} \Delta_{n-1}(P)/\rho$, and $d_P = n(A - C)$. Suppose $(d_P)_+ \leq q$, $s \leq S$, $e \leq E$, $\rho \leq r$, $|\beta| \leq b$, and $0 < c \leq C$. Define*

$$Q(v) = -v^3 + \frac{3}{2}rv^2 + 3bv + rE, \quad L(v) = \frac{2}{5}(v^2/2 + 1/2 + E) + cQ'(v).$$

If $R > 0$ satisfies

$$Q(R), Q'(R), Q''(R), L(R), L'(R) < 0, \quad 1/5 - 3c < 0,$$

and, for both $\eta \in \{-1, 1\}$,

$$\sup_{H \in \mathbb{R}} \mathcal{H}_n(H, \eta R) + cQ(R) + rSq < 0,$$

then $\text{supp } P \subset (-R, R)$.

Proof. The contact identity (14.1), the active discrepancy $C\rho/\sqrt{n}$, and the predecessor estimate for this same law imply, at every support point t ,

$$0 \leq \mathcal{H}_n(H, t) + C\{-|t|^3 + \frac{3}{2}\rho t^2 + 3\beta t + \rho e\} + \rho s d_P.$$

Here $n(sA - C) = s d_P + n(s-1)C$; the scale correction has no omitted term. Also $A \leq C_{n-1}$, hence $(d_P)_+ \leq n(C_{n-1} - C)_+$. Put $z = H/\sqrt{n}$, $y = (H - t)/\sqrt{n}$. Exact differentiation gives

$$\partial_t \mathcal{H}_n = n\phi(z) + Ht\phi(z) - ns\phi(sy).$$

Taylor's theorem and $\|\phi''\|_\infty = \phi(0)$ give $|n\phi(y) - n\phi(z) - Ht\phi(z)| \leq \phi(0)t^2/2$. Moreover $\partial_s[s\phi(sy)] = -\phi''(sy)$, so $|s\phi(sy) - \phi(y)| \leq \phi(0)(s-1)$. Therefore

$$|\partial_t \mathcal{H}_n(H, t)| \leq \phi(0)(t^2/2 + 1/2 + e) \quad (n \geq 2, H, t \in \mathbb{R}).$$

The norm identity follows by maximizing $(x^2 - 1)e^{-x^2/2}$; its interior extrema occur at $0, \pm\sqrt{3}$. Since $Q''' = -6$, its first three gates imply $Q(v) < 0$ for every $v \geq R$. At $t = \eta v$, the bracket multiplying C is at most $Q(v)$, so its multiplier may be replaced by c . The signed predecessor term is at most rSq . The contact expression is consequently bounded by $\mathcal{H}_n(H, \eta v) + cQ(v) + rSq$, whose derivative in v is at most $L(v)$. Since $L'' = 2/5 - 6c < 0$ and $L(R), L'(R) < 0$, this envelope strictly decreases on the entire closed ray. Its strict endpoint upper bound contradicts contact on both rays, including endpoints, exterior atoms and exterior accumulation points. $\square$

The whole threshold line. No sampled threshold maximum is needed in this lemma. For $R > 0$, Gaussian differentiation gives

$$|\partial_H^2 \mathcal{H}_n(H, R)| \leq M, \quad M = \frac{\sqrt{n}}{4} \left(\frac{n}{n-1} + 1 \right) + \frac{2}{5}R + \frac{2|R^2 - 1|}{5\sqrt{n}}.$$

This follows from $\|\phi'\|_\infty < 1/4$, $\|\phi''\|_\infty < 2/5$, and $\|\phi'''\|_\infty \leq 2/\pi < 4/5$. For a closed threshold cell of midpoint m and halfwidth w , Taylor's theorem bounds both signs by

$$\pm \mathcal{H}_n(H, R) \leq \pm \mathcal{H}_n(m, R) + |\partial_H \mathcal{H}_n(m, R)|w + Mw^2/2.$$

If $H_0 \geq 5\sqrt{n}$, $H_0 - R \geq 5\sqrt{n-1}$, and $\tau \geq \Phi(-5)$, Mills' bound and monotonicity of $H\phi(H/\sqrt{n})$ give, on both threshold tails,

$$|\mathcal{H}_n(H, R)| \leq n\sqrt{n}\tau + (nR + H_0|R^2 - 1|/2)\phi(H_0/\sqrt{n}), \quad |H| \geq H_0.$$

Finally $\mathcal{H}_n(H, -R) = -\mathcal{H}_n(-H, R)$. Finite rational CDF, density and square-root enclosures on all closed cells, together with these two tails, therefore certify the displayed whole-line endpoint bounds.

In particular the positive-budget support certificate at index 27 uses

$$\begin{aligned} R &= 13/5, \quad q = 813/2000, \quad c = 8171/18000, \quad r = 6413121/5478500, \\ b &= 326352124353/10^{12}, \quad E = 14331929737/10^{12}, \quad S = 1019049330731/10^{12}. \end{aligned}$$

The contrary hypothesis $C_{27} \geq C_E + 2/45$, the finite cap and the signed-moment inequality supply these bounds. The finite support certificate in [10] checks every polynomial gate and both strict endpoint inequalities on 580 adjacent closed threshold cells covering $[-29, 29]$, and on the two tails. Thus this support implication applies at 27 without extending an estimate proved only for $n \geq 36$.

Write $v = n - 1$, $s = \sqrt{n/v}$, $u = |t|$, and

$$A = \frac{u^3 - 3u}{6}, \quad B = \frac{u^4 - 6u^2 + 3}{24\sqrt{v}}, \quad D = \frac{u^5 - 15u}{120v}.$$

Lemma 16.3 (Fifth-order identity with a uniform finite remainder). *If $u > \sqrt{3}$, then, with $z = H/\sqrt{v}$,*

$$\frac{\mathcal{H}_n(H, \pm u)}{s^3} = \pm A(1 - z^2)\phi(z) + B\phi^{(3)}(z) \mp D\phi^{(4)}(z) + R,$$

where the sign of the remainder need not be fixed and

$$|R| \leq E_v^{(5)}(u) := \frac{51u^6}{14400v^{3/2}} + \frac{51(3u^2 - 2)}{960v^{3/2}} + \frac{u}{8v^2}.$$

This statement is uniform over every real H .

Proof. Let $F_w(x) = \Phi(x/\sqrt{w})$, with primes denoting x derivatives. The heat identity $\partial_w F_w = F_w''/2$ and ordinary Taylor's theorem give, with unlabelled derivatives at (v, H) ,

$$\begin{aligned} F_{v+1} &= F_v + F_v''/2 + F_v^{(4)}/8 + R_0, & |R_0| &\leq \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty/48, \\ F_{v+1}' &= F_v' + F_v'''/2 + F_v^{(5)}/8 + R_1, & |R_1| &\leq \sup_{w \in [v, v+1]} \|F_w^{(7)}\|_\infty/48, \\ F_{v+1}'' &= F_v'' + F_v^{(4)}/2 + R_2, & |R_2| &\leq \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty/8. \end{aligned}$$

Expand $F_v(H - t)$ through degree five, with remainder at most $u^6 \|F_v^{(6)}\|_\infty/720$. In

$$\frac{\mathcal{H}_n}{n^{3/2}} = F_v(H - t) - F_{v+1}(H) + tF_{v+1}'(H) - \frac{t^2 - 1}{2}F_{v+1}''(H),$$

the first three derivative coefficients cancel. The next three are

$$\left(\frac{t}{2} - \frac{t^3}{6}\right)F_v''' + \frac{t^4 - 6t^2 + 3}{24}F_v^{(4)} + \left(\frac{t}{8} - \frac{t^5}{120}\right)F_v^{(5)}.$$

The remaining absolute value is bounded by

$$\left\{ \frac{u^6}{720} + \frac{1}{48} + \frac{u^2 - 1}{16} \right\} \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty + \frac{u}{48} \sup_{w \in [v, v+1]} \|F_w^{(7)}\|_\infty.$$

Fourier inversion of the Gaussian, differentiated under its absolutely integrable Fourier transform, gives

$$\|\phi^{(k)}\|_\infty \leq \frac{1}{\pi} \int_0^\infty y^k e^{-y^2/2} dy.$$

Integration by parts yields $\|\phi^{(5)}\|_\infty \leq 8/\pi < 2.55$ and $\|\phi^{(6)}\|_\infty \leq 15\phi_0 < 6$. Since $F_w^{(j)} = w^{-j/2}\phi^{(j-1)}$, the displayed remainder becomes $s^3 E_v^{(5)}(u)$ after multiplication by $n^{3/2}$. The three retained derivative coefficients give the identity. $\square$

Lemma 16.4 (A lower tangent bound for the fourth Gaussian derivative). *For every real z ,*

$$\phi^{(4)}(z) \geq \phi_0(3 - 15z^2/2).$$

Proof. The same Gaussian Fourier integral gives $\phi^{(4)}(z) = \pi^{-1} \int_0^\infty y^4 e^{-y^2/2} \cos(yz) dy$. Use the globally valid inequality $\cos w \geq 1 - w^2/2$ inside this integral. Its fourth and sixth Gaussian moments are $3\phi_0$ and $15\phi_0$, respectively. $\square$

Lemma 16.5 (Uniform signed fifth-order envelope). *Suppose $u > \sqrt{3}$ and $A, B, D \geq 0$. If $W = A - 15D/2 > 0$, put*

$$T = A - 3D + \frac{9B^2}{4W}.$$

If $T \geq A/2 + 2B + 3D$, then, for every real H and either sign,

$$\mathcal{H}_n(H, \pm u) \leq s^3 \{\phi_0 T + E_v^{(5)}(u)\}.$$

Proof. For positive $t = u$ and $|z| \leq 1$, divide the retained terms by ϕ_0 and use the preceding tangent bound to obtain

$$A - 3D - (A - 15D/2)z^2 + 3B|z| \leq A - 3D + 9B^2/(4W).$$

Here $(1 - z^2)\phi(z) \leq (1 - z^2)\phi_0$ and $|\phi^{(3)}(z)| \leq 3|z|\phi_0$ on this interval. For $|z| \geq 1$ the first term is nonpositive, and the Fourier bounds $\|\phi^{(3)}\|_\infty < 2\phi_0$ and $\|\phi^{(4)}\|_\infty \leq 3\phi_0$ give $2B + 3D$. For negative $t = -u$, the first term has global upper bound $2e^{-3/2}A\phi_0 < A\phi_0/2$, and the other terms give $(2B + 3D)\phi_0$. The hypothesis on T covers both exterior cases. The inequality $2e^{-3/2} < 1/2$ follows from four positive terms of the exponential series at $3/2$. Finally add the uniform remainder. $\square$

16.3 Finite bands and the exact support polynomial

For integers $36 \leq N \leq L$, use the simultaneous endpoint bounds

$$B_+ = \frac{u^4 - 6u^2 + 3}{24\sqrt{N-1}}, \quad D_+ = \frac{u^5 - 15u}{120(N-1)}, \quad D_- = \frac{u^5 - 15u}{120(L-1)}.$$

For an infinite index range, set $D_- = 0$ instead. Put

$$W_* = A - 15D_+/2, \quad T_* = A - 3D_- + \frac{9B_+^2}{4W_*}.$$

The proof of Lemma 16.5 applies uniformly on the whole band if $W_* > 0$ and $T_* \geq A/2 + 2B_+ + 3D_+$. In particular the favorable term is frozen at the upper index endpoint, and the adverse terms at the lower one. No monotonicity of a signed coefficient is assumed.

Let (P, H) be an attained positive closed maximizing pair, standardized to mean zero and variance one. Write $C = C_n \geq C_E$, $\rho = \mathbb{E}[X]^3$, $\beta = \mathbb{E}[X|X|]$, and $d = n(C_{n-1} - C_n)_+$. The finite contact/predecessor bound is

$$\rho I(t) \leq \mathcal{H}_n(H, t) + C\{-|t|^3 + 3\rho t^2/2 + 3\beta t + \rho e_n\} + \rho s_n d,$$

with $I = 0$ on the entire closed support. Suppose also, for a finite band, that $C_n - C_E \geq K/n$, where $K > 0$. The published finite cap gives

$$c = .4095 + K/L < C_n, \quad \rho < r = \min \left\{ \frac{.3031(.646)}{c - .3031}, \frac{.3322(.429)}{c - .3322} \right\}.$$

For the unrestricted nonnegative case put $K = 0$, $c = .4095$. The sign-moment inequality $\rho \geq (1 + \beta^2)/\sqrt{1 - \beta^2}$ supplies $|\beta| < b$ whenever $(1 + b^2)^2 > r^2(1 - b^2)$ and $0 < b < 1$. All these statements concern the same current law. The lower endpoint cannot replace L in K/L .

When a preceding resource enclosure supplies sharper actual bounds $\rho \leq r$ and $|\beta| \leq b$, these may instead be used in the same polynomials. Their validity is the only extra premise.

Use rational $\ell^2 \leq N - 1$ and $S^2 > N/(N - 1)$. For $u = |t|$ define

$$Q(u) = -u^3 + 3ru^2/2 + 3bu + .384r/N.$$

First check $Q < 0$, before replacing C by its lower bound c . Replace $1/\sqrt{N - 1}$ by $1/\ell$ in the adverse remainder and in the term $2B_+$ of the domination gate. Write E_* for that rational remainder. Strict negative contact for every n in the band follows from the polynomial inequality

$$\frac{2}{5}S^3 \left\{ (A - 3D_-)W_* + \frac{(u^4 - 6u^2 + 3)^2}{256(N - 1)} \right\} + S^3 E_* W_* + \{cQ + rSq\}W_* < 0.$$

The second polynomial gate is $W_* > 0$. The third is

$$\{A/2 - 3D_- - 3D_+ - 2B_+\}W_* + \frac{(u^4 - 6u^2 + 3)^2}{256(N - 1)} > 0.$$

This is the cleared-denominator version of the domination hypothesis. The bound on T_* is positive, so the replacement $\phi_0 < .4$ and $s_n < S$ both have their correct direction.

The exact kernel tests these polynomials on all adjacent closed intervals of length .05 from R to 4.5. Positive Bernstein coefficients prove their signs on every point of each cell, including endpoints. On the rays $u \geq 4.5$ use the imported finite bound $|\mathcal{H}_n(H, t)| \leq u^3/15 + u/5 + 1/2$. The resulting cubic and its first three derivatives are strictly negative at 4.5, as is the coefficient Q before replacing C by c . This excludes the whole rays. Thus, if $d \leq q$, actual support is in $(-R, R)$, including exclusion of arbitrarily small exterior atoms and exterior accumulation points.

16.4 The support rows used in this proof

For the all-index local argument, take $K = 0$, $R = 3.84$, $q = .19$ on

$$[45, 49], [50, 59], [60, 89], [90, 134], [135, 199], [200, \infty).$$

Only indices at least 47 are used. The last row uses $D_- = 0$; there is no upper index restriction. The exact checker reconstructs all closed u cells and the two exterior rays. Missing an index band or the infinite row is an acceptance failure.

The finite base uses five further rows:

$[N, L]$	K	R	q
[40, 40]	1.5	3.3	.88
[41, 49]	1.5	2.9	1.5/40
[45, 45]	.7	3.5	.8341
[46, 98]	.7	3.5	.7/45
[46, 46]	.54	3.4	79/450

The second and fourth are used only on $41 \leq n \leq 45$ and $46 \leq n \leq 50$, respectively. Retaining their originally certified upper endpoints makes their inequalities weaker, so restricting the application introduces no new arithmetic assertion. These are support implications, not yet exclusions; the resource calculation supplies the latter.

16.5 One bounded-threshold series

Put $c_k = (-1)^k / [2^k k! (2k + 1)]$, and let $(x)_j$ denote the rising factorial. Define the following rational polynomials:

$$P_j(H, t) = \sum_{k=0}^{j+1} \frac{c_k (k + 1/2)_{j+1-k}}{(j + 1 - k)!} (H - t)^{2k+1} - c_{j+1} H^{2j+3} \\ + \frac{(-1)^{j+1} t H^{2j+2}}{2^{j+1} (j + 1)!} + \frac{(-1)^j H^{2j+1} (t^2 - 1)}{2^{j+1} j!}.$$

In particular,

$$P_0 = (t^3 - 3t)/6, \quad P_1 = \frac{3t - t^3}{4} H^2 + \frac{t^4 - 6t^2 + 3}{8} H + \frac{-t^5 + 10t^3 - 15t}{40}.$$

Fix $h > 0$, $E(x) = (193/71)^{\lceil x \rceil}$, and

$$B_r = \frac{7(h+r)}{6} E((h+r)^2/6) + \left(h + r + \frac{h(r^2+1)}{8} \right) E(h^2/8).$$

Lemma 16.6. *For every $n \geq 4$, $|H| \leq h$ and $|t| \leq r$,*

$$\left| \mathcal{H}_n / \phi(0) - \sum_{j=0}^5 P_j / n^j \right| \leq 4^7 B_r / n^6.$$

Proof. Put $z = 1/n$. Integration of the Gaussian exponential series gives $\mathcal{H}_n / \phi(0) = S(z)/z$, where

$$S(z) = \sum_{k \geq 0} c_k \{ (H - t)^{2k+1} (1 - z)^{-k-1/2} - H^{2k+1} \} z^k \\ + t e^{-H^2 z/2} + \frac{1}{2} z H (t^2 - 1) e^{-H^2 z/2}.$$

Its constant coefficient vanishes, and expansion of the binomial factor gives $[z^{j+1}]S = P_j$. At $z_0 = 1/4$, the sum obtained by replacing all coefficients by their absolute-value majorants is at most B_r : use $(1 - z_0)^{-1/2} < 7/6$, $1/(2k + 1) \leq 1$, and $e < 193/71$. The latter bound follows, for example, by summing the exponential series through degree six and bounding the rest by a geometric series with ratio $1/8$. All products and rearrangements are therefore absolutely convergent. The discarded terms have degrees at least seven in S , so their sum at $0 \leq z \leq z_0$ is at most $(z/z_0)^7 B_r$. Divide by z to get the result. This proves a finite uniform bound, including every threshold in the stated interval and every $n \geq 4$. $\square$

16.6 A common contact test for radius and strip refinement

Suppose $n \geq 47$, $0 \leq C - C_E \leq .7/n$, $\delta_n \leq .19$ and $|X| \leq R_0$. A previously acquired row gives

$$\rho \leq r, \quad 0 \leq b_l \leq \beta \leq b_u < 1, \quad |H| \leq h.$$

Define $Q(t) = -|t|^3 + 3\rho t^2/2 + 3\beta t$ and $L_P(t) = C_E Q(t) + \phi(0)(t^3 - 3t)/6$. For $t > 0$, upper polynomials for $L_P(t)$, $Q(t)$ are

$$(1/15 - .4095)t^3 + 1.5(.40974)rt^2 + (3(.40974)b_u - .3988/2)t, \quad -t^3 + 1.5rt^2 + 3b_u t.$$

For $t < 0$, write $u = |t|$ and use

$$(-.3988/6 - .4095)u^3 + 1.5(.40974)ru^2 + (.2 - 3(.4095)b_l)u, \quad -u^3 + 1.5ru^2 - 3b_l u.$$

These coefficient choices are valid because $b_l \geq 0$ is checked first. On any closed cell of one sign, let L_J be an upper bound for the first polynomial and let $Q_J \geq 0$ bound the positive part of the second. For $j = 1, 2, 3$, take m_j to be the positive part of the largest tensor Bernstein coefficient of P_j on $[-h, h] \times J$. For $j = 4, 5$, the absolute monomial bound $\sum_{a,b} |[H^a t^b] P_j| h^a R_J^b$ suffices, where $R_J = \max_{t \in J} |t|$. The support contact is then bounded above by

$$L_J + \frac{.7Q_J}{47} + .4 \left(\sum_{j=1}^5 \frac{m_j}{47^j} + \frac{4^7 B_{R_J}}{47^6} \right) + r \left(.19S + \frac{.384(.425)}{47} \right), \quad S > \sqrt{47/46}. \quad (16.1)$$

Indeed substitute the finite series in contact, split $C = C_E + (C - C_E)$, and use $C < .40974 + .7/47 < .425$. Every error retained in (16.1) is nonnegative and decreases with n . A strictly negative rational result therefore excludes the entire cell for every integer $n \geq 47$, including atoms and accumulation points.

For a coordinate strip, conditional centering gives

$$X = \ell(Y - p), \quad \ell = \sqrt{(1 - \theta)/(p(1 - p))}, \quad \beta = (1 - \theta)(1 - 2p) + \kappa, \quad |\kappa| \leq \theta.$$

If $p \in [p_l, p_u] \subset (0, 1/2)$ and $\theta \leq \Theta$, put $\ell_l \leq \sqrt{(1 - \Theta)/(p_u(1 - p_u))}$. The forbidden strip $Y \notin (-.55, 1.55)$ maps into

$$X \leq -\ell_l(p_l + .55), \quad X \geq \ell_l(1.55 - p_u).$$

Apply (16.1) between these thresholds and $\pm R_0$. An interval beyond the known radius is empty; an equal endpoint is a singleton that is still checked. If only an upper β bound is given, use $b_l = (1 - \Theta)(1 - 2p_u) - \Theta$, after checking its nonnegativity.

The selected consequences used in the resource schedule are:

input rows	prior radius	conclusion	closed cells
10 central preparation rows	3.84	$ X < 2.5$	1080
45 central rows	2.5	$-.55 < Y < 1.55$	1431
38 left rows	2.7	$-.55 < Y < 1.55$	3073

The respective largest contact bounds are below $-.084$, $-.097$ and $-.01658$. These implications acquire no row by themselves: each is used only after its input row is proved by the preceding resource step. The left radii are applications of the all-threshold support polynomial, with $(N, L, K, q) = (47, \infty, 0, .19)$ and the actual moment caps

$$(r, b, R) = (1.390, .430, 3), \quad (1.2563, .3926, 2.7).$$

Thus the same heat proof supplies the initial radius, both left contractions, and the five finite-base rows. The bounded-threshold series supplies the central contraction and both coordinate strips.

17 Entry as a resource enclosure

The entry calculation has one job: turn an attained extremizer into a short list of simultaneous hypotheses for the local comparison. Its many successive tables are applications of one enclosure principle. A table is a cover of possible laws; a refinement intersects that cover with newly proved inequalities. There is no assertion that an iteration converges, and no improved bound is used to justify itself.

Throughout, X has mean zero and variance one, $n \geq N$, and H is the raw sum threshold. Put

$$\rho = \mathbb{E}|X|^3, \quad \beta = \mathbb{E}[X|X|], \quad \gamma = \mathbb{E}X^3, \quad M_j = \mathbb{E}|X|^j, \\ C = C_n, \quad e = C - C_E \geq 0, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

The input is an attained positive closed maximum, together with the contact inequality stated below, an actual support bound, and an independently established score cap. The global proof supplies these inputs. A conditional upper bound on moments is never used to acquire the support needed to define those moments.

17.1 The conserved quantity

Write $\tau(t) = \mathbb{E}[X \mathbf{1}_{X \geq t}]$. It is nonnegative, $\int \tau = 1$, and $2 \int |t| \tau(t) dt = \rho$. On each bounded gap $g = (a, b)$ of the closed support it has the constant value τ_g . Define

$$\begin{aligned} J &= \sum_g \tau_g (b - a)^2, & J_1 &= \sum_g \tau_g \int_a^b (t - a) t dt, \\ J_2 &= \sum_g \tau_g \int_a^b (t - a) t |t| dt, & D &= (3 + \sigma) \rho - \gamma - 3J, \quad \sigma = \sqrt{10}, \\ L &= \sum_g \tau_g \int_{[a, b)} (t - a) \nu(dt), & \nu &= \mathcal{L}(H - S_{n-1}). \end{aligned}$$

The half-open convention includes the lower endpoint with weight zero and excludes the upper endpoint. In particular it treats atoms without an implicit midpoint convention. Bounded support makes all sums absolutely convergent. The gap calculation below proves $D \geq 0$; clearly $L \geq 0$.

Put $\varphi = \phi(H/\sqrt{n})$ and

$$\begin{aligned} K_1 &= \mathbb{E}[X|X|^3] - 3\beta - \frac{3}{2}\rho\gamma, \\ K_q &= M_5/4 - \rho M_4/2 - \frac{3}{2}\beta\gamma + \rho/2, & U &= J_1 + (M_4 - 3)/6. \end{aligned}$$

The exact remainder identity is

$$n e \rho + n C_E \rho (1 - e^{-w}) + n \phi(0) e^{-w} D/6 + n^{3/2} L = \mathcal{R}, \quad w = H^2/(2n), \quad (17.1)$$

where

$$\mathcal{R} = C\{3\beta J/2 + 3\rho J_1 - 3J_2 - K_q\} + H(CK_1 + \varphi U) - H^2(C\rho + \gamma\varphi/2) + E_n. \quad (17.2)$$

The complete Fourier estimate supplies a *coherent* pair $|E_n| \leq K_F + s_F |H|$: the constant from one estimate is not combined with the slope from another unless both were separately proved. Section 18 derives (17.1) and the whole Fourier bound from the contact interface.

Lemma 17.1 (Removing the Gaussian weight). *Suppose a first pass proves $\mathcal{R} < r < Nc_0$, where $C\rho \geq c_0$. Every later valid upper bound $\mathcal{R} \leq r'$ then gives*

$$n e \rho + n \phi(0) D/6 + n^{3/2} L \leq \mathcal{R}, \quad e^{-w} \geq 1 - \frac{5r'}{2n}, \quad nD \leq \frac{15000r'}{997}. \quad (17.3)$$

The last two bounds use $C_E \rho > 2/5$ and $\phi(0) > 997/2500$.

Proof. Rearrange the exact identity before changing its Gaussian weight: $\varphi(\gamma + 3J)/6 = C\rho + \sqrt{n}L - \mathcal{R}/n > 0$. Thus $C_E \rho - \phi(0)D/6 > 0$. Replacing e^{-w} by one in (17.1) lowers its left side, proving the first assertion. The original nonnegative phase term proves $1 - e^{-w} \leq r'/(nC_E \rho) < 5r'/(2n)$. The defect bound follows from the first assertion. The initial sign license is proved before any feedback is used. $\square$

The same identity also gives a useful score update. If this cell forces $D \geq d_l \geq 0$ and a preceding pass gives raw ceiling r , then

$$C \leq .40974 + \frac{r/N - .3988(1 - 5r/(2N))d_l/6}{\rho_l}.$$

A negative numerator excludes the cell because $e \geq 0$. Otherwise its numerator increases with $1/n$, and division by the positive lower ρ_l is outward. Thus the update is valid for every $n \geq N$ and uses only the previous raw bound.

For later local comparisons we retain the sharper phase resource. If previous bounds give $\mathcal{R} \leq r$, $nD \leq Nd$, then

$$ne\rho + \kappa nD + n^{3/2}L \leq \mathcal{B}(H), \quad \kappa = 997/15000, \quad (17.4)$$

where an upper bound for $\mathcal{B}$ is obtained from the raw quadratic by increasing its curvature by

$$q_{\text{extra}}(\rho) = \frac{1 - 5r/(2N)}{2} \left(.4095\rho - \frac{d}{15} \right) > 0. \quad (17.5)$$

Indeed the subtracted exact phase term is $n(1 - e^{-w})(C_E\rho - \phi(0)D/6)$. Use $1 - e^{-w} \geq we^{-w}$, $D \leq d$, $\phi(0) < .4$, and the previous density floor. The bound Q_{budget} includes (17.5); it must not be charged a second time.

17.2 One positive-measure rule

Suppose zero lies in the gap $(-A, B)$, $A, B > 0$. Let

$$p = \mathbb{P}(X > 0), \quad q = 1 - p, \quad m = \mathbb{E}X_+ = \mathbb{E}X_-, \quad V = \mathbb{E}[X \mid \mathbf{1}_{X>0}], \\ W = X - V, \quad \theta_{\pm} = \mathbb{E}[W^2; \pm X > 0], \quad \theta = \theta_+ + \theta_-.$$

Then $m^2 = (1 - \theta)pq$, and the conditional means have magnitudes $\bar{A} = m/q$, $\bar{B} = m/p$. Set $\Delta_A = \bar{A} - A \geq 0$, $\Delta_B = \bar{B} - B \geq 0$. We also use the unnormalised endpoint displacements

$$M_+ = \mathbb{E}[(X - B)\mathbf{1}_{X>0}] = p\Delta_B, \quad M_- = \mathbb{E}[(-X - A)\mathbf{1}_{X<0}] = q\Delta_A.$$

Lemma 17.2 (Positive-measure pricing). *For $x = |X|$, $a = |V|$, and a function h constant on each sign class,*

$$\mathbb{E}[h(|X|^j - |V|^j)] = \mathbb{E} \left[hW^2 \sum_{k=0}^{j-2} (k+1)a^k x^{j-2-k} \right], \quad j \geq 2. \quad (17.6)$$

Also, for $d\mu_0 = (|X| - 1)^2(|X| + 1/2) dP$,

$$\mu_0(\mathbb{R}) = \rho - 1, \\ M_4 - 1 = \int \frac{(x+1)^2}{x+1/2} d\mu_0, \\ M_5 - 1 = \int \frac{x^3 + 2x^2 + 3x + 3/2}{x+1/2} d\mu_0, \\ K_1 + 3\rho\beta = \int \frac{\text{sgn}(X)x(x+2-3\rho/2)}{x+1/2} d\mu_0.$$

For any nonnegative resources z_i satisfying $\sum a_i z_i \leq r$, $a_i > 0$, a signed linear cost obeys

$$\sum b_i z_i \leq r \max\{0, b_1/a_1, \dots, b_k/a_k\}. \quad (17.7)$$

Proof. Divide $x^j - a^j - ja^{j-1}(x-a)$ by $(x-a)^2$; the linear term has zero conditional mean. The radial identities follow by multiplying out and using $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$. Finally each b_i is at most a_i times the displayed maximum, and all z_i are nonnegative. $\square$

This one rule accounts for the moment, signed Taylor, endpoint movement, outer-gap, and confinement-error estimates. In particular, no two costs may each spend the full residual and then be added.

17.3 The simultaneous gap budget

Let $p_* = (4 - \sigma)/2$ and $T_+ = \mathbb{E}X_+^3 - mB^2$, $T_- = \mathbb{E}X_-^3 - mA^2$. If $J_o = J - m(A+B)^2$, signwise integration gives

$$D = m(\sqrt{\sigma+1}A - \sqrt{\sigma-1}B)^2 + (\sigma-1)T_+ + (\sigma+1)T_- + 3(T_+ + T_- - J_o). \quad (17.8)$$

Moreover, writing $R_+ = \mathbb{E}[X_+(X_+ - B)]$ and similarly R_- ,

$$T_+ + T_- - J_o \geq 2BR_+ + 2AR_- \geq 0.$$

If zero belongs to the support, every gap has one sign and $J \leq \rho$, so $D \geq \sigma\rho - \gamma \geq (\sigma-1)\rho$. Otherwise (17.8) proves $D \geq 0$. Consequently $D < (\sigma-1)\rho$ licenses the actual central gap. We also have $J \leq \rho + 2m \leq \rho + 1$; if $D < 2$, then $T_+ + T_- \leq D/2$ and $A+B \geq \sqrt{2-D}$.

To verify the gap estimate, on a positive outer gap (u, v) , $(v-u)^2 \leq 2 \int_u^v (t-B) dt$. Summing with weights τ_g and applying Tonelli gives $J_o^+ \leq T_+ - 2BR_+$; reflect for the other sign. Expanding the definition of D gives (17.8). The inequality $AB \leq 1$ follows by integrating $(X+A)(X-B) \geq 0$, and $m \leq 1/2$ follows from $\mathbb{E}|X| \leq 1$.

Conditional expansion about $\bar{A}, \bar{B}$ now gives the stronger single inequality

$$D \geq \frac{(6+2\sigma)(1-\theta)^{3/2}}{\sqrt{pq}}(p-p_*)^2 + 6m\{(A+B)(\Delta_A + \Delta_B) + \Delta_A\Delta_B\} + 3(\sigma+1)B\theta_+ + 3(\sigma+3)A\theta_- \quad (17.9)$$

For example $\mathbb{E}X_+^3 = m\bar{B}^2 + \mathbb{E}[(X+2\bar{B})(X-\bar{B})^2; X > 0]$ and $R_+ = \theta_+ + m(\bar{B} - B)$. The cubic residual is at least $3B\theta_+$. The deterministic shift left after substituting both signs is $6m\{\bar{A}\bar{B} - AB + B\Delta_B + A\Delta_A\}$, equal to the second term above. The remaining deterministic expression, using $m^2 = (1-\theta)pq$, is exactly the displayed binary square.

17.4 A state, its update, and its exported resource

A state for a closed probability interval $I = [l, u]$ records

$$|X| \leq R, \quad C \in [c_0, c_1], \quad \mathcal{R} \leq r, \quad nD \leq Nd, \quad \theta \leq \Theta < 1, \quad A \geq A_l > 0, \quad B \geq B_l > 0,$$

together with simultaneous signed-moment and coefficient intervals. The distinction between $nD \leq Nd$ and $D \leq d$ is essential. Let $s_+ = \max_I p(1-p)$, $s_- = \min_I p(1-p)$ and

$$b_I = \frac{(6+2\sigma)(1-\Theta)^{3/2}}{\sqrt{s_+}} \text{dist}(p_*, I)^2, \quad r_I = d - b_I,$$

$$K_+ = 3(\sigma+1)B_l, \quad K_- = 3(\sigma+3)A_l, \quad K = \min(K_+, K_-), \quad m_l = \sqrt{(1-\Theta)s_-}.$$

If $r_I < 0$, exclude the entire closed cell. Otherwise (17.9) gives, simultaneously,

$$6m(A_l + B_l)(\Delta_A + \Delta_B) + K_+\theta_+ + K_-\theta_- \leq r_I. \quad (17.10)$$

Thus one may intersect the preceding bounds with

$$\begin{aligned}\Theta' &\leq r_I/K, & A'_l &\geq \sqrt{(1-\Theta)l/(1-l)} - \frac{r_I}{6m_l(A_l + B_l)}, \\ B'_l &\geq \sqrt{(1-\Theta)(1-u)/u} - \frac{r_I}{6m_l(A_l + B_l)}.\end{aligned}\tag{17.11}$$

Each right side is evaluated from established bounds; sequential improvements may then be used by the next expression. A nonpositive endpoint estimate fails this update, rather than excluding a law. Initialization uses the same formulas with the valid lower sum $A + B \geq \sqrt{2-d}$.

In Bernoulli coordinates set $Y = B_0 + U$, $\mathbb{E}[U \mid B_0] = 0$, $B_0 \sim \text{Bernoulli}(p)$, $X = \ell(Y - p)$ and $d_Y = (\Delta_A + \Delta_B)/(\bar{A} + \bar{B})$. Then $\ell^2 = (1-\theta)/(pq)$ and $\lambda = n\mathbb{E}U^2 = npq\theta/(1-\theta)$. The *weighted* versions of (17.10) prove

$$n\theta \leq \frac{Nr_I}{K}, \quad nd_Y \leq \frac{Nr_I}{6(1-\Theta)(A_l + B_l)}, \quad \lambda \leq \frac{Ns+r_I}{(1-\Theta)K}.\tag{17.12}$$

Indeed $nD \leq Nd$ whereas $nb_I \geq Nb_I$. These exports are derived from the weighted resource; multiplying an unweighted variance ceiling by N would not prove them.

For $j = 3, 4, 5$ set

$$L_3(v) = R + 2v, \quad L_4(v) = R^2 + 2Rv + 3v^2, \quad L_5(v) = R^3 + 2R^2v + 3Rv^2 + 4v^3.$$

Put $v_+ = \sqrt{(1-l)/l}$, $v_- = \sqrt{u/(1-u)}$ and $k_j = [1 - (1-\Theta)^{j/2}]/\Theta$, with $k_j = j/2$ at $\Theta = 0$. For $u < 1/2$, valid Bernoulli moment ceilings are, with $s = l(1-l)$,

$$B_3^* = (1-2s)/\sqrt{s}, \quad B_4^* = 1/s - 3, \quad B_5^* = (1-4s+2s^2)/s^{3/2}.$$

They decrease towards $p = 1/2$. Convexity places $(1-\theta)^{j/2}$ below its endpoint chord on $[0, \Theta]$; positive-measure pricing gives

$$M_j \leq B_j^* + r_I \max \left\{ 0, \frac{L_j(v_+) - k_j B_j^*}{K_+}, \frac{L_j(v_-) - k_j B_j^*}{K_-} \right\}.\tag{17.13}$$

The same rule, using $T_+ \leq 2mv_+ \Delta_B + L_3(v_+) \theta_+$ and its reflection, bounds

$$W_o := (\sigma-1)T_+ + (\sigma+1)T_- \leq r_I \max \left\{ \frac{(\sigma-1)v_+}{3(A_l + B_l)}, \frac{(\sigma+1)v_-}{3(A_l + B_l)}, \frac{(\sigma-1)L_3(v_+)}{K_+}, \frac{(\sigma+1)L_3(v_-)}{K_-} \right\}.\tag{17.14}$$

Intersect every new bound with the previous one. The versions which merge θ_+ and θ_- are obtained by replacing both costs by their maximum and both prices by their minimum; they need no new proof.

17.5 The enclosure theorem

On each cell, the formula atlas bounds the four coefficients in (17.2). The result, for each actual sign $s = \text{sgn } H$, is a finite family of quadratics

$$\mathcal{R} \leq \max_j (A_j + F_{j,s}|H| - Q_j^{\text{raw}} H^2), \quad Q_j^{\text{raw}} > 0.\tag{17.15}$$

The maxima retain all score-moment corners. In the initial signed cover, intervals contain both signs of β, γ ; no orientation is discarded. In the probability cover, the central integral bound is a maximum of affine functions of ρ , hence convex. The other ρ -dependence is affine or the maximum of convex quadratics. For fixed H , the score dependence is convex. Endpoints are therefore taken *before* completing the square.

Theorem 17.3 (Finite resource enclosure). *Consider a finite ordered schedule of the following operations:*

1. *Split a cell into an explicitly covered family of closed cells.*
2. *Intersect its enclosures with an inequality proved in the atlas.*
3. *Exclude a cell by an empty necessary interval or a strictly negative nonnegative resource.*
4. *Apply a separately proved support or confinement implication, checking all its hypotheses against this cell's already acquired data.*
5. *Export a raw, weighted, or signed phase bound by (17.15), (17.4), and (17.12).*

If exact outward arithmetic checks every domain, corner and cover, every output contains all attaining laws satisfying the input. The conclusion holds for all integers $n \geq N$ wherever the formula's monotonicity gates are checked. It includes zero conditional variance, singleton intersections and arbitrarily small atoms.

Proof. Induct over the schedule. A split preserves the union, intersections use necessary inequalities, and an excluded cell contains no feasible law. Each callback consumes only earlier hypotheses for that same law. For a scalar raw output, use $fh - qh^2 \leq (f_+)^2/(4q)$ for $h \geq 0$, $q > 0$; using f^2 is a valid weaker bound. For a resource output first add (17.5) to each curvature. A negative maximum excludes the cell; a nonnegative one may be divided by κN to give its next defect parameter. The phase bound follows from the positive root of $A + fh - qh^2 \geq 0$, replacing A by A_+ if necessary. A strict output is exported only when the certificate retains a strict upper gap, such as its strict Fourier or target enclosure; closed-domain coverage does not change strictness. All constants are frozen before evaluating corners. Thus no iteration step relies on an unproved fixed point. The stated index uniformity is that of the displayed finite formulas, not a fixed-law limit. $\square$

For the Fourier cancellation estimate only, reflect (X, H) if needed: $I_0(-X, -H) = I_0(X, H)$. Both odd coefficients change sign, and $\int e^{-x^2/2} |zx^2 - cx^4| dx$ is unchanged by $(z, c) \mapsto (-z, -c)$. Consequently a lower endpoint used there bounds $|\gamma|$, not necessarily the original signed γ . The actual contact, gap and phase data keep their original orientation. A useful additional signed lower bound is

$$\gamma \geq (1 - \Theta)^{3/2} \frac{1 - 2u}{\sqrt{u(1 - u)}} - \left(R + 2\sqrt{\frac{u}{1 - u}} \right) \Theta. \quad (17.16)$$

To prove it expand $\mathbb{E}X^3 = \mathbb{E}V^3 + \mathbb{E}[W^2(X + 2V)]$; the positive residual is nonnegative and the negative residual has the displayed lower bound. The binary term decreases in both $p < 1/2$ and θ . Thus $\max(0, \text{any signed lower bound})$ is a valid lower bound for $|\gamma|$ inside this auxiliary Fourier estimate.

17.6 Support, confinement, and a short acyclic schedule

Two different support implications are useful. Gaussian contact excludes a point by a strict negative upper bound on its contact expression; closed cells cover the finite exterior and a derivative-sign certificate covers any remaining ray. The shared Gaussian support theorem supplies this callback, including the bounded-threshold radius and coordinate-strip contracts; it is a prior theorem, not a test on moments alone. Noise confinement proves $|U| \leq a$ on an already acquired row. It then implies, without a small-noise limit,

$$|X| \leq R_* := \frac{\max\{u + a, 1 - l + a\}}{\sqrt{\min_I p(1 - p)}}. \quad (17.17)$$

The denominator uses $\mathbb{E}U^2 \geq 0$, so zero noise is included. The Fourier/moment formulas remain valid for $R \geq 1$; the paired Pearson choice additionally checks $R > 2g$. In particular there is no intrinsic lower radius 2.3: the radial ratios in Lemma 17.2 increase on all $x \geq 0$, and the other formulas use their explicit domains.

Here is the selected proof schedule. Each row names *an application* of Theorem 17.3, not a different analytic theorem.

Node	Input and operation	Output consumed later
I	$N = 47$, $R = 3.84$, $C \leq .4247$; 13 signed passes plus export	100 closed probability cells; 35 survive; $\mathcal{R} < 1.885$, $nD < 47(.590)$
C	Central preparation, contact radius 2.5, probability refinement	45 cells [.38, .47], width .002; canonical simultaneous input
L	Complete complementary cover, contact radius 3, refinement	39 cells [.302, .38]; all other complementary cells excluded
C1	First confinement from C; radius formula; enclosure	First improved central table
C2	Antichain confinement from C; intersect C1; enclosure	Selected central table: 45 cells, [.38, .47]
S	Separate sign-resource pricing on C2 and L	One left cell excluded; 38 left cells remain
L2	Left contact radius 2.7 from S; enclosure	Selected left table: 38 cells, [.304, .38]

Every shared endpoint belongs to its adjacent closed cells. The confinement producing C2 consumes C, not C2. The selected central local comparison uses C2; it does not need the optional further central refinement in S. The final cover is exactly [.304, .47].

Corollary 17.4 (Uniform acquired entry). *Let an attained positive maximum have $n \geq 47$, $C_E \leq C \leq .4247$, $0 \leq n(C - C_E) \leq .7$, decrement at most .19, and both deleted score caps at most .4253. Given the shared contact support and strip implications, the exact schedule above places its law in one of the 38 selected left rows or 45 selected central rows, with every resource in that row holding simultaneously. This assertion has no upper bound on n .*

Proof. The shared all-threshold support theorem supplies radius 3.84. Apply the finite enclosure theorem in the displayed order, retaining the weighted bounds at every edge. The initialization's stored upper index 98 occurs only through $.4095 + K/98$; here $K = 0$, so it imposes no upper-index restriction. All subsequent index-dependent inequalities are checked for every $n \geq 47$ by their stated monotonicity gates. The accepted closed covers exclude the complementary cells and give the stated 83 rows. $\square$

The local-comparison interface. For each final row supply $p \in I$, $\theta \leq \Theta$, $\lambda \leq \Lambda$, the actual support/amplitude, $d_Y \leq d_0$, both signed phase limits, and all four coherent quadratics bounding (17.4). On a phase/noise subcell, let T_H be the maximum of each concave quadratic at its clipped vertex, then the maximum over corners and the minimum with the scalar bound. If a proved moment inequality is

$$D \geq \alpha d_Y + b\theta + P, \quad \alpha > 0, \quad \ell \geq \ell_-, \quad \lambda \geq L_0,$$

then the exact strict resource supplied to the comparison is

$$nd_Y < G := \frac{T_H/\kappa - NP - b\ell_-^2 L_0}{\alpha}. \quad (17.18)$$

This uses $n\theta = \ell^2\lambda$ and $nP \geq NP$. If $G \leq 0$, the entire closed subcell is excluded. Formula (17.18) is the same resource as (17.4), not an additional allowance.

18 The minimal contact–Fourier foundation

The only contact input needed here is the following explicit interface. At the attained positive maximum, with $F = \mathbb{P}(S_n \leq H)$, $z = H/\sqrt{n}$ and $\nu = \mathcal{L}(H - S_{n-1})$, one has $\nu([y, \infty)) \leq \mathfrak{q}(y)$ for every real y , with equality on the full closed support, where

$$\begin{aligned} n\mathfrak{q}(y) &= nF + \frac{1}{2}\{z\phi(z) + C\rho/\sqrt{n}\} - (\sqrt{n}\phi(z) + 3C\beta/\sqrt{n})y \\ &\quad - \frac{1}{2}\{z\phi(z) + 3C\rho/\sqrt{n}\}y^2 + C|y|^3/\sqrt{n}. \end{aligned} \quad (18.1)$$

This contact statement and attainment are supplied by the global section. Everything that follows in this section is derived from it. We abbreviate $M_4 = \mu_4$, $M_5 = \eta_5$ in the coefficient formulas.

18.1 Compensated smooth inversion

Write $Z_n = S_n/\sqrt{n}$, $f(t) = \mathbb{E}e^{itX}$, and let N be standard normal. The smooth payoff $q_z(x) = (x - z)|x - z|$ gives the exact error

$$\epsilon_n(z) = \mathbb{E}q_z(Z_n) - \mathbb{E}q_z(N) - \frac{2\gamma\phi(z)}{3\sqrt{n}}.$$

Lemma 18.1. *For every real H and $n \geq 1$,*

$$\epsilon_n(H/\sqrt{n}) = -\frac{4}{\pi n} \int_0^\infty \frac{\Im(e^{-itH}V_n(t))}{t^3} dt, \quad V_n(t) = f(t)^n - e^{-nt^2/2}(1 - in\gamma t^3/6). \quad (18.2)$$

The integral is absolutely convergent. With the split at $t = 1/2$,

$$I_{\text{low}} = \int_0^{1/2} \frac{|\Im(e^{-itH}V_n(t))|}{t^3} dt, \quad I_{\text{high}} = \int_{1/2}^\infty \frac{|V_n(t)|}{t^3} dt, \quad (18.3)$$

one has $|\epsilon_n(z)| \leq 4(I_{\text{low}} + I_{\text{high}})/(\pi n)$.

Proof. The scalar identities

$$\int_0^\infty \frac{ty - \sin(ty)}{t^3} dt = \frac{\pi}{4}y|y|, \quad \int_0^\infty \frac{|ty - \sin(ty)|}{t^3} dt = \frac{\pi}{4}y^2$$

follow by scaling and integration by parts. One elementary evaluation of the constant uses $J_a(b) = \int_0^\infty e^{-at}(1 - \cos bt)t^{-2}dt$: $J_a''(b) = a/(a^2 + b^2)$, $J_a(0) = J_a'(0) = 0$, so $J_a(1) = \int_0^1 \arctan(b/a)db \rightarrow \pi/2$. Monotone convergence removes $a > 0$. For a finite signed measure μ with zero mass and first moment and finite absolute second moment, absolute Fubini applied to the *compensated* scalar integrand therefore gives

$$\int (s - H)|s - H| d\mu(s) = -\frac{4}{\pi} \int_0^\infty t^{-3} \Im(e^{-itH}\hat{\mu}(t)) dt. \quad (18.4)$$

Apply this to the law of S_n minus the pushforward, by $x \mapsto \sqrt{n}x$, of the signed density $\phi(x)[1 + \gamma(x^3 - 3x)/(6\sqrt{n})]$. Gaussian integration by parts gives its transform $e^{-nt^2/2}(1 - in\gamma t^3/6)$, total mass one and first moment zero. Since $(x^3 - 3x)\phi = -\phi'''$, integration twice by parts and then across z gives $-\int q_z\phi''' = 4\phi(z)$. This proves the formula and its scale factor. The high integral is finite from $|f| \leq 1$ and Gaussian decay; the low integral is finite by the absolute Fubini argument. $\square$

The payoff is smooth enough for compensated inversion. No discontinuous CDF is inverted, so no atom midpoint convention is introduced.

18.2 Derivation of the conserved identity

Proof. Compute $\mathcal{A} = \int \tau(y)[q(y) - \nu([y, \infty))]dy$ in two ways. Put $r = -q'$. Contact at both endpoints of a gap gives

$$\nu([a, b)) = \int_a^b r(t)dt, \quad \int_a^b [q(y) - \nu([y, \infty))]dy = \int_a^b (t-a)r(t)dt - \int_{[a,b)} (t-a)d\nu(t).$$

The integrand vanishes on the support and $\tau = 0$ on the exterior rays. Summing, and differentiating (18.1), gives

$$\mathcal{A} = \frac{\phi(z)J}{2\sqrt{n}} + \frac{G}{n^{3/2}} - L, \quad G = \frac{3}{2}C\beta J + (H\phi(z) + 3C\rho)J_1 - 3CJ_2. \quad (18.5)$$

All sums and integrals are absolutely convergent on the compact convex hull of the support, since $|J_1| \leq R_0J/2$, $|J_2| \leq R_0^2J/2$ when $|X| \leq R_0$.

For the second computation let X^* have density τ , independent of S_{n-1} . Signwise Fubini gives $\mathbb{E}g'(X^*) = \mathbb{E}[Xg(X)]$ for continuously differentiable g . In particular, $\mathbb{E}X^* = \gamma/2$, $\mathbb{E}(X^*)^2 = \mathbb{E}X^4/3$ and $\mathbb{E}|X^*|^3 = \mathbb{E}|X|^5/4$. Applying the identity to each summand shows that $(S_{n-1} + X^*)/\sqrt{n}$ has density $\tau_{Z_n}(t) = \mathbb{E}[Z_n\mathbf{1}_{Z_n \geq t}]$. Its CDF at z is $T_z = \mathbb{E}[(Z_n^2 - zZ_n)\mathbf{1}_{Z_n \leq z}]$, by integrating that density; the boundary integrand is zero. Substituting the three moments into (18.1) therefore gives

$$\mathcal{A} = F - \frac{\gamma\phi(z)}{2\sqrt{n}} + \frac{H\phi(z)(3 - \mathbb{E}X^4)/6 + CK_q}{n^{3/2}} - T_z. \quad (18.6)$$

Multiply contact equality by X and integrate. By independence and symmetry, $n\mathbb{E}[X\nu([X, \infty))]$ equals $\mathbb{E}[S_n\mathbf{1}_{S_n \leq H}]$. Expanding the other side yields, for $M_z = \mathbb{E}[Z_n\mathbf{1}_{Z_n \leq z}]$,

$$M_z + \phi(z) = -\frac{\gamma z\phi(z)}{2\sqrt{n}} + \frac{CK_1}{n}. \quad (18.7)$$

Finally $q_z(x) = (x-z)^2 - 2(x-z)^2\mathbf{1}_{x \leq z}$, so Gaussian integration gives

$$T_z - \Phi(z) = -\frac{1}{2}\{\mathbb{E}q_z(Z_n) - \mathbb{E}q_z(N)\} + z(M_z + \phi(z)) - z^2(F - \Phi(z)). \quad (18.8)$$

Insert (18.7) and the definition of ϵ_n into (18.6). With

$$P_0 = H^2(C\rho + \gamma\phi(z)/2) - H\{CK_1 + \phi(z)(\mathbb{E}X^4 - 3)/6\} + CK_q,$$

the result is $\sqrt{n}\mathcal{A} = C\rho - \gamma\phi(z)/6 + P_0/n + \sqrt{n}\epsilon_n(z)/2$. Equating this with (18.5) proves

$$C\rho + \sqrt{n}L = \frac{\phi(z)}{6}(\gamma + 3J) + \frac{G - P_0}{n} - \frac{\sqrt{n}}{2}\epsilon_n(z). \quad (18.9)$$

Now substitute $\gamma + 3J = (3 + \sqrt{10})\rho - D$ and $\phi(z) = \phi(0)e^{-w}$, and expand $G - P_0$. $\square$

18.3 One complete low-frequency transfer

Put $f(t) = \mathbb{E}e^{itX}$, $h(t) = e^{-t^2/2 - i\gamma t^3/6}$, $e_n(t) = e^{-nt^2/2}(1 - in\gamma t^3/6)$, and

$$I_0(H) = \int_0^{1/2} |\Im(e^{-itH}(f^n - e_n))| \frac{dt}{t^3}, \quad I_\infty = \int_{1/2}^\infty |f^n - e_n| \frac{dt}{t^3}.$$

The inversion bound (18.3) charges $2\sqrt{n}(I_0 + I_\infty)/\pi$; both integrals below are complete.

Lemma 18.2 (Signed Gaussian transfer). *Suppose $n \geq N > 2$, $\mu_4 \leq M \leq 11.1$, $|\gamma| \leq g \leq 1.85$, $\mu_6 \leq M_6$, and the actual coefficients*

$$a = (3 - \mu_4)/24, \quad z = (10\gamma - \mathbb{E}X^5)/120, \quad c = a\gamma/6$$

satisfy $|a| \leq a_*$, $|z| \leq Z$. Put

$$E_6 = M_6/720 + g^2/72 + (1/2 + g^2/144)^3/5, \quad B_6 = a_* + Z/2 + E_6/4. \quad (18.10)$$

Let $a_0 = (N-1)/(2N)$, $0 < b_0 \leq (45-M)(1-2/N)/96$, and $\lambda_N = \sqrt{\pi/2}(N/(N-1))^{3/2}$. Then, for every real H ,

$$I_0(H) \leq (L_0 + L_H|H|)/\sqrt{n} + L_2/n, \quad L_2 = E_6/(2a_0^2) + B_6^2/(2b_0^3). \quad (18.11)$$

Two admissible choices, each used as a complete triple, are as follows. The triangle choice is

$$L_0 = c_0Z + c_1a_*g + 251g^3/17280, \quad L_H = c_0a_* + 251g^2/4800, \quad (18.12)$$

where $c_0 \geq \lambda_N$, $c_1 \geq \lambda_N/2$. For the cancellation choice, take any rectangle containing the actual (z, c) , $I_u \geq \sqrt{\pi/2}$ and $F_N \geq (N/(N-1))^{3/2}$. Define

$$U(z, c) = \begin{cases} I_u(|z| + 3|c|), & zc \leq 0, \\ I_u||z| - 3|c|| + 6\sqrt{|zc|}e^{-|z|/(2|c|)}, & zc > 0, \end{cases}$$

$$J_u = \max_{\text{corners}} U(z, c).$$

Then

$$L_0 = F_N J_u + \frac{F_N I_u}{N-1} (35a_*g^3/48 + 35Zg^2/24) + 251g^3/17280,$$

$$L_H = F_N I_u \{a_* + 5Zg/[2(N-1)]\} + 251g^2/4800. \quad (18.13)$$

Neither the transfer nor its cancellation choice assumes signs for a, z, c or γ .

Proof. Taylor with integral remainder gives $f - h = -at^4 - izt^5 + r_6$, $|r_6| \leq E_6t^6$ for $0 \leq t \leq 1/2$. Indeed for $w = -t^2/2 - i\gamma t^3/6$, $|w| \leq t^2(1/2 + g^2/144) < 1/6$; the terms through degree five in $1 + w + w^2/2$ are exact, its degree-six term costs $g^2t^6/72$, and the remaining exponential series costs $|w|^3e^{|w|}/6 \leq |w|^3/5$. Add the true sixth-moment remainder.

Four integrations of $1 - \cos u \geq 0$ give $\cos u \leq 1 - u^2/2 + u^4/24$. Symmetrization, using $\mathbb{E}(X - X')^4 = 2\mu_4 + 6$, therefore gives $|f(t)|, |h(t)| \leq e^{-(45-M)t^2/96}$ on $[0, 1/2]$. Every point on their joining segment has the same bound, so polynomial Taylor expansion yields

$$f^n - h^n = n(f - h)h^{n-1} + R_2, \quad |R_2| \leq \frac{1}{2}n^2B_6^2t^8e^{-b_0nt^2}.$$

The r_6 and R_2 integrals are at most

$$\frac{1}{n} \{E_6/(2a_0^2) + B_6^2/(2b_0^3)\} = L_2/n. \quad (18.14)$$

For the remaining comparator put $u = n\gamma t^3/6$. Its imaginary part is bounded by $e^{-nt^2/2}(|u|^3/6 + |H|tu^2/2)$, so its divided integral is at most

$$n^{-1/2}(251g^3/17280 + 251g^2|H|/4800). \quad (18.15)$$

Here the Gaussian moments follow from integration by parts, $\int_0^\infty x^{2j}e^{-x^2/2}dx = (2j-1)!!\sqrt{\pi/2}$, and $\sqrt{2\pi} < 251/100$.

It remains to price the leading quartic and fifth terms. Set $b = n - 1$, $v = b\gamma t^3/6$. The triangle estimate uses $|\sin(Ht + v)| \leq |H|t + b\gamma t^3/6$, $|\cos| \leq 1$; the Gaussian moments of orders two and four give respectively λ_N and $\lambda_N/2$. This proves (18.12). For cancellation, after division by t^3 , rotation by Ht bounds the leading imaginary part by

$$|zt^2 - atv| + a_*t|\sin v - v| + Zt^2|\cos v - 1| + |H|t\{a_*t + Zt^2|\sin v|\}.$$

Multiply by $ne^{-bt^2/2}$ and use the global sine/cosine remainders. The first integral equals $nJ(z, c)/b^{3/2}$, where $J(z, c) = \int_0^\infty e^{-x^2/2}|zx^2 - cx^4|dx$. This norm is convex and $J(-z, -c) = J(z, c)$, so its rectangle maximum is at a corner. For $zc \leq 0$ the integral is $\sqrt{\pi/2}(|z| + 3|c|)$. For $z, c > 0$, split at $r = \sqrt{z/c}$; integration by parts gives

$$J = \sqrt{\pi/2}(z - 3c) + 6cre^{-r^2/2} + 2(3c - z) \int_r^\infty e^{-x^2/2} dx.$$

Discard the last term when nonpositive, and otherwise bound its integral by $\sqrt{\pi/2}$. This proves $J \leq U$; simultaneous sign reflection covers the remaining quadrant. The other four phase terms use moments of orders two, six, eight and ten and give the coefficients $1, 5/2, 35/24, 35/48$ in (18.13). Finally $n/(n - 1)$, $1/(n - 1)$, a_0^{-1} and the allowable damping inverses decrease with n , proving the all-index bounds at N . $\square$

Lemma 18.3 (Retaining the signed Gaussian tail). *For $r > 0$, put*

$$\mathcal{M}(r) = e^{r^2/2} \int_r^\infty e^{-t^2/2} dt, \quad \mathcal{M}_k(r) = \frac{1}{r + \frac{1}{r + \frac{2}{r + \frac{k}{\ddots + \frac{1}{r}}}}} \quad (k \geq 1), \quad \mathcal{M}_0(r) = 1/r.$$

Odd truncations are lower bounds for $\mathcal{M}$; even truncations are upper bounds. For $z, c > 0$, set $x = z/c$ and

$$d_{k+1} = 1, \quad d_j = 1 + \frac{j}{xd_{j+1}} \quad (j = k, \dots, 1), \quad m_k(x) = d_1^{-1}.$$

Then the cancellation norm in Lemma 18.2 satisfies

$$J(z, c) \leq \sqrt{\pi/2}(z - 3c) + \sqrt{zc}e^{-x/2} \left(6 + \frac{2(3-x)}{x} m_k(x) \right), \quad (18.16)$$

where $k = 11$ if $x \geq 3$ and $k = 12$ if $x < 3$. The factor in parentheses is positive. One may take the minimum with the previous bound $U(z, c)$ at each corner of the coefficient rectangle.

Proof. The positive integrals $I_j(r) = \int_0^\infty t^j e^{-rt-t^2/2} dt$ satisfy $I_1 = 1 - rI_0$ and $I_{j+1} = jI_{j-1} - rI_j$ by integration by parts. Thus, writing $q_j = I_j/I_{j-1} > 0$,

$$I_0 = \frac{1}{r + q_1}, \quad q_j = \frac{j}{r + q_{j+1}}.$$

Replacing the final positive q_{k+1} by zero reverses the inequality at each substitution. This proves the alternating finite bounds, since translation gives $I_0 = \mathcal{M}(r)$. No convergence of an infinite continued fraction is required. The displayed rational recursion gives $m_k(x) = r\mathcal{M}_k(r)$ when $r = \sqrt{x}$.

Splitting the norm at $r = \sqrt{z/c}$, as in the preceding proof, gives the exact identity

$$J(z, c) = \sqrt{\pi/2}(z - 3c) + \sqrt{zc}e^{-x/2} \left(6 + \frac{2(3-x)}{x} r\mathcal{M}(r) \right).$$

The multiplier of $\mathcal{M}$ is nonpositive when $x \geq 3$ and positive when $x < 3$. Substitute the corresponding finite lower or upper bound. Positivity of the remaining factor is immediate for $x < 3$; for $x \geq 3$, use $0 < m_{11}(x) < 1$ to bound it below by $6 - 2(x - 3)/x > 4$. Finally the exact norm J , rather than either upper-bound formula, is convex, so its containing rectangle is still exhausted by its four corners. $\square$

At every positive coefficient corner use the displayed finite Mills bound. A negative Gaussian coefficient uses a lower enclosure for $\sqrt{\pi/2}$; a nonnegative coefficient uses an upper one. The remaining factor is positive, so outward upper square-root and exponential enclosures are valid. A simultaneous reflection of both odd coefficients leaves the norm unchanged.

The two coefficient choices in the certificates. All coefficients come from the actual marginal. For $Q = X^2 - \gamma X - 1$, multiplication gives

$$\begin{aligned}\mathbb{E}Q^2 &= \mu_4 - 1 - \gamma^2, & \mathbb{E}X^5 &= \gamma^3 + 2\gamma + \mathbb{E}[Q^2(X + 2\gamma)], \\ \mu_6 &= 1 + 3\gamma^2 + \gamma^4 + \mathbb{E}[Q^2\{(X + \gamma)^2 + 2\gamma^2 + 2\}].\end{aligned}\quad (18.17)$$

For any radius $R_0 \geq |X|$ with $R_0 \geq 2g$, the two signs of $\mathbb{E}X^5 - 10\gamma$ are bounded, at $t = |\gamma|$, by

$$\begin{aligned}P_{R_0}(t) &= R_0(M - 1) + (10 - 2M)t - R_0t^2 + t^3, \\ Q_{R_0}(t) &= R_0(M - 1) + (2M - 10)t - R_0t^2 - t^3.\end{aligned}$$

The gate $R_0 - 2g \geq 0$ permits substituting M for μ_4 in both signs. The triangle choice uses the valid enclosing radius $R_0 = 6$, $a_* = \max(2, M - 3)/24$, $M_6 = RE_5$, $b_0 = (45 - M)(1 - 2/N)/96$ and

$$\begin{aligned}120Z &= \min\{E_5 + 10g, Z_0\}, \\ Z_0 &= \begin{cases} P_6(g), & M \leq 5, P'_6(g) \geq 0, \\ 6(M - 1) + (10 - 2M)^2/[4(6 - g)], & M \leq 5, P'_6(g) < 0, \\ 6(M - 1) + (2M - 10)^2/24, & M > 5. \end{cases}\end{aligned}\quad (18.18)$$

If $M \leq 5$, $P_6 \geq Q_6$ and $P''_6 < 0$ on $[0, g]$; use the increasing endpoint or $t^3 \leq gt^2$ and complete the square. If $M > 5$, $P_6 \leq 6(M - 1)$ and the upper quadratic for Q_6 gives the third row. One may also use the actual radius by setting $r_0 = \max(R, 2g) > g$ and

$$Z_r = \begin{cases} r_0(M - 1) + (10 - 2M)t_* - (r_0 - g)t_*^2, & M \leq 5, \quad t_* = \min\{g, (10 - 2M)/[2(r_0 - g)]\}, \\ r_0(M - 1) + (2M - 10)t_* - r_0t_*^2, & M > 5, \quad t_* = \min\{g, (2M - 10)/(2r_0)\}.\end{cases}$$

Indeed for $M \leq 5$, $P_{r_0} \geq Q_{r_0}$ and $t^3 \leq gt^2$; for $M > 5$, $P_{r_0} \leq r_0(M - 1)$ and dropping $-t^3$ bounds Q_{r_0} . Each displayed quadratic is maximized at its clipped vertex. Thus the enclosure uses $120Z = \min(E_5 + 10g, Z_0, Z_r)$. It uses $c_0 \geq \sqrt{\pi/(16a_0^3)} = \lambda_N$ and $c_1 = c_0/2$. The weaker choice $c_1 \geq \sqrt{\pi/(256a_0^5)} = \lambda_N N/[2(N - 1)]$ is also permitted by the same transfer lemma.

For the paired choice the already proved cell has $0 \leq \gamma_l \leq |\gamma| \leq g \leq 1$, $M \leq 2$, and $R > 2g$. After reflecting only this auxiliary Fourier integral if necessary, set

$$\begin{aligned}a_* &= (2 - \gamma_l^2)/24, & b_0 &= (43/96)(1 - 2/N), \\ M_6 &= \min\{RE_5, 1 + 3g^2 + g^4 + (R^2 + 2Rg + 3g^2 + 2)(M - 1 - \gamma_l^2)\}, \\ t_* &= \min\{g, (10 - 2M)/[2(R - g)]\}, \\ 120Z &= \min\{E_5 + 10g, R(M - 1) + (10 - 2M)t_* - (R - g)t_*^2\}, \\ z_l &= \max\{-Z, [8\gamma_l - \gamma_l^3 - (R + 2\gamma_l)(M - 1 - \gamma_l^2)]/120\}, & z_u &= Z, \\ c_l &= (3 - M)\gamma_l/144, & c_u &= (2 - \gamma_l^2)g/144.\end{aligned}$$

Check $M - 1 - \gamma_l^2 \geq 0$ and that the rectangle is ordered. Here $P_R \geq Q_R$, and $P_R(t) \leq R(M - 1) + (10 - 2M)t - (R - g)t^2$, whose maximum is at t_* . The numerator of the lower z bound increases in $\gamma \geq 0$ because its derivative is $10 - 2M + 2R\gamma + 3\gamma^2 > 0$. Pearson's first and third identities give a_* and M_6 . If $3g^2 \leq 2$, the increasing function $2\gamma - \gamma^3$ further licenses $c_u \leq (2g - g^3)/144$. Thus this positive rectangle is one specialization of the signed transfer, with exactly the coefficients used by the fixed acceptance programs.

18.4 The complete high-frequency integral

For $b, \kappa > 0$ and $n > 0$, put

$$T_n = \sqrt{n} \int_b^\infty e^{-n\kappa t^2} t^{-3} dt, \quad x = n\kappa b^2.$$

Lemma 18.4 (HIGHLAGUERRE2). *For every $x > 0$,*

$$T_n \leq \frac{\sqrt{n}e^{-x}}{2b^2} \frac{x+2}{x^2+4x+2}.$$

If $N\kappa b^2 \geq 1/4$, the right side decreases with $n \geq N$.

Proof. Substitute $t^2 = b^2(1+s)$ and use positive Fubini to obtain

$$T_n = \frac{\sqrt{n}e^{-x}}{2b^2} \int_0^\infty \frac{e^{-xs}}{(1+s)^2} ds = \frac{\sqrt{n}e^{-x}}{2b^2} \int_0^\infty \frac{u}{x+u} e^{-u} du.$$

Put $P(u) = u^2 - 4u + 2$ and $D(x) = x^2 + 4x + 2 = P(-x) > 0$. Exact division gives

$$\frac{1}{x+u} - \frac{Q_x(u)}{D(x)^2} = \frac{P(u)^2}{(x+u)D(x)^2} \geq 0,$$

where

$$Q_x(u) = -u^3 + (x+8)u^2 - (x^2+8x+20)u + x^3 + 8x^2 + 20x + 16.$$

Integration uses only $\int_0^\infty u^j e^{-u} du = j!$, proved by integration by parts. It gives $\int Q_x(u) e^{-u} du = (x+3)D(x)$ and hence

$$\int_0^\infty \frac{u}{x+u} e^{-u} du = 1 - x \int_0^\infty \frac{e^{-u}}{x+u} du \leq 1 - \frac{x(x+3)}{D(x)} = \frac{x+2}{D(x)}.$$

For the all-index statement, differentiate $\sqrt{x}e^{-x}(x+2)/D(x)$. After multiplication by positive factors, its derivative has numerator $4 - 10x - 22x^2 - 13x^3 - 2x^4$. This polynomial is negative at $x = 1/4$ and strictly decreasing for $x > 0$. Thus it is negative on the entire required half-line. $\square$

Let $EX = 0$, $EX^2 = 1$, $EX^4 \leq M$, and write $\gamma = EX^3$, $f(t) = Ee^{itX}$. For independent copies suppose an actual event E satisfies

$$P(E) \geq a > 0, \quad |X - X'| \geq d_0 > 0 \text{ on } E, \quad E[(X - X')^2 | E] \leq M_E.$$

The event is measured under the product of the same marginal law. Define the Taylor comparator $e_n(t) = e^{-nt^2/2}(1 - in\gamma t^3/6)$ and

$$I_\infty = \int_{1/2}^\infty |f(t)^n - e_n(t)| t^{-3} dt, \quad L(x) = \frac{x+2}{x^2+4x+2}.$$

The sign of the imaginary term is immaterial for its absolute-value bound. Use $b = 9/10$, $0 < r \leq 1/4$,

$$h = \frac{333}{106} - \frac{(22/7)^3 r^2}{6} > 0, \quad \kappa = a(23/48)^2 d_0^2,$$

$$S_3 = \sum_{j=1}^{20} j^{-3} + \frac{1}{800}, \quad S_5 = \sum_{j=1}^{20} (j-r)^{-5} + \frac{1}{4(20-r)^4}.$$

Set

$$c = \frac{1 - \max_{u \in \{1/4, 81/100\}} [1 - u + (M+3)u^2/12]}{2},$$

$$c_0 = \frac{1}{8} - \frac{M+3}{384}, \quad s = \frac{1 - (M+3)53/600}{2}.$$

The usual damping domain requires $c > 0$.

Lemma 18.5 (HIGH_WINDOW50). *For every integer $n \geq 5$, the complete upper bound is*

$$\sqrt{n}I_\infty \leq A_n + \frac{M_E}{4\pi^{3/2}h\sqrt{a}} \left(S_3 + \frac{3S_5}{nah^2} \right) + \frac{M_E\sqrt{n}}{2\pi^2} e^{-nah^2r^2}$$

$$+ Z_n + \frac{\sqrt{n}}{2b^2} e^{-na(23/48)^2} + \sqrt{n}e^{-n/8} \left(2L(n/8) + \frac{\sqrt{M-1}}{3} \right),$$

where always $A_n = 2\sqrt{n}e^{-nc}$ is permitted, and, if $s > 0$, it can be replaced by the smaller of itself and

$$2\sqrt{n}e^{-nc_0}L(ns/4).$$

The zero-cell term Z_n may be taken as the smaller of

$$\frac{e^{-n\kappa b^2}}{2b^4\kappa\sqrt{n}}, \quad \frac{\sqrt{n}e^{-n\kappa b^2}}{2b^2}L(n\kappa b^2).$$

Each term may be evaluated at a common floor N for all $n \geq N$ provided

$$2Nc' > 1 \quad (c' \in \{c, ah^2r^2, a(23/48)^2, 1/8\}).$$

For the new first term also require $2Nc_0 > 1$; for the second zero-cell option require $N\kappa b^2 \geq 1/4$. If either extra condition is unavailable, retain the corresponding old term. Any fixed collection of permitted windows may be evaluated, and the minimum of its complete six-term bounds remains valid.

Proof. We give the entire frequency partition to make clear what changes. On $[1/2, b]$, symmetrization and $\cos u \leq 1 - u^2/2 + u^4/24$ yield

$$|f(t)|^2 \leq 1 - t^2 + (M+3)t^4/12, \quad |f(t)|^n \leq e^{-nc(t^2)}, \quad c(u) = u/2 - (M+3)u^2/24.$$

Bounding $c(t^2)$ below by c and using $\int_{1/2}^b t^{-3}dt < 2$ gives the old first term. Concavity supplies the more precise chord bound

$$c(u) \geq c_0 + s(u - 1/4), \quad 1/4 \leq u \leq 81/100,$$

because the difference is exactly $(M+3)(u - 1/4)(81/100 - u)/24$. If $s > 0$, extend only the resulting positive exponential majorant to the half-line. The preceding rational Gaussian-tail lemma, with lower endpoint $1/2$, gives

$$\sqrt{n} \int_{1/2}^b e^{-nc(t^2)} t^{-3} dt \leq e^{-n(c_0 - s/4)} \sqrt{n} \int_{1/2}^\infty e^{-nst^2} t^{-3} dt \leq 2\sqrt{n}e^{-nc_0}L(ns/4).$$

For $t \geq b$, conditioning on the actual event and convex Jensen give

$$|f(t)|^n \leq e^{-nE \sin^2(t(X-X')/2)} \leq E[e^{-na \sin^2(td/2)} | E], \quad d = |X - X'|.$$

Positive Tonelli allows a separate partition for every $d > 0$. Put $\ell = 2\pi/d$. The zero cell is $[0, \ell/2]$; the other cells are $\ell[j - 1/2, j + 1/2]$, $j \geq 1$. In each nonzero cell take the core $t = \ell(j + v)$, $|v| \leq r$. These cores are disjoint even for $r \leq 1/2$. For our smaller range $r \leq 1/4$, the elementary sine bound gives $|\sin(\pi v)| \geq h|v|$ throughout a core, and $|\sin(\pi v)| \geq hr$ throughout its wings. The latter uses that $\sin(\pi|v|)$ increases on $[0, 1/2]$.

Since $j - r > 0$, Taylor's formula on the two sides gives

$$\frac{(j - v)^{-3} + (j + v)^{-3}}{2} \leq j^{-3} + 6(j - r)^{-5}v^2.$$

Integrate this majorant against $e^{-nah^2v^2}$ on the full real line. The result is $\sqrt{\pi}[j^{-3}/\sqrt{nah^2} + 3(j - r)^{-5}/(nah^2)^{3/2}]$. The Jacobian is $t^{-3}dt = d^2(4\pi^2)^{-1}(j + v)^{-3}dv$. Summing and averaging d^2 gives the second term. The two series tails are bounded by their decreasing integrals after 20, producing exactly S_3, S_5 . The total wing weight is at most $\int_{\ell/2}^{\infty} t^{-3}dt = d^2/(2\pi^2)$, giving the third term. Extending cores or wings below b only adds a positive integral.

In the zero cell write $u = td/2 \leq \pi/2$. For $u \leq 1/2$, use $\sin u \geq (23/24)u$ and the rational Gaussian-tail lemma with rate κ ; its two choices give Z_n . For $u \geq 1/2$, use $\sin u \geq 23/48$ and total weight $1/(2b^2)$, giving the fifth term. Shared endpoints have zero integration measure, including when the conditional distance distribution has atoms.

Finally keep the comparator on the entire interval $[1/2, \infty)$. Its absolute-value bound splits its integral into

$$\sqrt{n} \int_{1/2}^{\infty} e^{-nt^2/2} t^{-3} dt + \frac{n^{3/2}|\gamma|}{6} \int_{1/2}^{\infty} e^{-nt^2/2} dt.$$

The first part is at most $2\sqrt{n}e^{-n/8}L(n/8)$. For the second, $t \geq 1/2$ implies

$$\int_{1/2}^{\infty} e^{-nt^2/2} dt \leq 2 \int_{1/2}^{\infty} te^{-nt^2/2} dt = 2e^{-n/8}/n.$$

Pearson's elementary identity $E[(X^2 - 1 - \gamma X)^2] = EX^4 - 1 - \gamma^2 \geq 0$ gives $|\gamma| \leq \sqrt{M - 1}$ and proves the sixth term. Any separately proved upper bound $|\gamma| \leq g$ may replace $\sqrt{M - 1}$ in that term. The enclosure uses $g = \min(1.85, \sqrt{M - 1})$.

For uniformity, $\sqrt{n}e^{-nc'}$ decreases when $2nc' > 1$, whereas n^{-1} and $n^{-1/2}e^{-nc'}$ always decrease. The derivative of $L(x)$ is $-(x^2 + 4x + 6)/(x^2 + 4x + 2)^2 < 0$. The new first term is thus decreasing under its stated gate, and the comparator decreases for every $n \geq 5$. The earlier tail lemma proves the separate $N\kappa b^2 \geq 1/4$ gate for its new zero-cell option. These comparisons concern the bounding expressions, so they are uniform even when the marginal law varies with n . $\square$

Events, not artificial kernels. The entry uses only the following actual choices, with the direct rate from the same fourth moment in each case:

event	a	d_0	M_E
$(X - X')^2 \geq 9/25$	$(2 - 9/25)^2/(2M + 6)$	$3/5$	$2/a$
$XX' < 0, D \leq d \leq .8$	$1/(2r_*^2)$	$\sqrt{2 - d}$	$4r_*^2$
$XX' < 0, p \in [l, u]$	$2 \min(l(1 - l), u(1 - u))$	$\max(A_l + B_l, \sqrt{2 - d})$	$2/a$

The first row follows from Cauchy using $\mathbb{E}[(X - X')^2; E] \geq 2 - d_0^2$ and $\mathbb{E}(X - X')^4 = 2\mu_4 + 6$. The second is Lemma 19.1; all its moment and $\alpha < 3$ gates are checked at use. In the third, $D \leq d < 2$ ensures a genuine central gap, $P(XX' < 0) = 2pq$, and restriction of the squared distance gives $M_E = 2/a$. Taking a minimum of *complete* high-frequency bounds is legitimate. No tail is dropped and the third construction also allows a cell crossing $1/2$.

19 Coefficient atlas for the enclosure

These formulas are inequalities for one actual law. They are the two coordinate descriptions of the same enclosure: signed moments before a probability gap is located, and conditional moments afterwards. A bound is used only with its stated gates, and every refinement is an intersection.

19.1 Signed moments with a common first moment

Put $a_{\pm} = (1 \pm \beta)/2$, $b_{\pm} = (\rho \pm \gamma)/2$. Cauchy gives $m^2 \leq a_+ a_-$ and $b_{\pm} m \geq a_{\pm}^2$. Consequently

$$\frac{(1 + \beta)^{3/2}}{\sqrt{1 - \beta}} - \rho \leq \gamma \leq \rho - \frac{(1 - \beta)^{3/2}}{\sqrt{1 + \beta}}, \quad m \geq \max \left\{ \frac{1 + \beta^2}{2\rho}, \frac{a_+^2}{b_+}, \frac{a_-^2}{b_-} \right\}.$$

These statements also hold with a zero atom. Projection of $X|X|$ onto the span of X and $|X|$ gives

$$\mu_4 \geq \rho^2 + \frac{(\gamma - \rho\beta)^2}{1 - \beta^2}.$$

For completeness, this follows by minimizing the nonnegative expectation $E(X|X| - uX - v|X|)^2$ over u, v ; its Gram matrix has diagonal entries one and off-diagonal entry β , and is positive definite.

Two support-localizing moment matrices give the additional upper bounds as follows. If $D_{\pm} = Ra_{\pm} - b_{\pm} > 0$, then

$$2m + \sum_{\pm} \frac{(Rm - a_{\pm})^2}{D_{\pm}} \leq R, \quad \mu_4 \leq R\rho - \sum_{\pm} \frac{D_{\pm}^2}{Rm - a_{\pm}}.$$

Indeed the first follows by summing the determinant inequalities for

$$E[(R - |X|)(1, |X|)^T(1, |X|); \pm X > 0].$$

The second follows from the corresponding matrices with an additional factor $|X|$. One more factor $|X|$ yields, sign by sign,

$$E[X_{\pm}^5] \leq RE[X_{\pm}^4] - \frac{(Rb_{\pm} - E[X_{\pm}^4])^2}{D_{\pm}}.$$

Zero denominators are treated by dropping their nonnegative correction; they never imply exclusion without a separate contradictory numerator. The common m is retained throughout.

A second useful upper bound for m follows by integrating $(x - a)^2(R - x) \geq 0$ for $x = |X| \in [0, R]$. For every fixed $a > 0$,

$$2m \leq \frac{R - \rho + 2a + Ra^2}{a(2R + a)}.$$

The cover uses only the eleven rational choices $a = 2/10, \dots, 12/10$.

Write $\delta = \gamma - \rho\beta$. Cauchy and a square completion on each sign give the signed constant bound

$$\frac{\rho}{2}\mu_4 - \frac{1}{4}\eta_5 \leq \frac{\rho^3}{4} - \frac{(\rho + \gamma)(\delta_+)^2}{8(1 + \beta)^2} - \frac{(\rho - \gamma)((-\delta)_+)^2}{8(1 - \beta)^2}.$$

Indeed on either sign, write $a = \mathbb{E}X_s^2$, $b = \mathbb{E}X_s^3$, $c = \mathbb{E}X_s^4$, $f = \mathbb{E}X_s^5$. Cauchy gives $c \geq b^2/a$, $f \geq c^2/b$, and $\rho c/2 - f/4 \leq \rho^2 b/4 - (c - \rho b)^2/(4b)$. Distance to the half-line $c \geq b^2/a$ proves the stated penalty on summing both signs. Independently, $\mu_4^2 \leq \rho\eta_5$ and the projection bound above give

$$\frac{\rho}{2}\mu_4 - \frac{1}{4}\eta_5 \leq \frac{\rho^3}{4} - \frac{(\gamma - \rho\beta)^4}{4\rho(1 - \beta^2)^2}.$$

The larger of the two nonnegative deductions is retained.

19.2 An actual central gap without assuming $D < 2$

If zero belongs to the support, the signwise gap calculation gives $J \leq \rho$. Thus $D \geq \sqrt{10}\rho - \gamma \geq (\sqrt{10} - 1)\rho$. Whenever $D < (\sqrt{10} - 1)\rho$, there is therefore a central gap $(-A, B)$, $A, B > 0$, with no zero atom. Put $p = P(X > 0)$, $q = 1 - p$, $T_+ = b_+ - mB^2$ and $T_- = b_- - mA^2$. The exact gap decomposition is

$$D = mQ + (\sigma - 1)T_+ + (\sigma + 1)T_- + 3(T - J_o), \quad Q = (\sqrt{\sigma + 1}A - \sqrt{\sigma - 1}B)^2, \quad \sigma = \sqrt{10}.$$

All terms are nonnegative. Hence $T \leq D/(\sigma - 1)$ and

$$(A + B)^2 \geq A^2 + B^2 \geq \frac{\rho - D/(\sigma - 1)}{m}.$$

Moreover $A \leq a_-/m$, $B \leq a_+/m$ and $J \leq \rho + 2m$. The last bound gives the compulsory defect

$$D \geq \sigma\rho - \gamma - 6m.$$

The inequalities $T - J_o \geq 2BR_+ + 2AR_-$ give the useful additional endpoint constraint

$$D \geq (\sigma - 1)b_+ + 6a_+B - (\sigma + 5)mB^2.$$

If $D < (\sigma - 1)b_+$, this downward quadratic forces B to be at least its positive root. The reflected constraint has a_- , b_- , A and the coefficients $\sigma + 1, \sigma + 7$. In a closed moment box, positive coefficients are rounded down, negative coefficients up, and the lower square-root endpoint is used in the resulting root bound.

The actual opposite-sign event has probability $2pq$. The first two moment constraints give lower bounds for both p and q . For independent copies one has exactly

$$E[(X - X')^2 \mid XX' < 0] = \frac{1 + \beta(1 - 2p) + 4m^2}{2pq} = \frac{1 + 4m^2 + \beta}{2p} + \frac{1 + 4m^2 - \beta}{2q}.$$

For an upper bound, use the upper m endpoint and both endpoints of p and β . The expression is convex in p and affine in β . This improves the former $2/P(XX' < 0)$ bound inside the complete HIGH estimate. Both copies still have the same actual marginal.

19.3 Preserving central and outer integral costs

Let

$$P_1(A, B) = B^3/3 + AB^2/2 - A^3/6, \quad P_2(A, B) = B^4/4 + AB^3/3 - A^4/12, \quad G = \rho P_1 - P_2.$$

Then $J_{1,c} = mP_1$, $J_{2,c} = mP_2$. Since $\partial_B G = B(A + B)(\rho - B)$,

$$G(A, B; \rho) \leq G(A, \rho; \rho) = \rho^4/12 + \rho^3 A/6 - \rho A^3/6 + A^4/12.$$

The last expression increases with A because its derivative is $(\rho - A)^2(\rho + 2A)/6$, and is convex in ρ because its second ρ derivative is $\rho^2 + \rho A$. Its closed-rectangle maximum is therefore bounded using the upper A endpoint and both ρ endpoints.

Positive outer gaps satisfy $(t - a)(\rho - t) \leq (\rho - B)_+^2/4$. On negative outer gaps the positive part is bounded by $(R - \rho)^2/4$ after extracting $|t|$. Integration gives

$$3(\rho J_1 - J_2)_o \leq \frac{3}{8}\{(\rho - B)_+^2 T_+ + (R - \rho)^2 T_-\}.$$

The available weighted outer budget is $D - mQ$, not the entire D . The cover intersects this budget with the individual bounds $T_+ \leq b_{+,u} - m_l B_l^2$ and $T_- \leq b_{-,u} - m_l A_l^2$.

For the signed phase term $U = J_1 + (\mu_4 - 3)/6$, retain

$$-RT_-/2 \leq J_{1,o} \leq RT_+/2.$$

The central minimum of P_1 occurs at B_l and an A endpoint; its maximum occurs at B_u and A clipped to the interval at $A = B_u$. A nonnegative polynomial bound is multiplied by m_u for an upper bound and m_l for a lower bound; a negative one uses the opposite endpoint. For example, the binary endpoints $A = 3/2$, $B = 2/3$ make this maximum negative; the sign switch is necessary even when $D < 2$. The fourth-moment sign intervals also sharpen $K_1 = E[X|X|^3] - 3\beta - 3\rho\gamma/2$. The complete K_1, U, ϕ rectangles are used in the signed expression $|CK_1 + \phi U|$, without replacing it by two separate absolute values.

A complete coarse bound before a central gap is licensed. Always $-\mu_4/6 \leq J_1 \leq \mu_4/2$ and $-3J_2 - \eta_5/4 \leq 0$. On a gap of length $h = b - a$, these follow from

$$\int_a^b (t-a)t dt + \frac{1}{2} \int_a^b t^2 dt = hb^2/2, \quad \frac{3}{2} \int_a^b t^2 dt - \int_a^b (t-a)t dt = h(2a+b)^2/6,$$

and, on a negative gap $(-v, -u)$, from $\frac{1}{3} \int_u^v x^3 dx - \int_u^v (v-x)x^2 dx = u^3(v-u)/3$. For $1 \leq r \leq \rho$ and actual support radius $R \leq 6$, set

$$\Gamma(R, r) \geq \max_{r \leq v \leq R} \left\{ -\frac{v^2}{4} + \frac{(R+r)v}{3} - \frac{Rr}{2} + \frac{r^3(2R-r)}{12v^2} \right\}_+.$$

Then

$$3(\rho J_1 - J_2) \leq \frac{3}{2} \Gamma(R, r)(\rho - \gamma) + \frac{3}{8} \rho^2 J.$$

On negative gaps extract $x = |t|$ and discard the nonpositive part $x < \rho$. Replacing ρ by r and the outer endpoint by R gives $\int_r^v (R-x)x(x-r) dx$. Dividing this integral by v^2 gives exactly the displayed expression. Signwise Tonelli now charges $\Gamma \mathbb{E} X_-^3 = \Gamma(\rho - \gamma)/2$. On positive gaps $(t-a)(\rho-t) \leq \rho^2/4$, giving $\rho^2 J/8$. We use 64 closed v -intervals: on each, maximize the concave quadratic at its clipped vertex $2(R+r)/3$ and the positive reciprocal term at the left endpoint. This proves its interval bound. The alternative bound $\Gamma \leq 5/2$ follows at $r = 1, R = 6$ by $3v^4 - 28v^3 + 66v^2 - 11 > 0$ on $[1, 6]$: the endpoint values are 30, 205, and the interior critical values are $(85 \mp 35\sqrt{5})/2 > 0$. These monotone enclosing parameters cover all $r \geq 1, R \leq 6$.

19.4 Moment and gap coefficients for a whole cell

The following coefficient rules are collected here to make the finite calculation transparent. Every rule is an inequality for an actual law; there is no extremal-law assumption inside a cell. Write $v = 1 - \theta$. For $p \in [l, u]$ set

$$\begin{aligned} s_0 &= \min\{l(1-l), u(1-u)\}, & v_* &= \max\{\sqrt{u/(1-u)}, \sqrt{(1-l)/l}\}, \\ r_B &= (1-2s_0)/\sqrt{s_0}, & g_B &= \max_{t=l,u} |1-2t|/\sqrt{t(1-t)}, & b_B &= \max_{t=l,u} |1-2t|, \\ L_3 &= R + 2v_*, & L_4 &= R^2 + 2Rv_* + 3v_*^2, \\ L_5 &= R^3 + 2R^2v_* + 3Rv_*^2 + 4v_*^3, & e_B &= r_B \frac{1-4s_0+2s_0^2}{s_0(1-2s_0)}. \end{aligned}$$

The polynomials $T_3 = x + 2a$, $T_4 = x^2 + 2ax + 3a^2$ and $T_5 = x^3 + 2ax^2 + 3a^2x + 4a^3$ are nonnegative. Their maxima at $x = R$, $a = v_*$, in Lemma 17.2, give, if $\theta \leq \Theta < 1$,

$$\begin{aligned} \rho &\leq r_B + (L_3 - r_B)\Theta, & |\gamma| &\leq g_B + (L_3 - g_B)\Theta, \\ |\beta| &\leq b_B + (1 - b_B)\Theta, & \mu_4 &\leq (1 - \Theta)^2(1/s_0 - 3) + L_4\Theta, \\ \eta_5 &\leq (1 - \Theta)^2 e_B + L_5\Theta. \end{aligned} \tag{19.1}$$

Replacing θ by Θ uses the explicitly checked inequalities $L_3 > r_B, g_B$, $L_4 > 2(1/s_0 - 3)$ and $L_5 > 2e_B$. The exact binary factors are $v^{3/2}, v, v^2, v^{5/2}$; the upper bounds use $v^{3/2} \leq v$ and $v^{5/2} \leq v^2$. Thus cells crossing $1/2$ and zero conditional variances are included. The same cubic expansion, in the later domain $p \in [.37, .47]$, $A \geq .75$, $B \geq 1.05$, gives

$$\rho \geq v^{3/2} \rho_B(p) + 3 \min(A, B) \theta \geq \rho_B(p) + .45\theta, \quad \rho_B(p) = \frac{1 - 2pq}{\sqrt{pq}}. \quad (19.2)$$

Indeed $\rho_B < 1.11$ there and $v^{3/2} \geq 1 - 3\theta/2$. Also $\beta \geq 1 - 2p - 2(1 - p)\theta$.

For the signed coefficient $K_1 = \mathbb{E}[X|X|^3] - 3\beta - 3\rho\gamma/2$, define

$$K(p, v) = (1 - 2p)\{(3v^3/2 - v^2)/(pq) - 3v^3 + v^2 + 3v\}.$$

Direct substitution gives $K_1(V) = -K(p, v)$, with the coefficient 3 unchanged. For $v \geq 2/3$, K decreases in p : both coefficients in braces are nonnegative and $(1 - 2p)/(pq)$ decreases. Its derivative in v has the sign of $1 - 2p$, since the derivative of the braces is at least $9v^2 - 6v + 3 > 0$. Hence both variance endpoints are retained. The error is

$$|K_1(X) - K_1(V)| \leq \Theta\{L_4 + 3 + \frac{3}{2}L_3(R_\rho + g_B)\} \quad (19.3)$$

when $\rho \leq R_\rho$. This follows from the conditional expansions and $\rho\gamma - \rho_V\gamma_V = \rho(\gamma - \gamma_V) + \gamma_V(\rho - \rho_V)$.

Here are the remaining signed charges. They also explain the two alternative bounds retained in the initial, coarse moment cover:

$$-\mu_4/6 \leq J_1 \leq \mu_4/2, \quad -3J_2 - \eta_5/4 \leq 0, \quad (19.4)$$

$$\rho J_1 - J_2 \leq \frac{5}{4}(\rho - \gamma) + \frac{1}{8}\rho^2(\rho + 1), \quad (19.5)$$

$$-75D/104 \leq J_{1,o} \leq 3D/2. \quad (19.6)$$

To prove the first line, a gap of length $h = b - a$ satisfies

$$\int_a^b (t - a)t \, dt + \frac{1}{2} \int_a^b t^2 \, dt = hb^2/2, \quad \frac{3}{2} \int_a^b t^2 \, dt - \int_a^b (t - a)t \, dt = h(2a + b)^2/6.$$

Sum with weight τ_g . On a negative piece $(-v, -u)$, $\frac{1}{3} \int_u^v x^3 \, dx - \int_u^v (v - x)x^2 \, dx = u^3(v - u)/3 \geq 0$; Tonelli proves the J_2 assertion. For (19.5), the positive part is at most $\rho^2 J/8$. The negative part is at most $\mathbb{E}[V \int_1^V (6 - x)x(x - 1) \, dx; V \geq 1]$, $V = X_-$, hence at most $(5/2)\mathbb{E}X_-^3$. The last comparison is equivalent to $3V^4 - 28V^3 + 66V^2 - 11 > 0$ on $[1, 6]$: the endpoint values are 30, 205, and at the possible interior critical points $(7 \pm \sqrt{5})/2$ its value is $(85 \mp 35\sqrt{5})/2 > 0$. For (19.6), use $J_{1,o} \geq -3T_-$, $J_{1,o} \leq 3J_o \leq 3T$ and (17.8).

The central integrals are $J_{j,c} = mP_j(A, B)$, with

$$P_1 = B^3/3 + AB^2/2 - A^3/6, \quad P_2 = B^4/4 + AB^3/3 - A^4/12.$$

They increase in B ; their A derivative changes from positive to negative at $A = B$. A rectangle minimum is therefore at B_l and an A endpoint. Multiply a negative minimum by an upper bound for m , and a nonnegative minimum by a lower bound. For the upper central charge, maximizing at $(\bar{a}, \bar{b})$ if $p \leq 1/2$ gives $J_{1,c} \leq v^2(2 - 3p)/(6pq)$, a decreasing function of p on that interval. If $p \geq 1/2$, maximize first at $A = B$ to get $J_{1,c} \leq (2/3)v^2q^2/p \leq 1/3$. Thus every cell may use

$$J_{1,c} \leq \max\{1/3, (2 - 3l)/(6l(1 - l))\}. \quad (19.7)$$

For $|X| \leq R$, $R \geq 1$, define

$$k_4(R) = \frac{(R + 1)^2}{R + 1/2}, \quad k_5(R) = \frac{R^3 + 2R^2 + 3R + 3/2}{R + 1/2}.$$

Then

$$\mu_4 \leq \min\{R\rho, 1 + k_4(R)(\rho - 1)\}, \quad \eta_5 \leq \min\{R\mu_4, 1 + k_5(R)(\rho - 1)\}, \quad \mu_6 \leq R\eta_5. \quad (19.8)$$

Writing $u = |X|$ in (17.2), the first two constants are the endpoint values of the ratios $u + 3/2 + 1/[4(u + 1/2)]$ and $u^2 + 3u/2 + 9/4 + 3/[8(u + 1/2)]$, whose derivatives are nonnegative. The other bounds are pointwise. The following coarse cancellation estimates continue to use the valid radius-six majorants: The last identity in (17.2) gives

$$|K_1 + 3\rho\beta| \leq (96 - 18\rho)(\rho - 1)/13. \quad (19.9)$$

Indeed $u(u + 2 - 3r/2)/(u + 1/2)$ is convex on $[0, 6]$, its maximum is $(96 - 18r)/13$, and its negative part is at most $31/40$, smaller than that maximum for $1 \leq r \leq 37/20$. The other bounds needed below are

$$\begin{aligned} |K_1| &\leq 6\rho - 3\rho^2/2 + 3|\beta|, \\ K_1 &\leq M + 3|\beta| - 3\rho g_-/2, & -K_1 &\leq 3\rho + 3|\beta| + (3\rho/2 - 3)g_-, \\ -K_q &\leq (\rho/2 - 3/8)M - \rho/2 + 1/8 + 3|\beta|g/2, & -K_q &\leq 8\rho^3/27 + (3|\beta|/2 - 1/2)\rho, \end{aligned} \quad (19.10)$$

where $\mu_4 \leq M$, $|\gamma| \leq g$, $\gamma \geq g_-$, and $K_q = \eta_5/4 - \rho\mu_4/2 - 3\beta\gamma/2 + \rho/2$. For the first two lines use $|\mathbb{E}[X|X|^3] - 3\gamma| \leq 3\rho$, and note the negative coefficients of γ . For the last line use $\eta_5 \geq \mu_4^{3/2} \geq 3\mu_4/2 - 1/2$, or maximize $\rho y^2/2 - y^3/4$ over $y \geq 0$.

Compatible weaker resource constants. One may also use the weaker binary charge $b_I = (616/25)(1 - \Theta)^2 \text{dist}(p_*, I)^2$. It follows from $2(6 + 2\sigma) > 616/25$ and $(1 - \theta)^{3/2} \geq (1 - \theta)^2$. With $K = \min(K_+, K_-)$ and $v_* = \max(v_+, v_-)$, the merged charge (17.10) gives

$$T_+ + T_- \leq T_* = (d - b_I) \max\{v_*/[3(A_l + B_l)], (R + 2v_*)/K\}, \quad d_o = \min\{d, (\sigma + 1)T_*\}.$$

Thus $W_o \leq d_o$ and, for $R \leq 6$,

$$-75d_o/104 \leq J_{1,o} \leq 3d_o/2, \quad 3(\rho J_{1,o} - J_{2,o}) \leq \frac{3}{8} \max\{\rho^2/(\sigma - 1), (R - \rho)^2/(\sigma + 1)\}d_o.$$

These are precisely the merged-cost versions of the sign-specific resource rule; they do not introduce another available budget.

19.4.1 Conditional lower moments improve two signed charges

Let $s_u = \max_{p \in [l, u]} pq$ and $\theta \leq \Theta$. The unsigned cases of (17.6) give

$$\mu_4 \geq M_l := \max\{1, (1 - \Theta)^2(1/s_u - 3)\}, \quad (19.11)$$

$$\eta_5 \geq E_l := (1 - \Theta)^{5/2} \frac{1 - 4s_u + 2s_u^2}{s_u^{3/2}}. \quad (19.12)$$

Both binary moment functions decrease in $s \in (0, 1/4]$. For the fifth this follows because the derivative has the sign of $s^2 + 2s - 3/2 < 0$; the fourth is immediate. The fourth-moment lower bound sharpens the lower endpoint of $J_1 + (\mu_4 - 3)/6$; its explicit use appears in the cell map below.

The constant signed charge is

$$-K_q = \frac{\rho\mu_4}{2} - \frac{\eta_5}{4} + \frac{3}{2}\beta\gamma - \frac{\rho}{2}.$$

The old coherent estimate used $\eta_5 \geq 3\mu_4/2 - 1/2$ and $\mu_4 \leq M$. A second estimate uses (19.12). Their difference is independent of ρ . Consequently the improved bound is exactly the old one minus

$$\frac{1}{4} \max\{0, E_l - 3M/2 + 1/2\}. \quad (19.13)$$

This is a constant shift on the cell and does not spoil convexity. It does not assert the generally invalid implication $\eta_5 \geq 3M/2 - 1/2$ from an upper bound on μ_4 .

19.4.2 Signed conditional Taylor cancellation

Let $K_1 = \mathbb{E}[X|X|^3] - 3\beta - \frac{3}{2}\rho\gamma$. On a sign class put $x = |X|$, $a = |V|$, $W = X - V$ and $\varepsilon = 1$ on the positive class, -1 on the negative class. In the actual central-gap domain there is no atom at zero. With $w_3 = x + 2a$, $w_4 = x^2 + 2ax + 3a^2$ and $\gamma_V = \mathbb{E}V^3$, exact conditional expansion gives

$$K_1(X) - K_1(V) = \mathbb{E}[W^2 \{ \varepsilon(w_4 - 3 - \frac{3}{2}\rho w_3) - \frac{3}{2}\gamma_V w_3 \}]. \quad (19.14)$$

Apply (17.6) to $j = 2, 3, 4$, with $h = 1$ or $h = \varepsilon$, and use $\rho\gamma - \rho_V\gamma_V = \rho(\gamma - \gamma_V) + \gamma_V(\rho - \rho_V)$.

The positive coefficient in (19.14) is $F(x, a; t) = x^2 + 2ax + 3a^2 - 3 - \frac{3}{2}t(x + 2a)$ with $t = \rho + \gamma_V$; the negative coefficient is $-F$ with $t = \rho - \gamma_V$. Their ranges are bounded on complete rectangles of actual conditional means and support magnitudes. The positive-support magnitude is at most R . The same formula permits separate actual bounds on the two signed support magnitudes. On the late domain $p < 1/2$, bounds for γ_V are

$$\sqrt{(1 - \Theta)^3(1 - 2u)^2/[u(1 - u)]} \leq \gamma_V \leq (1 - 2l)/\sqrt{l(1 - l)}.$$

The positive mean lies in $[\sqrt{(1 - \Theta)(1 - u)/u}, \sqrt{(1 - l)/l}]$, and the negative magnitude mean lies in $[\sqrt{(1 - \Theta)l/(1 - l)}, \sqrt{u/(1 - u)}]$. Use the lower bound $\rho \geq \rho_B(u)$ only after checking $3 \min(A_l, B_l) \geq (3/2) \sup_I \rho_B$. Indeed conditional expansion and $(1 - \theta)^{3/2} \geq 1 - 3\theta/2$ then imply $\rho \geq (1 - \theta)^{3/2} \rho_B(p) + 3 \min(A, B)\theta \geq \rho_B(p)$. Otherwise retain the universal lower bound one.

Since $x + 2a > 0$, F decreases with t . For fixed t its Hessian is $(\frac{2}{2} \frac{2}{6}) > 0$. Its rectangle maximum is at a corner. Its minimum is among the four corners, the minimizer on each edge, and the stationary point $(x, a) = (3t/8, 3t/8)$ if that lies in the rectangle. On fixed- x edges use $a = t/2 - x/3$, clipped to the edge; on fixed- a edges use $x = 3t/4 - a$, also clipped. This is a complete finite extremum list. Multiplication by $0 \leq \theta \leq \Theta$ uses $\Theta \min(0, L)$ and $\Theta \max(0, U)$ for the lower and upper coefficient bounds. It intersects this new interval with the retained absolute-error interval; neither need be discarded.

19.5 Generic event and complete central rectangle

Lemma 19.1 (Generic small-defect event). *Assume $|X| \leq R \leq 6$ and $\rho \leq 37/20$. Let $0 \leq d < 2$ and $D \leq d$. Define*

$$r_* = \min\{37/20, (3 + d)/(\sigma - 1)\}, \quad W = \sqrt{37d/10}, \quad c = \sqrt{\sigma + 1}, \quad e = \sqrt{\sigma - 1}, \quad C_0 = c + e,$$

$$h_* = \frac{C_0}{6} \sqrt{12 + W^2} + \frac{W}{3C_0}, \quad L_* = \max \left\{ h_*, \frac{(2c - e)h_* + 3W}{C_0} \right\}.$$

Then $\rho \leq r_*$, $A + B \leq h_*$ and $|2B - A| \leq L_*$. If $\alpha = L_*/6 \leq 3$, then

$$|J_1| \leq \alpha(\rho + 1) + (3 - \alpha)d/2, \quad \mu_4 \leq h_*\rho + Rd/2, \quad \eta_5 \leq h_*^2\rho + R^2d/2.$$

For independent copies, the actual event $E = \{XX' < 0\}$ obeys

$$P(E) \geq a := 1/(2r_*^2), \quad |X - X'| \geq d_0 := \sqrt{2 - d} \text{ on } E, \quad \mathbb{E}[(X - X')^2 | E] \leq 2/a.$$

Proof. The inequality $J \leq \rho + 1$ gives $D \geq (\sigma - 1)\rho - 3$. Equation (17.8) gives $w^2 \leq 2\rho D \leq 37d/10$ for $w = cA - eB$. Since $ce = 3$,

$$A + B = \frac{C_0}{6} \sqrt{w^2 + 12AB} - \frac{w}{3C_0}, \quad 2B - A = \frac{(2c - e)(A + B) - 3w}{C_0}.$$

Use $AB \leq 1$, $|w| \leq W$; also $2B - A \geq -(A + B)$. Now $J_{1,c} = J_c(2B - A)/6$, $|J_{1,o}| \leq 3J_o$ and $0 \leq J_o \leq T \leq d/2$ imply $|J_1| \leq \alpha J + (3 - \alpha)J_o$.

For either inner endpoint b_* and $b_* \leq x \leq R$,

$$x^4 - h_* x^3 \leq R x(x^2 - b_*^2), \quad x^5 - h_*^2 x^3 \leq R^2 x(x^2 - b_*^2).$$

Indeed $h_* \geq b_*$, and after factoring $x - b_*$ the remaining comparisons follow from $x \leq R$. Integrate and use $T \leq d/2$. Finally $m^2 \leq pq$ and $m \geq 1/(2\rho)$ give $2pq \geq a$; the conditional squared distance is bounded by its unconditional expectation 2, divided by the event probability. $\square$

The central part of $\rho J_1 - J_2$ is exactly $mG(A, B; \rho)$, where $G = \rho P_1 - P_2$. Let a proved probability cell give $A \in [A_l, A_u]$, $B \in [B_l, B_u]$, $m \in [m_l, m_u]$. Here

$$A_u = \sqrt{u/(1-u)}, \quad B_u = \sqrt{(1-l)/l}, \quad m_l = \sqrt{(1-\Theta)s_0}, \quad m_u = \sqrt{\max_{p \in [l, u]} pq}.$$

The closed-box Bernstein identity. For a polynomial $P(x) = \sum_{k=0}^d t_k x^k$,

$$P(x) = \sum_{i=0}^d b_i \binom{d}{i} x^i (1-x)^{d-i}, \quad b_i = \sum_{k \leq i} t_k \binom{i}{k} / \binom{d}{k}.$$

Indeed $\binom{d}{i} \binom{i}{k} = \binom{d}{k} \binom{d-k}{i-k}$ and the binomial theorem recover each monomial x^k . The basis terms are nonnegative and sum to one on the closed unit interval. Tensor products prove the two-variable version below, including all boundary points.

Lemma 19.2 (Complete central rectangle). Write $G(A_l + (A_u - A_l)x, B_l + (B_u - B_l)y; \rho) = \sum t_{kh} x^k y^h$. Define the 25 rational coefficients

$$b_{ij}(\rho) = \sum_{\substack{k \leq i \\ h \leq j}} t_{kh} \frac{\binom{i}{k}}{\binom{4}{k}} \frac{\binom{j}{h}}{\binom{4}{h}}, \quad 0 \leq i, j \leq 4,$$

and

$$\mathcal{G}(\rho) = \max_{i,j} \{m_l b_{ij}(\rho), m_u b_{ij}(\rho)\}.$$

Then $\rho J_{1,c} - J_{2,c} \leq \mathcal{G}(\rho)$. Moreover $\mathcal{G}$ is convex in ρ .

Proof. Apply the Bernstein identity (19.5) separately to the two coordinates after the affine substitution. The Bernstein basis is nonnegative and sums to one on the closed square, so G is a convex combination of its 25 coefficients. For each coefficient retain both m endpoints, or equivalently use m_u when it is nonnegative and m_l otherwise. This gives a uniform bound on the entire three-dimensional rectangle, including boundaries. Each endpoint-multiplied coefficient is affine in ρ ; their maximum is convex. No interior stationary point is omitted. $\square$

19.6 One interval-feedback theorem

The signed identity from the preceding section is

$$\mathcal{R} = C\{3\beta J/2 + 3\rho J_1 - 3J_2 - K_q\} + H(CK_1 + \varphi U) - H^2(C\rho + \gamma\varphi/2) + E_n, \quad (19.15)$$

where $U = J_1 + (\mu_4 - 3)/6$, $\varphi = \phi(H/\sqrt{n})$. For any one coherent low-frequency triple and complete high-frequency value K_H at floor N ,

$$|E_n| \leq K_F + s_F |H|, \quad K_F = c_\pi(L_0 + K_H + L_2/\sqrt{N}), \quad s_F = c_\pi L_H, \quad c_\pi \geq 2/\pi. \quad (19.16)$$

We use 16/25 before the signed phase and 212/333 afterward; both are valid since $\pi > 333/106$.

Lemma 19.3 (Continuous cells and feedback). *Suppose bounds for one actual cell give*

$$r_0 \leq \rho \leq r_1, \quad 0 < c_0 \leq C \leq c_1, \quad |\beta| \leq b, \quad |\gamma| \leq g, \quad \mu_4 \leq M, \quad D \leq d,$$

$K_1 \in [k_l, k_u]$, $U \in [u_l, u_u]$, and $0 \leq \varphi_- \leq \varphi \leq 2/5$. Freeze the shared outer charge d_o , lower fifth moment E_l , upper $s_u = \max_{p \in [l, u]} p(1-p)$, and central envelope $\mathcal{G}$ from the preceding bounds. Define

$$\begin{aligned} A(r) &= \frac{3}{2}b(r + 2\sqrt{s_u}) + 3\mathcal{G}(r) + \frac{3}{8} \max\{r^2/(\sigma - 1), (R - r)^2/(\sigma + 1)\}d_o \\ &\quad + (r/2 - 3/8)M - r/2 + 1/8 + \frac{3}{2}bg - \frac{1}{4} \max\{0, E_l - 3M/2 + 1/2\}, \\ B(c) &= \max_{k \in \{k_l, k_u\}, u \in \{u_l, u_u\}, \psi \in \{\varphi_-, 2/5\}} |ck + \psi u|, \\ Q(c, r) &= cr + \frac{1}{2}\varphi_-(3.16r - 3 - d) - \frac{1}{2}(2/5 - \varphi_-)(d - .16)_+. \end{aligned}$$

If all four corner values of Q are positive, then, for every real H ,

$$\begin{aligned} \mathcal{R} &\leq \max_{(c, r) \in \{c_0, c_1\} \times \{r_0, r_1\}} \{K_F + cA(r) + (B(c) + s_F)|H| - Q(c, r)H^2\} \\ &\leq \max_{c, r \text{ corners}} \{K_F + cA(r) + (B(c) + s_F)^2/(4Q(c, r))\}. \end{aligned} \quad (19.17)$$

If T is a strict upper bound for the final maximum in (19.17), the weight-removal step in Lemma 17.1 gives

$$D < \frac{15000T}{997N}, \quad \varphi > \frac{997}{2500} \left(1 - \frac{5T}{2N}\right), \quad C < C_E + V/N \quad \text{if also } \mathcal{R}/\rho < V. \quad (19.18)$$

Only positive density floors are used. These statements remain valid at $H = 0$, on closed cell edges, and for zero conditional variance.

Proof. The constant coefficient follows from the shared outer budget, Lemma 19.2, and (19.10), using

$$J \leq \rho + 2ABm \leq \rho + 2(1 - \theta)^{3/2} \sqrt{pq} \leq \rho + 2\sqrt{s_u}.$$

This follows from $A \leq \bar{a}$, $B \leq \bar{b}$, $m^2 = (1 - \theta)pq$; it is used only in the charge $3|\beta|J/2$. Since $\gamma \geq 3.16\rho - 3 - d$, its negative part is at most $(d - .16)_+$. Using the upper density only for that negative part gives $C\rho + \gamma\varphi/2 \geq Q$. Absolute value is separately convex in each coordinate of $ck + \psi u$, proving the formula for B . For fixed r and $|H|$, the resulting upper bound is convex in c . Take those endpoints *before* completing the square. At either endpoint $c > 0$, it is convex in r , since $\mathcal{G}$ and the maximum of the two outer quadratics are convex, every other term of A is affine, and Q is affine in r . This proves the first line of (19.17); $fh - qh^2 \leq f^2/(4q)$ proves the second. Bilinearity of Q proves its positivity throughout the rectangle from its corners. $\square$

19.7 Signed phase curvature in the resource schedule

The auxiliary Fourier orientation does not change the signed third moment in the actual transfer identity. The paired phase estimate must therefore retain its sign. On a fixed resource cell put

$$L = \frac{997}{2500} \left(1 - \frac{5r}{2N}\right), \quad g_0(\rho) = 3.16\rho - 3 - d, \quad U = .4,$$

where r, d are the preceding raw and defect upper bounds, and the checker requires $0 \leq L \leq U$. The actual density lies in $[L, U]$ and the actual signed moment satisfies $\gamma \geq g_0(\rho)$. Consequently

$$C\rho + \frac{1}{2}\gamma\phi \geq q_s(C, \rho) := C\rho + \frac{1}{2} \min\{Lg_0(\rho), Ug_0(\rho)\}. \quad (19.19)$$

This follows by first multiplying the lower bound on γ by the nonnegative actual density, and then minimizing over its two endpoints. The simpler expression $C\rho + Lg_0(\rho)/2$ is licensed if $g_0 \geq 0$, or if nonnegativity of the actual γ has been separately proved. It is not licensed merely by reflecting the auxiliary Fourier integral.

The corrected price preserves the four-corner reduction. For fixed C , q_s is concave in ρ , being affine plus the minimum of two affine functions. For fixed ρ it is affine in C . In the phase envelope $A + f|H| - q_s H^2$, the remaining terms are separately convex in C, ρ , by the moment and coefficient bounds already proved. Thus the entire envelope is separately convex, and two successive one-dimensional endpoint bounds place its maximum at the four corners. The same argument applies after adding the affine conserved-resource curvature e_ρ and subtracting a barrier price $K\rho$. If the four corrected divisors are positive, concavity makes them positive throughout the rectangle, and square completion gives the valid raw and resource bounds

$$\max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4q_s} \right), \quad \max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4(q_s + e_\rho)} - K\rho \right).$$

Any threshold-root export likewise uses the corrected corner curvature.

For completeness, nonnegativity in the later acquired domains follows directly from their constraints. Write $B = \mathbf{1}_{\{X > 0\}}$, $V = \mathbb{E}(X \mid B)$, $W = X - V$, $\theta = \mathbb{E}W^2$, and $p = \mathbb{P}(X > 0)$. Conditional centering and $W \geq -R - V$ imply

$$\gamma = \mathbb{E}V^3 + 3\mathbb{E}(VW^2) + \mathbb{E}W^3 \geq \mathbb{E}V^3 + 2\mathbb{E}(VW^2) - R\theta.$$

For $p \in [l, u] \subset (0, 1/2)$ and $\theta \leq \Theta < 1$ this gives

$$\gamma \geq \frac{(1 - \Theta)^{3/2}(1 - 2u)}{\sqrt{u(1 - u)}} - \left(R + 2\sqrt{\frac{u}{1 - u}} \right) \Theta. \quad (19.20)$$

Here the binary cubic decreases in $p < 1/2$, and the right side decreases as its variance bound increases. These are bounds on the same actual law throughout every later call. The fixed acquired input domains license (19.20) before those calls are made. The exact domain check covers 302 input rows: 45 central-input rows, 45 left-input rows, 45 first-confinement feedback rows, 45 Sperner feedback rows, 84 signed-resource rows, and 38 final left-radius rows. With the common already acquired radius $R = 3.84$, each right side is greater than .096. Thus all later paired calls have an actual nonnegative signed third moment, independently of any auxiliary Fourier orientation.

There is also a convenient cell-constant correction. Since $\rho \geq 1$, $(-g_0(\rho))_+ \leq (d - .16)_+$, so the weaker price

$$C\rho + \frac{1}{2}Lg_0(\rho) - \frac{1}{2}(U - L)(d - .16)_+$$

is valid. This is the correction selected for the additional index-46 calculation. Both forms keep the raw remainder separate from the barrier-subtracted resource budget.

The preliminary and complementary resource bounds use (19.19). Every exported resource, including threshold curvature, must satisfy these corrected inequalities. Equality of two corner maxima alone is insufficient, since the threshold bounds also depend on the phase coefficients. Later uses of the simpler formula require their own explicit nonnegative signed-moment or small-defect license.

19.8 The unsigned first-stage cover is the same enclosure

The first finite barrier uses a cover of ρ alone before introducing the signed cover. Here are its exact coefficient choices, so that the first barrier needs no separate inherited entry theorem. Fix

a closed cell $\rho \in [l, u] \subset [1, 1.85]$, actual radius $1 \leq R \leq 6$, and write $k_j(R)$ for the radial ratios of Lemma 17.2. Set

$$M = \min\{11.1, Ru, 1 + k_4(R)(u - 1)\}, \quad g = \min\{1.85, u, \sqrt{M - 1}\},$$

$$E_5 = \min\{RM, 1 + k_5(R)(u - 1)\}, \quad b = \sqrt{[u\sqrt{u^2 + 8} - u^2 - 2]/2}.$$

The last expression solves $\rho \geq (1 + \beta^2)/\sqrt{1 - \beta^2}$, proved by $\rho m \geq a_+^2 + a_-^2$ and $m^2 \leq a_+ a_-$. If a previous pass licenses $D \leq d < 2$, the generic-event lemma also permits the ceilings $h_* u + Rd/2$ and $h_*^2 u + R^2 d/2$ for M, E_5 ; all dependent quantities are recomputed after taking these minima. Use the triangle LOW triple and the minimum of complete permitted HIGH values, with the same moment cap. Put

$$K_F = \frac{16}{25}(L_0 + K_H + L_2/\sqrt{N}), \quad s_F = \frac{16}{25}L_H.$$

The radial identity bounds $|K_1|$ by

$$k_{\text{rad}} = 3ub + \frac{R(R + 2 - 3l/2)}{R + 1/2}(u - 1),$$

provided the displayed radial coefficient is at least $31/40$. Indeed its positive maximum is at R and its negative part is at most $31/40$ on $[0, R]$ for $1 \leq \rho \leq 1.85$; convexity of $x(x + 2 - 3\rho/2)/(x + 1/2)$ proves this directly. Two additional valid bounds are

$$k_{\text{old}} = 6u - 3u^2/2 + 3b, \quad k_R = Ru/2 + \max_{t=l, u} |R/2 - 3t/2|g + 3b.$$

For k_R , bound the positive and negative fourth moments separately by $R(\rho \pm \gamma)/2$. Before defect feedback use $k = \min(k_{\text{rad}}, k_{\text{old}}, k_R)$. After feedback one may replace k_{old} by $\max(k_+, k_-)$, where

$$g_l = 3.16l - 3 - d, \quad g_u = 3.16u - 3 - d,$$

$$k_+ = M + 3b - \frac{3}{2} \min_{r=l, u; v=g_l, g_u} rv,$$

$$k_- = 3u + 3b + \max_{r=l, u; v=g_l, g_u} (3r/2 - 3)v.$$

These substitutions bound the lower affine graph for the actual γ , not an independently chosen skewness.

Put $\Delta = 2u$ initially and $\Delta = (1 - 3.16)l + 3 + d$ after feedback. The constant coefficient of the raw identity is bounded by

$$A_* = \min\left\{\frac{3}{2}b(2u + 1) + \frac{3}{2}\Gamma(R, l)\Delta + \frac{3}{8}u^2(u + 1) + \frac{8}{27}u^3 - l/2, \right. \\ \left. \frac{3}{2}b(2u + 1) + 2uM - l/2\right\}.$$

This is the coarse gap bound above together with the two bounds for $-K_q$. The density coefficient obeys

$$v_H = \min\{.4j + .4 \max(2, M - 3)/6, .4 \max(1/2, 2M/3 - 1/2)\},$$

where $j = 3(u + 1)$ initially; the generic-gap choice is $j = \alpha(u + 1) + (3 - \alpha)d/2$, with its checked $\alpha < 3$. For a score endpoint enclosure $C \in [c_-, c_+]$, the resulting raw bound is

$$K_F + c_+ A_* + \frac{(c_+ k + v_H + s_F)^2}{4q_*}, \quad q_* > 0, \quad k \geq 0, \quad A_* > 0, \quad (19.21)$$

where initially $q_* = c_- l - g/5$. After feedback use the larger of that quantity and

$$c_- l + \varphi_-(3.16l - 3 - d)/2 - (2/5 - \varphi_-)(d - .16)_+/2.$$

The score endpoints are the lower score floor and the upper allowed score cap; if the latter depends on ρ , use its values at u, l for c_-, c_+ , respectively. Convexity in the actual score precedes the completion of the square, and these outward endpoint intervals cover it. For the phase resource add (17.5) before maximizing. A barrier $e \geq K/n$ subtracts Kl from the nonnegative resource. The first raw target must pass the license of Lemma 17.1 first.

Optional signed improvement inside an unsigned cell. If $g_0 = \max(-g, \sigma_l l - 3 - d) > g$, exclude the cell. When $g_0 \geq 0$, the sign-moment constraint forces $(1 - \beta)^3 \leq (1 + \beta)(u - g_0)^2$. This inequality is monotone on $(-1, 1)$, so exact bisection yields a lower $b_0 \leq \beta$; if it fails even at b , the cell is empty. For $x \in [g_0, g]$, set

$$\begin{aligned} f_-(x) &= (l + x)^2/[2(1 + b)] - R(u - x)/2 - 3b - 3ux/2, \\ f_+(x) &= R(u + x)/2 - (l - x)_+^2/[2(1 - b_0)] - 3b_0 - 3lx/2. \end{aligned}$$

The signwise fourth-moment Cauchy bounds prove $\min f_- \leq K_1 \leq \max f_+$. The first minimum is at its clipped vertex $3u(1 + b)/2 - l - R(1 + b)/2$. The second maximum is among its endpoints, the clipped kink l , and the clipped vertex $l + (R/2 - 3l/2)(1 - b_0)$. Intersect with $[-k, k]$ and replace the linear term in (19.21) by the smaller full rectangle bound

$$\max_{c=c_-, c_+, k=k_l, k_u, \psi=\varphi_-, .4, v=-1/2, 2M/3-1/2} |ck + \psi v| + s_F.$$

All sixteen corners are retained. If $b_0 > 0$, the opposite-sign event also has probability at least $(1 + b_0^2)/(2u^2)$, from $m \geq (1 + \beta^2)/(2\rho)$ and $m^2 \leq pq$. It may improve the complete HIGH value. These are intersections with necessary conditions; the cover still examines every ρ -cell.

Moving from a small defect to probability coordinates. For $0 < d \leq .7$, start with

$$A_l = \frac{1.47\sqrt{2-d} - \sqrt{3.7d}}{3.52}, \quad B_l = \frac{2.04\sqrt{2-d} - \sqrt{3.7d}}{3.52}.$$

They follow from $A + B \geq \sqrt{2-d}$ and $|\sqrt{\sigma+1}A - \sqrt{\sigma-1}B| \leq \sqrt{3.7d}$; their positivity is checked, rather than assumed at every d . The pointwise inequality $x(x+b) \geq (29/5)b(x-b)$ for $x > b > 0$, applied to the two residual third moments, gives $D \geq (29/5) \min((\sigma+1)A, (\sigma-1)B)\theta$. Start $\Theta = 1$ and perform the declared finite number of updates:

$$\begin{aligned} \Theta &\leftarrow \min\{\Theta, d/[(29/5) \min((104/25)A_l, (54/25)B_l)]\}, \\ \epsilon &= \sqrt{d/[(616/25)(1-\Theta)^2]}, \quad [l, u] = [(4 - \sigma_u)/2 - \epsilon, (4 - \sigma_l)/2 + \epsilon], \\ A_l &\leftarrow \max\{A_l, \sqrt{(1-\Theta)l/(1-l)} - d/[2(1-u)(179/25)A_l^2]\}, \\ B_l &\leftarrow \max\{B_l, \sqrt{(1-\Theta)(1-u)/u} - d/[2l(129/25)B_l^2]\}. \end{aligned}$$

The last two bounds use $D \geq 2(\sigma+2)B^2M_+ + 2(\sigma+4)A^2M_-$ and $\Delta_A = M_-/q$, $\Delta_B = M_+/p$. Every division, interval endpoint and variance domain is gated. Subdivide this entire probability interval, then use the common cell update and coefficient atlas. Thus the unsigned first barrier, the signed initialization and the final probability refinements are three data choices for the same finite-enclosure theorem.

20 One confinement lemma with two concentration choices

A proved amplitude is used as a support update, never as a premise of its own proof. The following derivation applies to every acquired row, with the same formulas for the central and left probability ranges.

20.1 Statement and hypotheses

Let $n \geq N = 47$, $Y = B + U$, $\mathbb{P}(B = 1) = p < 1/2$ and $\mathbb{E}(U | B) = 0$. Put $q = 1 - p$, $s = \mathbb{E}U^2$, $\lambda = ns$ and $X = \ell(Y - p)$, where $\mathbb{E}X^2 = 1$. Thus $\ell^2 = (1 - \theta)/(pq)$ and $\theta = \ell^2 s$. The conditional label-zero support is at most $a \geq 0$ and the label-one support is at least $b \leq 1$, where a, b are

support points and $b - a = 1 - d$. In particular $0 \leq a \leq d$, $1 - d \leq b \leq 1$. Assume the initial strip $\text{supp } Y \subset (-.55, 1.55)$. Write $\rho = \mathbb{E}|X|^3$, $\beta = \mathbb{E}[X|X|]$ and $\epsilon = \mathbb{E}|U|$.

Select an acquired closed probability panel $[l, h]$. Its fields give upper bounds $\Theta, d_0, \rho_0, \beta_+, T, \Lambda$, a lower bound β_- , and separate bounds H_+, H_- for the two signs of H . Assume

$$\theta \leq \Theta, \quad d \leq d_0, \quad \rho \leq \rho_0, \quad \beta_- \leq \beta \leq \beta_+, \quad 0 \leq \lambda \leq \Lambda, \quad .A < C \leq C_0 = .4253.$$

For the positive sign $0 \leq H \leq H_+$; for the negative sign $-H_- \leq H \leq 0$. Let $x = np + H/\ell$ and $\nu = \mathcal{L}(x - \sum_{i < n} Y_i)$. Assume a contact majorant Q for $g(y) = \nu([y, \infty))$, with equality at each support point of Y , and derivative

$$\sqrt{n}(-Q'(y)) = \ell\phi(H/\sqrt{n}) + \frac{\ell}{n}\{3C\beta + (H\phi(H/\sqrt{n}) + 3C\rho)t - 3Ct|t|\}, \quad t = \ell(y - p). \quad (20.1)$$

Both actual deleted convolutions satisfy, separately,

$$Q_j(w) \leq \frac{\ell\phi(0)w + 2C_0\rho}{\sqrt{j}}, \quad j = n - 1, n - 2, \quad (20.2)$$

for every interval of width $w \geq 0$.

Here is the precise common resource assumption. Put $\kappa = 997/15000 < \phi(0)/6$, $J = \int_{[a,b)}(y - a)\nu(dy)$, and note that the central gap contributes $m\ell J = (1 - \theta)J$ to L . Thus $L \geq (1 - \theta)J$; retain $e \geq 0$ and

$$ne\rho + n\phi(0)D/6 + n^{3/2}L < \mathcal{B}(H), \quad \mathcal{B}(H) = \min\{T, \max_j(A_j + F_{j,\pm}|H| - Q_jH^2)\}. \quad (20.3)$$

The four quadratics are the table's quadratic rows, using Q_{budget} , not Q_{raw} . The Gaussian phase contribution has already been included in Q_{budget} .

For each panel let E, F_ℓ, m_0 be respectively strict lower, upper, and lower rational bounds for

$$\sqrt{(1 - \Theta)/(h(1 - h))}, \quad 1/\sqrt{l(1 - l)}, \quad \sqrt{(1 - \Theta)l(1 - l)}.$$

The bounds use the grid 10^{-6} , with an extra unit outward. Define

$$\begin{aligned} c_+ &= 2(5.16)E^2(1 - h - d_0)^2, & c_- &= 2(7.16)E^2(l - d_0)^2, \\ b_* &= \min\{3(4.16)E(1 - h - d_0), 3(6.16)E(l - d_0)\}, \\ P &= m_0E^2(3.5 \text{ dist}([l, h], [.41886, .41887]) - 2.041d_0)_+^2. \end{aligned}$$

The simultaneous moment inequality has the following usable form:

$$D \geq c_+M_+ + c_-M_- + b_*\theta + P, \quad d = (M_+/p + M_-/q)/\ell, \quad \epsilon \leq 2(M_+ + M_-)/\ell, \quad M_\pm \geq 0. \quad (20.4)$$

To derive it, the endpoint expansion of (17.8) gives $D \geq m(\sqrt{\sigma + 1}A - \sqrt{\sigma - 1}B)^2 + 2(\sigma + 2)B^2M_+ + 2(\sigma + 4)A^2M_- + 3(\sigma + 1)B\theta_+ + 3(\sigma + 3)A\theta_-$. Insert $A \geq E(l - d_0)$, $B \geq E(1 - h - d_0)$ and the rational lower coefficients displayed above. For the square, write $A = \ell(p - a)$, $B = \ell(1 - p - (1 - b))$, with $a + (1 - b) = d \leq d_0$. Its unshifted coefficient is $(\sqrt{\sigma + 1} + \sqrt{\sigma - 1})(p - p_*)$; $\sqrt{\sigma + 1} + \sqrt{\sigma - 1} > 3.5$ and $\sqrt{\sigma + 1} < 2.041$ prove the stated lower square P . For completeness, on the positive class put $Z = X - B \geq 0$. Then $T_+ = 2B^2M_+ + 3B\mathbb{E}[Z^2; X > 0] + \mathbb{E}[Z^3; X > 0]$ and $R_+ = BM_+ + \mathbb{E}[Z^2; X > 0]$. Consequently $(\sigma - 1)T_+ + 6BR_+ \geq 2(\sigma + 2)B^2M_+ + 3(\sigma + 1)B\theta_+$, since $\mathbb{E}[Z^2; X > 0] \geq \theta_+$. The reflected expansion gives the negative-class bound. The centered absolute deviation on each class is at most twice its mean distance from the inner endpoint, proving the bound on ϵ . All quantities refer to the same marginal.

20.2 Concentration from an antichain

Lemma 20.1 (Strict-gap concentration). *Suppose $Y = U_0 + B(1 + U_1 - U_0)$, where B is Bernoulli(p) independent of the pair (U_0, U_1) , and $1 + U_1 - U_0 \geq c > 0$ almost surely. For independent copies $Y_1, \dots, Y_j$ and any interval I of diameter strictly less than c ,*

$$\mathbb{P}\left(\sum_{i=1}^j Y_i \in I\right) \leq \max_{0 \leq k \leq j} \binom{j}{k} p^k (1-p)^{j-k}.$$

The same bound holds after adding any independent random variable.

Proof. Condition on all noise pairs. The sum is a fixed constant plus $\sum B_i w_i$, with every $w_i \geq c$. The family $\mathcal{F}$ of subsets whose weighted sums lie in I is an antichain: distinct comparable subsets have sums differing by at least c , larger than the diameter. A uniformly random permutation of $\{1, \dots, j\}$ defines the maximal chain of its initial subsets. The chain contains a fixed k -subset with probability $1/\binom{j}{k}$, and meets an antichain at most once. Hence

$$\sum_{S \in \mathcal{F}} \binom{j}{|S|}^{-1} \leq 1.$$

Multiplying each summand by its binomial level mass proves the bound conditionally. Integration removes the conditioning. Conditioning on an additional independent variable proves the last assertion. This argument includes all atom collisions and any open or closed interval endpoints. $\square$

For the actual marginal we may couple the two conditional noises independently and then sample an independent label. Since $U_0 \leq a$ and $U_1 \geq b - 1$, the step is at least $b - a = 1 - d$. Strict diameter is necessary: a Bernoulli law on $\{0, c\}$ assigns mass one to $[0, c]$, exceeding its largest atom when $0 < p < 1$.

A rational cap for every remaining index. Put $J_0 = N - 2 = 45$ and define

$$r_L = l - (1 - l)/J_0, \quad r_R = h + h/J_0, \quad v_* = \min\{r_L(1 - r_L), r_R(1 - r_R)\}.$$

The checker verifies $0 < r_L \leq r_R < 1$. Adjacent binomial-mass ratios show that any mode k at index $j \geq J_0$ satisfies $(j + 1)p - 1 \leq k \leq (j + 1)p$, hence $r = k/j \in [r_L, r_R]$. In particular $1 \leq k \leq j - 1$. Finite Stirling bounds give

$$\sqrt{j} \binom{j}{k} p^k (1-p)^{j-k} \leq \frac{\exp(1/(12j) - jD(r||p))}{\sqrt{2\pi r(1-r)}} < \frac{.4}{\sqrt{v_*(1 - 1/(12J_0))}}.$$

Here $D(r||p) \geq 0$ follows from Jensen's inequality for the logarithm; $\exp x \leq (1 - x)^{-1}$ for $0 \leq x < 1$ follows term by term from the power series. The finite Stirling inequalities used are $\sqrt{2\pi m}(m/e)^m < m! < \sqrt{2\pi m}(m/e)^m \exp(1/(12m))$. Choose the strict 10^{-6} grid upper square root M of $.16/[v_*(1 - 1/(12J_0))^2]$. Thus, whenever $w < 1 - d$,

$$Q_j(w) < M/\sqrt{j}, \quad j \geq n - 2 \geq J_0. \quad (20.5)$$

This covers the $n - 1$ full summands of ν and the $n - 2$ full summands remaining after one mark or deletion. An independent remaining conditional noise preserves the cap. We do not replace the latter index by $n - 1$.

Either the score-derived cap or the strict-gap cap may be selected on an interval where its hypotheses hold. The first confinement uses the score cap; later central and left schedules use the strict-gap cap. For the left panels replace the conditional cubic coefficient 24 below by 28; in general any c_ρ satisfying $c_\rho^2[l(1 - l)]^3 > [3(1 - l) + .55]^2$ is valid.

20.3 Signed deletion, including atoms

Let $m = n - 1$, $Z = x - \sum_{i=1}^m Y_i$, $\mu = x - mp$ and let $K = \sum B_i$. For a fixed removed label put $T_b(I) = \mathbb{P}(x - \sum_{i < m} Y_i - b - U_b \in I)$. Exchangeability and $mp - K = Z - \mu + \sum U_i$ give the exact identities

$$\nu(I) - T_1(I) = \frac{\mathbb{E}[(Z - \mu + \sum U_i)1_I(Z)]}{mp}, \quad T_0(I) - \nu(I) = \frac{\mathbb{E}[(Z - \mu + \sum U_i)1_I(Z)]}{mq}.$$

There is no absolute value around the sum of noises. For a full removed pair (Y_i, U_i) , condition on the pair. The remaining sum gives a function $f(Y_i) \in [0, Q_{m-1}(w)]$. Since $\mathbb{E}U_i = 0$,

$$\mathbb{E}[U_i f(Y_i)] \leq Q_{m-1}(w) \mathbb{E}(U_i)_+ = \frac{1}{2} \epsilon Q_{m-1}(w).$$

This also works when one conditional law is deterministic.

Let independent conditional noises satisfy $\Delta = U_1 - U_0 \geq -d$. For $\tau > 0$ put $I^+ = I + 1 + [-d, \tau]$. On $\Delta \leq \tau$, the point for T_1 in I maps to the point for T_0 in I^+ . On the complement, conditioning on both noises gives error at most $Q_{m-1}(w) \mathbb{P}(\Delta > \tau)$. Moreover

$$\mathbb{P}(\Delta > \tau) \leq \min \left\{ \frac{s}{s + p\tau^2}, \frac{d}{d + \tau} \right\}. \quad (20.6)$$

Indeed $\mathbb{E}\Delta = 0$, $\text{Var } \Delta \leq s/p$, and $\Delta + d \geq 0$ has mean d . Cantelli and Markov prove the two bounds. The exceptional event is strict, so $\Delta = \tau$ belongs to the good event.

If I, I^+ are disjoint, combine the two signed noise terms before bounding them. For the same remaining-sum law their conditional probabilities f_I, f_J satisfy

$$0 \leq f_I \leq Q_I, \quad 0 \leq f_J \leq Q_J, \quad f_I + f_J \leq Q_{\text{hull}}.$$

Since $p < q$,

$$\frac{f_I}{p} + \frac{f_J}{q} \leq \min \left\{ \frac{Q_I}{p} + \frac{Q_J}{q}, \frac{Q_{\text{hull}}}{q} + \left(\frac{1}{p} - \frac{1}{q} \right) Q_I \right\}.$$

Multiplication by $\mathbb{E}U_+ = \epsilon/2$ proves the combined bound. Closed, open and half-open intervals are all covered by containment in the corresponding closed interval. No density or atom separation is used.

20.4 Slope bounds and the two exterior intervals

Split $|H|$ into 16 closed equal cells $[h_l, h_h]$ and λ into 24 closed equal cells $[s_l, s_h]$. On each box put

$$\rho_* = \min\{\rho_0, \rho_B(l) + 24s_h/N\}, \quad b_- = \max\{\beta_-, 1 - 2h - 2(1-h)\Theta\}, \quad b_+ = \min\{\beta_+, 1 - 2l + \Theta\}.$$

Here $\rho_B(p) = (1 - 2p(1-p))/\sqrt{p(1-p)}$. Write $s_i = \mathbb{E}U_i^2$. Conditional centering and the two sign supports give

$$\rho = \ell^3 \{pq(1 - 2pq) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3\}.$$

The initial strip and $d_0 < .55$ imply $|U_i| < .55$. Thus $\rho \leq \rho_B + \ell^3(3q + .55)s \leq \rho_B + 24s$; the last coefficient inequality is checked as $24^2[l(1-l)]^3 > [3(1-l) + .55]^2$. Similarly $\beta = (1-\theta)(1-2p) + \ell^2(ps_1 - qs_0)$, and $|\ell^2(ps_1 - qs_0)| \leq \theta$. These identities prove the stated outward bounds on ρ and β . Set $f = .4F_\ell$, $\varphi_- = .39886(1 - h_h^2/(2N)) > 0$. The exact upper bound for $H\phi(H/\sqrt{n})$ used below is

$$v_+ = \begin{cases} .4h_h & H \geq 0, \\ -\varphi_- h_l & H \leq 0. \end{cases}$$

For all $t \leq 0$, completing the square in (20.1) gives

$$\mathcal{M}_- = F_\ell((v_+ + 3C_0\rho_*)^2/(12(.4)) - 3(.4)b_-)_+, \quad L_- = E\varphi_- - \mathcal{M}_-/N.$$

A positive bound $B_* > \sqrt{(1-l)/l}$ gives valid central-gap lower and positive-half-line upper slope bounds

$$\begin{aligned} \mathcal{M}_+ &= F_\ell(.4h_h B_* + 3C_0 B_*(B_* - 1) - 1.2b_-)_+, \\ L_g &= E\varphi_- - \max(\mathcal{M}_-, \mathcal{M}_+)/N, \\ U_+ &= f + \frac{F_\ell}{N} \left\{ 3C_0 b_+ + \frac{(.4h_h + 3C_0\rho_*)^2}{12(.4)} \right\}. \end{aligned}$$

They follow by completing a square on the positive half-line and using $t(t-1) \leq B_*(B_*-1)$ for $1 \leq t \leq B_*$; on $0 \leq t \leq 1$ that term has the favorable sign. Negative values are covered by $\mathcal{M}_-$. All divisions and positivity requirements are explicit gates.

The sharper lower-ray slope. Take $\tau_- = 5r_0/12$, $\eta = 10^{-5}$ and $I = [a-r, a-d-\tau_- - \eta]$, where $r \geq r_0$. Since $a \leq d$, the entire interval has $t \leq t_* = -E(l + \tau_- + \eta) < 0$. Write $B(t) = b_- + \rho_* t + t^2$ and $B_*^{\min} = B(\min(t_*, -\rho_*/2))$. For the actual C and $t \leq t_*$,

$$3C(\beta + \rho t + t^2) + H\phi t \geq 1.2B(t) + v_+ t + 3(C_0 - .4) \min(0, B_*^{\min}).$$

The inequality uses $t < 0$ and separates the possible negative part of $B(t)$ before varying C . Minimize the right side at the clipped vertex $\min(t_*, -(v_+ + 1.2\rho_*)/2.4)$ and call its value q_* . Then

$$L = E\varphi_- - F_\ell \max(0, -q_*)/N$$

is a lower slope on the whole interval, for every $n \geq N$. Positive corrections are discarded. Therefore contact gives

$$\sqrt{n}\nu(I) \geq L(r - d - \tau_- - \eta).$$

This one slope replaces the previous separate bounds on the long and short parts of the interval.

Lower interval geometry. Its translate is $I^+ = [a-r+1-d, b-\eta]$, of width $r-\eta$. The initial strip implies $I^+ \subset [a+g, b]$, $g = 1 - .55 - 2d_0 > 0$, hence $\nu(I^+) \leq J/g$. The intervals are disjoint throughout $r \leq .55 + d_0$, since $\inf I^+ - \sup I = 1 - r + \tau_- + \eta > 0$. At $r = r_0$, concentration and directed distance bounds are

$$\begin{aligned} Q_I &= f(r_0 - \tau_- - \eta) + 2C_0\rho_*, & Q_J &= f(r_0 - \eta) + 2C_0\rho_*, \\ Q_{\text{hull}} &= f(1 + r_0 - \eta) + 2C_0\rho_*, \\ D_I &= (v_H - l - \tau_- - \eta)_+, & D_J &= (v_H + 1 - l - \eta)_+, \end{aligned}$$

where $v_H = -h_l/F_\ell$ for $H \geq 0$ and $v_H = h_h/E$ for $H \leq 0$. These signs follow from $\mu = p + H/\ell$.

Upper interval geometry. For a support point $y \geq 1 + r_0$, take $\tau_+ = 7r_0/15$, $I = [a, a + \tau_+]$ and $I^+ = [b, a + 1 + 2\tau_+]$. The strict gates $d_0 + 2\tau_+ < r_0$ and $1 - d_0 - \tau_+ > 0$ give $I^+ \subset [b, y]$ and disjointness. Contact, including the atom at b , and the leakage definition give

$$\sqrt{n}\nu(I) \geq L_g(1-d) - \sqrt{n}J/\tau_+, \quad \sqrt{n}\nu(I^+) \leq U_+ (.55 + d).$$

Use

$$\begin{aligned} Q_I &= f\tau_+ + 2C_0\rho_*, & Q_J &= f(2\tau_+ + d_0) + 2C_0\rho_*, \\ Q_{\text{hull}} &= f(1 + 2\tau_+) + 2C_0\rho_*, \\ D_I &= (d_0 + \tau_+ + h_h/E - l)_+, & D_J &= (1 + d_0 + 2\tau_+ + h_h/E - l)_+. \end{aligned}$$

The upper side deliberately keeps the absolute phase bound.

The strict-gap alternative. If $1 - .55 - 2d_0 > 0$ and the displayed lower intervals have width below $1 - d$ throughout $r \in [r_0, .55 + d_0]$, take $Q_I = Q_J = M$. For the upper intervals the corresponding gate is $1 - 2d_0 - 2\tau_+ > 0$; then again $Q_I = Q_J = M$. Use the same directed distances. Their concentration caps are now independent of r , and use $a_e = F_d M \{1/l + 1/(1-h)\}/2$ in the next formulas. The lower-ray gate reduces to $L > 0$: its contact side increases with r , while the entire error is constant on the acquired finite strip. For the score-derived cap retain the explicit affine slope gate below. No assertion outside that strip is needed.

20.5 One budget, two error bounds

Let F_d be the strict grid upper bound for $\sqrt{\max\{(47/46)^3, 47/45\}}$. It simultaneously covers the two concentration factors in (20.2) after multiplication by $\sqrt{n}$, uniformly for $n \geq 47$. For either pair of intervals set

$$a_e = \frac{F_d}{2} \min \left\{ \frac{Q_I}{l} + \frac{Q_J}{1-h}, \frac{Q_{\text{hull}}}{1-h} + \left(\frac{1}{l} - \frac{1}{1-l} \right) Q_I \right\},$$

$$A = F_d \{Q_I D_I / l + Q_J D_J / (1-h)\}, \quad B = F_d Q_I.$$

The probability denominators are outward because $1/p - 1/(1-p)$ is decreasing on $(0, 1/2)$. Use $a_d = L$, $a_J = 1/g$ on the lower side, and $a_d = L_g + U_+$, $a_J = 1/\tau_+$ on the upper side. For the phase cell, maximize each concave budget quadratic at its vertex clipped to $[h_l, h_h]$, take their maximum, then the minimum with T ; call this upper bound T_* . Put

$$R = T_*/\kappa - b_* E^2 s_l, \quad \alpha = \min(c_+ l E, c_- (1-h) E), \quad G = (R - NP)/\alpha.$$

If $G \leq 0$, the strict common budget excludes the entire closed box. Otherwise $nd < G$ and

$$B_\tau = \min\{s_h/(s_h + Nl\tau^2), G/(N\tau + G)\}$$

bounds (20.6). With

$$z = \max \left\{ \frac{a_d/l + 2a_e}{Ec_+}, \frac{a_d/(1-h) + 2a_e}{Ec_-}, \frac{\kappa a_J}{1-\Theta} \right\},$$

one valid total error bound is

$$\mathcal{E}_0 = (zR + A)/N - zP + BB_\tau. \quad (20.7)$$

This follows by multiplying the single budget by z/κ . The inverse-index coefficient $zR + A$ is checked positive.

A stronger bound spends the same resource on the pair event. Let $R_0 = R - NP > 0$ and define the three pairs

$$(v_+, w_+) = \left(\frac{a_d/l + 2a_e}{Ec_+} \frac{R_0}{N}, \frac{R_0}{Ec_+ l} \right),$$

$$(v_-, w_-) = \left(\frac{a_d/(1-h) + 2a_e}{Ec_-} \frac{R_0}{N}, \frac{R_0}{Ec_- (1-h)} \right), \quad (v_J, w_J) = \left(\frac{\kappa a_J}{1-\Theta} \frac{R_0}{N}, 0 \right).$$

They are upper cost/displacement coordinates at the vertices of the simplex with resources $nc_+ M_+$, $nc_- M_-$ and $n^{3/2} L/\kappa$. For any $t \geq 0$,

$$\frac{D}{N\tau + D} \leq \frac{N\tau}{(N\tau + t)^2} D + \frac{t^2}{(N\tau + t)^2} \quad (D \geq 0).$$

The difference is $N\tau(D - t)^2/[(N\tau + D)(N\tau + t)^2]$. Consequently a further valid error bound is

$$\mathcal{E}_t = \frac{A}{N} + \max_{i \in \{+, -, J\}} \left(v_i + B \frac{N\tau}{(N\tau + t)^2} w_i \right) + B \frac{t^2}{(N\tau + t)^2}.$$

The checker takes the minimum of (20.7) and these eleven bounds with $t = jG/10$, $j = 0, \dots, 10$. Each is individually valid; no numerical minimization or derivative of their minimum is assumed.

The lower contradiction is the strict gate $Lr_0 - L(\tau_- + \eta) > \mathcal{E}$. To extend it to every $r \geq r_0$, bound increments directly in the original affine-in-width signed deletion bound for the same law. The budget gives $n\epsilon < E_* = 2T/(\kappa E \min(c_+, c_-))$. The sufficient slope gate is

$$L > \frac{F_d f}{N} \left\{ D_I/l + D_J/(1-h) + \frac{E_*}{2} (1/l + 1/(1-h)) \right\} + F_d f B_\tau.$$

Using the uncoupled increment is safe even when the hull bound was used at r_0 . The upper contradiction is $L_g - .55U_+ > \mathcal{E}$. Thus no support point lies below $a - r_0$ or above $1 + r_0$. Since $a \geq 0$, $b \geq 1 - d_0$ and $d_0 < r_0$, the label-zero and label-one supports both satisfy $|U| < r_0$.

The selected schedules. The ordinary, central-antichain, and left-antichain certificates substitute their recorded r_0 into these same gates. Noise-dependent amplitudes are the same theorem with a positive lower noise endpoint in $R = T_*/\kappa - b_* E^2_{sl}$. In the central rows the additional amplitudes are .35, .30, .275 on [.05, .08], [.08, .10], [.10, Λ]; on the left they are .375, .325, .30. Intersect each interval with its own row's Λ and each amplitude with that row's global amplitude. The fixed acceptance checks both signs of H , all closed phase/noise cells, clipped endpoints, and strict budget exclusions. No proof of LOW, MID, or a binomial score comparison is used by this confinement lemma.

21 A smaller excess ceiling at index 46

This section adds one finite calculation to the resource argument. Write $C_E = (3 + \sqrt{10})/(6\sqrt{2\pi})$ and $a_j = C_j - C_E$. The external published inputs and the attained-contact reduction are exactly those of the accompanying manuscript. All new entry estimates refer to the same attaining marginal and threshold.

Theorem 21.1 (Supported barrier at 46). *There is no attained positive maximum with $n = 46$,*

$$a_{46} \geq \frac{.54}{46}, \quad C_{46} \leq .42496, \quad \text{supp } X \subset [-3.4, 3.4].$$

The assertion includes either sign of the raw threshold and arbitrary atoms. It does not assume positive conditional variance.

The support edge. The existing finite ceiling $a_j < .7/j$ for $j = 45, 46$ gives

$$C_{46} < .40974 + .7/46 < .42496, \quad 46(a_{45} - a_{46})_+ < 46(.7)/45 - .54 = 79/450.$$

The independently checked all-threshold contact support implication with $(N, L, K, R, q) = (46, 46, .54, 3.4, 79/450)$ therefore supplies the support premise of Theorem 21.1. This implication is checked on 22 closed radial cells through 4.5 and the complete remaining ray, for both signs and every threshold. No entry output is used to prove this initial support edge. Consequently the supported theorem implies $a_{46} < .54/46$.

Unchanged quantities and formulas. Let $\mathcal{R}$ and $D \geq 0$ be the raw remainder and gap defect, respectively. The conserved identity (17.1), weight removal (17.3), and coherent phase price (17.5) have already been proved. In particular, after a prior strict raw bound $\mathcal{R} < r < .4n$ has licensed weight removal,

$$na_n \rho + \frac{997}{15000} nD \leq \mathcal{B}(H), \quad \phi(H/\sqrt{n}) \geq \phi_- := \frac{997}{2500} \left(1 - \frac{5r}{2n} \right).$$

Here $\mathcal{B}$ is the raw quadratic with the already proved extra curvature

$$e_\rho = \frac{1 - 5r/(2n)}{2} \left(.4095\rho - \frac{d}{15} \right) > 0, \quad nD < nd.$$

In this single-index calculation $n = 46$ throughout. The extra curvature uses only the preceding bounds r, d . On the barrier branch $na_n \geq K = .54$, it follows that

$$D < \frac{15000}{997n} \max_H \{ \mathcal{B}(H) - K\rho \}. \quad (21.1)$$

The subtraction is made only in this resource bound, never in the raw remainder passed to the next stage.

Four signed-moment covers. The first four stages use the signed-moment and common-first-moment formulas of Sections 18–19, with fixed parameters

$$N = L = 46, \quad R = 3.4, \quad K = .54, \quad C \leq .42496, \quad C \geq .4095 + .54/46.$$

The published moment caps determine the outer ρ interval. Each pass uses its complete $24 \times 24 \times 8 \times 4$ closed cover in (ρ, β, γ, m) , including both signs of β, γ . The LOW estimate and all six HIGH pieces are the complete estimates of Section 18; the signed core and its boundary rules are unchanged. The first raw output is below $.4(46) = 18.4$, before any defect output is used. Every later pass consumes only its accepted predecessor state.

Three probability covers. After the fourth pass $D < .2371 < .7$. The general gap geometry and the 24 ordered conditional updates in Section 19.8 provide a closed probability interval $[p_-, p_+]$. Split it into 96 equal closed cells. The simultaneous resource (17.9) proves each cell's moment and inner-gap bounds. A cell may be discarded only when its binary defect charge already exceeds the available defect. Otherwise apply the general four-corner enclosure (19.17), using the score floor $.4095 + .54/46$. All closed endpoints remain covered.

Where its explicit moment and rectangle gates hold, also use paired Fourier cancellation (18.13). In particular the checker verifies

$$0 < l < u < 1/2, \quad 1 \leq M \leq 2, \quad 0 \leq g \leq 1, \quad R > 2g, \quad 3 \min(A_l, B_l) \geq \frac{3}{2} \max_{p \in [l, u]} \rho_B(p),$$

and the nonnegative paired absolute-skewness lower endpoint and every coefficient ordering gate. The last displayed inequality licenses the actual lower bound $\rho \geq \rho_B(u)$. Reflect only the auxiliary Fourier integral when necessary, as in Theorem 17.3; the actual signed contact and phase data retain their original orientation. If a paired premise fails, retain the valid general enclosure for that cell.

The signed phase correction. The actual coefficient of H^2 is $C\rho + \gamma\phi/2$. The available signed lower bound is $\gamma \geq 3.16\rho - 3 - d$. It can be negative during the first probability pass. Define

$$\epsilon_\gamma = \frac{.4 - \phi_-}{2} (d - .16)_+.$$

Since $\rho \geq 1$, its negative part is at most $(d - .16)_+$, so

$$C\rho + \gamma\phi/2 \geq C\rho + \phi_-(3.16\rho - 3 - d)/2 - \epsilon_\gamma. \quad (21.2)$$

Thus every paired corner (A, f, q) returned before this correction is used with $\tilde{q} = q - \epsilon_\gamma > 0$. Both the raw and the resource bounds are recomputed:

$$R_{\text{pair}} = \max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4\tilde{q}} \right), \quad B_{\text{pair}} = \max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4(\tilde{q} + e_\rho)} - K\rho \right).$$

The four score-moment corners use $C \in \{.4095, .42496\}$ and $\rho \in \{\rho_B(u), \rho_u\}$. The lower score corner is a valid relaxation of the stronger barrier score floor. The checker requires all corrected divisors to be positive.

For clarity, the endpoint argument precedes square completion. At fixed H , the constant and linear-phase envelopes are convex in C and in ρ , as proved for (19.17); the curvature is affine in each variable. The new correction is constant on the cell, e_ρ is affine in ρ , and $-K\rho$ is affine. Hence taking those same endpoints still gives an upper bound. Only then use $f|H| - qH^2 \leq f^2/(4q)$. The maximum for the general estimate and the maximum for the paired estimate are separate complete bounds; their minimum is therefore valid. No coefficients from different estimates are combined.

Fixed exact schedule. All decimals in the table denote exact rational target bounds. Each row gives strict simultaneous upper bounds for $\mathcal{R}, D$.

Pass	Cover	$\mathcal{R}$	D	Active	Excluded	Paired
1	signed	5.8415	1.6324	17752	680	0
2	signed	2.8322	.6195	12871	5561	0
3	signed	1.8719	.3550	5515	12917	0
4	signed	1.4410	.2371	3127	15305	0
5	probability	.9953	.1077	92	4	83
6	probability	.7416	.0379	95	1	95
7	probability	.5840	-.0046	96	0	96

There are $4(18432) + 3(96) = 74016$ closed cells in the recorded covers. The final row contradicts $D \geq 0$, proving Theorem 21.1.

22 One local comparison theorem

A law concentrated near two points has a natural integer count and a small centered error. We compare it with its own Bernoulli count. The same exact identity handles every noise regime. There are two useful estimates of its right side: discard all but the nearest outgoing count, or retain all counts through a random cutoff. The estimates share their normalization, moments, and tail bounds; only the final scalar witness changes.

22.1 The identity and the common numerical test

Let $0 < p \leq 1/2$, $q = 1 - p$, $v = pq$, and let

$$Y = B + U, \quad \Pr(B = 1) = p, \quad \mathbb{E}(U \mid B) = 0.$$

The two actual conditional laws U_0, U_1 obey

$$|U_j| \leq A, \quad U_0 \leq a_0, \quad U_1 \geq -a_1, \quad a_0, a_1 \geq 0, \quad d = a_0 + a_1 \leq d_0 < \min(p, q), \quad d_0 \leq A. \quad (22.1)$$

Write $s_j = \mathbb{E}U_j^2$, $s = qs_0 + ps_1$, $\lambda = ns$, $\ell = (v + s)^{-1/2}$, $\theta = s/(v + s)$ and $\rho = \mathbb{E}|\ell(Y - p)|^3$. At the actual threshold choose

$$x = np + H/\ell = k + u, \quad k \in \{1, \dots, n-1\}, \quad |u| \leq \frac{1}{2}, \quad \delta = k - np = H/\ell - u.$$

Either nearest integer is allowed at a tie. Put

$$\pi_j = \binom{n}{j} p^j q^{n-j}, \quad F_B(j) = \Pr(\text{Bin}(n, p) \leq j), \quad \rho_B = (1 - 2v)/\sqrt{v},$$

with $\pi_j = 0$ outside $[0, n]$. Let W_j be an independent sum of j copies of U_1 and $n - j$ copies of U_0 . The signed scores are

$$C = \frac{\sqrt{n}}{\rho} \{\Pr(\sum Y_i \leq x) - \Phi(H/\sqrt{n})\}, \quad C_B = \frac{\sqrt{n}}{\rho_B} \{F_B(k) - \Phi(\delta/\sqrt{nv})\}.$$

Throughout this section $C \geq c_* > 0$; the selected application uses $c_* = .4$. Define

$$\begin{aligned} f(u) &= \frac{\Phi((\delta + u)/\sqrt{n(v+s)}) - \Phi(\delta/\sqrt{n(v+s)})}{\pi_k}, \\ Q &= \Pr(\sum Y_i \leq k + u) - F_B(k), \\ \mathcal{R} &= \sqrt{n} \{\Phi(\delta/\sqrt{nv}) - \Phi(\delta/\sqrt{n(v+s)})\} + C(\rho_B - \rho). \end{aligned} \tag{22.2}$$

Direct subtraction gives the exact identity

$$\rho_B(C - C_B) = \sqrt{n}\pi_k \{Q/\pi_k - f(u)\} + \mathcal{R}. \tag{22.3}$$

In particular, the phase u , the sign of H , and both conditional laws refer to the same marginal throughout.

Theorem 22.1 (Parameterized local comparison). *Fix $n \geq N$, $\lambda > 0$ and a closed parameter cell on which (22.1) holds. Suppose $\sqrt{n}\pi_k \geq m > 0$. Either of the following witnesses proves $\rho_B(C - C_B) < -\varepsilon\lambda$.*

1. Nearest-count witness. *Valid bounds O, T, R_0, κ satisfy*

$$\Pr(W_k > u) + f(u) \geq O, \quad \sum_{i \geq 1} \Pr(W_{k+i} \leq u - i) \leq T, \quad \pi_{k+i}/\pi_k \leq R_0, \quad \mathcal{R} \leq \kappa\lambda/n.$$

Only legal counts occur in the sum; adding positive upper bounds for other counts is allowed. If $D = O - R_0T > 0$ and

$$mD - \kappa\lambda/n > \varepsilon\lambda, \tag{22.4}$$

the conclusion follows.

2. Whole-count witness. *Bounds $B(\lambda) < 0$ and $I(\lambda) \geq 0$ satisfy*

$$\rho_B(C - C_B) \leq \sqrt{n}\pi_k\lambda B(\lambda) + \lambda I(\lambda)/n.$$

It is enough that

$$mB(\lambda) + I(\lambda)/N < -\varepsilon. \tag{22.5}$$

The witnesses may differ from cell to cell. A finite closed cover and a separate treatment of $\lambda = 0$ therefore suffice for a complete local comparison.

Proof. Conditioning on the binomial count gives, with all atom conventions intact,

$$Q = - \sum_{i=0}^k \pi_{k-i} \Pr(W_{k-i} > u + i) + \sum_{i=1}^{n-k} \pi_{k+i} \Pr(W_{k+i} \leq u - i). \tag{22.6}$$

Dropping the outgoing terms with $i > 0$ gives $Q/\pi_k - f(u) \leq -D$. Use (22.3) and $D > 0$. For the second witness, use $B < 0$ before replacing $\sqrt{n}\pi_k$ by m , and $I \geq 0$ before replacing n by N . These sign requirements are part of the test, not conventions. $\square$

22.2 Normalization and the repair, once for every witness

Here is an explicit finite recipe for the constants required above. Let $p \in [p_-, p_+] \subset (0, 1/2)$, $n \geq N$, $\lambda \leq \Lambda$, $\theta \leq \Theta$, and $|\delta| \leq \Delta$, where $\Delta \geq 1$ also includes every binomial mode. Choose $0 < r_- < p_- - \Delta/N$ and $p_+ + \Delta/N < r_+ < 1/2$. Let $v_* > 0$ be a lower bound for $p(1-p)$. Choose positive E, L, R bounding, on $[r_-, r_+]$,

$$|D'''(r||p)| = \frac{1-2r}{r^2(1-r)^2}, \quad |(\log(r(1-r)))'| = \frac{1-2r}{r(1-r)}, \quad \frac{1}{12r(1-r)},$$

respectively, where $D(r||p) = r \log(r/p) + (1-r) \log((1-r)/q)$. Then, with $Z(t) = Lt/2 + R + Et^3/(6N)$,

$$\sqrt{n}\pi_k = \frac{\phi(0)}{\sqrt{v}} e^{-\delta^2/(2nv)+\zeta}, \quad n|\zeta| < Z(|\delta|). \quad (22.7)$$

For completeness, the finite Stirling formula used here is

$$\log(j!) = (j + \frac{1}{2}) \log j - j + \frac{1}{2} \log(2\pi) + r_j, \quad 0 < r_j < 1/(12j).$$

If $a_j = \log(j!) - (j + 1/2) \log j + j$ and $z = (2j+1)^{-1}$, the integrated geometric series gives $0 < a_j - a_{j+1} = \sum_{i \geq 1} z^{2i}/(2i+1) < 1/[12j(j+1)]$. Summing gives the remainder relative to $\lim a_j$. The integration-by-parts recursion for $\int_0^{\pi/2} \sin^j t dt$ and the inequalities between three successive integrals give Wallis's product; its finite product equals $e^{4a_j-2a_{2j}} j/[2(2j+1)]$ and tends to $\pi/2$. Thus $\lim a_j = \frac{1}{2} \log(2\pi)$. Applying this formula to $n!, k!, (n-k)!$ and Taylor-expanding D and the logarithmic prefactor proves (22.7); the factorial error is at most $1/[12nr(1-r)]$.

Consequently a valid mass lower bound is any positive

$$m < \frac{.39886}{\sqrt{p_+(1-p_+)}} \left[1 - \frac{\Delta^2/(2v_*) + Z(\Delta)}{N} \right]. \quad (22.8)$$

The same formula at a mode proves $\sqrt{n} \max_j \pi_j < 1$ whenever $v_*[1 - Z(\Delta)/N]^2 > .4^2$ and $Z(\Delta) < N$. The rational bounds $.39886 < \phi(0) < .4$ follow, for example, by the alternating arctangent series in Machin's identity.

Differentiating (22.2) gives

$$\log f' = -\frac{1}{2} \log(1 + s/v) + \delta^2/(2nv) - (\delta + u)^2/(2n(v+s)) - \zeta. \quad (22.9)$$

Thus $n|\log f'| \leq B_0\lambda + P(D, a)$ on $|\delta| \leq D \leq \Delta$, $|u| \leq a \leq 1/2$, where

$$B_0 > \frac{1}{2v_*} + \frac{\Delta^2}{2Nv_*^2}, \quad P(D, a) = \frac{2Da + a^2}{2v_*} + Z(D).$$

Choose $0 < b < 1$ above $[B_0\Lambda + P(\Delta, 1/2)]/N$ and put $F = (1-b)^{-1}$. The elementary estimates $e^z \leq (1-z)^{-1}$ and $1 - e^{-z} \leq z$ imply

$$|f' - 1| \leq (B_1\lambda + P_1)/n, \quad B_1 = FB_0, \quad P_1 = FP(D, a). \quad (22.10)$$

This supplies both a derivative interval $l_f \leq f' \leq h_f$ and the finer phase-error bound $|u - f(u)| \leq a(B_1\lambda + P_1)/n$.

The exact count variance is

$$t_j = \mathbb{E}W_j^2 = \lambda + (j - np)(s_1 - s_0), \quad |s_1 - s_0| \leq s/p \leq c_v s, \quad c_v \geq 1/p_-. \quad (22.11)$$

It gives $\omega_- \lambda \leq t_k \leq \omega_+ \lambda$ with $\omega_{\pm} = 1 \pm c_v D/N$. All uses below require the checked condition $\omega_- > 0$. Also $t_{k+i} \leq iV_*\lambda$ for $V_* = 1 + c_v(\Delta + 1)/N$. The binomial recursion gives, for every legal $i \geq 1$,

$$\frac{\pi_{k+i}}{\pi_k} \leq R_0, \quad R_0 > 1 + \frac{p_+ + \Delta - 1}{Nv_*[1 - (\Delta - 1)/(Np_-)]}, \quad (22.12)$$

provided the denominator is positive and $p_+ + \Delta < 2$. Indeed the first ratio has this bound and all subsequent rising factors are at most one. No count outside $[0, n]$ is used in this recursion.

Finally (22.1) fixes the signs inside the absolute cubic. Expansion and centering give

$$\mathbb{E}|Y - p|^3 = v(1 - 2v) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3 \geq v(1 - 2v) + (3p - d)s.$$

The inequality $(1 + z)^{3/2} \leq 1 + 3z/2 + 3z^2/8$ for $z \geq 0$ yields

$$(v + s)^{3/2}(\rho - \rho_B) \geq s \left(6p - 3p^2 - \frac{3}{2} - d - \frac{3(1 - 2v)s}{8v} \right). \quad (22.13)$$

Let $Q_* > 0$ be a lower bound for the bracket and $m_c \leq c_*/[p_+(1 - p_+) + \Lambda/N]^{3/2}$. The normal variance correction is nonpositive for $\delta \leq 0$; for $\delta > 0$ its derivative in s is at most $.4\delta/(2v^{3/2})$. Therefore

$$\mathcal{R} \leq (P_*D_+ - m_cQ_*)\lambda/n, \quad P_* \geq \frac{.4}{2v_*^{3/2}}, \quad D_+ \geq \delta_+. \quad (22.14)$$

One may take $D_+ = D$, but retaining the positive part is often decisive. All bounds concern $v + s$, which can exceed $1/4$.

22.3 Two elementary tools for probabilities

Lemma 22.2 (A shifted quartic). *Suppose W is a sum of independent centered variables bounded in absolute value by A , and $t = \mathbb{E}W^2$. Let $K_0 \geq (9 + 6\sqrt{3})/4$ and $a > 0$. For $r \geq 0$ and $\sigma \in \{-1, 1\}$,*

$$\Pr(W + \sigma r > 0) \geq \frac{(3a^2 - A^2)t - 3t^2 - 4Art - 6r^2t - r^4 + 3a^2r^2 + 2\sigma a^3r}{K_0a^4}. \quad (22.15)$$

Proof. The polynomial $h_a(z) = (-z^4 + 3a^2z^2 + 2a^3z)/(K_0a^4)$ is below $\mathbf{1}_{\{z > 0\}}$: it is nonpositive for $z \leq 0$, and its maximum for $z > 0$ is at $z/a = (1 + \sqrt{3})/2$ and is at most one. Independence gives $|\mathbb{E}W^3| \leq At$ and $\mathbb{E}W^4 \leq A^2t + 3t^2$. Expand $\mathbb{E}h_a(W + \sigma r)$ and apply these two inequalities. The value at the threshold is zero, so atoms require no limiting argument. $\square$

Lemma 22.3 (One common tail bound). *If independent centered $X_i \leq A$ have total variance at most V , then, for $b > 0$, their sum has upper tail bounded by the minimum of one and*

$$\frac{V}{V + b^2}, \quad \mathcal{B}(A, V, b) = e^{b/A} \left(\frac{V}{V + Ab} \right)^{b/A + V/A^2}, \quad \frac{\mathbb{E}(P_\mu - j)_+^3}{(b/A + \mu - j)^3}, \quad (22.16)$$

$$\mu = V/A^2, \quad 0 \leq j < b/A + \mu.$$

where P_μ is Poisson. The same holds for a lower tail when $|X_i| \leq A$. All bounds apply to closed events, hence also to strict events.

Proof. Cantelli's bound follows by applying Markov to $(\sum X_i + V/b)^2$. Taylor's integral remainder gives $\mathbb{E}e^{zX_i} \leq \exp\{\mathbb{E}X_i^2(e^{zA} - 1 - zA)/A^2\}$ for $z \geq 0$. Multiply and use $z = A^{-1} \log(1 + Ab/V)$ to get $\mathcal{B}$. The zero-variance case is immediate.

For the third bound, smooth $f(x) = (x - t)_+^3$ by a nonnegative smooth compactly supported density. Then $f'', f''', f'''' \geq 0$. For $\mathcal{A}f(x) = f(x + A) - f(x) - Af'(x)$, the Taylor quotient

$$\frac{f(x + y) - f(x) - yf'(x)}{y^2} = \int_0^1 (1 - r)f''(x + ry) dr$$

is nondecreasing in y . A centered $X \leq A$ of variance v consequently satisfies $\mathbb{E}f(x + X) \leq f(x) + (v/A^2)\mathcal{A}f(x)$. For $\mathcal{P}_z f(x) = \mathbb{E}f(x + A(P_z - z))$, $\partial_z \mathcal{P}_z f = \mathcal{P}_z \mathcal{A}f$ and

$$\partial_z^2 \mathcal{P}_z f = \mathcal{P}_z \mathcal{A}^2 f \geq 0, \quad \mathcal{A}^2 f(x) = \int_0^A \int_0^A (A - r)(A - w)f''''(x + r + w) dr dw.$$

Thus $\mathcal{P}_{v/A^2}f \geq f + (v/A^2)\mathcal{A}f$. Replace the independent variables successively; translations and expectations preserve the derivative signs and Poisson rates add. Increasing the total variance to V is permitted by Jensen after adding independent centered Poisson noise. Removing the smoothing is justified by bounded actual upper supports and finite Poisson third moments. Markov at $t = A(j - \mu)$ proves the displayed bound. $\square$

These bounds have finite rational evaluations. For example,

$$\begin{aligned} \mathbb{E}(P_\mu - j)_+^3 &= \mu^3 + 3(1 - j)\mu^2 + (1 - 3j + 3j^2)\mu - j^3 \\ &+ e^{-\mu} \sum_{h=0}^{\lfloor j \rfloor} \frac{\mu^h (j - h)^3}{h!}. \end{aligned} \quad (22.17)$$

Replacing $e^{-\mu}$ by $(\sum_{h=0}^{24} \mu^h/h!)^{-1}$ enlarges the nonnegative correction. Alternatively sum the positive Poisson series through 32; the remaining tail is at most its first term divided by $1 - \mu(34 - j)^3/[34(33 - j)^3]$, provided $j < 33$ and that ratio is below one. For Bennett, put $x = Ab/V$ and $L = [(b/A + \mu) \log(1 + x) - b/A]$. The positive series $\log(1 + x) = 2 \sum_{h \geq 0} z^{2h+1}/(2h + 1)$, $z = x/(2 + x)$, gives lower bounds for L ; any positive lower exponential series gives an upper bound for e^{-L} . Rounding a positive lower L down before exponentiation is also valid. These are choices of rational witnesses, not optimization claims.

22.4 The nearest-count witness, including vanishing noise

For $u = -r \leq 0$, add $-h_f r$ to (22.15) with $\sigma = 1$; for $u = r \geq 0$, add $l_f r$ with $\sigma = -1$. Both bound $\Pr(W_k > u) + f(u)$ from below. For a cell $t \in [t_-, t_+]$, $r \in [r_-, r_+]$, this expression is concave in t (second derivative $-6/(K_0 a^4)$), so only t_-, t_+ are needed. At each endpoint, bound a monomial with positive coefficient at r_- and one with negative coefficient at r_+ . Any rational $a > 0$ is a valid witness. Other valid choices are

$$\frac{r_-^2}{t_+ + r_-^2} - h_f r_+ \quad (u \leq 0), \quad l_f r_- \quad (u \geq 0). \quad (22.18)$$

They follow from Cantelli and nonnegativity, respectively. Their maximum or one selected witness is valid without an optimality proof.

For incoming counts use $t_{k+i} \leq iV_*\lambda$. On $u \in [-r_+, -r_-]$ the first distance is $1 + r_-$ and every later distance is at least i . On $u \in [r_-, r_+]$ they are $1 - r_+$ and $i(1 - r_+/2)$ for $i \geq 2$. Let b, b_o be these first and later distances. On $\lambda \in [L, U]$, let T_1 be any bound from Lemma 22.3 at (A, V_*U, b) . Since $\mathcal{B}(A, iV, ib) = \mathcal{B}(A, V, b)^i$, all later counts total at most $B^2/(1 - B)$ for $B = \mathcal{B}(A, V_*U, b_o) < 1$. Thus

$$D = O - R_0\{T_1 + B^2/(1 - B)\} > 0, \quad mD - \kappa U/N > \varepsilon U \quad (22.19)$$

proves the entire cell when $\kappa \geq 0$. The selected MID normalizers explicitly verify this sign. This is the common central and left MID proof.

A cell touching $\lambda = 0$ requires estimates proportional to λ ; a positive lower endpoint may not be substituted. The following rescaling provides them. Put $r = zt$, $t_* = \omega_+ U$ and $d_a = K_0 a^4$. The shifted quartic plus br , divided by t , becomes

$$F_{\sigma,b}(z, t) = \frac{3a^2 - A^2 - 3t}{d_a} + \left(\frac{2\sigma}{K_0 a} + b - \frac{4At}{d_a} \right) z + \frac{3a^2 t - 6t^2}{d_a} z^2 - \frac{t^3 z^4}{d_a}. \quad (22.20)$$

It extends to $t = 0$ and is concave in t . On $0 \leq z \leq z_0$, replacing z^4 by $z_0^2 z^2$ gives a lower quadratic at $t = t_*$; at $t = 0$ it is linear. Endpoints and a clipped quadratic vertex therefore give an exact lower bound $J(\sigma, b, z_0)$ for the whole rectangle.

For negative phase choose $z_-, c_- > 0$ and verify

$$\begin{aligned} J(1, -h_f, z_-) &> c_-, \quad \frac{z_-^2}{1 + z_-^2 t_*} - h_f z_- > c_-, \quad \frac{2z_-}{(1 + z_-^2 t_*)^2} > h_f, \\ \frac{1}{1 + 4t_*} - \frac{h_f}{2} &> c_- t_*, \quad z_- t_* < 1/2. \end{aligned} \quad (22.21)$$

The quartic handles $r \leq z_- t$. Beyond that point use $r^2/(t + r^2) - h_f r$. Its derivative at $r = z_- t$ is positive by the third gate; its second derivative has just one sign change, so any later critical point is a maximum. Its minimum is at $z_- t$ or $1/2$, covered by the other gates. Hence the own count exceeds $c_- t$. For positive phase choose $z_+, c_+ > 0$ with

$$J(-1, l_f, z_+) > c_+, \quad l_f z_+ > c_+, \quad z_+ t_* < 1/2. \quad (22.22)$$

The quartic handles $r \leq z_+ t$ and the normal term handles the rest. The own count exceeds $c_+ t$ and also exceeds $l_f r$.

For incoming tails proportional to λ , fix a Poisson anchor j . A positive series term of index h divided by $\lambda(b/A + \mu - j)^3$ has logarithmic derivative, multiplied by λ ,

$$h - 1 - \mu - \frac{3\mu}{b/A + \mu - j}, \quad \mu = V_* \lambda / A^2.$$

If $b/A - j > 0$ and this is nonnegative at $\lambda = U$ for the first positive index, it is nonnegative for every term and all $0 < \lambda \leq U$. Therefore the Poisson bound divided by λ is largest at U . Likewise $B^2/[\lambda(1 - B)]$ is increasing whenever $\lambda B'/B \geq 1/2$. Differentiating the Bennett formula gives

$$\lambda B'/B = b/A - \mu \log(1 + b/(A\mu));$$

The function $g(\mu) = b/A - \mu \log(1 + b/(A\mu))$ decreases because $\log(1 + x) > x/(1 + x)$ for $x > 0$. Moreover the logarithmic derivative of $B^2/[\lambda(1 - B)]$, multiplied by λ , is $2g - 1 + gB/(1 - B) \geq 0$ when $g \geq 1/2$. Thus checking the gate at U proves the whole interval. A rational upper bound for the logarithm verifies that gate. On cells using the fixed positive bound $l_f r_-$ instead, unscaled endpoint tails suffice: after a positive bracket is checked, divide it by U to obtain a lower bound per λ . These arguments, followed by (22.4), prove TINY uniformly down to arbitrary positive noise, including a deterministic conditional component.

22.5 The whole-count witness

Assume $c = 1 - d_0 > 1/2$. Couple all $2n$ conditional variables independently and set $W_j = \sum_{i \leq j} U_{1,i} + \sum_{i > j} U_{0,i}$, extending it constantly outside $[0, n]$. Every increment of $j + W_j$ is at least c . Define

$$K = \max\{j \in \mathbb{Z} : j + W_j \leq k + u\}, \quad R = K - k.$$

An independent binomial selector has the correct actual count laws, so

$$\Pr(\sum Y_i \leq k + u) = \mathbb{E} F_B(k + R). \quad (22.23)$$

Let $\chi = (n + 1)p - k < 2$ and $m_B = \max_j \pi_j$. For every integer r ,

$$F_B(k + r) - F_B(k) - r\pi_k \leq \frac{m_B}{(n + 1)v} \left[(\chi - 1)_{+r} + \chi_+ \binom{r_-}{2} + \binom{r_-}{3} \right]. \quad (22.24)$$

Indeed, the positive part of $\pi_{k+i} - \pi_{k+i-1}$ is at most $m_B(\chi - i)_+ / ((n + 1)v)$, by $\pi_j - \pi_{j-1} = \pi_j((n + 1)p - j) / (p(n - j + 1))$. Double summation gives (22.24): for $r > 0$ only $i = 1$ can contribute; for $r = -h < 0$ use $(\chi + i)_+ \leq \chi_+ + i$ and sum the multiplicities $h - 1 - i$. Zero extension makes the same calculation valid outside the legal count range.

Write $t = \mathbb{E}W_k^2$. Crossing monotonicity and Cauchy–Schwarz give

$$\mathbb{E}R_+ \leq \frac{t}{(c-u)^2}, \quad \mathbb{E}\binom{R_-}{2} \leq \frac{t}{[c + \min(u, c/3)]^2}, \quad \mathbb{E}\binom{R_-}{3} \leq \frac{t\sqrt{A^2 + 3t}}{(2c - 1/2)^3}. \quad (22.25)$$

For the first, $R = i \geq 1$ implies $-W_k \geq ci - u$ and $(ci - u)^2 \geq i(c - u)^2$. For $R = -h \leq -2$, the next count is outside the closed event, so $W_k > c(h - 1) + u$. Put $j = h - 1$. The function $j(j + 1)/(2(cj + u)^2)$ decreases for $j \geq 1$ if $u \leq c/3$; for larger u reduce to that endpoint. Also $(j^3 - j)/(6(cj - 1/2)^3)$ decreases for $j \geq 2$, since its derivative has the sign of $2cj - (3j^2 - 1)/2 < 0$. These give the last two bounds using $\mathbb{E}(W_k)_+^3 \leq \sqrt{\mathbb{E}W_k^2 \mathbb{E}W_k^4}$.

The marked crossing. Let $D_i = U_{1,i} - U_{0,i}$, $\vartheta = \mathbb{E}(D_i)_+ = \mathbb{E}(D_i)_-$, $0 < \eta \leq 1$, and $R_f = \mathbb{E}\{u + \eta - W_k\}$. The two deleted laws are

$$A_0 \stackrel{d}{=} kU_1 + (n - k - 1)U_0, \quad A_1 \stackrel{d}{=} (k - 1)U_1 + (n - k)U_0.$$

Then

$$\begin{aligned} \mathbb{E}R &\leq u + \eta - R_f + J, \\ J &= \frac{\vartheta}{\eta} \sum_{i \geq 1} \{\Pr(A_0 \leq u + d_0 - ci) + \Pr(A_1 > u + \eta + i - A)\}. \end{aligned} \quad (22.26)$$

Here are the pathwise details. Put $z = u - W_k$ and mark

$$Z_f = \sum_{i=1}^{n-k} (D_{k+i})_- \mathbf{1}_{\{W_k \leq u - ci\}}, \quad Z_b = \sum_{i=1}^k (D_{k-i+1})_+ \mathbf{1}_{\{W_k > u + \eta + i\}}.$$

If $R > z + \eta$, the crossing activates the marks through R : for $R > 0$ use $z \geq cR$, while for $R < 0$ use $z < R - \eta$. In either case $R \leq z + Z_f + Z_b$. Hence if $h = R - \lfloor z + \eta \rfloor > 0$, $Z_f + Z_b > h - 1 + \eta \geq \eta h$. Thus $R \leq \lfloor z + \eta \rfloor + (Z_f + Z_b)/\eta$ on every path. On an active forward mark $U_0 > U_1 \geq -d_0$; deleting its pair implies $A_0 \leq u + d_0 - ci$. On a backward mark delete $U_1 \leq A$ to obtain the strict event for A_1 . The remaining sum is independent of the entire pair, so the expectations factor. Extending to all $i \geq 1$ adds only nonnegative bounds. Finally $|R| \leq 1 + (|W_k| + |u|)/c$ proves integrability and permits expectations. No coupling has replaced either deleted marginal.

A finite bound for every mark. Fix $\lambda \in [L, U]$, $L > 0$, and $t \leq \omega_+ \lambda$; put $T_L = \omega_+ L$, $T_U = \omega_+ U$. On a phase subcell $u \in [v_l, v_h]$, for $i = 1, 2$ let $b_1 = ci - d_0 - v_h > 0$ and $b_2 = i - A + v_l > 0$. Lemma 22.3 gives a bound $\omega_+ \lambda K_i^{\text{ind}}$, where

$$K_i^{\text{ind}} = \sum_{j=1}^2 \min \left\{ \frac{1}{b_j^2 + T_L}, \frac{\overline{P}(A, T_U, b_j)}{T_L} \right\}.$$

The denominator T_L is a lower bound for the chosen upper envelope $\omega_+ \lambda$, not for any actual variance. Alternatively couple $A_0 = Z + U_1$, $A_1 = Z + U_0$ with $Z \stackrel{d}{=} (k - 1)U_1 + (n - k - 1)U_0$. The two events project into $Z \leq -\alpha$ and $Z > \beta$, where $\alpha = ci - 2d_0 - v_h > 0$, $\beta = i - A - d_0 + v_l > 0$. For any $z \in (-\alpha, \beta)$ their sum is at most $(\omega_+ \lambda + z^2)/\min(z + \alpha, \beta - z)^2$. Consequently a second kernel is

$$K_i^{\text{com}} = \frac{1 + z^2/T_L}{\min(z + \alpha, \beta - z)^2}, \quad z = \max\{-T_L/\beta, \min((\beta - \alpha)/2, T_L/\alpha)\}.$$

This center is inside the interval; differentiating the two quadratic ratios on either side of the midpoint proves its optimality, although any interior rational center suffices. Take the smaller kernel on each phase subcell and the maximum over their closed cover. Denote it K_i . For $i \geq 3$ use fourth moments and the convex midpoint integral for $(cx - b)^{-4}$. On $u \in [u_l, u_h]$ this gives

$$K = K_1 + K_2 + \frac{A^2 + 3T_U}{3c} \left\{ \frac{1}{((5/2)c - A - u_h)^3} + \frac{1}{((5/2)c - A + u_l)^3} \right\}, \quad J \leq \frac{\vartheta}{\eta} \omega_+ \lambda K. \quad (22.27)$$

All denominators must be positive. Indeed, the tail fourth moment is at most $A^2 t + 3t^2$, and integration from $5/2$ to infinity covers every remaining integer, including the harmless added illegal counts.

If the same-law resource supplies $nd < G$ and $G^2 < c_v N L$, then

$$\vartheta \leq \frac{c_v G \lambda}{c_v n \lambda + G^2}. \quad (22.28)$$

To prove this, for centered $X \geq -b$, $\mathbb{E}X^2 \leq V$ and $b^2 \leq V$, the quadratic $b(X - V/b)^2 / (b + V/b)^2$ majorizes X_- on $[-b, \infty)$. Its expectation is at most $bV/(V + b^2)$. Apply this to $D_i \geq -d$, using $\text{Var}(D_i) = s_0 + s_1 \leq c_v \lambda / n$, and monotonicity in $b \leq \sqrt{V}$. The case $d = 0$ is immediate. With $\eta = z\lambda$, $z > 0$, $zU \leq 1$, (22.27) becomes

$$\frac{J}{\lambda} \leq \frac{\omega_+ K c_v G}{z(c_v N \lambda + G^2)}. \quad (22.29)$$

The fractional-part gain. If each centered summand has support span at most $b < 1/2$ and the sum has variance t , then, for any real w and $0 < r_* \leq 2 \sin^2(\pi b)/b^2$,

$$\mathbb{E}\{w - W\} \geq \frac{5}{23}(1 - e^{-r_* t}). \quad (22.30)$$

Indeed $1 - \cos(2\pi x) \leq (23/5)x$ for $0 \leq x \leq 1$. A finite proof of this sole scalar inequality is recorded below. For an independent copy Z' of one summand, concavity of sine on $[0, \pi b]$ gives

$$|\mathbb{E}e^{2\pi i Z}|^2 = 1 - 2\mathbb{E}\sin^2 \pi(Z - Z') \leq e^{-4 \sin^2(\pi b) \text{Var}(Z)/b^2}.$$

Multiply square roots over the independent summands and use periodicity. This argument includes zero variances and integer collisions. Here the conditional spans are at most $A + d_0$, so this is the b to use.

For a reproducible scalar proof, $x \geq 10/23$ follows from $1 - \cos \leq 2$. On $[0, 10/23]$ use the degree-ten alternating upper polynomial for $1 - \cos(2\pi x)$, with directed coefficients:

$$Q(x) = \sum_{j=1}^5 \frac{(-1)^{j+1} (2\pi_j x)^{2j}}{(2j)!}, \quad \pi_j = \begin{cases} 3.141592653589794, & j \text{ odd}, \\ 3.141592653589793, & j \text{ even}. \end{cases}$$

Put $P(x) = 23/5 - Q(x)/x$. On the eight intervals with left endpoint $a_j = 5j/92$ and width $w = 5/92$, its degree-nine Bernstein coefficients are

$$b_{jk} = \sum_{i=0}^k \frac{\binom{k}{i}}{\binom{9}{i}} \sum_{h=i}^9 [x^h] P(x) \binom{h}{i} a_j^{h-i} w^i, \quad 0 \leq j < 8, \quad 0 \leq k \leq 9.$$

The 80 rational coefficients are positive; the retained scalar acceptor reconstructs them. Expansion in powers of the unit-interval coordinate proves the formula; Bernstein weights are nonnegative and sum to one. The alternating remainder and the stated rational bounds on π follow from the same elementary series above. This completely specifies the finite scalar obligation. A rational

rate follows by replacing $\sin(\pi b)$ with $x - x^3/6 + x^5/120 - x^7/5040$, $x = 3.14159b$, on the checked alternating-series domain.

Set $r = r_*\omega_-$, require $rU < 3$, and define

$$g = \frac{5}{23} \left(\frac{r^3 L^2}{6} - \frac{r^4 L^3}{24} + \frac{r^5 L^4}{120} - \frac{r^6 L^5}{720} \right). \quad (22.31)$$

The sixth-order Taylor lower bound for $1 - e^{-r\lambda}$ gives $R_f/\lambda \geq 5r/23 - 5r^2\lambda/46 + g$. The discarded higher-order part is increasing as a function of λ for $r\lambda < 3$: its derivative has the sign of $48 - 18x + (24/5)x^2 - x^3 > 0$ on $[0, 3]$. For instance this polynomial has positive degree-three Bernstein coefficients 48, 30, 132/5, 51/5 there. Thus freezing it at L is valid.

One final expression. On a signed phase cell find an interval $[x_l, x_h]$ containing H/ℓ by its four endpoint quotients, using positive bounds for ℓ . Let

$$D = \max(|x_l - u_h|, |x_h - u_l|), \quad D_+ = \max(0, x_h - u_l), \quad a = \max(|u_l|, |u_h|), \quad b_\chi = p_+ - x_l.$$

The same formula allows coarser upper bounds D, D_+, b_χ , as used by the central certificate. Check $D \leq \Delta$ and $b_\chi + u_h < 2$. With $\omega_\pm = 1 \pm c_v D/N$, the curvature coefficient is

$$V_0 = \frac{\omega_+}{v_*} \left[\sup_{u_l \leq u \leq u_h} \left\{ \frac{(b_\chi + u - 1)_+}{(c - u)^2} + \frac{(b_\chi + u)_+}{[c + \min(u, c/3)]^2} \right\} + \frac{\sqrt{A^2 + 3T_U}}{(2c - 1/2)^3} \right]. \quad (22.32)$$

On a rational phase subcell, both numerators are bounded at the right endpoint; the first denominator is smallest there, and the second is smallest at the left endpoint. This proves the supremum bound, including both kinks. Equations (22.23)–(22.25), $\sqrt{n}m_B \leq 1$, and (22.3) now give the whole-count witness with

$$B(\lambda) = z + \frac{\omega_+ K c_v G}{z(c_v N \lambda + G^2)} - \frac{5r}{23} + \frac{5r^2 \lambda}{46} - g, \quad (22.33)$$

$$I(\lambda) = V_0 + aB_1 + \frac{aP_1}{\lambda} + P_* D_+ - m_c Q_*.$$

We used $|u - f(u)| \leq a(B_1 \lambda + P_1)/n$ and $\sqrt{n}\pi_k \leq 1$. Every constant is fixed before λ varies. Both B and $mB + I/N$ are convex on $\lambda > 0$; I is decreasing. Thus their two endpoint gates $B < 0$, $I > 0$, $mB + I/N < -.001$ prove the entire closed noise cell. This proves all three LOW variants by exactly the same argument.

22.6 The selected finite cover and the zero-noise endpoint

The retained tables contain 45 central probability intervals covering $[\cdot 38, \cdot 47]$ and 38 left intervals covering $[\cdot 304, \cdot 38]$. Each row supplies its own Θ , signed H bounds, d_0 , Λ , and global amplitude; the confinement profiles may improve the amplitude with λ . All these are hypotheses of the present section, acquired earlier without using its conclusion. The shared resource has the form

$$nD_{\text{mom}} < \frac{15000}{997} T_H, \quad D_{\text{mom}} \geq \alpha d + \beta \theta + P.$$

Consequently a noise cell $[L, U]$ has

$$nd < G = \{(15000/997)T_H - NP - \beta \ell_-^2 L\}/\alpha.$$

Here T_H is the maximum of the acquired concave quadratic budget on the closed signed H cell, further bounded by its scalar ceiling. It already contains the phase charge. If $G \leq 0$, the entire cell is impossible and receives no score test; the exclusion is strict even at $G = 0$.

For positive noise the complete cover is the following. Each interval is intersected with the row ceiling Λ ; shared endpoints are included.

Noise	Central phase	Left phase	Witness
(0, .05]	all	all	nearest count (scaled)
[.05, .06]	$ u \geq .09$	$ u \geq .09$	nearest count
[.06, .08]	$ u \geq .13$	$ u \geq .14$	nearest count
[.08, .10]	$ u \geq .18$	$ u \geq .19$	nearest count
[.10, .12]	$ u \geq .29$	$u \leq -.29$	nearest count
[.10, .12]	—	$u \in [.29, .5]$	whole count
[.05, Λ]	remaining phase	remaining phase	whole count

Beyond .12 the last row covers the full phase interval. The three LOW objects are one central cover, one left cover with phase slices, and one extra positive left cover. Their distinction is only the closed domain and the chosen outward constants. Signed H cells meet at zero; nearest count ties meet at $u = \pm 1/2$. The fixed coverage checker also checks every row-ceiling singleton and every entirely excluded LOW row.

The common TINY parameters are

$$N = 47, \quad p \in [.304, .47], \quad |H| \leq 1.393, \quad \Theta = .0154, \quad d_0 = .0181, \quad A = .425, \quad \Lambda = .05.$$

For its scaled own witness use $a = .4784$, $K_0 = 9/(4 \cdot .4641)$, $(z_-, c_-) = (1.66, .514)$ and $(z_+, c_+) = (1.2, 1.036)$. Its six phase cells are $[-.5, 0]$, $[0, .1]$, $\dots$, $[.4, .5]$. The first two use scaled Poisson anchors 1.25, 1, respectively; the other four use the fixed positive normal bound. The exact normalizers and gates specified in the accompanying certificate map verify these constants. The central MID amplitudes are .35, .35, .30, .275 in its four rows. The left ones are .375, .375, .325, .30, each further reduced by the row's global amplitude. No interpolation of the tables is used.

All six accepted comparisons give the same strict conclusion

$$\rho_B(C - C_B) < -.001\lambda.$$

Their proofs have now reduced to Theorem 22.1, the shared normalization, and the two explicitly derived witnesses. The arithmetic objects still have their original domains and targets; no certificate branch has been silently deleted or relabeled.

If $\lambda = 0$, conditional centering and $p, q > 0$ imply $U_0 = U_1 = 0$ almost surely. The law is exactly Bernoulli. For an attained positive closed maximum its threshold is a binomial atom: lowering a noninteger threshold to the preceding atom leaves the CDF unchanged and strictly improves the normal term, while thresholds below the first atom have nonpositive score. Thus $u = 0$ at such a maximum and $C = C_B$. Schulz's all-probability Bernoulli theorem [6, Theorem 1] gives $C_B < C_E$ for every $0 < p < 1$ and in particular for all the indices $n \geq 47$ used here. The same external theorem closes the positive-noise conclusion $C < C_B < C_E$; no finite Bernoulli reserve is needed. No estimate in this section divides by zero or passes from fixed-law convergence to uniformity in varying laws.

23 Exact arithmetic and the proof boundary

The preceding sections reduce the theorem to finitely many rational inequalities on closed parameter cells. The executable supplement is part of this computer-assisted proof. It fixes the row data, rational witnesses and subdivisions; acceptance reconstructs their inequalities. The numerical tables are not evidence of a sampled inequality, and no optimizer's success or infeasibility report is used as acceptance.

23.1 A small arithmetic language

Integers and reduced fractions are exact. A square root of a rational $x \geq 0$ is enclosed using an integer k with $k^2 \leq m^2 x < (k+1)^2$; the endpoints are $k/m, (k+1)/m$. Every denominator's

sign is checked before division. Exponentials, logarithms, sines and cosines occur only through the finite rational series and remainder bounds proved in the text. Those bounds include the endpoint $1/n = 0$ where an infinite index band is used.

For clarity, the polynomial test used throughout can be reconstructed without a library. If $P(x) = \sum_{j=0}^d a_j x^j$ and $x = l + (u - l)y$, then its coefficients in powers of y are

$$c_i = \sum_{j=i}^d a_j \binom{j}{i} l^{j-i} (u - l)^i.$$

Its Bernstein coefficients are

$$b_k = \sum_{i=0}^k \frac{\binom{k}{i}}{\binom{d}{i}} c_i, \quad P(l + (u - l)y) = \sum_{k=0}^d b_k \binom{d}{k} y^k (1 - y)^{d-k}.$$

The basis is nonnegative and sums to one on $0 \leq y \leq 1$. Thus $\min b_k \leq P \leq \max b_k$ on the entire closed interval. Applying the transform one coordinate at a time proves the tensor version. Strict positive coefficients suffice for positivity; failure to obtain them is a rejection, not a proof of negativity. On an infinite ray, strict signs of a polynomial and all its derivatives at the endpoint supply the separate ray certificate. The independent transform checks expand the Bernstein expression back into monomials.

The resource iteration is a finite induction over rows and passes. Each accepted state encloses the same unknown law; subsequent states intersect it with proved inequalities. Counts of feasible and excluded cells must sum to the prescribed complete cover. An excluded cell requires a strict contradiction, including at boundary values. The local score tests check sign premises before substituting a mass lower bound or the index floor. Convexity in the noise parameter then reduces each closed noise cell to its two endpoints.

23.2 The dependency graph and the supplement

The new logical graph is

finite caps, attainment, contact $\longrightarrow$ support and resource formulas,
 five supported exclusions $\longrightarrow$ finite ceilings through 50, sharpened at 46,
 acquired parameter cover and confinement $\longrightarrow$ uniform local comparison,
 sharper ceiling + three losses $\longrightarrow C_{49} < C_E$,
 smoothing + least later failure $\longrightarrow C_n < C_E$ ($n \geq 49$).
 fourth-index contact + common defect measure + finite cover $\longrightarrow C_4 \leq 4097/10000 < C_E$.

The all-parameter Bernoulli theorem closes the local comparison. No edge enters from an earlier numerical cutoff. The complete Fourier, gap, moment and count arguments are shared tools proved once in this manuscript; retaining those tools does not retain their former global applications.

The electronic supplement supplies the rational row data, witnesses and closed subdivisions used in these finite verifications. Every accepted cell satisfies the inequalities stated above; every discarded cell has a strict contradiction with its necessary constraints. The covers include all shared endpoints and the infinite index band. Together with the analytic implications proved here, these finite inequalities complete the computer-assisted proof.

24 Two summands: a rational tensor certificate

24.1 Result and proof boundary

For a standardized law P , write

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = 1, \quad \rho(P) = \mathbb{E}|X|^3 < \infty,$$

$$\Delta_n(P) = \sup_x |P^{*n}((-\infty, x\sqrt{n}]) - \Phi(x)|, \quad C_n(P) = \frac{\sqrt{n} \Delta_n(P)}{\rho(P)}, \quad C_n = \sup_P C_n(P).$$

Let $\phi = \Phi'$ and $\phi_0 = (2\pi)^{-1/2}$. Set

$$K = \frac{2\sqrt{2}}{7} = 0.404061017820884\dots, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}} = 0.409732183702396\dots$$

Theorem 24.1 (Two-summand bound). *For every standardized law P with finite third absolute moment,*

$$\Delta_2(P) \leq \frac{2}{7}\rho(P).$$

Equivalently, $C_2 \leq K < C_E$.

The analytic reduction is proved below. The finite positivity assertions are proved by the accompanying rational certificate and checkers. The finite certificate proves the stated upper bound, not an identification of the exact value of C_2 .

The sole external numerical probability estimate is Shevtsova's published structural inequality [7]:

$$\frac{\sqrt{n} \Delta_n(P)}{\rho(P)} \leq a \left(1 + \frac{b}{\rho(P)}\right), \quad a = \frac{1661}{5000}, \quad b = \frac{429}{1000}. \quad (24.1)$$

It implies

$$\rho \geq 2 \implies C_n(P) \leq a(1 + b/2) < K. \quad (24.2)$$

The last comparison is rational after squaring, since $K^2 = 8/49$.

24.2 Reduction of a putative maximizer to $[-6, 6]$

Lemma 24.2 (Attainment under the contrary hypothesis). *If $C_2 > K$, the supremum is attained by a standardized law P with $\rho(P) < 2$.*

Proof. By (24.2), the supremum may be restricted to standardized laws with $\rho \leq 2$. This class is weakly compact: the second moment gives tightness, and the third-moment bound gives uniform integrability of the first two moments, so their equalities survive weak limits. The third-moment bound is closed by lower semicontinuity. Convolution preserves weak convergence. The Kolmogorov distance to the continuous cdf Φ is continuous under weak convergence, as follows from the Lévy inequalities and uniform continuity of Φ . Since $\rho \geq 1$ is lower semicontinuous, the quotient is upper semicontinuous and attains its maximum. A maximizer with $\rho = 2$ would contradict (24.2). $\square$

Reflect a maximizing law if needed. Upper semicontinuity of the two signed cdf discrepancies and their vanishing at infinity give a finite positive maximizing point. Write

$$t = x\sqrt{2}, \quad F = P^{*2}((-\infty, t]), \quad D = F - \Phi(x) > 0,$$

$$C = \frac{\sqrt{2}D}{\rho} = C_2, \quad \lambda = \frac{D}{\rho}, \quad m = \mathbb{E}[X|X|].$$

Here $|m| \leq 1$. The negative branch uses the left limit of the cdf; reflection converts it to this positive right-continuous branch.

Lemma 24.3 (Explicit contact inequality). *Define*

$$Q(y) = \lambda|y|^3 - \frac{x\phi(x) + 3D}{2}y^2 - (\sqrt{2}\phi(x) + 3\lambda m)y + 2F + \frac{x\phi(x) + D}{2}. \quad (24.3)$$

Then

$$2P((-\infty, t - y]) \leq Q(y) \quad (y \in \mathbb{R}), \quad Q(y) = 2P((-\infty, t - y]) \quad (y \in \text{supp } P). \quad (24.4)$$

Proof. Contaminate the raw law, $R_\epsilon = (1 - \epsilon)P + \epsilon\delta_y$, and then standardize. Its mean and variance are

$$\mu_\epsilon = \epsilon y, \quad \sigma_\epsilon^2 = 1 + \epsilon(y^2 - 1) - \epsilon^2 y^2.$$

Evaluate the standardized sum at $x_\epsilon = (t - 2\mu_\epsilon)/(\sqrt{2}\sigma_\epsilon)$. The event in the raw variables is the fixed event $Y_1 + Y_2 \leq t$, so its probability is a polynomial in ϵ ; no differentiation across a moving cdf jump is used. At zero,

$$x'_0 = -\sqrt{2}y - \frac{x}{2}(y^2 - 1), \quad \rho'_0 = |y|^3 - 3my + \frac{\rho}{2} - \frac{3\rho}{2}y^2,$$

$$\left. \frac{d}{d\epsilon} R_\epsilon^{*2}((-\infty, t]) \right|_0 = 2\{P((-\infty, t - y]) - F\}.$$

The derivative of the selected normalized score is nonpositive by global maximality. Thus $2\{P((-\infty, t - y]) - F\} - \phi(x)x'_0 - \lambda\rho'_0 \leq 0$, which is the inequality in (24.4). Its two sides have the same P -integral $2F$. The nonnegative difference vanishes P -almost surely. It is lower semicontinuous because the cdf expression on the left is upper semicontinuous. Its zero set therefore contains $\text{supp } P$. $\square$

Lemma 24.4 (Six-unit support bound). *Under $C_2 > K$, every maximizing witness in Lemma 24.2 satisfies $\text{supp } P \subset (-6, 6)$.*

Proof. Equation (24.1) gives $C < 1/2$ and $C\rho < a(2 + b) < 1$. Hence $\lambda > 2/7$, $D < 1/\sqrt{2} < 5/7$, and $\lambda < 5/14$. Using $\phi(x) < 2/5$, $|x\phi(x)| < 1/4$, $\sqrt{2} < 10/7$, $|m| \leq 1$, and $F, D \geq 0$, we obtain

$$Q(y) \geq P_6(|y|), \quad P_6(r) = \frac{2}{7}r^3 - \frac{67}{56}r^2 - \frac{23}{14}r - \frac{1}{8}.$$

The exact identity

$$P_6(6 + u) - 2 = \frac{2}{7}u^3 + \frac{221}{56}u^2 + \frac{104}{7}u + \frac{373}{56} > 0 \quad (u \geq 0)$$

contradicts $Q \leq 2$ on the support whenever $|y| \geq 6$. $\square$

No finite-support assumption appears in this reduction.

24.3 The two moment parameters

Let $\gamma = \mathbb{E}X^3$. Every standardized law satisfies

$$1 \leq \rho, \quad |\gamma| \leq \rho, \quad \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0. \quad (24.5)$$

For completeness, the last inequality follows from the elementary defect factorization of Theorem 2.1. Put

$$s = \sqrt{\gamma^2 + 4}, \quad a_0 = (s - \gamma)/2, \quad b_0 = (s + \gamma)/2, \quad c_\gamma = \gamma(\gamma^2 + 6)/s^3.$$

The function

$$w_\gamma(y) = |y|^3 - c_\gamma y^3 - 12y^2/s^3 + 6\gamma y/s^3 + 4/s^3$$

factors on the two half-lines as

$$w_\gamma(y) = \frac{2a_0^2}{s^3}(y - b_0)^2\{2 + (a_0 + 3b_0)y\} \quad (y \geq 0),$$

$$w_\gamma(-z) = \frac{2b_0^2}{s^3}(z - a_0)^2\{2 + (3a_0 + b_0)z\} \quad (z \geq 0).$$

Thus $w_\gamma \geq 0$ and $\mathbb{E}w_\gamma(X) = \rho - (\gamma^2 + 2)/\sqrt{\gamma^2 + 4} \geq 0$. Squaring gives (24.5). In the remaining problem $\rho \leq 2$, so it suffices to consider the compact domain

$$\mathcal{D} = \{(\gamma, \rho) \in [-2, 2] \times [1, 2] : \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0\}. \quad (24.6)$$

Only necessity of these moment constraints is used; no assertion that they characterize all laws on $[-6, 6]$ is needed.

24.4 The tensor majorants

Define

$$v_0(y) = 1, \quad v_1(y) = y, \quad v_2(y) = y^2, \quad v_3(y) = y^3, \quad v_4(y) = |y|^3, \quad \mu = (1, 0, 1, \gamma, \rho).$$

For rational coefficients $c = (c_{ij})_{0 \leq i \leq j \leq 4}$ set

$$M_c(u, v) = \sum_i c_{ii} v_i(u) v_i(v) + \sum_{i < j} c_{ij} \{v_i(u) v_j(v) + v_j(u) v_i(v)\}, \quad (24.7)$$

$$q_c(\gamma, \rho) = \sum_i c_{ii} \mu_i^2 + 2 \sum_{i < j} c_{ij} \mu_i \mu_j. \quad (24.8)$$

Independence and the common distribution give the exact identity

$$\mathbb{E}M_c(X_1, X_2) = q_c(\gamma, \rho). \quad (24.9)$$

On each quadrant M_c is a polynomial of degree at most three in each coordinate. Its expectation is a polynomial of total degree at most two in (γ, ρ) .

For $j = 0, \dots, 69$, put

$$a_j = -\frac{5}{2} + \frac{j}{20}, \quad b_j = a_j + \frac{1}{20}.$$

The certificate supplies one vector $c^{(j)}$ for each interval. It proves

$$M_{c^{(j)}}(u, v) \geq \mathbf{1}_{\{u+v \leq b_j\}} \quad (u, v \in [-6, 6]), \quad (24.10)$$

$$q_{c^{(j)}}(\gamma, \rho) \leq \Phi(a_j/\sqrt{2}) + \frac{2}{7}\rho \quad ((\gamma, \rho) \in \mathcal{D}). \quad (24.11)$$

The rational certificate contains exactly 70 threshold rows and 1050 coefficients; coefficients are ordered lexicographically by (i, j) with $i \leq j$.

Lemma 24.5 (Certificate implication). *Equations (24.10)–(24.11) imply $\Delta_2(P) \leq 2\rho(P)/7$ for every standardized law on $[-6, 6]$ with $\rho \leq 2$.*

Proof. If $t \in [a_j, b_j]$, monotonicity, (24.9), and the certificate give

$$\begin{aligned} P(X_1 + X_2 \leq t) - \Phi(t/\sqrt{2}) &\leq \mathbb{E}M_{c^{(j)}}(X_1, X_2) - \Phi(a_j/\sqrt{2}) \\ &\leq \frac{2}{7}\rho. \end{aligned}$$

For $t \leq -5/2$, Cantelli's inequality gives $P(X_1 + X_2 \leq t) \leq 2/(2 + t^2) \leq 8/33 < 2/7 \leq 2\rho/7$. For $t \geq 1$, $1 - \Phi(t/\sqrt{2}) \leq 1 - \Phi(1/\sqrt{2}) < 2/7$. The last strict inequality is checked by the rational Gaussian enclosure below. Thus the positive branch is bounded at every real threshold. Applying the same result to $-X$ bounds $\Phi(t/\sqrt{2}) - P(X_1 + X_2 < t)$, and hence also the negative right-cdf error. Atoms and exact sum collisions are included, not removed by genericity. $\square$

24.5 Exact checking of the continuous domains

24.5.1 Piecewise-polynomial positivity

Each of the four quadrants of $[-6, 6]^2$ is intersected with the two closed half-planes $u + v \leq b_j$ and $u + v \geq b_j$. Each resulting rational polygon is triangulated. On a triangle, the checker proves nonnegativity of $M_{c(j)} - 1$ on the first side or $M_{c(j)}$ on the second. The shared boundary belongs to both closed regions.

Write a degree-six polynomial in barycentric Bernstein form on a triangle:

$$p = \sum_{\alpha_0 + \alpha_1 + \alpha_2 = 6} B_\alpha \frac{6!}{\alpha_0! \alpha_1! \alpha_2!} \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}.$$

If every $B_\alpha \geq 0$, then $p \geq 0$ on the whole triangle. Otherwise the longest edge is bisected and both children are checked. A resource limit or negative vertex value is a rejection, never an acceptance.

The first checker constructs the coefficients from a monomial expansion and subdivides them using rational de Casteljau identities. The second checker independently reconstructs the homogeneous polynomial on *every* child. If D_c clears polynomial denominators and D_v clears vertex denominators, write $U = D_v u$, $V = D_v v$, and $S = \lambda_0 + \lambda_1 + \lambda_2$. For $p(u, v) = \sum a_{rs} u^r v^s$ the second checker expands

$$\sum_{r,s} (D_c a_{rs}) D_v^{6-r-s} U^r V^s S^{6-r-s}.$$

All its coefficients are integers. They are nonnegative exactly when the corresponding Bernstein coefficients are nonnegative, because the omitted denominators and multinomial factors are positive. This computation does not use the first checker's subdivision formula.

24.5.2 Moment-parameter positivity

The scalar expression $\Phi(a_j/\sqrt{2}) + 2\rho/7 - q_{c(j)}(\gamma, \rho)$ is enclosed from below on rational rectangles using degree-two Bernstein coefficients in each variable. A rectangle outside $\mathcal{D}$ is discarded. This test is exact: if g_* is the minimum of γ^2 on its γ interval and r_+ is its upper ρ endpoint, the rectangle is outside whenever

$$r_+^2(g_* + 4) - (g_* + 2)^2 < 0.$$

Indeed, $r^2(w + 4) - (w + 2)^2$ increases in $r \geq 0$ and decreases in $w \geq 0$ for $r \leq 2$. All other rectangles are certified or subdivided.

24.5.3 The normal cdf

For $|t| \leq 5/2$,

$$\Phi(t/\sqrt{2}) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{4^k k! (2k+1)}. \quad (24.12)$$

For $t \geq 0$ the term magnitudes decrease: their successive ratio is at most $(25/16)(2k+1)/((k+1)(2k+3)) < 1$. Alternating partial sums therefore give rational enclosures after bounding

$1/(2\sqrt{\pi})$. Machin's arctangent identity and integer square roots supply those bounds. The second checker uses different rational prefactor bounds and different truncation orders. Negative arguments are handled by symmetry. No platform implementation of Φ , exponential, sine, or cosine is used for acceptance.

24.5.4 The complete finite certificate

The supplied certificate has exactly seventy consecutive threshold rows and 1050 rational coefficients. Complete exact verification establishes both the pointwise majorant and moment-parameter inequalities on their stated closed domains. The two independent positivity calculations described above verify the same finite obligations.

24.6 Completion of the two-summand proof

Proof of Theorem 24.1. Suppose $C_2 > K$. Lemma 24.2 supplies a global maximizing law with $\rho < 2$. Lemma 24.4 confines it to $(-6, 6)$. The complete rational verification establishes (24.10)–(24.11); Lemma 24.5 then gives $C_2(P) \leq K$, a contradiction. Thus $C_2 \leq K$. Finally, exact rational enclosures give $K < 0.405 < 0.409 < C_E$ using rational bounds and squaring. $\square$

The bounded-support theorem in Lemma 24.5 treats arbitrary probability measures, not merely finitely supported measures or an assumed number of atoms. The passage to unrestricted laws uses global extremality, not a truncation approximation.

25 Three summands: independence-null certificates

25.1 Result and external input

For a standardized law P , let

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = 1, \quad \rho(P) = \mathbb{E}|X|^3 < \infty,$$

$$\Delta_n(P) = \sup_x |P^{*n}((-\infty, x\sqrt{n}]) - \Phi(x)|, \quad C_n = \sup_P \frac{\sqrt{n} \Delta_n(P)}{\rho(P)}.$$

Write $\phi = \Phi'$ and set

$$\Lambda = \frac{59}{250}, \quad K = \sqrt{3}\Lambda = 0.408763990586255\dots, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

Theorem 25.1 (Three summands). *Every standardized law with finite third absolute moment satisfies*

$$\boxed{\Delta_3(P) \leq \Lambda\rho(P)}.$$

Consequently $C_3 \leq 59\sqrt{3}/250 < C_E$.

The analytic reduction and the finite-certificate implication are proved below. The polynomial inequalities are verified by the accompanying exact checker and rational data. This upper bound does not identify the exact value of C_3 .

The sole external numerical probability bound is Shevtsova's structural estimate [7]:

$$C_n(P) \leq A \left(1 + \frac{B}{\rho(P)} \right), \quad A = \frac{1661}{5000}, \quad B = \frac{429}{1000}. \quad (25.1)$$

It implies

$$\rho(P) \geq \frac{15}{8} \implies C_n(P) \leq \frac{159456}{390625} < K. \quad (25.2)$$

The last inequality is rational after squaring. No result about the optimal two-point constant, no near-lattice transfer, and no effective eventual threshold is used.

25.2 A bounded witness would attain any violation

Lemma 25.2 (Attainment under the contrary hypothesis). *If $C_3 > K$, a standardized law P with $\rho(P) < 15/8$ attains C_3 .*

Proof. The laws with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^3 \leq 15/8$ form a weakly compact set. Tightness follows from the moment bound. For $j = 1, 2$, the uniform tail estimate

$$\mathbb{E}[|X|^j \mathbf{1}_{\{|X| > R\}}] \leq \frac{15}{8R^{3-j}}$$

preserves the first two moment equalities under weak limits; lower semicontinuity preserves the third-moment bound. Convolution preserves weak convergence. The Kolmogorov distance to the fixed continuous cdf Φ is continuous under weak convergence: the Lévy inequalities change the cdf argument by at most an arbitrarily small amount, whose cost is bounded by the uniform modulus of continuity of Φ . Since $\rho \geq 1$ is lower semicontinuous, Δ_3/ρ is upper semicontinuous on this compact set. It attains its supremum there. Bound (25.2) excludes all remaining laws and excludes $\rho = 15/8$ at a violating maximizer. $\square$

After reflecting a maximizing law if necessary, its maximum is attained at a positive right-cdf discrepancy. Indeed, the functions $F(x) - \Phi(x)$ and $\Phi(x) - F(x-)$ are upper semicontinuous and vanish at infinity; reflection interchanges these two signed discrepancies. Put

$$\begin{aligned} t &= x\sqrt{3}, \quad F = P^{*3}((-\infty, t]), \quad D = F - \Phi(x) > 0, \\ C &= C_3 = \frac{\sqrt{3}D}{\rho}, \quad \lambda = \frac{D}{\rho}, \quad m = \mathbb{E}[X|X|], \quad |m| \leq 1. \end{aligned}$$

Lemma 25.3 (Explicit contamination contact inequality). *Define*

$$\begin{aligned} Q(y) &= \lambda|y|^3 - \frac{x\phi(x) + 3D}{2}y^2 - (\sqrt{3}\phi(x) + 3\lambda m)y \\ &\quad + 3F + \frac{x\phi(x) + D}{2}. \end{aligned} \tag{25.3}$$

Then

$$3P^{*2}((-\infty, t - y]) \leq Q(y) \quad (y \in \mathbb{R}), \tag{25.4}$$

with equality throughout $\text{supp } P$.

Proof. For fixed y , contaminate the raw law by $R_\epsilon = (1 - \epsilon)P + \epsilon\delta_y$. Its mean and variance are

$$\mu_\epsilon = \epsilon y, \quad \sigma_\epsilon^2 = 1 + \epsilon(y^2 - 1) - \epsilon^2 y^2.$$

Standardize R_ϵ and evaluate at

$$x_\epsilon = \frac{t - 3\mu_\epsilon}{\sqrt{3}\sigma_\epsilon}.$$

This choice keeps the event in the raw variables exactly $Y_1 + Y_2 + Y_3 \leq t$, even at cdf jumps. Its probability is a polynomial in ϵ . Direct differentiation gives

$$x'_0 = -\sqrt{3}y - \frac{x}{2}(y^2 - 1),$$

$$\rho'_0 = |y|^3 - 3my + \frac{\rho}{2} - \frac{3\rho}{2}y^2, \quad F'_0 = 3\{P^{*2}((-\infty, t - y]) - F\}.$$

Differentiation of the third absolute moment is justified by the integrable quadratic majorant for the derivative of $|z - \epsilon y|^3$. Global maximality makes the right derivative of the selected normalized score nonpositive. Substitution and rearrangement give (25.4). Both sides have integral $3F$ under P : for the polynomial this follows from $\lambda\rho = D$, $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. The nonnegative gap is therefore zero P -almost surely. The function $y \mapsto P^{*2}((-\infty, t - y])$ is upper semicontinuous, so the gap is lower semicontinuous. It must vanish everywhere on $\text{supp } P$. $\square$

Lemma 25.4 (Uniform support bound). *A violating maximizer in Lemma 25.2 satisfies $\text{supp } P \subset (-6, 6)$.*

Proof. The structural bound and $1 \leq \rho < 15/8$ give

$$\frac{2}{5} < C < \frac{19}{40}, \quad C\rho < \frac{77}{100}.$$

Using $12/7 < \sqrt{3} < 7/4$ and $\sqrt{3} > 17/10$, we obtain

$$D < \frac{9}{20}, \quad \frac{8}{35} < \lambda < \frac{2}{7}.$$

Also $\phi(x) < 2/5$ and $|x\phi(x)| < 1/4$. The first follows from $\pi > 25/8$, and the second from the maximum $(2\pi e)^{-1/2} < 1/4$. Thus (25.3) implies, with $r = |y|$,

$$Q(y) \geq \frac{8}{35}r^3 - \frac{4}{5}r^2 - \frac{8}{5}r - \frac{1}{8}.$$

For $r = 6 + u$, the difference of the right side and 3 equals

$$\frac{8}{35}u^3 + \frac{116}{35}u^2 + \frac{472}{35}u + \frac{2197}{280} > 0 \quad (u \geq 0).$$

But (25.4) is an equality on the support, and its left side is at most 3. This excludes every support point with $|y| \geq 6$. $\square$

Remark 25.5. This is an extremizer reduction, not a truncation of arbitrary laws. The original theorem assumes only a finite third absolute moment. Higher moments are used below only after the hypothetical violating maximizer has been proved bounded.

25.3 The admissible two-dimensional moment region

Write $\gamma = \mathbb{E}X^3$. The defect identity in Theorem 2.1 gives an exact necessary condition. For completeness, the elementary proof is included. Set

$$s = \sqrt{\gamma^2 + 4}, \quad a = \frac{s - \gamma}{2}, \quad b = \frac{s + \gamma}{2}, \quad c = \frac{\gamma(\gamma^2 + 6)}{s^3},$$

so $ab = 1$. Define

$$w_\gamma(y) = |y|^3 - cy^3 - \frac{12y^2}{s^3} + \frac{6\gamma y}{s^3} + \frac{4}{s^3}.$$

For $y, z \geq 0$ direct multiplication gives

$$w_\gamma(y) = \frac{2a^2}{s^3}(y - b)^2\{2 + (a + 3b)y\},$$

$$w_\gamma(-z) = \frac{2b^2}{s^3}(z - a)^2\{2 + (3a + b)z\}.$$

Therefore

$$0 \leq \mathbb{E}w_\gamma(X) = \rho - \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}}.$$

Together with $|\gamma| \leq \rho$ and $\rho \geq 1$, every relevant law belongs to

$$\mathcal{D} = \left\{ (\gamma, \rho) : |\gamma| \leq \frac{15}{8}, \quad 1 \leq \rho \leq \frac{15}{8}, \quad \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0 \right\}. \quad (25.5)$$

This is an outer domain: every point need not be realizable by a law. The certificate checks all of it, so non-realizable points cannot cause an invalid acceptance.

25.4 Independence-null controls

For a single coordinate define the features

$$(v_0, v_1, v_2, v_3, v_4, v_5, v_6)(y) = (1, y, y^2 - 1, y^3, |y|^3, y^4, y^5). \quad (25.6)$$

Let $\mathcal{I}$ consist of the nondecreasing triples $\alpha = (i, j, k)$ with $0 \leq i \leq j \leq k \leq 6$ such that either $k < 5$, or at least one entry is 1 or 2. There are 59 such triples, ordered lexicographically in the certificate files. For each α , let

$$V_\alpha(x_1, x_2, x_3) = \frac{1}{|\text{Perm}(\alpha)|} \sum_{(i,j,k) \in \text{Perm}(\alpha)} v_i(x_1) v_j(x_2) v_k(x_3),$$

where the sum is over the *distinct* permutations of the index triple. For $c \in \mathbb{Q}^{59}$ put

$$M_c = \sum_{\alpha \in \mathcal{I}} c_\alpha V_\alpha. \quad (25.7)$$

Lemma 25.6 (Exact expectation without higher-moment parameters). *For independent standardized identically distributed bounded variables,*

$$\mathbb{E} M_c(X_1, X_2, X_3) = q_c(\gamma, \rho) := \sum_{\substack{\alpha \in \mathcal{I} \\ \text{entries only } 0,3,4}} c_\alpha \gamma^{\#3(\alpha)} \rho^{\#4(\alpha)}. \quad (25.8)$$

In particular, q_c has total degree at most three and has no unknown fourth- or fifth-moment parameters.

Proof. The expectations of v_1 and v_2 are zero. Independence makes every product containing either factor have expectation zero. By the definition of $\mathcal{I}$, every product involving v_5 or v_6 contains such a zero-mean factor in another coordinate. The only remaining products use v_0, v_3, v_4 , whose means are $1, \gamma, \rho$. $\square$

This adds 24 coefficients to the 35 low-feature tensor coefficients. In the resulting family, 49 basis elements have zero expectation and only 10 contribute monomials in (γ, ρ) . For example, all permutations of

$$X_1^4 X_2, \quad X_1^5 (X_2^2 - 1), \quad X_1^5 X_2 |X_3|^3$$

have mean zero. These are exact independence identities, not assumptions that the fourth or fifth moment has a prescribed value.

The general principle is useful beyond three summands. For any integrable function of the other coordinates,

$$\begin{aligned} \mathbb{E}\{X_i A(X_1, \dots, \hat{X}_i, \dots, X_n)\} &= 0, \\ \mathbb{E}\{(X_i^2 - 1) B(X_1, \dots, \hat{X}_i, \dots, X_n)\} &= 0. \end{aligned}$$

Such terms may reshape a pointwise majorant without changing its i.i.d. expectation. For the present proof all functions are polynomials on a bounded cube, so integrability is automatic.

25.5 Finite-certificate implication

Proposition 25.7 (Continuous-domain certificate). *For each row j of the supplied rational certificate, let $[a_j, b_j]$ be its threshold interval and $c^{(j)} \in \mathbb{Q}^{59}$ its coefficients. The rows form an exact consecutive partition of $[-7/2, 3/2]$ and satisfy*

$$M_{c^{(j)}}(u, v, w) \geq \mathbf{1}_{\{u+v+w \leq b_j\}} \quad ((u, v, w) \in [-7, 7]^3), \quad (25.9)$$

$$q_{c^{(j)}}(\gamma, \rho) \leq \Phi(a_j/\sqrt{3}) + \Lambda\rho \quad ((\gamma, \rho) \in \mathcal{D}). \quad (25.10)$$

These assertions are verified by the exact algorithm in Section 25.6.

Proof of Theorem 25.1. Suppose $C_3 > K$. Lemmas 25.2–25.4 give a globally maximizing violating law on $(-6, 6)$, with $\rho < 15/8$, and a positive maximizing raw threshold t . If $t \in [a_j, b_j]$, then independence and Proposition 25.7 give

$$\begin{aligned} P(X_1 + X_2 + X_3 \leq t) - \Phi(t/\sqrt{3}) &\leq \mathbb{E}M_{c(j)}(X_1, X_2, X_3) - \Phi(a_j/\sqrt{3}) \\ &\leq \Lambda\rho. \end{aligned}$$

The larger certificate cube $[-7, 7]^3$ contains the entire witness support. For $t \leq -7/2$, Cantelli's inequality for the variance-three sum yields

$$P(X_1 + X_2 + X_3 \leq t) - \Phi(t/\sqrt{3}) \leq \frac{3}{3+t^2} \leq \frac{12}{61} < \Lambda \leq \Lambda\rho.$$

For $t \geq 3/2$, the discrepancy is at most

$$1 - \Phi(\sqrt{3}/2) < \Lambda \leq \Lambda\rho.$$

All these inequalities contradict the choice of the maximizing witness. The reduction by reflection already handles the other signed discrepancy; no atom or boundary event is discarded. $\square$

25.6 Exact verification algorithm

Each coefficient is rational. Verification reconstructs the polynomial from its defining features and requires complete threshold coverage. Every unresolved positivity test is rejected.

25.6.1 Piecewise polynomial representation and symmetry

On each sign orthant, $|y|^3$ is y^3 or $-y^3$. Thus M_c is a polynomial of degree at most five in each variable. The checker expands this polynomial from the feature definition, not from a supplied power expansion. Symmetry permits restriction to $u \leq v \leq w$. The four initial sign boxes are

$$[-7, 0]^3, \quad [-7, 0]^2 \times [0, 7], \quad [-7, 0] \times [0, 7]^2, \quad [0, 7]^3.$$

A dyadic box $[L_1, H_1] \times [L_2, H_2] \times [L_3, H_3]$ may be discarded only if $L_i > H_j$ for some $i < j$. Such a box has no ordered point. All other accepted inequalities are proved on the whole box, a safe enlargement.

25.6.2 An affine half-space Bernstein test

For a box, let B_α be the tensor Bernstein basis of degree five in three variables, so $B_\alpha \geq 0$ and $\sum B_\alpha = 1$. Write

$$M_c = \sum_{\alpha} b_{\alpha} B_{\alpha}, \quad u + v + w - b_j = \sum_{\alpha} s_{\alpha} B_{\alpha},$$

where

$$s_{\alpha} = \sum_{i=1}^3 \left(L_i + \frac{\alpha_i}{5} (H_i - L_i) \right) - b_j.$$

Lemma 25.8 (Exact affine test). *If a real $\ell \leq 0$ satisfies $b_{\alpha} \geq 1 + \ell s_{\alpha}$ for all α , then $M_c \geq 1$ on the portion $u + v + w \leq b_j$. If $\ell \geq 0$ and $b_{\alpha} \geq \ell s_{\alpha}$ for all α , then $M_c \geq 0$ on the portion $u + v + w \geq b_j$.*

Proof. Average the coefficient inequalities with the nonnegative Bernstein weights. In each case the sign restriction makes $\ell(u + v + w - b_j)$ nonnegative on the specified half-space. $\square$

Existence of ℓ is checked by intersecting rational half-lines. A coefficient with $s_\alpha = 0$ requires $b_\alpha \geq 1$ or $b_\alpha \geq 0$. The remaining coefficients give upper or lower rational bounds on ℓ . Only signed integer products are needed to compare these bounds.

A box wholly inside the event is checked by $\min b_\alpha \geq 1$; a box strictly outside by $\min b_\alpha \geq 0$. A crossing box uses the affine tests. An unresolved box is bisected along a widest coordinate. Dyadic subdivision uses the exact degree-five de Casteljau identities

$$b_i^{\text{left}} = 2^{-i} \sum_{j=0}^i \binom{i}{j} b_j, \quad b_i^{\text{right}} = 2^{-(5-i)} \sum_{j=i}^5 \binom{5-i}{j-i} b_j.$$

After multiplying the common denominator by 32, all new numerators are integers. Ties at the event plane remain on the included side, and that plane is required to have value at least one.

25.6.3 Moment region and Gaussian enclosure

The scalar polynomial

$$\underline{\Phi}(a_j/\sqrt{3}) + \Lambda\rho - q_{c(j)}(\gamma, \rho)$$

is checked by rational tensor Bernstein coefficients of degree three on moment rectangles. Here $\underline{\Phi}$ is a rigorous rational lower bound. For $s = \gamma^2$ the domain constraint is

$$h(s, \rho) = \rho^2(s + 4) - (s + 2)^2 \geq 0.$$

When $\rho \leq 15/8 < 2$, this expression decreases in $s \geq 0$ and increases in $\rho \geq 0$. If its maximum at the smallest possible γ^2 and largest possible ρ on a rectangle is negative, that rectangle is outside the domain and is discarded. Otherwise Bernstein positivity proves the scalar inequality or the rectangle is bisected.

For rational t in the required range,

$$\Phi(t/\sqrt{3}) = \frac{1}{2} + \frac{1}{\sqrt{6\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{6^k k! (2k+1)}.$$

The checker uses consecutive alternating partial sums through indices 34, 35. For $|t| \leq 4$ the magnitudes decrease, so these enclose the sum. Rational Machin bounds enclose π , and integer square roots enclose $\sqrt{6\pi}$ to rational precision. No floating-point normal cdf enters acceptance.

25.6.4 The complete finite certificate

The certificate contains 111 consecutive threshold intervals and 6,549 rational coefficients. Exact integer calculations reconstruct the polynomial expansions and verify the continuous cube and moment inequalities. Undecided cells are rejected.

The initial expansions are also checked on a complete 6^3 tensor interpolation grid in each sign box. Since the degree is at most five in each coordinate, agreement on this grid proves the polynomial identity. This is an independent identity check, not a second verification of all positivity inequalities.

26 Four summands: an unrestricted bound

Theorem 26.1. *For every standardized real law with finite third absolute moment,*

$$R_4(P) \leq \frac{4097}{10000} < C_E.$$

In particular, $C_4 \leq 4097/10000$. No restriction on the number of support points is imposed.

Put $T = 4097/10000$ and $c_0 = 256/625$. The proof has two steps. A hypothetical unrestricted maximizer above T lies in a compact moment class. A finite rational certificate bounds every law in that class by T . The three-point law introduced below is a reference law, not a reduction of the admissible laws to three points.

26.1 The canonical reference and its excess

For a standardized X , write

$$\vartheta = \mathbb{E}[X|X|], \quad u = \mathbb{E}X_+ = \mathbb{E}X_- > 0, \quad v_{\pm} = \frac{1 \pm \vartheta}{2},$$

$$a = \frac{v_-}{u}, \quad b = \frac{v_+}{u}, \quad w = \frac{4u^2}{1 - \vartheta^2}.$$

Both $v_{\pm}$ are positive. Cauchy's inequality on the two half-lines gives $0 < w \leq 1$. Define

$$Q = \frac{u}{a}\delta_{-a} + (1 - w)\delta_0 + \frac{u}{b}\delta_b, \quad r = \rho(Q) = \frac{1 + \vartheta^2}{\sqrt{w(1 - \vartheta^2)}}. \quad (26.1)$$

This is a probability law of mean zero and variance one. Its contacts are

$$a = \sqrt{\frac{1 - \vartheta}{w(1 + \vartheta)}}, \quad b = \sqrt{\frac{1 + \vartheta}{w(1 - \vartheta)}}.$$

Lemma 26.2. *Set*

$$W(X) = X_+(X_+ - b)^2 + X_-(X_- - a)^2, \quad \eta = \mathbb{E}W = \rho(P) - r.$$

Then $\eta \geq 0$, and $\eta = 0$ if and only if $P = Q$. Moreover,

$$\begin{aligned} \mathbb{E}X^3 &= b - a + \mathbb{E}[\text{sgn}(X)W(X)], \\ \mathbb{E}X^4 &= a^2 - ab + b^2 + \mathbb{E}[(|X| + 2c(X))W(X)], \end{aligned} \quad (26.2)$$

whenever the fourth moment is finite, where $c(X) = b$ on $X > 0$ and $c(X) = a$ on $X < 0$.

Proof. On a tagged half-line put $y = |X|$ and $c = a$ or b . The identities

$$y^3 = y(y - c)^2 + 2cy^2 - c^2y, \quad y^4 = (y + 2c)y(y - c)^2 + 3c^2y^2 - 2c^3y$$

integrate using $\mathbb{E}[y; \text{tag}] = u$ and $\mathbb{E}[y^2; \text{tag}] = uc$. They give all three formulas. If $\eta = 0$, the support is contained in $\{-a, 0, b\}$; the first two moments fix its masses as in (26.1). The converse is immediate. $\square$

The excess η is not the two-point defect d of Part I. Its zero set permits an atom at zero; the reference is two-point precisely when $w = 1$.

26.2 Localization of an unrestricted extremizer

Write $K_3 = 59\sqrt{3}/250$. Theorem 25.1 gives $C_3 \leq K_3$, and

$$c_0^2 - K_3^2 = \frac{1069}{1562500} > 0, \quad c_0 < T.$$

Lemma 26.3. *If $C_4 > c_0$, an unrestricted maximizing pair (P, H) can be chosen as a positive closed discrepancy. It satisfies*

$$\text{supp } P \subset (-101/25, 101/25), \quad \rho(P) < 46/25, \quad \frac{625}{2116} < w \leq 1, \quad |\vartheta| < \frac{321}{500}. \quad (26.3)$$

Proof. Attainment is Lemma 14.1. Shevtsova's estimate (13.1) gives, at this pair,

$$\rho < \frac{(3031/10000)(323/500)}{c_0 - 3031/10000} = \frac{979013}{532500} < \frac{46}{25}.$$

This bound is obtained before any support or fourth-moment assumption. Put $C = C_4$, $z = H/2$, $F = F_{4,P}(H)$ and $G = F_{3,P}$. Lemma 14.2 gives

$$\begin{aligned} \rho I(y) &= 8\{G(H-y) - F\} + 4y\phi(z) + z\phi(z)(y^2 - 1) \\ &\quad + \frac{3}{2}C\rho(y^2 - 1) - C\{|y|^3 - \rho - 3\vartheta y\}. \end{aligned} \quad (26.4)$$

Here $I \leq 0$ everywhere and $I = 0$ at every point of the closed support, including accumulation points. Using the all-law predecessor bound and $F = \Phi(z) + C\rho/2$, with $s = \sqrt{4/3}$, yields

$$\begin{aligned} \rho I(y) &\leq \mathcal{H}_4(z, y) + 4\rho(K_3 - C) + 4\rho(s - 1)K_3 \\ &\quad + C\{-|y|^3 + \frac{3}{2}\rho y^2 + 3\vartheta y - \rho/2\}, \end{aligned} \quad (26.5)$$

where

$$\mathcal{H}_4(z, y) = 8 \left[\Phi \left(\frac{z - y/2}{\sqrt{3/4}} \right) - \Phi(z) \right] + 4y\phi(z) + z\phi(z)(y^2 - 1).$$

We use the following elementary uniform estimate:

$$|\mathcal{H}_4(z, y)| < \frac{\phi(0)}{6}|y|^3 + \frac{|y|}{5} + \frac{1}{15}. \quad (26.6)$$

Indeed, $\phi(0) < 2/5$, $\sup |\phi''| = \phi(0)$ and $\sup |(x\phi(x))'| = \phi(0)$. Also

$$\sup_x |x(3 - x^2)\phi(x)| < 3/5.$$

For the last inequality, set $t = x^2$. On $0 \leq t \leq 3$,

$$\frac{9}{4}(1 + t + t^2/2 + t^3/6) - t(3 - t)^2 \geq \frac{3}{4} [7(t - 9/14)^2 + 3/28] > 0.$$

Compare with e^t and use $\phi(0)^2 < 4/25$. On $t \geq 3$, use $|x|^3\phi(x) < 1/2$, which follows from $e > 8/3$. For $f(\lambda) = \Phi((z - y/2)/\sqrt{1 - \lambda})$,

$$f'(0) = \frac{1}{2}(z - y/2)\phi(z - y/2), \quad |f''(\lambda)| < 4/15 \quad (0 \leq \lambda \leq 1/4).$$

Thus $8\{f(1/4) - f(0)\} = (z - y/2)\phi(z - y/2) + E$, with $|E| < 1/15$. A second-order translation of Φ gives the cubic term in (26.6); the remaining difference $(z - y/2)\phi(z - y/2) - z\phi(z)$ is bounded by $|y|/5$.

By (26.1), $r \leq \rho < 46/25$ and $w \leq 1$. At $|\vartheta| = 321/500$,

$$\frac{(1 + \vartheta^2)^2}{1 - \vartheta^2} = \frac{124637947681}{36739750000} > (46/25)^2.$$

Monotonicity in ϑ^2 proves $|\vartheta| < 321/500$. Drop the negative term $4\rho(K_3 - C)$ in (26.5). Since $4(s - 1) < 5/8$, its remaining C -bracket is bounded, for $r_y = |y| \geq 4$, by

$$B(r_y) = -r_y^3 + \frac{69}{25}r_y^2 + \frac{963}{500}r_y + \frac{23}{100}.$$

The values of B, B', B'', B''' at 4 are $-5953/500, -11997/500, -462/25, -6$; hence $B < 0$ on that ray. Only now substitute $C > c_0$ in the negative bracket. Equation (26.6) gives

$$\rho I(y) < P(r_y) := -\frac{643}{1875}r_y^3 + \frac{17664}{15625}r_y^2 + \frac{77257}{78125}r_y + \frac{7541}{46875}.$$

At $101/25$ the values of P, P', P'', P''' are respectively

$$-\frac{51557}{9765625}, \quad -\frac{520966}{78125}, \quad -\frac{94558}{15625}, \quad -\frac{1286}{625}.$$

They are negative, so $P < 0$ thereafter. Contact excludes every support point with $|y| \geq 101/25$. Finally $r^2 \geq 1/w$ gives $w > 625/2116$. This proves (26.3). $\square$

26.3 One defect measure for all frequencies

For $\eta > 0$, let $\nu = W dP/\eta$, viewed as a probability on tagged points (σ, y) , $\sigma \in \{-1, 1\}$, $y > 0$. Write $c_+ = b$, $c_- = a$, $p_0 = 1 - w$, and

$$k_0(\sigma, y) = \frac{1}{c_\sigma^2 y}, \quad k_2(\sigma, y) = \frac{1}{(y - c_\sigma)^2}, \quad k_1(\sigma, y) = \frac{1}{y - c_\sigma}.$$

The contacts have zero ν mass. Singular expressions are integrated only off the contacts; the following identities prove integrability.

Lemma 26.4. *With $g(q) = \varphi_Q(q)$ and $r_q(\sigma, y) = \exp(i\sigma q \cdot)[0, c_\sigma, c_\sigma, y]$,*

$$\begin{aligned} \varphi_P(q) - g(q) &= \eta \int r_q d\nu, \\ \eta \int k_0 d\nu &= p_0 - P\{0\} \leq p_0, \\ \eta \int_\sigma k_2 d\nu &= u - c_\sigma P\{\sigma c_\sigma\} \leq u, \quad \int_\sigma k_1 d\nu = 0. \end{aligned} \quad (26.7)$$

The same ν occurs at every frequency, and $|r_q| \leq |q|^3/6$. For $\eta = 0$, $P = Q$ and every characteristic-function difference is zero, without defining ν .

Proof. Hermite interpolation at $0, c, c$ has remainder $y(y - c)^2 \exp(i\sigma q \cdot)[0, c, c, y]$. On each tag, the linear and quadratic terms have the same integrals against P and Q , by their matching first two tagged moments. Both interpolating polynomials have constant term 1. Their constant contributions agree only after summing the tags and adding the atom at zero, since both total masses are 1. Hermite–Genocchi gives the response bound. Multiplication of the three costs by $W = y(y - c)^2$ gives $(y - c)^2/c^2$, y , and $y(y - c)$, respectively. Their integrals give (26.7), including the omitted contact masses. Absolute integrability of k_1 follows from Cauchy and the k_2 bound. Lemma 26.2 handles $\eta = 0$. $\square$

Fix a threshold H and finite coefficients $\kappa_j(H)$, defined below. For the bounded class $|X| \leq R_* = 17/4$, all suprema below are over $\sigma \in \{-1, 1\}$ and $0 < y \leq R_*$, off the contact. The linear response is

$$A_H(\sigma, y) = \Re \sum_{j=0}^{N-1} 4\kappa_j(H) g(q_j)^3 r_{q_j}(\sigma, y).$$

For any fixed $\lambda_0, \lambda_+, \lambda_- \geq 0$ and $\theta_+, \theta_- \in \mathbb{R}$, a bound

$$S_H \geq \sup_{\sigma, y} \{A_H - \lambda_0 k_0 - \lambda_\sigma k_2 - \theta_\sigma k_1\} \quad (26.8)$$

implies

$$\eta \int A_H d\nu \leq \Gamma_H(\eta) := p_0 \lambda_0 + u(\lambda_+ + \lambda_-) + S_H \eta. \quad (26.9)$$

Upper bounds for p_0, u may replace them. The two equality prices retain their signs. In particular, S_H and Γ_H need not be nonnegative. The prices are fixed before taking the supremum over the whole tagged domain. This preserves the cancellation between frequencies; separate absolute estimates for each r_q do not.

26.4 The fourth-power remainder

For complex $\psi = g + \delta$,

$$\psi^4 - g^4 = 4g^3\delta + \delta^2(\psi^2 + 2\psi g + 3g^2). \quad (26.10)$$

Suppose $|g| \leq A \leq 1$, $|\psi| \leq 1$ and $|\delta| \leq E$. Two complete bounds for the remainder are

$$E^2(p^2 + 2pA + 3A^2), \quad p = \min(1, A + E), \quad v(4A^2 + 2 - v), \quad v = \min(E^2, 2A^2 + 1). \quad (26.11)$$

The first follows directly from (26.10). For the second, use

$$\psi^4 - g^4 - 4g^3\delta = 12\delta^2 \int_0^1 (1-t)(g+t\delta)^2 dt$$

and $|g+t\delta|^2 = (1-t)|g|^2 + t|\psi|^2 - t(1-t)|\delta|^2$.

A sharper two-disk bound is also finite algebra. Rotate $g = a \geq 0$ and put $\delta = x + iy$, $s = |\delta|^2$. Then

$$|\psi^4 - g^4 - 4g^3\delta|^2 = s^2\{36a^4 + 4a^2s + s^2 + 8ax(6a^2 + s) + 24a^2x^2\}.$$

For fixed a, s , the expression in braces is convex in x . Its maximum on the two disks occurs at a collinear endpoint or on $|\psi| = 1$. Put $r_+ = \min(E, 1)$, $m = \min(A, 1 - r_+)$, $r_- = \min(E, 1 + A)$. The collinear bounds are

$$r_+^2(6m^2 + 4mr_+ + r_+^2), \quad r_-^2(6A^2 - 4Ar_- + r_-^2), \quad \begin{cases} r_-^4, & r_- \leq 1, \\ r_-^2(3r_-^2 - 8r_- + 6), & r_- \geq 1. \end{cases}$$

On the unit circle it remains to maximize

$$Q_A(s) = s^2\{18A^4 + 12A^2 + 6 - (12A^2 + 8)s + 3s^2\} \quad (26.12)$$

on $[(1-A)^2, \min(E^2, (1+A)^2)]$. Its endpoints and the sole interior local maximum

$$s_- = (3A^2 + 2 - A\sqrt{4 - 3A^2})/2$$

suffice. For $A = 0$ the interval is degenerate or empty. Taking the maximum of the collinear values and the square root of this last maximum gives the lens bound. To see that no $a < A$ is lost, note that the unit-circle expression is convex in a^2 ; its lower feasible endpoint is already collinear. The collinear expressions increase with their admissible radius after maximization over a . Thus these are whole-disk bounds, not sampled values. Their minimum with (26.11) is valid.

On a closed excess interval $[L, U]$, an affine radius $e_j(\eta) = A_j + B_j\eta$ is used only when it is nonnegative at both ends and bounds $|\delta_j|$ throughout. Freeze $p_j = \min(1, a_j + \max\{e_j(L), e_j(U)\})$ in the first bound of (26.11). This gives a quadratic in η with nonnegative leading coefficient. Alternatively, freezing E on the whole cell gives a constant disk or lens bound. A choice between complete bounds is allowed; a coefficientwise minimum is not.

26.5 A physical Prawitz comparison

We use the closed-CDF Prawitz majorant [5], in the normalization of Lemma 2.4. To distinguish the factor from the kernel K used there, put

$$\widehat{K}(u) = 2K(u) = (1-u) + iR(u), \quad R(u) = (1-u) \cot(\pi u) + 1/\pi.$$

For $Z = (X_1 + \cdots + X_4)/2$,

$$F_Z(x) \leq J_P(x) := \frac{1}{2} + \int_0^1 \Re\{\widehat{K}(u)e^{-6iux}\varphi_P(3u)^4\} du. \quad (26.13)$$

The real integrand is regular at zero; no convergence of the complex integral by itself is asserted. With $D = x - Z$ and $S(u) = uR(u)$ it is

$$F_x(u) = (1 - u)\mathbb{E} \cos(6Du) + S(u)\mathbb{E}[6D \operatorname{sinc}(6Du)].$$

The cotangent partial fractions, extended to the endpoints, give

$$0 \leq S \leq 1/3, \quad |S'| < 2, \quad |S''| < 5.$$

Here is a direct derivative check. Separate the first pole:

$$S = \frac{1}{\pi} - \frac{2}{\pi}(A + MG), \quad A = \frac{u^2}{1+u}, \quad M = u^2(1-u), \quad G = \sum_{j \geq 2} \frac{1}{j^2 - u^2}.$$

On $[0, 1]$, $|A'| \leq 3/4$, $|A''| \leq 2$, $|M| \leq 1/6$, $|M'| \leq 1$ and $|M''| \leq 4$. The derivative series converge uniformly, and

$$G \leq 3/4, \quad |G'| \leq 10/27 < 1/2, \quad |G''| \leq 106/135 < 7/6.$$

The first sum telescopes. For the other two use $j^2 - u^2 \geq 3j^2/4$ and $\sum_{j \geq 2} j^{-k} \leq 2^{-k} + \int_2^\infty t^{-k} dt$, with $k = 4, 6$. The product rule gives $|S'| < 19/18 < 2$ and $|S''| < 223/54 < 5$. For $u > 1/2$, put $v = 1 - u$ and use $v \cot(\pi v) \leq 1/\pi$; for $u \leq 1/2$ the cotangent is nonnegative. This proves $R \geq 0$. The positive partial-fraction terms then give $0 \leq S \leq 1/\pi < 1/3$. Since $\operatorname{sinc}(v) = \int_0^1 \cos(tv) dt$,

$$|F_x''(u)| \leq 42\mathbb{E}|D| + 108\mathbb{E}D^2 + 24\mathbb{E}|D|^3. \quad (26.14)$$

These differentiations are justified by the displayed moments.

Now assume $|X| \leq R_* = 17/4$ and $\rho < 46/25$. The fourth moment follows from these hypotheses; it is not an additional input. Write $\kappa = \mathbb{E}X^3$, $M_4 = \mathbb{E}X^4$. Lemma 26.2 gives the useful cell bounds

$$\begin{aligned} |\kappa| &\leq \min\{\rho, |b - a| + \eta\}, \\ M_4 &\leq \min\{R_*\rho, a^2 - ab + b^2 + (R_* + 2\max(a, b))\eta\}. \end{aligned} \quad (26.15)$$

For $x = H/2$, choose whole-cell bounds κ_* and $M_{4,*}$ from these formulas and put

$$D_2 = x^2 + 1, \quad D_4 = x^4 + 6x^2 + 2|x|\kappa_* + (M_{4,*} + 9)/4.$$

Then $\mathbb{E}|D| \leq \sqrt{D_2}$ and $\mathbb{E}|D|^3 \leq \sqrt{D_2 D_4}$. Uniformly on $|x| \leq 2$, the simpler bounds $\mathbb{E}D^2 \leq 5$, $\mathbb{E}D^4 < 54$, $\mathbb{E}|D| < 9/4$, $\mathbb{E}|D|^3 < 33/2$ give $|F_x''| < 2061/2$.

Take $N = 256$, $u_j = (j + 1/2)/N$, $q_j = 3u_j$ and

$$\kappa_j(H) = \widehat{K}(u_j)e^{-iq_j H}/N, \quad C_H(Q) = 1 + 2\Re \sum_j \kappa_j(H)g(q_j)^4 - 2\Phi(H/2).$$

Composite midpoint quadrature gives one raw numerator reserve

$$Q_H = \min \left\{ \frac{687}{8N^2}, \frac{42\sqrt{D_2} + 108D_2 + 24\sqrt{D_2 D_4}}{12N^2} \right\}. \quad (26.16)$$

It controls the integral of the original law P once. No separate reference, linear-transfer or remainder quadrature error is added.

The raw thresholds are $H_\ell = -4 + \ell/8$, $0 \leq \ell \leq 57$. To fill the intervals between them, bound the characteristic function on entire frequency cells, not just at the midpoints. If $a_j \geq |g(q_j)|$ and e_j bounds $|\varphi_P(q_j) - g(q_j)|$ on the whole excess cell, then

$$P_j = \min\{1, a_j + e_j + 3/(2N)\}$$

bounds $|\varphi_P(3u)|$ throughout $j/N \leq u \leq (j+1)/N$, since $\mathbb{E}|X| \leq 1$. Two complete kernel bounds are

$$u^2|\widehat{K}(u)| \leq u^2(1-u) + u/3, \quad u^2|\widehat{K}(u)| \leq u^2(1-u) + u(1-u)/2.$$

The second follows by retaining the first term of the cotangent partial fractions: $S(u) \leq (1-u)(1+2u)/(\pi(1+u)) < (1-u)/2$. Let d_j be the smaller of the two exact integrals over the j th frequency cell. Their separate totals are $1/4$ and $1/6$. Thus, for the same whole geometry and excess cell,

$$|J_P''(x)| \leq B = 36 \sum_j d_j P_j^4.$$

Since $\sup |\Phi''| < 1/4$, the smooth physical function $H \mapsto 2\{J_P(H/2) - \Phi(H/2)\}$ needs the single interpolation reserve

$$E_H = \frac{h^2}{16}(B + 1/4), \quad h = 1/8. \quad (26.17)$$

It is not an interpolation of the discontinuous CDF or of a minimum of computed bounds. The elementary cube, mass and unit radii

$$\eta q_j^3/6, \quad \max(a, b)^2 p_0 q_j^2, \quad 1 + a_j$$

suffice for this curvature calculation. The mass bound follows from $|r_q| \leq q^2/y$ and the first budget in (26.7).

For $H \leq -4$, Cantelli gives $F_Z(H/2) \leq 1/5$. For $H \geq 25/8$, $\Phi(H/2) > 4/5$. The latter inequality follows, for example, by integrating the alternating Gaussian Taylor polynomial through degree six on $[0, 25/16]$ and using $\phi(0) > 19947/50000$. Consequently both threshold tails have positive normalized score at most $2/5$.

26.6 The low-excess certificate

On a geometry rectangle and a closed excess cell $[L, U]$, let M_H bound $C_H(Q) - Tr$ for the entire rectangle. Let Γ_H be obtained from (26.9), and let

$$R_2(\eta) = r_0 + r_1\eta + r_2\eta^2, \quad r_2 \geq 0,$$

bound the sum of the fourth-power remainders weighted by $|\kappa_j(H)|$. Combining (26.10) at the same H gives

$$2\{F_{4,P}(H) - \Phi(H/2)\} - T\rho \leq P_H(\eta) := M_H + Q_H + E_H + 2\Gamma_H(\eta) + 2R_2(\eta) - T\eta. \quad (26.18)$$

When used at grid points, E_H reserves interpolation to adjacent thresholds. The leading coefficient is nonnegative. Endpoint nonpositivity therefore proves the bound on the entire closed excess cell. Equivalently its degree-two Bernstein coefficients are

$$P_H(L), \quad \frac{P_H(L) + P_H(U)}{2} - \frac{p_2(U-L)^2}{2}, \quad P_H(U),$$

where p_2 is the quadratic coefficient. For a quantitative signed score, with $r_L \leq r$ and $\rho < 46/25$, use

$$T + \max_{\eta=L,U} \left\{ \frac{P_H(\eta)}{46/25}, \frac{P_H(\eta)}{r_L + \eta} \right\}.$$

Both denominator faces are needed when the numerator is negative. The cap branch is convex; for the other branch the derivative numerator has derivative $P_H''(\eta)(r_L + \eta) \geq 0$, so an interior critical point can only be a minimum.

Here are the finite enclosures that produce these polynomials. The center is enclosed by signed Taylor expansion in (w, ϑ) or in the contacts (a, b) along probability-preserving paths. For either pair of coordinates i, j ,

$$\partial_i C_H = 8\Re \sum_j \kappa_j g^3 \partial_i g, \quad |\partial_{ik} C_H| \leq 8 \sum_j |\kappa_j| (3D_i D_k + D_{ik}),$$

where D_i, D_{ik} bound the derivatives of the explicit three-atom characteristic function. The first-order terms retain their signs; the second-order error bounds the whole rectangle. The function $r(w, \vartheta)$ is jointly convex. Indeed, writing $r = w^{-1/2} f(\vartheta)$, $f = (1 + \vartheta^2)/\sqrt{1 - \vartheta^2}$, the Hessian determinant has the positive factor

$$3ff'' - (f')^2 = \frac{9 + 9\vartheta^2 + 15\vartheta^4 - \vartheta^6}{(1 - \vartheta^2)^3}.$$

Subtracting its supporting plane therefore encloses $C_H - Tr$ without an absolute Hessian charge for r .

For the signed response, put $R_* = 17/4$, $y_k = k/64$, $0 \leq k \leq 272$, and use their nearest-node closed cells, clipped to $[0, R_*]$. Hermite–Genocchi gives

$$\begin{aligned} r_q(\sigma, y) &= \sum_{m=0}^{60} (i\sigma)^{m+3} b_m h_m + E_q, & |E_q| &\leq \frac{q^3 (qR_*)^{61}}{6 \cdot 61!}, \\ b_0 &= q^3/6, & b_m &= b_{m-1} q R_*/(m+3), \\ h_0 &= 1, & h_m &= (y/R_*) h_{m-1} + (m+1)(c/R_*)^m. \end{aligned} \quad (26.19)$$

The bound $|\partial_y r_q| \leq q^4/24$ adds the half-cell reserve $q^4/(48 \cdot 64)$. The signed frequency coefficients are summed before bounding their polynomial on a y cell. If g_0, c'_0 are nominal values, $e_q \geq |g - g_0|$ is the geometry error, $p_q \geq |g|$ and $a_{0,q} \geq |g_0|$, then

$$|g^3 - g_0^3| \leq e_q(p_q^2 + p_q a_{0,q} + a_{0,q}^2), \quad |\partial_c r_q| \leq q^4/12.$$

These give a whole-rectangle correction for $g^3 r_q - g_0^3 r_{q,0}$, with no second copy of the series or mesh reserve. Thus the priced supremum bounds actual references throughout the rectangle, not only its nominal center.

The prices used in the finite certificate are chosen from

$$\begin{aligned} \lambda_0 &\in \{0, 1/1000, 1/400, 1/200, 1/100, 1/50, 1/25, 2/25, 4/25, 8/25, 16/25\}, \\ \lambda_{\pm} &\in \{0, 1/2000, 1/1000, 1/400, 1/200, 1/100, 1/50\}, \\ \theta_{\pm} &\in \{0, \pm 1/200, \pm 1/100, \pm 1/50\}. \end{aligned}$$

For each fixed choice, all 273 cells on each of the two tags enter the supremum. When $\lambda_{\sigma} = 0$, only $\theta_{\sigma} = 0$ is used; otherwise the reciprocal contact rays would give an infinite supremum. On a cell $[y_L, y_U]$, with $c \in [c_L, c_U]$, $k_0 \geq 1/(c_U^2 y_U)$. The remaining cost is the concave quadratic $-\lambda_{\sigma} z^2 - \theta_{\sigma} z$ in $z = 1/(y - c)$. Its endpoints and admissible vertex suffice; when the cell crosses the contact both reciprocal rays are included. No division at the contact or at zero is used for the response series.

The same budgets yield whole-cell radius models by pricing $|r_q|$ or its signed coordinate projections. They are combined with the three elementary radii above and the complete two-disk remainder. An independent absolute branch uses

$$|\psi^4 - g^4| \leq E(p^3 + p^2 A + p A^2 + A^3), \quad p = \min(1, A + E),$$

or the complete linear-plus-remainder bound. It is evaluated as a whole alternative. In every branch, the actual reference modulus, the actual normalization $\rho = r + \eta$, one midpoint reserve and one physical threshold reserve are retained. The final maximum over H is taken only after assembling each complete row.

26.7 Closing the large excess

Let $p = P(X \geq 0)$ and $q = 1 - p$; zero belongs to the nonnegative class. Condition on exactly j nonnegative observations among four. For $Z = (X_1 + \dots + X_4)/2$, the conditional moments are

$$\mu_j = \frac{u(j - 4p)}{2pq}, \quad V_j = \frac{1}{4} \left[j \left(\frac{v_+}{p} - \frac{u^2}{p^2} \right) + (4 - j) \left(\frac{v_-}{q} - \frac{u^2}{q^2} \right) \right].$$

Closed-event Cantelli gives, for every x ,

$$F_Z(x) \leq \sum_{j=0}^4 \binom{4}{j} p^j q^{4-j} A_j(x), \quad A_j(x) = \begin{cases} 1, & x \geq \mu_j, \\ V_j / (V_j + (\mu_j - x)^2), & x < \mu_j. \end{cases} \quad (26.20)$$

This includes variance zero and atoms at the threshold. The necessary sign-mass interval is

$$w(1 - \vartheta)/2 \leq p \leq 1 - w(1 + \vartheta)/2.$$

It is covered by closed cells. On a threshold cell $[x_L, x_U]$, bound $A_j(x_U)$ and subtract a lower bound for $\Phi(x_L)$. The five degree-four Bernstein coefficients bound the binomial mixture jointly; its masses are not maximized independently. Each retained sign-mass cell supplies a nonnegative raw numerator bound d_i and a correlated lower bound $r_i \leq r$, so its score is at most $d_i/(r_i + \eta)$.

If r_B is the whole-box lower bound for r , put

$$c_C^0 = \max \left\{ 0, \max_i (d_i/T - r_i), \frac{2}{5T} - r_B \right\}, \quad c_C = 10^{-4} \lceil 10^4 c_C^0 \rceil + 10^{-4}.$$

This closes every $\eta \geq c_C$ by a strict bound below T , including the threshold tails. Independently, Shevtsova closes every $\rho \geq 979013/533000$:

$$\frac{3031}{10000} \left(1 + \frac{323}{500\rho} \right) \leq T.$$

Hence $c_S = \max\{0, 979013/533000 - r_B\}$ is another closed sufficient cutoff. Take $c_B = \min(c_C, c_S)$, selecting one complete route. The low-excess interval $[0, c_B]$ and the large-excess range $[c_B, \infty)$ include their common endpoint.

26.8 The finite cover and conclusion

Proposition 26.5 (Complete bounded-law certificate). *For every standardized law P with*

$$|X| \leq 17/4, \quad \rho(P) < 46/25, \quad 625/2116 \leq w \leq 1, \quad |\vartheta| \leq 13/20,$$

and every real H ,

$$\frac{2\{F_{4,P}(H) - \Phi(H/2)\}}{\rho(P)} \leq T.$$

The same assertion holds for the genuine reflected law $-P$.

Proof. Divide $[625/2116, 1]$ into 20 equal closed bands and $[0, 13/20]$ into 13 closed bands of width $1/20$. There are 260 rectangles. The minimum of $r^2 = (1 + \vartheta^2)^2/[w(1 - \vartheta^2)]$ is attained at the largest w and smallest $|\vartheta|$. A rectangle with $\min r^2 \geq (46/25)^2$ is empty for the strict physical cap. For every other rectangle construct its own large-excess cutoff. If $c_B = 0$, the large-excess bound covers all $\eta \geq 0$ directly. If $c_B > 0$, divide $[0, c_B]$ into 16 equal closed intervals. At each interval check all 58 thresholds, their 57 joins, the two tails, and both actual reflected orientations, using the preceding inequalities. A parent is accepted by its complete bound or by complete bounds for all four children obtained by bisecting both coordinates. Each child has its own cutoff; no part of another child's domain or numerical conclusion is used in its place.

The exact certificate gives the following complete cover:

Closed parent rectangles	260
Empty by the strict moment cap	54
Nonempty, bounded directly	129
Bounded by all four children, absolute route	43
Bounded by all four children, full comparison portfolios	34
Uncovered rectangles	0
Uniform upper bound	4097/10000

The electronic supplement accompanying this revision supplies the four-summand certificate: the finite rational inequalities for all 260 rectangles, the complete child portfolios and their reconstruction. All elementary operations are outward rational enclosures; square roots are checked by integer squaring. Trigonometric, Gaussian and cotangent values are enclosed by finite series with analytic remainders, using the arithmetic language of Section 23. The response uses (26.19), not divided quotients at contacts. The complete arithmetic reconstruction reproduces the whole cover and its upper bounds; no sampled value or optimizer's status supplies an inequality.

For completeness, acceptance has a direct mathematical interpretation. Every actual canonical point belongs to a closed parent. In a nonempty parent or child, the large-excess inequalities cover $\eta \geq c_B$, while the low-excess inequalities cover all of $[0, c_B]$, including $\eta = 0$. The same unknown law supplies the single ν in every priced frequency response. Quadrature and physical interpolation extend the finite rows to every central threshold; the tails cover the rest. Thus the accepted rational inequalities prove the proposition for all actual laws, including bounded infinite supports and zero atoms. $\square$

Proof of Theorem 26.1. Suppose $C_4 > T$. Since $T > c_0$, Lemma 26.3 gives an unrestricted positive closed maximizing pair in the class of Proposition 26.5. The proposition bounds its score by T , a contradiction. Both discrepancy signs are covered:

$$\Phi(x) - F_Z(x) \leq \Phi(x) - F_Z(x-) = F_{-Z}(-x) - \Phi(-x).$$

This is the closed-CDF reflection identity, with the atom retained. Finally, elementary Machin bounds give $\pi < 3927/1250$ and hence $\phi(0) > 19947/50000$; also $\sqrt{10} > 15811/5000$. Therefore

$$C_E > \frac{204862339}{500000000} = \frac{4097}{10000} + \frac{12339}{500000000} > T.$$

No equality case $C_4 = T$ is excluded. $\square$

27 A sharper fourth-index bound for three-point laws

We prove the fourth-summand bound for every nondegenerate law supported on at most three points, with arbitrary masses and spacing. The support bound used by the finite verification is derived for a hypothetical restricted maximizer. No reduction of unrestricted maximizers to three points is assumed.

Theorem 27.1. *For centered variance-one probability laws supported on at most three points, put $\rho(P) = \mathbb{E}_P|X|^3$ and*

$$C_{4,\leq 3} = \sup_P \frac{2}{\rho(P)} \sup_x |\mathbb{P}_P\{(X_1 + \cdots + X_4)/2 \leq x\} - \Phi(x)|.$$

Then

$$C_{4,\leq 3} < \frac{409}{1000} < C_E.$$

Both inequalities are strict, including the first supremum over laws. Nonzero affine transformations give the same conclusion for every nondegenerate law on at most three points.

27.1 The external input and critical-level attainment

For all centered variance-one iid laws with finite ρ , use

$$R_n(P) := \frac{\sqrt{n} \Delta_n(P)}{\rho} \leq \frac{3031}{10000} \left(1 + \frac{323}{500\rho} \right). \quad (27.1)$$

This is Shevtsova's structural estimate in *On the absolute constants in the Berry–Esseen inequality and its structural and nonuniform improvements*, Informatics and its Applications 7(1) (2013), 124–125. In the notation of that structural estimate, substituting $X_i/\sqrt{n}$ gives $\ell_n = \rho/\sqrt{n}$ and $\tau_n = 1/\sqrt{n}$. Only this finite estimate is used. In particular, neither $C_3 \leq C_E$, a sharp Bernoulli theorem, nor the project's large-index bound is needed for this proof.

Jensen gives $\rho \geq 1$, and hence

$$R_n(P) \leq \frac{2494513}{5000000} < \frac{1}{2}.$$

At $\rho = 37/20$, the right side of (27.1) is strictly below 409/1000. More specifically, if $R_n(P) > 40899/100000$, then

$$\rho < \frac{979013}{529450} < \frac{37}{20}. \quad (2)$$

Suppose $C_{4,\leq 3} \geq 409/1000$. Take an optimizing sequence with ratios above 40899/100000, and reflect each law if necessary so that the chosen discrepancy is positive and upper closed. Reflection is legitimate because a negative open-CDF edge at x becomes a positive closed edge at $-x$; continuity of Φ equates the relevant suprema.

The laws are tight, and (2) gives uniform integrability of their first two moments: $\mathbb{E}[X^2; |X| > M] \leq (37/20)/M$. A weakly converging subsequence therefore has mean zero and variance one. The class of laws supported on at most three points is weakly closed: four disjoint open neighborhoods of positive limiting mass would, by Portmanteau, force four distinct support points in all sufficiently late approximants. Lower semicontinuity gives $\rho(P) \leq \liminf \rho(P_j)$; uniform integrability of third moments is not assumed.

For the raw sum threshold H , Cantelli gives $\mathbb{P}(S_4 \leq H) \leq 4/(4 + H^2) \leq 1/5$ when $H \leq -4$. When $H \geq 2$, $1 - \Phi(H/2) \leq 1 - \Phi(1) < 1/5$. Thus positive scores greater than 2/5 have owners in $[-4, 2]$. The elementary last normal bound follows, for instance, from $\phi(0) > 39/100$ and $\int_0^1 e^{-t^2/2} dt \geq 5/6$. Each fixed-law positive discrepancy attains its positive supremum: a CDF minus a continuous function is upper semicontinuous and tends to zero in both tails.

Pass to a subsequence with $H_j \rightarrow H$ and $\rho(P_j) \rightarrow r$. Weak convergence of the actual fourfold products and moving-threshold Portmanteau give

$$\limsup_j \mathbb{P}_{P_j}(S_4 \leq H_j) \leq \mathbb{P}_P(S_4 \leq H).$$

The limiting positive discrepancy is therefore at least $C_{4,\leq 3}r/2$. Since $1 \leq \rho(P) \leq r$, its normalized score is at least $C_{4,\leq 3}$. The definition of the supremum forces equality, and also forces $r = \rho(P)$. We have obtained an actual restricted maximizing pair (P, H) with

$$c = C_{4,\leq 3} \geq 409/1000, \quad \rho < 37/20, \quad c < 1/2. \quad (3)$$

This is attainment only in the stated closed support-cardinality class.

27.2 Existing-atom variations give the needed compact support

Write $z = H/2$, $F = \mathbb{P}_P(S_4 \leq H)$, $\beta = \mathbb{E}_P[X|X|]$, and $G(y) = \mathbb{P}_P(S_3 \leq H - y)$, with the same iid marginal throughout. For each existing atom y , contaminate by $P_\epsilon = (1 - \epsilon)P + \epsilon\delta_y$, then center and scale. This variation remains supported on at most three points. Hold the *raw* event

$S_4 \leq H$ fixed. Its probability is a polynomial in ϵ ; no derivative of an atomic CDF in its threshold is taken. The required derivatives at zero are

$$\begin{aligned} F' &= 4(G(y) - F), & z' &= -2y - z(y^2 - 1)/2, \\ \rho' &= |y|^3 - 3\beta y + \rho/2 - 3\rho y^2/2. \end{aligned} \quad (4)$$

Nonpositive score derivative implies $Q(y) \geq 4G(y)$ at every existing atom, where

$$\begin{aligned} Q(y) &= \frac{c}{2}|y|^3 - \left(\frac{z\phi(z)}{2} + \frac{3c\rho}{4}\right)y^2 - \left(2\phi(z) + \frac{3c\beta}{2}\right)y \\ &\quad + 4F + \frac{z\phi(z) + c\rho/2}{2}. \end{aligned} \quad (5)$$

Direct integration gives $\mathbb{E}Q(X) = 4F = 4\mathbb{E}G(X)$, so equality holds at every atom. No contact inequality at a newly introduced fourth atom is asserted or used. In particular, $Q(y) \leq 4$ on the actual support.

We have $|\beta| \leq 1$, $\phi(z) < 2/5$ and $|z\phi(z)| < 1/4$ on the whole real line. The latter follows by maximizing $x\phi(x)$ at $x = 1$, using $e^{1/2} > 13/8$ and $\phi(0) < 2/5$. By (3), for $r = |y|$,

$$Q(y) - 4 \geq L(r) := \frac{409}{2000}r^3 - \frac{131}{160}r^2 - \frac{31}{20}r - \frac{33}{8}. \quad (6)$$

The complete exterior is covered by the exact identity

$$L(6+s) = \frac{159}{125} + \frac{10711}{1000}s + \frac{11449}{4000}s^2 + \frac{409}{2000}s^3 > 0 \quad (s \geq 0). \quad (7)$$

Consequently the restricted maximizing law has support in $(-6, 6)$. This reduction is independent of any predecessor-row bound.

27.3 The complete bounded three-point parameter space

It remains to prove the following finite-dimensional statement.

Lemma 27.2. *If P has at most three support points, mean zero, variance one, support in $[-6, 6]$ and $\rho \leq 37/20$, then $R_4(P) < 409/1000$.*

Affine invariance permits raw support $\{0, v, 1\}$, with masses

$$p_0 = a, \quad p_1 = (1-a)b, \quad p_2 = (1-a)(1-b), \quad 0 \leq a, b \leq 1. \quad (8)$$

The endpoints 0 and 1 are actual support points. Three-point laws have $0 < a, b < 1$ and $0 < v < 1$; two-point laws are represented by $b = 0$ and $v = 1/2$. A degenerate standardized law is impossible. Reflection and checking both discrepancy signs reduce v to $[0, 1/2]$.

Put $\mu = p_1v + p_2$, $V = \text{Var } Y$, and $M = \mathbb{E}|Y - \mu|^3$. The exact positive formulas are

$$V = p_0p_1v^2 + p_0p_2 + p_1p_2(1-v)^2, \quad (9)$$

$$M = p_0\mu^3 + p_1|v - \mu|^3 + p_2(1 - \mu)^3, \quad \rho = M/V^{3/2}. \quad (10)$$

Since both endpoints are present, the support assumption implies

$$V \geq \max\{\mu, 1 - \mu\}^2/36. \quad (11)$$

We also retain $0 < V \leq 1/4$ and $\rho \geq 1$.

The fifteen fourfold count vectors are $\mathcal{A} = \{(i, j, k) \in \mathbb{N}_0^3 : i + j + k = 4\}$. Their sum locations and actual product probabilities are

$$h_{ijk} = jv + k, \quad w_{ijk} = \frac{24}{i!j!k!} p_0^i p_1^j p_2^k. \quad (12)$$

All order changes in $0 < v < 1/2$ occur at $v = 1/4, 1/3$; $v = 1/2$ is another possible collision boundary. This follows by solving $(j - j')v = k' - k$ for all pairs in $\mathcal{A}$, retaining precisely the ratios in the stated range. The acceptance code independently orders counts at the rational midpoints $1/8, 7/24, 5/12$.

For each of the three closed chambers, let $A_1, \dots, A_{15}$ be that order. Every upper-closed positive discrepancy and lower-open negative discrepancy is covered by the thirty expressions

$$\frac{2}{\rho} \left[\sum_{r \leq t} w_{A_r} - \Phi \left(\frac{h_{A_t} - 4\mu}{2\sqrt{V}} \right) \right], \quad \frac{2}{\rho} \left[\Phi \left(\frac{h_{A_t} - 4\mu}{2\sqrt{V}} \right) - \sum_{r < t} w_{A_r} \right]. \quad (13)$$

At a collision boundary, use the last member of a tied block for the positive closed value and the first for the negative left value. Both are in (13). Thus no mass is lost at a coincident sum or zero-probability occupancy. Between successive sum atoms, the relevant extrema lie at these closed or left edges, and the two infinite tails tend to zero.

27.4 Leaf inequalities and exact domain coverage

A certificate cell is a closed box in $(a, b, t) \in [0, 1]^3$. The chamber coordinate is respectively

$$v = t/4, \quad v = 1/4 + t/12, \quad v = 1/3 + t/6. \quad (14)$$

Its dyadic endpoints in a, b, t are exact. They are mapped to outward enclosures in v . Subscripts l, h below denote valid lower and upper enclosures, not independently adjustable law moments.

The mean decreases with a, b and increases with v :

$$\mu = (1 - a)(1 - b + bv), \quad \mu_l = (1 - a_h)(1 - b_h + b_h v_l), \quad \mu_h = (1 - a_l)(1 - b_l + b_l v_h). \quad (15)$$

Nonnegative products in (9) give V_l, V_h . With $d = \max(0, \mu_l - v_h, v_l - \mu_h)$, (10) gives

$$M_l = (p_0)_l \mu_l^3 + (p_1)_l d^3 + (p_2)_l (1 - \mu_h)^3, \quad \rho_l = \max\{1, M_l/V_h^{3/2}\}. \quad (16)$$

A leaf may be rejected as infeasible if $\rho_l > 37/20$ or if $V_h < \max\{\mu_l, 1 - \mu_h\}^2/36$.

For each fixed prefix in (13), its probability increases with a and with b . Increasing a replaces a draw in $\{v, 1\}$ by 0; increasing b replaces a draw at 1 by v . An independent coordinatewise coupling therefore decreases every sum. Thus the upper prefix value is bounded by its value at (a_h, b_h) , and the lower prefix by its value at (a_l, b_l) . The same argument applies to strict prefixes, since the ordering is fixed within the chamber. At chamber boundaries the fixed prefix polynomial is the one-sided limit; monotonicity persists by continuity in the masses. This is a common-marginal multinomial calculation, not a coordinatewise optimization with unrelated laws.

The numerator $h_{A_t} - 4\mu$ is enclosed directly using (15). Its sign determines whether the lower or upper endpoint of $2\sqrt{V}$ minimizes or maximizes its ratio. If $V_l = 0$, the corresponding negative lower phase or positive upper phase is conservatively infinite. A finite Gaussian table bounds Φ at both enclosed phases. After retaining the positive part of either discrepancy, division by $\rho_l \geq 1$ gives a valid upper bound in (13). A score leaf is accepted only if every such bound is strictly below 409/1000.

The certificate consists of three finite binary subdivision trees, one for each chamber. An internal node splits one dyadic coordinate at its exact midpoint, and both closed children must be verified. Each leaf satisfies either a strict constraint exclusion or the score inequality above. The three roots cover the entire three chambers. Induction on these trees therefore proves complete closed-domain coverage, including all collisions and shared boundaries.

27.5 Gaussian enclosures and outward interval arithmetic

For the Gaussian table, Machin's identity $\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$ and alternating rational sums certify

$$\frac{3989422804014326}{10^{16}} < \phi(0) < \frac{3989422804014328}{10^{16}}.$$

At $x = i/1024$, $0 \leq i \leq 4096$, integrate the degree-80 Taylor polynomial of $e^{-s^2/2}$. Its remainder lies between zero and the first omitted term, with the known negative sign. Multiplication by the displayed positive bounds encloses $\Phi(x)$; the table stores outward multiples of 2^{-50} . Its 4097 entries are rechecked from their defining rational expressions. Monotonicity brackets arbitrary phases by adjacent entries, reflection handles negative phases, and outside $[-4, 4]$ the valid outer bounds 0 and 1 are retained.

The interval calculation assumes IEEE-754 binary64 basic arithmetic with rounding to nearest. Each operation is enlarged outward to the adjacent representable endpoints, without contraction or reassociation. A square-root approximation is only an initial guess: its endpoints must also satisfy outward squared-endpoint tests. No Gaussian special function or library exponential is used to decide acceptance. These enclosure conditions and the complete closed forest certify all leaf inequalities.

Lemma 27.2 follows. It contradicts the actual maximizing pair (3), whose support was proved bounded by (7), proving Theorem 27.1. Finally

$$C_E^2 > \frac{6643}{39600} > \left(\frac{409}{1000} \right)^2,$$

using $\sqrt{10} > 79/25$ and $\pi < 22/7$. The bound in this section is sharper than the unrestricted estimate of Theorem 26.1, but applies only on at most three points. Its restricted first variations do not reduce the unrestricted optimization to this class; that optimization was treated separately in Section 26.

A Auxiliary Bernstein coefficients and binomial localization

A.1 The exact Bernstein data

For a polynomial $f(x) = \sum_{j=0}^9 a_j x^j$, the Bernstein coefficients of degree nine are

$$b_k = \sum_{j=0}^k a_j \frac{\binom{k}{j}}{\binom{9}{j}}.$$

After affine substitution to the two half-intervals, the coefficients used in Lemma 4.1 are listed below:

j	(b_{0j}, b_{1j})
0	(243/1250, 2079582664341761/1197443957149440000)
1	(5623/45000, 1709192148941903/1347124451793120000)
2	(1334869/17010000, 450741032531969/206258289269760000)
3	(13004779/272160000, 10866565157710711/2958390952957440000)
4	(146380277/5239080000, 198519592136938909/37719484650207360000)
5	(28525339727/1838917080000, 125666379478652267/18859742325103680000)
6	(487000027957/60071291280000, 48022819751421589/6286580775034560000)
7	(37879182239513/9336794987520000, 10619877339829403/1347124451793120000)
8	(11879475383308237/5388497807172480000, 1147322830325941/168390556474140000)
9	(2079582664341761/1197443957149440000, 29384282680087/9355030915230000)

Every entry is positive. The Bernstein convex-hull property therefore proves positivity of the polynomial on both halves without numerical evaluation.

A.2 Binomial localization

Only the following previously established result is used in passing from the local beta integral to the global supremum over lattice indices.

Proposition A.1 (Schulz's binomial localization, normalized). *Let $n \in \mathbb{N}$, $p \in [1/6, 5/6]$, $q = 1 - p$, $D = p^2 + q^2$, and $\rho(p) = D/\sqrt{pq}$. For an integer k put*

$$z_k = \frac{k - np}{\sqrt{npq}},$$

$$R_{n,p}(k) := \frac{\sqrt{n}}{\rho(p)} |B_{n,p}(k) - \Phi(z_k)|, \quad L_{n,p}(k) := \frac{\sqrt{n}}{\rho(p)} |B_{n,p}(k-1) - \Phi(z_k)|, \quad (\text{A.1})$$

where $B_{n,p}(-1) = 0$. Define

$$\widehat{G}(p) := \frac{\phi_0\{3 + |p - q|\}}{6D}, \quad 0 < c_0 := \frac{2}{\sqrt{2\pi} C_E} - 1 = 12\sqrt{10} - 37 < \frac{237}{250} = 0.948. \quad (\text{A.2})$$

Then, with $k_- := \lfloor np \rfloor$ and $k_+ := \lceil np \rceil$,

$$\frac{\sqrt{n} \Delta_n(Q_p)}{\rho(p)} \leq \max\{R_{n,p}(k_-), L_{n,p}(k_-), R_{n,p}(k_+), L_{n,p}(k_+), c_0 \widehat{G}(p)\}. \quad (\text{A.3})$$

Moreover, for every $n \in \mathbb{N}$ and $p \in [1/3, 2/3]$,

$$\frac{\sqrt{n} \Delta_n(Q_p)}{\rho(p)} < \widehat{G}(p). \quad (\text{A.4})$$

Source and normalization. The unnormalized statement (A.3) is Proposition 6.1 of [6], p. 31 (the proposition is restated at the beginning of Chapter 7); (A.4) is the sharper assertion in Theorem 1 there, p. 1. Schulz writes $\sigma = \sqrt{npq}$ and bounds the noncentral alternative in Proposition 6.1 by

$$c_0 \frac{3 + |p - q|}{6\sqrt{2\pi} \sigma}.$$

The rational bound in (A.2) is exact: after moving 37 and dividing by 12 it reduces to $\sqrt{10} < 9487/3000$, whose square is $10 < 90003169/9000000$. Multiplication by $\sqrt{n}/\rho(p)$ gives exactly $c_0\hat{G}(p)$ because

$$\rho(p) = \frac{D}{\sqrt{pq}}, \quad \frac{\sqrt{n}}{\rho(p)\sigma} = \frac{1}{D}.$$

The same calculation without c_0 converts the sharper part of Schulz's Theorem 1 into (A.4). For $p \leq 1/2$,

$$\hat{G}(p) = \frac{\phi_0(2-p)}{3D} = G(p), \quad (\text{A.5})$$

while reflection $p \leftrightarrow q$, $k \leftrightarrow n - k$ exchanges the two edge orientations. Thus (A.1)–(A.4) are exactly the cited claims, not an asymptotic reformulation of them. $\square$

This citation is used solely for lattice-index localization. The n^{-1} coefficient and its uniform $o(n^{-1})$ remainder are proved in Lemma 6.1; no unquantified Edgeworth expansion is used.

B Exact coefficient ledger

With the scaled notation in the proof of Lemma 9.2, exact Gaussian integration gives

$$\begin{aligned} \tilde{A}_1 &= \eta + \frac{2-p}{3}, \\ \tilde{\delta}_2 &= \frac{6\eta^2 + 6\eta + p^2 - p + 1}{12}, \\ \tilde{A}_3 &= \frac{1}{135}(45\eta^3 + 45p\eta^2 + 45\eta^2 + 45p\eta - 4p^3 + 6p^2 + 6p - 4), \\ \tilde{\delta}_4 &= \frac{1}{288}(36\eta^4 + 96p\eta^3 + 24\eta^3 + 12p^2\eta^2 + 132p\eta^2 - 24\eta^2 \\ &\quad + 12p^2\eta + 36p\eta - 12\eta + p^4 - 2p^3 + 3p^2 - 2p + 1), \\ \tilde{A}_5 &= \frac{1}{2835}(189\eta^5 + 945p\eta^4 + 630p^2\eta^3 + 1260p\eta^3 - 315\eta^3 \\ &\quad - 42p^3\eta^2 + 1008p^2\eta^2 + 63p\eta^2 - 42\eta^2 - 42p^3\eta + 378p^2\eta \\ &\quad - 252p\eta + 84\eta + 8p^5 - 20p^4 + 8p^3 + 8p^2 - 20p + 8). \end{aligned}$$

Substitution into (9.25) yields (9.16). This ledger is short enough to be checked independently without reproducing the full polynomials E_4 and E_5 .

C Arithmetic supplement to the elementary bound $C \leq 9/20$

This early bound is retained as a short historical supplement. Its mechanism is classical smoothing, comparison with a Rademacher law, positive elementary estimates, and a finite interval proof. It does not require the asymptotic or extremizer arguments of the main text.

Let $X, X_1, X_2, \dots$ be iid, with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\rho = \mathbb{E}|X|^3 < \infty$. Put

$$\Delta_n = \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{X_1 + \dots + X_n}{\sqrt{n}} \leq x\right) - \Phi(x) \right|, \quad C = \sup_{n, X} \frac{\sqrt{n} \Delta_n}{\rho}.$$

Theorem C.1. *The interval certificate described below gives $C \leq 9/20 = 0.45$.*

Proof. The finite reduction. The one-summand estimate of Bentkus–Kirša [1] and two estimates of Shevtsova [7, 4] give

$$\Delta_1 \leq 0.3704\rho, \quad \Delta_n \leq \frac{0.3322(\rho + 0.429)}{\sqrt{n}}, \quad \Delta_n \leq C_E \frac{\rho}{\sqrt{n}} + \frac{3\rho^2}{n}, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

All terminating decimals here are exact rationals. Since $\rho \geq 1$, set

$$\rho_0 = \frac{0.3322 \cdot 0.429}{0.45 - 0.3322} = \frac{712569}{589000}.$$

The second estimate settles $\rho \geq \rho_0$. The third settles $n \geq 8124$ when $1 \leq \rho \leq \rho_0$, because $C_E + 3\rho_0/\sqrt{8124} < 9/20$. For example, rational enclosures give

$$C_E < \frac{409732184}{10^9}, \quad \sqrt{8124} > \frac{90133234}{10^6},$$

and

$$\frac{9}{20} - \frac{409732184}{10^9} - \frac{3(712569/589000)}{90133234/10^6} = \frac{3121000875001}{3318029676625000000} > 0.$$

The square-root enclosure follows by squaring; the enclosure of π needed for the first comparison follows from Machin's formula and the alternating arctangent series. It remains to prove the bound on

$$2 \leq n \leq 8123, \quad 1 \leq \rho \leq \rho_0. \quad (\text{C.1})$$

A distribution-free smoothing bound. Write

$$\varphi(u) = \mathbb{E}e^{iuX}, \quad f_n(t) = \varphi(t/\sqrt{n})^n.$$

$$K(s) = \frac{1-s}{2} + \frac{i}{2} \left((1-s) \cot(\pi s) + \frac{1}{\pi} \right), \quad L(s) = K(s) - \frac{i}{2\pi s}.$$

For $T > 0$ and $0 < t_0 < 1$, Prawitz's inequality [5] bounds Δ_n by

$$\begin{aligned} U_n(\rho; T, t_0) &= \int_0^{t_0} 2|K(s)|R_{n,\rho}(Ts) \, ds + \int_{t_0}^1 2|K(s)|M_{n,\rho}(Ts) \, ds \\ &\quad + \int_0^{t_0} 2|L(s)|e^{-T^2s^2/2} \, ds + \frac{1}{2\pi} E_1(T^2t_0^2/2), \end{aligned} \quad (\text{C.2})$$

where $E_1(x) = \int_x^\infty e^{-y} dy/y$, $R_{n,\rho}(t) \geq |f_n(t) - e^{-t^2/2}|$, and $M_{n,\rho}(t) \geq |f_n(t)|$.

These majorants are explicit. The Mattner–Shevtsova comparison [2] is

$$|\varphi(u) - \cos u| \leq \frac{B(\rho)|u|^3}{6}, \quad B(\rho) = \sqrt{\frac{\rho^2 + \rho\sqrt{\rho^2 + 8}}{2}} - 2.$$

Put $u = t/\sqrt{n}$, $c = |\cos u|$, $\delta = B(\rho)|u|^3/6$, and $a = \min(1, c + \delta)$. The identity for a difference of powers gives

$$D(n, \rho, t) = \min \left(2, n\delta, \delta \sum_{j=0}^{n-1} a^j c^{n-1-j} \right), \quad R_{n,\rho}(t) = \min \left(2, |\cos^n u - e^{-t^2/2}| + D(n, \rho, t) \right). \quad (\text{C.3})$$

For the modulus we take the smaller of $\min(1, c^n + D(n, \rho, t))$ and Shevtsova's characteristic-function majorant. With $T = q\sqrt{n}/(\rho + 1)$, the latter is

$$M_P(n, \rho, Ts) = \left(1 - \frac{2\psi(qs)}{(\rho + 1)^2} \right)^{n/2}.$$

Here

$$\psi(z) = \begin{cases} z^2/2 - \kappa z^3, & 0 \leq z \leq \vartheta_0, \\ 1 - \cos z, & \vartheta_0 < z \leq 2\pi, \\ 0, & z > 2\pi, \end{cases} \quad \kappa = \frac{\cos \vartheta_0 - 1 + \vartheta_0^2/2}{\vartheta_0^3},$$

and ϑ_0 is the unique root in $(\pi, 2\pi)$ of $z^2 + 2z \sin z + 6(\cos z - 1) = 0$. Elementary differentiation gives $0 \leq \psi \leq 2$; the base of M_P therefore lies in $[0, 1]$. The resulting R and M bound every admissible law with moment ρ , not merely the Rademacher law.

Stable bounds at the endpoints. The following identities remove the apparent singularities:

$$2B(\rho)^2 = (\rho^2 - 1) \left(1 + \frac{\rho^2 + 9}{\rho\sqrt{\rho^2 + 8 + 3}} \right), \quad \ell(s) := \frac{1}{\pi s} - \cot(\pi s) = \frac{2s}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 - s^2}. \quad (\text{C.4})$$

The series is positive and increasing. Its omitted tail after m terms is at most

$$\frac{2s}{\pi m(1 - s^2/(m+1)^2)}.$$

Also $2|L(s)| = (1-s)\sqrt{1+\ell(s)^2}$, while the exact identity

$$2|K(1-h)| = h\sqrt{1+\ell(h)^2}$$

gives, for $0 < H \leq 1/2$,

$$\int_{1-H}^1 2|K(s)|M_{n,\rho}(Ts) \, ds \leq \frac{H^2}{2} \sqrt{1+\ell(H)^2}. \quad (\text{C.5})$$

At the other endpoint, if $0 < \eta, \lambda \leq \min(t_0, 1/2)$ and $T\eta/\sqrt{n} \leq 1$, elementary Taylor bounds give

$$\int_0^\eta 2|K|R_{n,\rho}(Ts) \, ds \leq \frac{B(\rho)T^3\eta^3}{9\sqrt{n}} + \frac{T^4\eta^4}{12n}, \quad \int_0^\lambda 2|L|e^{-T^2s^2/2} \, ds \leq \lambda\sqrt{1+4/\pi^2}. \quad (\text{C.6})$$

For $0 \leq u < \pi/2$, the Gaussian comparison in (C.3) can be evaluated as the nonnegative product

$$e^{-nu^2/2} - \cos^n u = (e^{-u^2/2} - \cos u) \sum_{j=0}^{n-1} e^{-ju^2/2} \cos^{n-1-j} u.$$

Outside this range the signed power $\cos^n u$ is enclosed directly. Thus odd sample sizes require no change of sign convention.

Why the finite arithmetic proves the theorem. For a closed rational cell $I = [\rho_-, \rho_+]$, choose rational q, t_0, η, λ and use $T(\rho) = q\sqrt{n}/(\rho + 1)$. Require

$$q > 0, \quad 0 < t_0 < 1, \quad 0 < \eta, \lambda \leq \min(t_0, 1/2), \quad \frac{q\eta}{\rho_- + 1} \leq 1.$$

The last inequality makes (C.6) valid uniformly on I . Each remaining proper integral in (C.2) is bounded by an upper sum

$$\sum_j (s_{j+1} - s_j) b_j, \quad b_j \geq \sup_{\rho \in I, s \in [s_j, s_{j+1}]} G(\rho, s)$$

for its nonnegative integrand G . The b_j are obtained with outward rounding: the interval assigned to every operation contains all its exact values on the whole rectangle. Equations (C.4)–(C.6) bound the omitted endpoint pieces. A cell crossing a branch point of ψ may safely use the lower bound $\psi \geq 0$.

For the first endpoint use $B(\rho_+)$ and $T_+ = q\sqrt{n}/(\rho_- + 1)$. For the exponential-integral tail use $T_- = q\sqrt{n}/(\rho_+ + 1)$, since E_1 is decreasing. Let $\widehat{U}_{n,I}$ be the sum of these upper bounds. Then, throughout the entire closed cell,

$$\frac{\sqrt{n} \Delta_n}{\rho} \leq \frac{\sqrt{n}}{\rho_-} \widehat{U}_{n,I}. \quad (\text{C.7})$$

The certificate supplies a finite rational cover of $[1, \rho_0]$ for every integer $n = 2, \dots, 8123$, including all shared endpoints. Its 886 998 cells satisfy

$$\max_{n,I} \frac{\sqrt{n}}{\rho_-} \widehat{U}_{n,I} \leq \frac{449999}{10^6} < \frac{9}{20}.$$

This is the finite computational step of the proof. It controls continuous intervals of ρ and of the integration variable, not sampled points. Equation (C.7) proves (C.1), and the initial reductions prove $C \leq 9/20$ for every sample size. $\square$

Historical note. The original certificate records completion on 6 August 2026; the public source archive begins on 25 September 2026. The full argument and interval certificate are available in Rodziewicz’s certificate repository. Stronger universal bounds, including 0.4395 by Xiao–Li and 0.423 by Lin, are now reported in their respective repositories. The present appendix is retained for its elementary mathematical mechanism, not as the current best bound.

AI assistance declaration

Artificial-intelligence systems were used in developing the mathematical arguments and proofs, constructing and checking computational certificates, and drafting and revising this manuscript. This included assistance from ChatGPT and Codex using the GPT-5.6 Sol, GPT-6.1 Sol and GPT-6 Astra models. Kacper Rodziewicz and Bartosz Kołodziejek remain responsible for the mathematical claims and the final text.

References

- [1] V. Bentkus and K. Kirša, Estimates of the closeness of a distribution to the normal law, *Lithuanian Mathematical Journal* **29** (1989), 321–332.
- [2] L. Mattner and I. Shevtsova, An optimal Berry–Esseen type theorem for integrals of smooth functions, *ALEA* **16** (2019), 487–530. <https://doi.org/10.30757/ALEA.v16-19>.
- [3] I. Shevtsova, On the accuracy of the approximation of the complex exponent by the first terms of its Taylor expansion with applications, *Journal of Mathematical Analysis and Applications* **418**(1) (2014), 185–210. <https://doi.org/10.1016/j.jmaa.2014.03.075>.
- [4] I. Shevtsova, Moment-type estimates with asymptotically optimal structure for the accuracy of the normal approximation, *Annales Mathematicae et Informaticae* **39** (2012), 241–307. https://ami.uni-eszterhazy.hu/uploads/papers/finalpdf/AMI_39_from241to307.pdf.
- [5] H. Prawitz, Limits for a distribution, if the characteristic function is given in a finite domain, *Skandinavisk Aktuarietidskrift* (1972), 138–154. <https://doi.org/10.1080/03461238.1972.10404645>.
- [6] J. Schulz, *The optimal Berry–Esseen constant in the binomial case*, doctoral dissertation, Universität Trier, 2016. <https://d-nb.info/1197702695/34>.
- [7] I. G. Shevtsova, On the absolute constants in the Berry–Esseen inequality and its structural and nonuniform improvements, *Informatika i Ee Primeneniya* **7**(1) (2013), 124–125. <https://www.mathnet.ru/eng/ia252>.
- [8] H. He and G. Cheng, *The Berry–Esseen Bound is Sharp for All Sufficiently Large Sample Sizes*, arXiv:2609.06358v1 (2026). <https://arxiv.org/abs/2609.06358v1>.
- [9] H. He and G. Cheng, *The Berry–Esseen Constant Conjecture is Eventually True*, arXiv:2609.06358v3 (2026), revised 30 September. <https://arxiv.org/abs/2609.06358v3>.
- [10] K. Rodziewicz and B. Kołodziejek, *Effective finite-index continuation and certificates for the iid Berry–Esseen problem*, supplementary proof material (2026). <https://github.com/rodziewicz/berry-esseen-045-certificate/tree/044e93ec97f377752924fe32bd08e0bc6f4334b1/moment-defects>.
- [11] K. Rodziewicz and B. Kołodziejek, *Finite continuation at indices 20–24 for the iid Berry–Esseen problem*, supplementary certificate material, 16 October 2026. Conditional on the retained analytic and historical premises specified in the accompanying report. <https://github.com/rodziewicz/berry-esseen-045-certificate/tree/514f89e53005a13680518b2127f0e09cd1bd52f0/moment-defects/continuations/cutoff20-2026-10-16>.
- [12] A. Zolotukhin, S. Nagaev and V. Chebotarev, On a bound of the absolute constant in the Berry–Esseen inequality for i.i.d. Bernoulli random variables, *Modern Stochastics: Theory and Applications* **5** (2018), 385–410.
- [13] K. Rodziewicz, *An elementary interval certificate for the Berry–Esseen bound 0.45*, certificate archive (2026). <https://github.com/rodziewicz/berry-esseen-045-certificate>.