

PHASE RETRIEVAL FOR STATIONARY SCHRÖDINGER EVOLUTIONS

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ABSTRACT. We prove three phase-retrieval results for stationary one-dimensional Schrödinger evolutions. Masuda's unique-continuation theorem yields phase retrieval for finite-energy solutions and a broad class of semibounded potentials which may grow at infinity. For real potentials in the Faddeev class $L_1^1(\mathbb{R})$, measurements on all of spacetime determine arbitrary L^2 initial data. If the potential is smooth and compactly supported to the left of a point, measurements on the exterior half-line to its right suffice.

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1. INTRODUCTION AND MAIN RESULTS

Let $V : \mathbb{R} \rightarrow \mathbb{R}$, let $H = -\partial_x^2 + V$ be a self-adjoint Schrödinger operator, and write

$$u(t) = e^{-itH}u_0, \quad v(t) = e^{-itH}v_0.$$

We ask whether equality of $|u|$ and $|v|$ determines the initial data up to one constant in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This note gives three answers for time-independent potentials. We state first the argument based on Masuda's unique-continuation theorem.

Theorem A (Masuda class). *Let $V \in L_{\text{loc}}^\infty(\mathbb{R}; \mathbb{R})$. Assume that*

$$H_0 = -\partial_x^2 + V, \quad D(H_0) = C_c^\infty(\mathbb{R}),$$

is essentially self-adjoint and its self-adjoint closure is bounded from below. Let u and v be weak solutions of

$$if_t + f_{xx} = Vf, \quad u, v \in C(\mathbb{R}; L^2(\mathbb{R})).$$

Suppose that, on a nonempty open interval $I \subset \mathbb{R}$,

$$u, v \in C(I; H^1(\mathbb{R})), \quad |u(t, x)| = |v(t, x)| \quad \text{for almost every } (t, x) \in I \times \mathbb{R}.$$

Then $v(t) = \zeta u(t)$ in $L^2(\mathbb{R})$ for every $t \in \mathbb{R}$ and some $\zeta \in \mathbb{T}$.

The second theorem permits arbitrary L^2 data. Set

$$L_1^1(\mathbb{R}) = \left\{ V : \mathbb{R} \rightarrow \mathbb{C} : \int_{\mathbb{R}} (1 + |x|) |V(x)| dx < \infty \right\}.$$

Theorem B (Stationary potentials in the Faddeev class). *Let $V \in L_1^1(\mathbb{R})$ be real, let $H = -\partial_x^2 + V$ be its self-adjoint quadratic-form realization, and let $u_0, v_0 \in L^2(\mathbb{R})$. If*

$$|e^{-itH}u_0(x)| = |e^{-itH}v_0(x)| \quad \text{for almost every } (t, x) \in \mathbb{R}^2,$$

then $v_0 = \zeta u_0$ for some $\zeta \in \mathbb{T}$.

For a smooth compactly supported potential, observation on the free side of the potential suffices.

Theorem C (Observation on one exterior half-line). *Let $V \in C_c^\infty(\mathbb{R}; \mathbb{R})$, let $H = -\partial_x^2 + V$, and choose $R \in \mathbb{R}$ so that $\text{supp } V \subset (-\infty, R]$. If $u_0, v_0 \in L^2(\mathbb{R})$ and*

$$|e^{-itH}u_0(x)| = |e^{-itH}v_0(x)| \quad \text{for almost every } (t, x) \in \mathbb{R} \times (R, \infty),$$

then $v_0 = \zeta u_0$ for some $\zeta \in \mathbb{T}$.

The first theorem allows locally bounded semibounded potentials which may grow at infinity, but it assumes finite energy on the observation interval. The second assumes spatial decay and observations on all of spacetime, but it allows arbitrary L^2 data, negative eigenvalues, and a zero-energy resonance. The third strengthens the observation geometry under the more restrictive hypothesis $V \in C_c^\infty$.

The proofs use three different facts. Equality of densities determines the one-dimensional probability current, after which Masuda's theorem propagates a local phase globally. In the Faddeev class, the temporal Fourier transform separates pairs of spectral energies by their difference; a four-dimensional first-order product system then separates the remaining energy sum. On an exterior half-line, the spectral expansion is free, and a Cauchy-transform argument recovers its rank-one spectral kernel.

All nonsmooth identities below are distributional unless another meaning is specified. We use

$$\widehat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

and take Hilbert-space inner products to be linear in the first variable. The value of a spectral representative on a null set is immaterial.

1.1. LLM Usage. The current file is to be treated as an auxiliary file to [arxiv.org/2609.30511](https://arxiv.org/abs/2609.30511). Its main ideas and proofs are human-devised and guided, but it is completely machine-written.

2. PHASE RETRIEVAL IN MASUDA'S CLASS

Throughout this section V satisfies the hypotheses of Theorem A. Masuda's theorem [4] is stated with a null exceptional set Q whose complement is connected and open and on which V is locally bounded. In one dimension this complement must equal $\mathbb{R}$: a connected open subset of $\mathbb{R}$ is an interval, and a proper interval cannot have full measure. Thus the exceptional-set hypothesis reduces to $V \in L^\infty_{\text{loc}}(\mathbb{R})$. Masuda also allows a real C^2 magnetic potential, but we only use the nonmagnetic case.

A Masuda weak solution of

$$(2.1) \quad if_t + f_{xx} = Vf$$

is a function $f \in C(\mathbb{R}; L^2(\mathbb{R}))$ satisfying (2.1) in distributions. Thus, for every $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} f(t, x) (-i\varphi_t(t, x) + \varphi_{xx}(t, x) - V(x)\varphi(t, x)) dx dt = 0,$$

with the usual complex-bilinear distributional pairing. A preliminary lemma in Masuda's proof identifies every such weak solution with the unitary orbit

$$f(t) = e^{-i(t-t_0)H} f(t_0).$$

The same time-mollification argument on a compact subinterval gives local uniqueness; hence a weak solution defined only on an open time interval has a unique global unitary extension from any one of its time slices. Under the assumptions above, Masuda proves the following statement.

Proposition 2.1 (Masuda). *If a Masuda weak solution of (2.1) vanishes on a nonempty open subset of $\mathbb{R}_t \times \mathbb{R}_x$, then it vanishes identically.*

We now prove Theorem A. Thus $I \subset \mathbb{R}$ is a nonempty open interval, u, v are Masuda weak solutions, and

$$(2.2) \quad u, v \in C(I; H^1(\mathbb{R})).$$

The density hypothesis is

$$(2.3) \quad |u(t, x)| = |v(t, x)| \quad \text{for almost every } (t, x) \in I \times \mathbb{R}.$$

Equality of the densities first determines the difference of the currents. The one-dimensional integrability of that difference is used below.

2.1. Density and current. For $f \in H^1(\mathbb{R})$ define

$$\rho_f = |f|^2, \quad j_f = \text{Im}(\bar{f}f_x).$$

Then $j_f \in L^1(\mathbb{R})$ and

$$\|j_f\|_1 \leq \|f\|_2 \|f_x\|_2.$$

Lemma 2.2 (Recovery of the current). *Under the hypotheses of Theorem A,*

$$(2.4) \quad j_u = j_v \quad \text{almost everywhere on } I \times \mathbb{R}.$$

Consequently,

$$(2.5) \quad \bar{u} u_x = \bar{v} v_x \quad \text{almost everywhere on } I \times \mathbb{R}.$$

Proof. Every solution satisfying (2.2) obeys the continuity equation

$$(2.6) \quad \partial_t |f|^2 + 2\partial_x \operatorname{Im}(\bar{f} f_x) = 0 \quad \text{in } \mathcal{D}'(I \times \mathbb{R}).$$

For a smooth solution, solve (2.1) for the time derivative: $f_t = i f_{xx} - i V f$. Since V is real,

$$\begin{aligned} \partial_t |f|^2 &= f_t \bar{f} + f \bar{f}_t \\ &= i(f_{xx} \bar{f} - f \overline{f_{xx}}) \\ &= -2\partial_x \operatorname{Im}(\bar{f} f_x). \end{aligned}$$

For a solution satisfying (2.2), regularize the equation in (t, x) and repeat this calculation. On compact spatial intervals, $V \in L^2$, while the one-dimensional embedding $H^1 \subset L^\infty$ gives $V f \in L^2_{\text{loc}}$. Thus the potential terms converge and cancel. The products containing f_x converge in L^1 by Cauchy–Schwarz and (2.2). Passing to the limit proves (2.6).

Subtract (2.6) for u and v . By (2.3),

$$(2.7) \quad \partial_x (j_u - j_v) = 0 \quad \text{in } \mathcal{D}'(I \times \mathbb{R}).$$

We show that the constant in x is zero. For $\eta \in C_c^\infty(I)$ put

$$J_\eta(x) = \int_I \eta(t) (j_u - j_v)(t, x) dt.$$

The estimate

$$\|j_f(t)\|_1 \leq \|f(t)\|_2 \|f_x(t)\|_2$$

and the $C(I; H^1)$ assumption give $J_\eta \in L^1(\mathbb{R})$. Testing (2.7) against $\eta(t)\phi(x)$ shows that $J'_\eta = 0$ in $\mathcal{D}'(\mathbb{R})$. A distribution on $\mathbb{R}$ with zero derivative is a constant distribution: for example, it annihilates every test function of integral zero, each of which is the derivative of a compactly supported test function. Since J_η is represented by an L^1 function, this constant must be zero. Thus

$$\int_I \eta(t) (j_u - j_v)(t, x) dt = 0$$

for almost every x and every temporal test function η . Hence (2.4) holds.

Finally,

$$\operatorname{Re}(\bar{f} f_x) = \frac{1}{2} \partial_x |f|^2$$

in distributions. The real parts of $\bar{u} u_x$ and $\bar{v} v_x$ agree by (2.3), and their imaginary parts agree by (2.4). Both products belong locally to L^1 , so the distributional identity is the almost-everywhere identity (2.5). $\square$

2.2. Local constancy of the relative phase. Every $H^1(\mathbb{R})$ function has a continuous representative, since

$$|f(x) - f(y)| \leq |x - y|^{1/2} \|f_x\|_2.$$

Together with the L^2 condition, this representative tends to zero at infinity. The embedding $H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$ is continuous. Thus (2.2) gives jointly continuous representatives of u and v :

continuity in t holds uniformly in x , and continuity in x holds on every time slice. Set

$$\Omega = \{(t, x) \in I \times \mathbb{R} : u(t, x) \neq 0\}.$$

By (2.3), v has the same nonzero set. The set Ω is open.

Lemma 2.3 (Relative phase). *On every connected component of Ω there is a constant $\zeta \in \mathbb{T}$ such that $v = \zeta u$.*

Proof. Fix an open rectangle $J \times O$ whose closure is contained in Ω . There is a constant $c_0 > 0$ such that

$$|u(t, x)| \geq c_0 \quad ((t, x) \in J \times O).$$

Define $q = v/u$ there. The map $z \mapsto 1/z$ is smooth on $\{|z| \geq c_0\}$, so the Sobolev chain and product rules give $q \in C(J; H^1(O))$. Equation (2.3) and continuity imply $|q| = 1$ everywhere on the rectangle. For almost every $t \in J$, the quotient rule in $H^1(O)$ and (2.5) give

$$\begin{aligned} 0 &= \bar{v}v_x - \bar{u}u_x \\ &= |u|^2 \bar{q} q_x \quad \text{almost everywhere on } O. \end{aligned}$$

Hence $q(t, \cdot)$ is constant on O for almost every t . Since $q \in C(J; H^1(O)) \subset C(J \times O)$, approximation by such times gives the same conclusion for every $t \in J$. We may therefore write

$$(2.8) \quad v(t, x) = c(t)u(t, x) \quad ((t, x) \in J \times O), \quad |c(t)| = 1.$$

It remains to prove that c is independent of t . Choose a nonnegative, nonzero function $\chi \in C_c^\infty(O)$ and set

$$A(t) = \int_{\mathbb{R}} \chi(x) \overline{u(t, x)} v(t, x) dx, \quad B(t) = \int_{\mathbb{R}} \chi(x) |u(t, x)|^2 dx.$$

The lower bound for $|u|$ on $\text{supp } \chi$ gives $B(t) > 0$ on J , and (2.8) gives

$$(2.9) \quad A(t) = c(t)B(t).$$

We compute their derivatives. Formally,

$$u_t = iu_{xx} - iVu, \quad v_t = iv_{xx} - iVv.$$

Therefore

$$\begin{aligned} A' &= \int \chi(\bar{u}_t v + \bar{u} v_t) \\ &= i \int \chi(\bar{u} v_{xx} - \bar{u}_{xx} v), \end{aligned}$$

because the two potential terms cancel. Integrating each second derivative once transfers the remaining derivative to χ and gives, for almost every $t \in J$,

$$(2.10) \quad A'(t) = i \int_{\mathbb{R}} \chi'(x) (\overline{u_x(t, x)} v(t, x) - \overline{u(t, x)} v_x(t, x)) dx,$$

$$(2.11) \quad B'(t) = i \int_{\mathbb{R}} \chi'(x) (\overline{u_x(t, x)} u(t, x) - \overline{u(t, x)} u_x(t, x)) dx.$$

The same calculation with $v = u$ gives (2.11). All displayed integrands belong to L^1 by Cauchy-Schwarz. For the weak solutions, perform the calculation after local regularization. The terms

$V\bar{u}v$ are locally integrable because $V \in L^2_{\text{loc}}$ and u, v are locally bounded. The formulas show that the distributional derivatives A' and B' belong to $L^1_{\text{loc}}(J)$; hence A and B have locally absolutely continuous representatives.

On $\text{supp } \chi$, equation (2.8) and its weak spatial derivative give $v = cu$ and $v_x = cu_x$. Therefore

$$A'(t) = c(t)B'(t) \quad \text{for almost every } t \in J.$$

Since $B > 0$, equation (2.9) shows that $c = A/B$ is locally absolutely continuous. Differentiating (2.9) gives

$$c'(t)B(t) = A'(t) - c(t)B'(t) = 0$$

for almost every t . Hence c is constant on J .

Every point of Ω lies in a rectangle of the type just considered, so q is locally constant on Ω . Let Ω_0 be a connected component and fix one value ζ attained there. The set $\{(t, x) \in \Omega_0 : q(t, x) = \zeta\}$ is nonempty, open, and closed relative to Ω_0 . Connectedness implies that it is all of Ω_0 . This proves the lemma. $\square$

2.3. Conclusion of the proof.

Proof of Theorem A. If u vanishes on $I \times \mathbb{R}$, Proposition 2.1 gives $u \equiv 0$. Equation (2.3) then shows that v vanishes on the same nonempty open set, so Proposition 2.1 also gives $v \equiv 0$. The conclusion holds with $\zeta = 1$.

Suppose that Ω is nonempty. Choose one connected component Ω_0 . By Lemma 2.3, there is a $\zeta \in \mathbb{T}$ such that

$$v = \zeta u \quad \text{on } \Omega_0.$$

The equation is linear and u, v have the same potential. Hence the difference

$$w = v - \zeta u$$

belongs to $C(\mathbb{R}; L^2)$ and is a Masuda weak solution of (2.1). It vanishes on the nonempty open subset Ω_0 of spacetime. Proposition 2.1 gives $w \equiv 0$ on $\mathbb{R}^2$. Thus $v(t) = \zeta u(t)$ in $L^2(\mathbb{R})$ for every $t \in \mathbb{R}$. $\square$

3. STATIONARY POTENTIALS IN THE FADDEEV CLASS

We prove Theorem B. Let $H = -\partial_x^2 + V$ be the self-adjoint quadratic-form realization, and choose jointly measurable representatives of

$$u(t) = e^{-itH}u_0, \quad v(t) = e^{-itH}v_0.$$

The hypothesis is

$$(3.1) \quad |u(t, x)| = |v(t, x)| \quad \text{for almost every } (t, x) \in \mathbb{R}^2.$$

Strong continuity of the unitary group and

$$\| |a|^2 - |b|^2 \|_1 \leq \|a - b\|_2 (\|a\|_2 + \|b\|_2)$$

show that $t \mapsto |e^{-itH}f|^2$ is continuous from $\mathbb{R}$ to $L^1(\mathbb{R})$. Thus (3.1), which initially holds for almost every time, implies equality in L^1_x for every t . The proof uses scattering at both infinities.

A zero-energy resonance is allowed. The rank-one structure of the spectral density matrix gives integrability on almost every fixed energy-gap curve, while the product system in Lemma 3.4 uses no derivative of V .

3.1. Scattering representation. We record the standard one-dimensional facts used in the proof. They are valid for real $V \in L^1_1(\mathbb{R})$; see, for example, [1, 2, 3].

Proposition 3.1 (Standard scattering representation). *Let $V \in L^1_1(\mathbb{R})$ be real. Then the following statements hold.*

- (1) *The singular continuous spectrum of H is empty, the absolutely continuous spectrum is $[0, \infty)$ with multiplicity two, and the negative spectrum consists of finitely many simple eigenvalues*

$$E_j = -\kappa_j^2, \quad 0 < \kappa_1 < \cdots < \kappa_N.$$

There is no eigenvalue in $[0, \infty)$.

- (2) *There are real normalized eigenfunctions ϕ_j and physical generalized eigenfunctions $e_\alpha(\cdot, k)$, $\alpha \in \{1, 2\}$, $k > 0$, such that*

$$\mathcal{F} : L^2(\mathbb{R}) \longrightarrow \mathbb{C}^N \oplus L^2(\mathbb{R}_+; \mathbb{C}^2)$$

is unitary and diagonalizes H . Thus, if

$$\mathcal{F}f = (c_1, \dots, c_N; g_1, g_2),$$

then

$$(3.2) \quad \begin{aligned} (e^{-itH}f)(x) &= \sum_{j=1}^N c_j e^{-itE_j} \phi_j(x) \\ &\quad + \sum_{\alpha=1}^2 \int_0^\infty e^{-itk^2} g_\alpha(k) e_\alpha(x, k) dk \end{aligned}$$

in L^2_x .

- (3) *For each $k > 0$, the two functions $e_1(\cdot, k), e_2(\cdot, k)$ form a basis of the solution space of*

$$-y'' + Vy = k^2 y.$$

On compact x -sets, uniformly for $k > 0$,

$$(3.3) \quad |e_\alpha(x, k)| \lesssim 1, \quad |\partial_x e_\alpha(x, k)| \lesssim 1 + k.$$

More precisely, for each $\sigma \in \{+, -\}$ there are coefficients $c_{\alpha, \varepsilon}^\sigma(k)$, $\varepsilon \in \{+1, -1\}$, and remainders $r_\alpha^\sigma(x, k)$ such that

$$(3.4) \quad e_\alpha(x, k) = \sum_{\varepsilon=\pm 1} c_{\alpha, \varepsilon}^\sigma(k) e^{i\varepsilon kx} + r_\alpha^\sigma(x, k), \quad x \rightarrow \sigma\infty,$$

and

$$\sup_{k>0} |c_{\alpha, \varepsilon}^\sigma(k)| < \infty, \quad \lim_{x \rightarrow \sigma\infty} \sup_{k>0} |r_\alpha^\sigma(x, k)| = 0.$$

- (4) *For every negative eigenfunction,*

$$(3.5) \quad \phi_j(x) = a_j^+ e^{-\kappa_j x} (1 + o(1)) \quad (x \rightarrow +\infty),$$

and there is an analogous expansion at $-\infty$, with nonzero leading coefficients.

Let $f_{\pm}(x, k) \sim e^{\pm ikx}$ as $x \rightarrow \pm\infty$ be the Jost solutions. We choose the physical incidence eigenfunctions in the proposition to be

$$e_1(x, k) = (2\pi)^{-1/2} T(k) f_+(x, k), \quad e_2(x, k) = (2\pi)^{-1/2} T(k) f_-(x, k).$$

Thus the uniform assertion in part (3) concerns the physical incidence eigenfunctions, not an unnormalized Jost solution on the opposite half-line. The right Jost Volterra equation and the transformation-operator estimate give

$$\sup_{k>0} |f_+(x, k) e^{-ikx} - 1| \longrightarrow 0 \quad (x \rightarrow +\infty),$$

and the analogous assertion holds for f_- at $-\infty$. Multiplication by the transmission coefficient, followed by the scattering identity

$$T(k) f_+(x, k) = f_-(x, -k) + R_-(k) f_-(x, k),$$

transfers the bounds to the left because $|T(k)|, |R_-(k)| \leq 1$. The other channel is analogous; here one also uses $f_{\pm}(x, -k) = \overline{f_{\pm}(x, k)}$ for real V . This proves (3.4). The same Volterra estimates, together with the differential equation, give the local derivative bound in (3.3). This also shows directly why a zero resonance causes no loss of uniformity for the physical eigenfunctions.

Only this proposition is imported from scattering theory. Everything below is an elementary consequence of it, one-dimensional ODE theory, and Fourier analysis.

3.2. A fixed-gap estimate. Let

$$\delta = k^2 - \ell^2, \quad s = k^2 + \ell^2, \quad k, \ell > 0.$$

The inverse change of variables is

$$k = \sqrt{\frac{s + \delta}{2}}, \quad \ell = \sqrt{\frac{s - \delta}{2}}, \quad s > |\delta|,$$

and

$$(3.6) \quad dk d\ell = \frac{d\delta ds}{4\sqrt{s^2 - \delta^2}}.$$

The next estimate is the reason arbitrary L^2 initial states are admissible.

Lemma 3.2 (Rank-one shell estimate). *Let $f, g \in L^2(\mathbb{R}_+)$ and let $I \subset \mathbb{R}$ be a bounded interval. Then*

$$(3.7) \quad \iint_{\{k^2 - \ell^2 \in I\}} |f(k)g(\ell)| dk d\ell \leq C_I \|f\|_2 \|g\|_2.$$

Consequently, for almost every $\delta \in \mathbb{R}$,

$$(3.8) \quad s \mapsto \frac{|f(\sqrt{(s + \delta)/2})g(\sqrt{(s - \delta)/2})|}{4\sqrt{s^2 - \delta^2}}$$

belongs to $L^1(|\delta|, \infty)$. The resulting L^1 norm is locally integrable as a function of δ .

Proof. Set

$$K_I(k, \ell) = \mathbf{1}_I(k^2 - \ell^2), \quad k, \ell > 0.$$

Choose $M > 0$ with $I \subset [-M, M]$. For fixed k , the condition $K_I(k, \ell) \neq 0$ implies

$$\max(0, k^2 - M) \leq \ell^2 \leq k^2 + M.$$

If $k^2 \leq M$, the length of this interval in the ℓ variable is at most $\sqrt{k^2 + M} \leq \sqrt{2M}$. If $k^2 > M$, its length is

$$\begin{aligned} \sqrt{k^2 + M} - \sqrt{k^2 - M} &= \frac{2M}{\sqrt{k^2 + M} + \sqrt{k^2 - M}} \\ &\leq \sqrt{2M}. \end{aligned}$$

Consequently

$$\sup_{k>0} \int_0^\infty K_I(k, \ell) d\ell \leq \sqrt{2M}.$$

Interchanging k and ℓ gives the same estimate for the other Schur integral. Schur's test states that a nonnegative kernel K satisfying

$$\sup_k \int K(k, \ell) d\ell \leq A, \quad \sup_\ell \int K(k, \ell) dk \leq B$$

defines an L^2 operator of norm at most $\sqrt{AB}$. We therefore have $\|K_I\|_{L^2 \rightarrow L^2} \leq \sqrt{2M}$ and

$$\begin{aligned} \iint K_I(k, \ell) |f(k)g(\ell)| dk d\ell &= \int_0^\infty |f(k)|(K_I|g|)(k) dk \\ &\leq \|f\|_2 \|K_I|g|\|_2 \leq \sqrt{2M} \|f\|_2 \|g\|_2. \end{aligned}$$

This proves (3.7).

For $(k, \ell) \in (0, \infty)^2$, the map $(k, \ell) \mapsto (\delta, s) = (k^2 - \ell^2, k^2 + \ell^2)$ is a C^1 diffeomorphism onto $\{(\delta, s) : s > |\delta|\}$. The change-of-variables formula (3.6) gives, for every bounded interval I ,

$$\begin{aligned} &\iint_{\{k^2 - \ell^2 \in I\}} |f(k)g(\ell)| dk d\ell \\ &= \int_I \int_{|\delta|}^\infty \frac{|f(\sqrt{(s+\delta)/2})g(\sqrt{(s-\delta)/2})|}{4\sqrt{s^2 - \delta^2}} ds d\delta. \end{aligned}$$

The left side is finite by (3.7). Tonelli's theorem now shows that the inner integral is finite for almost every $\delta \in I$, and that these inner integrals form an $L^1(I)$ function. A countable exhaustion of $\mathbb{R}$ by bounded intervals proves both remaining assertions. $\square$

Remark 3.3. It would not be enough merely to observe that the spectral density-matrix kernel belongs to $L^2(\mathbb{R}_+^2)$. A general L^2 kernel need not have an integrable restriction to almost every hyperbola $k^2 - \ell^2 = \delta$. Lemma 3.2 uses the finite sum of rank-one terms in an essential way.

We will also use the following consequence of the one-dimensional change of variables. Suppose $g \in L^2(\mathbb{R}_+)$ and $E_j = -\kappa_j^2$. A bound-continuous term has temporal frequency $k^2 + \kappa_j^2$. Its density in that frequency variable is locally integrable, because under $\delta = k^2 + \kappa_j^2$,

$$\frac{|g(\sqrt{\delta - \kappa_j^2})|}{2\sqrt{\delta - \kappa_j^2}} d\delta = |g(k)| dk,$$

and g is integrable on bounded k -intervals. Bound-bound terms, by contrast, are point masses in the temporal-frequency variable.

3.3. The product system. The following elementary system is the central structural observation.

Lemma 3.4 (Product system). *Let $V \in L^1_{\text{loc}}(\mathbb{R})$, let $E, F \in \mathbb{C}$, and suppose that $y, z \in W^{2,1}_{\text{loc}}(\mathbb{R})$ satisfy*

$$-y'' + Vy = Ey, \quad -z'' + Vz = Fz$$

almost everywhere. Put

$$\delta = E - F, \quad s = E + F,$$

and define

$$(3.9) \quad A = yz, \quad B = y'z + yz', \quad C = y'z - yz', \quad D = y'z'.$$

Then A, B, C, D are locally absolutely continuous and satisfy

$$(3.10) \quad \boxed{\begin{aligned} A' &= B, \\ B' &= (2V - s)A + 2D, \\ C' &= -\delta A, \\ D' &= \left(V - \frac{s}{2}\right)B + \frac{\delta}{2}C. \end{aligned}}$$

Proof. The functions y, z and their first derivatives are locally absolutely continuous, so the ordinary product rule is valid almost everywhere. Since

$$y'' = (V - E)y, \quad z'' = (V - F)z,$$

we obtain

$$A' = y'z + yz' = B.$$

Differentiating B gives

$$\begin{aligned} B' &= y''z + 2y'z' + yz'' \\ &= (V - E)yz + 2y'z' + (V - F)yz \\ &= (2V - E - F)A + 2D = (2V - s)A + 2D. \end{aligned}$$

Similarly,

$$\begin{aligned} C' &= y''z - yz'' \\ &= (V - E)yz - (V - F)yz = (F - E)A = -\delta A. \end{aligned}$$

Finally,

$$D' = y''z' + y'z'' = (V - E)y'z' + (V - F)y'z'.$$

The definitions of B and C imply

$$yz' = \frac{B - C}{2}, \quad y'z = \frac{B + C}{2}.$$

Since $E = (s + \delta)/2$ and $F = (s - \delta)/2$, substitution gives

$$\begin{aligned} D' &= \left(V - \frac{s + \delta}{2}\right) \frac{B - C}{2} + \left(V - \frac{s - \delta}{2}\right) \frac{B + C}{2} \\ &= \left(V - \frac{s}{2}\right) B + \frac{\delta}{2} C. \end{aligned}$$

Every right-hand side is locally integrable. Hence all four components are locally absolutely continuous and the system holds everywhere in its integral form. $\square$

For fixed E let $\mathcal{S}_E$ denote the solution space of $-y'' + Vy = Ey$. Fix $x_0 \in \mathbb{R}$. Existence and uniqueness for the integral system associated with the Schrödinger equation show that

$$y \mapsto (y(x_0), y'(x_0))$$

is an isomorphism from $\mathcal{S}_E$ onto $\mathbb{C}^2$. In particular, $\dim \mathcal{S}_E = 2$. The tensor product $\mathcal{S}_E \otimes \mathcal{S}_F$ is therefore four-dimensional. At x_0 , a simple tensor $y \otimes z$ has coordinates

$$yz, \quad y'z, \quad yz', \quad y'z'.$$

The product-state coordinates are obtained from these by the invertible linear change

$$A = yz, \quad B = y'z + yz', \quad C = y'z - yz', \quad D = y'z',$$

whose inverse is

$$yz = A, \quad y'z = \frac{B + C}{2}, \quad yz' = \frac{B - C}{2}, \quad y'z' = D.$$

The first-order system (3.10) has unique solutions for arbitrary initial data in $\mathbb{C}^4$. It follows that the map

$$(3.11) \quad \mathcal{S}_E \otimes \mathcal{S}_F \longrightarrow \{\text{solutions of (3.10)}\}$$

induced by (3.9) is an isomorphism.

Lemma 3.5 (Independence of unequal-energy products). *Suppose $E \neq F$. If y_1, y_2 form a basis of $\mathcal{S}_E$ and z_1, z_2 form a basis of $\mathcal{S}_F$, then the four functions*

$$y_i z_j, \quad 1 \leq i, j \leq 2,$$

are linearly independent.

Proof. Suppose

$$\sum_{i,j=1}^2 c_{ij} y_i(x) z_j(x) = 0 \quad \text{for every } x.$$

The tensor $T = \sum_{i,j} c_{ij} y_i \otimes z_j$ determines, through (3.11), a solution (A, B, C, D) of (3.10). The displayed relation says $A = 0$. Therefore $B = A' = 0$. The second equation of the system becomes $0 = 2D$, so $D = 0$. The fourth equation then becomes $0 = \delta C/2$. Since $\delta = E - F \neq 0$, it follows that $C = 0$. Thus the product state is the zero solution. The map (3.11) is injective, so $T = 0$. Because $\{y_i \otimes z_j : 1 \leq i, j \leq 2\}$ is a basis of $\mathcal{S}_E \otimes \mathcal{S}_F$, every coefficient c_{ij} vanishes. $\square$

We also record the elementary negative-energy ODE fact used below.

Lemma 3.6 (Negative-energy dichotomy). *Let $V \in L^1(\mathbb{R})$ be real and let $G = -\kappa^2 < 0$. At $+\infty$ the solution space $\mathcal{S}_G$ has a basis $y_{G,+}^d, y_{G,+}^g$ satisfying*

$$\begin{aligned} y_{G,+}^d(x) &= e^{-\kappa x}(1 + o(1)), & (y_{G,+}^d)'(x) &= -\kappa e^{-\kappa x}(1 + o(1)), \\ y_{G,+}^g(x) &= e^{\kappa x}(1 + o(1)), & (y_{G,+}^g)'(x) &= \kappa e^{\kappa x}(1 + o(1)). \end{aligned}$$

At $-\infty$ there is a basis $y_{G,-}^d, y_{G,-}^g$ with

$$\begin{aligned} y_{G,-}^d(x) &= e^{\kappa x}(1 + o(1)), & (y_{G,-}^d)'(x) &= \kappa e^{\kappa x}(1 + o(1)), \\ y_{G,-}^g(x) &= e^{-\kappa x}(1 + o(1)), & (y_{G,-}^g)'(x) &= -\kappa e^{-\kappa x}(1 + o(1)). \end{aligned}$$

In particular, if the lines

$$L_{G,+}^d := \text{span}\{y_{G,+}^d\}, \quad L_{G,-}^d := \text{span}\{y_{G,-}^d\}$$

coincide, then G is an L^2 eigenvalue of H .

Proof. The equation at energy $G = -\kappa^2$ is

$$y'' - \kappa^2 y = Vy.$$

On a right half-line, the solution which is asymptotic to $e^{-\kappa x}$ is obtained from the Volterra equation

$$(3.12) \quad y(x) = e^{-\kappa x} + \frac{1}{\kappa} \int_x^\infty \sinh(\kappa(t-x))V(t)y(t) dt.$$

To see why only $V \in L^1$ is needed, write $m(x) = e^{\kappa x}y(x)$. Then (3.12) becomes

$$m(x) = 1 + \frac{1}{2\kappa} \int_x^\infty (1 - e^{-2\kappa(t-x)})V(t)m(t) dt.$$

The kernel in parentheses has absolute value at most 2. Successive approximation on a sufficiently far right half-line, followed by continuation to the left, gives a unique bounded solution with $m(x) \rightarrow 1$. Differentiating the Volterra equation gives $m'(x) \rightarrow 0$. Therefore

$$y_{G,+}^d(x) = e^{-\kappa x}(1 + o(1)), \quad (y_{G,+}^d)'(x) = -\kappa e^{-\kappa x}(1 + o(1)).$$

In particular, $y_{G,+}^d$ has no zeros for all sufficiently large x . Fix such an x_0 . Reduction of order gives a second solution

$$(3.13) \quad y_{G,+}^g(x) = 2\kappa y_{G,+}^d(x) \int_{x_0}^x \frac{dt}{(y_{G,+}^d(t))^2}.$$

Since $(y_{G,+}^d(t))^{-2} = e^{2\kappa t}(1 + o(1))$, integration yields

$$\int_{x_0}^x \frac{dt}{(y_{G,+}^d(t))^2} = \frac{e^{2\kappa x}}{2\kappa}(1 + o(1)).$$

Substitution in (3.13), followed by differentiation, gives the asserted growing asymptotics. The Wronskian of the two solutions is $2\kappa \neq 0$, so they form a basis. ODE uniqueness extends both solutions to all of $\mathbb{R}$.

Replacing x by $-x$ and $V(x)$ by $V(-x)$ gives the corresponding decaying and growing basis at $-\infty$.

Suppose now that $L_{G,+}^d = L_{G,-}^d$. A nonzero solution y spanning this common line satisfies, for some constants $C, R > 0$,

$$|y(x)| + |y'(x)| \leq Ce^{-\kappa|x|} \quad (|x| \geq R).$$

Hence $y, y' \in L^2(\mathbb{R})$, so $y \in H^1(\mathbb{R})$. Since $V \in L^1(\mathbb{R})$ and $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} |V(x)| |y(x)|^2 dx \leq \|V\|_1 \|y\|_\infty^2 < \infty.$$

Thus y belongs to the form domain of the quadratic-form realization of H . The distributional equation gives, first for $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} y' \overline{\varphi'} + Vy \overline{\varphi} = G \int_{\mathbb{R}} y \overline{\varphi}.$$

Both sides are continuous in the form norm, so the identity extends to the whole form domain. The representation theorem for closed semibounded forms then gives $y \in D(H)$ and $Hy = Gy$. Hence G is an L^2 eigenvalue. $\square$

3.4. Fixed-gap injectivity. We now prove the analytic core of the argument. Fix $\delta \neq 0$. For $s > |\delta|$, put

$$E(s) = \frac{s + \delta}{2}, \quad F(s) = \frac{s - \delta}{2},$$

and let

$$X_{\alpha\beta}(x, s) = (A_{\alpha\beta}, B_{\alpha\beta}, C_{\alpha\beta}, D_{\alpha\beta})(x, s)$$

be the product state associated with

$$y = e_\alpha(\cdot, \sqrt{E(s)}), \quad z = \overline{e_\beta(\cdot, \sqrt{F(s)})}.$$

Since V is real, the second factor is again a solution of the Schrödinger equation at energy $F(s)$.

There may also be bound-continuous products with the same energy difference δ . They form the following finite family. If $\delta > \kappa_j^2$, set

$$k_j = \sqrt{\delta - \kappa_j^2}, \quad s_j = \delta - 2\kappa_j^2,$$

and take the two product states associated with $e_\alpha(\cdot, k_j)\phi_j$, $\alpha = 1, 2$. Their energies are k_j^2 and $-\kappa_j^2$, so their difference is δ and their sum is s_j . If $\delta < -\kappa_j^2$, set

$$k_j = \sqrt{-\delta - \kappa_j^2}, \quad s_j = -\delta - 2\kappa_j^2,$$

and take the states associated with $\phi_j \overline{e_\alpha(\cdot, k_j)}$. These have energy difference δ as well. Enumerate all compatible states as $X_\rho(x)$, denote their energy sums by s_ρ , and denote their scalar coefficients by a_ρ . If several contributions produce the same state, combine their coefficients first.

Proposition 3.7 (Fixed-gap injectivity). *Suppose $q_{\alpha\beta} \in L^1((|\delta|, \infty))$ and*

$$(3.14) \quad W(x) := \sum_{\alpha, \beta=1}^2 \int_{|\delta|}^{\infty} q_{\alpha\beta}(s) A_{\alpha\beta}(x, s) ds + \sum_{\rho} a_\rho A_\rho(x) = 0$$

for every $x \in \mathbb{R}$. Then every $q_{\alpha\beta}$ vanishes almost everywhere and every a_ρ vanishes.

Proof. The proof has four steps.

Step 1: the energy-sum resolvent. The continuum sums satisfy $s > |\delta|$. There are only finitely many bound-continuous sums s_ρ . Hence the set of all sums occurring in (3.14) is bounded from below. Choose a real number λ strictly below this set. In particular, every denominator below is nonzero and has one sign. Define

$$(3.15) \quad Z_\lambda(x) := \sum_{\alpha,\beta=1}^2 \int_{|\delta|}^\infty \frac{q_{\alpha\beta}(s)}{s-\lambda} X_{\alpha\beta}(x,s) ds + \sum_\rho \frac{a_\rho}{s_\rho-\lambda} X_\rho(x).$$

Write its components as $Z_\lambda = (A_\lambda, B_\lambda, C_\lambda, D_\lambda)$.

Let $K \subset \mathbb{R}$ be compact. The energies $E(s)$ and $F(s)$ are positive and at most $s + |\delta|$. The local scattering estimates (3.3) therefore give, uniformly for $x \in K$ and $s > |\delta|$,

$$(3.16) \quad |A_{\alpha\beta}| \lesssim_K 1, \quad |B_{\alpha\beta}| + |C_{\alpha\beta}| \lesssim_K 1 + \sqrt{s}, \quad |D_{\alpha\beta}| \lesssim_K 1 + s.$$

For example, D is the product of one derivative at energy $E(s)$ and one at energy $F(s)$, so

$$|D| \lesssim_K (1 + \sqrt{E(s)})(1 + \sqrt{F(s)}) \lesssim_K 1 + s.$$

Because λ is fixed below the spectral sums,

$$\frac{1}{|s-\lambda|} \lesssim_\lambda \frac{1}{1+s}.$$

After division by $s - \lambda$, the bounds for the four components are respectively $O(1)$, $O((1+s)^{-1/2})$, $O((1+s)^{-1/2})$, and $O(1)$. Each is bounded by a constant times $|q_{\alpha\beta}(s)|$. Since every $q_{\alpha\beta}$ is integrable, all components in (3.15) converge absolutely and uniformly on K . The finite atomic sum causes no convergence issue.

Direct differentiation under the untruncated integral would require more than $q \in L^1$, because derivatives of D grow with s . We avoid this. Choose $\chi_R \in C_c^\infty(|\delta|, \infty)$ with $0 \leq \chi_R \leq 1$ and $\chi_R(s) \rightarrow 1$ pointwise. Insert χ_R in every continuum integral in (3.15), retain all atomic terms, and call the resulting state $Z_{\lambda,R}$. Let W_R be the same truncation of (3.14). Every truncated integral may be differentiated distributionally by pairing with a compactly supported test function in x and using Fubini's theorem. Applying (3.10) and the identity

$$\frac{s}{s-\lambda} = 1 + \frac{\lambda}{s-\lambda}$$

gives

$$(3.17) \quad A'_{\lambda,R} = B_{\lambda,R},$$

$$(3.18) \quad B'_{\lambda,R} = (2V - \lambda)A_{\lambda,R} + 2D_{\lambda,R} - W_R,$$

$$(3.19) \quad C'_{\lambda,R} = -\delta A_{\lambda,R},$$

$$(3.20) \quad D'_{\lambda,R} = \left(V - \frac{\lambda}{2}\right) B_{\lambda,R} + \frac{\delta}{2} C_{\lambda,R} - \frac{1}{2} W'_R.$$

The uniform local bound for A and dominated convergence give $W_R \rightarrow W = 0$ locally uniformly. Therefore, for every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle W'_R, \varphi \rangle = - \int_{\mathbb{R}} W_R(x) \varphi'(x) dx \longrightarrow 0.$$

Thus $W'_R \rightarrow 0$ in distributions. The component bounds established above also give local uniform convergence $Z_{\lambda,R} \rightarrow Z_\lambda$. Products such as $VB_{\lambda,R}$ converge in L^1_{loc} , because $V \in L^1_{\text{loc}}$ and $B_{\lambda,R}$ converges locally uniformly. We may therefore pass to the limit in each of the four distributional identities and obtain

$$(3.21) \quad \begin{cases} A'_\lambda = B_\lambda, \\ B'_\lambda = (2V - \lambda)A_\lambda + 2D_\lambda, \\ C'_\lambda = -\delta A_\lambda, \\ D'_\lambda = \left(V - \frac{\lambda}{2}\right)B_\lambda + \frac{\delta}{2}C_\lambda. \end{cases}$$

Initially these identities hold distributionally, while all four components are continuous. The first and third equations already show that A_λ and C_λ are locally absolutely continuous. The right side of the fourth equation belongs to L^1_{loc} : VB_λ is locally integrable and the other terms are continuous. Hence D_λ is locally absolutely continuous. The right side of the second equation is then locally integrable, so B_λ is locally absolutely continuous as well. Thus Z_λ is an ordinary solution of the product system with auxiliary energies

$$(3.22) \quad E_\lambda = \frac{\lambda + \delta}{2}, \quad F_\lambda = \frac{\lambda - \delta}{2}.$$

Step 2: decay of the scalar component. We claim that

$$(3.23) \quad A_\lambda(x) \longrightarrow 0 \quad (x \rightarrow \pm\infty).$$

Every atomic term is a product of a fixed positive-energy scattering solution and a negative-energy eigenfunction. The scattering factor is bounded at both ends, while the eigenfunction decays exponentially there. Hence every atomic contribution to $A_\lambda(x)$ tends to zero as $x \rightarrow \pm\infty$.

Consider a continuum product at one of the two ends. Write both scattering solutions as the sum of the two plane waves in (3.4) and a remainder. Multiplication gives four plane-wave terms, with phases

$$(3.24) \quad x(\varepsilon k(s) + \eta \ell(s)), \quad \varepsilon, \eta \in \{-1, 1\},$$

and terms containing at least one remainder. The plane-wave coefficients are uniformly bounded. Therefore their amplitudes after insertion into A_λ are bounded by

$$C \frac{|q_{\alpha\beta}(s)|}{|s - \lambda|},$$

which belongs to $L^1(|\delta|, \infty)$. The scattering remainders tend to zero uniformly in the energy parameter. Their products are bounded by the same integrable majorant, so dominated convergence makes their integrals tend to zero.

It remains to treat each plane-wave integral. Define

$$p_+(s) = k(s) + \ell(s), \quad p_-(s) = k(s) - \ell(s).$$

Since $k^2 - \ell^2 = \delta$,

$$p_-(s) = \frac{\delta}{p_+(s)}.$$

The function p_+ is strictly increasing: both $k(s)$ and $\ell(s)$ are strictly increasing. Since $\delta \neq 0$, p_- is strictly monotone as well. Their derivatives do not vanish on the open interval $(|\delta|, \infty)$. Thus each is a C^1 diffeomorphism onto its image. If $f \in L^1(ds)$, the substitution $p = p_\pm(s)$ gives

$$\int_{|\delta|}^{\infty} f(s) e^{ixp_\pm(s)} ds = \int_{p_\pm((|\delta|, \infty))} f(s(p)) \left| \frac{ds}{dp} \right| e^{ixp} dp.$$

The new amplitude is integrable because its $L^1(dp)$ norm equals $\|f\|_{L^1(ds)}$. The Riemann–Lebesgue lemma makes the last integral tend to zero as $|x| \rightarrow \infty$. The four phases in (3.24) are $\pm p_+$ and $\pm p_-$. Summing them proves (3.23) at both ends.

Step 3: uniqueness at a negative auxiliary sum. Choose λ sufficiently negative that

$$(3.25) \quad \lambda < -|\delta|, \quad \frac{\lambda - |\delta|}{2} < \inf \sigma(H).$$

Then $E_\lambda, F_\lambda < 0$, and the lower of these two energies lies below the spectrum of H .

Assume first that $\delta > 0$. Write

$$E_\lambda = -\alpha^2, \quad F_\lambda = -\beta^2, \quad 0 < \alpha < \beta.$$

By Lemma 3.6, choose decaying and growing bases at $+\infty$ for both energy spaces. In these bases the tensor represented by Z_λ has a unique expansion

$$\begin{aligned} T = & c_{dd} y_{E_\lambda, +}^d \otimes y_{F_\lambda, +}^d + c_{gd} y_{E_\lambda, +}^g \otimes y_{F_\lambda, +}^d \\ & + c_{dg} y_{E_\lambda, +}^d \otimes y_{F_\lambda, +}^g + c_{gg} y_{E_\lambda, +}^g \otimes y_{F_\lambda, +}^g. \end{aligned}$$

Its scalar component A_λ is the same linear combination of the four ordinary products. Their asymptotic rates are, in the displayed order,

$$e^{-(\alpha+\beta)x}, \quad e^{-(\beta-\alpha)x}, \quad e^{(\beta-\alpha)x}, \quad e^{(\alpha+\beta)x}.$$

Divide $A_\lambda(x)$ by $e^{(\alpha+\beta)x}$. Every term except the last tends to zero, while the last tends to c_{gg} . Since $A_\lambda(x) \rightarrow 0$, we obtain $c_{gg} = 0$. Now divide the remaining expansion by $e^{(\beta-\alpha)x}$. The first two terms tend to zero and the third tends to c_{dg} , so $c_{dg} = 0$. The coefficients c_{dd} and c_{gd} are not forced to vanish at this end because their products decay. We have shown that

$$T \in \mathcal{S}_{E_\lambda} \otimes L_{F_\lambda, +}^d.$$

where $L_{F_\lambda, +}^d$ is the one-dimensional space of solutions at energy F_λ which decay at $+\infty$. Repeating the argument at $-\infty$ places the same tensor in

$$\mathcal{S}_{E_\lambda} \otimes L_{F_\lambda, -}^d.$$

The two lines $L_{F_\lambda, +}^d$ and $L_{F_\lambda, -}^d$ are distinct by Lemma 3.6; otherwise F_λ would be an L^2 eigenvalue, contrary to $F_\lambda < \inf \sigma(H)$. Distinct one-dimensional subspaces of the two-dimensional space $\mathcal{S}_{F_\lambda}$ have zero intersection. To verify the tensor statement, choose a basis z_+, z_- of $\mathcal{S}_{F_\lambda}$ with z_+ spanning $L_{F_\lambda, +}^d$ and z_- spanning $L_{F_\lambda, -}^d$. A tensor in the first product space has the form $Y \otimes z_+$, while one in the second has the form $\tilde{Y} \otimes z_-$. Uniqueness of coordinates in the basis z_+, z_- forces both tensors to vanish if they are equal. Hence the two tensor-product subspaces have zero intersection, so $T = 0$ and therefore $Z_\lambda = 0$. If $\delta < 0$, the same argument applies after interchanging E_λ and F_λ .

We have proved

$$(3.26) \quad Z_\lambda = 0 \quad \text{for every sufficiently negative real } \lambda.$$

Step 4: Cauchy uniqueness. For fixed x , define the finite complex measure

$$(3.27) \quad d\mu_x(s) := \sum_{\alpha,\beta=1}^2 q_{\alpha\beta}(s) A_{\alpha\beta}(x, s) ds + \sum_{\rho} a_{\rho} A_{\rho}(x) \delta_{s_{\rho}}.$$

Choose $m \in \mathbb{R}$ strictly below every spectral sum occurring in (3.14); in particular $\text{supp } \mu_x \subset [m, \infty)$. Set

$$F_x(z) := \int_{[m, \infty)} \frac{d\mu_x(s)}{s - z}, \quad z \in \mathbb{C} \setminus [m, \infty).$$

For every compact subset of the slit plane, the denominator is uniformly separated from zero on the support of the measure. Differentiation under the integral is therefore justified, so F_x is holomorphic on the connected domain $\mathbb{C} \setminus [m, \infty)$. For real $\lambda < m$, its defining integral is exactly the first component $A_\lambda(x)$ of (3.15). By (3.26), it vanishes on a nonempty real interval. The holomorphic identity theorem gives $F_x \equiv 0$ on the slit plane.

We recall why this forces the measure to vanish. For $a < b$ at which μ_x has no atoms, the Stieltjes inversion formula gives

$$\mu_x((a, b)) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b (F_x(t + i\varepsilon) - F_x(t - i\varepsilon)) dt.$$

The right side is zero. Applying this separately to the real and imaginary parts and then using regularity of finite Borel measures gives $\mu_x = 0$.

For a fixed x , the equality $\mu_x = 0$ says that its absolutely continuous density vanishes for almost every s , and that each of its atomic masses is zero. The exceptional set in s may depend on x . Choose a countable dense set $D \subset \mathbb{R}$. The union of the exceptional sets over $x \in D$ is still null. For every s outside this union,

$$\sum_{\alpha,\beta=1}^2 q_{\alpha\beta}(s) A_{\alpha\beta}(x, s) = 0 \quad (x \in D).$$

For fixed s , each product is continuous in x . Density of D extends the identity to every $x \in \mathbb{R}$. Lemma 3.5 gives $q_{\alpha\beta}(s) = 0$ for every α, β . At an atom, distinct bound energies give distinct values of s for fixed δ . More explicitly, if $\delta > 0$ and the second energy is $E_j = -\kappa_j^2$, then

$$E = \delta - \kappa_j^2 > 0, \quad s = \delta - 2\kappa_j^2, \quad |s| < \delta.$$

Thus this atom lies strictly below the continuum threshold $s = \delta$, and different j give different atoms. A zero residue has the form

$$\phi_j(x) \sum_{\alpha=1}^2 b_{\alpha} e_{\alpha}(x, k) = 0.$$

The eigenfunction ϕ_j is nonzero on some interval, and the two continuum solutions are independent there, so $b_1 = b_2 = 0$. The case $\delta < 0$ is symmetric. Thus all atomic coefficients vanish as well. $\square$

3.5. The temporal Fourier transform of the density. Let

$$\mathcal{F}u_0 = (c_1, \dots, c_N; g_1, g_2), \quad \mathcal{F}v_0 = (d_1, \dots, d_N; h_1, h_2).$$

Choose measurable representatives of the four L^2 functions; their values on null sets will play no role.

Step 1: approximation of the spectral density. Define

$$(3.28) \quad Q_{\alpha\beta}(k, \ell) = g_\alpha(k) \overline{g_\beta(\ell)} - h_\alpha(k) \overline{h_\beta(\ell)}.$$

This is a finite sum of rank-one L^2 kernels. Formally, the continuous–continuous part of the density difference is

$$(3.29) \quad \sum_{\alpha, \beta=1}^2 \int_0^\infty \int_0^\infty e^{-it(k^2 - \ell^2)} Q_{\alpha\beta}(k, \ell) \\ (3.30) \quad \times e_\alpha(x, k) \overline{e_\beta(x, \ell)} dk d\ell.$$

The temporal frequency is the energy difference $\delta = k^2 - \ell^2$. The second variable $s = k^2 + \ell^2$ parametrizes the corresponding fixed-gap curve. The purpose of the next approximation is to make this formal expansion valid for L^2 spectral coefficients.

The equality of moduli says that

$$(3.31) \quad \mathcal{D}(t, x) := |u(t, x)|^2 - |v(t, x)|^2 = 0$$

as a distribution in (t, x) . To justify the spectral expansion, choose $g_{\alpha,n}, h_{\alpha,n} \in C_c^\infty((0, \infty))$ converging to g_α, h_α in L^2 , while leaving the bound-state coefficients fixed. Let u_n, v_n be the corresponding evolutions. Unitarity gives

$$\sup_{t \in \mathbb{R}} \|u_n(t) - u(t)\|_2 = \|u_{0,n} - u_0\|_2 \longrightarrow 0.$$

For any $a, b \in L^2$, the identity $|a|^2 - |b|^2 = (a - b)\bar{a} + b(\bar{a} - \bar{b})$ and Cauchy–Schwarz give

$$\||a|^2 - |b|^2\|_1 \leq \|a - b\|_2 (\|a\|_2 + \|b\|_2).$$

Consequently, uniformly in t ,

$$(3.32) \quad \||u_n(t)|^2 - |u(t)|^2\|_{L_x^1} \leq \|u_n(t) - u(t)\|_2 (\|u_n(t)\|_2 + \|u(t)\|_2) \longrightarrow 0,$$

and the same holds for v_n .

Fix a spatial test function $\varphi \in C_c^\infty(\mathbb{R})$ and a temporal frequency cutoff $\chi \in C_c^\infty((-M, M))$. On $\text{supp } \varphi$, the physical eigenfunctions are uniformly bounded. Thus the difference between the (α, β) continuous–continuous forms for u_n and u is bounded, up to a constant depending on φ , by

$$\begin{aligned} & \iint_{|k^2 - \ell^2| < M} |g_{\alpha,n}(k) \overline{g_{\beta,n}(\ell)} - g_\alpha(k) \overline{g_\beta(\ell)}| dk d\ell \\ & \leq \iint_{|k^2 - \ell^2| < M} |g_{\alpha,n}(k) - g_\alpha(k)| |g_{\beta,n}(\ell)| dk d\ell \\ & \quad + \iint_{|k^2 - \ell^2| < M} |g_\alpha(k)| |g_{\beta,n}(\ell) - g_\beta(\ell)| dk d\ell. \end{aligned}$$

Lemma 3.2 bounds this by a constant times

$$(3.33) \quad \sqrt{2M} (\|g_{\alpha,n} - g_\alpha\|_2 \|g_{\beta,n}\|_2 + \|g_\alpha\|_2 \|g_{\beta,n} - g_\beta\|_2),$$

which tends to zero. The same estimate applies to h .

Let $\check{\chi}$ be the inverse Fourier transform of χ . Since $\check{\chi}$ is Schwartz, it belongs to $L^1(\mathbb{R}_t)$. Equation (3.32) therefore permits passage to the limit in

$$\iint \check{\chi}(t) \varphi(x) (|u_n(t, x)|^2 - |v_n(t, x)|^2) dx dt.$$

For the smooth compactly supported spectral data, Fubini's theorem gives the spectral expansion directly. Estimate (3.33) shows that its continuous–continuous part converges to (3.30) after testing in (t, x) . A mixed term contains only one continuum integral. The support of $\chi(\pm(k^2 + \kappa_j^2))$ is a bounded k -interval, on which an L^2 function is in L^1 by Cauchy–Schwarz. This justifies the mixed terms as well.

Step 2: disintegration by the energy difference. In the continuous–continuous term, apply the change of variables

$$\delta = k^2 - \ell^2, \quad s = k^2 + \ell^2.$$

The shell estimate and Tonelli's theorem show that this change is legitimate for almost every δ . Formula (3.6) gives

$$(3.34) \quad q_{\alpha\beta, \delta}(s) = \frac{Q_{\alpha\beta} \left(\sqrt{(s + \delta)/2}, \sqrt{(s - \delta)/2} \right)}{4\sqrt{s^2 - \delta^2}}.$$

Define

$$G_{\text{cc}}(\delta, x) := \sum_{\alpha, \beta=1}^2 \int_{|\delta|}^{\infty} q_{\alpha\beta, \delta}(s) A_{\alpha\beta}(x, s) ds.$$

For the mixed terms, if $\delta > \kappa_j^2$, put

$$k_j^+(\delta) = \sqrt{\delta - \kappa_j^2}, \quad a_{j\alpha}^+(\delta) = \frac{g_{\alpha}(k_j^+(\delta)) \bar{c}_j - h_{\alpha}(k_j^+(\delta)) \bar{d}_j}{2k_j^+(\delta)},$$

and if $\delta < -\kappa_j^2$, put

$$k_j^-(\delta) = \sqrt{-\delta - \kappa_j^2}, \quad a_{j\alpha}^-(\delta) = \frac{\overline{c_j g_{\alpha}(k_j^-(\delta))} - \overline{d_j h_{\alpha}(k_j^-(\delta))}}{2k_j^-(\delta)}.$$

These formulas follow directly from the mixed spectral phases. The product of a continuum term with the conjugate of the j th bound term has phase

$$e^{-it(k^2 + \kappa_j^2)},$$

so it contributes at $\delta = k^2 + \kappa_j^2 > \kappa_j^2$. Since $d\delta = 2k dk$, its density contains $1/(2k)$. The conjugate ordering has phase $e^{it(k^2 + \kappa_j^2)}$, contributes at $\delta = -(k^2 + \kappa_j^2)$, and has the same Jacobian. Set

$$\begin{aligned} G_{\text{mix}}(\delta, x) := & \sum_{j, \alpha} \mathbf{1}_{\{\delta > \kappa_j^2\}} a_{j\alpha}^+(\delta) e_{\alpha}(x, k_j^+(\delta)) \phi_j(x) \\ & + \sum_{j, \alpha} \mathbf{1}_{\{\delta < -\kappa_j^2\}} a_{j\alpha}^-(\delta) \phi_j(x) \overline{e_{\alpha}(x, k_j^-(\delta))}, \end{aligned}$$

and $G_{\text{ac}} = G_{\text{cc}} + G_{\text{mix}}$.

With the standard jointly measurable choice of the physical eigenfunctions, these formulas define strongly measurable $C(K)$ -valued functions for every compact K . To verify continuity in x , let

$x_n \rightarrow x$ in K . Each product of generalized eigenfunctions converges pointwise, and the local bound (3.3) supplies an integrable majorant on every fixed shell. Dominated convergence gives convergence in the defining integrals.

Let $I \subset \mathbb{R}$ be bounded. By the local scattering bound and the triangle inequality,

$$\int_I \|G_{cc}(\delta, \cdot)\|_{C(K)} d\delta \lesssim_K \sum_{\alpha, \beta=1}^2 \iint_{\{k^2 - \ell^2 \in I\}} (|g_\alpha(k)g_\beta(\ell)| + |h_\alpha(k)h_\beta(\ell)|) dk d\ell.$$

Lemma 3.2 bounds the right side by

$$C_{I,K} \sum_{\alpha, \beta=1}^2 (\|g_\alpha\|_2 \|g_\beta\|_2 + \|h_\alpha\|_2 \|h_\beta\|_2).$$

For a positive mixed term, the substitution $\delta = k^2 + \kappa_j^2$ gives

$$\begin{aligned} & \int_I \left\| \mathbf{1}_{\{\delta > \kappa_j^2\}} a_{j\alpha}^+(\delta) e_\alpha(\cdot, k_j^+(\delta)) \phi_j \right\|_{C(K)} d\delta \\ & \lesssim_K (|c_j| + |d_j|) \int_{J_{I,j}} (|g_\alpha(k)| + |h_\alpha(k)|) dk, \end{aligned}$$

where $J_{I,j}$ is a bounded k -interval. This is finite by Cauchy–Schwarz. The negative mixed term is identical. Therefore

$$\int_I \|G_{\text{mix}}(\delta, \cdot)\|_{C(K)} d\delta < \infty.$$

Consequently

$$(3.35) \quad G_{ac} \in L_{\text{loc}}^1(\mathbb{R}; C(K)) \quad \text{for every compact } K \subset \mathbb{R}.$$

Fix $\varphi \in C_c^\infty(\mathbb{R})$. The estimates above permit Fubini’s theorem, so the continuous–continuous and mixed parts of the temporal Fourier transform of (3.31), after pairing in x with φ , form the absolutely continuous measure

$$\left(\int_{\mathbb{R}} G_{ac}(\delta, x) \varphi(x) dx \right) d\delta.$$

A bound–bound product has one fixed temporal frequency $E_j - E_m$ and hence contributes a multiple of the Dirac mass $\delta_{E_j - E_m}$. There are only finitely many such terms. Thus the remaining part of the Fourier transform is a finite pure-point measure. Since (3.31) is the zero distribution, its Fourier transform is the zero distribution. The Lebesgue decomposition of a locally finite measure into absolutely continuous and singular parts is unique. Therefore the displayed scalar density vanishes for almost every δ . To choose the exceptional set independently of x , take the exhausting compact intervals $K_m = [-m, m]$ and, for each m , a countable subset of $C_c^\infty((-m, m))$ dense in $L^1(K_m)$. After removing the union of the corresponding null sets, as well as the null sets on which the chosen $C(K_m)$ -valued representatives are undefined, the integral of $G_{ac}(\delta, \cdot)$ against every member of every such dense set is zero. For a fixed admissible δ , the function $G_{ac}(\delta, \cdot)$ is continuous on K_m . If it were nonzero at one point, continuity would provide an L^1 test function for which the integral is nonzero. Approximation of that test function by the chosen countable dense family would give the same conclusion for one member of the family, a contradiction. Hence

$$(3.36) \quad G_{ac}(\delta, x) = 0 \quad \text{for every } x \in \mathbb{R}$$

for almost every $\delta \neq 0$. Intersect this full-measure set with the one on which every $q_{\alpha\beta,\delta}$ is integrable.

For each remaining δ , equation (3.36) is precisely a relation of the form (3.14). The mixed products are its finitely many atomic states in the energy-sum variable: their sums are $s_j^+(\delta) = \delta - 2\kappa_j^2$ for $\delta > \kappa_j^2$ and $s_j^-(\delta) = -\delta - 2\kappa_j^2$ for $\delta < -\kappa_j^2$, both strictly inside the continuum threshold $|s| < |\delta|$. Proposition 3.7 therefore gives $q_{\alpha\beta,\delta} = 0$ almost everywhere in s and $a_{j\alpha}^\pm(\delta) = 0$ wherever defined. This conclusion holds for almost every $\delta \neq 0$. Under the diffeomorphism $(k, \ell) \mapsto (\delta, s)$, a null set in the (δ, s) plane pulls back to a null set in $(0, \infty)^2$, because the inverse map is locally Lipschitz away from the boundary. Formula (3.34) now gives

$$(3.37) \quad Q_{\alpha\beta}(k, \ell) = 0 \quad \text{for every } \alpha, \beta \text{ and almost every } (k, \ell) \in \mathbb{R}_+^2.$$

The omitted set $k^2 = \ell^2$ is the diagonal and has two-dimensional measure zero. Since $k \mapsto \pm(k^2 + \kappa_j^2)$ is a diffeomorphism away from $k = 0$, the vanishing of the mixed atomic coefficients gives

$$(3.38) \quad \overline{c_j g_\alpha(k)} = d_j \overline{h_\alpha(k)},$$

$$(3.39) \quad g_\alpha(k) \overline{c_j} = h_\alpha(k) \overline{d_j}$$

for every j, α and almost every $k > 0$.

Step 3: the bound-state block. The part just analyzed has an L_{loc}^1 temporal-frequency density, so it has no point masses. The pure-point part of the temporal Fourier transform of (3.31) is therefore the finite sum of the bound-bound terms. Dirac masses at distinct gaps are linearly independent as distributions: one tests with a smooth function supported near one gap and vanishing near all the others. Hence, for each fixed gap γ and every spatial test function φ ,

$$\int_{\mathbb{R}} \varphi(x) \sum_{E_j - E_m = \gamma} (c_j \overline{c_m} - d_j \overline{d_m}) \phi_j(x) \phi_m(x) dx = 0.$$

The finite sum in the integrand is continuous. Its vanishing as a distribution therefore gives the pointwise identity

$$(3.40) \quad \sum_{E_j - E_m = \gamma} (c_j \overline{c_m} - d_j \overline{d_m}) \phi_j(x) \phi_m(x) = 0.$$

By (3.5), the summands have nonzero leading terms with rates

$$e^{-(\kappa_j + \kappa_m)x} \quad (x \rightarrow +\infty).$$

For a fixed gap

$$\gamma = E_j - E_m = \kappa_m^2 - \kappa_j^2,$$

these rates are distinct. Indeed, knowing γ and $S = \kappa_j + \kappa_m$ determines

$$\kappa_m - \kappa_j = \frac{\gamma}{S},$$

and hence determines the ordered pair (κ_j, κ_m) . There are finitely many terms in each fixed-gap sum. Choose the smallest value of $\kappa_j + \kappa_m$, multiply (3.40) by $e^{(\kappa_j + \kappa_m)x}$, and let $x \rightarrow +\infty$. Every faster-decaying term tends to zero, while the selected term tends to its coefficient times the nonzero product of the leading tail constants. Its coefficient must vanish. Remove that term and repeat.

The finiteness of the sum yields

$$(3.41) \quad c_j \overline{c_m} = d_j \overline{d_m} \quad \text{for all } j, m.$$

3.6. Recovery of the initial state. Put

$$U = \mathcal{F}u_0, \quad W = \mathcal{F}v_0$$

in

$$\mathcal{H} = \mathbb{C}^N \oplus L^2(\mathbb{R}_+; \mathbb{C}^2).$$

For $Y \in \mathcal{H}$, define the rank-one operator generated by U by

$$(U \otimes \overline{U})Y = \langle Y, U \rangle_{\mathcal{H}} U.$$

Its kernel has four blocks: bound–bound, bound–continuous, continuous–bound, and continuous–continuous. Equation (3.41) identifies the first block; (3.38)–(3.39) identify the two off-diagonal blocks; and (3.37) identifies the final block. Therefore

$$(3.42) \quad U \otimes \overline{U} = W \otimes \overline{W}$$

as operators on $\mathcal{H}$.

If $U = 0$, the left side is zero. Taking the inner product of the right side with W gives $\|W\|_{\mathcal{H}}^4 = 0$, so $W = 0$; take $\zeta = 1$. Suppose $U \neq 0$. The range of $U \otimes \overline{U}$ is $\text{span}\{U\}$. Equality (3.42) shows that its range also equals $\text{span}\{W\}$, so $W = \zeta U$ for a nonzero scalar ζ . Substitution in (3.42) gives

$$U \otimes \overline{U} = |\zeta|^2 U \otimes \overline{U}.$$

Since the operator on the left is nonzero, $|\zeta| = 1$. In either case, since $\mathcal{F}$ is unitary,

$$v_0 = \zeta u_0.$$

This proves Theorem B.

4. OBSERVATION ON ONE EXTERIOR HALF-LINE

We prove Theorem C. We first isolate the Fourier statement used in the proof. For $F \in L^2(\mathbb{R})$ set

$$\mathcal{E}_t F = \int_{\mathbb{R}} F(k) e^{iky - ik^2 t} dk$$

as an L_y^2 inverse Fourier transform.

Lemma 4.1 (Free half-line model). *Let $0 < \kappa_1 < \dots < \kappa_N$, let $a_j, b_j \in \mathbb{C}$, and let $F, G \in L^2(\mathbb{R})$. Suppose*

$$\begin{aligned} U(t, y) &= \sum_{j=1}^N a_j e^{-\kappa_j y} e^{i\kappa_j^2 t} + \mathcal{E}_t F(y), \\ W(t, y) &= \sum_{j=1}^N b_j e^{-\kappa_j y} e^{i\kappa_j^2 t} + \mathcal{E}_t G(y) \end{aligned}$$

in $L^2((0, \infty)_y)$ for every t , and assume $|U(t, y)| = |W(t, y)|$ for almost every $(t, y) \in \mathbb{R} \times (0, \infty)$. Then there is $\zeta \in \mathbb{T}$ such that

$$G = \zeta F \quad \text{a.e. on } \mathbb{R}, \quad b_j = \zeta a_j \quad (1 \leq j \leq N).$$

The assertion includes $N = 0$.

Proof. Step 1: time continuity. Plancherel's theorem gives $\|\mathcal{E}_t F\|_{L^2(\mathbb{R})} = \sqrt{2\pi} \|F\|_2$, independently of t . Moreover, dominated convergence in Fourier space shows that $t \mapsto \mathcal{E}_t F$ is continuous into $L^2(\mathbb{R})$. The finite bound-state sum is also continuous into $L^2(0, \infty)$. Hence $t \mapsto U(t), W(t)$ are continuous into $L^2(0, \infty)$. The elementary estimate

$$\||a|^2 - |b|^2\|_1 \leq \|a - b\|_2 (\|a\|_2 + \|b\|_2)$$

shows that $t \mapsto |U(t)|^2 - |W(t)|^2$ is continuous into $L^1(0, \infty)$. It vanishes for almost every t by hypothesis, and therefore vanishes for every t .

Step 2: a Cauchy-transform uniqueness statement. Let $J \subset \mathbb{R}$ be a bounded open interval, let $m \in L^1(J)$, and suppose

$$C_m(z) = \int_J \frac{m(\eta)}{z - i\eta} d\eta, \quad z \in \mathbb{C} \setminus iJ,$$

extends holomorphically through iJ , then $m = 0$ almost everywhere on J . Indeed,

$$H(w) := iC_m(iw) = \int_J \frac{m(\eta)}{w - \eta} d\eta$$

extends through J . Let $K \Subset J$ and $\phi \in C_c^\infty(K)$. For $\varepsilon > 0$ small enough, Fubini's theorem and

$$\frac{1}{x + i\varepsilon - \eta} - \frac{1}{x - i\varepsilon - \eta} = \frac{-2i\varepsilon}{(x - \eta)^2 + \varepsilon^2}$$

give

$$\begin{aligned} & \int_K (H(x + i\varepsilon) - H(x - i\varepsilon)) \phi(x) dx \\ &= -2\pi i \int_J m(\eta) (P_\varepsilon * \phi)(\eta) d\eta, \quad P_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \end{aligned}$$

Because the extended function H is uniformly continuous on a compact neighborhood of K , the left side tends to zero. The Poisson kernels form an approximate identity, so $P_\varepsilon * \phi \rightarrow \phi$ uniformly. Since $m \in L^1(J)$, the right side tends to $-2\pi i \int_J m(\eta) \phi(\eta) d\eta$. Thus m vanishes as a distribution on every $K \Subset J$, and hence almost everywhere on J .

Step 3: the time-space transform. Fix $\chi \in C_c^\infty(\mathbb{R})$ and define

$$\check{\chi}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \chi(\omega) d\omega.$$

The Fourier inversion formula gives

$$(4.1) \quad \int_{\mathbb{R}} \check{\chi}(t) e^{i\omega t} dt = \chi(\omega).$$

For $\Re z > 0$, multiply $|U|^2 - |W|^2$ by $\check{\chi}(t) e^{-zy}$ and integrate in (t, y) . This integral is absolutely convergent. Indeed, if $a = \Re z > 0$, then

$$\int_0^\infty e^{-ay} |\mathcal{E}_t F(y)|^2 dy \leq \|\mathcal{E}_t F\|_2^2 = 2\pi \|F\|_2^2$$

uniformly in t , and the same estimate holds for G . Each bound-state sum has a uniformly bounded exponentially weighted L_y^2 norm. Since $\check{\chi} \in L^1(\mathbb{R})$, integration in t is legitimate.

For Schwartz F, G , expand the density difference into free–free, bound–free, free–bound, and bound–bound terms. For example, the free–free term is

$$\iint_{\mathbb{R}^2} q(k, \ell) e^{i(k-\ell)y} e^{i(\ell^2-k^2)t} dk d\ell.$$

Using (4.1) and

$$\int_0^\infty e^{-zy} e^{i(k-\ell)y} dy = \frac{1}{z - i(k-\ell)}$$

produces Φ_χ . A bound–free term has spatial factor $e^{-(\kappa_j+ik)y}$ and temporal frequency $k^2 + \kappa_j^2$, producing $\Psi_{j,\chi}^+$. The opposite ordering produces $\Psi_{j,\chi}^-$, and two bound states produce R_χ . Thus

$$(4.2) \quad 0 = \Phi_\chi(z) + \sum_{j=1}^N (\Psi_{j,\chi}^+(z) + \Psi_{j,\chi}^-(z)) + R_\chi(z),$$

where

$$\begin{aligned} q(k, \ell) &= F(k)\overline{F(\ell)} - G(k)\overline{G(\ell)}, \\ r_j^+(k) &= a_j\overline{F(k)} - b_j\overline{G(k)}, \\ r_j^-(k) &= \overline{a_j}F(k) - \overline{b_j}G(k), \\ d_{jm} &= a_j\overline{a_m} - b_j\overline{b_m}, \end{aligned}$$

and

$$\begin{aligned} \Phi_\chi(z) &= \iint_{\mathbb{R}^2} q(k, \ell) \frac{\chi(\ell^2 - k^2)}{z - i(k - \ell)} dk d\ell, \\ \Psi_{j,\chi}^+(z) &= \int_{\mathbb{R}} r_j^+(k) \frac{\chi(k^2 + \kappa_j^2)}{z + \kappa_j + ik} dk, \\ \Psi_{j,\chi}^-(z) &= \int_{\mathbb{R}} r_j^-(k) \frac{\chi(-(k^2 + \kappa_j^2))}{z + \kappa_j - ik} dk, \\ R_\chi(z) &= \sum_{j,m=1}^N d_{jm} \frac{\chi(\kappa_j^2 - \kappa_m^2)}{z + \kappa_j + \kappa_m}. \end{aligned}$$

To justify (4.2) for arbitrary L^2 data, choose Schwartz functions F_n, G_n converging to F, G in L^2 . Let U_n, W_n be the corresponding functions with the bound-state coefficients unchanged. Plancherel gives convergence uniformly in t in L_y^2 , and the inequality

$$\| |p|^2 - |q|^2 \|_1 \leq \|p - q\|_2 (\|p\|_2 + \|q\|_2)$$

shows that the transformed density of U_n, W_n converges to that of U, W . The factor $e^{-\Re(z)y}$ only improves this estimate, and $\check{\chi} \in L_t^1$ permits integration in time.

It remains to pass to the limit on the spectral side. If $\text{supp } \chi \subset [-M, M]$, the kernel $\mathbf{1}_{\{|\ell^2 - k^2| \leq M\}}$ has uniformly bounded integrals in each variable. Schur's test therefore gives

$$\iint |F(k)F(\ell)| |\chi(\ell^2 - k^2)| dk d\ell \lesssim_\chi \|F\|_2^2,$$

and the corresponding bilinear estimate controls the difference between the kernels formed from F_n and F . The same statement holds for G . The mixed factors have compact k -support and are controlled by Cauchy–Schwarz. Thus every term in (4.2) converges to the displayed formula.

Step 4: recovery of the continuous spectral kernel. We separate the free–free term from the others by analytic continuation. The numerator of $\Psi_{j,\chi}^+$ has compact k -support. If $|\Re z| < \kappa_j/2$, then

$$|z + \kappa_j + ik| \geq \Re(z + \kappa_j) > \kappa_j/2.$$

The integrand and all its z -derivatives are therefore dominated by integrable functions on compact substrips. Thus $\Psi_{j,\chi}^+$ is holomorphic there. The same argument applies to $\Psi_{j,\chi}^-$. The poles of R_χ are the negative real numbers $-(\kappa_j + \kappa_m)$, so R_χ is holomorphic in a strip around the imaginary axis. Equation (4.2) now defines a holomorphic continuation of Φ_χ through that axis. Put

$$M_\chi(\xi) = \int_{\mathbb{R}} q(k, k - \xi) \chi(\xi^2 - 2k\xi) dk.$$

The preceding Schur estimate and Fubini’s theorem give $M_\chi \in L^1(\mathbb{R})$ and

$$\Phi_\chi(z) = \int_{\mathbb{R}} \frac{M_\chi(\xi)}{z - i\xi} d\xi.$$

For a bounded interval J , split this integral into its parts over J and $\mathbb{R} \setminus J$. The latter is holomorphic near iJ , so the Cauchy-transform fact applies to the former. An exhaustion by bounded intervals gives $M_\chi = 0$ almost everywhere for every χ .

This implies $q = 0$ almost everywhere on $\mathbb{R}^2$. We include the measure-theoretic step. Away from $\xi = 0$, the change of variables

$$\xi = k - \ell, \quad \omega = \ell^2 - k^2$$

has Jacobian $2|\xi|$ and inverse

$$k = \frac{\xi^2 - \omega}{2\xi}, \quad \ell = -\frac{\xi^2 + \omega}{2\xi}.$$

Let $I \Subset \mathbb{R} \setminus \{0\}$ and $J \Subset \mathbb{R}$. For $\theta \in C_c^\infty(I)$ and $\eta \in C_c^\infty(J)$, the identity $M_\eta = 0$ gives

$$0 = \int_I \theta(\xi) \int_{\mathbb{R}} q(k, k - \xi) \eta(\xi^2 - 2k\xi) dk d\xi.$$

On the support of the integrand, use (ξ, ω) as variables. Since $dk d\xi = (2|\xi|)^{-1} d\omega d\xi$, this becomes

$$\int_{I \times J} \frac{q(k(\xi, \omega), \ell(\xi, \omega))}{2|\xi|} \theta(\xi) \eta(\omega) d\xi d\omega = 0.$$

The pullback of q belongs to $L^2(I \times J)$, and $1/(2|\xi|)$ is bounded above and below there. Finite sums of the tensors $\theta(\xi)\eta(\omega)$ are dense in $L^2(I \times J)$. The weighted pullback, and hence the pullback itself, vanishes almost everywhere. Choose countable families of such intervals exhausting $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$. The inverse change of variables then gives $q(k, \ell) = 0$ away from the diagonal $k = \ell$. The diagonal has Lebesgue measure zero, so $q = 0$ almost everywhere on $\mathbb{R}^2$.

Consequently the Hilbert–Schmidt kernels of the two rank-one operators coincide:

$$F(k) \overline{F(\ell)} = G(k) \overline{G(\ell)} \quad \text{for almost every } (k, \ell).$$

If $F = 0$, Fubini's theorem shows that $G(k)\overline{G(\ell)} = 0$ almost everywhere, hence $G = 0$. If $F \neq 0$, the two nonzero rank-one operators have the same range, so $G = \zeta F$ for some nonzero scalar ζ . Substitution in the kernel identity gives $|\zeta|^2 = 1$. Thus $|\zeta| = 1$.

Step 5: recovery of the bound-state coefficients. First suppose $F \neq 0$. Replace W by $\bar{\zeta}W$; then $G = F$. We must prove $b_j = a_j$ in this normalization. Set

$$\Lambda = \{\kappa_j^2 - \kappa_m^2 : 1 \leq j, m \leq N\}.$$

Choose $\chi \in C_c^\infty((0, \infty) \setminus \Lambda)$. In (4.2), the free-free, negative-frequency mixed, and bound-bound terms vanish. Hence

$$(4.3) \quad \sum_{j=1}^N C_{j,\chi}(z) = 0, \quad C_{j,\chi}(z) = \int_{\mathbb{R}} \frac{(a_j - b_j)\overline{F(k)}\chi(k^2 + \kappa_j^2)}{z + \kappa_j + ik} dk.$$

The function $C_{j,\chi}$ is holomorphic on the half-plane $\Re z > -\kappa_j$. The sum in (4.3) vanishes on $\Re z > 0$, so the identity theorem makes it vanish wherever all terms are holomorphic.

We begin with $j = 1$. Every $C_{m,\chi}$ with $m > 1$ is holomorphic on a neighborhood of the line $\Re z = -\kappa_1$, because $-\kappa_1 > -\kappa_m$. On the right of that line,

$$C_{1,\chi}(z) = - \sum_{m=2}^N C_{m,\chi}(z).$$

The right side extends through the line, and therefore so does $C_{1,\chi}$. Put $w = z + \kappa_1$ and change k to $-\eta$. Then $C_{1,\chi}(w - \kappa_1)$ is the Cauchy transform of the compactly supported L^1 function

$$\eta \mapsto (a_1 - b_1)\overline{F(-\eta)}\chi(\eta^2 + \kappa_1^2).$$

The Cauchy-transform fact gives

$$(a_1 - b_1)\overline{F(k)}\chi(k^2 + \kappa_1^2) = 0 \quad \text{for almost every } k.$$

Thus $C_{1,\chi} = 0$. Remove it from (4.3). The same argument at $\Re z = -\kappa_2$ gives the result for $j = 2$. Induction gives

$$(a_j - b_j)\overline{F(k)}\chi(k^2 + \kappa_j^2) = 0 \quad \text{for almost every } k$$

for every j . Varying χ shows that $(a_j - b_j)F(k) = 0$ outside the finite set $\{k : k^2 + \kappa_j^2 \in \Lambda\}$. Since $F \neq 0$, one has $a_j = b_j$.

If $F = G = 0$, expansion of the modulus identity gives

$$\sum_{j,m=1}^N d_{jm} e^{-(\kappa_j + \kappa_m)y} e^{i(\kappa_j^2 - \kappa_m^2)t} = 0.$$

The left side is continuous, so the almost-everywhere identity holds for all (t, y) . Fix $y > 0$ and group terms having the same temporal frequency $\omega = \kappa_j^2 - \kappa_m^2$. Distinct exponentials $e^{i\omega t}$ are linearly independent: evaluating the first M derivatives at one time gives a Vandermonde system. Hence, for every ω ,

$$\sum_{\kappa_j^2 - \kappa_m^2 = \omega} d_{jm} e^{-(\kappa_j + \kappa_m)y} = 0 \quad (y > 0).$$

Within one frequency class, the numbers $\kappa_j + \kappa_m$ are distinct. Indeed, the difference of squares and the sum determine the ordered pair:

$$\kappa_j - \kappa_m = \frac{\omega}{\kappa_j + \kappa_m}.$$

A second Vandermonde argument, now in y , gives $d_{jm} = 0$ for all j, m . Hence $(a_1, \dots, a_N) \otimes (\bar{a}_1, \dots, \bar{a}_N)$ equals the corresponding rank-one matrix for b , and $b = \zeta a$ for some $\zeta \in \mathbb{T}$. This completes the proof. $\square$

Proof of Theorem C. Translation reduces the proof to $R = 0$. Standard one-dimensional scattering theory gives finitely many simple negative eigenvalues $-\kappa_1^2, \dots, -\kappa_N^2$, real normalized eigenfunctions ϕ_j , and two continuum channels $e_1(\cdot, k), e_2(\cdot, k)$, $k > 0$. Since $V = 0$ on $(0, \infty)$, square integrability excludes the growing exponential and therefore

$$\phi_j(x) = \alpha_j e^{-\kappa_j x}, \quad \alpha_j \neq 0.$$

The coefficient cannot vanish, since the eigenfunction and its derivative would then vanish on $(0, \infty)$ and ODE uniqueness would give $\phi_j \equiv 0$. The continuum functions may be normalized so that

$$e_1(x, k) = T(k) e^{ikx}, \quad e_2(x, k) = e^{-ikx} + R_2(k) e^{ikx}, \quad x > 0.$$

The coefficients T, R_2 are bounded, and $T(k) \neq 0$ for $k > 0$. Indeed, $T(k_0) = 0$ would make $e_1(\cdot, k_0)$ and its derivative vanish on $(0, \infty)$; ODE uniqueness would then give $e_1 \equiv 0$, contradicting its nonzero incoming asymptotic at $-\infty$.

Write the spectral coefficients of u_0 as $c_j; g_1, g_2$ and those of v_0 as $d_j; h_1, h_2$. Suppressing one common normalization constant, the spectral expansion on $x > 0$ is

$$\begin{aligned} e^{-itH} u_0(x) &= \sum_{j=1}^N c_j \alpha_j e^{-\kappa_j x} e^{i\kappa_j^2 t} \\ &\quad + \int_0^\infty e^{-ik^2 t} [T(k) g_1(k) e^{ikx} + g_2(k) e^{-ikx} + R_2(k) g_2(k) e^{ikx}] dk. \end{aligned}$$

Combine the two coefficients of e^{ikx} and change k to $-k$ in the coefficient of e^{-ikx} . The continuum integral becomes $\int_{\mathbb{R}} F_u(k) e^{ikx - ik^2 t} dk$, where

$$F_u(k) = \begin{cases} T(k) g_1(k) + R_2(k) g_2(k), & k > 0, \\ g_2(-k), & k < 0. \end{cases}$$

Define F_v analogously. The boundedness of T and R_2 implies $F_u, F_v \in L^2(\mathbb{R})$. Thus the restrictions of the two evolutions to the free half-line have exactly the form in Lemma 4.1. The half-line modulus identity and Lemma 4.1 give

$$F_v = \zeta F_u, \quad d_j \alpha_j = \zeta c_j \alpha_j.$$

Thus $d_j = \zeta c_j$. On the negative frequency half-line, $F_v = \zeta F_u$ gives $h_2 = \zeta g_2$. On the positive frequency half-line it then gives

$$T(k)(h_1(k) - \zeta g_1(k)) = 0 \quad \text{a.e. } k > 0.$$

Since $T(k) \neq 0$, $h_1 = \zeta g_1$. All spectral coefficients agree up to ζ , and unitarity of the spectral transform gives $v_0 = \zeta u_0$. $\square$

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