hexagon:2610.00143v4

A note on counterexamples to homological conjectures

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October 8, 2026

Abstract

Building on OpenAI's constructions, we extend the base field to any field containing an element of infinite multiplicative order. Over each such field, we construct finite-dimensional algebras giving counterexamples to both Tachikawa conjectures, the Auslander–Reiten and Gorenstein-projective conjectures, the Nakayama, generalized Nakayama, and strong Nakayama conjectures, the Auslander–Gorenstein conjecture, the little and big finitistic dimension conjectures, and the Wakamatsu tilting conjecture.

AI use and scholarly credit

Starting from OpenAI's Result No. 199, we used GPT-6-Astra-Pro to review the constructions and explore extensions to more general base fields and further homological conjectures. The resulting arguments underwent a separate adversarial review using GPT-6-Astra. The review prompts requested a fresh derivation of each lemma, proposition, and theorem, checks of the cited results and their scope, and suggestions for filling gaps and improving the exposition. The manuscript was then revised using the built-in mathematical writing skill.

By sharing this manuscript, I do not seek to assert ownership of the results or claim priority. My intention is to make this AI-generated work available for independent review. Once its correctness has been independently verified, the paper may be cited directly. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work. I also welcome further research, including extensions, new applications, and related questions. Additional references and detailed research materials are available from me at the email address given at the end of the paper.

1 The field of definition

All algebras in this note are associative and unital. Modules are finite-dimensional left modules unless specified otherwise. For a field k, write $D = \mathrm{Hom}_k(-, k)$. The Jacobson radical of an algebra R is denoted $\mathrm{rad}\, R$; the radical and socle of a module are the intersection of its maximal submodules and its largest semisimple submodule, respectively. We write pd_R and id_R for projective and injective dimension. A finite-dimensional k-algebra A is *symmetric* if $A \cong DA$ as bimodules. Equivalently, it has a linear form $t : A \to k$ for which $t(ab) = t(ba)$ and the pairing $(a, b) \mapsto t(ab)$ is nondegenerate. Such an algebra is self-injective. A module M is *self-orthogonal* if $\mathrm{Ext}^i_A(M, M) = 0$ for every $i > 0$.

2020 Mathematics Subject Classification. Primary 16E30; Secondary 16D50, 16E10. *Keywords.* Symmetric algebra, self-orthogonal module, Auslander–Reiten conjecture, Tachikawa conjectures, twisted trace.

The Auslander–Reiten conjecture asserts that $\operatorname{Ext}^i_R(N, N \oplus R) = 0$ for all $i > 0$ forces N to be projective. For self-injective algebras this is the second Tachikawa conjecture. OpenAI's constructions [1, 2] give counterexamples over the rational function field $\mathbb{F}_2(q, H_1, H_2)$ in three independent indeterminates. We follow the derived-induction construction of [2], replacing its ten-dimensional algebra and its untwisted character map. The replacement map takes values in a twisted dual bimodule; its factorization obstruction is the vanishing trace of an involution on each corner.

An algebra is *split* if its semisimple quotient is a product of full matrix algebras over k. A module is *Gorenstein-projective* if it is a cokernel in a doubly infinite exact complex of finite projectives that remains exact under $\operatorname{Hom}_R(-, R)$.

Theorem 1.1. *Let k be a field containing an element $q \in k^\times$ of infinite multiplicative order. There exist a finite-dimensional split symmetric k-algebra A and a finite-dimensional nonprojective A-module M such that*

$$\operatorname{Ext}^i_A(M, M) = 0 \qquad (i > 0).$$

The algebra A has eight isomorphism classes of simple modules, is noncommutative, and satisfies $(\operatorname{rad} A)^4 \neq 0$. There also exist a finite-dimensional lower triangular matrix k-algebra Λ and a nonprojective Gorenstein-projective module Z with

$$\operatorname{Ext}^i_\Lambda(Z, Z \oplus \Lambda) = 0 \qquad (i > 0).$$

The proof of Theorem 1.1 ends in Section 7.

Corollary 1.2. *Theorem 1.1 holds over every field of characteristic zero and over every positive-characteristic field not algebraic over its prime field.*

Proof. In characteristic zero take $q = 2$. In characteristic $p > 0$, choose an element q transcendental over $\mathbb{F}_p$. Each has infinite multiplicative order. □

Section 8 gives direct witnesses for the homological conjectures using A, Λ, and the endomorphism algebra of $A \oplus M$. The Auslander–Reiten conjecture holds for Igusa–Todorov algebras [6, Theorem 3.2], quantum complete intersections [7, Theorem 4.2], and split radical-cube-zero algebras [8, Theorem 1.1].

2 Complete resolutions and stable maps

For an algebra R, put $R^e = R \otimes_k R^{\mathrm{op}}$, where R^{op} is the opposite algebra, so left R^e-modules are R-bimodules. Unmarked tensor products are over k. Write $\operatorname{add} V$ for the direct summands of finite sums of copies of a module V. The space $\underline{\operatorname{Hom}}_R(U, V)$ is ordinary Hom modulo maps factoring through finite projectives; put $\underline{\operatorname{End}}_R(U) = \underline{\operatorname{Hom}}_R(U, U)$.

Complexes are homologically indexed: $d_j : P_j \to P_{j-1}$. For integers j, r, we use

$$P[r]_j = P_{j-r}, \qquad d_{P[r]} = (-1)^r d_P, \qquad c_j P = \operatorname{coker}(d_{j+1} : P_{j+1} \to P_j).$$

For complexes P, Q, the full Hom complex has

$$\operatorname{Hom}^a_R(P, Q) = \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(P_j, Q_{j-a}), \qquad \delta f = d_Q f - (-1)^a f d_P. \tag{1}$$

Its cohomology is denoted H^a, and $\operatorname{End}^\bullet_R(P) = \operatorname{Hom}^\bullet_R(P, P)$. A module in a complex argument is placed in degree zero. For a map f, the notation $f[r]$ denotes its shifted map;

juxtaposition of composable maps means composition from right to left. For a chain map $f : P \to Q$,

$$\mathrm{Cone}(f)_j = Q_j \oplus P_{j-1}, \qquad d_{\mathrm{Cone}(f)} = \begin{pmatrix} d_Q & f \\ 0 & -d_P \end{pmatrix}.$$

An exact complex of finite projectives is *totally acyclic* when its Hom complex into R is exact. It is a complete resolution of $c_0 P$.

Lemma 2.1. *If P is totally acyclic and Q is an exact complex of finite projectives, then for every integer a,*

$$H^a \operatorname{Hom}_R(P, Q) \cong \underline{\operatorname{Hom}}_R(c_a P, c_0 Q) \cong H^a \operatorname{Hom}_R(P, c_0 Q). \tag{2}$$

The first identification respects shifted composition. For $a > 0$ these groups equal $\operatorname{Ext}^a_R(c_0 P, c_0 Q)$.

Proof. For a finite projective J, the complex $\operatorname{Hom}_R(P, J)$ is a summand of a finite sum of copies of $\operatorname{Hom}_R(P, R)$, so it is exact. If $b : c_j P \to J$, its composite with $P_j \twoheadrightarrow c_j P$ is a cocycle in degree j. Exactness expresses this composite as a map $P_{j-1} \to J$ composed with d_j, up to the differential sign. Thus b extends to P_{j-1}.

For any finite module N, a degree-a cocycle in $\operatorname{Hom}_R(P, N)$ is a map $P_a \to N$ vanishing on $\operatorname{im} d_{a+1}$, hence a map $c_a P \to N$. A boundary extends through the inclusion $c_a P \hookrightarrow P_{a-1}$ and therefore factors through a finite projective. Conversely, if $c_a P \to N$ factors through a finite projective J, the preceding extension property extends its first factor to P_{a-1}, so it is a boundary.

A map $c_0 P \to c_0 Q$ lifts to $P_0 \to Q_0$ and successively to positive degrees by projectivity. For the downward extension, the already defined map $d_Q f_j$ vanishes on $\operatorname{im} d^P_{j+1}$, hence defines a map $c_j P \to Q_{j-1}$. The extension property supplies f_{j-1}. This constructs a chain map in all degrees.

If the cokernel map factors through J, extend its first factor to P_{-1} and lift its second factor to Q_0. Their composite supplies a homotopy component removing the cokernel map. For the residual chain map inducing zero on cokernels, set $h_{-1} = 0$. Its degree-zero map has image in $\operatorname{im} d^Q_1$ and therefore lifts to $h_0 : P_0 \to Q_1$. Inductively for $j > 0$, the chain-map identity and the equation already solved in degree $j-1$ show that $f_j - h_{j-1} d^P_j$ has image in $\ker d^Q_j = \operatorname{im} d^Q_{j+1}$. Projectivity of P_j gives h_j.

For the downward induction, suppose the homotopy equation holds in degree $j+1$. Then $f_j - d^Q_{j+1} h_j$ vanishes on $\operatorname{im} d^P_{j+1}$, and hence defines a map $c_j P \to Q_j$. Since Q_j is projective, the extension property extends this map to $h_{j-1} : P_{j-1} \to Q_j$. Beginning at $j = -1$ constructs all negative components. We obtain

$$f_j = d^Q_{j+1} h_j + h_{j-1} d^P_j \qquad (j \in \mathbb{Z}),$$

a full nullhomotopy. Replacing P by $P[-a]$ proves the assertion in degree a, with the signs of (1). Cokernel maps preserve composition. The nonnegative tail of P is a projective resolution, which proves the positive Ext assertion. □

Over a symmetric algebra R, finite projectives are injective. Splicing a projective resolution with a projective-injective coresolution gives a complete resolution of every finite module. Its stable shift is $U[r] = c_{-r} P$; thus $[1] = \Omega^{-1}$, where ΩU is the kernel of a projective cover of U, a surjection from a finite projective module whose kernel lies in its radical. Short exact sequences give stable triangles and long exact stable Hom sequences. In particular,

$$H^a \operatorname{Hom}_R(P, V) = \underline{\operatorname{Hom}}_R(U, V[a]).$$

The following form of Tate duality is also recorded in [3, Section 2].

Lemma 2.2. *For symmetric R there are natural isomorphisms*

$$D\underline{\mathrm{Hom}}_R(U,V) \cong \underline{\mathrm{Hom}}_R(V,U[-1]).$$

There is a linear form $\lambda_U : \underline{\mathrm{Hom}}_R(U,U[-1]) \to k$ such that the resulting perfect pairing is $(f,g) \mapsto \lambda_U(gf)$.

Proof. For $J = R$, evaluation gives $\mathrm{Hom}_R(R,V) = V$, and the symmetric bimodule isomorphism $R \cong DR$ identifies $\mathrm{Hom}_R(V,R)$ with DV. These identifications extend to finite sums of R and then to their summands, giving the natural isomorphism

$$D\,\mathrm{Hom}_R(J,V) \cong \mathrm{Hom}_R(V,J)$$

for every finite projective J. Let P be a complete resolution of U. Dualizing the three terms that compute $H^0\,\mathrm{Hom}_R(P,V)$ gives the homology at $\mathrm{Hom}_R(V,P_0)$ of

$$\mathrm{Hom}_R(V,P_1) \longrightarrow \mathrm{Hom}_R(V,P_0) \longrightarrow \mathrm{Hom}_R(V,P_{-1}).$$

Since $\ker d_0 = \mathrm{im}\, d_1 \cong c_1P$, this homology is

$$\mathrm{Hom}_R(V,c_1P)/\,\mathrm{im}\big(\mathrm{Hom}_R(V,P_1) \to \mathrm{Hom}_R(V,c_1P)\big).$$

Every factorization through a projective lifts through $P_1 \twoheadrightarrow c_1P$, so this is the asserted stable Hom space. Define λ_U by pairing with 1_U. Naturality in V identifies the pairing with composition. Shifting gives the same assertion for all stable degrees. □

We use $\mathrm{D}^b(R)$ for the derived category of bounded complexes of finite modules. A *perfect* object is one represented by a bounded complex of finite projectives. A bounded derived object has a bounded-below projective model with finite terms and bounded homology. Such a model computes derived maps by chain maps modulo homotopy [4, Proposition 1.4 and Section 3.2(a)].

If $P_j = 0$ for $j < N$, Q is acyclic, and $f : P \to Q$ is a chain map, set $h_{N-1} = 0$. Once h_{j-1} has been chosen, the map $f_j - h_{j-1}d_j^P$ takes values in $\ker d_j^Q$. The surjection $Q_{j+1} \twoheadrightarrow \ker d_j^Q$ and projectivity of P_j give $h_j : P_j \to Q_{j+1}$ with

$$f_j = d_{j+1}^Q h_j + h_{j-1}d_j^P.$$

Induction constructs a nullhomotopy. The same argument after every shift makes the full Hom complex into an acyclic target exact. Applying Hom to the cone of a quasi-isomorphism therefore gives an isomorphism on homotopy classes. A roof representing a derived map can consequently be replaced by a chain map from P, uniquely up to homotopy.

To construct a projective model of a bounded complex, filter it by the subcomplexes supported in degrees at most j. Each successive quotient is a single module in degree j. Starting with a projective resolution of the lowest term, choose a projective resolution of each successive quotient, use the comparison above to represent the connecting morphism to the previous model shifted by [1] by a chain map, and take its cone shifted by $[-1]$. There are finitely many stages; each cone remains bounded below with finite projective terms. Dually, a bounded-above complex of injectives computes derived maps into it.

Lemma 2.3. *For symmetric R and such a projective model K, the stable object*

$$\langle K\rangle = (c_nK)[n] \qquad (n \text{ above its homology support})$$

is independent of n, functorial on derived maps, and compatible with shifts and cones. A derived map with zero stable image factors through a perfect object.

Proof. In the exact tail, the sequences $0 \to c_{n+1}K \to K_n \to c_nK \to 0$ identify adjacent choices. A chain homotopy changes a cokernel map by a projective factorization. High cokernels of a degreewise split short exact sequence form a short exact sequence; its connecting map is induced, with the shift signs, by the cross differential. This proves compatibility with triangles and cones.

For the last assertion, represent the map by $f : P \to Q$. In sufficiently high degree, write a factorization $c_nP \to J \to c_nQ$ of $c_n(f)$ through a finite projective J. Extend the first map along $c_nP \hookrightarrow P_{n-1}$ and lift the second along $Q_n \twoheadrightarrow c_nQ$. Their composite is $h_{n-1} : P_{n-1} \to Q_n$. Subtracting $dh+hd$ with this component makes the induced map on c_nP zero. The remaining degree-n map therefore lands in $\operatorname{im} d^Q_{n+1}$ and lifts to $h_n : P_n \to Q_{n+1}$. Continuing upwards constructs h_j successively and makes the homotopic map f' zero in every degree $j \geq n$. The chain equation then says $f'_{n-1}d^P_n = 0$. Consequently f' factors through the bounded projective subcomplex of Q supported in degrees at most $n-1$. □

For a symmetric algebra E, the algebra E^e is symmetric as well: the tensor product of the symmetric forms on E and E^{op} is a nondegenerate symmetric form on E^e. We use finite bimodules projective separately on the left and right. Their bimodule syzygies and cosyzygies can be formed using finite E^e-projective-injective modules. The defining short exact sequences split on each E-side: for syzygies their quotient is projective, and for cosyzygies their submodule is projective and hence injective over the symmetric algebra E.

For an E-module V, write $LV = L \otimes_E V$ and $L(f) = 1_L \otimes f$ for a module map f. Evaluation $L \mapsto LV$ preserves these sidewise split sequences. An E^e-free module is a sum of copies of $E \otimes E$, and $(E \otimes E) \otimes_E V \cong E \otimes V$ is left E-free. Thus evaluation sends E^e-projectives to E-projectives and induces a functor on stable categories. Applying it to the defining sequences shows that it commutes with stable shifts and their connecting maps. If L is projective on both sides, then $L \otimes_E -$ is exact and sends an E-projective to an E-projective: the latter is a summand of a finite sum of copies of the left module L.

3 A signed ten-dimensional algebra

Fix $q \in k^\times$ of infinite order. Let C have basis

$$e, f, x, y, z, u, v, t, j, n$$

with corner spaces

$$eCe = \langle e, x, y, z\rangle_k, \quad eCf = \langle u, v\rangle_k, \quad fCe = \langle t, j\rangle_k, \quad fCf = \langle f, n\rangle_k.$$

The brackets denote k-linear span. The elements e, f are orthogonal idempotents with sum 1, acting according to these corners. The nonzero products of the other basis elements are

$$\begin{gathered} xy = -qz, \quad yx = z, \quad xu = v, \quad yu = -v, \\ tx = j, \quad ty = -q^2 j, \quad ut = y + qx, \quad vt = -qz, \\ uj = z, \quad tu = (q-1)n, \quad nt = -qj, \quad un = v. \end{gathered} \tag{3}$$

All omitted products of these eight elements are zero. For any displayed basis, a star denotes its dual basis in the k-linear dual: each functional is one on the corresponding basis vector and zero on the others.

Lemma 3.1. *The table defines an associative algebra with radical spanned by the eight nonidempotent basis elements. Its radical has fifth power zero and fourth power nonzero. The assignment*

$$\gamma(e, f, z, u, j) = (e, f, z, u, j), \qquad \gamma(x, y, v, t, n) = (-x, -y, -v, -t, -n) \tag{4}$$

defines an involutive automorphism. Write Tr_k *for the ordinary linear trace, the sum of diagonal entries in a* k*-basis. For* $a, b \in \{e, f\}$*,*

$$\mathrm{Tr}_k(\gamma|_{aCb}) = 0. \tag{5}$$

Proof. Give u, t degree one, x, y, n degree two, v, j degree three, and z degree four. Products in (3) preserve this grading. A radical triple of total degree greater than four has both products zero. The composable radical triples of total degree at most four are

$$utu, tut, xut, yut, ntu, unt, txu, tyu, utx, uty, tun.$$

In this order both bracketings have values

$$(q-1)v, -q(q-1)j, -qz, qz, 0, -qz, 0, 0, z, -q^2z, 0.$$

Triples involving idempotents follow from the corners. Thus multiplication is associative. The positive-degree ideal is nilpotent with quotient k^2, so it is the radical. Moreover $utut = q(1-q)z \neq 0$.

The signs in (4) multiply correctly in each of the twelve nonzero products, and preserve zero products and corners. The eigenvalue lists on the four corners are $(1, -1, -1, 1)$, $(1, -1)$, $(-1, 1)$, and $(1, -1)$, whose sums vanish in every characteristic. □

Let s be the one-dimensional left C-module on which f acts as 1 and all other basis elements act as zero. For $i \in \mathbb{Z}$ set $\ell_i = x + q^i y$. Write ρ_a for right multiplication by a between the appropriate left ideals, and λ_a for left multiplication between right ideals.

Proposition 3.2. *A projective resolution* R *of* s *is*

$$R_0 = Cf, \quad R_i = Ce \ (i \geq 1), \qquad d_1 = \rho_u, \quad d_{i+2} = \rho_{\ell_i} \ (i \geq 0). \tag{6}$$

It satisfies

$$H^a \operatorname{Hom}_C(R, s) = \begin{cases} k & a = 0, \\ 0 & a \neq 0, \end{cases} \qquad H^a \operatorname{Hom}_C(R, C) = \begin{cases} Ds & a = 2, \\ 0 & a \neq 2, \end{cases}$$

where the second identification is one of right C*-modules.*

Proof. On the basis (e, x, y, z, t, j) of Ce, the images under the differentials are

$$\rho_u = (u, v, -v, 0, (q-1)n, 0), \qquad \rho_{\ell_i} = (\ell_i, -q^{i+1}z, z, 0, (1-q^{i+2})j, 0).$$

Since $q-1$ and $1-q^{i+2}$ are nonzero for $i \geq 0$, the image and kernel of ρ_u are $\langle u, v, n\rangle_k$ and $\langle \ell_0, z, j\rangle_k$. Those of ρ_{ℓ_i} are $\langle \ell_i, z, j\rangle_k$ and $\langle \ell_{i+1}, z, j\rangle_k$. The first image is $\mathrm{rad}(Cf)$, proving exactness. Applying $\operatorname{Hom}_C(-, s)$ leaves only the degree-zero term k.

Applying $\operatorname{Hom}_C(-, C)$ gives

$$fC \xrightarrow{\lambda_u} eC \xrightarrow{\lambda_{\ell_0}} eC \xrightarrow{\lambda_{\ell_1}} eC \longrightarrow \cdots.$$

On (e, x, y, z, u, v),

$$\lambda_{\ell_i} = (\ell_i, q^i z, -qz, 0, (1-q^i)v, 0).$$

The first map sends (f, t, j, n) to $(u, y + qx, z, v)$ and is injective. At $i = 0$ the image is $\langle \ell_0, z\rangle_k$ and the kernel is $\langle \ell_{-1}, z, u, v\rangle_k$. For $i \geq 1$ the image and kernel are $\langle \ell_i, z, v\rangle_k$ and $\langle \ell_{i-1}, z, v\rangle_k$. The sole cohomology class is $\overline{v}$ in degree two. It satisfies $vf = v$, and its only nonzero radical right product is $vt = -qz$, a boundary. This identifies the right module as Ds. □

Proposition 3.3. *The dual regular module DC has projective dimension at most two on both sides. More precisely,*

$$D(eC) \cong Ce, \qquad D(Ce) \cong eC,$$

and there are exact sequences

$$0 \to Cf \xrightarrow{\rho_t} Ce \xrightarrow{\rho_{\ell_{-2}}} Ce \xrightarrow{e \mapsto j^*} D(fC) \to 0, \tag{7}$$

$$0 \to fC \xrightarrow{\lambda_u} eC \xrightarrow{\lambda_{\ell_0}} eC \xrightarrow{e \mapsto v^*} D(Cf) \to 0. \tag{8}$$

Proof. The coefficient of z means the scalar multiplying z in the displayed basis of C. Taking this coefficient in the product defines a pairing $eC \times Ce \to k$. In the ordered bases (e, x, y, z, u, v) and (e, x, y, z, t, j) its matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -q & 0 \end{pmatrix}, \qquad \det = -q^2.$$

It is nondegenerate. For $a \in eC$, $b \in Ce$, and $c \in C$, associativity gives equality of the z-coefficients of $(ac)b$ and $a(cb)$. Thus the adjoints of the pairing are a left-module isomorphism $Ce \to D(eC)$ and a right-module isomorphism $eC \to D(Ce)$.

The maps labelled $e \mapsto j^*$ and $e \mapsto v^*$ are module maps because $ej^* = j^*$ in $D(fC)$ and $v^*e = v^*$ in $D(Cf)$. They are determined by the image of the generator e. The last map in (7) has images $(j^*, t^*, -q^2t^*, 0, -qn^*, f^*)$ and kernel $\langle \ell_{-2}, z \rangle_k$. The middle map has that image and kernel $\langle y + qx, z, t, j \rangle_k$. The first map has the independent images $(t, y + qx, -qz, -qj)$. In (8) the last map has images $(v^*, u^*, -u^*, 0, n^*, f^*)$ and kernel $\langle \ell_0, z \rangle_k$. The middle map has that image and kernel $\langle y + qx, z, u, v \rangle_k$, the image of the injective first map. □

For any finite-dimensional algebra R, its trivial extension is $R \ltimes DR$, with multiplication

$$(a, \phi)(b, \psi) = (ab, a\psi + \phi b), \qquad (a\phi b)(c) = \phi(bca).$$

It is symmetric: $(a, \phi) \mapsto \phi(1)$ gives the pairing $((a, \phi), (b, \psi)) \mapsto \phi(b) + \psi(a)$. Set

$$T = C \ltimes DC, \qquad B = C \otimes C \subset E = T \otimes T, \qquad X = s \otimes s. \tag{9}$$

The C-module s is viewed as a T-module via $T \to C$, and similarly for X. The algebra E is symmetric of dimension 400. Tensoring the resolutions in Proposition 3.3 gives

$$\mathrm{pd}_B({}_BE) \le 4, \qquad \mathrm{pd}_{B^{\mathrm{op}}}(E_B) \le 4, \qquad \mathrm{id}_B({}_BB) \le 4. \tag{10}$$

For the first two bounds, $T = C \oplus DC$ has projective dimension at most two as a left and as a right C-module. The total tensor product over k of two corresponding length-two resolutions has length four and resolves $T \otimes T$ over $C \otimes C$. Its terms are projective because an external tensor product of finite projectives is a summand of a finite free B-module. Tensoring over a field preserves these resolutions: a complex of vector spaces splits into its homology and contractible two-term summands. The same construction gives a right projective resolution of $DB = DC \otimes DC$ of length at most four. Dualizing it gives the last bound in (10).

4 Self-extensions and twisting weights

The Yoneda algebra $\operatorname{Ext}^*_T(s,s) = \bigoplus_{a\geq 0} \operatorname{Ext}^a_T(s,s)$ has the product obtained by splicing extensions; $|\tau|$ denotes the cohomological degree of a homogeneous class τ.

Proposition 4.1. *There is a derived triangle of T-modules*

$$s[2] \longrightarrow \mathcal{J} := T \otimes_C R \longrightarrow s \xrightarrow{\tau} s[3], \tag{11}$$

and $\operatorname{Ext}^*_T(s,s) = k[\tau]$ *with* $|\tau| = 3$.

Proof. Tensor $0 \to DC \to T \to C \to 0$ with the projective terms of R. The resulting exact sequence has left complex $DC \otimes_C R$. The evaluation isomorphism

$$DC \otimes_C R_j \cong D\operatorname{Hom}_C(R_j, C), \qquad \phi \otimes v \mapsto (f \mapsto \phi(f(v)))$$

and Proposition 3.2 identify its only homology as s in degree two. The square-zero ideal DC annihilates $DC \otimes_C R$, so this identifies the homology as a T-module through $T \to C$. A complex with homology in just one degree is isomorphic in the derived category to that homology placed in that degree: replace the degree-two term first by its cycles and then quotient by its boundaries, obtaining quasi-isomorphisms. The quotient complex is R, viewed as a T-complex through $T \to C$, and its augmentation is a quasi-isomorphism to s. The short exact sequence of complexes therefore gives (11). Its middle term is a bounded-below projective model, and adjunction gives

$$\operatorname{Hom}_{\mathrm{D}^b(T)}(\mathcal{J}, s[i]) = H^i \operatorname{Hom}_C(R, s) = \begin{cases} k & i = 0, \\ 0 & i \neq 0. \end{cases}$$

The long exact Hom sequence gives $\operatorname{Ext}^1_T(s,s) = \operatorname{Ext}^2_T(s,s) = 0$ and, for $i \geq 2$, an isomorphism

$$\operatorname{Ext}^{i-3}_T(s,s) \longrightarrow \operatorname{Ext}^i_T(s,s), \qquad \alpha \longmapsto \alpha[3]\tau,$$

with negative ordinary Ext zero. Hence τ^m is a basis in degree $3m$ for every $m \geq 0$. □

Write $\mathcal{H}^a = \underline{\operatorname{Hom}}_E(X, X[a])$ for $a \in \mathbb{Z}$ and $\mathcal{H}^{\geq 0} = \bigoplus_{a\geq 0} \mathcal{H}^a$, and let $\tau_1 = \tau \otimes 1$, $\tau_2 = 1 \otimes \tau$ be the external classes. Products mean shifted composition: for f of degree r and g of degree s, $fg = f[s] \circ g$. Let $\mathcal{P}_m$ be the span of the ordered monomials $\tau_1^a \tau_2^b$ with $a, b \geq 0$ and $a + b = m$; put $\mathcal{P}_m = 0$ for $m < 0$.

Proposition 4.2. *The module X is nonprojective, and*

$$\mathcal{H}^{\geq 0} = k\langle \tau_1, \tau_2 \rangle / (\tau_1\tau_2 + \tau_2\tau_1), \qquad \mathcal{H}^{3m} = \mathcal{P}_m, \quad \mathcal{H}^{-3m-1} = D\mathcal{P}_m \quad (m \geq 0). \tag{12}$$

All other stable degrees are zero. For $a, b \geq 0$, let $\eta_{a,b} \in D\mathcal{P}_{a+b}$ be the coefficient functional of $\tau_1^a \tau_2^b$ in the ordered monomial basis. With the duality identification specified in the proof, left multiplication is

$$\tau_1 \eta_{a,b} = \begin{cases} (-1)^b \eta_{a-1,b} & a > 0, \\ 0 & a = 0, \end{cases} \qquad \tau_2 \eta_{a,b} = \begin{cases} \eta_{a,b-1} & b > 0, \\ 0 & b = 0. \end{cases} \tag{13}$$

Proof. The tensor product of two projective resolutions of s, with $|v|$ denoting the homological degree of a homogeneous element and differential $d(v \otimes w) = dv \otimes w + (-1)^{|v|} v \otimes dw$, resolves X. In each total degree the tensor resolution has finitely many terms. The natural termwise Hom isomorphisms identify its Hom complex into $s \otimes s$ with the tensor product

over k of the two Hom complexes into s. Splitting complexes of vector spaces into their homology and contractible summands shows that its cohomology is the tensor product of their cohomologies.

For homogeneous comparison maps f, g of cohomological degrees r, s, respectively, the external tensor map is

$$(f \otimes g)(v \otimes w) = (-1)^{s|v|} f(v) \otimes g(w).$$

Composing two such maps gives the sign $(-1)^{sr'}$ when the second map on the first factor has degree r'. Thus multiplication on cohomology is the graded tensor product multiplication. In particular $\tau_2\tau_1 = -\tau_1\tau_2$, while the classes $\tau_1^a\tau_2^b$ correspond to the tensor basis $\tau^a \otimes \tau^b$. The degree-three group is nonzero, so X is nonprojective. Since $\mathrm{End}_E(X) = k$, its stable degree-zero endomorphism ring is k as well.

Lemma 2.2 gives the negative degrees. Choose a nonzero linear form $\lambda : \mathcal{H}^{-1} \to k$ and identify a negative class $\beta \in \mathcal{H}^{-3m-1}$ with the functional

$$p \longmapsto \lambda(p[-3m-1]\beta), \qquad p \in \mathcal{P}_m.$$

Associativity makes postcomposition by τ_i transpose to right multiplication by τ_i on ordered monomials. Right multiplication by τ_1 introduces $(-1)^b$; right multiplication by τ_2 introduces no sign. This gives (13), including its zero boundary at $m = 0$. □

For $H \in k^\times$ define automorphisms of T by

$$h_H(c, \phi) = (c, H\phi), \qquad \widetilde{\gamma}(c, \phi) = (\gamma(c), \phi \circ \gamma^{-1}).$$

They commute. Put

$$\sigma_H = (\widetilde{\gamma} h_H) \otimes (\widetilde{\gamma} h_H), \qquad U_H = E_{\sigma_H}. \tag{14}$$

The notation E_{σ_H} means usual left action and right action $a \cdot b = a\sigma_H(b)$. Its tensor functor is restriction along σ_H^{-1}, via $a \otimes v \mapsto \sigma_H^{-1}(a)v$. Let $\chi : E \to k$ be the character of X. Since $\chi\sigma_H = \chi$, this fixes $U_H X = X$ and an action on every $\mathcal{H}^a$.

Lemma 4.3. *Let G_a be the invertible operator on $\mathcal{H}^a$ induced by the twist $E_{\widetilde{\gamma}\otimes\widetilde{\gamma}}$. For all $H \neq 0$ and $m \geq 0$,*

$$U_H|_{\mathcal{H}^{3m}} = H^{-m} G_{3m}, \qquad U_H|_{\mathcal{H}^{-3m-1}} = H^{m+2} G_{-3m-1}. \tag{15}$$

Proof. First consider the twist by h_H. Write $h_H(\tau)$ for the action of the tensor functor $T_{h_H} \otimes_T -$ on the class τ. The comparison on $\mathcal{J}$ is $a \otimes r \mapsto h_H(a) \otimes r$. Since h_H fixes C, this is well defined. It induces multiplication by H on the homology $s[2]$ of $DC \otimes_C R$ and the identity on the quotient s, so its connecting square gives $\tau = H h_H(\tau)$, or $h_H(\tau) = H^{-1}\tau$. On the two-factor algebra this yields the scalar H^{-m} in degree $3m$.

Put $p = f \otimes f$ and $\zeta = f^* \otimes f^* \in Ep$. The map $X \to Ep$, $1 \mapsto \zeta$, has image in $\mathrm{rad}(E)p = \Omega X$ and is nonzero stably. Indeed, every map $X \to E$ has image in the left socle. Under the symmetric bimodule isomorphism $E \cong DE$, the left socle consists of functionals vanishing on $\mathrm{rad}\, E$. These are also annihilated by $\mathrm{rad}\, E$ on the right. A map $E \to \mathrm{rad}(E)p$ is right multiplication by a radical element, hence kills this socle. All projective factorizations of $X \to \Omega X$ are therefore zero.

The map $1 \mapsto \zeta$ spans $\mathcal{H}^{-1}$. Comparing the twisted projective cover via $h_H \otimes h_H$ sends ζ to $H^2\zeta$. For $p \in \mathcal{P}_m$ and $\beta \in \mathcal{H}^{-3m-1}$, their shifted composite lies in $\mathcal{H}^{-1}$. The twist preserves this composite, acts on p by H^{-m}, and acts on $\mathcal{H}^{-1}$ by H^2. Writing $(h_H \otimes h_H)(\beta)$ for the induced two-factor twist, and using the functional λ from Proposition 4.2, we obtain

$$\lambda(p[-3m-1](h_H \otimes h_H)(\beta)) = H^{m+2}\lambda(p[-3m-1]\beta).$$

Nondegeneracy gives $(h_H \otimes h_H)(\beta) = H^{m+2}\beta$. Finally compose with the fixed commuting twist induced by $\widetilde{\gamma} \otimes \widetilde{\gamma}$. □

5 Derived induction and a twisted trace

The two-sided bar model K for $E \otimes_B^{\mathbf{L}} E$ has

$$K_i = E \otimes B^{\otimes i} \otimes E \quad (i \geq 0), \qquad K_i = 0 \quad (i < 0),$$

and differential, for $i \geq 1$,

$$\begin{aligned} d_i(a \otimes b_1 \otimes \cdots \otimes b_i \otimes c) = {} & ab_1 \otimes b_2 \otimes \cdots \otimes b_i \otimes c \\ & + \sum_{r=1}^{i-1} (-1)^r a \otimes \cdots \otimes b_r b_{r+1} \otimes \cdots \otimes c \\ & + (-1)^i a \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes b_i c. \end{aligned}$$

Multiplication in degree zero gives $m : K \to E$.

Lemma 5.1. *The stable image $\mathcal{I} = \langle K \rangle$ has a finite representative projective on both E-sides, and there is such a representative Y of $\mathcal{I}[-1]$. It satisfies*

$$W^a := \underline{\mathrm{Hom}}_E(X, YX[a]) = \begin{cases} k & a = -3, 0, \\ 0 & a \notin \{-3, 0\}. \end{cases} \tag{16}$$

Proof. The bar terms are finite E^e-free modules. On the left, compare the bar resolution of ${}_BE$ with a length-four projective resolution furnished by (10). Inducing preserves the homotopy equivalence. Thus K, as a left E-complex, is homotopy equivalent to a projective complex in degrees $0, \ldots, 4$. For the right side, use the right bar resolution of E_B and a right projective resolution of length four, and induce by $- \otimes_B E$. Both induced comparisons remain chain homotopy equivalences because tensoring preserves the equations defining a homotopy.

In either comparison the composite through the length-four complex is zero in degree five, so the homotopy there gives

$$1_{K_5} = d_6 h_5 + h_4 d_5.$$

The map d_5 descends to $c_5 K \to K_4$. Composing this with h_4 and then $K_5 \twoheadrightarrow c_5 K$ gives the identity on $c_5 K$. Thus $c_5 K$ is a direct summand of K_4 on each side. The same comparison makes the homology of K vanish above degree four, so Lemma 2.3 gives $\mathcal{I} = (c_5 K)[5]$.

Starting with $N_0 = c_5 K$, take four sequences

$$0 \to N_r \longrightarrow \mathrm{Hom}_k(E^e, N_r) \to N_{r+1} \to 0, \qquad n \mapsto (a \mapsto an), \qquad 0 \leq r < 4.$$

The middle term has E^e-action by right multiplication on its input, and is finite injective, hence projective. Each sequence splits on both E-sides. Taking $Y = N_4$ gives $Y \simeq \mathcal{I}[-1]$.

Evaluation of the bar model at X is homotopy equivalent to $\mathcal{J} \otimes \mathcal{J}$: before inducing, both models resolve X over B. Represent the triangle (11) by a two-cell projective model and tensor two copies. After shifting by $[-4]$, the three filtration quotients represent

$$X, \qquad X[-2] \oplus X[-2], \qquad X[-4].$$

To construct this filtration, choose a bounded-below T-projective resolution P_s of s. Represent the connecting map $\tau : s \to s[3]$ by a degree-three cocycle on P_s. The triangle (11) is then represented by a complex with underlying graded module $P_s[2] \oplus P_s$ and differential

$$\begin{pmatrix} d & t \\ 0 & d \end{pmatrix}, \qquad t_j : (P_s)_j \longrightarrow (P_s)_{j-3}, \qquad dt + td = 0,$$

with the sign of t chosen to induce τ. This is the shifted mapping-cone model of the triangle. Its tensor square has bottom layer $P_s[2] \otimes P_s[2]$, middle layers $P_s \otimes P_s[2]$ and $P_s[2] \otimes P_s$ in that order, and top layer $P_s \otimes P_s$. Shifting all layers by $[-4]$ gives the three quotients displayed above.

Write ^{t} for matrix transpose. In this middle-layer order the bottom connecting map is (τ_1, τ_2), and the top map projected to the middle layers is $(\tau_2, \tau_1)^{\mathsf{t}}$. Indeed, on $v \otimes w$ the first cross differential is $t(v) \otimes w$, and the second is $(-1)^{|v|} v \otimes t(w)$. The latter sign equals $(-1)^{3|v|}$, the sign of the external degree-three map in Proposition 4.2. The shifts by two change no parity. Their composite is therefore $\tau_1\tau_2 + \tau_2\tau_1 = 0$.

For $m \geq 0$ the left multiplication sequence

$$0 \to \mathcal{P}_{m-1} \xrightarrow{(\tau_2,\tau_1)^{\mathsf{t}}} \mathcal{P}_m^{\oplus 2} \xrightarrow{(\tau_1,\tau_2)} \mathcal{P}_{m+1} \to 0 \tag{17}$$

is exact. Every monomial of degree $m+1$ is divisible on the left by τ_1 or τ_2, so the row is onto. Left multiplication by τ_2 sends distinct ordered monomials to distinct ordered monomials with coefficients ± 1, so the column is injective. The relation in (12) makes the composite zero. Since $\dim \mathcal{P}_m = m+1$, the row kernel has dimension $2(m+1) - (m+2) = m = \dim \mathcal{P}_{m-1}$.

The corresponding right multiplication sequence is exact by the same monomial argument. By (13), its vector-space dual, with the two middle slots exchanged, is the sequence of left multiplications in negative Tate degrees. Thus it is exact as well. At the boundary, $\mathcal{H}^{-1} = k$ maps to $\mathcal{H}^2 = 0$, and $\mathcal{H}^0 = k$ has zero incoming group $\mathcal{H}^{-3}$.

For $l \in \mathbb{Z}$, write

$$d_l : (\mathcal{H}^{l-2})^{\oplus 2} \to \mathcal{H}^{l+1}, \quad d_l(u,v) = \tau_1 u + \tau_2 v, \quad d'_l : \mathcal{H}^{l-4} \to (\mathcal{H}^{l-1})^{\oplus 2}, \quad d'_l(w) = (\tau_2 w, \tau_1 w).$$

The exact sequences just proved, including $\mathcal{H}^{-1} \to \mathcal{H}^2 = 0$, give

$$\ker d_l = \operatorname{im} d'_{l-1}, \qquad \operatorname{coker} d_{l-1} = \begin{cases} k & l = 0, \\ 0 & l \neq 0, \end{cases} \quad \ker d'_l = \begin{cases} k & l = 3, \\ 0 & l \neq 3. \end{cases} \tag{18}$$

Let $\mathcal{L}_1$ be the stable image of the bottom two layers and put $N^l = \underline{\mathrm{Hom}}_E(X, \mathcal{L}_1[l])$. The first triangle gives

$$0 \to \operatorname{coker} d_{l-1} \to N^l \xrightarrow{q_l} \ker d_l \to 0.$$

The filtration gives a second triangle

$$\mathcal{L}_1 \longrightarrow (\mathcal{I}X)[-4] \longrightarrow X[-4] \longrightarrow \mathcal{L}_1[1].$$

Let $r_l : \mathcal{H}^{l-4} \to N^{l+1}$ be its connecting map after applying $\underline{\mathrm{Hom}}_E(X, -[l])$. Composing the top attaching map with the projection of $\mathcal{L}_1[1]$ onto its middle-layer quotient gives the column computed above. Hence

$$q_{l+1} r_l = d'_l.$$

If $l \neq 3$, (18) makes d'_l injective, and therefore r_l injective. When $l = 3$, the first triangle gives $N^4 = 0$, since both $\operatorname{coker} d_3$ and $\ker d_4 = \operatorname{im} d'_3$ vanish. Thus $\ker r_3 = \mathcal{H}^{-1} = k$.

For the cokernel, $q_l(\operatorname{im} r_{l-1}) = \operatorname{im} d'_{l-1} = \ker d_l$. If $l \neq 0$, the map $q_l : N^l \to \ker d_l$ is an isomorphism by (18), so r_{l-1} is onto. At $l = 0$, the first triangle gives $N^0 = k$, and the source of r_{-1} is $\mathcal{H}^{-5} = 0$. The second triangle gives the short exact sequences

$$0 \longrightarrow \operatorname{coker} r_{l-1} \longrightarrow \underline{\mathrm{Hom}}_E(X, (\mathcal{I}X)[l-4]) \longrightarrow \ker r_l \longrightarrow 0.$$

The middle group is consequently k in degrees $l = 0, 3$ and zero otherwise. Substituting $l = a + 3$, since $YX = (\mathcal{I}X)[-1]$, proves (16). $\square$

Put $\theta = \gamma \otimes \gamma \in \mathrm{Aut}(B)$ and let $\chi_B = \chi|_B$. The twisted dual $(DB)_\theta$ has left action $(a\lambda)(c) = \lambda(ca)$ and right action $\lambda \cdot b = \lambda\theta(b)$, where $(\lambda b)(c) = \lambda(bc)$ denotes the usual dual action. Since $\chi_B\theta = \chi_B$, the formula

$$\eta_\theta : B \to (DB)_\theta, \qquad a \mapsto \chi_B(a)\chi_B \tag{19}$$

defines a bimodule map.

Lemma 5.2. *If a bimodule map $B \to (DB)_\theta$ factors in $\mathrm{D}^b(B^e)$ through a perfect object, it factors by ordinary bimodule maps through a module in $\mathrm{add}(DB \otimes B)$, with the outer bimodule actions on this tensor product.*

Proof. For a bimodule L, the natural embedding

$$j_L : L \to \mathrm{Hom}_k(B, L), \quad j_L(z)(c) = cz, \qquad (a\phi b)(c) = \phi(ca)b$$

is a bimodule map. Iterate its cokernel four times. For $L = B \otimes B$, this is the left coinduced coresolution of B tensored with the right regular module. The four coinduced terms belong to $\mathrm{add}(DB \otimes B)$: their left factors are of the form $\mathrm{Hom}_k(B, V) = DB \otimes V$.

Denote the successive terms of this finite coresolution by $J^r(L)$, where $0 \le r \le 3$ are the coinduced terms and $J^4(L)$ is the fourth cokernel. The embeddings and cokernel maps are natural in L, so this is a complex of additive functors with natural augmentation

$$0 \longrightarrow L \longrightarrow J^0(L) \longrightarrow \cdots \longrightarrow J^4(L) \longrightarrow 0.$$

Apply the same coinduction construction to left B-modules, and write $J^r(B)$ for its terms on the left regular module. For $L = B \otimes B$, the coresolution above is this left-module coresolution tensored with the right regular module. Dimension shifting and (10) give

$$\mathrm{Ext}^1_B(V, J^4(B)) \cong \mathrm{Ext}^5_B(V, B) = 0$$

for every finite left B-module V, so the final left factor is injective. Every finite injective left module is in $\mathrm{add}\, DB$. Additivity then gives $J^r(L) \in \mathrm{add}(DB \otimes B)$ for every projective bimodule L and $0 \le r \le 4$.

For a bounded complex P of finite projective bimodules, form the total complex

$$P'_n = \bigoplus_{j-r=n} J^r(P_j), \qquad 0 \le r \le 4.$$

On $J^r(P_j)$ its differential is $J^r(d_j^P)$ plus $(-1)^j$ times the coresolution differential. Naturality makes the two unsigned differentials commute, and the signs make their cross terms cancel. The natural map $P \to P'$ is a quasi-isomorphism: filter its cone by the finitely many horizontal indices j; every successive quotient is a shift of an exact augmented coresolution. All terms of P' are in $\mathrm{add}(DB \otimes B)$.

For each such term V and $i > 0$,

$$\mathrm{Ext}^i_{B^e}(B, V) = 0, \qquad \mathrm{Ext}^i_{B^e}(V, (DB)_\theta) = 0.$$

For $V = DB \otimes B$, identify V with $\mathrm{Hom}_k(B, B)$ with the actions in the displayed formula for j_L. Evaluation at 1 gives

$$\mathrm{Hom}_{B^e}(M, \mathrm{Hom}_k(B, B)) \cong \mathrm{Hom}_{B^{\mathrm{op}}}(M, B).$$

The inverse sends a right-linear map $u : M \to B$ to $m \mapsto [c \mapsto u(cm)]$. A B^e-projective resolution restricts to a right B-projective resolution. Taking $M = B$ therefore identifies the first Ext group with $\mathrm{Ext}^i_{B^{\mathrm{op}}}(B, B) = 0$.

For the second assertion, tensor a left B-projective resolution of DB with the right regular module B. This is a B^e-projective resolution of $DB \otimes B$. Evaluation on the right-hand unit identifies its Hom complex into $(DB)_\theta$ with the left-module Hom complex into DB, which is injective. This proves the second vanishing. Finite sums and summands preserve both assertions.

Let $Q_B \to B$ be a B^e-projective resolution in nonnegative homological degrees and let $(DB)_\theta \to I$ be an injective resolution in nonpositive homological degrees. The two Ext vanishings give quasi-isomorphisms

$$\begin{aligned} \mathrm{Hom}^\bullet_{B^e}(B, P') &\longrightarrow \mathrm{Hom}^\bullet_{B^e}(Q_B, P'), \\ \mathrm{Hom}^\bullet_{B^e}(P', (DB)_\theta) &\longrightarrow \mathrm{Hom}^\bullet_{B^e}(P', I). \end{aligned}$$

To verify this first for a single term of P', positive cohomology vanishes by the displayed Ext equalities, degree zero is ordinary Hom, and negative cohomology is zero for modules. Filter P' by its finitely many degrees. The degreewise split short exact sequences of this filtration remain short exact after Hom, so induction proves the two assertions for P'. The bounded-direction comparison argument in Section 2 identifies the complexes on the right with the respective derived Hom complexes.

Both derived arrows can thus be represented by chain maps $B[0] \to P' \to (DB)_\theta[0]$. Their composite is the ordinary module map given by the two degree-zero components, and hence factors through $P'_0 \in \mathrm{add}(DB \otimes B)$. Since degree-zero derived Hom between modules is ordinary Hom, this is the original bimodule map. □

Proposition 5.3. *The map η_θ does not factor through a perfect B^e-complex.*

Proof. By Lemma 5.2, it suffices to consider ordinary factorizations through $DB \otimes B$. Identify that module with $\mathrm{Hom}_k(B, B)$ by $\lambda \otimes c \mapsto [x \mapsto \lambda(x)c]$. For a map $\alpha : B \to DB \otimes B$, the balance condition on $\alpha(1)$ says that its associated operator is right B-linear, hence left multiplication by some $d \in B$. If $(c_i), (c_i^*)$ are dual bases,

$$\alpha(1) = \sum_i c_i^* \otimes dc_i.$$

For $\beta : DB \otimes B \to (DB)_\theta$, the map $\lambda \mapsto \beta(\lambda \otimes 1)$ is a left-module endomorphism of DB, hence right multiplication by some $b' \in B$. Right linearity then gives

$$\beta(\lambda \otimes c) = \lambda b' \theta(c).$$

For $p = f \otimes f$, and with L_a, R_a denoting left and right multiplication on B, respectively, we obtain

$$(\beta\alpha)(1)(p) = \sum_i c_i^*(b'\theta(d)\theta(c_i)p) = \mathrm{Tr}_k(L_{b'\theta(d)}\theta R_p). \tag{20}$$

Write $b'\theta(d)$ as a radical element plus a linear combination of the four vertex idempotents $a \in \{e \otimes e, e \otimes f, f \otimes e, f \otimes f\}$. Left multiplication by the radical term, composed with θR_p, strictly raises the radical filtration and therefore has trace zero. The contribution at a is the trace of θ on aBp, the product of the two corresponding corner traces of γ. It vanishes by (5). Thus every composite (20) vanishes. The same holds for finite sums and their direct summands. But $\eta_\theta(1)(p) = 1$. □

Proposition 5.4. *For every $H \in k^\times$ there is an ordinary bimodule map $g_H : U_H \to Y$ whose evaluation represents a nonzero element of W^0 in (16).*

Proof. Set $S = \mathrm{Hom}_k(X, X)$, with action $(atb)(x) = a\,t(bx)$, and let $\pi : E \to S$ be the representation. Thus $S = k$ with both actions given by χ. Write $\overline{m} : \mathcal{I} \to E$ for the stable image of m and $\phi = \pi\overline{m}$. The element $\zeta = f^* \otimes f^*$ satisfies $a\zeta b = \chi(a)\chi(b)\zeta$, so $b_0 : S \to U_H$, $1 \mapsto \zeta$, is a bimodule map.

Suppose $b_0\phi$ were zero stably. Lemma 2.3 makes the derived map $b_0\pi m : K \to U_H$ factor through a perfect E^e-complex. On restriction to B^e the intermediate object remains perfect: a free E^e-module restricts to $E \otimes E$, whose projective dimension over B^e is at most eight by (10). The same follows for finite sums, summands, and bounded complexes.

Let $\mathrm{Bar}(B)$ be the bar complex defined above with both outer copies of E replaced by B and augmentation given by multiplication. Inclusion of the outer factors gives $\mathrm{Bar}(B) \to K$ as a B^e-chain map. Since $\sigma_H|_B = \theta$, the four summands of $\mathrm{Res}_{B^e} U_H$ are the right-θ twists of B, $DC \otimes C$, $C \otimes DC$, and DB. Projecting to the last summand, the composite from $\mathrm{Bar}(B)$ is η_θ composed with the augmentation: in degree zero it sends $a \otimes c$ to $\chi_B(ac)\chi_B$. This contradicts Proposition 5.3. Hence $b_0\phi \neq 0$ stably.

By Lemma 2.2 over E^e, the transpose of

$$\phi_* : \underline{\mathrm{Hom}}_{E^e}(U_H[1], \mathcal{I}) \to \underline{\mathrm{Hom}}_{E^e}(U_H[1], S)$$

is $b \mapsto b\phi$ from $\underline{\mathrm{Hom}}_{E^e}(S, U_H)$ to $\underline{\mathrm{Hom}}_{E^e}(\mathcal{I}, U_H)$. It is nonzero on b_0, so choose β with $\phi\beta \neq 0$.

For every finite bimodule L projective on both sides, ordinary adjunction is given by

$$f \longmapsto \widehat{f}, \qquad \widehat{f}(l \otimes x) = f(l)(x), \qquad \mathrm{Hom}_{E^e}(L, S) \cong \mathrm{Hom}_E(LX, X).$$

The bimodule identity $f(alb)(x) = af(l)(bx)$ makes $\widehat{f}$ balanced and left E-linear. Passing to projective-factorization quotients gives

$$\underline{\mathrm{Hom}}_{E^e}(L, S) \cong \underline{\mathrm{Hom}}_E(LX, X). \tag{21}$$

If f factors through an E^e-projective module J, then $\widehat{f}$ factors through JX, which is E-projective by Section 2. Conversely, embed L into a finite E^e-projective-injective J. This embedding splits on the right, so $LX \to JX$ is injective. If $LX \to X$ factors through an E-projective P, injectivity of P extends the first factor to JX. Adjunction gives $J \to S$ extending the original map. This proves (21).

Choose side-projective representatives of $\mathcal{I}$ and $U_H[1]$ using the syzygies and cosyzygies from Section 2. The adjunction is natural in these source modules: for an ordinary map $b : L' \to L$,

$$\widehat{fb}(l' \otimes x) = f(b(l'))(x) = \widehat{f}\big((b \otimes 1_X)(l' \otimes x)\big).$$

It therefore remains natural after passing to stable maps. The adjoint of $\pi\overline{m}$ is $\overline{m}X$, because π is the representation on X. Consequently the nonzero class $\phi\beta$ corresponds to

$$X[1] = U_H[1]X \xrightarrow{\beta X} \mathcal{I}X \xrightarrow{\overline{m}X} X.$$

This composite is nonzero, so βX is nonzero. Shift β by $[-1]$, identify $\mathcal{I}[-1]$ with the fixed Y, and choose an ordinary representative g_H. □

6 The triangular module

Take $H_1 = q$, $H_2 = 1$, and put $U_i = U_{H_i}$ for $i = 1, 2$. Let $g_i : U_i \to Y$ be scalar multiples of the maps g_{H_i} from Proposition 5.4, chosen so that both evaluated stable maps equal one

fixed nonzero $w \in W^0$. Choose a finite free E^e-module Q surjecting onto Y, with map π_Q, and define F by

$$0 \to F \to U_1 \oplus U_2 \oplus Q \xrightarrow{(g_1,-g_2,\pi_Q)} Y \to 0. \tag{22}$$

It splits on each E-side, so F is finite and projective on both sides. Let $\mathcal{V}^a = \underline{\mathrm{Hom}}_E(X, FX[a])$. Evaluation gives

$$(\mathcal{H}^{a-1})^{\oplus 2} \to W^{a-1} \to \mathcal{V}^a \xrightarrow{p^a} (\mathcal{H}^a)^{\oplus 2} \to W^a. \tag{23}$$

At degree zero the last map is $(c,d) \mapsto (c-d)w$. Thus $\mathcal{V}^0 = kv_0$ with $p^0(v_0) = (1,1)$. For $g, h \in \mathcal{H}^a$, define

$$\Delta^a(g,h) = F(g)v_0 - v_0[a]h \in \mathcal{V}^a,$$

where $F(g)$ uses the exact functor $F \otimes_E -$ and its stable shift comparison.

Lemma 6.1. *The map Δ^0 is onto; Δ^a is an isomorphism for $a > 0$, $a \le -3$, and $a = -1$; and $\Delta^{-2} : 0 \to k$ is injective.*

Proof. By (23), p^a is an isomorphism for $a > 0$ and $a \le -4$. At $a = 1$ this uses surjectivity onto W^0. The remaining degrees give

$$\mathcal{V}^{-3} = 0, \quad \mathcal{V}^{-2} = k, \quad \mathcal{V}^{-1} = k^2, \quad \mathcal{V}^0 = kv_0, \quad \mathcal{V}^1 = 0.$$

Naturality of the bimodule projections $F \to U_i$ gives

$$p^a \Delta^a(g,h) = (U_1(g) - h, U_2(g) - h).$$

On $\mathcal{H}^{3m}$, $m > 0$, this is the block matrix

$$\begin{pmatrix} q^{-m}G_{3m} & -1 \\ G_{3m} & -1 \end{pmatrix};$$

subtracting its two rows leaves the invertible block $(q^{-m} - 1)G_{3m}$. On $\mathcal{H}^{-3m-1}$, $m \ge 0$, the corresponding block is $(q^{m+2} - 1)G_{-3m-1}$, again invertible. In the other stated degrees the groups are zero. Finally $\Delta^0(c,d) = (c-d)v_0$, and $\mathcal{H}^{-2} = 0$. □

Form

$$\Lambda = \begin{pmatrix} E & 0 \\ F & E \end{pmatrix}. \tag{24}$$

A left module is a triple $(L, N, \alpha : F \otimes_E L \to N)$, with action $\left(\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right)(x,y) = (ax, \alpha(b \otimes x) + dy)$. Define the column functors

$$\mathsf{D}_0(L) = (L, FL, 1), \qquad \mathsf{D}_1(L) = (0, L, 0).$$

Exactness of D_0 follows from right projectivity of F, while exactness of D_1 is componentwise. The columns $\mathsf{D}_0(E)$ and $\mathsf{D}_1(E)$ are the left ideals of Λ given by its two diagonal idempotents. Thus both functors preserve finite projectives. A morphism of triples from (L, N, α) to (L', N', α') is a pair of maps $a : L \to L'$, $b : N \to N'$ satisfying $b\alpha = \alpha'(1_F \otimes a)$. Substitution gives

$$\begin{aligned} &\mathrm{Hom}_\Lambda(\mathsf{D}_i L, \mathsf{D}_i L') = \mathrm{Hom}_E(L, L') \quad (i = 0, 1), \\ &\mathrm{Hom}_\Lambda(\mathsf{D}_0 L, \mathsf{D}_1 L') = 0, \qquad \mathrm{Hom}_\Lambda(\mathsf{D}_1 L, \mathsf{D}_0 L') = \mathrm{Hom}_E(L, FL'). \end{aligned} \tag{25}$$

Put $p = f \otimes f$ and $\zeta = f^* \otimes f^*$. Choose a complete resolution P of X with $P_{-1} = Ep$ and connecting embedding $i_X : X \to Ep$, $1 \mapsto \zeta$. Lift an ordinary representative $\widetilde{v} : X \to FX$ of v_0 to a chain map $P \to FP$ using Lemma 2.1. Through (25) this is a chain map $V : \mathsf{D}_1(P) \to \mathsf{D}_0(P)$. Put

$$P_Z = \mathrm{Cone}(V), \qquad Z = c_0 P_Z.$$

Proposition 6.2. *The complex P_Z is totally acyclic, Z is finite and nonprojective, and*

$$H^a \operatorname{Hom}_\Lambda(P_Z, Z) = \begin{cases} k & a = -1, 0, \\ 0 & a \notin \{-1, 0\}. \end{cases} \tag{26}$$

Proof. Both column complexes are exact and projective. Their Hom complexes into $\Lambda = \mathsf{D}_0(E) \oplus \mathsf{D}_1(E)$ are sums of $\operatorname{Hom}_E(P, E)$, $\operatorname{Hom}_E(P, F)$, and zero. These are exact because F is left projective. Their cone is therefore totally acyclic.

Taking its degree-zero cokernel, the upper component of Z is X and the lower component is

$$(FX \oplus Ep)/\{(\widetilde{v}(x), -i_X(x)) : x \in X\}.$$

Indeed, the lower component of the cone in degree zero is $FP_0 \oplus P_{-1}$. The degree-one differential sends $(a, b) \in FP_1 \oplus P_0$ to

$$(d_{FP}a + V_0 b, -d_P b).$$

Quotienting first by the image of FP_1 identifies the first component with FX. The remaining relation depends only on the image x of b in $X = c_0 P$ and becomes $(\widetilde{v}(x), -i_X(x))$. The structure map is induced by inclusion of FX. Hence Z is finite-dimensional. The upper-component functor sends projective Λ-modules to projective E-modules. Since X is nonprojective, so is Z.

Write $A_0 = \mathsf{D}_0(P)$ and $B_0 = \mathsf{D}_1(P)$. In the graded decomposition $P_Z = A_0 \oplus B_0[1]$, the Hom block from A_0 to $B_0[1]$ vanishes. Thus there is an exact sequence of full Hom complexes

$$0 \to \operatorname{Hom}^\bullet_\Lambda(B_0[1], A_0) \to \operatorname{End}^\bullet_\Lambda(P_Z) \to \operatorname{End}^\bullet_\Lambda(A_0) \oplus \operatorname{End}^\bullet_\Lambda(B_0[1]) \to 0.$$

The cross-block degree a reindexes as $\operatorname{Hom}^{a-1}_E(P, FP)$. Its differential under that reindexing is $df + (-1)^a fd$, the differential in degree $a-1$ of the unshifted Hom complex. Its cohomology is therefore $\mathcal{V}^{a-1}$.

To identify the connecting map, a degree-a cocycle h on B_0 corresponds to $(-1)^a h$ on $B_0[1]$. The cross block of the Hom differential of a diagonal lift $(g, (-1)^a h)$ is $(-1)^a(Vh - F(g)V)$. On cohomology this is $(-1)^{a+1}\Delta^a$. Hence

$$0 \to \operatorname{coker} \Delta^{a-1} \to H^a \operatorname{End}_\Lambda(P_Z) \to \ker \Delta^a \to 0.$$

Lemma 6.1 makes both ends zero for $a > 0$ and $a \leq -2$, including $a = 1$ by surjectivity of Δ^0 and $a = -2$ by injectivity of Δ^{-2}. In degree -1 the cokernel of Δ^{-2} is k; in degree zero the kernel of Δ^0 is k. The other end is zero in each case. Lemma 2.1 identifies these groups with (26). □

7 Transfer to a symmetric algebra

Proposition 7.1. *Let R be a finite-dimensional algebra and P a totally acyclic complex of finite projectives. Suppose $Z = c_0 P$ is nonprojective and*

$$H^a \operatorname{Hom}_R(P, Z) = 0 \qquad (a > 0 \text{ or } a \leq -2).$$

Then $A = R \ltimes DR$ is symmetric and $M = A \otimes_R Z$ is a finite nonprojective self-orthogonal A-module.

Proof. Put $I = DR \otimes_R P$. For every term, evaluation gives

$$DR \otimes_R P_j \cong D\operatorname{Hom}_R(P_j, R), \qquad \operatorname{Hom}_R(Z, DR \otimes_R P_j) \cong D\operatorname{Hom}_R(P_j, Z).$$

For $P_j = R$, the first isomorphism is $DR \otimes_R R = DR$ and the second is the natural isomorphism $\operatorname{Hom}_R(Z, DR) = DZ$. Finite sums and direct summands give the formulas for every P_j, naturally in both P_j and Z. A left-projective P_j makes $\operatorname{Hom}_R(P_j, R)$ right projective, so its dual is left injective. The differentials of I are dual to those of $\operatorname{Hom}_R(P, R)$, with the Hom degree reversed. Total acyclicity therefore makes I exact with injective terms. The complex $A \otimes_R P$ is $P \oplus I$ on restriction to R, hence exact with cokernel M.

Right exactness of tensor gives $N = DR \otimes_R Z = c_0 I$. Exactness of I identifies this cokernel with $\ker(I_{-1} \to I_{-2})$, yielding an injective coresolution

$$0 \longrightarrow N \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots.$$

Its cochain term in degree $i \geq 0$ after applying $\operatorname{Hom}_R(Z, -)$ is

$$\operatorname{Hom}_R(Z, I_{-i-1}) \cong D\operatorname{Hom}_R(P_{-i-1}, Z).$$

For $i > 0$, both adjacent terms are also in this tail. The cohomology is therefore the dual of the cohomology of $\operatorname{Hom}_R(P, Z)$ in degree $-i-1$; changing a differential by its usual sign leaves its image and kernel unchanged. Consequently,

$$\operatorname{Ext}^i_R(Z, N) = DH^{-i-1}\operatorname{Hom}_R(P, Z) = 0.$$

Each $A \otimes_R P_j$ is A-projective because P_j is a summand of a finite free R-module. Its nonnegative tail is therefore a projective resolution of M. On restriction to R, the module M is $Z \oplus N$, and termwise tensor–Hom adjunction on this resolution gives

$$\operatorname{Ext}^i_A(M, M) = \operatorname{Ext}^i_R(Z, Z \oplus N) = 0 \qquad (i > 0).$$

The trivial-extension form proves symmetry. If M were a summand of A^n, tensoring that split inclusion and retraction along the quotient $A \to R$ would make $R \otimes_A M \cong Z$ a summand of R^n, a contradiction. □

Proof of Theorem 1.1. Take Λ, Z, P_Z from Section 6. Total acyclicity and (26) give the asserted Ext vanishing over Λ. Proposition 7.1 gives $A = \Lambda \ltimes D\Lambda$ and $M = A \otimes_\Lambda Z$ with the required symmetric-algebra properties.

The constructions of Y, F, Z use finitely many kernels, cokernels, cosyzygies, and choices in finite Hom spaces, so all objects are finite-dimensional over k. The radical quotients of C, T, E, Λ, A are respectively k^2, k^2, k^4, k^8, k^8: for the tensor product this uses the nilpotent ideal $\operatorname{rad} T \otimes T + T \otimes \operatorname{rad} T$; for the triangular and trivial extensions it uses their nilpotent off-diagonal and dual ideals. Thus A has eight simple modules.

Let $\pi_C : T \to C$ be $(c, \phi) \mapsto c$, and let $\chi_T : T \to k$ be the character of s, so $a \cdot 1 = \chi_T(a)$ on the one-dimensional module s. The map

$$E = T \otimes T \longrightarrow C, \qquad a \otimes b \longmapsto \pi_C(a)\chi_T(b)$$

is unital and multiplicative, and it is onto since $(c, 0) \otimes 1$ maps to c. Composing with the quotient $A \twoheadrightarrow \Lambda$ and the upper-left diagonal projection $\Lambda \twoheadrightarrow E$ gives a surjective algebra homomorphism $A \twoheadrightarrow C$.

A surjection of finite-dimensional algebras maps the radical onto the radical: its radical image is nilpotent, and the quotient by that image is a quotient of a semisimple algebra, hence semisimple. In C, the elements u, t lie in the radical, and $utut = q(1-q)z \neq 0$. Their radical lifts to A therefore have nonzero fourfold product. Also $eu = u \neq ue = 0$ makes C noncommutative. Both conclusions follow for A. □

For a finite-dimensional k-algebra R, finite left R-modules U, V, and a field extension K/k, projective resolutions with finite-dimensional terms give, for every $i \geq 0$,

$$K \otimes_k \operatorname{Ext}^i_R(U, V) \cong \operatorname{Ext}^i_{K\otimes R}(K \otimes U, K \otimes V).$$

Indeed, for a finite projective term P_j the natural map

$$K \otimes_k \operatorname{Hom}_R(P_j, V) \longrightarrow \operatorname{Hom}_{K\otimes_k R}(K \otimes_k P_j, K \otimes_k V)$$

is an isomorphism, first for $P_j = R$ and then for finite sums and summands. Tensoring with the field K is exact, so taking cohomology proves the displayed Ext isomorphism.

To see that scalar extension preserves nonprojectivity, choose a finite projective surjection $P_0 \to U$ with kernel L. If U is nonprojective, the sequence defines a nonzero class in $\operatorname{Ext}^1_R(U, L)$. Its scalar extension remains nonzero, since the map from a k-vector space to its extension of scalars is injective. Hence $K \otimes_k U$ is nonprojective. Finite-dimensional duality also commutes with scalar extension, so symmetry is preserved. In particular, the construction with $q = 2$ is defined over $\mathbb{Q}$ itself, and the construction with $q = t$ is defined over $\mathbb{F}_p(t)$ for every prime p.

8 Endomorphism algebras and Auslander's condition

The Gorenstein-projective conjecture asserts that every self-orthogonal Gorenstein-projective module is projective. The module Z in Theorem 1.1 is a counterexample over Λ. The module M is another over A, since every finite module over a symmetric algebra has a totally acyclic complete resolution.

Keep A, M from Theorem 1.1, and put

$$G = A \oplus M, \qquad \Gamma = \operatorname{End}_A(G)^{\mathrm{op}}, \qquad H = \operatorname{Hom}_A(G, -).$$

The left Γ-action on $H(V)$ is precomposition. Regard G as an (A, Γ)-bimodule by $g \cdot f^{\mathrm{op}} = f(g)$. The dominant dimension of an algebra is the least index of a nonprojective term in the minimal injective coresolution of its left regular module, and is infinite if all terms are projective.

Proposition 8.1. *The algebra Γ is non-self-injective and has infinite dominant dimension. There is a nonzero simple left module S with $\operatorname{Ext}^i_\Gamma(S, \Gamma) = 0$ for all $i \geq 0$. Moreover,*

$$\operatorname{Ext}^i_\Gamma(D\Gamma, \Gamma) = 0 \quad (i > 0), \qquad D\Gamma \text{ is nonprojective.} \tag{27}$$

Its little and big left finitistic dimensions are infinite.

Proof. The symmetric property and self-orthogonality give $\operatorname{Ext}^{>0}_A(G, G) = 0$. Choose an injective coresolution $0 \to G \to J^0 \to J^1 \to \cdots$ with finite terms. Every J^r lies in $\operatorname{add} A$, since finite injectives are summands of finite sums of $DA \cong A$. Applying H gives an exact sequence

$$0 \to \Gamma \to H(J^0) \to H(J^1) \to \cdots. \tag{28}$$

These terms are projective since $J^r \in \operatorname{add} G$ and $H(G) = \Gamma$ as a left Γ-module. For injectivity, let $\iota : A \to G$ and $\pi_A : G \to A$ be the summand inclusion and projection, and put $e = (\iota\pi_A)^{\mathrm{op}} \in \Gamma$. Evaluation and projection give isomorphisms

$$\begin{aligned} e\Gamma &\longrightarrow G, & ef^{\mathrm{op}} &\longmapsto f(\iota(1)), \\ \Gamma e &\longrightarrow \operatorname{Hom}_A(G, A), & f^{\mathrm{op}}e &\longmapsto \pi_A f. \end{aligned}$$

The first is right Γ-linear; the second is left Γ-linear. The corner $e\Gamma e$ is A, because $\mathrm{End}_A(A) = A^{\mathrm{op}}$. These maps respect the corresponding left and right A-actions.

The symmetric isomorphism $A \cong DA$ and evaluation at 1 give

$$H(A) \cong H(DA) \longrightarrow DG, \qquad u \longmapsto [g \mapsto u(g)(1)].$$

The inverse of the last map sends $\lambda \in DG$ to $g \mapsto [a \mapsto \lambda(ag)]$. This is left Γ-linear, so $H(A) \cong D(e\Gamma)$ is injective. Every $H(J^r)$ is a summand of a finite sum of copies of $H(A)$ and is therefore injective.

The minimal injective coresolution can be extracted term by term from (28). The injective envelope of Γ embeds as a summand of its first injective term: extend the inclusion of Γ to its envelope, use essentiality to get a monomorphism, and split it using injectivity of the envelope. After removing this summand, its complementary injective summand and its image in the next term form a split two-term complex. Repeat on the cokernel. Thus each minimal injective term is a summand of the corresponding projective-injective term in (28), proving infinite dominant dimension.

For finite sums of G, every Γ-linear map between their images under H is a matrix of endomorphisms of G: evaluation on 1_G identifies the rank-one case, with the opposite multiplication in Γ accounting for precomposition. Inserting the inclusion and projection of each direct summand extends this identification to $\operatorname{add} G$. Hence H is fully faithful there.

If Γ were self-injective, the projective $H(M)$ would be injective. For an embedding $M \to J$ with $J \in \operatorname{add} A$, applying H would give a split monomorphism. Full faithfulness would split the original embedding, making M injective and hence projective, a contradiction.

There is therefore an indecomposable injective module J that is not projective. Its socle S is simple and essential. A nonzero map $S \to H(J^r)$ extends to $J \to H(J^r)$; essentiality makes the extension injective, and injectivity of J makes it split. That would make J projective. Thus $\mathrm{Hom}_\Gamma(S, H(J^r)) = 0$ for all r, proving the claimed Ext vanishing, including degree zero.

The same idempotent identifications give isomorphisms of left A-modules

$$G \otimes_\Gamma D\Gamma \cong e\Gamma \otimes_\Gamma D\Gamma \cong eD\Gamma \cong D(\Gamma e) \cong D\,\mathrm{Hom}_A(G, A) \cong G.$$

Here $e\Gamma \otimes_\Gamma D\Gamma \to eD\Gamma$ is multiplication, and $eD\Gamma \to D(\Gamma e)$ is restriction of a functional. Its inverse extends $\mu \in D(\Gamma e)$ by $x \mapsto \mu(xe)$. The final isomorphism is the dual of $\mathrm{Hom}_A(G, A) \cong DG$ just proved, followed by the finite-dimensional double-dual identification. Apply tensor–Hom adjunction to (28). It identifies $\mathrm{Hom}_\Gamma(D\Gamma, H(J^\bullet))$ with $\mathrm{Hom}_A(G, J^\bullet)$, proving (27). If $D\Gamma$ were projective, every indecomposable injective would be projective; counting the finitely many indecomposable projective and injective types would make Γ self-injective. Thus $D\Gamma$ is nonprojective. Its injectivity in the second variable also gives $\mathrm{Ext}^{>0}_\Gamma(D\Gamma, D\Gamma) = 0$.

Finally write the minimal coresolution in short exact pieces

$$0 \to C_n \to I^n \to C_{n+1} \to 0, \qquad C_0 = \Gamma.$$

All I^n are projective. The first sequence is nonsplit, since otherwise Γ would be injective. Its class in $\mathrm{Ext}^1_\Gamma(C_1, \Gamma)$ is nonzero. For $n \geq 2$, the long exact sequence for $0 \to C_{n-1} \to I^{n-1} \to C_n \to 0$, with I^{n-1} projective, gives

$$\mathrm{Ext}^n_\Gamma(C_n, \Gamma) \cong \mathrm{Ext}^{n-1}_\Gamma(C_{n-1}, \Gamma).$$

Iterating reaches the nonzero class in $\mathrm{Ext}^1_\Gamma(C_1, \Gamma)$. On the other hand,

$$0 \longrightarrow \Gamma \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow C_n \longrightarrow 0$$

is a projective resolution of C_n of length n. Hence $\mathrm{pd}_\Gamma C_n = n$. The supremum over finite modules of finite projective dimension (the little finitistic dimension) is infinite, and so is the supremum over all modules of finite projective dimension (the big one). $\square$

This gives direct witnesses for the Nakayama conjecture (infinite dominant dimension without self-injectivity), the generalized Nakayama conjecture (the injective J missing from the regular coresolution), and the strong Nakayama conjecture (the nonzero simple S with all $\operatorname{Ext}^i_\Gamma(S,\Gamma)$ zero). Equation (27) gives the first Tachikawa conjecture and, using the injectivity of $D\Gamma$, another Auslander–Reiten witness on the same algebra. The endomorphism identity is the symmetric case of [5, Proposition 2.3].

The Auslander–Gorenstein conjecture asserts that an algebra whose minimal injective coresolution satisfies $\operatorname{pd} I^n \le n$ for every $n \ge 0$ has finite injective dimension on both sides. A module W is Wakamatsu tilting if it is self-orthogonal and the regular module has a coresolution by $\operatorname{add} W$ that stays exact under $\operatorname{Hom}_\Gamma(-,W)$. A tilting module has finite projective dimension, is self-orthogonal, and admits a finite such coresolution of the regular module. The Wakamatsu tilting conjecture asserts that every Wakamatsu tilting module of finite projective dimension is tilting.

Corollary 8.2. *The algebra Γ is a counterexample to the Auslander–Gorenstein conjecture: its minimal injective coresolution has projective terms, but $\operatorname{id}_\Gamma \Gamma = \infty$. The sum W of one representative of each indecomposable projective-injective left Γ-module is Wakamatsu tilting of projective dimension zero, but is not tilting.*

Proof. Every term I^n is projective, giving the condition. If the minimal coresolution terminated at I^d, the final surjection onto I^d would split because I^d is projective. Its kernel would be a summand of I^{d-1}, hence both projective and injective. Repeating this argument backwards would make Γ injective. Thus $\operatorname{id}_\Gamma \Gamma = \infty$.

The module W is projective and injective, hence self-orthogonal. Sequence (28) has terms in $\operatorname{add} W$ and remains exact under $\operatorname{Hom}_\Gamma(-,W)$. Thus W is Wakamatsu tilting. A finite coresolution by $\operatorname{add} W$ would again split backwards, so W is not tilting. □

Auslander's condition (AC) says that for each finite module U there is an integer b_U such that, for every finite V, eventual vanishing of $\operatorname{Ext}^i_R(U,V)$ implies vanishing for every $i > b_U$ [9]. The uniform condition requires a bound independent of U.

Proposition 8.3. *For every $n \ge 2$ and $i > 0$,*

$$\dim_k \operatorname{Ext}^i_A(M,\Omega^n M) = \begin{cases} \dim_k \underline{\operatorname{End}}_A(M) & i = n-1, n, \\ 0 & i \notin \{n-1, n\}. \end{cases}$$

In particular A fails AC and its uniform version.

Proof. Self-orthogonality and symmetric stable duality show that $\underline{\operatorname{Hom}}_A(M, M[j])$ vanishes outside $j = -1, 0$, and has dimension $\dim_k \underline{\operatorname{End}}_A(M)$ in those two degrees. This dimension is positive because the identity of a nonprojective module cannot factor through a projective. Since $\Omega^n M = M[-n]$ stably,

$$\operatorname{Ext}^i_A(M,\Omega^n M) = \underline{\operatorname{Hom}}_A(M, M[i-n]) \qquad (i > 0).$$

Each of these sequences eventually vanishes, with last nonzero degree n. No fixed b_M is possible. □

9 Finite-order parameters

Let $q \in k^\times$ have finite order $d \ge 1$, and define C, s by the same multiplication table. The associativity proof remains valid. A module's top is its quotient by its radical; a projective cover has the same top.

Proposition 9.1. *If $d \geq 2$, the minimal projective resolution of s has a period of length d from its second syzygy onward, and for $i > 0$,*

$$\dim_k \operatorname{Ext}^i_C(s,s) = \begin{cases} 1 & i = md \text{ or } md+1 \text{ for some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

If $q = 1$, then $\dim_k \operatorname{Ext}^1_C(s,s) = 1$.

Proof. For $q \neq 1$ put $K_i = \langle \ell_i, z, j \rangle_k \subset Ce$, with indices taken modulo d. The first two syzygies of s are $\operatorname{rad}(Cf)$ and K_0. For $i \not\equiv -2 \pmod d$, right multiplication by ℓ_i is a projective cover $Ce \to K_i$, with kernel K_{i+1}. At $i = -2$, the radical of K_{-2} is kz, so its top has one simple at each vertex. Its cover is

$$Ce \oplus Cf \to K_{-2}, \qquad (a,b) \mapsto a\ell_{-2} + bj.$$

For the kernel calculation, write

$$a = a_e e + a_x x + a_y y + a_z z + a_t t + a_j j \in Ce, \qquad b = b_f f + b_u u + b_v v + b_n n \in Cf,$$

where the subscripted coefficients are scalars in k. Multiplication gives

$$a\ell_{-2} + bj = a_e \ell_{-2} + (a_y - q^{-1}a_x + b_u)z + b_f j.$$

Thus the kernel is specified by $a_e = b_f = 0$ and $b_u = q^{-1}a_x - a_y$. Its seven free coordinates give the following basis of L:

$$(\ell_{-1}, 0),\ (-y, u),\ (z, 0),\ (0, v),\ (t, 0),\ (j, 0),\ (0, n).$$

The radical of L is spanned by all these vectors except $(-y, u)$ and $(t, 0)$. A cover of L is therefore

$$Ce \oplus Cf \to L, \qquad (a,b) \mapsto (-ay + bt, au).$$

Its image spans L: the Ce-generator yields $(-y, u)$, (qz, v), $(0, -v)$, and $(q^2 j, (q-1)n)$; the Cf-generator yields $(t, 0)$, $(q\ell_{-1}, 0)$, $(-qz, 0)$, and $(-qj, 0)$. In the same coordinates this map has components

$$\begin{aligned} -ay + bt &= qb_u x + (b_u - a_e)y + b_f t + q(a_x - b_v)z + (q^2 a_t - qb_n)j, \\ au &= a_e u + (a_x - a_y)v + (q-1)a_t n. \end{aligned}$$

Since $q \neq 0, 1$, both components vanish exactly when

$$a_e = a_t = b_f = b_u = b_n = 0, \qquad a_y = a_x = b_v,$$

with a_x, a_z, a_j free. The kernel therefore has basis $(\ell_0, v), (z, 0), (j, 0)$ and projects isomorphically onto K_0 in the first coordinate. This is a module isomorphism: its inverse is the graph of the map $K_0 \to Cf$ sending ℓ_0 to v and z, j to zero. On ℓ_0 the only nonzero radical actions are

$$x\ell_0 = -qz, \qquad y\ell_0 = z, \qquad t\ell_0 = (1 - q^2)j,$$

whose images are zero, just as every radical element kills v. On z, j, the only additional nonzero radical product is $uj = z$, again sent to zero. The idempotent actions agree as well.

The successive syzygies thus cycle through $K_0, K_1, \ldots, K_{d-2}, L, K_0$. Only the covers of K_{d-2} and L contain Cf, first in degrees d and $d+1$. All differentials in a minimal resolution induce zero on Hom into the simple s, proving (29). For $q = 1$, $\operatorname{rad}(Cf) = \langle u, v, n \rangle_k$ has radical kv and top generated by u, n at the two vertices. Its cover has one Cf summand, giving the stated first Ext group. □

For $d \geq 2$ and an indeterminate z_0, the dimensions in (29) have generating series

$$\sum_{i \geq 0} \dim_k \operatorname{Ext}^i_C(s,s) z_0^i = 1 + \frac{z_0^d + z_0^{d+1}}{1 - z_0^d}.$$

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