

A note on counterexamples to homological conjectures

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(Research generated and reviewed using GPT-6-Astra)

October 7, 2026

Public discussion draft; not intended for journal submission.

Abstract

Building on OpenAI's constructions, we extend the base field to any field containing an element of infinite multiplicative order. Over each such field, we construct finite-dimensional algebras giving counterexamples to both Tachikawa conjectures, the Auslander–Reiten and Gorenstein-projective conjectures, the Nakayama, generalized Nakayama, and strong Nakayama conjectures, the Auslander–Gorenstein conjecture, the little and big finitistic dimension conjectures, and the Wakamatsu tilting conjecture.

1 The field of definition

All algebras in this note are associative and unital. Modules are finite-dimensional left modules unless specified otherwise. For a field k , write $D = \text{Hom}_k(-, k)$. The Jacobson radical of an algebra R is denoted $\text{rad } R$; the radical and socle of a module are the intersection of its maximal submodules and its largest semisimple submodule, respectively. We write pd_R and id_R for projective and injective dimension. A finite-dimensional k -algebra A is *symmetric* if $A \cong DA$ as bimodules. Equivalently, it has a linear form $t : A \rightarrow k$ for which $t(ab) = t(ba)$ and the pairing $(a, b) \mapsto t(ab)$ is nondegenerate. Such an algebra is self-injective. A module M is *self-orthogonal* if $\text{Ext}_A^i(M, M) = 0$ for every $i > 0$.

The Auslander–Reiten conjecture asserts that $\text{Ext}_R^i(N, N \oplus R) = 0$ for all $i > 0$ forces N to be projective. For self-injective algebras this is the second Tachikawa conjecture. OpenAI's constructions [1, 2] give counterexamples over the rational function field $\mathbb{F}_2(q, H_1, H_2)$ in three independent indeterminates. We follow the derived-induction construction of [2], replacing its ten-dimensional algebra and its untwisted character map. The replacement map takes values in a twisted dual bimodule; its factorization obstruction is the vanishing trace of an involution on each corner.

An algebra is *split* if its semisimple quotient is a product of full matrix algebras over k . A module is *Gorenstein-projective* if it is a cokernel in a doubly infinite exact complex of finite projectives that remains exact under $\text{Hom}_R(-, R)$.

Theorem 1.1. *Let k be a field containing an element $q \in k^\times$ of infinite multiplicative order. There exist a finite-dimensional split symmetric k -algebra A and a finite-dimensional nonprojective A -module M such that*

$$\text{Ext}_A^i(M, M) = 0 \quad (i > 0).$$

2020 Mathematics Subject Classification. Primary 16E30; Secondary 16D50, 16E10. *Keywords.* Symmetric algebra, self-orthogonal module, Auslander–Reiten conjecture, Tachikawa conjectures, twisted trace.

The algebra A has eight isomorphism classes of simple modules, is noncommutative, and satisfies $(\text{rad } A)^4 \neq 0$. There also exist a finite-dimensional triangular k -algebra Λ and a nonprojective Gorenstein-projective module Z with

$$\text{Ext}_\Lambda^i(Z, Z \oplus \Lambda) = 0 \quad (i > 0).$$

The proof of Theorem 1.1 ends in Section 7.

Corollary 1.2. *Theorem 1.1 holds over every field of characteristic zero and over every positive-characteristic field not algebraic over its prime field.*

Proof. In characteristic zero take $q = 2$. In characteristic $p > 0$, choose an element q transcendental over $\mathbb{F}_p$. Each has infinite multiplicative order. $\square$

For a field algebraic over $\mathbb{F}_p$, every nonzero element has finite order. Section 9 describes the corresponding change in the simple-module resolution.

Section 8 gives direct witnesses for the homological conjectures using A , Λ , and the endomorphism algebra of $A \oplus M$. The Auslander–Reiten conjecture holds for Igusa–Todorov algebras [6, Theorem 3.2], quantum complete intersections [7, Theorem 4.2], and split radical-cube-zero algebras [8, Theorem 1.1].

2 Complete resolutions and stable maps

For an algebra R , put $R^e = R \otimes_k R^{\text{op}}$, where R^{op} is the opposite algebra, so left R^e -modules are R -bimodules. Unmarked tensor products are over k . Write $\text{add } V$ for the direct summands of finite sums of copies of a module V . The space $\underline{\text{Hom}}_R(U, V)$ is ordinary Hom modulo maps factoring through finite projectives; put $\underline{\text{End}}_R(U) = \underline{\text{Hom}}_R(U, U)$.

Complexes are homologically indexed: $d_j : P_j \rightarrow P_{j-1}$. For integers j, r , we use

$$P[r]_j = P_{j-r}, \quad d_{P[r]} = (-1)^r d_P, \quad c_j P = \text{coker}(d_{j+1} : P_{j+1} \rightarrow P_j).$$

For complexes P, Q , the full Hom complex has

$$\text{Hom}_R^a(P, Q) = \prod_{j \in \mathbb{Z}} \text{Hom}_R(P_j, Q_{j-a}), \quad \delta f = d_Q f - (-1)^a f d_P. \quad (1)$$

Its cohomology is denoted H^a , and $\text{End}_R^\bullet(P) = \text{Hom}_R^\bullet(P, P)$. A module in a complex argument is placed in degree zero. For a map f , the notation $f[r]$ denotes its shifted map; juxtaposition of composable maps means composition from right to left. For a chain map $f : P \rightarrow Q$,

$$\text{Cone}(f)_j = Q_j \oplus P_{j-1}, \quad d_{\text{Cone}(f)} = \begin{pmatrix} d_Q & f \\ 0 & -d_P \end{pmatrix}.$$

An exact complex of finite projectives is *totally acyclic* when its Hom complex into R is exact. It is a complete resolution of $c_0 P$.

Lemma 2.1. *If P is totally acyclic and Q is an exact complex of finite projectives, then for every integer a ,*

$$H^a \text{Hom}_R(P, Q) \cong \underline{\text{Hom}}_R(c_a P, c_0 Q) \cong H^a \text{Hom}_R(P, c_0 Q). \quad (2)$$

The first identification respects shifted composition. For $a > 0$ these groups equal $\text{Ext}_R^a(c_0 P, c_0 Q)$.

Proof. For a finite projective J , exactness of $\mathrm{Hom}_R(P, J)$ implies that every map $c_j P = \mathrm{im} d_j \rightarrow J$ extends to P_{j-1} . Thus the cycles of $\mathrm{Hom}_R(P, N)^a$ are maps $c_a P \rightarrow N$, and its boundaries are precisely those factoring through projectives.

A map $c_0 P \rightarrow c_0 Q$ lifts to $P_0 \rightarrow Q_0$ and successively to positive degrees by projectivity. For the downward extension, the already defined map $d_Q f_j$ vanishes on $\mathrm{im} d_{j+1}^P$, hence defines a map $c_j P \rightarrow Q_{j-1}$. The extension property supplies f_{j-1} . This constructs a chain map in all degrees.

If the cokernel map factors through J , extend its first factor to P_{-1} and lift its second factor to Q_0 . Their composite supplies a homotopy component removing the cokernel map. For a chain map inducing zero on cokernels, solve $f_j = d_Q h_j + h_{j-1} d_P$ upwards by projectivity and downwards by the extension property. These equations give a full nullhomotopy. Replacing P by $P[-a]$ proves the assertion in degree a , with the signs of (1). Cokernel maps preserve composition. The nonnegative tail of P is a projective resolution, which proves the positive Ext assertion. $\square$

Over a symmetric algebra R , finite projectives are injective. Splicing a projective resolution with a projective-injective coresolution gives a complete resolution of every finite module. Its stable shift is $U[r] = c_{-r} P$; thus $[1] = \Omega^{-1}$, where ΩU is the kernel of a projective cover of U , a surjection from a finite projective module whose kernel lies in its radical. Short exact sequences give stable triangles and long exact stable Hom sequences. In particular,

$$H^a \mathrm{Hom}_R(P, V) = \underline{\mathrm{Hom}}_R(U, V[a]).$$

The following form of Tate duality is also recorded in [3, Section 2].

Lemma 2.2. *For symmetric R there are natural isomorphisms*

$$D\underline{\mathrm{Hom}}_R(U, V) \cong \underline{\mathrm{Hom}}_R(V, U[-1]).$$

There is a linear form $\lambda_U : \underline{\mathrm{Hom}}_R(U, U[-1]) \rightarrow k$ such that the resulting perfect pairing is $(f, g) \mapsto \lambda_U(gf)$.

Proof. For a finite projective J , the symmetric form identifies $D\underline{\mathrm{Hom}}_R(J, V)$ with $\mathrm{Hom}_R(V, J)$. Applying this termwise to a complete resolution P of U , the dual of degree-zero cohomology is

$$\mathrm{Hom}_R(V, c_1 P) / \mathrm{im}(\mathrm{Hom}_R(V, P_1) \rightarrow \mathrm{Hom}_R(V, c_1 P)).$$

Every factorization through a projective lifts through $P_1 \twoheadrightarrow c_1 P$, so this is the asserted stable Hom space. Define λ_U by pairing with 1_U . Naturality in V identifies the pairing with composition. Shifting gives the same assertion for all stable degrees. $\square$

We use $D^b(R)$ for the derived category of bounded complexes of finite modules. A *perfect* object is one represented by a bounded complex of finite projectives. A bounded derived object has a bounded-below projective model with finite terms and bounded homology. Such a model computes derived maps by chain maps modulo homotopy: maps to an acyclic complex are nullhomotopic by successive lifting from the lowest degree [4, Proposition 1.4 and Section 3.2].

Lemma 2.3. *For symmetric R and such a projective model K , the stable object*

$$\langle K \rangle = (c_n K)[n] \quad (n \text{ above its homology support})$$

is independent of n , functorial on derived maps, and compatible with shifts and cones. A derived map with zero stable image factors through a perfect object.

Proof. In the exact tail, the sequences $0 \rightarrow c_{n+1}K \rightarrow K_n \rightarrow c_nK \rightarrow 0$ identify adjacent choices. A chain homotopy changes a cokernel map by a projective factorization. High cokernels of a degreewise split short exact sequence form a short exact sequence; its connecting map is induced, with the shift signs, by the cross differential. This proves compatibility with triangles and cones.

For the last assertion, represent the map by $f : P \rightarrow Q$. In sufficiently high degree, factor $c_n(f)$ through a finite projective J . Extend to P_{n-1} using injectivity of J and lift to Q_n using projectivity. Successive lifts in the exact tail give a homotopy making $f'_j = 0$ for $j \geq n$. The chain equation then says $f'_{n-1}d_n^P = 0$. Consequently f' factors through the bounded projective subcomplex of Q supported in degrees at most $n - 1$. $\square$

For a symmetric algebra E , the algebra E^e is symmetric as well. We use finite bimodules projective separately on the left and right. Their bimodule syzygies and cosyzygies can be formed using finite E^e -projective-injective modules. The defining short exact sequences split on each E -side. For an E -module V , write $LV = L \otimes_E V$ and $L(f) = 1_L \otimes f$ for a module map f . Evaluation $L \mapsto LV$ therefore preserves these sequences, and sends the middle E^e -projective modules to E -projectives. It commutes with stable shifts and their connecting maps.

3 A signed ten-dimensional algebra

Fix $q \in k^\times$ of infinite order. Let C have basis

$$e, f, x, y, z, u, v, t, j, n$$

with corner spaces

$$eCe = \langle e, x, y, z \rangle_k, \quad eCf = \langle u, v \rangle_k, \quad fCe = \langle t, j \rangle_k, \quad fCf = \langle f, n \rangle_k.$$

The brackets denote k -linear span. The elements e, f are orthogonal idempotents with sum 1, acting according to these corners. The nonzero products of the other basis elements are

$$\begin{aligned} xy &= -qz, & yx &= z, & xu &= v, & yu &= -v, \\ tx &= j, & ty &= -q^2j, & ut &= y + qx, & vt &= -qz, \\ uj &= z, & tu &= (q - 1)n, & nt &= -qj, & un &= v. \end{aligned} \tag{3}$$

All omitted products of these eight elements are zero. For any displayed basis, a star denotes its dual basis in the k -linear dual: each functional is one on the corresponding basis vector and zero on the others.

Lemma 3.1. *The table defines an associative algebra with radical spanned by the eight nonidempotent basis elements. Its radical has fifth power zero and fourth power nonzero. The assignment*

$$\gamma(e, f, z, u, j) = (e, f, z, u, j), \quad \gamma(x, y, v, t, n) = (-x, -y, -v, -t, -n) \tag{4}$$

defines an involutive automorphism. For $a, b \in \{e, f\}$,

$$\mathrm{Tr}_k(\gamma|_{aCb}) = 0. \tag{5}$$

Here Tr_k is the ordinary linear trace, the sum of diagonal matrix entries in any k -basis.

Proof. Give u, t degree one, x, y, n degree two, v, j degree three, and z degree four. Products in (3) preserve this grading. A radical triple of total degree greater than four has both products zero. The composable radical triples of total degree at most four are

$$utu, tut, xut, yut, ntu, unt, txu, tyu, utx, uty, tun.$$

In this order both bracketings have values

$$(q-1)v, -q(q-1)j, -qz, qz, 0, -qz, 0, 0, z, -q^2z, 0.$$

Triples involving idempotents follow from the corners. Thus multiplication is associative. The positive-degree ideal is nilpotent with quotient k^2 , so it is the radical. Moreover $utut = q(1-q)z \neq 0$.

The signs in (4) multiply correctly in each of the twelve nonzero products, and preserve zero products and corners. The eigenvalue lists on the four corners are $(1, -1, -1, 1)$, $(1, -1)$, $(-1, 1)$, and $(1, -1)$, whose sums vanish in every characteristic. $\square$

Let s be the one-dimensional left C -module on which f acts as 1 and all other basis elements act as zero. For $i \in \mathbb{Z}$ set $\ell_i = x + q^i y$. Write ρ_a for right multiplication by a between the appropriate left ideals, and λ_a for left multiplication between right ideals.

Proposition 3.2. *A projective resolution R of s is*

$$R_0 = Cf, \quad R_i = Ce \ (i \geq 1), \quad d_1 = \rho_u, \quad d_{i+2} = \rho_{\ell_i} \ (i \geq 0). \quad (6)$$

It satisfies

$$H^a \operatorname{Hom}_C(R, s) = \begin{cases} k & a = 0, \\ 0 & a \neq 0, \end{cases} \quad H^a \operatorname{Hom}_C(R, C) = \begin{cases} Ds & a = 2, \\ 0 & a \neq 2, \end{cases}$$

where the second identification is one of right C -modules.

Proof. On the basis (e, x, y, z, t, j) of Ce , the images under the differentials are

$$\rho_u = (u, v, -v, 0, (q-1)n, 0), \quad \rho_{\ell_i} = (\ell_i, -q^{i+1}z, z, 0, (1-q^{i+2})j, 0).$$

Since $q-1$ and $1-q^{i+2}$ are nonzero for $i \geq 0$, the image and kernel of ρ_u are $\langle u, v, n \rangle_k$ and $\langle \ell_0, z, j \rangle_k$. Those of ρ_{ℓ_i} are $\langle \ell_i, z, j \rangle_k$ and $\langle \ell_{i+1}, z, j \rangle_k$. The first image is $\operatorname{rad}(Cf)$, proving exactness. Applying $\operatorname{Hom}_C(-, s)$ leaves only the degree-zero term k .

Applying $\operatorname{Hom}_C(-, C)$ gives

$$fC \xrightarrow{\lambda_u} eC \xrightarrow{\lambda_{\ell_0}} eC \xrightarrow{\lambda_{\ell_1}} eC \longrightarrow \dots$$

On (e, x, y, z, u, v) ,

$$\lambda_{\ell_i} = (\ell_i, q^i z, -qz, 0, (1-q^i)v, 0).$$

The first map sends (f, t, j, n) to $(u, y + qx, z, v)$ and is injective. At $i = 0$ the image is $\langle \ell_0, z \rangle_k$ and the kernel is $\langle \ell_{-1}, z, u, v \rangle_k$. For $i \geq 1$ the image and kernel are $\langle \ell_i, z, v \rangle_k$ and $\langle \ell_{i-1}, z, v \rangle_k$. The sole cohomology class is $\bar{v}$ in degree two. It satisfies $vf = v$, and its only nonzero radical right product is $vt = -qz$, a boundary. This identifies the right module as Ds . $\square$

Proposition 3.3. *The dual regular module DC has projective dimension at most two on both sides. More precisely,*

$$D(eC) \cong Ce, \quad D(Ce) \cong eC,$$

and there are exact sequences

$$0 \rightarrow Cf \xrightarrow{\rho_t} Ce \xrightarrow{\rho_{\ell-2}} Ce \xrightarrow{e \mapsto j^*} D(fC) \rightarrow 0, \quad (7)$$

$$0 \rightarrow fC \xrightarrow{\lambda_u} eC \xrightarrow{\lambda_{\ell_0}} eC \xrightarrow{e \mapsto v^*} D(Cf) \rightarrow 0. \quad (8)$$

Proof. The coefficient of z in a product defines a pairing $eC \times Ce \rightarrow k$. In the bases above it has determinant $-q^2$, so its adjoints give the two module isomorphisms. The last map in (7) has images $(j^*, t^*, -q^2 t^*, 0, -qn^*, f^*)$ and kernel $\langle \ell_{-2}, z \rangle_k$. The middle map has that image and kernel $\langle y + qx, z, t, j \rangle_k$. The first map has the independent images $(t, y + qx, -qz, -qj)$. In (8) the last map has images $(v^*, u^*, -u^*, 0, n^*, f^*)$ and kernel $\langle \ell_0, z \rangle_k$. The middle map has that image and kernel $\langle y + qx, z, u, v \rangle_k$, the image of the injective first map. $\square$

For any finite-dimensional algebra R , its trivial extension is $R \ltimes DR$, with multiplication

$$(a, \phi)(b, \psi) = (ab, a\psi + \phi b), \quad (a\phi b)(c) = \phi(bca).$$

It is symmetric: $(a, \phi) \mapsto \phi(1)$ gives the pairing $((a, \phi), (b, \psi)) \mapsto \phi(b) + \psi(a)$. Set

$$T = C \ltimes DC, \quad B = C \otimes C \subset E = T \otimes T, \quad X = s \otimes s. \quad (9)$$

The C -module s is viewed as a T -module via $T \rightarrow C$, and similarly for X . The algebra E is symmetric of dimension 400. Tensoring the resolutions in Proposition 3.3 gives

$$\text{pd}_B({}_B E) \leq 4, \quad \text{pd}_{B^{\text{op}}}(E_B) \leq 4, \quad \text{id}_B({}_B B) \leq 4. \quad (10)$$

The last inequality follows by dualizing a right projective resolution of $DB = DC \otimes DC$.

4 Self-extensions and twisting weights

The Yoneda algebra $\text{Ext}_T^*(s, s) = \bigoplus_{a \geq 0} \text{Ext}_T^a(s, s)$ has the product obtained by splicing extensions; $|\tau|$ denotes the cohomological degree of a homogeneous class τ .

Proposition 4.1. *There is a derived triangle of T -modules*

$$s[2] \longrightarrow \mathcal{J} := T \otimes_C R \longrightarrow s \xrightarrow{\tau} s[3], \quad (11)$$

and $\text{Ext}_T^*(s, s) = k[\tau]$ with $|\tau| = 3$.

Proof. Tensor $0 \rightarrow DC \rightarrow T \rightarrow C \rightarrow 0$ with the projective terms of R . The resulting exact sequence has left complex $DC \otimes_C R$. The evaluation isomorphism

$$DC \otimes_C R_j \cong D \text{Hom}_C(R_j, C), \quad \phi \otimes v \mapsto (f \mapsto \phi(f(v)))$$

and Proposition 3.2 identify its only homology as s in degree two. Good truncation gives (11). Its middle term is a bounded-below projective model, and adjunction gives

$$\text{Hom}_{D^b(T)}(\mathcal{J}, s[i]) = H^i \text{Hom}_C(R, s) = \begin{cases} k & i = 0, \\ 0 & i \neq 0. \end{cases}$$

The long exact Hom sequence gives $\text{Ext}_T^1(s, s) = \text{Ext}_T^2(s, s) = 0$ and, for $i \geq 2$, an isomorphism

$$\text{Ext}_T^{i-3}(s, s) \longrightarrow \text{Ext}_T^i(s, s), \quad \alpha \longmapsto \alpha[3]\tau,$$

with negative ordinary Ext zero. Hence τ^m is a basis in degree $3m$ for every $m \geq 0$. $\square$

Write $\mathcal{H}^a = \underline{\text{Hom}}_E(X, X[a])$ for $a \in \mathbb{Z}$ and $\mathcal{H}^{\geq 0} = \bigoplus_{a \geq 0} \mathcal{H}^a$, and let $\tau_1 = \tau \otimes 1$, $\tau_2 = 1 \otimes \tau$ be the external classes. Products mean shifted composition: for f of degree r and g of degree s , $fg = f[s] \circ g$. Let $\mathcal{P}_m$ be the span of the ordered monomials $\tau_1^a \tau_2^b$ with $a, b \geq 0$ and $a + b = m$; put $\mathcal{P}_m = 0$ for $m < 0$.

Proposition 4.2. *The module X is nonprojective, and*

$$\mathcal{H}^{\geq 0} = k\langle \tau_1, \tau_2 \rangle / (\tau_1 \tau_2 + \tau_2 \tau_1), \quad \mathcal{H}^{3m} = \mathcal{P}_m, \quad \mathcal{H}^{-3m-1} = D\mathcal{P}_m \quad (m \geq 0). \quad (12)$$

All other stable degrees are zero. Under suitable dual bases $\eta_{a,b}$ of $D\mathcal{P}_{a+b}$, left multiplication is

$$\tau_1 \eta_{a,b} = \begin{cases} (-1)^b \eta_{a-1,b} & a > 0, \\ 0 & a = 0, \end{cases} \quad \tau_2 \eta_{a,b} = \begin{cases} \eta_{a,b-1} & b > 0, \\ 0 & b = 0. \end{cases} \quad (13)$$

Proof. The tensor product of two projective resolutions of s , with $|v|$ denoting the homological degree of a homogeneous element and differential $d(v \otimes w) = dv \otimes w + (-1)^{|v|} v \otimes dw$, resolves X . Künneth and tensor products of comparison maps identify its Yoneda algebra with the graded tensor product of two copies of $k[\tau]$. Interchanging the two degree-three maps gives $(-1)^9 = -1$, proving the relation and the ordered monomial basis. The degree-three group is nonzero, so X is nonprojective. Since $\text{End}_E(X) = k$, its stable degree-zero endomorphism ring is k as well.

Lemma 2.2 gives the negative degrees. Choose a nonzero linear form $\lambda : \mathcal{H}^{-1} \rightarrow k$ and identify a negative class $\beta \in \mathcal{H}^{-3m-1}$ with the functional

$$p \mapsto \lambda(p[-3m-1]\beta), \quad p \in \mathcal{P}_m.$$

Associativity makes postcomposition by τ_i transpose to right multiplication by τ_i on ordered monomials. Right multiplication by τ_1 introduces $(-1)^b$; right multiplication by τ_2 introduces no sign. This gives (13), including its zero boundary at $m = 0$. $\square$

For $H \in k^\times$ define automorphisms of T by

$$h_H(c, \phi) = (c, H\phi), \quad \tilde{\gamma}(c, \phi) = (\gamma(c), \phi \circ \gamma^{-1}).$$

They commute. Put

$$\sigma_H = (\tilde{\gamma} h_H) \otimes (\tilde{\gamma} h_H), \quad U_H = E_{\sigma_H}. \quad (14)$$

The notation E_{σ_H} means usual left action and right action $a \cdot b = a\sigma_H(b)$. Its tensor functor is restriction along σ_H^{-1} , via $a \otimes v \mapsto \sigma_H^{-1}(a)v$. Let $\chi : E \rightarrow k$ be the character of X . Since $\chi\sigma_H = \chi$, this fixes $U_H X = X$ and an action on every $\mathcal{H}^a$.

Lemma 4.3. *Let G_a be the invertible operator on $\mathcal{H}^a$ induced by the twist $E_{\tilde{\gamma} \otimes \tilde{\gamma}}$. For all $H \neq 0$ and $m \geq 0$,*

$$U_H|_{\mathcal{H}^{3m}} = H^{-m} G_{3m}, \quad U_H|_{\mathcal{H}^{-3m-1}} = H^{m+2} G_{-3m-1}. \quad (15)$$

Proof. First twist only by h_H . Applying h_H on the coefficient of $\mathcal{J}$ compares the twisted triangle (11) with the original one. It is the identity on the quotient s and multiplication by H on the homology $s[2]$ of $DC \otimes_C R$. Naturality of the connecting map gives $h_H(\tau) = H^{-1}\tau$. On the two-factor algebra this yields the scalar H^{-m} in degree $3m$.

Put $p = f \otimes f$ and $\zeta = f^* \otimes f^* \in Ep$. The map $X \rightarrow Ep$, $1 \mapsto \zeta$, has image in $\text{rad}(E)p = \Omega X$ and is nonzero stably. Indeed, every map $X \rightarrow E$ has image in the left socle. Under the symmetric bimodule isomorphism $E \cong DE$, the left socle consists of functionals vanishing on $\text{rad } E$. These are also annihilated by $\text{rad } E$ on the right. A map $E \rightarrow \text{rad}(E)p$ is

right multiplication by a radical element, hence kills this socle. All projective factorizations of $X \rightarrow \Omega X$ are therefore zero.

The map $1 \mapsto \zeta$ spans $\mathcal{H}^{-1}$. Comparing the twisted projective cover via $h_H \otimes h_H$ sends ζ to $H^2\zeta$. The perfect composition pairing in Proposition 4.2 now gives weight H^{m+2} on $D\mathcal{P}_m$. Finally compose with the fixed commuting twist induced by $\tilde{\gamma} \otimes \tilde{\gamma}$. $\square$

5 Derived induction and a twisted trace

The two-sided bar model K for $E \otimes_B^{\mathbf{L}} E$ has

$$K_i = E \otimes B^{\otimes i} \otimes E \quad (i \geq 0), \quad K_i = 0 \quad (i < 0),$$

and differential, for $i \geq 1$,

$$\begin{aligned} d_i(a \otimes b_1 \otimes \cdots \otimes b_i \otimes c) &= ab_1 \otimes b_2 \otimes \cdots \otimes b_i \otimes c \\ &\quad + \sum_{r=1}^{i-1} (-1)^r a \otimes \cdots \otimes b_r b_{r+1} \otimes \cdots \otimes c \\ &\quad + (-1)^i a \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes b_i c. \end{aligned}$$

Multiplication in degree zero gives $m : K \rightarrow E$.

Lemma 5.1. *The stable image $\mathcal{I} = \langle K \rangle$ has a finite representative projective on both E -sides, and there is such a representative Y of $\mathcal{I}[-1]$. It satisfies*

$$W^a := \underline{\mathrm{Hom}}_E(X, YX[a]) = \begin{cases} k & a = -3, 0, \\ 0 & a \notin \{-3, 0\}. \end{cases} \quad (16)$$

Proof. The bar terms are finite E^e -free modules. On the left, compare the bar resolution of ${}_B E$ with a length-four projective resolution furnished by (10). Inducing preserves the homotopy equivalence. Thus K , as a left E -complex, is homotopy equivalent to a projective complex in degrees $0, \dots, 4$. The right version is identical using E_B . In degree five a contracting homotopy gives $1_{K_5} = d_6 h_5 + h_4 d_5$. Passing to $c_5 K$ factors its identity through the corresponding one-sided projective K_4 . Hence $c_5 K$ is projective on both sides and $\mathcal{I} = (c_5 K)[5]$.

Starting with $N_0 = c_5 K$, take four sequences

$$0 \rightarrow N_r \rightarrow \mathrm{Hom}_k(E^e, N_r) \rightarrow N_{r+1} \rightarrow 0, \quad n \mapsto (a \mapsto an), \quad 0 \leq r < 4.$$

The middle term has E^e -action by right multiplication on its input, and is finite injective, hence projective. Each sequence splits on both E -sides. Taking $Y = N_4$ gives $Y \simeq \mathcal{I}[-1]$.

Evaluation of the bar model at X is homotopy equivalent to $\mathcal{J} \otimes \mathcal{J}$: before inducing, both models resolve X over B . Represent the triangle (11) by a two-cell projective model and tensor two copies. After shifting by $[-4]$, the three filtration quotients represent

$$X, \quad X[-2] \oplus X[-2], \quad X[-4].$$

Write t for matrix transpose. With middle terms ordered by the first and second attaching class, the adjacent connecting maps are (τ_1, τ_2) and $(\tau_2, \tau_1)^t$. To see their signs, write a two-cell differential as $\begin{pmatrix} d & t \\ 0 & d \end{pmatrix}$ with t a degree-one cross differential. In its tensor square the cross map on the second factor is multiplied by $(-1)^{|v|}$. Since the subcell represents $s[2]$, the two-cell shifts are even, and both resulting column entries have the displayed positive sign. Their composite is $\tau_1 \tau_2 + \tau_2 \tau_1 = 0$.

For $m \geq 0$ the left multiplication sequence

$$0 \rightarrow \mathcal{P}_{m-1} \xrightarrow{(\tau_2, \tau_1)^t} \mathcal{P}_m^{\oplus 2} \xrightarrow{(\tau_1, \tau_2)} \mathcal{P}_{m+1} \rightarrow 0 \quad (17)$$

is exact. The row is onto, the column is injective, and their composite is zero; the middle dimensions are $2(m+1) - (m+2) = m$. The negative-degree calculation is the dual of the right multiplication sequence, with the middle slots exchanged, using (13).

For $l \in \mathbb{Z}$, write

$$d_l : (\mathcal{H}^{l-2})^{\oplus 2} \rightarrow \mathcal{H}^{l+1}, \quad d_l(u, v) = \tau_1 u + \tau_2 v, \quad d'_l : \mathcal{H}^{l-4} \rightarrow (\mathcal{H}^{l-1})^{\oplus 2}, \quad d'_l(w) = (\tau_2 w, \tau_1 w).$$

The exact sequences just proved, including $\mathcal{H}^{-1} \rightarrow \mathcal{H}^2 = 0$, give

$$\ker d_l = \operatorname{im} d'_{l-1}, \quad \operatorname{coker} d_{l-1} = \begin{cases} k & l = 0, \\ 0 & l \neq 0, \end{cases} \quad \ker d'_l = \begin{cases} k & l = 3, \\ 0 & l \neq 3. \end{cases} \quad (18)$$

Let $\mathcal{L}_1$ be the stable image of the bottom two layers and put $N^l = \underline{\operatorname{Hom}}_E(X, \mathcal{L}_1[l])$. The first triangle gives

$$0 \rightarrow \operatorname{coker} d_{l-1} \rightarrow N^l \xrightarrow{q_l} \ker d_l \rightarrow 0.$$

The top connecting map $r_l : \mathcal{H}^{l-4} \rightarrow N^{l+1}$ satisfies $q_{l+1} r_l = d'_l$. Thus $\ker r_l = 0$ unless $l = 3$; there $N^4 = 0$ and the kernel is k . Also $q_l(\operatorname{im} r_{l-1}) = \ker d_l$. For $l \neq 0$, q_l is an isomorphism, so r_{l-1} is onto. At $l = 0$, $N^0 = k$ and the source $\mathcal{H}^{-5}$ is zero. The second triangle therefore gives stable Hom equal to k precisely in degrees 0, 3 for $(\mathcal{I}X)[-4]$. Since $YX = (\mathcal{I}X)[-1]$, this is (16). $\square$

Put $\theta = \gamma \otimes \gamma \in \operatorname{Aut}(B)$ and let $\chi_B = \chi|_B$. The twisted dual $(DB)_\theta$ has left action $(a\lambda)(c) = \lambda(ca)$ and right action $\lambda \cdot b = \lambda\theta(b)$, where $(\lambda b)(c) = \lambda(bc)$ denotes the usual dual action. Since $\chi_B \theta = \chi_B$, the formula

$$\eta_\theta : B \rightarrow (DB)_\theta, \quad a \mapsto \chi_B(a)\chi_B \quad (19)$$

defines a bimodule map.

Lemma 5.2. *If a bimodule map $B \rightarrow (DB)_\theta$ factors in $\operatorname{D}^b(B^e)$ through a perfect object, it factors by ordinary bimodule maps through a module in $\operatorname{add}(DB \otimes B)$, with the outer bimodule actions on this tensor product.*

Proof. For a bimodule L , the natural embedding

$$j_L : L \rightarrow \operatorname{Hom}_k(B, L), \quad j_L(z)(c) = cz, \quad (a\phi b)(c) = \phi(ca)b$$

is a bimodule map. Iterate its cokernel four times. For $L = B \otimes B$, this is the left coinduced coresolution of B tensored with the right regular module. The four coinduced terms belong to $\operatorname{add}(DB \otimes B)$. The fourth cokernel does as well: its left factor is injective by $\operatorname{id}_B B \leq 4$ and dimension shifting. These additive functors preserve split summands, so the construction applies to all finite projective bimodules.

Apply this finite functorial coresolution to the terms of a bounded projective intermediate complex P . Signed totalization gives a bounded complex P' with terms in $\operatorname{add}(DB \otimes B)$ and a quasi-isomorphism $P \rightarrow P'$. For each such term V and $i > 0$,

$$\operatorname{Ext}_{B^e}^i(B, V) = 0, \quad \operatorname{Ext}_{B^e}^i(V, (DB)_\theta) = 0.$$

For $V = DB \otimes B$, the first assertion follows from coinduction and the projectivity of the right regular module B . For the second, tensor a left projective resolution of DB with the

right regular module, and apply Hom into $(DB)_\theta$. It reduces to left-module Hom into DB , which is injective. Sums and summands preserve both assertions.

Using a projective resolution of B and an injective coresolution of $(DB)_\theta$, finite filtrations by the terms of P' show that both derived arrows have chain-map representatives $B[0] \rightarrow P' \rightarrow (DB)_\theta[0]$. Their composite factors through P'_0 . Degree-zero derived Hom between modules is ordinary Hom, so this composite is the specified map. $\square$

Proposition 5.3. *The map η_θ does not factor through a perfect B^e -complex.*

Proof. By Lemma 5.2, it suffices to consider ordinary factorizations through $DB \otimes B$. Identify that module with $\text{Hom}_k(B, B)$ by $\lambda \otimes c \mapsto [x \mapsto \lambda(x)c]$. For a map $\alpha : B \rightarrow DB \otimes B$, the balance condition on $\alpha(1)$ says that its associated operator is right B -linear, hence left multiplication by some $d \in B$. If $(c_i), (c_i^*)$ are dual bases,

$$\alpha(1) = \sum_i c_i^* \otimes dc_i.$$

For $\beta : DB \otimes B \rightarrow (DB)_\theta$, the map $\lambda \mapsto \beta(\lambda \otimes 1)$ is a left-module endomorphism of DB , hence right multiplication by some $b' \in B$. Right linearity then gives

$$\beta(\lambda \otimes c) = \lambda b' \theta(c).$$

For $p = f \otimes f$, and with L_a, R_a denoting left and right multiplication on B , respectively, we obtain

$$(\beta\alpha)(1)(p) = \sum_i c_i^* (b' \theta(d) \theta(c_i) p) = \text{Tr}_k(L_{b' \theta(d)} \theta R_p). \quad (20)$$

Write $b' \theta(d)$ as a radical element plus a linear combination of the four vertex idempotents $a \in \{e \otimes e, e \otimes f, f \otimes e, f \otimes f\}$. Left multiplication by the radical term, composed with θR_p , strictly raises the radical filtration and therefore has trace zero. The contribution at a is the trace of θ on aBp , the product of the two corresponding corner traces of γ . It vanishes by (5). Thus every composite (20) vanishes. The same holds for finite sums and their direct summands. But $\eta_\theta(1)(p) = 1$. $\square$

Proposition 5.4. *For every $H \in k^\times$ there is an ordinary bimodule map $g_H : U_H \rightarrow Y$ whose evaluation represents a nonzero element of W^0 in (16). The same Y works for every H .*

Proof. Set $S = \text{Hom}_k(X, X)$, with action $(atb)(x) = at(bx)$, and let $\pi : E \rightarrow S$ be the representation. Thus $S = k$ with both actions given by χ . Write $\overline{m} : \mathcal{I} \rightarrow E$ for the stable image of m and $\phi = \pi \overline{m}$. The element $\zeta = f^* \otimes f^*$ satisfies $a\zeta b = \chi(a)\chi(b)\zeta$, so $b_0 : S \rightarrow U_H$, $1 \mapsto \zeta$, is a bimodule map.

Suppose $b_0 \phi$ were zero stably. Lemma 2.3 makes the derived map $b_0 \pi m : K \rightarrow U_H$ factor through a perfect E^e -complex. On restriction to B^e the intermediate object remains perfect: a free E^e -module restricts to $E \otimes E$, whose projective dimension over B^e is at most eight by (10). The same follows for finite sums, summands, and bounded complexes.

Let $\text{Bar}(B)$ be the bar complex defined above with both outer copies of E replaced by B and augmentation given by multiplication. Inclusion of the outer factors gives $\text{Bar}(B) \rightarrow K$ as a B^e -chain map. Since $\sigma_H|_B = \theta$, the four summands of $\text{Res}_{B^e} U_H$ are the right- θ twists of B , $DC \otimes C$, $C \otimes DC$, and DB . Projecting to the last summand, the composite from $\text{Bar}(B)$ is η_θ composed with the augmentation: in degree zero it sends $a \otimes c$ to $\chi_B(ac)\chi_B$. This contradicts Proposition 5.3. Hence $b_0 \phi \neq 0$ stably.

By Lemma 2.2 over E^e , the transpose of

$$\phi_* : \underline{\text{Hom}}_{E^e}(U_H[1], \mathcal{I}) \rightarrow \underline{\text{Hom}}_{E^e}(U_H[1], S)$$

is $b \mapsto b\phi$ from $\underline{\text{Hom}}_{E^e}(S, U_H)$ to $\underline{\text{Hom}}_{E^e}(\mathcal{I}, U_H)$. It is nonzero on b_0 , so choose β with $\phi\beta \neq 0$.

For every finite bimodule L projective on both sides, ordinary adjunction descends to

$$\underline{\text{Hom}}_{E^e}(L, S) \cong \underline{\text{Hom}}_E(LX, X). \quad (21)$$

One factorization ideal maps into the other by evaluation. Conversely, embed L into a finite E^e -projective-injective J . This embedding splits on the right, so $LX \rightarrow JX$ is injective. If $LX \rightarrow X$ factors through an E -projective P , injectivity of P extends the first factor to JX . Adjunction gives $J \rightarrow S$ extending the original map. This proves (21).

Applying it to $U_H[1]$, the nonzero $\phi\beta$ corresponds to the evaluated composite $X[1] \rightarrow \mathcal{I}X \rightarrow X$. Thus βX is nonzero. Shift β by $[-1]$, identify $\mathcal{I}[-1]$ with the fixed Y , and choose an ordinary representative g_H . $\square$

6 The triangular module

Take $H_1 = q$, $H_2 = 1$, and put $U_i = U_{H_i}$ for $i = 1, 2$. Let $g_i : U_i \rightarrow Y$ be scalar multiples of the maps g_{H_i} from Proposition 5.4, chosen so that both evaluated stable maps equal one fixed nonzero $w \in W^0$. Choose a finite free E^e -module Q surjecting onto Y , with map π_Q , and define F by

$$0 \rightarrow F \rightarrow U_1 \oplus U_2 \oplus Q \xrightarrow{(g_1, -g_2, \pi_Q)} Y \rightarrow 0. \quad (22)$$

It splits on each E -side, so F is finite and projective on both sides. Let $\mathcal{V}^a = \underline{\text{Hom}}_E(X, FX[a])$. Evaluation gives

$$(\mathcal{H}^{a-1})^{\oplus 2} \rightarrow W^{a-1} \rightarrow \mathcal{V}^a \xrightarrow{p^a} (\mathcal{H}^a)^2 \rightarrow W^a. \quad (23)$$

At degree zero the last map is $(c, d) \mapsto (c - d)w$. Thus $\mathcal{V}^0 = kv_0$ with $p^0(v_0) = (1, 1)$. For $g, h \in \mathcal{H}^a$, define

$$\Delta^a(g, h) = F(g)v_0 - v_0[a]h \in \mathcal{V}^a,$$

where $F(g)$ uses the exact functor $F \otimes_E -$ and its stable shift comparison.

Lemma 6.1. *The map Δ^0 is onto; Δ^a is an isomorphism for $a > 0$, $a \leq -3$, and $a = -1$; and $\Delta^{-2} : 0 \rightarrow k$ is injective.*

Proof. By (23), p^a is an isomorphism for $a > 0$ and $a \leq -4$. At $a = 1$ this uses surjectivity onto W^0 . The remaining degrees give

$$\mathcal{V}^{-3} = 0, \quad \mathcal{V}^{-2} = k, \quad \mathcal{V}^{-1} = k^2, \quad \mathcal{V}^0 = kv_0, \quad \mathcal{V}^1 = 0.$$

Naturality of the bimodule projections $F \rightarrow U_i$ gives

$$p^a \Delta^a(g, h) = (U_1(g) - h, U_2(g) - h).$$

On $\mathcal{H}^{3m}$, $m > 0$, this is the block matrix

$$\begin{pmatrix} q^{-m}G_{3m} & -1 \\ G_{3m} & -1 \end{pmatrix};$$

subtracting its two rows leaves the invertible block $(q^{-m} - 1)G_{3m}$. On $\mathcal{H}^{-3m-1}$, $m \geq 0$, the corresponding block is $(q^{m+2} - 1)G_{-3m-1}$, again invertible. In the other stated degrees the groups are zero. Finally $\Delta^0(c, d) = (c - d)v_0$, and $\mathcal{H}^{-2} = 0$. $\square$

Form

$$\Lambda = \begin{pmatrix} E & 0 \\ F & E \end{pmatrix}. \quad (24)$$

A left module is a triple $(L, N, \alpha : F \otimes_E L \rightarrow N)$, with action $\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}(x, y) = (ax, \alpha(b \otimes x) + dy)$. The two exact column functors are

$$D_0(L) = (L, FL, 1), \quad D_1(L) = (0, L, 0).$$

They preserve projectives, and module maps give

$$\begin{aligned} \operatorname{Hom}_\Lambda(D_i L, D_i L') &= \operatorname{Hom}_E(L, L') \quad (i = 0, 1), \\ \operatorname{Hom}_\Lambda(D_0 L, D_1 L') &= 0, \quad \operatorname{Hom}_\Lambda(D_1 L, D_0 L') = \operatorname{Hom}_E(L, FL'). \end{aligned} \quad (25)$$

Put $p = f \otimes f$ and $\zeta = f^* \otimes f^*$. Choose a complete resolution P of X with $P_{-1} = Ep$ and connecting embedding $i_X : X \rightarrow Ep$, $1 \mapsto \zeta$. Lift an ordinary representative $\tilde{v} : X \rightarrow FX$ of v_0 to a chain map $P \rightarrow FP$ using Lemma 2.1. Through (25) this is a chain map $V : D_1(P) \rightarrow D_0(P)$. Put

$$P_Z = \operatorname{Cone}(V), \quad Z = c_0 P_Z.$$

Proposition 6.2. *The complex P_Z is totally acyclic, Z is finite and nonprojective, and*

$$H^a \operatorname{Hom}_\Lambda(P_Z, Z) = \begin{cases} k & a = -1, 0, \\ 0 & a \notin \{-1, 0\}. \end{cases} \quad (26)$$

Proof. Both column complexes are exact and projective. Their Hom complexes into $\Lambda = D_0(E) \oplus D_1(E)$ are sums of $\operatorname{Hom}_E(P, E)$, $\operatorname{Hom}_E(P, F)$, and zero. These are exact because F is left projective. Their cone is therefore totally acyclic.

Taking its degree-zero cokernel, the upper component of Z is X and the lower component is

$$(FX \oplus Ep) / \{(\tilde{v}(x), -i_X(x)) : x \in X\}.$$

The structure map is induced by inclusion of FX . Hence Z is finite-dimensional. The upper-component functor sends projective Λ -modules to projective E -modules. Since X is nonprojective, so is Z .

Write $A_0 = D_0(P)$ and $B_0 = D_1(P)$. In the graded decomposition $P_Z = A_0 \oplus B_0[1]$, the Hom block from A_0 to $B_0[1]$ vanishes. Thus there is an exact sequence of full Hom complexes

$$0 \rightarrow \operatorname{Hom}_\Lambda^\bullet(B_0[1], A_0) \rightarrow \operatorname{End}_\Lambda^\bullet(P_Z) \rightarrow \operatorname{End}_\Lambda^\bullet(A_0) \oplus \operatorname{End}_\Lambda^\bullet(B_0[1]) \rightarrow 0.$$

The cross-block degree a reindexes as $\operatorname{Hom}_E^{a-1}(P, FP)$. Its differential under that reindexing is $df + (-1)^a fd$, exactly the differential in degree $a - 1$ of the unshifted Hom complex. Its cohomology is therefore $\mathcal{V}^{a-1}$.

To identify the connecting map, a degree- a cocycle h on B_0 corresponds to $(-1)^a h$ on $B_0[1]$. The cross block of the Hom differential of a diagonal lift $(g, (-1)^a h)$ is $(-1)^a (Vh - F(g)V)$. On cohomology this is $(-1)^{a+1} \Delta^a$. Hence

$$0 \rightarrow \operatorname{coker} \Delta^{a-1} \rightarrow H^a \operatorname{End}_\Lambda(P_Z) \rightarrow \ker \Delta^a \rightarrow 0.$$

Lemma 6.1 makes both ends zero for $a > 0$ and $a \leq -2$, including $a = 1$ by surjectivity of Δ^0 and $a = -2$ by injectivity of Δ^{-2} . In degree -1 the cokernel of Δ^{-2} is k ; in degree zero the kernel of Δ^0 is k . The other end vanishes in both cases. Lemma 2.1 identifies these groups with (26). $\square$

7 Transfer to a symmetric algebra

Proposition 7.1. *Let R be a finite-dimensional algebra and P a totally acyclic complex of finite projectives. Suppose $Z = c_0 P$ is nonprojective and*

$$H^a \operatorname{Hom}_R(P, Z) = 0 \quad (a > 0 \text{ or } a \leq -2).$$

Then $A = R \ltimes DR$ is symmetric and $M = A \otimes_R Z$ is a finite nonprojective self-orthogonal A -module.

Proof. Put $I = DR \otimes_R P$. For every term, evaluation gives

$$DR \otimes_R P_j \cong D \operatorname{Hom}_R(P_j, R), \quad \operatorname{Hom}_R(Z, DR \otimes_R P_j) \cong D \operatorname{Hom}_R(P_j, Z).$$

These identifications reverse the Hom degree; the usual dual-complex signs do not change kernels and images. Total acyclicity makes I exact, and its terms are injective. The complex $A \otimes_R P$ is $P \oplus I$ on restriction to R , hence exact, with projective A -terms and cokernel M .

Let $N = DR \otimes_R Z = c_0 I$. The exact negative tail gives an injective coresolution $0 \rightarrow N \rightarrow I_{-1} \rightarrow I_{-2} \rightarrow \cdots$. Consequently, for $i > 0$,

$$\operatorname{Ext}_R^i(Z, N) = DH^{-i-1} \operatorname{Hom}_R(P, Z) = 0.$$

Adjunction on the projective resolution $A \otimes_R P_{\geq 0}$ gives

$$\operatorname{Ext}_A^i(M, M) = \operatorname{Ext}_R^i(Z, Z \oplus N) = 0 \quad (i > 0).$$

The trivial-extension form proves symmetry. If M were a summand of A^n , tensoring that split inclusion and retraction along the quotient $A \rightarrow R$ would make $R \otimes_A M \cong Z$ a summand of R^n , a contradiction. $\square$

Proof of Theorem 1.1. Take Λ, Z, P_Z from Section 6. Total acyclicity and (26) give the asserted Ext vanishing over Λ . Proposition 7.1 gives $A = \Lambda \ltimes D\Lambda$ and $M = A \otimes_\Lambda Z$ with the required symmetric-algebra properties.

The constructions of Y, F, Z use finitely many kernels, cokernels, cosyzygies, and choices in finite Hom spaces, so all objects are finite-dimensional over k . The radical quotients of C, T, E, Λ, A are respectively k^2, k^2, k^4, k^8, k^8 : for the tensor product this uses the nilpotent ideal $\operatorname{rad} T \otimes T + T \otimes \operatorname{rad} T$; for the triangular and trivial extensions it uses their nilpotent off-diagonal and dual ideals. Thus A has eight simple modules. Projecting successively onto a diagonal block, one tensor factor, and C gives a surjective algebra map $A \rightarrow C$. Since $eu = u \neq ue = 0$ and $utut = q(1 - q)z \neq 0$, the algebra A is noncommutative and $(\operatorname{rad} A)^4 \neq 0$. $\square$

For any field extension K/k , projective resolutions with finite-dimensional terms give

$$K \otimes_k \operatorname{Ext}_R^i(U, V) \cong \operatorname{Ext}_{K \otimes R}^i(K \otimes U, K \otimes V).$$

Nonprojectivity is preserved: the nonzero class of a nonsplit finite projective presentation remains nonzero under faithful scalar extension. Finite-dimensional duality also commutes with scalar extension, so symmetry is preserved. In particular, the construction with $q = 2$ is defined over $\mathbb{Q}$ itself, and the construction with $q = t$ is defined over $\mathbb{F}_p(t)$ for every prime p .

8 Endomorphism algebras and Auslander's condition

The Gorenstein-projective conjecture asserts that every self-orthogonal Gorenstein-projective module is projective. The module Z in Theorem 1.1 is a counterexample over Λ . The module M is another over A , since every finite module over a symmetric algebra has a totally acyclic complete resolution.

Keep A, M from Theorem 1.1, and put

$$G = A \oplus M, \quad \Gamma = \text{End}_A(G)^{\text{op}}, \quad H = \text{Hom}_A(G, -).$$

The left Γ -action on $H(V)$ is precomposition. Regard G as an (A, Γ) -bimodule by $g \cdot f^{\text{op}} = f(g)$. The dominant dimension of an algebra is the least index of a nonprojective term in the minimal injective coresolution of its left regular module, and is infinite if all terms are projective.

Proposition 8.1. *The algebra Γ is non-self-injective and has infinite dominant dimension. There is a nonzero simple left module S with $\text{Ext}_\Gamma^i(S, \Gamma) = 0$ for all $i \geq 0$. Moreover,*

$$\text{Ext}_\Gamma^i(D\Gamma, \Gamma) = 0 \quad (i > 0), \quad D\Gamma \text{ is nonprojective.} \quad (27)$$

Its little and big left finitistic dimensions are infinite.

Proof. The symmetric property and self-orthogonality give $\text{Ext}_A^{>0}(G, G) = 0$. Choose an injective coresolution $0 \rightarrow G \rightarrow J^0 \rightarrow J^1 \rightarrow \cdots$ with finite terms. Every J^r lies in $\text{add } A$, and applying H gives an exact sequence

$$0 \rightarrow \Gamma \rightarrow H(J^0) \rightarrow H(J^1) \rightarrow \cdots. \quad (28)$$

These terms are projective since $J^r \in \text{add } G$. They are injective as well: if $e \in \Gamma$ is the idempotent corresponding to the summand A of G , then $G_\Gamma \cong e\Gamma$ and

$$H(A) \cong H(DA) \cong DG.$$

Thus (28) is an injective coresolution with projective terms. The minimal one is termwise a summand, giving infinite dominant dimension.

The functor H is fully faithful on $\text{add } G$: for sums of G this is the matrix form of evaluation on 1_G , and it passes to summands. If Γ were self-injective, the projective $H(M)$ would be injective. For an embedding $M \rightarrow J$ with $J \in \text{add } A$, applying H would give a split monomorphism. Full faithfulness would split the original embedding, making M injective and hence projective, a contradiction.

There is therefore an indecomposable injective module J that is not projective. Its socle S is simple and essential. A nonzero map $S \rightarrow H(J^r)$ extends to $J \rightarrow H(J^r)$; essentiality makes the extension injective, and injectivity of J makes it split. That would make J projective. Thus $\text{Hom}_\Gamma(S, H(J^r)) = 0$ for all r , proving the claimed Ext vanishing, including degree zero.

The idempotent identifications also give $\Gamma e \cong \text{Hom}_A(G, A)$, and symmetry supplies an isomorphism of left A -modules

$$G \otimes_\Gamma D\Gamma \cong eD\Gamma \cong D(\Gamma e) \cong D\text{Hom}_A(G, A) \cong G.$$

Apply tensor–Hom adjunction to (28). It identifies $\text{Hom}_\Gamma(D\Gamma, H(J^\bullet))$ with $\text{Hom}_A(G, J^\bullet)$, proving (27). If $D\Gamma$ were projective, every indecomposable injective would be projective; counting the finitely many indecomposable projective and injective types would make Γ

self-injective. Thus $D\Gamma$ is nonprojective. Its injectivity in the second variable also gives $\text{Ext}_\Gamma^{>0}(D\Gamma, D\Gamma) = 0$.

Finally write the minimal coresolution in short exact pieces

$$0 \rightarrow C_n \rightarrow I^n \rightarrow C_{n+1} \rightarrow 0, \quad C_0 = \Gamma.$$

All I^n are projective. The first sequence is nonsplit, since otherwise Γ would be injective. Its class in $\text{Ext}_\Gamma^1(C_1, \Gamma)$ is nonzero. Dimension shifting in the first variable gives $\text{Ext}_\Gamma^n(C_n, \Gamma) \neq 0$ for $n \geq 1$, while the finite truncated sequence gives $\text{pd } C_n \leq n$. Thus $\text{pd } C_n = n$. The supremum over finite modules of finite projective dimension (the little finitistic dimension) is infinite, and so is the supremum over all modules of finite projective dimension (the big one). $\square$

This gives direct witnesses for the Nakayama conjecture (infinite dominant dimension without self-injectivity), the generalized Nakayama conjecture (the injective J missing from the regular coresolution), and the strong Nakayama conjecture (the nonzero simple S with all $\text{Ext}_\Gamma^i(S, \Gamma)$ zero). Equation (27) gives the first Tachikawa conjecture and, using the injectivity of $D\Gamma$, another Auslander–Reiten witness on the same algebra. The endomorphism identity is the symmetric case of [5, Proposition 2.3].

The Auslander–Gorenstein conjecture asserts that an algebra whose minimal injective coresolution satisfies $\text{pd } I^n \leq n$ for every $n \geq 0$ has finite injective dimension on both sides. A module W is Wakamatsu tilting if it is self-orthogonal and the regular module has a coresolution by add W that stays exact under $\text{Hom}_\Gamma(-, W)$. A tilting module has finite projective dimension, is self-orthogonal, and admits a finite such coresolution of the regular module. The Wakamatsu tilting conjecture asserts that every Wakamatsu tilting module of finite projective dimension is tilting.

Corollary 8.2. *The algebra Γ is a counterexample to the Auslander–Gorenstein conjecture: its minimal injective coresolution has projective terms, but $\text{id}_\Gamma \Gamma = \infty$. The sum W of one representative of each indecomposable projective-injective left Γ -module is Wakamatsu tilting of projective dimension zero, but is not tilting.*

Proof. Every term I^n is projective, giving the condition. A finite such coresolution would split backwards and make Γ injective. The module W is projective and injective, hence self-orthogonal. Sequence (28) has terms in add W and remains exact under $\text{Hom}_\Gamma(-, W)$. Thus W is Wakamatsu tilting. A finite coresolution by add W would again split backwards, so W is not tilting. $\square$

Auslander’s condition (AC) says that for each finite module U there is an integer b_U such that, for every finite V , eventual vanishing of $\text{Ext}_R^i(U, V)$ implies vanishing for every $i > b_U$ [9]. The uniform condition requires a bound independent of U .

Proposition 8.3. *For every $n \geq 2$ and $i > 0$,*

$$\dim_k \text{Ext}_A^i(M, \Omega^n M) = \begin{cases} \dim_k \underline{\text{End}}_A(M) & i = n - 1, n, \\ 0 & i \notin \{n - 1, n\}. \end{cases}$$

In particular A fails AC and its uniform version.

Proof. Self-orthogonality and symmetric stable duality show that $\underline{\text{Hom}}_A(M, M[j])$ vanishes outside $j = -1, 0$, and has dimension $\dim_k \underline{\text{End}}_A(M)$ in those two degrees. This dimension is positive because the identity of a nonprojective module cannot factor through a projective. Since $\Omega^n M = M[-n]$ stably,

$$\text{Ext}_A^i(M, \Omega^n M) = \underline{\text{Hom}}_A(M, M[i - n]) \quad (i > 0).$$

Each of these sequences eventually vanishes, with last nonzero degree n . No fixed b_M is possible. $\square$

9 Finite-order parameters

In this section alone allow $q \in k^\times$ to have finite order $d \geq 1$, and define C, s by the same table. The associativity proof remains valid. A module's top is its quotient by its radical; a projective cover has the same top.

Proposition 9.1. *If $d \geq 2$, the minimal projective resolution of s has a period of length d from its second syzygy onward, and for $i > 0$,*

$$\dim_k \operatorname{Ext}_C^i(s, s) = \begin{cases} 1 & i = md \text{ or } md + 1 \text{ for some } m \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

If $q = 1$, then $\dim_k \operatorname{Ext}_C^1(s, s) = 1$.

Proof. For $q \neq 1$ put $K_i = \langle \ell_i, z, j \rangle_k \subset Ce$, with indices taken modulo d . The first two syzygies of s are $\operatorname{rad}(Cf)$ and K_0 . For $i \not\equiv -2 \pmod{d}$, right multiplication by ℓ_i is a projective cover $Ce \rightarrow K_i$, with kernel K_{i+1} . At $i = -2$, the radical of K_{-2} is kz , so its top has one simple at each vertex. Its cover is

$$Ce \oplus Cf \rightarrow K_{-2}, \quad (a, b) \mapsto a\ell_{-2} + bj.$$

Its kernel L has basis

$$(\ell_{-1}, 0), (-y, u), (z, 0), (0, v), (t, 0), (j, 0), (0, n).$$

The radical of L is spanned by all these vectors except $(-y, u)$ and $(t, 0)$. A cover of L is therefore

$$Ce \oplus Cf \rightarrow L, \quad (a, b) \mapsto (-ay + bt, au).$$

Its image spans L : the Ce -generator yields $(-y, u)$, (qz, v) , $(0, -v)$, and $(q^2j, (q-1)n)$; the Cf -generator yields $(t, 0)$, $(q\ell_{-1}, 0)$, $(-qz, 0)$, and $(-qj, 0)$. Its kernel has basis $(\ell_0, v), (z, 0), (j, 0)$ and projects isomorphically onto K_0 in the first coordinate. Indeed this kernel is the graph of the C -linear map $K_0 \rightarrow Cf$ sending ℓ_0 to v and z, j to zero; the action table verifies linearity on the generators.

The successive syzygies thus cycle through $K_0, K_1, \dots, K_{d-2}, L, K_0$. Only the covers of K_{d-2} and L contain Cf , first in degrees d and $d+1$. All differentials in a minimal resolution induce zero on Hom into the simple s , proving (29). For $q = 1$, $\operatorname{rad}(Cf) = \langle u, v, n \rangle_k$ has radical kv and top generated by u, n at the two vertices. Its cover has one Cf summand, giving the stated first Ext group. $\square$

For $d \geq 2$ and an indeterminate z_0 , the dimensions in (29) have generating series

$$\sum_{i \geq 0} \dim_k \operatorname{Ext}_C^i(s, s) z_0^i = 1 + \frac{z_0^d + z_0^{d+1}}{1 - z_0^d}.$$

The construction over fields algebraic over a finite field remains open.

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