
Noncommutative Buzano-Dragomir Inequality and Applications

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Abstract: Dragomir [*Bull. Aust. Math. Soc.*, 2016] showed that the Buzano inequality holds for orthogonal projections on Hilbert spaces. Dragomir [*Linear Multilinear Algebra*, 2016] also derived the most general form of the Buzano inequality for bounded linear operators on Hilbert spaces. We show that Dragomir result extends to adjointable morphisms on Hilbert C^* -modules. Using this generalization, we derive bounds for the roots of polynomials over commutative unital C^* -algebras. We formulate the notion of noncommutative numerical range and derive numerical radius bounds for self-adjoint morphisms on Hilbert C^* -modules over unital C^* -algebras. We formulate several open problems, including the noncommutative Toeplitz-Hausdorff, von Neumann inequality, Ando inequality, Berger power dilation, Kittaneh inequality, Crouzeix problems.

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1. INTRODUCTION

More than half-century ago, Buzano derived the following generalization of the *Cauchy-Schwarz inequality* [18].

Theorem 1.1. [18, 36, 91] (**Buzano Inequality**) *Let $\mathcal{H}$ be a complex Hilbert space (with the inner product linear in the first variable and conjugate linear in the second variable). Then*

$$(1) \quad |\langle \tau, h \rangle \langle h, \omega \rangle| \leq \frac{\|h\|^2 (|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|)}{2} \leq \|h\|^2 \|\tau\| \|\omega\|, \quad \forall h, \tau, \omega \in \mathcal{H}.$$

In other words,

$$\left| \left\langle \tau, \frac{h}{\|h\|} \right\rangle \left\langle \frac{h}{\|h\|}, \omega \right\rangle \right| \leq \frac{|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|}{2} \leq \|\tau\| \|\omega\|, \quad \forall \tau, \omega \in \mathcal{H}, \forall h \in \mathcal{H} \setminus \{0\}.$$

Further, we have following.

- (i) Let $\tau, \omega \in \mathcal{H} \setminus \{0\}$ be such that $\langle \tau, \omega \rangle = 0$. Choose $\alpha, \beta \in \mathbb{C}$ with $|\beta| = 1$. Define

$$h_{\alpha, \beta} := \alpha \left(\frac{\tau}{\|\tau\|} + \beta \frac{\omega}{\|\omega\|} \right).$$

Then

$$|\langle \tau, h_{\alpha, \beta} \rangle \langle h_{\alpha, \beta}, \omega \rangle| = \frac{\|h_{\alpha, \beta}\|^2 (|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|)}{2}.$$

- (ii) Let $\tau, \omega \in \mathcal{H} \setminus \{0\}$ be such that $\langle \tau, \omega \rangle \neq 0$. For $\alpha \in \mathbb{C}$, define

$$h_{\alpha} := \alpha \left(\frac{\tau}{\|\tau\|} + \frac{\langle \tau, \omega \rangle}{|\langle \tau, \omega \rangle|} \frac{\omega}{\|\omega\|} \right).$$

Then

$$|\langle \tau, h_{\alpha} \rangle \langle h_{\alpha}, \omega \rangle| = \frac{\|h_{\alpha}\|^2 (|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|)}{2}.$$

Ten years ago, Dragomir made the following far reaching generalization of Theorem 1.1 [29].

Theorem 1.2. [29] (**Buzano-Dragomir Inequality**) Let $\mathcal{H}$ be a complex Hilbert space and $P : \mathcal{H} \rightarrow \mathcal{H}$ be an orthogonal projection. Then

$$(2) \quad |\langle P\tau, \omega \rangle| \leq \frac{|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|}{2}, \quad \forall \tau, \omega \in \mathcal{H}.$$

Now note that for any nonzero $h \in \mathcal{H}$, the map

$$\mathcal{H} \ni \tau \mapsto \left\langle \tau, \frac{h}{\|h\|} \right\rangle \frac{h}{\|h\|} \in \mathcal{H}$$

is a rank one orthogonal projection. Thus Inequality (2) recovers Inequality (1). Inequality (1) has various applications, we describe few. Recall that [44, 46, 50, 56, 93] the **numerical range** of a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is defined as

$$W_{\mathbb{C}}(T) := \{\langle Th, h \rangle : h \in \mathcal{H}, \|h\| = 1\}$$

and the **numerical radius** of T is defined as

$$w_{\mathbb{C}}(T) := \sup_{h \in \mathcal{H}, \|h\|=1} |\langle Th, h \rangle|.$$

For $\tau, \omega \in \mathcal{H}$, define the rank one operator

$$\tau \otimes \omega : \mathcal{H} \ni h \mapsto (\tau \otimes \omega)h := \langle h, \omega \rangle \tau \in \mathcal{H}.$$

In 1993, Fujii and Kubo derived following result, using Inequality (1) [36].

Theorem 1.3. [36] Let $\mathcal{H}$ be a complex Hilbert space and $\tau, \omega \in \mathcal{H}$. Then

$$w_{\mathbb{C}}(\tau \otimes \omega) = \frac{|\langle \tau, \omega \rangle| + \|\tau\| \|\omega\|}{2}.$$

Let $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$. A direct observation reveals that the zeros of p are the eigenvalues of the **Frobenius companion matrix**

$$C_p := \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

and the eigenvalues of C_p are the same as the zeros of p . Let R be the right shift matrix defined by

$$R := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathbb{C}).$$

Define

$$\mathbf{a} := \begin{pmatrix} \overline{a_{n-1}} \\ \overline{a_{n-2}} \\ \overline{a_{n-3}} \\ \vdots \\ \overline{a_2} \\ \overline{a_1} \\ \overline{a_0} \end{pmatrix}, \quad \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$(3) \quad C_p = R - \mathbf{e}_1 \otimes \mathbf{a}.$$

Fujii and Kubo derived following bound for the zeros of p using Inequality 1 and the representation (3) [36].

Theorem 1.4. [36] (*Fujii-Kubo polynomial root upper bound*) Let $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$. If λ is a zero of p , then

$$(4) \quad |\lambda| \leq w_{\mathbb{C}}(R) + \frac{\sqrt{\sum_{j=0}^{n-1} |a_j|^2} + |a_{n-1}|}{2} = \cos\left(\frac{\pi}{n+1}\right) + \frac{\sqrt{\sum_{j=0}^{n-1} |a_j|^2} + |a_{n-1}|}{2}.$$

Let $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$ with $a_0 \neq 0$. Define

$$q(z) := \frac{1}{a_0} z^n p\left(\frac{1}{z}\right) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0} z + \cdots + \frac{a_1}{a_0} z^{n-1} + z^n \in \mathbb{C}[z].$$

We note that $\lambda \in \mathbb{C} \setminus \{0\}$ satisfies $p(\lambda) = 0$ if and only if $q(1/\lambda) = 0$ [56]. Frobenius companion matrix of q is

$$C_q := \begin{pmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & -\frac{a_3}{a_0} & \cdots & -\frac{a_{n-1}}{a_0} & -\frac{a_n}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

By applying Theorem 1.4 to q , we get following result.

Theorem 1.5. [36] (*Fujii-Kubo polynomial root lower bound*) Let $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z]$ with $a_0 \neq 0$. If λ is a zero of p , then

$$|\lambda| \geq \frac{2|a_0|}{2|a_0| \cos\left(\frac{\pi}{n+1}\right) + \sqrt{1 + \sum_{j=1}^{n-1} |a_j|^2} + |a_1|}.$$

For any bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$, the Cauchy-Schwarz inequality gives

$$(5) \quad w_{\mathbb{C}}(T) \leq \|T\|.$$

In 2007, Dragomir strengthened Inequality (5) using Buzano inequality [30].

Theorem 1.6. [30] (*Dragomir Numerical Radius Inequality*) Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator. Then

$$w_{\mathbb{C}}(T) \leq \frac{1}{\sqrt{2}} \sqrt{w_{\mathbb{C}}(T^2) + \|T\|^2} \leq \|T\|.$$

Our fundamental motivation comes from the following question: What is the noncommutative analogue of Theorem 1.2? This is then naturally connected with the notion of Hilbert C^* -modules which are first introduced by Kaplansky [58] for modules over commutative C^* -algebras and later developed for modules over arbitrary C^* -algebras by Paschke [81] and Rieffel [88].

Definition 1.7. [58, 81, 88] Let $\mathcal{A}$ be a unital C^* -algebra. A right module $\mathcal{E}$ over $\mathcal{A}$ is said to be a **semi-inner product C^* -module** if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ such that the following hold.

- (i) $\langle x, x \rangle \geq 0$, $\forall x \in \mathcal{E}$.
- (ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, $\forall x, y, z \in \mathcal{E}$.
- (iii) $\langle x, ya \rangle = \langle x, y \rangle a$, $\forall x, y \in \mathcal{E}$, $\forall a \in \mathcal{A}$.
- (iv) $\langle x, y \rangle = \langle y, x \rangle^*$, $\forall x, y \in \mathcal{E}$.

Semi-inner product C^* -modules satisfy noncommutative Cauchy-Schwarz inequality.

Theorem 1.8. [68] (*Kaplansky-Paschke-Rieffel Inequality*) Let $\mathcal{E}$ be a semi-inner product C^* -module. Then

$$\langle y, x \rangle \langle x, y \rangle \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\|, \quad \forall x, y \in \mathcal{E}.$$

In particular,

$$\|\langle y, x \rangle\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\|, \quad \forall x, y \in \mathcal{E}.$$

Definition 1.9. [68] A semi-inner product C^* -module $\mathcal{E}$ is said to be an **inner product C^* -module** if $x \in \mathcal{E}$ satisfies $\langle x, x \rangle = 0$, then $x = 0$.

Definition 1.10. [68] A inner product C^* -module $\mathcal{E}$ is said to be **Hilbert C^* -module** if $\mathcal{E}$ is complete w.r.t. the norm $\|x\| := \sqrt{\|\langle x, x \rangle\|}$, $\forall x \in \mathcal{E}$.

In 2012, Khosravi, Drnovsek and Moslehian derived noncommutative version of Inequality (1) [61].

Theorem 1.11. [61] (**Buzano-Khosravi-Drnovsek-Moslehian Inequality**) Let $\mathcal{E}$ be a Hilbert C^* -module. Then

$$(6) \quad \|\langle \omega, x \rangle \langle x, \tau \rangle\| \leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\| \|\omega\|}{2} \leq \|\tau\| \|\omega\|, \quad \forall \tau, \omega \in \mathcal{E}, \forall x \in \mathcal{E} \text{ with } \langle x, x \rangle = 1.$$

In this article, we show that Theorem 1.2 extends to orthogonal projections on Hilbert C^* -modules (which will also generalize Theorem 1.11). We also derive modular analogue of Theorems 1.4 and 1.6.

2. NONCOMMUTATIVE BUZANO-DRAGOMIR INEQUALITY

We start by deriving noncommutative analogue of Theorem 1.2.

Theorem 2.1. (**Noncommutative Buzano-Dragomir Inequality**) Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $P : \mathcal{E} \rightarrow \mathcal{E}$ be an orthogonal projection. Then

$$\|\langle \tau, P\omega \rangle\| \leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\| \|\omega\|}{2}, \quad \forall \tau, \omega \in \mathcal{E}.$$

Proof. We find

$$\|2\langle \tau, P\omega \rangle - \langle \tau, \omega \rangle\| = \|\langle \tau, 2P\omega - \omega \rangle\| \leq \|\tau\| \|2P\omega - \omega\|.$$

We now find

$$\begin{aligned} \|2P\omega - \omega\|^2 &= \|\langle 2P\omega - \omega, 2P\omega - \omega \rangle\| \\ &= \|4\langle P\omega, P\omega \rangle - 2\langle P\omega, \omega \rangle - 2\langle \omega, P\omega \rangle + \langle \omega, \omega \rangle\| \\ &= \|\langle \omega, \omega \rangle\| \\ &= \|\omega\|^2. \end{aligned}$$

Therefore

$$2\|\langle \tau, P\omega \rangle\| - \|\langle \tau, \omega \rangle\| \leq \|2\langle \tau, P\omega \rangle - \langle \tau, \omega \rangle\| \leq \|\tau\| \|\omega\|.$$

□

Corollary 2.2. Theorem 1.11 follows from Theorem 2.1.

Proof. Let $\mathcal{E}$ be a Hilbert C^* -module and $x \in \mathcal{E}$ with $\langle x, x \rangle = 1$. Define

$$P : \mathcal{E} \ni y \mapsto Py := x\langle x, y \rangle \in \mathcal{E}.$$

Then P is an orthogonal projection. Theorem 2.1 gives

$$\begin{aligned} \|\langle \tau, x \rangle \langle x, \omega \rangle\| &= \|\langle \tau, x\langle x, \omega \rangle \rangle\| = \|\langle \tau, P\omega \rangle\| \\ &\leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\| \|\omega\|}{2}, \quad \forall \tau, \omega \in \mathcal{E}. \end{aligned}$$

□

In 2016, Dragomir further improved Theorem 1.2 for bounded linear operators [28].

Theorem 2.3. [28] (*Buzano-Dragomir Inequality*) Let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator. Then

$$|\langle T\tau, T\omega \rangle| \leq \frac{\|T\|^2(|\langle \tau, \omega \rangle| + \|\tau\|\|\omega\|)}{2}, \quad \forall \tau, \omega \in \mathcal{H}.$$

We now derive following noncommutative version of Theorem 2.3.

Theorem 2.4. (*Noncommutative Buzano-Dragomir Inequality*) Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism. Then

$$\|\langle T\tau, T\omega \rangle\| \leq \|T\|^2 \|\langle \tau, \omega \rangle\| + \left\| \langle \tau, \tau \rangle \|T\|^2 - \langle T\tau, T\tau \rangle \right\|^{\frac{1}{2}} \left\| \langle \omega, \omega \rangle \|T\|^2 - \langle T\omega, T\omega \rangle \right\|^{\frac{1}{2}}, \quad \forall \tau, \omega \in \mathcal{E}.$$

Proof. Our proof is motivated by the Hilbert space argument due to Dragomir [28]. Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism and $\tau, \omega \in \mathcal{E}$. Define

$$[\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto [x, y] := \langle x, y \rangle \|T\|^2 - \langle Tx, Ty \rangle \in \mathcal{A}.$$

Then $[\cdot, \cdot]$ is a semi-inner product. By applying Theorem 1.8 to this semi-inner product, we get

$$\|[x, y]\|^2 \leq \|[x, x]\| \|[y, y]\|, \quad \forall x, y \in \mathcal{E}.$$

In particular,

$$\|[\tau, \omega]\|^2 \leq \|[\tau, \tau]\| \|[\omega, \omega]\|.$$

Previous inequality gives

$$\left\| \langle \tau, \omega \rangle \|T\|^2 - \langle T\tau, T\omega \rangle \right\|^2 \leq \left\| \langle \tau, \tau \rangle \|T\|^2 - \langle T\tau, T\tau \rangle \right\| \left\| \langle \omega, \omega \rangle \|T\|^2 - \langle T\omega, T\omega \rangle \right\|.$$

Therefore

$$\begin{aligned} \|\langle T\tau, T\omega \rangle\| - \|T\|^2 \|\langle \tau, \omega \rangle\| &\leq \left\| \langle \tau, \omega \rangle \|T\|^2 - \langle T\tau, T\omega \rangle \right\| \\ &\leq \left\| \langle \tau, \tau \rangle \|T\|^2 - \langle T\tau, T\tau \rangle \right\|^{\frac{1}{2}} \left\| \langle \omega, \omega \rangle \|T\|^2 - \langle T\omega, T\omega \rangle \right\|^{\frac{1}{2}}. \end{aligned}$$

□

We are unable to simplify Theorem 2.4. We therefore ask following problem.

Problem 2.5. Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism. Whether

$$\|\langle T\tau, T\omega \rangle\| \leq \frac{\|T\|^2(\|\langle \tau, \omega \rangle\| + \|\tau\|\|\omega\|)}{2}, \quad \forall \tau, \omega \in \mathcal{E}?$$

Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Similar to Hilbert space case, we define the **noncommutative numerical range** of an adjointable morphism $T : \mathcal{E} \rightarrow \mathcal{E}$ is defined as

$$W_{\mathcal{A}}(T) := \{\langle x, Tx \rangle : x \in \mathcal{E}, \langle x, x \rangle = 1\} \subseteq \mathcal{A}$$

and the **noncommutative numerical radius** T is defined as

$$w_{\mathcal{A}}(T) := \sup_{x \in \mathcal{E}, \langle x, x \rangle = 1} \|\langle x, Tx \rangle\| \geq 0.$$

Note that our definition of numerical range and radius on Hilbert C*-modules differ from the existing same notion on Hilbert C*-modules [76, 86, 105]. We give various examples. In all following examples, $\mathcal{A}$ is a unital C*-algebra.

Example 2.6. Let $b \in \mathcal{A}$. Then

$$W_{\mathcal{A}}(b) = \{a^*ba : a \in \mathcal{A}, a^*a = 1\}$$

and

$$\begin{aligned} w_{\mathcal{A}}(b) &= \sup\{\|a^*ba\| : a \in \mathcal{A}, a^*a = 1\} \\ &= \|b\|. \end{aligned}$$

If $\mathcal{A}$ is commutative, then $W_{\mathcal{A}}(b) = \{b\}$.

Example 2.7. Let $d \in \mathbb{N}$ and I_d be the identity matrix of size d by d . Then we have

$$W_{\mathcal{A}}(I_d) = \{\langle x, x \rangle : x \in \mathcal{A}^d, \langle x, x \rangle = 1\} = \{1\}$$

and

$$w_{\mathcal{A}}(I_d) = 1.$$

Example 2.8. Define

$$M := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} W_{\mathcal{A}}(M) &= \{\langle x, Mx \rangle : x \in \mathcal{A}^2, \langle x, x \rangle = 1\} \\ &= \{a^*b : a, b \in \mathcal{A}, a^*a + b^*b = 1\} \end{aligned}$$

and

$$w_{\mathcal{A}}(M) = \sup\{\|a^*b\| : a, b \in \mathcal{A}, a^*a + b^*b = 1\}.$$

We note that

$$a^*bb^*a \leq \frac{a^*a + b^*b}{2}, \quad \forall a, b \in \mathcal{A}.$$

Therefore

$$\begin{aligned} w_{\mathcal{A}}(M) &= \sup\{\|a^*b\| : a, b \in \mathcal{A}, a^*a + b^*b = 1\} \\ &\leq \frac{1}{\sqrt{2}} \sup\{\sqrt{\|a^*a + b^*b\|} : a, b \in \mathcal{A}, a^*a + b^*b = 1\} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Example 2.9. Let $d \in \mathbb{N}$ and $b_1, \dots, b_d \in \mathcal{A}$. Define $\text{diag}((b_j)_{j=1}^d)$ as the diagonal matrix with diagonal entries $b_1, \dots, b_d$. Then

$$W_{\mathcal{A}}(\text{diag}((b_j)_{j=1}^d)) = \left\{ \sum_{j=1}^d a_j^* b_j a_j : (a_j)_{j=1}^d \in \mathcal{A}^d, \sum_{j=1}^d a_j^* a_j = 1 \right\}$$

and

$$w_{\mathcal{A}}(\text{diag}((b_j)_{j=1}^d)) = \sup \left\{ \left\| \sum_{j=1}^d a_j^* b_j a_j \right\| : (a_j)_{j=1}^d \in \mathcal{A}^d, \sum_{j=1}^d a_j^* a_j = 1 \right\} \\ \leq \sum_{j=1}^d \|b_j\|.$$

The set $W_{\mathcal{A}}(\text{diag}((b_j)_{j=1}^d))$ is the C^* -convex hull of $\{b_j\}_{j=1}^d$ [71].

Example 2.10. Let $R \in \mathbb{M}_d(\mathcal{A})$ be the right shift matrix defined by

$$R := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Thus the action of R is

$$R(a_j)_{j=1}^d = (0, a_1, a_2, \dots, a_{d-1}), \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

We then have

$$\langle (a_j)_{j=1}^d, R(a_j)_{j=1}^d \rangle = a_2^* a_1 + \cdots + a_d^* a_{d-1}, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

Hence

$$\begin{aligned} \|\langle (a_j)_{j=1}^d, R(a_j)_{j=1}^d \rangle\| &= \|a_2^* a_1 + \cdots + a_d^* a_{d-1}\| \\ &\leq \left\| \sum_{j=2}^d a_j^* a_j \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^{d-1} a_k^* a_k \right\|^{\frac{1}{2}} \\ &\leq \left\| \sum_{j=1}^d a_j^* a_j \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^d a_k^* a_k \right\|^{\frac{1}{2}} \\ &= \left\| \sum_{j=1}^d a_j^* a_j \right\|, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d. \end{aligned}$$

Therefore

$$W_{\mathcal{A}}(R) = \left\{ a_2^* a_1 + \cdots + a_d^* a_{d-1} : (a_j)_{j=1}^d \in \mathcal{A}^d, \sum_{j=1}^d a_j^* a_j = 1 \right\}$$

and

$$w_{\mathcal{A}}(R) = \sup \left\{ \|a_2^* a_1 + \cdots + a_d^* a_{d-1}\| : (a_j)_{j=1}^d \in \mathcal{A}^d, \sum_{j=1}^d a_j^* a_j = 1 \right\} \\ \leq 1.$$

Example 2.11. Let $\mathcal{A}$ be a unital C^* -algebra and $\ell^2(\mathcal{A})$ be the standard Hilbert C^* -module defined by

$$\ell^2(\mathcal{A}) := \left\{ \{a_n\}_{n=1}^\infty : a_n \in \mathcal{A}, \forall n \in \mathbb{N}, \sum_{n=1}^\infty a_n^* a_n \in \mathcal{A} \right\}$$

equipped with inner product

$$\langle \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \rangle := \sum_{n=1}^\infty a_n^* b_n, \quad \forall \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \ell^2(\mathcal{A})$$

and norm

$$\|\{a_n\}_{n=1}^\infty\| = \left\| \sum_{n=1}^\infty a_n^* a_n \right\|^{\frac{1}{2}}, \quad \forall \{a_n\}_{n=1}^\infty \in \ell^2(\mathcal{A}).$$

Let L be the left shift morphism defined by

$$L : \ell^2(\mathcal{A}) \ni \{a_n\}_{n=1}^\infty \mapsto L\{a_n\}_{n=1}^\infty := \{a_{n+1}\}_{n=1}^\infty \in \ell^2(\mathcal{A}).$$

Then

$$W_{\mathcal{A}}(L) = \left\{ \sum_{n=1}^\infty a_n^* a_{n+1} : \{a_n\}_{n=1}^\infty \in \ell^2(\mathcal{A}), \sum_{n=1}^\infty a_n^* a_n = 1 \right\}$$

and

$$w_{\mathcal{A}}(L) = \sup \left\{ \left\| \sum_{n=1}^\infty a_n^* a_{n+1} \right\| : \{a_n\}_{n=1}^\infty \in \ell^2(\mathcal{A}), \sum_{n=1}^\infty a_n^* a_n = 1 \right\} \leq 1.$$

Example 2.12. Define

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$W_{\mathcal{A}}(A) = \{x^* a x + x^* b y + y^* c x + y^* d y : x, y \in \mathcal{A}, x^* x + y^* y = 1\}$$

and

$$w_{\mathcal{A}}(A) = \sup \{\|x^* a x + x^* b y + y^* c x + y^* d y\| : x, y \in \mathcal{A}, x^* x + y^* y = 1\} \leq \|a\| + \|b\| + \|c\| + \|d\|.$$

Given $r > 0$ and $v \in \mathcal{A}$, the closed disc centered at $v \in \mathcal{A}$ of radius r is defined as

$$\mathbb{D}_{\mathcal{A}}[v, r] := \{z \in \mathcal{A} : \|z - v\| \leq r\}.$$

Following are basic properties of noncommutative numerical range and radius.

Proposition 2.13. Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $\mathcal{A}^+$ be the set of all positive elements in $\mathcal{A}$. Let $T, S : \mathcal{E} \rightarrow \mathcal{E}$ be adjointable morphisms and $a, b \in \mathcal{A}$. Let I be the identity morphism on $\mathcal{E}$.

- (i) $W_{\mathcal{A}}(T) \subseteq \mathbb{D}_{\mathcal{A}}[0, \|T\|]$.
- (ii) $0 \leq w_{\mathcal{A}}(T) \leq \|T\|$.
- (iii) $W_{\mathcal{A}}(T + S) \subseteq W_{\mathcal{A}}(T) + W_{\mathcal{A}}(S)$.
- (iv) $w_{\mathcal{A}}(S + T) \leq w_{\mathcal{A}}(S) + w_{\mathcal{A}}(T)$.

- (v) $W_{\mathcal{A}}(Ta) = W_{\mathcal{A}}(T)a$.
- (vi) $w_{\mathcal{A}}(Ta) \leq \|a\|w_{\mathcal{A}}(T)$.
- (vii) $W_{\mathcal{A}}(T^*) = (W_{\mathcal{A}}(T))^*$.
- (viii) $w_{\mathcal{A}}(T^*) = w_{\mathcal{A}}(T)$.
- (ix) $W_{\mathcal{A}}(Ta + bI) = W_{\mathcal{A}}(T)a + b$.
- (x) $W_{\mathcal{A}}(\operatorname{Re}(T)) = \operatorname{Re}(W_{\mathcal{A}}(T))$ and $W_{\mathcal{A}}(\operatorname{Im}(T)) = \operatorname{Im}(W_{\mathcal{A}}(T))$.
- (xi) If $T \geq 0$, then $W_{\mathcal{A}}(T) \subseteq \mathcal{A}^+$.
- (xii) Let $\mathcal{E}_0$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ such that $\mathcal{E}$ is orthogonally complementable in $\mathcal{E}_0$. If an adjointable morphism $V : \mathcal{E}_0 \rightarrow \mathcal{E}_0$ is a dilation of T , then $W_{\mathcal{A}}(T) \subseteq W_{\mathcal{A}}(V)$.
- (xiii) Let $\mathcal{E}_1$ be an orthogonally complementable closed submodule in $\mathcal{E}$. Let $P : \mathcal{E} \rightarrow \mathcal{E}_1$ be an onto orthogonal projection. Then $W_{\mathcal{A}}(PT|_{\mathcal{E}_1}) \subseteq W_{\mathcal{A}}(T)$.
- (xiv) $W_{\mathcal{A}}(U^*TU) = W_{\mathcal{A}}(T)$ for every unitary morphism $U : \mathcal{E} \rightarrow \mathcal{E}$.
- (xv) Let $a \in \mathcal{A}$ be such that there exists a $x \in \mathcal{E}$ with $Tx = xa$ and $\langle x, x \rangle = 1$. Then $a \in W_{\mathcal{A}}(T)$. In other words,

$$\sigma_1(T) := \{a \in \mathcal{A} : \exists x \in \mathcal{E}, Tx = xa, \langle x, x \rangle = 1\} \subseteq W_{\mathcal{A}}(T)$$

and

$$\|a\| \leq w_{\mathcal{A}}(T), \quad \forall a \in \sigma_1(T).$$

(xvi)

$$w_{\mathcal{A}}(ST + TS) \leq \frac{w_{\mathcal{A}}((S + T)^2) + w_{\mathcal{A}}((S - T)^2)}{2}.$$

In particular, if $ST = TS$, then

$$w_{\mathcal{A}}(ST) \leq \frac{w_{\mathcal{A}}((S + T)^2) + w_{\mathcal{A}}((S - T)^2)}{4}.$$

- (xvii) If a sequence $\{T_n\}_{n=1}^{\infty}$ of adjointable morphisms on $\mathcal{E}$ converges to an adjointable morphism T on $\mathcal{E}$ in the morphism norm, then the sequence $\{w_{\mathcal{A}}(T_n)\}_{n=1}^{\infty}$ converges to $w_{\mathcal{A}}(T)$ in the norm.

Using generalized polarization identity, it is known that $w_{\mathbb{C}}(T) \geq \|T\|/2$ [46]. We are unable to derive noncommutative version of this result. For $\tau, \omega \in \mathcal{E}$, define the morphism

$$\tau \otimes \omega : \mathcal{E} \ni x \mapsto (\tau \otimes \omega)x := \tau \langle \omega, x \rangle \in \mathcal{E}.$$

Following is modular version of Theorem 1.3.

Theorem 2.14. Let $\mathcal{E}$ be a Hilbert C^* -module over a commutative unital C^* -algebra $\mathcal{A}$ and $\tau, \omega \in \mathcal{E}$. Then

$$(7) \quad w_{\mathcal{A}}(\tau \otimes \omega) \leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\|\|\omega\|}{2} \leq \|\tau\|\|\omega\|.$$

In particular,

$$w_{\mathcal{A}}(\tau \otimes \tau) \leq \|\tau\|^2.$$

Proof. Let $\tau, \omega \in \mathcal{E}$. Let $x \in \mathcal{E}$ with $\langle x, x \rangle = 1$. Then using commutativity of C^* -algebra,

$$\begin{aligned} \|\langle x, (\tau \otimes \omega)x \rangle\| &= \|\langle x, \tau \langle \omega, x \rangle \rangle\| = \|\langle x, \tau \rangle \langle \omega, x \rangle\| \\ &= \|\langle \omega, x \rangle \langle x, \omega \rangle\| \leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\|\|\omega\|}{2}. \end{aligned}$$

Therefore

$$w_{\mathcal{A}}(\tau \otimes \omega) \leq \frac{\|\langle \tau, \omega \rangle\| + \|\tau\| \|\omega\|}{2}.$$

□

Unlike the Hilbert space case, we are unable to derive equality in Inequality (7). Let $\mathcal{A}$ be a unital C^* -algebra and $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathcal{A}[z]$. Define the **modular Frobenius companion matrix**

$$C_p := \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathcal{A}).$$

Given $n \in \mathbb{N}$, we consider $\mathcal{A}^n$ with the standard inner product

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle := \sum_{j=1}^n a_j^* b_j, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence the norm on $\mathcal{A}^n$ is

$$\|(a_j)_{j=1}^n\| = \left\| \sum_{j=1}^n a_j^* a_j \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

We then have the following result.

Theorem 2.15. *Let $\mathcal{A}$ be a commutative unital C^* -algebra and $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathcal{A}[z]$. If $b \in \mathcal{A}$ satisfies $p(b) = 0$, then $b \in W_{\mathcal{A}}(C_p)$.*

Proof. Define

$$x := \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix} \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}}.$$

Then $\langle x, x \rangle = 1$. Using commutativity of C^* -algebra, we have

$$\begin{aligned}
 \langle x, C_p x \rangle &= \\
 &\left\langle \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix} \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}}, \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix} \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \right\rangle \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left\langle \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix} \right\rangle \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left\langle \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} -\sum_{j=0}^{n-1} a_j b^j \\ b^{n-1} \\ b^{n-2} \\ \vdots \\ b^3 \\ b^2 \\ b \end{pmatrix} \right\rangle \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left\langle \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} b^n \\ b^{n-1} \\ b^{n-2} \\ \vdots \\ b^3 \\ b^2 \\ b \end{pmatrix} \right\rangle \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left\langle \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} b^{n-1} \\ b^{n-2} \\ b^{n-3} \\ \vdots \\ b^2 \\ b \\ 1 \end{pmatrix} \right\rangle b \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right) b \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \\
 &= \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right) \left(\sum_{j=0}^{n-1} (b^*)^j b^j \right)^{\frac{-1}{2}} b \\
 &= b.
 \end{aligned}$$

Therefore $b = \langle x, C_p x \rangle \in W_{\mathcal{A}}(C_p)$. $\square$

We now derive modular analogue of Theorem 1.4.

Theorem 2.16. *Let $\mathcal{A}$ be a commutative unital C^* -algebra and $p(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathcal{A}[z]$. If $b \in \mathcal{A}$ satisfies $p(b) = 0$, then*

$$\begin{aligned} \|b\| &\leq w_{\mathcal{A}}(R) + \frac{\sqrt{\left\| \sum_{j=0}^{n-1} a_j^* a_j \right\|} + \|a_{n-1}\|}{2} \\ &\leq w_{\mathcal{A}}(R) + \frac{\sqrt{\sum_{j=0}^{n-1} \|a_j\|^2} + \|a_{n-1}\|}{2}, \end{aligned}$$

where R is the right shift matrix defined by

$$R := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathcal{A}).$$

Proof. Define

$$\mathbf{a} := \begin{pmatrix} (a_{n-1})^* \\ (a_{n-2})^* \\ (a_{n-3})^* \\ \vdots \\ (a_2)^* \\ (a_1)^* \\ (a_0)^* \end{pmatrix} \in \mathcal{A}^n, \quad \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{A}^n.$$

Then

$$(8) \quad C_p = R - \mathbf{e}_1 \otimes \mathbf{a}.$$

Using (8) and Inequality (7), we get

$$\begin{aligned} \|b\| &\leq w_{\mathcal{A}}(C_p) \\ &= w_{\mathcal{A}}(R - \mathbf{e}_1 \otimes \mathbf{a}) \\ &\leq w_{\mathcal{A}}(R) + w_{\mathcal{A}}(\mathbf{e}_1 \otimes \mathbf{a}) \\ &\leq w_{\mathcal{A}}(R) + \frac{\sqrt{\left\| \sum_{j=0}^{n-1} a_j^* a_j \right\|} + \|a_{n-1}\|}{2} \\ &\leq w_{\mathcal{A}}(R) + \frac{\sqrt{\sum_{j=0}^{n-1} \|a_j\|^2} + \|a_{n-1}\|}{2}. \end{aligned}$$

$\square$

Let $\mathcal{A}$ be a commutative unital C^* -algebra. Let $p(z) := a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n \in \mathcal{A}[z]$ with a_0 invertible. Define

$$q(z) := \frac{1}{a_0}z^n p\left(\frac{1}{z}\right) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0}z + \cdots + \frac{a_1}{a_0}z^{n-1} + z^n \in \mathcal{A}[z].$$

We note that an invertible element $b \in \mathcal{A}$ satisfies $p(b) = 0$ if and only if $q(1/b) = 0$. Frobenius companion matrix of q is

$$C_q := \begin{pmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & -\frac{a_3}{a_0} & \cdots & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{M}_n(\mathcal{A}).$$

By applying Theorem 4 to q , we get following result.

Theorem 2.17. *Let $\mathcal{A}$ be a commutative unital C^* -algebra and $p(z) := a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n \in \mathcal{A}[z]$ with a_0 invertible. If an invertible element $b \in \mathcal{A}$ satisfies $p(b) = 0$, then*

$$\frac{1}{\|b^{-1}\|} \geq \frac{2}{2w_{\mathcal{A}}(R) + \sqrt{\left\| (a_0^{-1})^* a_0^{-1} + \sum_{j=1}^{n-1} (a_j a_0^{-1})^* a_j a_0^{-1} \right\|^2} + \|a_1 a_0^{-1}\|}.$$

Equality term in (4) comes from the numerical radius of right shift matrix obtained by Davidson and Holbrook [25] (also see [48, 59]). We are unable to do this for the right shift matrix over C^* -algebras. We now derive Theorem 1.6 for Hilbert C^* -modules.

Theorem 2.18. (Modular Dragomir Numerical Radius Inequality) *Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $T : \mathcal{E} \rightarrow \mathcal{E}$ be a self-adjoint morphism. Then*

$$w_{\mathcal{A}}(T) \leq \frac{1}{\sqrt{2}} \sqrt{w_{\mathcal{A}}(T^2) + \|T\|^2} \leq \frac{1}{\sqrt{2}} \sqrt{\|T^2\| + \|T\|^2} \leq \|T\|.$$

Proof. Let $x \in \mathcal{E}$ with $\langle x, x \rangle = 1$. Then using Inequality (6) and the self-adjointness of T , we get

$$\begin{aligned} \|\langle x, Tx \rangle\|^2 &= \|\langle x, Tx \rangle \langle x, Tx \rangle^*\| = \|\langle x, Tx \rangle \langle Tx, x \rangle\| \\ &= \|\langle Tx, x \rangle \langle x, Tx \rangle\| \leq \frac{\|\langle Tx, Tx \rangle\| + \|Tx\| \|Tx\|}{2} \\ &= \frac{\|\langle x, T^2 x \rangle\| + \|Tx\| \|T^* x\|}{2} \leq \frac{\|\langle x, T^2 x \rangle\| + \|T\| \|x\| \|T\| \|x\|}{2} \\ &= \frac{\|\langle x, T^2 x \rangle\| + \|T\|^2}{2} \leq \frac{\sup_{y \in \mathcal{E}, \langle y, y \rangle = 1} \|\langle y, T^2 y \rangle\| + \|T\|^2}{2} \\ &= \frac{w_{\mathcal{A}}(T^2) + \|T\|^2}{2}. \end{aligned}$$

Therefore

$$w_{\mathcal{A}}(T) = \sup_{x \in \mathcal{E}, \langle x, x \rangle = 1} \|\langle x, Tx \rangle\| \leq \frac{1}{\sqrt{2}} \sqrt{w_{\mathcal{A}}(T^2) + \|T\|^2} \leq \frac{1}{\sqrt{2}} \sqrt{\|T^2\| + \|T\|^2} \leq \|T\|.$$

□

Note that we derived Theorem 2.18 only for self-adjoint morphisms. We are unable to derive Theorem 2.18 for arbitrary adjointable morphisms. We also note that in the case of self-adjoint operator T on Hilbert spaces, we have $w_{\mathbb{C}}(T) = \|T\|$ [46] which we can't say for self-adjoint morphisms on Hilbert C^* -modules.

3. NONCOMMUTATIVE TOEPLITZ-HAUSDORFF PROBLEM

Motivated from several breakthrough results in numerical range and radius, we formulate following problems. What is noncommutative

- (1) Toeplitz-Hausdorff Theorem (fundamental theorem of the numerical range) [26, 34, 45–47, 50, 51, 74, 78, 85, 92, 97]?
- (2) Wintner spectral inclusion theorem for numerical range [8, 46, 102]?
- (3) Halmos-Bernau-Smithies-Berger power inequality [11, 82]?
- (4) Alpin-Chien-Yeh inequality [6]?
- (5) Kittaneh inequalities [63, 64]?
- (6) Yamazaki inequality [104]?
- (7) Kittaneh-Moslehian-Yamazaki inequality [66]?
- (8) Abu-Omar-Kittaneh inequality [3]?
- (9) Bhunia-Bag-Paul inequality [12]?
- (10) elliptical range theorem [17, 70]?
- (11) Berger power dilation for numerical contraction [9, 10, 46, 95]?
- (12) Ando theorem for numerical contraction [7]?
- (13) Furuta-Nakamoto theorem for numerical contractions [37]?
- (14) El-Haddad-Kittaneh numerical radius inequality [33]?
- (15) description of numerical range of 3×3 matrices [60]?
- (16) description of numerical range of 4×4 matrices [39]?
- (17) Holbrook numerical radius inequalities for the product of commuting matrices [54]?
- (18) Okubo-Ando numerical radius inequalities for the product of commuting matrices [55, 77, 80]?
- (19) Bouldin numerical radius inequalities for the product of commuting matrices [14, 15]?
- (20) Poncelet property for numerical ranges [4, 41, 103]?
- (21) Anderson theorem [31, 44, 96]?
- (22) Kippenhahn boundary generating curve theorem [35, 62]?
- (23) Kippenhahn corner-point theorem [35, 62]?
- (24) Hildebrandt normal eigenvalue theorem [16, 53]?
- (25) Hildebrandt intersection theorem [38, 101]?
- (26) Marcus-Shure description for the numerical range of zero-one matrices [72] and Davidson-Holbrook estimate for the numerical radius of zero-one matrices [25]?
- (27) Johnson numerical range inclusion theorem [57]?
- (28) Choi-Li constrained numerical range theorem [20]?
- (29) Wang-Wu-Gau result for Crawford number [100]?
- (30) Abu-Omar-Kittaneh inequality (involving generalized Aluthge transform) [1]?
- (31) Gau-Wu numerical radius equality characterizations [43]?
- (32) Gau-Wu theorem for compact operators [42]?
- (33) Perron-Frobenius type results on the numerical range [73]?

- (34) Helton-Spitkovsky characterization of shapes of numerical ranges [52]?
- (35) Radjabalipour-Radjavi result on numerical ranges [84]?
- (36) Pollack theorem on numerical ranges [83]?
- (37) Anderson result on numerical ranges [5]?
- (38) Abu-Omar-Kittaneh numerical radius inequalities for the product of matrices [2]?
- (39) Kittaneh-Moradi inequality [65]?
- (40) Lancaster characterization for the closedness of numerical range [67]?
- (41) Gau-Wu characterization for numerical ranges of completely non-unitary contractions [40]?
- (42) Mees-Atherton domains containing numerical ranges [75]?
- (43) Chien-Tam characterizations of circularity of numerical ranges [19]?
- (44) Shiu theorem on the growth of numerical range of powers of operator [90]?

4. NONCOMMUTATIVE CROUZEIX INEQUALITY, VON NEUMANN INEQUALITY AND ANDO INEQUALITY PROBLEMS

Breakthrough Crouzeix theorem says following.

Theorem 4.1. [13, 22–24, 87] (**Crouzeix Theorem**) For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathbb{C})$, we have

$$(\textbf{Crouzeix-Palencia Inequality}) \quad \|p(M)\| \leq (1 + \sqrt{2}) \sup \{|p(z)| : z \in W_{\mathbb{C}}(M)\}, \quad \forall p \in \mathbb{C}[z].$$

We formulate following problem.

Problem 4.2. (Noncommutative Crouzeix Problem) Let $\mathcal{A}$ be a unital C^* -algebra. Whether there is a universal constant $R_{\mathcal{A}}$ (which may depend upon $\mathcal{A}$) satisfying following: For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathcal{A})$, we have

$$\|p(M)\| \leq R_{\mathcal{A}} \sup \{\|p(z)\| : z \in W_{\mathcal{A}}(M)\}, \quad \forall p \in \mathcal{A}[z].$$

An inequality which is very close to Crouzeix inequality is the von Neumann inequality.

Theorem 4.3. [27, 79, 94, 95, 99] (**von Neumann Theorem**) For every $d \in \mathbb{N}$ and for every matrix $M \in \mathbb{M}_d(\mathbb{C})$ with $\|M\| \leq 1$, we have

$$(\textbf{von Neumann Inequality}) \quad \|p(M)\| \leq \sup \{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}, \quad \forall p \in \mathbb{C}[z].$$

Based on Theorem 4.3 we formulate following problems.

Problem 4.4. (Noncommutative von Neumann Inequality Problem) Let $\mathcal{A}$ be a unital C^* -algebra. Whether there is a universal constant $R_{\mathcal{A}}$ (which may depend upon $\mathcal{A}$) satisfying following: For every $d \in \mathbb{N}$ and for every $A \in \mathbb{M}_d(\mathcal{A})$ with $\|A\| \leq 1$, we have

$$\|p(A)\| \leq R_{\mathcal{A}} \sup \{\|p(z)\| : z \in \mathcal{A}, \|z\| \leq 1\}, \quad \forall p \in \mathcal{A}[z].$$

Theorem 4.3 has been extended by Ando for two variables.

Theorem 4.5. [7, 95] (**Ando Theorem**) For every $d \in \mathbb{N}$ and for all matrices $M, N \in \mathbb{M}_d(\mathbb{C})$ with $\|M\| \leq 1$, $\|N\| \leq 1$ and $MN = NM$, we have

$$(\textbf{Ando Inequality}) \quad \|p(M, N)\| \leq \sup \{|p(z, w)| : z, w \in \mathbb{C}, |z| \leq 1, |w| \leq 1\}, \quad \forall p \in \mathbb{C}[z, w].$$

Based on Theorem 4.5, we formulate following problem.

Problem 4.6. (Noncommutative Ando Inequality Problem) Let $\mathcal{A}$ be a unital C^* -algebra. Whether there is a universal constant $R_{\mathcal{A}}$ (which may depend upon $\mathcal{A}$) satisfying following: For every $d \in \mathbb{N}$ and for all $A, B \in \mathbb{M}_d(\mathcal{A})$ with $\|A\| \leq 1$, $\|B\| \leq 1$ and $AB = BA$, we have

$$\|p(A, B)\| \leq \sup \{ \|p(z, w)\| : z, w \in \mathcal{A}, \|z\| \leq 1, \|w\| \leq 1 \}, \quad \forall p \in \mathcal{A}[z, w].$$

Note that Ando theorem cannot be extended to more than two commuting matrices [21, 98]. We observe that the Halmos dilation [49], Egervary dilation [32, 69] and Sz.-Nagy dilation (Schaffer construction) [89] carry over to adjointable morphisms on Hilbert C^* -modules. This observation will give following result.

Theorem 4.7. (Noncommutative Sz.-Nagy Dilation) Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism such that $\|T\| \leq 1$. Then there exist a Hilbert C^* -module $\mathcal{E}_0$ which contains $\mathcal{E}$ isometrically, $\mathcal{E}$ is orthogonally complementable in $\mathcal{E}_0$, $P_{\mathcal{E}} : \mathcal{E}_0 \rightarrow \mathcal{E}$ is an onto orthogonal projection and unitary morphism $U : \mathcal{E}_0 \rightarrow \mathcal{E}_0$ such that

$$T^n x = P_{\mathcal{E}} U^n x, \quad \forall n \in \mathbb{N}, \forall x \in \mathcal{E}.$$

Corollary 4.8. Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism such that $\|T\| \leq 1$. Let $\mathcal{E}_0$, $P_{\mathcal{E}}$ and U be as in Theorem 4.7. Then

$$\|p(T)\| \leq \|p(U)\|, \quad \forall p(z) = a_0 + za_1 + \cdots + z^n a_n \in \mathcal{A}[z].$$

Corollary 4.9. Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $T : \mathcal{E} \rightarrow \mathcal{E}$ be an adjointable morphism such that $\|T\| \leq 1$. Let $x \in \mathcal{E}$ be such that $Tx = x$. Then $T^*x = x$.

Proof. Let $x \in \mathcal{E}$ be such that $Tx = x$. Let $\mathcal{E}_0$, $P_{\mathcal{E}}$ and U be as in Theorem 4.7. We then have

$$x = Tx = P_{\mathcal{E}} Ux, \quad \langle Ux, Ux \rangle = \langle x, x \rangle, \quad P_{\mathcal{E}} x = x.$$

We see that

$$\begin{aligned} \langle Ux - x, Ux - x \rangle &= 2\langle x, x \rangle - \langle Ux, x \rangle - \langle x, Ux \rangle \\ &= 2\langle x, x \rangle - \langle Ux, P_{\mathcal{E}} x \rangle - \langle P_{\mathcal{E}} x, Ux \rangle \\ &= 2\langle x, x \rangle - \langle P_{\mathcal{E}} Ux, x \rangle - \langle x, P_{\mathcal{E}} Ux \rangle \\ &= 2\langle x, x \rangle - \langle Tx, x \rangle - \langle x, Tx \rangle \\ &= 2\langle x, x \rangle - \langle x, x \rangle - \langle x, x \rangle = 0. \end{aligned}$$

Therefore

$$Ux = x \implies x = U^*x.$$

Hence

$$x = U^*x = P_{\mathcal{E}} U^*x = T^*x.$$

□

We also observe that the proof of Ando dilation [7] for commuting operators on Hilbert space will not carry over to commuting adjointable morphisms on Hilbert C^* -modules (mainly because submodules need not be orthogonally complementable and C^* -algebras need not have invariant basis number property).

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