

A note of refinements to an algebra of infinite little finitistic dimension

Qiyue Tang

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Abstract

Building on OpenAI's *An algebra of infinite little finitistic dimension*, we refine the construction and extend it to every field.

1 Introduction

For a finite-dimensional algebra A over a field k , write $\text{pd}_A N$ for the projective dimension of a left A -module N , and set

$$\text{fin. dim } A = \sup\{\text{pd}_A N : N \text{ is a finite-dimensional left } A\text{-module, } \text{pd}_A N < \infty\}.$$

The finitistic-dimension problem asks whether this supremum is finite for every finite-dimensional algebra; see [2, 11]. OpenAI's construction [7] gives a complex algebra for which the supremum is infinite. We retain its group, localization, tensor realization, and square-zero extension constructions, and refine their field of definition, test modules, and dimension estimates.

For an algebra A , let $J_A = \text{rad } A$ denote its Jacobson radical. The Loewy length of a nonzero module N is the least $h \geq 1$ with $J_A^h N = 0$. An algebra is split basic if A/J_A is a finite product of copies of k . We call a module a brick when its endomorphism ring is a division ring; the complex bricks below have endomorphism ring $\mathbb{C}$.

Theorem 1.1. *For every field k there are a finite-dimensional split basic k -algebra A with $J_A^7 = 0$, integers $b_* \geq 0$ and $l \geq 0$, and finite-dimensional left A -modules N_m , $m \geq 1$, such that*

$$(b_* + 5)(m - 1) + 1 \leq \text{pd}_A N_m \leq (b_* + 5)(m - 1) + l + 3.$$

The modules have Loewy length three, their endomorphism rings are division rings, and $\text{Hom}_A(N_m, N_n) = 0$ for $m \neq n$. In particular, $\text{fin. dim } A = \infty$. For $k = \mathbb{C}$ they can be chosen with

$$\dim_{\mathbb{C}} N_m = 3 \cdot 2^{3m}, \quad \text{End}_A(N_m) = \mathbb{C}.$$

Every primitive vertex idempotent ε in this construction satisfies $\varepsilon A \varepsilon = k \varepsilon$.

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The proof follows the original construction in [7]. A finitely presented group supplies a group algebra R , a central idempotent e , and an automorphism α . For a left R -module Y , the operation $H(Y) = {}_\alpha(eY)$ takes the summand eY and lets $r \in R$ act by $\alpha(r)$. We construct simple modules that vanish under precisely the m th iteration of H . A three-vertex algebra B encodes R , and a bounded bimodule complex P realizes H together with a shift by three. A chain algebra records the differential of P in an ordinary bimodule X over an algebra D of finite global dimension. The square-zero extension of D by X converts finite tensor extinction into finite projective dimension.

All algebras are associative and unital, and algebra homomorphisms preserve k and the unit. Modules are left modules unless designated right. The notation $D^b(A\text{-mod})$ means the bounded derived category of finite-dimensional left A -modules; $K^b(\text{proj-}A)$ means the homotopy category of bounded finitely generated projective right A -module complexes. The symbol $\otimes^L$ denotes derived tensor product. For a complex K and an integer n , write $d_K^n : K^n \rightarrow K^{n+1}$ for its differential and $H^n(K) = \ker d_K^n / \text{im } d_K^{n-1}$ for its cohomology. For integers n, s , the shift convention is

$$K[s]^n = K^{n+s}, \quad d_{K[s]}^n = (-1)^s d_K^{n+s}.$$

A module is regarded as a complex concentrated in degree zero, so its shift by s has cohomology in degree $-s$. Path products and endomorphism products are composed from right to left.

2 A selection process with unbounded extinction

Let R be a unital k -algebra, let $e \in R$ be a central idempotent, and let $\alpha : R \rightarrow R$ be a unital k -algebra endomorphism. For a left R -module Y , define

$$H(Y) = {}_\alpha(eY). \quad (2.1)$$

Here eY is an R -submodule, and the subscript denotes restriction of scalars along α : on the vector space eY , an element $r \in R$ acts by the original operator $\alpha(r)$. On a module map $f : Y \rightarrow Z$, the functor acts by the restriction $f|_{eY} : eY \rightarrow eZ$, with both actions twisted by α . It is additive and preserves finite dimensionality. For $j \geq 0$, write $H^{\circ j}$ for its j -fold iterate, with $H^{\circ 0}$ the identity functor; superscripts without a circle on H will denote cohomology of a complex.

Proposition 2.1. *For every field k , there exist a finitely presented unital k -algebra R , a central idempotent $e \in R$, and an automorphism α of R such that, for every integer $m \geq 1$, a finite-dimensional simple left R -module Y_m satisfies*

$$H^{\circ j}(Y_m) \neq 0 \quad (0 \leq j < m), \quad H^{\circ m}(Y_m) = 0. \quad (2.2)$$

Fix a prime p invertible in k : take $p = 2$ if $\text{char } k \neq 2$, and $p = 3$ otherwise. We construct a finitely presented group G with central elements of order p and take its group algebra $R = kG$.

The group presentation is modeled on matrix groups of Abels type. The commutator transfer and centrality argument parallel [9, Proposition 4.9, equations (4.9)–(4.10)].

2.1 The group and its finite presentation

Write $[x, y] = xyx^{-1}y^{-1}$. Define a group G with commuting generators T_ℓ , $\ell \in \{1, 2, 3\}$, and generators

$$U_i(r), \quad V_i(r), \quad W_{ij}(r) \quad (i, j \in \{1, 2, 3\}, i \neq j, r \in \mathbb{Z}).$$

Each generator type S has a weight $\lambda_S \in \mathbb{Z}^3$, given by

$$\lambda_{U_i} = -\varepsilon_i, \quad \lambda_{V_i} = \varepsilon_i, \quad \lambda_{W_{ij}} = \varepsilon_i - \varepsilon_j, \quad (2.3)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is the standard basis of $\mathbb{Z}^3$. Write $\lambda_{S,\ell}$ for the ℓ th coordinate of λ_S . Impose the conjugation rules

$$T_\ell S(r) T_\ell^{-1} = S(r + \lambda_{S,\ell}). \quad (2.4)$$

The remaining relations, with all arguments ranging independently over $\mathbb{Z}$, are

- (i) $U_i(r)^p = 1$;
- (ii) generators of types U_i, U_j , of types V_i, V_j , and of types U_i, V_j commute whenever $i \neq j$;
- (iii) W_{ij} commutes with U_h for $h \neq i$ and with V_h for $h \neq j$;
- (iv) for $i \neq j$,
$$[U_i(r), W_{ij}(s)] = U_j(r+s), \quad [W_{ij}(s), V_j(r)] = V_i(s+r). \quad (2.5)$$

Lemma 2.2. *The group G is finitely presented.*

Proof. For $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, put $T^n = T_1^{n_1} T_2^{n_2} T_3^{n_3}$ and regard a weight as the map $\lambda_S(n) = \sum_\ell \lambda_{S,\ell} n_\ell$. Retain only the fifteen generators

$$T_\ell, \quad u_i = U_i(0), \quad v_i = V_i(0), \quad w_{ij} = W_{ij}(0) \quad (i \neq j).$$

Besides commutation of the T_ℓ , require the following finite set of centralizer relations. Each of u_i, v_i commutes with T_h for $h \neq i$, and w_{ij} commutes with $T_i T_j$ and with T_h for the unique $h \notin \{i, j\}$. These are precisely centralization by a basis of each kernel lattice:

$$\begin{aligned} \ker \lambda_{U_i} &= \ker \lambda_{V_i} = \bigoplus_{h \neq i} \mathbb{Z} \varepsilon_h, \\ \ker \lambda_{W_{ij}} &= \mathbb{Z}(\varepsilon_i + \varepsilon_j) \oplus \mathbb{Z} \varepsilon_h \quad (h \notin \{i, j\}). \end{aligned}$$

Since the T_ℓ commute, the kernel lattice also centralizes every T -conjugate of its argument-zero generator. Thus one may define

$$U_i(r) = T_i^{-r} u_i T_i^r, \quad V_i(r) = T_i^r v_i T_i^{-r}, \quad W_{ij}(r) = T_i^r w_{ij} T_i^{-r}. \quad (2.6)$$

Equivalently, $S(r) = T^n S(0) T^{-n}$ for any n with $\lambda_S(n) = r$. The kernel relations make this independent of the choice of n and imply all of (2.4).

It remains to impose the argument-zero instance of each of the remaining relation patterns. For every pair of input types occurring in those patterns, the two weights give a surjection $\mathbb{Z}^3 \rightarrow \mathbb{Z}^2$. The following table supplies integral preimages of an arbitrary pair (r, s) ; unmentioned coordinates are zero.

Input types	Index condition	Coordinates of n
U_i, U_j	$i \neq j$	$n_i = -r, n_j = -s$
V_i, V_j	$i \neq j$	$n_i = r, n_j = s$
U_i, V_j	$i \neq j$	$n_i = -r, n_j = s$
W_{ij}, U_j	$i \neq j$	$n_j = -s, n_i = r - s$
W_{ij}, U_h	$h \notin \{i, j\}$	$n_i = r, n_j = 0, n_h = -s$
W_{ij}, V_i	$i \neq j$	$n_i = s, n_j = s - r$
W_{ij}, V_h	$h \notin \{i, j\}$	$n_i = r, n_j = 0, n_h = s$
U_i, W_{ij}	$i \neq j$	$n_i = -r, n_j = -r - s$
W_{ij}, V_j	$i \neq j$	$n_j = s, n_i = r + s$

Conjugating a zero-argument commutation relation by the corresponding T^n therefore gives every required instance. For the two relations in (2.5), the output weight is the sum of the input weights:

$$-\varepsilon_i + (\varepsilon_i - \varepsilon_j) = -\varepsilon_j, \quad (\varepsilon_i - \varepsilon_j) + \varepsilon_j = \varepsilon_i.$$

Hence the same conjugation also gives the correct output argument $r + s$. The power relation propagates from argument zero because each individual weight is surjective onto $\mathbb{Z}$.

We have obtained all the defining relations from finitely many relators on fifteen generators. Conversely, the chosen kernel centralizers and zero-argument relators hold in the original presentation by (2.4). The two presentations are therefore equivalent. $\square$

2.2 Central elements and an automorphism

Central elements shifted by an automorphism will determine the extinction time.

Lemma 2.3. *For each $N \in \mathbb{Z}$, the expression*

$$z_N = [U_i(a), V_i(b)], \quad a + b = N, \quad (2.7)$$

is independent of i, a, b . Each z_N is central and satisfies $z_N^p = 1$. There is an automorphism α_G of G fixing all $T_\ell, V_i(r), W_{ij}(r)$ and sending $U_i(r)$ to $U_i(r+1)$; it satisfies $\alpha_G(z_N) = z_{N+1}$.

Proof. Fix distinct i, j and integers r, q, s , and set

$$x = U_i(r), \quad y = W_{ij}(q), \quad v = V_j(s), \quad u = [x, y] = U_j(r+q), \quad w = [y, v] = V_i(q+s).$$

The defining commutations give $[x, v] = [w, u] = [w, v] = 1$. Using $xy = uyx$ and $yv = wvy$, reorder xyv in the two ways

$$xyv = uwyv = (xwx^{-1})vuyx.$$

Cancellation gives $uwv = [x, w]wvu$, and consequently

$$[x, w] = uwvu^{-1}v^{-1}w^{-1} = uvu^{-1}v^{-1} = [u, v].$$

The last equality uses only the commutation of w with u and v . We have proved

$$[U_i(r), V_i(q+s)] = [U_j(r+q), V_j(s)]. \quad (2.8)$$

Given $a+b=c+d=N$ at distinct internal indices, choose $r=a$, $s=d$, and $q=c-a=b-d$ in (2.8). This compares the two expressions in (2.7). Two splittings at the same internal index can be compared through another index, proving the claimed independence.

To show that z_N commutes with $U_h(r)$ or $V_h(r)$, express z_N using an index $i \neq h$. Both factors of that commutator commute with the tested generator. To commute with $W_{hj}(r)$, use the index outside $\{h, j\}$, so that again both factors commute with it. Write $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. Conjugation by T_ℓ changes the arguments (a, b) in (2.7) to $(a - \delta_{\ell i}, b + \delta_{\ell i})$. Their sum is unchanged, so the splitting independence proves invariance under T_ℓ . Thus z_N is central. Since $U_i(a)^p = 1$, centrality also gives

$$1 = [U_i(a)^p, V_i(b)] = [U_i(a), V_i(b)]^p = z_N^p.$$

Translation of every U -argument by 1 preserves the conjugation, power, and commutation relations. The first relation in (2.5) becomes

$$[U_i(r+1), W_{ij}(s)] = U_j(r+s+1),$$

which is another defining instance; the second relation is unchanged. This defines an endomorphism α_G . Translation by -1 preserves the relations for the same reason and is its two-sided inverse. Formula (2.7) then gives $\alpha_G(z_N) = z_{N+1}$. $\square$

2.3 Finite quotients detecting independent central elements

Lemma 2.4. *For every $m \geq 1$, there is a finite quotient F_m of G in which the images of $z_0, \dots, z_{m-1}$ generate a central subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$.*

Proof. Let $\mathbb{F}_p$ be the field with p elements and put $S_m = \mathbb{F}_p[t]/(t^m - 1)$, using t also for its residue class. This ring is finite, and t is a unit with inverse t^{m-1} , so t^r is defined for every $r \in \mathbb{Z}$. Use 5×5 matrices over S_m , indexed by $0, 1, 2, 3, 4$. Let I_5 be the identity matrix and E_{ab} the matrix with entry 1 at (a, b) and 0 elsewhere. Send T_ℓ to the diagonal matrix with t in position ℓ and 1 elsewhere, and set

$$U_i(r) \mapsto I_5 + t^r E_{0i}, \quad V_i(r) \mapsto I_5 + t^r E_{i4}, \quad W_{ij}(r) \mapsto I_5 + t^r E_{ij}. \quad (2.9)$$

Every displayed elementary matrix is invertible, with $(I_5 + aE_{ij})^{-1} = I_5 - aE_{ij}$ for $i \neq j$. Also $(I_5 + aE_{ij})^p = I_5$ in characteristic p .

We check the relations. Diagonal conjugation multiplies E_{ab} by the ratio of the diagonal entries at a and b , which gives exactly (2.3). The identity $E_{ab}E_{cd} = \delta_{bc}E_{ad}$ shows that all products in both orders vanish for each required commuting pair: two units E_{0i} , two units E_{i4} , the pair E_{0i}, E_{j4} for $i \neq j$, the pair E_{ij}, E_{0h} for $h \neq i$, and the pair E_{ij}, E_{h4} for $h \neq j$. The required power relations hold by the same square-zero calculation. For $i \neq j$, the nonzero product in the first noncommutation relation is $E_{0i}E_{ij} = E_{0j}$, and in the second it is $E_{ij}E_{j4} = E_{i4}$. The higher products vanish, giving

$$\begin{aligned} [I_5 + t^r E_{0i}, I_5 + t^s E_{ij}] &= I_5 + t^{r+s} E_{0j}, \\ [I_5 + t^s E_{ij}, I_5 + t^r E_{j4}] &= I_5 + t^{s+r} E_{i4}. \end{aligned}$$

Thus (2.9) defines a homomorphism from G into the finite group $\mathrm{GL}_5(S_m)$ of invertible 5×5 matrices over S_m . Let F_m be its image.

The image of z_N is $I_5 + t^N E_{04}$. It is central in F_m : the unit E_{04} has zero product in both orders with every off-diagonal unit in (2.9), and the diagonal matrices have entry 1 at both 0 and 4. For $c_0, \dots, c_{m-1} \in \{0, \dots, p-1\}$,

$$\prod_{q=0}^{m-1} (I_5 + t^q E_{04})^{c_q} = I_5 + \left(\sum_{q=0}^{m-1} c_q t^q \right) E_{04}.$$

The classes $1, t, \dots, t^{m-1}$ are an $\mathbb{F}_p$ -basis of S_m , because the defining polynomial is monic of degree m . Thus this product is the identity only when all c_q are zero. The claimed independence and orders follow. $\square$

2.4 The algebra and its test modules

Proof of Proposition 2.1. Take $R = kG$. A finite group presentation gives a finite algebra presentation by adjoining inverses to its generators and imposing the inverse equations and group relators. Extend α_G linearly to $\alpha : R \rightarrow R$. For $q \in \mathbb{Z}$ define the central idempotents

$$e_q = p^{-1} \sum_{a=0}^{p-1} z_q^a, \quad e = e_0.$$

Then $\alpha(e_q) = e_{q+1}$. Use the same symbols for their images in $k[F_m]$, and put

$$p_m = \left(\prod_{q=0}^{m-2} e_q \right) (1 - e_{m-1}),$$

with empty product 1 when $m = 1$. The coefficient of the group basis element z_{m-1} in p_m is $-p^{-m} \neq 0$. Indeed, the term $\prod_{q=0}^{m-2} e_q$ has no coefficient at z_{m-1} , whereas the product with e_{m-1} contributes p^{-m} there. The independence in Lemma 2.4 makes this contribution unique. Thus $k[F_m]p_m$ is nonzero. For $0 \leq q < m-1$, the identities $e_q p_m = p_m$ and $e_{m-1} p_m = 0$ show that e_q acts as identity and e_{m-1} as zero on this left ideal and on each quotient, since these idempotents are central. Choose a simple quotient Y_m . It is finite-dimensional and remains simple under the surjection $R \rightarrow k[F_m]$.

For every R -module Y and $j \geq 0$, restriction of scalars gives

$$H^{\circ j}(Y) \cong_{\alpha^j} (e_0 e_1 \cdots e_{j-1} Y). \quad (2.10)$$

The product is 1 for $j = 0$. At stage j , selection by e is the original action of $\alpha^j(e) = e_j$, and the next pullback makes r act as $\alpha^{j+1}(r)$. This proves the formula by induction, and its application to Y_m proves (2.2). $\square$

3 Localization and a fixed lifted diagram

Let k be a field, let R be a finitely presented unital k -algebra, let $e \in R$ be a central idempotent, and let $\alpha : R \rightarrow R$ be a unital k -algebra endomorphism. Retain $H(Y) = {}_{\alpha}(eY)$, now allowing α to be an endomorphism. Centrality of e still makes eY stable under this action.

We first encode the presentation of R in a finite-dimensional directed algebra. This is a three-vertex version of the homogenized quiver construction in [6, Section 1, Theorem 1.1 and Proposition 1.2]. The associated Verdier quotient carries the required R -action.

3.1 A quadratic presentation and its directed encoding

Choose a presentation

$$R = k\langle x_1, \dots, x_d \rangle / (p_1, \dots, p_t)$$

in which each p_j has degree at most two, allowing constant and linear terms. Degree means the maximum word length among the nonzero monomials, with the zero polynomial assigned degree $-\infty$. Such a presentation exists: in any finite presentation, introduce finitely many auxiliary generators for the successive partial products in its finitely many monomials, together with their binary-product definitions. The resulting relations have degree at most two, and eliminating the auxiliary generators recovers the original presentation.

Consider the quiver with vertices $0, 1, 2$ and arrow sets

$$\mathcal{A}_{01} = \{s, a_1, \dots, a_d\}, \quad \mathcal{A}_{12} = \{s', a'_1, \dots, a'_d\}, \quad \mathcal{A} = \mathcal{A}_{01} \sqcup \mathcal{A}_{12},$$

where the subscripts specify the source and target. Impose the relations

$$a'_h s = s' a_h \quad (1 \leq h \leq d), \quad \tilde{p}_j = 0 \quad (1 \leq j \leq t), \quad (3.1)$$

where homogenization replaces

$$x_h x_l \longmapsto a'_h a_l, \quad x_h \longmapsto s' a_h, \quad 1 \longmapsto s' s.$$

Let B be the resulting path algebra with relations, and let $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ be its vertex idempotents. Write $\mathcal{R}$ for a fixed vector-space basis of the span of all the relations in (3.1), viewed in the free space of paths of length two. Thus each $\rho \in \mathcal{R}$ has a unique expression

$$\rho = \sum_{\substack{a \in \mathcal{A}_{01} \\ b \in \mathcal{A}_{12}}} \rho_{ba} ba.$$

The basis $\mathcal{R}$ generates the same relation ideal. There are no paths of length three, so B is finite-dimensional.

The global dimension $\text{gl.dim } \Lambda$ is the supremum of the projective dimensions of all left Λ -modules; the right global dimension is $\text{gl.dim}(\Lambda^{\text{op}})$, where Λ^{op} is the algebra with reversed multiplication.

Lemma 3.1. *Let Λ be a finite-dimensional k -algebra with complete orthogonal idempotents $\varepsilon_0, \dots, \varepsilon_{n-1}$ such that*

$$1 = \sum_{i=0}^{n-1} \varepsilon_i, \quad \varepsilon_i \Lambda \varepsilon_i = k \varepsilon_i, \quad \varepsilon_j \Lambda \varepsilon_i = 0 \quad (j < i).$$

Then Λ has global dimension at most $n - 1$ on both sides. In particular, the algebra B above has global dimension at most two on both sides.

Proof. Say that a left module M is supported at indices at least p if $\varepsilon_j M = 0$ for $j < p$. There is a projective surjection

$$\bigoplus_{i=p}^{n-1} \Lambda \varepsilon_i \otimes_k \varepsilon_i M \longrightarrow M, \quad c \otimes v \longmapsto cv.$$

The source is projective even when M is not finitely generated, since each summand is a direct sum of copies of $\Lambda \varepsilon_i$. Its components below p vanish, and its component at p maps isomorphically onto $\varepsilon_p M$: only the summand $i = p$ contributes there. Its kernel is therefore supported at indices at least $p + 1$.

Starting with $p = 0$ and repeating this construction, the $(n - 1)$ st kernel is supported only at the last vertex. Every such module is projective, since $\Lambda \varepsilon_{n-1} = k \varepsilon_{n-1}$. This gives the left dimension bound, including the case $n = 1$. The same proof for Λ^{op} , with the reverse ordering of the idempotents, gives the right bound. The path directions and the homogeneous quadratic relations of B give precisely the displayed hypotheses with $n = 3$. $\square$

For a finite-dimensional left R -module Y , define $M(Y)$ to be the left B -module with Y at every vertex, with s, s' acting as the identity, and with a_h, a'_h acting as x_h . The relations of B hold because they evaluate to the defining relations of R .

3.2 The quotient action and evaluation

Let

$$K = \mathbf{K}^b(\text{proj-}B), \quad E_i = \mathbf{e}_i B,$$

where $i = 0, 1, 2$. Here K is the homotopy category of bounded complexes of finitely generated projective *right* B -modules. With our path convention, an arrow $a : i \rightarrow j$ belongs to $\mathbf{e}_j B \mathbf{e}_i$. Left multiplication by a is consequently a right-module map $E_i \rightarrow E_j$. Composing these maps agrees with multiplying paths in the stated order.

For a chain map $u : T \rightarrow T'$, its mapping cone has terms $\text{Cone}(u)^n = (T')^n \oplus T^{n+1}$ and differential $(y, x) \mapsto (d_{T'} y + u(x), -d_T x)$. Its isomorphism class in K depends only on the homotopy class of u . A thick subcategory is a full triangulated subcategory closed under direct summands. Let $\mathcal{S}$ be the thick subcategory of K generated by the cones of $s : E_0 \rightarrow E_1$ and $s' : E_1 \rightarrow E_2$, and form

$$q : K \longrightarrow Q = K/\mathcal{S}.$$

We use the same notation for objects and arrows after applying q . The maps s and s' are isomorphisms in Q . Set

$$\bar{x}_h = s^{-1} a_h \in \text{End}_Q(E_0).$$

The relation $a'_h s = s' a_h$ gives $a'_h = s' a_h s^{-1}$, whence

$$(s's)^{-1} a'_h a_l = s^{-1} a_h s^{-1} a_l = \bar{x}_h \bar{x}_l.$$

Likewise $(s's)^{-1} s' a_h = \bar{x}_h$ and $(s's)^{-1} s' s = \text{id}_{E_0}$. The homogenized relations therefore give a unital homomorphism

$$\theta : R \longrightarrow \text{End}_Q(E_0), \quad x_h \longmapsto \bar{x}_h. \quad (3.2)$$

For each finite-dimensional Y , tensoring defines an exact functor

$$\text{ev}_Y : K \longrightarrow \mathbf{D}^b(k), \quad T \longmapsto T \otimes_B M(Y).$$

Here $\mathbf{D}^b(k)$ denotes the bounded derived category of finite-dimensional k -vector spaces; termwise projectivity of T computes the derived tensor product. Under the natural identifications

$$E_i \otimes_B M(Y) \cong \mathbf{e}_i M(Y) = Y,$$

an arrow map evaluates to its action on $M(Y)$. Thus ev_Y takes s, s' to identities and their cones to zero. The kernel of this exact functor is thick, so it contains $\mathcal{S}$. The universal property of the Verdier quotient [10, Tag 05RJ] therefore gives a factorization through Q . We retain the notation ev_Y for this factorization. On E_0 the homomorphism (3.2) evaluates to the given R -action on Y .

3.3 An odd double of a formal summand

The idempotent $\theta(e)$ evaluates to the projection of Y onto eY , but it need not split inside Q . We therefore seek an object of Q whose evaluation is eY plus a shifted copy. The following lemma gives such an object for a copy shifted by three.

For an additive category $\mathcal{T}$, its idempotent completion $\text{Kar}(\mathcal{T})$ has objects (T, p) , where $p^2 = p \in \text{End}(T)$, and morphisms $u : (T, p) \rightarrow (T', p')$ satisfying $u = p' u p$. The identity of (T, p) is p , and the embedding $T \mapsto (T, \text{id}_T)$ is fully faithful. If $\mathcal{T}$ has a shift functor, it extends by $(T, p)[1] = (T[1], p[1])$.

For the general triangulated structure on the idempotent completion, see [1, Theorem 1.5]. The following argument uses triangles in $\mathcal{T}$.

Lemma 3.2. *Let $\mathcal{T}$ be a triangulated category, let $L \in \mathcal{T}$, and let $\epsilon \in \text{End}_{\mathcal{T}}(L)$ be idempotent. Put $U = (L, \epsilon) \in \text{Kar}(\mathcal{T})$. Both $U \oplus U[1]$ and $U \oplus U[3]$ are isomorphic in $\text{Kar}(\mathcal{T})$ to objects of $\mathcal{T}$.*

Proof. First consider a triangle from $\mathcal{T}$,

$$A \xrightarrow{f} B \xrightarrow{g} T \xrightarrow{h} A[1],$$

and suppose that formal decompositions $A = I \oplus A_0$, $B = I \oplus B_0$ identify f with the identity on I and zero elsewhere. Write pr_{B_0} for the projection $B \rightarrow B_0$ and $\text{in}_{A_0[1]}$ for the inclusion $A_0[1] \rightarrow A[1]$. For any $Z \in \text{Kar}(\mathcal{T})$, the long exact sequence $\text{Hom}(Z, -)$ still holds for this triangle: Z is a retract of an object of $\mathcal{T}$, and its Hom sequence is a direct summand of the exact sequence for that object.

The equalities $gf = 0$ and $f[1]h = 0$ give maps $g_0 : B_0 \rightarrow T$ and $h_0 : T \rightarrow A_0[1]$ with

$$g = g_0 \text{pr}_{B_0}, \quad h = \text{in}_{A_0[1]} h_0.$$

Writing $(g_0)_*$ and $(h_0)_*$ for postcomposition by these maps, exactness gives, for every such Z , a short exact sequence

$$0 \longrightarrow \text{Hom}(Z, B_0) \xrightarrow{(g_0)_*} \text{Hom}(Z, T) \xrightarrow{(h_0)_*} \text{Hom}(Z, A_0[1]) \longrightarrow 0. \quad (3.3)$$

Indeed, a map into B_0 killed by g_0 factors through f after inclusion into B , and projection back to B_0 makes it zero. At the other end, every map into $A_0[1]$ is killed by $f[1]$ after inclusion into $A[1]$, so it lifts through h . Taking $Z = A_0[1]$ supplies a section σ of h_0 . It follows from (3.3) that

$$(g_0, \sigma) : B_0 \oplus A_0[1] \longrightarrow T$$

induces isomorphisms on $\text{Hom}(Z, -)$ for every Z . Explicitly, a map $t : Z \rightarrow T$ is the sum of $\sigma h_0 t$ and a unique map factoring through g_0 , by (3.3). The same exact sequence shows uniqueness of this decomposition. Taking $Z = T$ gives a right inverse to (g_0, σ) ; injectivity with $Z = B_0 \oplus A_0[1]$ shows that this inverse is also a left inverse.

Apply this calculation to a triangle on $1 - \epsilon : L \rightarrow L$. Its identity summand is $(L, 1 - \epsilon)$, and its zero summand on both sides is U . Its cone is an object $S \in \mathcal{T}$ with $S \cong U \oplus U[1]$ in the completion. Under this identification, consider the map

$$S[1] \cong U[1] \oplus U[2] \xrightarrow{\begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix}} U \oplus U[1] \cong S.$$

Fullness of $\mathcal{T} \hookrightarrow \text{Kar}(\mathcal{T})$ makes this a morphism between the old objects $S[1]$ and S . Its cone is again in $\mathcal{T}$. The preceding calculation, with common identity summand $U[1]$, source complement $U[2]$, and target complement U , identifies that cone with $U \oplus U[3]$. $\square$

Apply Lemma 3.2 with $\mathcal{T} = Q$, $L = E_0$, and $\epsilon = \theta(e)$, and write

$$U = (E_0, \theta(e)).$$

Choose $V_0 \in Q$ and an isomorphism $V_0 \cong U \oplus U[3]$ in $\text{Kar}(Q)$ as in Lemma 3.2. Centrality of e gives a unital action of R on U by

$$r \longmapsto \theta(e\alpha(r)e) \in \text{End}_{\text{Kar}(Q)}(U).$$

Its unit is $\theta(e) = \text{id}_U$, and its products agree with those in R because e commutes with every $\alpha(r)$. Use the same action on $U[3]$ and transport the resulting action on $U \oplus U[3]$ to V_0 .

A *B-diagram* in an additive k -linear category consists of objects at the three vertices and arrow maps satisfying the relations of B . The action above gives such a diagram in Q : place V_0 at each vertex, let s, s' act as identities, and let a_h, a'_h act by the endomorphism associated to x_h .

The category $\text{D}^b(k)$ splits idempotents: every bounded complex of vector spaces is isomorphic to its cohomology with zero differential, where splitting can be done degree by degree. Thus evaluation extends additively to $\text{Kar}(Q)$. It takes U to eY , and the preceding action of r to the restriction of $\alpha(r)$ to eY . Consequently the evaluated B -diagram has

$$eY \oplus (eY)[3] \tag{3.4}$$

at every vertex, identities on s, s' , and the action of $\alpha(x_h)$ on both summands for a_h, a'_h .

3.4 Clearing denominators for the finite diagram

To represent this quotient diagram in the homotopy category, use the denominator class

$$\Sigma = \{u : T \rightarrow T' \text{ in } K : \text{Cone}(u) \in \mathcal{S}\}.$$

By [10, Tag 05RG], Σ is a multiplicative system compatible with the triangulated structure, and Q is its Verdier localization. The right-fraction rules used below also follow directly from

triangles. Identities belong to Σ , and composition preserves Σ by the octahedral axiom. If $u : Y' \rightarrow Y$ belongs to Σ and $f : X \rightarrow Y$ is any map, complete u to a triangle

$$Y' \xrightarrow{u} Y \xrightarrow{c} T \longrightarrow Y'[1], \quad T \in \mathcal{S}.$$

Complete cf to a triangle $X' \xrightarrow{v} X \xrightarrow{cf} T \rightarrow X'[1]$. Then $v \in \Sigma$, and exactness of $\text{Hom}(X', -)$ gives $g : X' \rightarrow Y'$ with $fv = ug$. This is the required Ore square.

For the cancellation rule, suppose that $u : Y \rightarrow Y'$ belongs to Σ and $ug = 0$ for $g : X \rightarrow Y$. A triangle $T[-1] \rightarrow Y \xrightarrow{u} Y' \rightarrow T$ shows that g factors through a map $b : X \rightarrow T[-1]$ with $T \in \mathcal{S}$. Completing b to a triangle gives a denominator $v : X' \rightarrow X$ with $bv = 0$, hence $gv = 0$. Applying this to differences gives the corresponding rule for equality of morphisms.

These rules construct the localization by roofs

$$X \xleftarrow{u} X' \xrightarrow{f} Y, \quad u \in \Sigma,$$

representing $q(f)q(u)^{-1}$; equality is tested after a common source refinement. In particular, every morphism $q(X) \rightarrow q(Y)$ has this form, and

$$q(g) = 0 \implies gv = 0 \text{ for some } v : X' \rightarrow X \text{ in } \Sigma \quad (3.5)$$

for $g : X \rightarrow Y$. For the last assertion, compare the roofs (id_X, g) and $(\text{id}_X, 0)$ using [10, Tag 04VJ]. Finite families can be treated with one denominator by collecting their maps into a single map to a finite direct sum of targets [10, Tag 04VI].

Lemma 3.3. *Every B-diagram in Q is isomorphic to the image under q of a B-diagram in K .*

Proof. Write the given objects as W_i and its arrow maps as t_a . Choose $C_i \in K$ and isomorphisms $\psi_i : q(C_i) \rightarrow W_i$. First collect the maps

$$(\psi_2^{-1}t_b\psi_1)_{b \in \mathcal{A}_{12}} : q(C_1) \longrightarrow \bigoplus_{b \in \mathcal{A}_{12}} q(C_2).$$

Represent this tuple by one roof, with denominator $u_1 : D_1 \rightarrow C_1$ and numerator components $f_b : D_1 \rightarrow C_2$. Set $D_2 = C_2$ and

$$\phi_1 = \psi_1 q(u_1), \quad \phi_2 = \psi_2.$$

Then $\phi_2 q(f_b) = t_b \phi_1$ for every $b \in \mathcal{A}_{12}$.

Next collect the adjusted maps

$$(\phi_1^{-1}t_a\psi_0)_{a \in \mathcal{A}_{01}} : q(C_0) \longrightarrow \bigoplus_{a \in \mathcal{A}_{01}} q(D_1).$$

A single roof gives a denominator $u_0 : D_0 \rightarrow C_0$ and maps $f_a : D_0 \rightarrow D_1$. Put $\phi_0 = \psi_0 q(u_0)$. All the arrow squares now commute:

$$\phi_j q(f_a) = t_a \phi_i \quad (a : i \rightarrow j). \quad (3.6)$$

For $\rho \in \mathcal{R}$, form the morphism in K

$$g_\rho = \sum_{b,a} \rho_{ba} f_b f_a : D_0 \longrightarrow D_2.$$

The relations for t_a and (3.6) give $\phi_2 q(g_\rho) = 0$, so $q(g_\rho) = 0$. Apply (3.5) to the tuple $(g_\rho)_\rho$. There is one denominator $v : D'_0 \rightarrow D_0$ such that $g_\rho v = 0$ in K for every ρ . Replace D_0 by

D'_0 , every map f_a with source vertex 0 by $f_a v$, and ϕ_0 by $\phi_0 q(v)$. The arrow squares still commute, and each relation becomes

$$\sum_{b,a} \rho_{ba} f_b(f_a v) = g_\rho v = 0 \quad \text{in } K.$$

There are no arrows into vertex zero and no longer paths requiring further compatibility. Thus the new data form a B -diagram in K , and the three ϕ_i give an isomorphism from its image to the entire prescribed diagram in Q . $\square$

3.5 The chain data for tensor realization

The lifted relations are null-homotopic. Their homotopies give the chain data for tensor realization.

Proposition 3.4 (Fixed lifted diagram). *For the fixed data R, e, α and the algebra B above, there are bounded complexes D_0, D_1, D_2 of finitely generated projective right B -modules, chain maps*

$$f_a : D_i \longrightarrow D_j \quad (a : i \rightarrow j),$$

and right- B -linear maps $h_\rho : D_0 \rightarrow D_2$ of degree -1 for $\rho \in \mathcal{R}$, such that

$$d_{D_2} h_\rho + h_\rho d_{D_0} = \sum_{b,a} \rho_{ba} f_b f_a. \quad (3.7)$$

For every finite-dimensional left R -module Y , the diagram whose objects and arrows in $D^b(k)$ are

$$D_i \otimes_B M(Y), \quad f_a \otimes_B \text{id}_{M(Y)} \quad (3.8)$$

is isomorphic to the diagram having $eY \oplus (eY)[3]$ at all three vertices, identities on s, s' , and the action of $\alpha(x_h)$ on both summands for a_h, a'_h . All the complexes, maps, and homotopies in this statement are fixed before Y is chosen.

Proof. Apply Lemma 3.3 to the diagram on V_0 . Choose bounded right-projective complex representatives D_i and chain-map representatives f_a . Each relation is zero in K , hence its representative is null-homotopic. Choose one right-linear homotopy h_ρ for each of the finitely many relations. This gives (3.7).

Apply ev_Y to the arrow squares (3.6) and the identification $V_0 \cong U \oplus U[3]$. Equation (3.4) gives the required diagram isomorphism. $\square$

4 Realization by a bimodule complex

We retain the algebra B , its relation basis $\mathcal{R}$, and the module $M(Y)$ from the preceding section.

Theorem 4.1. *Let k be a field, let R be a finitely presented k -algebra, let $e \in R$ be a central idempotent, and let $\alpha : R \rightarrow R$ be a unital k -algebra endomorphism. Use the finite-dimensional algebra B and the modules $M(Y)$ constructed in the preceding section, and put $H(Y) = {}_\alpha(eY)$. There is a single bounded complex P of finite-dimensional B -bimodules, termwise projective over $B^e = B \otimes_k B^{\text{op}}$, such that for every finite-dimensional left R -module Y one has*

$$P \otimes_B^{\mathbf{L}} M(Y) \simeq M(H(Y)) \oplus M(H(Y))[3] \quad \text{in } D^b(B\text{-mod}). \quad (4.1)$$

To prove the theorem, we first replace the diagram of [Proposition 3.4](#) by a complex of B -bimodules. The three-column construction below incorporates the relation homotopies; compare the general homotopy-action framework in [\[4\]](#).

Lemma 4.2. *Let D_0, D_1, D_2 be bounded complexes of finitely generated projective right B -modules. Suppose that each arrow $a : i \rightarrow j$ of B is assigned a chain map $f_a : D_i \rightarrow D_j$, and that each relation*

$$\rho = \sum_{b,a} \rho_{ba} ba \in \mathcal{R}$$

is assigned a right B -linear map $h_\rho : D_0 \rightarrow D_2$ of degree -1 satisfying

$$d_{D_2} h_\rho + h_\rho d_{D_0} = \sum_{b,a} \rho_{ba} f_b f_a.$$

There is a bounded complex P of finite-dimensional B -bimodules, termwise projective as a right B -module, together with homotopy equivalences of right-module complexes

$$\iota_i : D_i \longrightarrow \mathbf{e}_i P \quad (i = 0, 1, 2),$$

such that, for every arrow $a : i \rightarrow j$, the two maps $a\iota_i$ and $\iota_j f_a$ from D_i to $\mathbf{e}_j P$ are chain homotopic.

Proof. All tensor products in this construction are over k . Form the bimodule complexes

$$\begin{aligned} L'_0 &= \bigoplus_{i=0}^2 B\mathbf{e}_i \otimes_k D_i, \\ L'_1 &= \bigoplus_{a:i \rightarrow j} B\mathbf{e}_j \otimes_k D_i, \\ L'_2 &= \bigoplus_{\rho \in \mathcal{R}} B\mathbf{e}_2 \otimes_k D_0. \end{aligned}$$

The left B -action is on the first tensor factor and the right B -action is on the second. Let d_0, d_1, d_2 denote the respective internal differentials of these three complexes.

Define $p : L'_1 \rightarrow L'_0$ on the summand indexed by $a : i \rightarrow j$ by

$$p(c \otimes v) = (ca \otimes v)_i - (c \otimes f_a(v))_j.$$

Here subscripts indicate summands of L'_0 . Define $q : L'_2 \rightarrow L'_1$ on the summand indexed by ρ by

$$q(c \otimes v) = \sum_{b,a} \rho_{ba} ((cb \otimes v)_a + (c \otimes f_a(v))_b).$$

These are bimodule chain maps of degree zero. Define also the bimodule map $h : L'_2 \rightarrow L'_0$ of degree -1 by

$$h(c \otimes v) = (c \otimes h_\rho(v))_2$$

on the relation summand ρ . Expanding the composite pq gives

$$\begin{aligned} pq(c \otimes v) &= \sum_{b,a} \rho_{ba} ((cba \otimes v)_0 - (cb \otimes f_a(v))_1 \\ &\quad + (cb \otimes f_a(v))_1 - (c \otimes f_b f_a(v))_2) \\ &= - \sum_{b,a} \rho_{ba} (c \otimes f_b f_a(v))_2. \end{aligned}$$

The last equality uses $\rho = 0$ in B .

Set

$$P = L'_0 \oplus L'_1[1] \oplus L'_2[2]$$

as a graded bimodule, and equip it with the differential

$$d_P = \begin{pmatrix} d_0 & p & h \\ 0 & -d_1 & q \\ 0 & 0 & d_2 \end{pmatrix}.$$

For example, $P^n = (L'_0)^n \oplus (L'_1)^{n+1} \oplus (L'_2)^{n+2}$, so p, q, h each have total degree one in this matrix. The diagonal signs agree with our shift convention. Since p and q commute with the internal differentials, the $(0, 1)$ and $(1, 2)$ entries of d_P^2 vanish. Its remaining off-diagonal entry is

$$d_0 h + p q + h d_2 = 0,$$

by the displayed calculation and the identity for h_ρ . The diagonal entries vanish as well. Thus P is a complex of bimodules.

Every term $B\mathbf{e}_j \otimes_k D_i^n$, viewed as a right B -module, is a finite direct sum of copies of D_i^n . Each D_i^n is a direct summand of a finite free right B -module. Tensoring its splitting with $B\mathbf{e}_j$ exhibits $B\mathbf{e}_j \otimes_k D_i^n$ as a direct summand of finitely many copies of $B\mathbf{e}_j \otimes_k B$. As left B^e -modules,

$$B^e(\mathbf{e}_j \otimes 1) \xrightarrow{\sim} B\mathbf{e}_j \otimes_k B, \quad b\mathbf{e}_j \otimes c^{\text{op}} \mapsto b\mathbf{e}_j \otimes c.$$

Here $(a \otimes d^{\text{op}})(v \otimes c) = av \otimes cd$, so the displayed map respects the action. The left-hand side is an idempotent summand of B^e , proving the required B^e -projectivity. All terms of P are finite dimensional and projective on both sides. There are finitely many summands, and the complexes D_i are bounded; hence P is bounded.

Define

$$\iota_i(v) = (\mathbf{e}_i \otimes v)_i \in \mathbf{e}_i L'_0 \subseteq \mathbf{e}_i P.$$

This is a chain map and is split injective as a map of graded right modules. We prove that its quotient is acyclic.

Call j the *source index* of a summand whose second tensor factor is D_j , in any of the three columns. Give $\mathbf{e}_i P$ the decreasing filtration

$$F^r(\mathbf{e}_i P) = \text{the sum of its summands with source index at least } r \quad (0 \leq r \leq 3).$$

These are subcomplexes. Indeed, the internal differentials and the path-multiplication parts of p, q preserve the source index; the f_a terms increase it; and h increases it from 0 to 2. In particular $F^0 = \mathbf{e}_i P$ and $F^3 = 0$. The quotient by $\iota_i(D_i)$ inherits this finite filtration.

On the associated graded, only the internal differentials and the path-multiplication parts remain. The source- i part consists solely of $\iota_i(D_i)$, since the only path from i to itself is the trivial path. It disappears in the quotient. Source indices larger than i give zero. For a finite set Ω , let k^Ω denote the vector space with basis indexed by Ω . If $j < i$ are adjacent vertices, the source- j part is the signed total complex obtained by tensoring D_j with the exact sequence

$$0 \longrightarrow k^{\{a:j \rightarrow i\}} \xrightarrow{a \mapsto a} \mathbf{e}_i B\mathbf{e}_j \longrightarrow 0.$$

There are no relations of length one, so the displayed map is an isomorphism.

For the remaining pair $(j, i) = (0, 2)$, that part is the signed total complex of D_0 tensored with

$$0 \longrightarrow k^{\mathcal{R}} \longrightarrow k^{\{(b,a):a:0 \rightarrow 1, b:1 \rightarrow 2\}} \longrightarrow \mathbf{e}_2 B\mathbf{e}_0 \longrightarrow 0.$$

The three nonzero horizontal terms have degrees $-2, -1, 0$. The first map sends the basis vector indexed by ρ to $\sum_{b,a} \rho_{ba}(b, a)$; the second sends (b, a) to ba . The first map is injective because $\mathcal{R}$ is a basis of the relation space. Its image is the kernel of the second map: all defining relations have length two, and no path of positive length can enter vertex 0 or leave vertex 2. Thus the ideal they generate, in this path space, is exactly their linear span. The second map is surjective because length-two paths span $\mathbf{e}_2 B \mathbf{e}_0$.

These exact complexes of vector spaces are split, and tensoring with a bounded complex D_j gives an acyclic total complex. All associated graded pieces of the quotient are therefore acyclic. Induction through the finite filtration proves that the quotient itself is acyclic. Thus ι_i is a quasi-isomorphism. Its cone is a bounded acyclic complex of projective right modules, and is contractible: starting in the largest nonzero degree, split the surjection onto its projective last term and continue downward. It follows that ι_i is a homotopy equivalence.

Finally, for $a : i \rightarrow j$ define a right-linear map of degree -1

$$H_a : D_i \longrightarrow \mathbf{e}_j P, \quad H_a(v) = (\mathbf{e}_j \otimes v)_a \in \mathbf{e}_j L'_1[1].$$

The internal differential on this summand is $-d_{D_i}$; its contribution cancels that of $H_a d_{D_i}$. The remaining component is p , giving

$$d_P H_a + H_a d_{D_i} = a \iota_i - \iota_j f_a.$$

This is the required arrow homotopy. □

Proof of Theorem 4.1. Choose the fixed complexes D_i , arrow maps f_a , and relation homotopies h_ρ of Proposition 3.4, and apply Lemma 4.2. Since P and the D_i are bounded complexes of projective right modules, their ordinary tensor products with $M(Y)$ compute the derived tensor products. If chain maps u, v satisfy $u - v = dh + hd$, tensoring this identity with $M(Y)$ gives the same identity with homotopy $h \otimes \text{id}$. Applying this to the homotopy inverses of ι_i and to the arrow homotopies preserves both comparisons after tensoring. Thus the vertex diagram of

$$T_Y := P \otimes_B^{\mathbf{L}} M(Y)$$

in the derived category of k -vector spaces is isomorphic to the evaluated diagram of Proposition 3.4.

At each vertex this diagram is $eY \oplus eY[3]$. The arrows s, s' act by identity maps, and a_h, a'_h act by $\alpha(x_h)$ on each summand. Applying cohomology, with its induced arrow maps, therefore gives isomorphisms of left B -modules

$$H^n(T_Y) \cong \begin{cases} M(H(Y)), & n = 0, -3, \\ 0, & n \neq 0, -3. \end{cases}$$

Using these cohomology identifications, the canonical truncation triangle is

$$M(H(Y))[3] \longrightarrow T_Y \longrightarrow M(H(Y)) \longrightarrow M(H(Y))[4].$$

The connecting morphism belongs to

$$\text{Hom}_{\mathbf{D}^b(B\text{-mod})}(M(H(Y)), M(H(Y))[4]) = \text{Ext}_B^4(M(H(Y)), M(H(Y))) = 0,$$

where the vanishing follows from the bound $\text{gl.dim } B \leq 2$ in Lemma 3.1. Exactness of $\text{Hom}(M(H(Y)), -)$ gives a section of the map $T_Y \rightarrow M(H(Y))$, since the connecting morphism is zero. Together with the first arrow, this section gives $M(H(Y))[3] \oplus M(H(Y)) \rightarrow T_Y$. It induces isomorphisms in degrees -3 and 0 , and all other cohomology groups vanish. It is therefore an isomorphism in the derived category, proving (4.1). □

Corollary 4.3. Put $\Phi = P \otimes_B^{\mathbf{L}} -$. For every integer $j \geq 0$ and every finite-dimensional left R -module Y , there is an isomorphism

$$\Phi^j(M(Y)) \simeq \bigoplus_{q=0}^j (M(H^{\circ j}(Y))[3q])^{\oplus \binom{j}{q}}.$$

Proof. The case $j = 0$ is immediate. For the induction step, apply Φ to an isomorphism for j ; derived tensor preserves isomorphisms, finite direct sums, and shifts. Apply (4.1) with $H^{\circ j}(Y)$ in place of Y to each summand. The summand indexed by q contributes shifts $3q$ and $3(q+1)$, each with multiplicity $\binom{j}{q}$. Thus the multiplicity at shift $3q$ in the next iterate is

$$\binom{j}{q} + \binom{j}{q-1} = \binom{j+1}{q},$$

where $\binom{j}{q} = 0$ for $q < 0$ or $q > j$. These nonnegative integers count copies in a direct sum. Each $H^{\circ j}(Y)$ is finite-dimensional. □

5 Simulation by an ordinary bimodule

We encode the differential of P in an ordinary bimodule. Two derived tensor steps will recover $P \otimes_B^{\mathbf{L}} -$ up to a fixed shift.

Choose integers $a_* \leq 0 \leq b_*$ such that $P^n = 0$ outside $[a_*, b_*]$, enlarging the interval if necessary, and put $l = b_* - a_*$. Let C be the algebra of the quiver

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} \dots \xrightarrow{c_l} l$$

modulo all paths of length two. Write ε_i for its vertex idempotents, and let W be the simple right C -module at vertex l . There is a projective resolution of right C -modules

$$0 \longrightarrow \varepsilon_0 C \xrightarrow{c_1} \varepsilon_1 C \xrightarrow{c_2} \dots \xrightarrow{c_l} \varepsilon_l C \longrightarrow W \longrightarrow 0. \quad (5.1)$$

Indeed, for $i > 0$, the right projective $\varepsilon_i C$ has basis ε_i, c_i , and its radical is the right simple at $i - 1$. Left multiplication by c_i maps ε_{i-1} to c_i and, when $i > 1$, maps c_{i-1} to zero. This proves exactness, including injectivity at the first term. For $l = 0$, the resolution consists of the isomorphism $\varepsilon_0 C \rightarrow W$.

Define a $(B \otimes_k C, B)$ -bimodule O by

$$\varepsilon_i O = P^{a_*+i}, \quad c_i|_{\varepsilon_{i-1} O} = d_P^{a_*+i-1}.$$

Here the two B -actions are those on the terms of P . The relation $d_P^2 = 0$ makes this a representation of C , and the bimodule linearity of d_P makes the C -action commute with both B -actions. Thus O is an ordinary finite-dimensional bimodule. Set

$$D = B \times (B \otimes_k C), \quad T = B \otimes_k W, \quad X = O \oplus T.$$

Regard O as having its left action from the second factor of D and its right action from the first. Regard T as having its left action from the first factor and its right action from the second, with the regular B -actions on its B -factor. These prescriptions define an ordinary finite-dimensional D -bimodule X .

The two summands of X move complexes between the factors of D , as shown in Figure 1. Thus two tensor steps return to the first factor; the next proposition identifies their composite.

$$\begin{array}{ccc}
& O \otimes_B^{\mathbf{L}} - & \\
\text{D}^b(B\text{-mod}) & \xrightarrow{\quad} & \text{D}^b((B \otimes_k C)\text{-mod}) \\
& \xleftarrow{\quad} & \\
& T \otimes_{B \otimes_k C}^{\mathbf{L}} - &
\end{array}$$

Figure 1: The two factors of D and the tensor functors supplied by the off-diagonal summands of X . Starting on the left, the composite is $P[b_*] \otimes_B^{\mathbf{L}} -$ by [Proposition 5.1](#).

Proposition 5.1. *The algebra D has finite global dimension on both sides. Let $F = X \otimes_D^{\mathbf{L}} -$. For every complex Z of left B -modules, regarded as a complex on the first factor of D , there is an isomorphism*

$$F^2(Z) \simeq P[b_*] \otimes_B^{\mathbf{L}} Z, \quad T \otimes_{B \otimes_k C}^{\mathbf{L}} O \simeq P[b_*]. \quad (5.2)$$

The second isomorphism is an isomorphism of complexes of B -bimodules, and the first is supported on the first factor of D .

Proof. The vertices of $B \otimes_k C$ are pairs (i, j) , where $0 \leq i \leq 2$ and $0 \leq j \leq l$. Its diagonal vertex corners are k . A nonzero component between distinct vertices moves in the product of the two directed orders. Assign height $i + j$ to vertex (i, j) . Every nonzero off-diagonal component increases height; the algebra on each height layer is a product of copies of k . If a left module M is supported in heights at least r , where $0 \leq r \leq l + 2$, multiplication gives the projective surjection

$$\bigoplus_{i+j \geq r} (B \otimes_k C)(\mathbf{e}_i \otimes \varepsilon_j) \otimes_k (\mathbf{e}_i \otimes \varepsilon_j) M \longrightarrow M.$$

At each vertex of height r , only its own summand contributes, and the map is an isomorphism there. The kernel is therefore supported in heights at least $r+1$. After $l+2$ steps it is supported at the last height, where it is projective. Reversing the order proves the right-module bound. Thus

$$\text{gl. dim}(B \otimes_k C), \quad \text{gl. dim}((B \otimes_k C)^{\text{op}}) \leq l + 2.$$

The same bound holds for D and D^{op} , since $\text{gl. dim } B, \text{gl. dim } B^{\text{op}} \leq 2$.

Tensor (5.1) over k with the regular B -bimodule, placing its last projective term in degree zero. This gives a bounded resolution $Q_T \rightarrow T$ by right $B \otimes_k C$ -projectives, preserving the left B -action. Also, O is projective as a right B -module, being the finite direct sum of the terms of P . Thus $Q_X = O \oplus Q_T$, with the prescribed block actions, is a bounded right- D -projective resolution of X retaining its left D -action. Ordinary tensor with Q_X computes F on every complex. On the first factor, the block supports therefore give

$$F^2(Z) \simeq Q_T \otimes_{B \otimes_k C} (O \otimes_B Z) \cong (Q_T \otimes_{B \otimes_k C} O) \otimes_B Z.$$

In degree n , for $-l \leq n \leq 0$, tensoring Q_T with O gives

$$(B \otimes_k \varepsilon_{l+n} C) \otimes_{B \otimes_k C} O \cong \varepsilon_{l+n} O = P^{a_*+l+n} = P^{b_*+n}.$$

The induced differential is d_P . In comparison, the standard differential of $P[b_*]$ is $(-1)^{b_*} d_P$. Multiplication by $(-1)^{b_* n}$ in degree n defines a chain isomorphism to $P[b_*]$; it respects both B -actions. Since this complex is termwise right- B -projective, its ordinary tensor with any Z computes the derived tensor. This proves the second isomorphism in (5.2), and hence the first. $\square$

6 A square-zero extension and projective dimension

For a D -bimodule X , the square-zero extension $D \ltimes X$ is the vector space $D \oplus X$ with multiplication $(d, x)(d', x') = (dd', dx' + xd')$. A derived bar decomposition relates tensor extinction to projective dimension.

6.1 The derived bar decomposition

The splitting below also follows from the graded-component decomposition of Minamoto and Yamaura [5, Lemma 4.13(4) and the proof of Theorem 4.17], after forgetting the grading and passing to opposite algebras. The direct proof identifies each derived tensor factor and its shift.

Proposition 6.1. *Let D be a finite-dimensional k -algebra, let X be a finite-dimensional D -bimodule, and let $A = D \ltimes X$. Let N be a left D -module, inflated to A through the projection $A \rightarrow D$, and put $F = X \otimes_D^{\mathbf{L}} -$. In the unbounded derived category $\mathbf{D}(D\text{-Mod})$ of all left D -modules there is an isomorphism*

$$D \otimes_A^{\mathbf{L}} N \simeq \bigoplus_{r \geq 0} (F^r N)[r]. \quad (6.1)$$

Here $F^0 N = N$, and every F^r denotes an iteration of the derived tensor functor.

Proof. Use the ordinary unnormalized bar resolution of N over the field k . It is a free left A -resolution with degree $-n$ term $A \otimes_k A^{\otimes_k n} \otimes_k N$. Its augmentation is a quasi-isomorphism: the augmented bar complex has the usual k -linear contraction obtained by inserting the unit. After tensoring with the right A -module D , it yields a model $\mathcal{B}$ for $D \otimes_A^{\mathbf{L}} N$, with

$$\mathcal{B}^{-n} = D \otimes_k A^{\otimes_k n} \otimes_k N \quad (n \geq 0).$$

Write $\pi : A \rightarrow D$ for the projection. For $n \geq 1$, the differential on $d \otimes a_1 \otimes \cdots \otimes a_n \otimes v$ is

$$\begin{aligned} & d\pi(a_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes v \\ & + \sum_{i=1}^{n-1} (-1)^i d \otimes a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \otimes v \\ & + (-1)^n d \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n v. \end{aligned}$$

For $n = 1$ the middle sum is empty.

Expand each internal copy of A as $D \oplus X$. Every nonzero face preserves the number of entries from X . Indeed, a product involving exactly one X -entry remains in X , a product of two X -entries is zero, and an endpoint action by X is zero on both D and N . Thus there is a decomposition of complexes

$$\mathcal{B} = \bigoplus_{r \geq 0} \mathcal{B}_r,$$

where $\mathcal{B}_r$ consists of words with precisely r entries from X .

For fixed r , write $i_0, \dots, i_r$ for the lengths of the blocks of D -entries before, between, and after the X -entries. The corresponding tensor is

$$D \otimes_k D^{\otimes_k i_0} \otimes_k X \otimes_k D^{\otimes_k i_1} \otimes_k \cdots \otimes_k X \otimes_k D^{\otimes_k i_r} \otimes_k N, \quad (6.2)$$

in degree $-r - \sum_{j=0}^r i_j$. We identify this with the multiple two-sided bar complex for

$$D \otimes_D^{\mathbf{L}} \underbrace{X \otimes_D^{\mathbf{L}} \cdots \otimes_D^{\mathbf{L}} X}_{r \text{ factors}} \otimes_D^{\mathbf{L}} N, \quad (6.3)$$

shifted by r .

First consider the signs. Within block j , a bar face has alternating-sign offset

$$j + \sum_{h < j} i_h.$$

In the ordinary multibar total differential, ordered from left to right, the corresponding offset is $\sum_{h < j} i_h$; shifting the total complex by r adds the factor $(-1)^r$. Rescale the component (6.2) by

$$(-1)^{\sum_{j=0}^r (r-j)i_j}.$$

When a face reduces i_j by one, this factor changes by $(-1)^{r-j}$, precisely the discrepancy between the two differentials. This gives the required chain isomorphism. When $r \geq 1$ and $i_j = 0$, there is no nonzero face in that block: the prospective face multiplies two X -entries or applies an endpoint X -action. For $r = 0$, this is the ordinary two-sided bar complex for $D \otimes_D^L N$. At $i_0 = 0$ its differential is absent; the map $D \otimes_k N \rightarrow N$ is the separate augmentation.

To identify the multibar with (6.3), we resolve each successive left D -complex by free modules. For a bounded-above complex Z of left D -modules, the total left-module bar resolution has terms obtained from

$$D \otimes_k D^{\otimes_k i} \otimes_k Z^q$$

in degree $q - i$. Its terms are free left D -modules because k is a field, and its augmentation to Z is a quasi-isomorphism. For each total degree there are only finitely many possible i , since Z is bounded above. Thus it is a bounded-above complex of free modules and can be used to compute a derived tensor.

Starting at the right, resolve N in this way and tensor with X . Resolve the resulting left D -complex in the same way and tensor with the next X , continuing for the fixed finite number r of factors. A final bar resolution with the leading factor D gives exactly the multibar (6.2), before its shift. All exterior D -actions are preserved. This establishes its identification with (6.3), which is isomorphic to $F^r N$.

Finally, in degree $-n$ of $\mathcal{B}$ only $0 \leq r \leq n$ occur, and for each such r there are finitely many tuples satisfying $r + \sum_j i_j = n$. All totalizations above therefore use direct sums that are finite in each degree. Direct sums are exact for modules, so the componentwise quasi-isomorphisms give (6.1). $\square$

6.2 Detecting finite projective dimension and lower bounds

Corollary 6.2. *In the setting of Proposition 6.1, suppose that D has finite global dimension on both sides and that N is finite-dimensional.*

- (i) *If $F^t N \simeq 0$ for some $t \geq 1$, then $\mathrm{pd}_A N < \infty$.*
- (ii) *If $F^r N \not\simeq 0$, then $\mathrm{pd}_A N \geq r$, with infinite projective dimension allowed.*

Proof. Every fixed iterate $F^r N$ has bounded, finite-dimensional cohomology. For example, if g bounds the right global dimension of D , then for every finite-dimensional left D -module V the groups $\mathrm{Tor}_i^D(X, V)$ are finite-dimensional and vanish for $i > g$. Applying this observation to successive truncation triangles shows inductively that

$$H^q(F^r N) = 0 \quad \text{unless} \quad -rg \leq q \leq 0.$$

For the first assertion, vanishing at t implies vanishing at every $r \geq t$. By (6.1), $D \otimes_A^{\mathbf{L}} N$ then has bounded cohomology. Since X is a nilpotent ideal, it is contained in $\text{rad } A$, and

$$\text{rad } A = (\text{rad } D) \oplus X, \quad A/\text{rad } A \cong D/\text{rad } D =: S.$$

Derived associativity gives

$$S \otimes_A^{\mathbf{L}} N \simeq S \otimes_D^{\mathbf{L}} (D \otimes_A^{\mathbf{L}} N).$$

The right D -module S has a finite projective resolution, so this complex also has bounded cohomology.

Choose a minimal projective resolution of N over the finite-dimensional algebra A , using successive projective covers. Its terms are finitely generated, and every differential has image in the radical of its target. Tensoring with $S = A/\text{rad } A$ therefore annihilates every differential. If Q is a nonzero finitely generated projective A -module, then

$$S \otimes_A Q \cong Q/(\text{rad } A)Q \neq 0$$

by Nakayama's Lemma. Consequently bounded cohomology of $S \otimes_A^{\mathbf{L}} N$ forces the minimal resolution to have only finitely many nonzero terms. This proves $\text{pd}_A N < \infty$.

For the second assertion, choose $q \leq 0$ with $H^q(F^r N) \neq 0$. Its direct summand in (6.1) gives

$$H^{q-r}(D \otimes_A^{\mathbf{L}} N) \neq 0, \quad q - r \leq -r.$$

A projective A -resolution of N of length strictly less than r would make the derived tensor zero in this degree. Hence $\text{pd}_A N \geq r$, as asserted. $\square$

With D of finite global dimension, for a bounded complex Z of left D -modules with finite-dimensional cohomology, $\text{pd}_D Z$ is the least integer d such that Z is represented by a bounded projective complex zero below degree $-d$; set $\text{pd}_D 0 = -\infty$. In particular $\text{pd}_D(Z[s]) = \text{pd}_D Z + s$.

Proposition 6.3. *Under the hypotheses of Corollary 6.2, if $F^t N = 0$ for some $t \geq 1$, then*

$$\text{pd}_A N = \max_{0 \leq r < t} \{\text{pd}_D(F^r N) + r\}.$$

Proof. Put $S = D/\text{rad } D = A/\text{rad } A$. A minimal bounded projective complex representing Z has radical differentials, so

$$\text{pd}_D Z = \sup\{n \in \mathbb{Z} : H^{-n}(S \otimes_D^{\mathbf{L}} Z) \neq 0\}.$$

Such a complex exists by finite global dimension, followed by splitting off contractible projective summands. Reduction modulo the radical kills the remaining differentials and detects every nonzero projective term by Nakayama's lemma. By Corollary 6.2, N has finite projective dimension. Applying $S \otimes_D^{\mathbf{L}} -$ to (6.1) gives

$$S \otimes_A^{\mathbf{L}} N \simeq \bigoplus_{r=0}^{t-1} (S \otimes_D^{\mathbf{L}} F^r N)[r].$$

For each surviving r , cohomology in degree $-n$ of its summand is $H^{-n+r}(S \otimes_D^{\mathbf{L}} F^r N)$. The largest such n is therefore $\text{pd}_D(F^r N) + r$. Cohomology commutes with this finite direct sum, so the largest nonzero degree detected over A is the maximum of these values. The minimal-resolution formula over A gives the result. $\square$

This is the inflated-module case of [5, Corollary 4.11], with sides reversed.

6.3 Tensor iterates and linear bounds

Fix R, e, α over k as in Proposition 2.1, construct B, P by Theorem 4.1, and choose $[a_*, b_*]$, C, W, O, T, D, X as in Section 5. Put $A = D \rtimes X$. For a finite-dimensional R -module V , regard $M(V)$ as supported on the first factor of D . Put $E = B \otimes_k C$ and $Q(V) = O \otimes_B M(V)$, a left E -module. Since O_B is projective, this ordinary tensor computes its derived tensor. Moreover $Q(V) \neq 0$ for $V \neq 0$: the nonzero projective module O_B is a direct sum of vertex projectives, each of which evaluates to V . The module O is nonzero because otherwise $P = 0$, whereas (4.1) applied to Y_2 has nonzero right-hand side.

For $j \geq 0$, induction from (5.2) and (4.1) gives

$$F^{2j}M(V) \simeq \bigoplus_{q=0}^j M(H^{\circ j}V)[jb_* + 3q]^{\oplus \binom{j}{q}}, \quad (6.4)$$

$$F^{2j+1}M(V) \simeq \bigoplus_{q=0}^j Q(H^{\circ j}V)[jb_* + 3q]^{\oplus \binom{j}{q}}. \quad (6.5)$$

Here the binomial coefficient counts direct summands. Take any Y_m satisfying (2.2), and inflate $N_m = M(Y_m)$ along $A \rightarrow D \rightarrow B$. Then

$$F^r N_m \neq 0 \quad (0 \leq r < 2m), \quad F^{2m} N_m = 0.$$

For $0 \leq j < m$, Proposition 6.3 therefore gives

$$\begin{aligned} \text{pd}_A N_m = \max_{0 \leq j < m} \{ & (b_* + 5)j + \text{pd}_B M(H^{\circ j}Y_m), \\ & (b_* + 5)j + 1 + \text{pd}_E Q(H^{\circ j}Y_m) \}. \end{aligned} \quad (6.6)$$

For an even iterate, the shift and its bar degree add to $jb_* + 3q + 2j$; for an odd iterate they add to $jb_* + 3q + 2j + 1$. Both are maximal at $q = j$, giving (6.6). For $j < m$, both modules in that formula are nonzero. Their projective dimensions lie in $[0, 2]$ and $[0, l + 2]$, respectively. The odd term at $j = m - 1$ gives the lower bound. Since $b_* + 5 > 0$, every term is at most $(b_* + 5)(m - 1) + l + 3$, giving

$$(b_* + 5)(m - 1) + 1 \leq \text{pd}_A N_m \leq (b_* + 5)(m - 1) + l + 3. \quad (6.7)$$

7 Radicals and test modules

Proposition 7.1. *The algebra A above is split basic, $J_A^7 = 0$, and $A/J_A \cong k^{3l+6}$. Its primitive vertex corners are k . For every nonzero finite-dimensional R -module Y , the inflated module $M(Y)$ has Loewy length three. For R -modules Y, Z ,*

$$\text{Hom}_A(M(Y), M(Z)) \cong \text{Hom}_R(Y, Z).$$

Proof. The arrow ideals give $J_B^3 = 0$, $J_C^2 = 0$, and split semisimple quotients. Thus

$$J_E = J_B \otimes_k C + B \otimes_k J_C, \quad J_E^4 = 0.$$

Indeed the displayed sum is nilpotent and its quotient is a product of copies of k ; any product of four terms contains three J_B factors or two J_C factors. Also $J_D = J_B \times J_E$ and

$$J_A = J_D \oplus X, \quad J_A^n = J_D^n + \sum_{i+j=n-1} J_D^i X J_D^j \quad (n \geq 1).$$

A word containing two X factors vanishes because $XD X = 0$. The summand O has left E -action and right B -action, so $J_D^i O J_D^j = 0$ if $i \geq 4$ or $j \geq 3$. For T the bounds are reversed. Thus every term for $n = 7$ vanishes. There are $3 + 3(l + 1) = 3l + 6$ vertices. Since X joins distinct factors of D , it contributes nothing to a primitive vertex corner; each such corner of A is k .

The action on $M(Y)$ factors through B , and the image of J_A is J_B . The path s' acts as identity from vertex 0 to vertex 2. Therefore $J_A^2 M(Y) \neq 0$ and $J_A^3 M(Y) = 0$. A B -module map $M(Y) \rightarrow M(Z)$ has three components. Commutation with s, s' makes them equal, and commutation with the arrows a_h makes their common value R -linear. Inflation preserves these morphisms, proving the Hom identity. $\square$

7.1 Explicit complex representations

For $k = \mathbb{C}$ use $p = 2$. For $m \geq 1$ set $S_m = \mathbb{F}_2[t]/(t^m - 1)$. Each element has a unique representative $\sum_{r=0}^{m-1} a_r t^r$. Define $\lambda_m : S_m \rightarrow \mathbb{F}_2$ by $\lambda_m(\sum a_r t^r) = a_{m-1}$. On the complex vector space

$$Y_m^{\text{exp}} = \bigoplus_{x \in S_m^3} \mathbb{C} v_x$$

the vectors v_x form a basis indexed by S_m^3 . Let I be the 3×3 identity matrix. Write e_i for the i th coordinate vector in S_m^3 , E_{ij} for the 3×3 matrix unit. Define, for $i, j, \ell \in \{1, 2, 3\}$, $i \neq j$, and $r \in \mathbb{Z}$,

$$\begin{aligned} T_\ell v_x &= v_{x+(t-1)x_\ell e_\ell}, & U_i(r) v_x &= (-1)^{\lambda_m(t^r x_i)} v_x, \\ V_i(r) v_x &= v_{x+t^r e_i}, & W_{ij}(r) v_x &= v_{(I+t^r E_{ij})x}. \end{aligned}$$

The exponent of -1 is read as 0 or 1.

Proposition 7.2. *These operators define a simple R -module of dimension 2^{3m} satisfying (2.2).*

Proof. Conjugation by T_ℓ changes the parameter of U_i by $-\delta_{\ell i}$, that of V_i by $\delta_{\ell i}$, and that of W_{ij} by $\delta_{\ell i} - \delta_{\ell j}$. Computing on v_x gives

$$[U_i(r), W_{ij}(s)] = U_j(r+s), \quad [W_{ij}(r), V_j(s)] = V_i(r+s),$$

and $[U_i(r), V_i(s)]$ acts as $(-1)^{\lambda_m(t^{r+s})}$. The remaining commutations follow from their coordinate supports, and $U_i(r)^2 = 1$. Thus all defining relations hold. The elements $z_0, \dots, z_{m-2}$ act as 1 and z_{m-1} as -1 , giving the required extinction.

The pairing $(a, b) \mapsto \lambda_m(ab)$ on S_m is nondegenerate: in the basis $1, t, \dots, t^{m-1}$ its matrix has entry 1 exactly when the two exponents sum to $m - 1$ modulo m , a permutation matrix. For $a \in S_m^3$, products of the operators $U_i(r)$ therefore give the diagonal operator $U(a)v_y = (-1)^{\lambda_m(\sum_i a_i y_i)} v_y$. For fixed $x \in S_m^3$, the operator

$$2^{-3m} \sum_{a \in S_m^3} (-1)^{\lambda_m(\sum_i a_i x_i)} U(a)$$

acts on v_y by the scalar $2^{-3m} \sum_a (-1)^{\lambda_m(\sum_i a_i (x_i + y_i))}$. If $y = x$, every term is 1. If $y \neq x$, nondegeneracy supplies a_0 with $\lambda_m(\sum_i (a_0)_i (x_i + y_i)) = 1$; translation $a \mapsto a + a_0$ pairs terms of opposite signs. The operator is therefore the projection onto $\mathbb{C} v_x$. A nonzero vector in an invariant subspace has a nonzero coefficient at some v_x , so this projection puts v_x in the subspace. The elements $1, t, \dots, t^{m-1}$ span S_m additively, and products of the $V_i(r)$ translate x to any element of S_m^3 . The subspace is consequently all of Y_m^{exp} . $\square$

Proof of Theorem 1.1. Use the simple modules from Proposition 2.1, or those from Proposition 7.2 over $\mathbb{C}$. Schur's lemma makes their endomorphism rings division rings, and over $\mathbb{C}$ these rings are $\mathbb{C}$. Different extinction times imply that the simple modules are nonisomorphic, so their mutual Hom spaces vanish. Proposition 7.1 transports these conclusions and gives Loewy length three and all assertions about A . Each encoding has three copies of its input space, giving $3 \cdot 2^{3m}$ in the explicit case. Finally (6.7) proves the claimed bounds and $\text{fin. dim } A = \infty$. $\square$

7.2 Size and other odd shifts

With d generators in the quadratic presentation of R and relation basis $\mathcal{R}$, counting vertices, arrows, and length-two paths gives

$$\dim_k B = 3 + 2(d+1) + (d+1)^2 - |\mathcal{R}|.$$

Since $\dim_k C = 2l+1$ and $\dim_k W = 1$, the construction gives

$$\dim_k A = (2l+3) \dim_k B + \sum_n \dim_k P^n.$$

The support and the terms of P depend on the chosen lifts.

Proposition 7.3. *For every odd integer $s \geq 3$, the realization theorem remains valid with [3] replaced by $[s]$, with a complex P chosen for that s .*

Proof. In the notation of Lemma 3.2, suppose an object S_{s-2} represents $U \oplus U[s-2]$, and S_1 represents $U \oplus U[1]$. Take the map $S_{s-2}[1] \rightarrow S_1$ that is identity on their common $U[1]$ and zero on the other summands. Fullness of the embedding into the idempotent completion supplies the map in the original category. Its cone represents $U \oplus U[s]$. Starting with S_1 proves the assertion for every odd s . The same finite diagram lifting and rectification apply. The final truncation triangle splits because $\text{Ext}_B^{s+1}(M(HY), M(HY)) = 0$ for $s \geq 3$. $\square$

For $\Phi = P \otimes_B^{\mathbf{L}} -$ as above and $j \geq 0$, let $\ker \Phi^j$ be the full subcategory of objects Z with $\Phi^j(Z) \simeq 0$. These are thick subcategories of $\mathbf{D}^b(B\text{-mod})$: they are closed under shifts, cones, and direct summands because Φ is an exact additive functor. These kernels form a strictly increasing chain $\ker \Phi^j \subsetneq \ker \Phi^{j+1}$ for $j \geq 0$, where Φ^0 is the identity. Indeed $M(Y_{j+1})$ lies in the latter kernel and not in the former, by the iteration formula.

8 Left–right asymmetry

The big finitistic dimension $\text{Fin. dim } A$ is defined as $\text{fin. dim } A$ with all left modules allowed in the supremum. Thus $\text{Fin. dim } A \geq \text{fin. dim } A$. Injective modules generate when the smallest triangulated subcategory of the unbounded derived category containing them and closed under arbitrary coproducts is the whole category. Rickard's theorem [8, Theorem 4.3], applied to the opposite algebra, implies that injective left modules do not generate for the algebra A of Theorem 1.1.

The following consequence over $\mathbb{C}$ is retained from [7].

Corollary 8.1. *There is a finite-dimensional complex algebra Λ such that*

$$\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda = \infty, \quad \text{fin. dim } (\Lambda^{\text{op}}) = \text{Fin. dim } (\Lambda^{\text{op}}) = 0.$$

Injective left Λ -modules do not generate, whereas injective right Λ -modules generate.

Proof. Take $k = \mathbb{C}$. The algebra A^{op} is basic by Proposition 7.1. Choose its primitive vertex idempotents $e_1, \dots, e_n$, and let $S_i = e_i A^{\text{op}} / e_i J_{A^{\text{op}}}$ be the corresponding simple right modules. Set

$$E_{\text{aux}} = \prod_{i=1}^n \mathbb{C}[\epsilon_i] / (\epsilon_i^2), \quad U_{\text{aux}} = \bigoplus_{i=1}^n S_i, \quad \tilde{\Lambda} = \begin{pmatrix} A^{\text{op}} & 0 \\ U_{\text{aux}} & E_{\text{aux}} \end{pmatrix}.$$

The left E_{aux} -action on U_{aux} factors through $E_{\text{aux}} / J_{E_{\text{aux}}} = \mathbb{C}^n$, with its i th factor acting on S_i . This is [3, Construction 2.1]. Set $\Lambda = \tilde{\Lambda}^{\text{op}}$. We give the module calculations underlying [3, Propositions 3.2–3.3 and 4.4].

A projective right A^{op} -module, placed on the old vertices with zero component on the new vertices, is projective over $\tilde{\Lambda}$. Inflation preserves exactness and radical differentials, so an inflated minimal projective resolution remains minimal. Its length is unchanged. The finite-dimensional left A -modules N_m therefore give finite-dimensional right $\tilde{\Lambda}$ -modules of the same unbounded finite projective dimensions. Equivalently, $\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda = \infty$.

For each i , let f_i be the idempotent of the i th factor of E_{aux} . As a right $\tilde{\Lambda}$ -module, $f_i \tilde{\Lambda}$ has radical $S_i \oplus \mathbb{C} \epsilon_i$ and one-dimensional top supported at f_i . Its radical is annihilated by $J_{\tilde{\Lambda}}$. Let L_i be the simple right Λ -module at f_i , let $\hat{S}_i$ be the dual of S_i supported on the old vertices, and put $I_i = \text{Hom}_{\mathbb{C}}(f_i \tilde{\Lambda}, \mathbb{C})$, an injective right Λ -module. Dualizing the radical sequence over $\mathbb{C}$ gives the inclusion u_i and quotient map v_i in the exact sequence

$$0 \longrightarrow L_i \xrightarrow{u_i} I_i \xrightarrow{v_i} \hat{S}_i \oplus L_i \longrightarrow 0.$$

The modules $\hat{S}_i, L_i$ exhaust the simple right Λ -modules. In particular, each simple right module is a quotient of an injective module.

This quotient property forces every right Λ -module of finite projective dimension to be projective. Indeed, finite-dimensional algebras have projective covers for arbitrary modules. If a module had finite projective dimension $d > 0$, its minimal resolution would end in an inclusion $0 \neq K \hookrightarrow P$ with K projective and $K \subseteq PJ_{\Lambda}$. Nilpotence of J_{Λ} implies $K/KJ_{\Lambda} \neq 0$, so choose a simple quotient $q : K \rightarrow S$. Choose an injective module mapping onto S . Projectivity of K lifts q to this injective module, and injectivity extends that lift to P . The resulting map $P \rightarrow S$ restricts to q on K but annihilates PJ_{Λ} , a contradiction. Thus $\text{fin. dim}(\Lambda^{\text{op}}) = \text{Fin. dim}(\Lambda^{\text{op}}) = 0$.

To check injective generation on the right, fix i and let pr_{L_i} and $\text{pr}_{\hat{S}_i}$ be the projections of $\hat{S}_i \oplus L_i$. Set $\delta_i = u_i \text{pr}_{L_i} v_i : I_i \rightarrow I_i$. The equality $v_i u_i = 0$ implies

$$\delta_i^2 = 0, \quad \text{im } \delta_i = u_i L_i, \quad \ker \delta_i = v_i^{-1}(\hat{S}_i \oplus 0).$$

Let K_i have term I_i in degrees $n \leq 0$, zero in degrees $n > 0$, and differential δ_i in every negative degree. Define a zero-differential complex H_i by

$$H_i^n = \begin{cases} \hat{S}_i, & n < 0, \\ \hat{S}_i \oplus L_i, & n = 0, \\ 0, & n > 0. \end{cases}$$

The maps $K_i^n \rightarrow H_i^n$ given by $\text{pr}_{\hat{S}_i} v_i$ for $n < 0$ and by v_i for $n = 0$ form a chain map, since $v_i \delta_i = 0$. In negative degrees its induced map is the isomorphism $\ker \delta_i / u_i L_i \rightarrow \hat{S}_i$; in degree zero it is $I_i / u_i L_i \rightarrow \hat{S}_i \oplus L_i$. Hence $K_i \simeq H_i$.

For $a \geq 0$, let $K_{i,a}$ be the subcomplex of K_i supported in degrees $-a, \dots, 0$, with the inherited differentials. Each is a bounded complex of injectives. The inclusions $K_{i,a} \rightarrow K_{i,a+1}$ have colimit K_i . Let shift send summand a of $\bigoplus_{a \geq 0} K_{i,a}$ into summand $a+1$ by that inclusion. There is a degreewise exact sequence of complexes

$$0 \longrightarrow \bigoplus_{a \geq 0} K_{i,a} \xrightarrow{1\text{-shift}} \bigoplus_{a \geq 0} K_{i,a} \longrightarrow K_i \longrightarrow 0,$$

The associated triangle puts K_i in the triangulated subcategory generated by injectives and coproducts. Such a subcategory is closed under direct summands: the same telescope construction applied to a repeated idempotent on an object gives its image. Since H_i has zero differential, both $\hat{S}_i$ and L_i are direct summands of H_i . Thus all simple right modules belong to this subcategory. A composition series then puts the regular module Λ in it; the regular module generates the unbounded derived category. Injective right Λ -modules therefore generate. Finally, Rickard’s theorem and $\text{Fin. dim } \Lambda = \infty$ imply that injective left Λ -modules do not generate. $\square$

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Qiyue Tang
 Zhili College, Tsinghua University
 Beijing 100084, China
Email: tangqy24@mails.tsinghua.edu.cn