

# A note of refinements to an algebra of infinite little finitistic dimension

Qiyue Tang

Generated by GPT-6-Astra

October 7, 2026

This paper is not intended for journal submission.

## Abstract

Building on OpenAI's *An algebra of infinite little finitistic dimension*, we simplify the exposition and extend the construction to every field.

## 1 Introduction

For a finite-dimensional algebra  $A$  over a field  $k$ , write  $\text{pd}_A N$  for the projective dimension of a left  $A$ -module  $N$ , and set

$$\text{fin. dim } A = \sup\{\text{pd}_A N : N \text{ is a finite-dimensional left } A\text{-module, } \text{pd}_A N < \infty\}.$$

The finitistic-dimension problem asks whether this supremum is finite for every finite-dimensional algebra; see [2, 11]. OpenAI's construction [7] gives a complex algebra for which the supremum is infinite. We retain its group, localization, tensor realization, and square-zero extension constructions, and refine their field of definition, test modules, and dimension estimates.

For an algebra  $A$ , let  $J_A = \text{rad } A$  denote its Jacobson radical. The Loewy length of a nonzero module  $N$  is the least  $h \geq 1$  with  $J_A^h N = 0$ . An algebra is split basic if  $A/J_A$  is a finite product of copies of  $k$ . We call a module a brick when its endomorphism ring is a division ring; the complex bricks below have endomorphism ring  $\mathbb{C}$ .

**Theorem 1.1.** *For every field  $k$  there are a finite-dimensional split basic  $k$ -algebra  $A$  with  $J_A^7 = 0$ , integers  $b_* \geq 0$  and  $l \geq 0$ , and finite-dimensional left  $A$ -modules  $N_m$ ,  $m \geq 1$ , such that*

$$(b_* + 5)(m - 1) + 1 \leq \text{pd}_A N_m \leq (b_* + 5)(m - 1) + l + 3.$$

*The modules have Loewy length three, their endomorphism rings are division rings, and  $\text{Hom}_A(N_m, N_n) = 0$  for  $m \neq n$ . In particular,  $\text{fin. dim } A = \infty$ . For  $k = \mathbb{C}$  they can be chosen with*

$$\dim_{\mathbb{C}} N_m = 3 \cdot 2^{3m}, \quad \text{End}_A(N_m) = \mathbb{C}.$$

*Every primitive vertex idempotent  $\varepsilon$  in this construction satisfies  $\varepsilon A \varepsilon = k \varepsilon$ .*

---

2020 Mathematics Subject Classification. Primary 16E10; Secondary 16G20. Keywords. Finitistic dimension, trivial extension, derived tensor product, brick.

The proof follows the original construction in [7]. A finitely presented group supplies a group algebra  $R$ , a central idempotent  $e$ , and an automorphism  $\alpha$ . The operation  $H(Y) = {}_\alpha(eY)$  first takes the summand  $eY$  and then lets  $r \in R$  act by  $\alpha(r)$ . We construct simple modules that vanish under precisely the  $m$ th iteration of  $H$ . A three-vertex algebra  $B$  encodes  $R$ , and a bounded bimodule complex  $P$  realizes  $H$  together with a shift by three. A chain algebra records the differential of  $P$  in an ordinary bimodule  $X$  over an algebra  $D$  of finite global dimension. The square-zero extension  $A = D \ltimes X$  converts finite tensor extinction into finite projective dimension.

Sections 3–4 retain the localization and realization proofs of [7], with condensed exposition. Section 2 extends the selection construction to arbitrary fields. Sections 5–6 sharpen the dimension bounds, and Section 7 proves the module and radical assertions of Theorem 1.1.

All algebras are associative and unital, and algebra homomorphisms preserve  $k$  and the unit. Modules are left modules unless designated right. The notation  $D^b(A\text{-mod})$  means the bounded derived category of finite-dimensional left  $A$ -modules;  $K^b(\text{proj-}A)$  means the homotopy category of bounded finitely generated projective right  $A$ -module complexes. The symbol  $\otimes^{\mathbf{L}}$  denotes derived tensor product. Complexes use cohomological grading and

$$K[s]^n = K^{n+s}, \quad d_{K[s]}^n = (-1)^s d_K^{n+s}.$$

Thus a module shifted by  $s$  has cohomology in degree  $-s$ . Path products and endomorphism products are composed from right to left.

## 2 A selection process with unbounded extinction

Let  $R$  be a unital  $k$ -algebra, let  $e \in R$  be a central idempotent, and let  $\alpha : R \rightarrow R$  be a unital  $k$ -algebra endomorphism. For a left  $R$ -module  $Y$ , define

$$H(Y) = {}_\alpha(eY). \quad (2.1)$$

Here  $eY$  is an  $R$ -submodule, and the subscript denotes restriction of scalars along  $\alpha$ : on the vector space  $eY$ , an element  $r \in R$  acts by the original operator  $\alpha(r)$ . This construction defines an additive functor on left  $R$ -modules and preserves finite dimensionality. We write  $H^{\circ j}$  for its  $j$ -fold iterate, with  $H^{\circ 0}$  the identity functor; superscripts without a circle on  $H$  will denote cohomology of a complex.

**Proposition 2.1.** *For every field  $k$ , there exist a finitely presented unital  $k$ -algebra  $R$ , a central idempotent  $e \in R$ , and an automorphism  $\alpha$  of  $R$  such that, for every integer  $m \geq 1$ , a finite-dimensional simple left  $R$ -module  $Y_m$  satisfies*

$$H^{\circ j}(Y_m) \neq 0 \quad (0 \leq j < m), \quad H^{\circ m}(Y_m) = 0. \quad (2.2)$$

*The algebra, idempotent, and automorphism are independent of  $m$ .*

Fix a prime  $p$  invertible in  $k$ : take  $p = 2$  if  $\text{char } k \neq 2$ , and  $p = 3$  otherwise. We use  $R = kG$ , where  $G$  has central elements  $z_N$  of order  $p$ , indexed by  $N \in \mathbb{Z}$ .

The auxiliary group used below is motivated by finitely presented matrix groups of Abels type. In particular, the commutator transfer and centrality argument below parallels the proof of Proposition 4.9, equations (4.9)–(4.10), in [9]. Our presentation and the finite quotients used for selection are specified here, and all required properties are proved directly.

## 2.1 The group and its finite presentation

Write  $[x, y] = xyx^{-1}y^{-1}$  and let  $I = \{1, 2, 3\}$ . Define a group  $G$  with commuting generators  $T_\ell$ ,  $\ell \in I$ , and generators

$$U_i(r), \quad V_i(r), \quad W_{ij}(r) \quad (i, j \in I, i \neq j, r \in \mathbb{Z}).$$

Each generator type  $S$  has a weight  $\lambda_S \in \mathbb{Z}^3$ , given by

$$\lambda_{U_i} = -\varepsilon_i, \quad \lambda_{V_i} = \varepsilon_i, \quad \lambda_{W_{ij}} = \varepsilon_i - \varepsilon_j, \quad (2.3)$$

where  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  is the standard basis. Impose the conjugation rules

$$T_\ell S(r) T_\ell^{-1} = S(r + \lambda_{S, \ell}). \quad (2.4)$$

The remaining relations, with all arguments ranging independently over  $\mathbb{Z}$ , are

- (i)  $U_i(r)^p = 1$ ;
  - (ii) generators of types  $U_i, U_j$ , of types  $V_i, V_j$ , and of types  $U_i, V_j$  commute whenever  $i \neq j$ ;
  - (iii)  $W_{ij}$  commutes with  $U_h$  for  $h \neq i$  and with  $V_h$  for  $h \neq j$ ;
  - (iv) for  $i \neq j$ ,
- $$[U_i(r), W_{ij}(s)] = U_j(r + s), \quad [W_{ij}(s), V_j(r)] = V_i(s + r). \quad (2.5)$$

**Lemma 2.2.** *The group  $G$  is finitely presented.*

*Proof.* For  $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ , put  $T^n = T_1^{n_1} T_2^{n_2} T_3^{n_3}$  and regard a weight as the map  $\lambda_S(n) = \sum_\ell \lambda_{S, \ell} n_\ell$ . Retain only the fifteen generators

$$T_\ell, \quad u_i = U_i(0), \quad v_i = V_i(0), \quad w_{ij} = W_{ij}(0) \quad (i \neq j).$$

Besides commutation of the  $T_\ell$ , require the following finite set of centralizer relations. Each of  $u_i, v_i$  commutes with  $T_h$  for  $h \neq i$ , and  $w_{ij}$  commutes with  $T_i T_j$  and with  $T_h$  for the unique  $h \notin \{i, j\}$ . These are precisely centralization by a basis of each kernel lattice:

$$\begin{aligned} \ker \lambda_{U_i} &= \ker \lambda_{V_i} = \bigoplus_{h \neq i} \mathbb{Z} \varepsilon_h, \\ \ker \lambda_{W_{ij}} &= \mathbb{Z}(\varepsilon_i + \varepsilon_j) \oplus \mathbb{Z} \varepsilon_h \quad (h \notin \{i, j\}). \end{aligned}$$

Since the  $T_\ell$  commute, the kernel lattice also centralizes every  $T$ -conjugate of its argument-zero generator. Thus one may define

$$U_i(r) = T_i^{-r} u_i T_i^r, \quad V_i(r) = T_i^r v_i T_i^{-r}, \quad W_{ij}(r) = T_i^r w_{ij} T_i^{-r}. \quad (2.6)$$

Equivalently,  $S(r) = T^n S(0) T^{-n}$  for any  $n$  with  $\lambda_S(n) = r$ . The kernel relations make this independent of the choice of  $n$  and imply all of (2.4).

It remains to impose the argument-zero instance of each of the remaining relation patterns. For every pair of input types actually occurring in those patterns, the two weights give a surjection  $\mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ . The following table supplies integral preimages of an arbitrary pair  $(r, s)$ ; unmentioned coordinates are zero.

| Input types   | Index condition     | Coordinates of $n$           |
|---------------|---------------------|------------------------------|
| $U_i, U_j$    | $i \neq j$          | $n_i = -r, n_j = -s$         |
| $V_i, V_j$    | $i \neq j$          | $n_i = r, n_j = s$           |
| $U_i, V_j$    | $i \neq j$          | $n_i = -r, n_j = s$          |
| $W_{ij}, U_j$ | $i \neq j$          | $n_j = -s, n_i = r - s$      |
| $W_{ij}, U_h$ | $h \notin \{i, j\}$ | $n_i = r, n_j = 0, n_h = -s$ |
| $W_{ij}, V_i$ | $i \neq j$          | $n_i = s, n_j = s - r$       |
| $W_{ij}, V_h$ | $h \notin \{i, j\}$ | $n_i = r, n_j = 0, n_h = s$  |
| $U_i, W_{ij}$ | $i \neq j$          | $n_i = -r, n_j = -r - s$     |
| $W_{ij}, V_j$ | $i \neq j$          | $n_j = s, n_i = r + s$       |

Conjugating a zero-argument commutation relation by the corresponding  $T^n$  therefore gives every required instance. For the two relations in (2.5), the output weight is the sum of the input weights:

$$-\varepsilon_i + (\varepsilon_i - \varepsilon_j) = -\varepsilon_j, \quad (\varepsilon_i - \varepsilon_j) + \varepsilon_j = \varepsilon_i.$$

Hence the same conjugation also gives the correct output argument  $r + s$ . The power relation propagates from argument zero because each individual weight is surjective onto  $\mathbb{Z}$ .

We have obtained all the defining relations from finitely many relators on fifteen generators. Conversely, the chosen kernel centralizers and zero-argument relators hold in the original presentation by (2.4). The two presentations are therefore equivalent.  $\square$

## 2.2 Central elements and an automorphism

The finite presentation is now fixed. We next identify central elements that the automorphism will shift; their independent images in finite quotients will let us prescribe the extinction time.

**Lemma 2.3.** *For each  $N \in \mathbb{Z}$ , the expression*

$$z_N = [U_i(a), V_i(b)], \quad a + b = N, \quad (2.7)$$

*is independent of  $i, a, b$ . Each  $z_N$  is central and satisfies  $z_N^p = 1$ . There is an automorphism  $\alpha_G$  of  $G$  fixing all  $T_\ell, V_i(r), W_{ij}(r)$  and sending  $U_i(r)$  to  $U_i(r+1)$ ; it satisfies  $\alpha_G(z_N) = z_{N+1}$ .*

*Proof.* Fix distinct  $i, j$  and integers  $r, q, s$ , and set

$$x = U_i(r), \quad y = W_{ij}(q), \quad v = V_j(s), \quad u = [x, y] = U_j(r+q), \quad w = [y, v] = V_i(q+s).$$

The defining commutations give  $[x, v] = [w, u] = [w, v] = 1$ . Using  $xy = uyx$  and  $yv = wvy$ , reorder  $xyv$  in the two ways

$$xyv = uwv yx = (xwx^{-1})vuyx.$$

With  $c = [x, w]$ , cancellation gives  $uwv = cwvu$ , and consequently

$$c = uwvu^{-1}v^{-1}w^{-1} = uvu^{-1}v^{-1} = [u, v].$$

The last equality uses only the commutation of  $w$  with  $u$  and  $v$ . We have proved

$$[U_i(r), V_i(q+s)] = [U_j(r+q), V_j(s)]. \quad (2.8)$$

Given  $a+b=c+d=N$  at distinct internal indices, choose  $r=a$ ,  $s=d$ , and  $q=c-a=b-d$  in (2.8). This compares the two expressions in (2.7). Two splittings at the same internal index can be compared through another index, proving the claimed independence.

To show that  $z_N$  commutes with  $U_h(r)$  or  $V_h(r)$ , express  $z_N$  using an index  $i \neq h$ . Both factors of that commutator commute with the tested generator. To commute with  $W_{hj}(r)$ , use the index outside  $\{h, j\}$ , so that again both factors commute with it. Finally, conjugation by  $T_\ell$  changes the arguments  $(a, b)$  in (2.7) to  $(a - \delta_{\ell i}, b + \delta_{\ell i})$ . Their sum is unchanged, so the splitting independence proves invariance under  $T_\ell$ . Thus  $z_N$  is central. Since  $U_i(a)^p = 1$ , centrality also gives

$$1 = [U_i(a)^p, V_i(b)] = [U_i(a), V_i(b)]^p = z_N^p.$$

Translation of every  $U$ -argument by 1 preserves the conjugation, power, and commutation relations. The first relation in (2.5) becomes

$$[U_i(r+1), W_{ij}(s)] = U_j(r+s+1),$$

which is another defining instance; the second relation is unchanged. This defines an endomorphism  $\alpha_G$ . Translation by  $-1$  preserves the relations for the same reason and is its two-sided inverse. Formula (2.7) then gives  $\alpha_G(z_N) = z_{N+1}$ .  $\square$

### 2.3 Finite quotients detecting independent central elements

**Lemma 2.4.** *For every  $m \geq 1$ , there is a finite quotient  $F_m$  of  $G$  in which the images of  $z_0, \dots, z_{m-1}$  generate a central subgroup isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^m$ .*

*Proof.* Let  $S_m = \mathbb{F}_p[t]/(t^m - 1)$ . This ring is finite, and  $t$  is a unit with inverse  $t^{m-1}$ , so  $t^r$  is defined for every  $r \in \mathbb{Z}$ . Use matrices indexed by  $0, 1, 2, 3, 4$ . Send  $T_\ell$  to the diagonal matrix with  $t$  in position  $\ell$  and 1 elsewhere, and set

$$U_i(r) \mapsto I_5 + t^r E_{0i}, \quad V_i(r) \mapsto I_5 + t^r E_{i4}, \quad W_{ij}(r) \mapsto I_5 + t^r E_{ij}. \quad (2.9)$$

Here  $E_{ab}$  denotes a matrix unit. Every displayed elementary matrix is invertible, with  $(I_5 + aE_{ij})^{-1} = I_5 - aE_{ij}$  for  $i \neq j$ . Also  $(I_5 + aE_{ij})^p = I_5$  in characteristic  $p$ .

We check the relations. Diagonal conjugation multiplies  $E_{ab}$  by the ratio of the diagonal entries at  $a$  and  $b$ , which gives exactly (2.3). The identity  $E_{ab}E_{cd} = \delta_{bc}E_{ad}$  shows that all products in both orders vanish for each required commuting pair: two units  $E_{0i}$ , two units  $E_{i4}$ , the pair  $E_{0i}, E_{j4}$  for  $i \neq j$ , the pair  $E_{ij}, E_{0h}$  for  $h \neq i$ , and the pair  $E_{ij}, E_{h4}$  for  $h \neq j$ . The required power relations hold by the same square-zero calculation. For  $i \neq j$ , the nonzero product in the first noncommutation relation is  $E_{0i}E_{ij} = E_{0j}$ , and in the second it is  $E_{ij}E_{j4} = E_{i4}$ . The higher products vanish, giving

$$\begin{aligned} [I_5 + t^r E_{0i}, I_5 + t^s E_{ij}] &= I_5 + t^{r+s} E_{0j}, \\ [I_5 + t^s E_{ij}, I_5 + t^r E_{j4}] &= I_5 + t^{s+r} E_{i4}. \end{aligned}$$

Thus (2.9) defines a homomorphism from  $G$  into the finite group  $\mathrm{GL}_5(S_m)$ . Let  $F_m$  be its image.

The image of  $z_N$  is  $I_5 + t^N E_{04}$ . It is central in  $F_m$ : the unit  $E_{04}$  has zero product in both orders with every off-diagonal unit in (2.9), and the diagonal matrices have entry 1 at both 0 and 4. For  $c_0, \dots, c_{m-1} \in \{0, \dots, p-1\}$ ,

$$\prod_{q=0}^{m-1} (I_5 + t^q E_{04})^{c_q} = I_5 + \left( \sum_{q=0}^{m-1} c_q t^q \right) E_{04}.$$

The classes  $1, t, \dots, t^{m-1}$  are an  $\mathbb{F}_p$ -basis of  $S_m$ , because the defining polynomial is monic of degree  $m$ . Thus this product is the identity only when all  $c_q$  are zero. This argument also applies when  $p$  divides  $m$  and  $S_m$  is nonreduced. The claimed independence and orders follow.  $\square$

## 2.4 The algebra and its test modules

*Proof of Proposition 2.1.* Take  $R = kG$ . A finite group presentation gives a finite algebra presentation by adjoining inverses to its generators and imposing the inverse equations and group relators. Extend  $\alpha_G$  linearly to  $\alpha : R \rightarrow R$ . For  $q \in \mathbb{Z}$  define the central idempotents

$$e_q = p^{-1} \sum_{a=0}^{p-1} z_q^a, \quad e = e_0.$$

Then  $\alpha(e_q) = e_{q+1}$ . Use the same symbols for their images in  $k[F_m]$ , and put

$$p_m = \left( \prod_{q=0}^{m-2} e_q \right) (1 - e_{m-1}),$$

with empty product 1 when  $m = 1$ . The coefficient of the group basis element  $z_{m-1}$  in  $p_m$  is  $-p^{-m} \neq 0$ , by independence of the  $m$  central generators. Thus  $k[F_m]p_m$  is a nonzero finite-dimensional module. Choose a simple quotient  $Y_m$  of it and regard it as an  $R$ -module. The operators  $e_0, \dots, e_{m-2}$  act as identity and  $e_{m-1}$  acts as zero on this quotient.

For every  $R$ -module  $Y$  and  $j \geq 0$ , restriction of scalars gives

$$H^{\circ j}(Y) \cong_{\alpha^j} (e_0 e_1 \cdots e_{j-1} Y). \quad (2.10)$$

Indeed, at stage  $j$ , selection by  $e$  is the original action of  $\alpha^j(e) = e_j$ , and the next pullback makes  $r$  act as  $\alpha^{j+1}(r)$ . This proves the formula by induction, and its application to  $Y_m$  proves (2.2).  $\square$

## 3 Localization and a fixed lifted diagram

Let  $k$  be a field, let  $R$  be a finitely presented unital  $k$ -algebra, let  $e \in R$  be a central idempotent, and let  $\alpha : R \rightarrow R$  be a unital  $k$ -algebra endomorphism. As in the preceding section,

$$H(Y) = \alpha(eY)$$

for a left  $R$ -module  $Y$ : the underlying vector space is  $eY$ , and  $r \in R$  acts there by  $\alpha(r)$ . Centrality of  $e$  ensures that this action preserves  $eY$ . The construction in this section uses these fixed data and applies to every finite-dimensional  $Y$ .

We first encode the presentation of  $R$  in a finite-dimensional directed algebra. This is a three-vertex version of the homogenized quiver construction in [6, Section 1, Theorem 1.1 and Proposition 1.2]. We then work directly with a Verdier quotient and its evaluation functors.

### 3.1 A quadratic presentation and its directed encoding

Choose a presentation

$$R = k\langle x_1, \dots, x_d \rangle / (p_1, \dots, p_t)$$

in which each  $p_j$  has degree at most two, allowing constant and linear terms. Such a presentation exists: in any finite presentation, introduce finitely many auxiliary generators for the successive partial products in its finitely many monomials, together with their binary-product definitions. The resulting relations have degree at most two, and eliminating the auxiliary generators recovers the original presentation.

Consider the quiver with vertices  $0, 1, 2$  and arrow sets

$$\mathcal{A}_{01} = \{s, a_1, \dots, a_d\}, \quad \mathcal{A}_{12} = \{s', a'_1, \dots, a'_d\}, \quad \mathcal{A} = \mathcal{A}_{01} \sqcup \mathcal{A}_{12},$$

where the subscripts specify the source and target. Products of arrows are written in the order of composition, from right to left. Impose the relations

$$a'_h s = s' a_h \quad (1 \leq h \leq d), \quad \tilde{p}_j = 0 \quad (1 \leq j \leq t), \quad (3.1)$$

where homogenization replaces

$$x_h x_l \mapsto a'_h a_l, \quad x_h \mapsto s' a_h, \quad 1 \mapsto s' s.$$

Let  $B$  be the resulting path algebra with relations, and let  $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$  be its vertex idempotents. Write  $\mathcal{R}$  for a fixed vector-space basis of the span of all the relations in (3.1), viewed in the free space of paths of length two. Thus each  $\rho \in \mathcal{R}$  has a unique expression

$$\rho = \sum_{\substack{a \in \mathcal{A}_{01} \\ b \in \mathcal{A}_{12}}} \rho_{ba} ba.$$

The basis  $\mathcal{R}$  generates the same relation ideal. There are no paths of length three, so  $B$  is finite-dimensional.

The following elementary bound will also be used for other directed algebras later.

**Lemma 3.1.** *Let  $\Lambda$  be a finite-dimensional  $k$ -algebra with complete orthogonal idempotents  $\varepsilon_0, \dots, \varepsilon_{n-1}$  such that*

$$1 = \sum_{i=0}^{n-1} \varepsilon_i, \quad \varepsilon_i \Lambda \varepsilon_i = k \varepsilon_i, \quad \varepsilon_j \Lambda \varepsilon_i = 0 \quad (j < i).$$

*Then  $\Lambda$  has global dimension at most  $n - 1$  on both sides. In particular, the algebra  $B$  above has global dimension at most two on both sides.*

*Proof.* Say that a left module  $M$  is supported at indices at least  $p$  if  $\varepsilon_j M = 0$  for  $j < p$ . There is a projective surjection

$$\bigoplus_{i=p}^{n-1} \Lambda \varepsilon_i \otimes_k \varepsilon_i M \longrightarrow M, \quad c \otimes v \longmapsto cv.$$

The source is projective even when  $M$  is not finitely generated, since each summand is a direct sum of copies of  $\Lambda \varepsilon_i$ . Its components below  $p$  vanish, and its component at  $p$  maps isomorphically onto  $\varepsilon_p M$ : only the summand  $i = p$  contributes there. Its kernel is therefore supported at indices at least  $p + 1$ .

Starting with  $p = 0$  and repeating this construction, the  $(n - 1)$ st kernel is supported only at the last vertex. Every such module is projective, since  $\Lambda \varepsilon_{n-1} = k \varepsilon_{n-1}$ . This gives the left dimension bound, including the case  $n = 1$ . The same proof for  $\Lambda^{\text{op}}$ , with the reverse ordering of the idempotents, gives the right bound. The path directions and the homogeneous quadratic relations of  $B$  give precisely the displayed hypotheses with  $n = 3$ .  $\square$

For a finite-dimensional left  $R$ -module  $Y$ , define  $M(Y)$  to be the left  $B$ -module with  $Y$  at every vertex, with  $s, s'$  acting as the identity, and with  $a_h, a'_h$  acting as  $x_h$ . The relations of  $B$  hold because they evaluate to the defining relations of  $R$ .

### 3.2 The quotient action and evaluation

Let

$$K = \mathbf{K}^b(\text{proj-}B), \quad E_i = \mathbf{e}_i B,$$

so  $K$  is the homotopy category of bounded complexes of finitely generated projective *right*  $B$ -modules. With our path convention, an arrow  $a : i \rightarrow j$  belongs to  $\mathbf{e}_j B \mathbf{e}_i$ . Left multiplication by  $a$  is consequently a right-module map  $E_i \rightarrow E_j$ . Composing these maps agrees with multiplying paths in the stated order.

Let  $\mathcal{S}$  be the thick subcategory of  $K$  generated by the cones of  $s : E_0 \rightarrow E_1$  and  $s' : E_1 \rightarrow E_2$ , and form

$$q : K \longrightarrow Q = K/\mathcal{S}.$$

We use the same notation for objects and arrows after applying  $q$ . The maps  $s$  and  $s'$  are isomorphisms in  $Q$ . Put  $L = E_0$  and

$$\bar{x}_h = s^{-1}a_h \in \text{End}_Q(L).$$

The relation  $a'_h s = s' a_h$  gives  $a'_h = s' a_h s^{-1}$ , whence

$$(s's)^{-1}a'_h a_l = s^{-1}a_h s^{-1}a_l = \bar{x}_h \bar{x}_l.$$

Likewise  $(s's)^{-1}s' a_h = \bar{x}_h$  and  $(s's)^{-1}s's = \text{id}_L$ . The homogenized relations therefore give a unital homomorphism

$$\theta : R \longrightarrow \text{End}_Q(L), \quad x_h \longmapsto \bar{x}_h. \quad (3.2)$$

For each finite-dimensional  $Y$ , tensoring defines an exact functor

$$\text{ev}_Y : K \longrightarrow \mathbf{D}^b(k), \quad T \longmapsto T \otimes_B M(Y).$$

Here  $\mathbf{D}^b(k)$  denotes the bounded derived category of finite-dimensional  $k$ -vector spaces; termwise projectivity of  $T$  computes the derived tensor product. Under the natural identifications

$$E_i \otimes_B M(Y) \cong \mathbf{e}_i M(Y) = Y,$$

an arrow map evaluates to its action on  $M(Y)$ . Thus  $\text{ev}_Y$  takes  $s, s'$  to identities and their cones to zero. The kernel of an exact functor is thick, so it contains  $\mathcal{S}$ . Equivalently, every map whose cone belongs to  $\mathcal{S}$  evaluates to an isomorphism. The functor consequently factors through  $Q$ . We retain the notation  $\text{ev}_Y$  for this factorization. On  $L$  the homomorphism (3.2) evaluates to the given  $R$ -action on  $Y$ .

### 3.3 An odd double of a formal summand

The idempotent  $\theta(e)$  evaluates to the projection of  $Y$  onto  $eY$ , but it need not split inside  $Q$ . We therefore seek an object of  $Q$  whose evaluation is  $eY$  plus a shifted copy. The following lemma gives such an object for a copy shifted by three.

For an additive category  $\mathcal{T}$ , its idempotent completion  $\text{Kar}(\mathcal{T})$  has objects  $(T, p)$ , where  $p^2 = p \in \text{End}(T)$ , and morphisms  $u : (T, p) \rightarrow (T', p')$  satisfying  $u = p' u p$ . The embedding  $T \mapsto (T, \text{id}_T)$  is fully faithful. If  $\mathcal{T}$  has a shift functor, it extends by  $(T, p)[1] = (T[1], p[1])$ .

We need the following consequence of the triangle axioms. It is compatible with the general idempotent-completion result of Balmer and Schlichting [1, Theorem 1.5]; the argument below uses only triangles already present in  $\mathcal{T}$ .

**Lemma 3.2.** *Let  $\mathcal{T}$  be a triangulated category, let  $L \in \mathcal{T}$ , and let  $\epsilon \in \text{End}_{\mathcal{T}}(L)$  be idempotent. Put  $U = (L, \epsilon) \in \text{Kar}(\mathcal{T})$ . Both  $U \oplus U[1]$  and  $U \oplus U[3]$  are isomorphic in  $\text{Kar}(\mathcal{T})$  to objects of  $\mathcal{T}$ .*

*Proof.* First consider a triangle from  $\mathcal{T}$ ,

$$A \xrightarrow{f} B \xrightarrow{g} T \xrightarrow{h} A[1],$$

and suppose that formal decompositions  $A = I \oplus A_0$ ,  $B = I \oplus B_0$  identify  $f$  with the identity on  $I$  and zero elsewhere. For any  $Z \in \text{Kar}(\mathcal{T})$ , the long exact sequence  $\text{Hom}(Z, -)$  still holds for this triangle:  $Z$  is a retract of an object of  $\mathcal{T}$ , and its Hom sequence is a direct summand of the exact sequence for that object.

The equalities  $gf = 0$  and  $f[1]h = 0$  give factorizations

$$g = g_0 \text{pr}_{B_0}, \quad h = \text{in}_{A_0[1]} h_0.$$

Exactness gives, for every such  $Z$ , a short exact sequence

$$0 \longrightarrow \text{Hom}(Z, B_0) \xrightarrow{(g_0)_*} \text{Hom}(Z, T) \xrightarrow{(h_0)_*} \text{Hom}(Z, A_0[1]) \longrightarrow 0. \quad (3.3)$$

Indeed, a map into  $B_0$  killed by  $g_0$  factors through  $f$  after inclusion into  $B$ , and projection back to  $B_0$  makes it zero. At the other end, every map into  $A_0[1]$  is killed by  $f[1]$  after inclusion into  $A[1]$ , so it lifts through  $h$ . Taking  $Z = A_0[1]$  supplies a section  $\sigma$  of  $h_0$ . It follows from (3.3) that

$$(g_0, \sigma) : B_0 \oplus A_0[1] \longrightarrow T$$

induces isomorphisms on  $\text{Hom}(Z, -)$  for every  $Z$ , hence is an isomorphism.

Apply this calculation to a triangle on  $1 - \epsilon : L \rightarrow L$ . Its identity summand is  $(L, 1 - \epsilon)$ , and its zero summand on both sides is  $U$ . Its cone is an object  $S \in \mathcal{T}$  with  $S \cong U \oplus U[1]$  in the completion. Under this identification, consider the map

$$S[1] \cong U[1] \oplus U[2] \xrightarrow{\begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix}} U \oplus U[1] \cong S.$$

Fullness of  $\mathcal{T} \hookrightarrow \text{Kar}(\mathcal{T})$  makes this a morphism between the old objects  $S[1]$  and  $S$ . Its cone is again in  $\mathcal{T}$ . The preceding calculation, with common identity summand  $U[1]$ , source complement  $U[2]$ , and target complement  $U$ , identifies that cone with  $U \oplus U[3]$ .  $\square$

Apply Lemma 3.2 in  $Q$  to  $\epsilon = \theta(e)$  and write

$$U = (L, \theta(e)), \quad V = U \oplus U[3].$$

Fix once and for all an object  $V_0 \in Q$  and an isomorphism  $V_0 \cong V$  in  $\text{Kar}(Q)$  supplied by Lemma 3.2. Centrality of  $e$  gives a unital action of  $R$  on  $U$  by

$$r \longmapsto \theta(e\alpha(r)e) \in \text{End}_{\text{Kar}(Q)}(U).$$

Its unit is  $\theta(e) = \text{id}_U$ , and its products agree with those in  $R$  because  $e$  commutes with every  $\alpha(r)$ . Use the same action on  $U[3]$  and transport the resulting action on  $V$  to  $V_0$ .

There is now a  $B$ -diagram in  $Q$  with  $V_0$  at all three vertices:  $s, s'$  act as identities, and  $a_h, a'_h$  act by the endomorphism just defined for  $x_h$ . Its relations hold by the unital  $R$ -action. Here a  $B$ -diagram in an additive  $k$ -linear category means objects at the three vertices and arrow maps satisfying the relations of  $B$  in that category.

The category  $\text{D}^b(k)$  splits idempotents: every bounded complex of vector spaces is isomorphic to its cohomology with zero differential, where splitting can be done degree by degree. Thus evaluation extends additively to  $\text{Kar}(Q)$ . It takes  $U$  to  $eY$ , and the preceding action of  $r$  to the restriction of  $\alpha(r)$  to  $eY$ . Consequently the evaluated  $B$ -diagram has

$$eY \oplus (eY)[3] \quad (3.4)$$

at every vertex, identities on  $s, s'$ , and the action of  $\alpha(x_h)$  on both summands for  $a_h, a'_h$ .

### 3.4 Clearing denominators for the finite diagram

The quotient diagram is now fixed and has the required evaluation on every  $Y$ . We next represent it by one diagram in the homotopy category, so that the same chain data will work under all those evaluations.

We give the lifting argument together with the fraction rules it uses. Let

$$\Sigma = \{u \in \text{Mor}(K) : \text{Cone}(u) \in \mathcal{S}\}.$$

The Verdier quotient is the localization of  $K$  at  $\Sigma$ . The following triangle arguments give its right-fraction description; see also [10, Section 13.6]. Identities belong to  $\Sigma$ , and composition preserves  $\Sigma$  by the octahedral axiom. If  $u : Y' \rightarrow Y$  belongs to  $\Sigma$  and  $f : X \rightarrow Y$  is any map, complete  $u$  to a triangle

$$Y' \xrightarrow{u} Y \xrightarrow{c} T \longrightarrow Y'[1], \quad T \in \mathcal{S}.$$

Complete  $cf$  to a triangle  $X' \xrightarrow{v} X \xrightarrow{cf} T \rightarrow X'[1]$ . Then  $v \in \Sigma$ , and exactness of  $\text{Hom}(X', -)$  gives  $g : X' \rightarrow Y'$  with  $fv = ug$ . This is the required Ore square.

For the cancellation rule, suppose that  $u : Y \rightarrow Y'$  belongs to  $\Sigma$  and  $ug = 0$  for  $g : X \rightarrow Y$ . A triangle  $T[-1] \rightarrow Y \xrightarrow{u} Y' \rightarrow T$  shows that  $g$  factors through a map  $b : X \rightarrow T[-1]$  with  $T \in \mathcal{S}$ . Completing  $b$  to a triangle gives a denominator  $v : X' \rightarrow X$  with  $bv = 0$ , hence  $gv = 0$ . Applying this to differences gives the corresponding rule for equality of morphisms.

These rules construct the localization by roofs

$$X \xleftarrow{u} X' \xrightarrow{f} Y, \quad u \in \Sigma,$$

representing  $q(f)q(u)^{-1}$ ; equality is tested after a common source refinement. In particular, every morphism  $q(X) \rightarrow q(Y)$  has this form, and

$$q(g) = 0 \implies gv = 0 \text{ for some } v : X' \rightarrow X \text{ in } \Sigma \quad (3.5)$$

for  $g : X \rightarrow Y$ . For the last assertion compare the roofs  $(\text{id}_X, g)$  and  $(\text{id}_X, 0)$ . Finite families can be treated with one denominator by collecting their maps into a single map to a finite direct sum of targets.

**Lemma 3.3.** *Every  $B$ -diagram in  $Q$  is isomorphic to the image under  $q$  of a  $B$ -diagram in  $K$ .*

*Proof.* Write the given objects as  $W_i$  and its arrow maps as  $t_a$ . Choose  $C_i \in K$  and isomorphisms  $\psi_i : q(C_i) \rightarrow W_i$ . First collect the maps

$$(\psi_2^{-1}t_b\psi_1)_{b \in \mathcal{A}_{12}} : q(C_1) \longrightarrow \bigoplus_{b \in \mathcal{A}_{12}} q(C_2).$$

Represent this tuple by one roof, with denominator  $u_1 : D_1 \rightarrow C_1$  and numerator components  $f_b : D_1 \rightarrow C_2$ . Set  $D_2 = C_2$  and

$$\phi_1 = \psi_1 q(u_1), \quad \phi_2 = \psi_2.$$

Then  $\phi_2 q(f_b) = t_b \phi_1$  for every  $b \in \mathcal{A}_{12}$ .

Next collect the adjusted maps

$$(\phi_1^{-1}t_a\psi_0)_{a \in \mathcal{A}_{01}} : q(C_0) \longrightarrow \bigoplus_{a \in \mathcal{A}_{01}} q(D_1).$$

A single roof gives a denominator  $u_0 : D_0 \rightarrow C_0$  and maps  $f_a : D_0 \rightarrow D_1$ . Put  $\phi_0 = \psi_0 q(u_0)$ . All the arrow squares now commute:

$$\phi_j q(f_a) = t_a \phi_i \quad (a : i \rightarrow j). \quad (3.6)$$

For  $\rho \in \mathcal{R}$ , form the morphism in  $K$

$$g_\rho = \sum_{b,a} \rho_{ba} f_b f_a : D_0 \longrightarrow D_2.$$

The relations for  $t_a$  and (3.6) give  $\phi_2 q(g_\rho) = 0$ , so  $q(g_\rho) = 0$ . Apply (3.5) to the tuple  $(g_\rho)_\rho$ . There is one denominator  $v : D'_0 \rightarrow D_0$  such that  $g_\rho v = 0$  in  $K$  for every  $\rho$ . Replace  $D_0$  by  $D'_0$ , every map  $f_a$  with source vertex 0 by  $f_a v$ , and  $\phi_0$  by  $\phi_0 q(v)$ . The arrow squares still commute, and each relation becomes

$$\sum_{b,a} \rho_{ba} f_b (f_a v) = g_\rho v = 0 \quad \text{in } K.$$

There are no arrows into vertex zero and no longer paths requiring further compatibility. Thus the new data form a  $B$ -diagram in  $K$ , and the three  $\phi_i$  give an isomorphism from its image to the entire prescribed diagram in  $Q$ .  $\square$

### 3.5 The chain data for tensor realization

Apply Lemma 3.3 to the diagram on  $V_0$  constructed above. The resulting relations hold in the homotopy category. Choosing representatives converts them into a finite list of specified null-homotopies, which are the input for the next section.

**Proposition 3.4** (Fixed lifted diagram). *For the fixed data  $R, e, \alpha$  and the algebra  $B$  above, there are bounded complexes  $D_0, D_1, D_2$  of finitely generated projective right  $B$ -modules, chain maps*

$$f_a : D_i \longrightarrow D_j \quad (a : i \rightarrow j),$$

*and right- $B$ -linear maps  $h_\rho : D_0 \rightarrow D_2$  of degree  $-1$  for  $\rho \in \mathcal{R}$ , such that*

$$d_{D_2} h_\rho + h_\rho d_{D_0} = \sum_{b,a} \rho_{ba} f_b f_a. \quad (3.7)$$

*For every finite-dimensional left  $R$ -module  $Y$ , the diagram whose objects and arrows in  $D^b(k)$  are*

$$D_i \otimes_B M(Y), \quad f_a \otimes_B \text{id}_{M(Y)} \quad (3.8)$$

*is isomorphic to the diagram having  $eY \oplus (eY)[3]$  at all three vertices, identities on  $s, s'$ , and the action of  $\alpha(x_h)$  on both summands for  $a_h, a'_h$ . All the complexes, maps, and homotopies in this statement are fixed before  $Y$  is chosen.*

*Proof.* The presentation, the quotient, the two cones, their isomorphisms in the completion, and the action on  $V_0$  have all been chosen using only  $R, e, \alpha$ . Apply the finite lifting construction once to this fixed diagram. Its objects lie in  $K$ , so choose bounded right-projective complex representatives  $D_i$  and chain-map representatives  $f_a$ . Each relation is zero in  $K$ , hence its representative is null-homotopic. Choose one right-linear homotopy  $h_\rho$  for each of the finitely many relations. This gives (3.7).

The isomorphisms  $\phi_i$  from the lifting construction are fixed morphisms in  $Q$  and satisfy all the fixed arrow squares (3.6). Every  $\text{ev}_Y$  factors through this same quotient. Applying it to those squares and then to the fixed identification  $V_0 \cong V$  gives the asserted isomorphism of diagrams, by (3.4).  $\square$

## 4 Realization by a bimodule complex

We retain the algebra  $B$ , its relation basis  $\mathcal{R}$ , and the module  $M(Y)$  from the preceding section. The fixed lifted diagram will now give a single tensor functor realizing the selection process.

**Theorem 4.1.** *Let  $k$  be a field, let  $R$  be a finitely presented  $k$ -algebra, let  $e \in R$  be a central idempotent, and let  $\alpha : R \rightarrow R$  be a unital  $k$ -algebra endomorphism. Use the finite-dimensional algebra  $B$  and the modules  $M(Y)$  constructed in the preceding section, and put  $H(Y) = \alpha(eY)$ . There is a single bounded complex  $P$  of finite-dimensional  $B$ -bimodules, termwise projective over  $B^e = B \otimes_k B^{\text{op}}$ , such that for every finite-dimensional left  $R$ -module  $Y$  one has*

$$P \otimes_B^{\mathbf{L}} M(Y) \simeq M(H(Y)) \oplus M(H(Y))[3] \quad \text{in } \mathbf{D}^b(B\text{-mod}). \quad (4.1)$$

All choices defining  $P$  are made independently of  $Y$ .

To prove the theorem, we first replace the diagram of [Proposition 3.4](#) by a complex of  $B$ -bimodules. The three-column construction below incorporates the relation homotopies; compare the general homotopy-action framework in [\[4\]](#).

**Lemma 4.2.** *Let  $D_0, D_1, D_2$  be bounded complexes of finitely generated projective right  $B$ -modules. Suppose that each arrow  $a : i \rightarrow j$  of  $B$  is assigned a chain map  $f_a : D_i \rightarrow D_j$ , and that each relation*

$$\rho = \sum_{b,a} \rho_{ba} ba \in \mathcal{R}$$

*is assigned a right  $B$ -linear map  $h_\rho : D_0 \rightarrow D_2$  of degree  $-1$  satisfying*

$$d_{D_2} h_\rho + h_\rho d_{D_0} = f_\rho, \quad f_\rho := \sum_{b,a} \rho_{ba} f_b f_a.$$

*There is a bounded complex  $P$  of finite-dimensional  $B$ -bimodules, termwise projective as a right  $B$ -module, together with homotopy equivalences of right-module complexes*

$$\iota_i : D_i \longrightarrow \mathbf{e}_i P \quad (i = 0, 1, 2),$$

*such that, for every arrow  $a : i \rightarrow j$ , the two maps  $a\iota_i$  and  $\iota_j f_a$  from  $D_i$  to  $\mathbf{e}_j P$  are chain homotopic.*

*Proof.* We construct the differential, compare its vertex complexes with the given ones, and finally compare the arrow actions. All tensor products in this construction are over  $k$ . Form the complexes of bimodules

$$\begin{aligned} L'_0 &= \bigoplus_{i=0}^2 B\mathbf{e}_i \otimes_k D_i, \\ L'_1 &= \bigoplus_{a:i \rightarrow j} B\mathbf{e}_j \otimes_k D_i, \\ L'_2 &= \bigoplus_{\rho \in \mathcal{R}} B\mathbf{e}_2 \otimes_k D_0. \end{aligned}$$

The left  $B$ -action is on the first tensor factor and the right  $B$ -action is on the second. Let  $d_0, d_1, d_2$  denote the respective internal differentials of these three complexes.

On the summand of  $L'_1$  indexed by  $a : i \rightarrow j$ , define

$$p(c \otimes v) = (ca \otimes v)_i - (c \otimes f_a(v))_j.$$

Here subscripts indicate summands of  $L'_0$ . On the summand of  $L'_2$  indexed by  $\rho$ , define

$$q(c \otimes v) = \sum_{b,a} \rho_{ba}((cb \otimes v)_a + (c \otimes f_a(v))_b).$$

These are bimodule chain maps of degree zero. Define also the bimodule map  $h : L'_2 \rightarrow L'_0$  of degree  $-1$  by

$$h(c \otimes v) = (c \otimes h_\rho(v))_2$$

on the relation summand  $\rho$ . Expanding the composite  $pq$  gives

$$\begin{aligned} pq(c \otimes v) &= \sum_{b,a} \rho_{ba}((cba \otimes v)_0 - (cb \otimes f_a(v))_1 \\ &\quad + (cb \otimes f_a(v))_1 - (c \otimes f_b f_a(v))_2) \\ &= -(c \otimes f_\rho(v))_2. \end{aligned}$$

The last equality uses  $\rho = 0$  in  $B$ .

Set

$$P = L'_0 \oplus L'_1[1] \oplus L'_2[2]$$

as a graded bimodule, and equip it with the differential

$$d_P = \begin{pmatrix} d_0 & p & h \\ 0 & -d_1 & q \\ 0 & 0 & d_2 \end{pmatrix}.$$

For example,  $P^n = (L'_0)^n \oplus (L'_1)^{n+1} \oplus (L'_2)^{n+2}$ , so  $p, q, h$  each have total degree one in this matrix. The diagonal signs agree with our shift convention. Since  $p$  and  $q$  commute with the internal differentials, the  $(0, 1)$  and  $(1, 2)$  entries of  $d_P^2$  vanish. Its remaining off-diagonal entry is

$$d_0 h + pq + h d_2 = 0,$$

by the displayed calculation and the identity for  $h_\rho$ . The diagonal entries vanish as well. Thus  $P$  is a complex of bimodules.

Every term  $B\mathbf{e}_j \otimes_k D_i^n$ , viewed as a right  $B$ -module, is a finite direct sum of copies of  $D_i^n$ . Each  $D_i^n$  is a direct summand of a finite free right  $B$ -module. Thus  $B\mathbf{e}_j \otimes_k D_i^n$  is projective over  $B^e = B \otimes_k B^{\text{op}}$  as well. All terms of  $P$  are finite dimensional and projective on both sides. There are finitely many summands, and the complexes  $D_i$  are bounded; hence  $P$  is bounded.

It remains to recover the given vertex complexes and arrow maps from this strict bimodule complex.

Define

$$\iota_i(v) = (\mathbf{e}_i \otimes v)_i \in \mathbf{e}_i L'_0 \subseteq \mathbf{e}_i P.$$

This is a chain map and is split injective as a map of graded right modules. We prove that its quotient is acyclic.

Call  $j$  the *source index* of a summand whose second tensor factor is  $D_j$ , in any of the three columns. Give  $\mathbf{e}_i P$  the decreasing filtration

$$F^r(\mathbf{e}_i P) = \text{the sum of its summands with source index at least } r \quad (0 \leq r \leq 3).$$

These are subcomplexes. Indeed, the internal differentials and the path-multiplication parts of  $p, q$  preserve the source index; the  $f_a$  terms increase it; and  $h$  increases it from 0 to 2. In particular  $F^0 = \mathbf{e}_i P$  and  $F^3 = 0$ . The quotient by  $\iota_i(D_i)$  inherits this finite filtration.

On the associated graded, only the internal differentials and the path-multiplication parts remain. The source- $i$  part consists solely of  $\iota_i(D_i)$ , since the only path from  $i$  to itself is the trivial path. It disappears in the quotient. Source indices larger than  $i$  give zero. If  $j < i$  are adjacent vertices, the source- $j$  part is the signed total complex obtained by tensoring  $D_j$  with the exact sequence

$$0 \longrightarrow k^{\{a:j \rightarrow i\}} \xrightarrow{a \mapsto a} \mathbf{e}_i B \mathbf{e}_j \longrightarrow 0.$$

There are no relations of length one, so the displayed map is an isomorphism.

For the remaining pair  $(j, i) = (0, 2)$ , that part is the signed total complex of  $D_0$  tensored with

$$0 \longrightarrow k^{\mathcal{R}} \longrightarrow k^{\{(b,a):a:0 \rightarrow 1, b:1 \rightarrow 2\}} \longrightarrow \mathbf{e}_2 B \mathbf{e}_0 \longrightarrow 0.$$

The three nonzero horizontal terms have degrees  $-2, -1, 0$ . The first map sends the basis vector indexed by  $\rho$  to  $\sum_{b,a} \rho_{ba}(b, a)$ ; the second sends  $(b, a)$  to  $ba$ . The first map is injective because  $\mathcal{R}$  is a basis of the relation space. Its image is the kernel of the second map: all defining relations have length two, and no path of positive length can enter vertex 0 or leave vertex 2. Thus the ideal they generate, in this path space, is exactly their linear span. The second map is surjective because length-two paths span  $\mathbf{e}_2 B \mathbf{e}_0$ .

These exact complexes of vector spaces are split, and tensoring with a bounded complex  $D_j$  gives an acyclic total complex. All associated graded pieces of the quotient are therefore acyclic. Induction through the finite filtration proves that the quotient itself is acyclic. Thus  $\iota_i$  is a quasi-isomorphism. Its cone is a bounded acyclic complex of projective right modules, and is contractible: starting in the largest nonzero degree, split the surjection onto its projective last term and continue downward. It follows that  $\iota_i$  is a homotopy equivalence.

Finally, for  $a : i \rightarrow j$  define a right-linear map of degree  $-1$

$$H_a : D_i \longrightarrow \mathbf{e}_j P, \quad H_a(v) = (\mathbf{e}_j \otimes v)_a \in \mathbf{e}_j L'_1[1].$$

The internal differential on this summand is  $-d_{D_i}$ ; its contribution cancels that of  $H_a d_{D_i}$ . The remaining component is  $p$ , giving

$$d_P H_a + H_a d_{D_i} = a \iota_i - \iota_j f_a.$$

This is the required arrow homotopy. □

*Proof of Theorem 4.1.* Choose the fixed complexes  $D_i$ , arrow maps  $f_a$ , and relation homotopies  $h_\rho$  of Proposition 3.4, and apply Lemma 4.2. Since  $P$  and the  $D_i$  are bounded complexes of projective right modules, their ordinary tensor products with  $M(Y)$  compute the derived tensor products. Moreover, tensoring preserves the homotopy equivalences  $\iota_i$  and their arrow homotopies. Thus the vertex diagram of

$$T_Y := P \otimes_B^{\mathbf{L}} M(Y)$$

in the derived category of  $k$ -vector spaces is isomorphic to the evaluated diagram of Proposition 3.4.

At each vertex this diagram is  $eY \oplus eY[3]$ . The arrows  $s, s'$  act by identity maps, and  $a_h, a'_h$  act by  $\alpha(x_h)$  on each summand. Applying cohomology, with its induced arrow maps, therefore gives isomorphisms of left  $B$ -modules

$$H^n(T_Y) \cong \begin{cases} M(H(Y)), & n = 0, -3, \\ 0, & n \neq 0, -3. \end{cases}$$

Write  $A_{-3} = H^{-3}(T_Y)$  and  $A_0 = H^0(T_Y)$ . The canonical truncation triangle is

$$A_{-3}[3] \longrightarrow T_Y \longrightarrow A_0 \xrightarrow{\delta} A_{-3}[4].$$

The connecting morphism belongs to

$$\mathrm{Hom}_{\mathcal{D}^b(B\text{-mod})}(A_0, A_{-3}[4]) = \mathrm{Ext}_B^4(A_0, A_{-3}) = 0,$$

where the vanishing follows from the bound  $\mathrm{gl.dim} B \leq 2$  in [Lemma 3.1](#). The triangle consequently splits, giving  $T_Y \simeq A_0 \oplus A_{-3}[3]$ , which is [\(4.1\)](#).  $\square$

**Corollary 4.3.** *Put  $\Phi = P \otimes_B^{\mathbf{L}} -$ . For every integer  $j \geq 0$  and every finite-dimensional left  $R$ -module  $Y$ , there is an isomorphism*

$$\Phi^j(M(Y)) \simeq \bigoplus_{q=0}^j (M(H^{\circ j}(Y))[3q])^{\oplus \binom{j}{q}}.$$

*Proof.* The case  $j = 0$  is immediate. For the induction step, apply  $\Phi$  to an isomorphism for  $j$ ; derived tensor preserves isomorphisms, finite direct sums, and shifts. Apply [\(4.1\)](#) with  $H^{\circ j}(Y)$  in place of  $Y$  to each summand, and combine the two resulting multiplicities using Pascal's identity. Each  $H^{\circ j}(Y)$  is finite dimensional.  $\square$

## 5 Simulation by an ordinary bimodule

We now construct a larger finite-dimensional algebra and an ordinary bimodule whose second derived tensor iterate recovers  $P \otimes_B^{\mathbf{L}} -$ , up to a fixed shift. All choices in this section depend only on the fixed complex  $P$  of [Theorem 4.1](#).

Choose integers  $a_* \leq 0 \leq b_*$  such that  $P^n = 0$  outside  $[a_*, b_*]$ , enlarging the interval if necessary, and put  $l = b_* - a_*$ . Let  $C$  be the algebra of the quiver

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} \dots \xrightarrow{c_l} l$$

modulo all paths of length two. Write  $\varepsilon_i$  for its vertex idempotents, and let  $W$  be the simple *right*  $C$ -module at vertex  $l$ . There is a projective resolution of right  $C$ -modules

$$0 \longrightarrow \varepsilon_0 C \xrightarrow{c_1} \varepsilon_1 C \xrightarrow{c_2} \dots \xrightarrow{c_l} \varepsilon_l C \longrightarrow W \longrightarrow 0. \quad (5.1)$$

Indeed, for  $i > 0$ , the right projective  $\varepsilon_i C$  has basis  $\varepsilon_i, c_i$ , and its radical is the right simple at  $i - 1$ . Left multiplication by  $c_i$  maps  $\varepsilon_{i-1}$  to  $c_i$  and, when  $i > 1$ , maps  $c_{i-1}$  to zero. This proves exactness, including injectivity at the first term. For  $l = 0$ , the resolution consists of the isomorphism  $\varepsilon_0 C \rightarrow W$ .

Define a  $(B \otimes_k C, B)$ -bimodule  $O$  by

$$\varepsilon_i O = P^{a_*+i}, \quad c_i|_{\varepsilon_{i-1} O} = d_P^{a_*+i-1}.$$

Here the two  $B$ -actions are those on the terms of  $P$ . The relation  $d_P^2 = 0$  makes this a representation of  $C$ , and the bimodule linearity of  $d_P$  makes the  $C$ -action commute with both  $B$ -actions. Thus  $O$  is an ordinary finite-dimensional bimodule. Set

$$D = B \times (B \otimes_k C), \quad T = B \otimes_k W, \quad X = O \oplus T.$$

Regard  $O$  as having its left action from the second factor of  $D$  and its right action from the first. Regard  $T$  as having its left action from the first factor and its right action from the second, with the regular  $B$ -actions on its  $B$ -factor. These prescriptions define an ordinary finite-dimensional  $D$ -bimodule  $X$ .

The two summands of  $X$  move complexes between the factors of  $D$ , as shown in [Figure 1](#). Thus two tensor steps return to the first factor; the next proposition identifies their composite.

$$\begin{array}{ccc}
& O \otimes_B^{\mathbf{L}} - & \\
\text{D}^b(B\text{-mod}) & \xrightarrow{\quad} & \text{D}^b((B \otimes_k C)\text{-mod}) \\
& \xleftarrow{\quad} & \\
& T \otimes_{B \otimes_k C}^{\mathbf{L}} - &
\end{array}$$

Figure 1: The two factors of  $D$  and the tensor functors supplied by the off-diagonal summands of  $X$ . Starting on the left, the composite is  $P[b_*] \otimes_B^{\mathbf{L}} -$  by [Proposition 5.1](#).

**Proposition 5.1.** *The algebra  $D$  has finite global dimension on both sides. Let  $F = X \otimes_D^{\mathbf{L}} -$ . For every complex  $Z$  of left  $B$ -modules, regarded as a complex on the first factor of  $D$ , there is an isomorphism*

$$F^2(Z) \simeq P[b_*] \otimes_B^{\mathbf{L}} Z, \quad T \otimes_{B \otimes_k C}^{\mathbf{L}} O \simeq P[b_*]. \quad (5.2)$$

The second isomorphism is an isomorphism of complexes of  $B$ -bimodules, and the first is supported on the first factor of  $D$ .

*Proof.* The vertices of  $B \otimes_k C$  are pairs  $(i, j)$ , where  $0 \leq i \leq 2$  and  $0 \leq j \leq l$ . Its diagonal vertex corners are  $k$ . A nonzero component between distinct vertices moves in the product of the two directed orders. Assign height  $i + j$  to vertex  $(i, j)$ . Every nonzero off-diagonal component increases height; the algebra on each height layer is a product of copies of  $k$ . The proof of [Lemma 3.1](#), applied one layer at a time, gives

$$\text{gl. dim}(B \otimes_k C), \quad \text{gl. dim}((B \otimes_k C)^{\text{op}}) \leq l + 2.$$

The same bound holds for  $D$  and  $D^{\text{op}}$ , since  $\text{gl. dim } B, \text{gl. dim } B^{\text{op}} \leq 2$ .

Tensor [\(5.1\)](#) over  $k$  with the regular  $B$ -bimodule, placing its last projective term in degree zero. This gives a bounded resolution  $Q_T \rightarrow T$  by right  $B \otimes_k C$ -projectives, preserving the left  $B$ -action. Also,  $O$  is projective as a right  $B$ -module, being the finite direct sum of the terms of  $P$ . Thus  $Q_X = O \oplus Q_T$ , with the prescribed block actions, is a bounded right- $D$ -projective resolution of  $X$  retaining its left  $D$ -action. Ordinary tensor with  $Q_X$  computes  $F$  on every complex. On the first factor, the block supports therefore give

$$F^2(Z) \simeq Q_T \otimes_{B \otimes_k C} (O \otimes_B Z) \cong (Q_T \otimes_{B \otimes_k C} O) \otimes_B Z.$$

In degree  $n$ , for  $-l \leq n \leq 0$ , tensoring  $Q_T$  with  $O$  gives

$$(B \otimes_k \varepsilon_{l+n} C) \otimes_{B \otimes_k C} O \cong \varepsilon_{l+n} O = P^{a_*+l+n} = P^{b_*+n}.$$

The induced differential is  $d_P$ . In comparison, the standard differential of  $P[b_*]$  is  $(-1)^{b_*} d_P$ . Multiplication by  $(-1)^{b_* n}$  in degree  $n$  defines a chain isomorphism to  $P[b_*]$ ; it respects both  $B$ -actions. Since this complex is termwise right- $B$ -projective, its ordinary tensor with any  $Z$  computes the derived tensor. This proves the second isomorphism in [\(5.2\)](#), and hence the first.  $\square$

## 6 A square-zero extension and projective dimension

We first establish the homological statement for an arbitrary split square-zero extension over a field. Its derived tensor powers cannot in general be replaced by ordinary tensor powers. The decomposition below will let us prove two separate facts: eventual extinction gives a finite projective resolution, while survival of the  $r$ th iterate rules out any projective resolution of length less than  $r$ .

## 6.1 The derived bar decomposition

The splitting below also follows from the graded-component decomposition of Minamoto and Yamaura [5, Lemma 4.13(4) and the proof of Theorem 4.17], after forgetting the grading and passing to opposite algebras. The direct proof identifies each derived tensor factor and its shift.

**Proposition 6.1.** *Let  $D$  be a finite-dimensional  $k$ -algebra, let  $X$  be a finite-dimensional  $D$ -bimodule, and let  $A = D \ltimes X$  be the algebra with multiplication*

$$(d, x)(d', x') = (dd', dx' + xd').$$

*Let  $N$  be a left  $D$ -module, inflated to  $A$  through the projection  $A \rightarrow D$ , and put  $F = X \otimes_D^{\mathbf{L}} -$ . In the derived category of left  $D$ -modules there is an isomorphism*

$$D \otimes_A^{\mathbf{L}} N \simeq \bigoplus_{r \geq 0} (F^r N)[r]. \quad (6.1)$$

*Here  $F^0 N = N$ , and every  $F^r$  denotes an iteration of the derived tensor functor.*

*Proof.* Use the ordinary unnormalized bar resolution of  $N$  over the field  $k$ . It is a free left  $A$ -resolution with degree  $-n$  term  $A \otimes_k A^{\otimes_k n} \otimes_k N$ . Its augmentation is a quasi-isomorphism: the augmented bar complex has the usual  $k$ -linear contraction obtained by inserting the unit. After tensoring with the right  $A$ -module  $D$ , it yields a model  $\mathcal{B}$  for  $D \otimes_A^{\mathbf{L}} N$ , with

$$\mathcal{B}^{-n} = D \otimes_k A^{\otimes_k n} \otimes_k N \quad (n \geq 0).$$

The differential is the alternating sum of the faces that multiply adjacent algebra entries or apply the two endpoint actions.

Expand each internal copy of  $A$  as  $D \oplus X$ . Every nonzero face preserves the number of entries from  $X$ . Indeed, a product involving exactly one  $X$ -entry remains in  $X$ , a product of two  $X$ -entries is zero, and an endpoint action by  $X$  is zero on both  $D$  and  $N$ . Thus there is an actual decomposition of complexes

$$\mathcal{B} = \bigoplus_{r \geq 0} \mathcal{B}_r,$$

where  $\mathcal{B}_r$  consists of words with precisely  $r$  entries from  $X$ .

For fixed  $r$ , write  $i_0, \dots, i_r$  for the lengths of the blocks of  $D$ -entries before, between, and after the  $X$ -entries. The corresponding tensor is

$$D \otimes_k D^{\otimes_k i_0} \otimes_k X \otimes_k D^{\otimes_k i_1} \otimes_k \cdots \otimes_k X \otimes_k D^{\otimes_k i_r} \otimes_k N, \quad (6.2)$$

in degree  $-r - \sum_{j=0}^r i_j$ . We identify this with the multiple two-sided bar complex for

$$D \otimes_D^{\mathbf{L}} \underbrace{X \otimes_D^{\mathbf{L}} \cdots \otimes_D^{\mathbf{L}} X}_{r \text{ factors}} \otimes_D^{\mathbf{L}} N, \quad (6.3)$$

shifted by  $r$ .

First consider the signs. Within block  $j$ , a bar face has alternating-sign offset

$$j + \sum_{h < j} i_h.$$

In the ordinary multibar total differential, ordered from left to right, the corresponding offset is  $\sum_{h < j} i_h$ ; shifting the total complex by  $r$  adds the factor  $(-1)^r$ . Rescale the component (6.2) by

$$(-1)^{\sum_{j=0}^r (r-j)i_j}.$$

When a face reduces  $i_j$  by one, this factor changes by  $(-1)^{r-j}$ , precisely the discrepancy between the two differentials. This gives the required chain isomorphism. When  $r \geq 1$  and  $i_j = 0$ , there is no nonzero face in that block: the prospective face multiplies two  $X$ -entries or applies an endpoint  $X$ -action. For  $r = 0$ , this is the ordinary two-sided bar complex for  $D \otimes_D^{\mathbf{L}} N$ . At  $i_0 = 0$  its differential is absent; the map  $D \otimes_k N \rightarrow N$  is the separate augmentation.

We next justify that the multibar computes the genuinely derived tensor in (6.3), without any flatness assumption on  $X$  over  $D$ . For a bounded-above complex  $Z$  of left  $D$ -modules, the total left-module bar resolution has terms obtained from

$$D \otimes_k D^{\otimes_k i} \otimes_k Z^q$$

in degree  $q - i$ . Its terms are free left  $D$ -modules because  $k$  is a field, and its augmentation to  $Z$  is a quasi-isomorphism. For each total degree there are only finitely many possible  $i$ , since  $Z$  is bounded above. Thus it is a bounded-above complex of free modules and can be used to compute a derived tensor.

Starting at the right, resolve  $N$  in this way and tensor with  $X$ . Resolve the resulting left  $D$ -complex in the same way and tensor with the next  $X$ , continuing for the fixed finite number  $r$  of factors. A final bar resolution with the leading factor  $D$  gives exactly the multibar (6.2), before its shift. All exterior  $D$ -actions are preserved. This establishes its identification with (6.3), which is isomorphic to  $F^r N$ .

Finally, in degree  $-n$  of  $\mathcal{B}$  only  $0 \leq r \leq n$  occur, and for each such  $r$  there are finitely many tuples satisfying  $r + \sum_j i_j = n$ . All totalizations above therefore use direct sums that are finite in each degree. There is no passage to products or completed tensor powers. Direct sums are exact for modules, so the componentwise quasi-isomorphisms give (6.1). The argument uses no connectedness assumption on  $D$ ; in particular it applies to the product algebra of Section 5.  $\square$

## 6.2 Detecting finite projective dimension and lower bounds

**Corollary 6.2.** *In the setting of Proposition 6.1, suppose that  $D$  has finite global dimension on both sides and that  $N$  is finite-dimensional.*

- (i) *If  $F^t N \simeq 0$  for some  $t \geq 1$ , then  $\mathrm{pd}_A N < \infty$ .*
- (ii) *If  $F^r N \not\simeq 0$ , then  $\mathrm{pd}_A N \geq r$ , with infinite projective dimension allowed.*

*Proof.* Every fixed iterate  $F^r N$  has bounded, finite-dimensional cohomology. For example, if  $g$  bounds the right global dimension of  $D$ , then for every finite-dimensional left  $D$ -module  $V$  the groups  $\mathrm{Tor}_i^D(X, V)$  are finite-dimensional and vanish for  $i > g$ . Applying this observation to successive truncation triangles shows inductively that

$$H^q(F^r N) = 0 \quad \text{unless} \quad -rg \leq q \leq 0.$$

In particular, all iterates lie in  $D^{\leq 0}(D)$ .

For the first assertion, vanishing at  $t$  implies vanishing at every  $r \geq t$ . By (6.1),  $Z = D \otimes_A^{\mathbf{L}} N$  then has bounded cohomology. Since  $X$  is a nilpotent ideal, it is contained in  $\mathrm{rad} A$ , and

$$\mathrm{rad} A = (\mathrm{rad} D) \oplus X, \quad A/\mathrm{rad} A \cong D/\mathrm{rad} D =: S.$$

Derived associativity gives

$$S \otimes_A^{\mathbf{L}} N \simeq S \otimes_D^{\mathbf{L}} (D \otimes_A^{\mathbf{L}} N) = S \otimes_D^{\mathbf{L}} Z.$$

The right  $D$ -module  $S$  has a finite projective resolution, so this complex also has bounded cohomology.

Choose a minimal projective resolution of  $N$  over the finite-dimensional algebra  $A$ , using successive projective covers. Its terms are finitely generated, and every differential has image in the radical of its target. Tensoring with  $S = A/\text{rad } A$  therefore annihilates every differential. If  $Q$  is a nonzero finitely generated projective  $A$ -module, then

$$S \otimes_A Q \cong Q/(\text{rad } A)Q \neq 0$$

by Nakayama's Lemma. Consequently bounded cohomology of  $S \otimes_A^{\mathbf{L}} N$  forces the minimal resolution to have only finitely many nonzero terms. This proves  $\text{pd}_A N < \infty$ .

For the second assertion, choose  $q \leq 0$  with  $H^q(F^r N) \neq 0$ . Its direct summand in (6.1) gives

$$H^{q-r}(D \otimes_A^{\mathbf{L}} N) \neq 0, \quad q - r \leq -r.$$

A projective  $A$ -resolution of  $N$  of length strictly less than  $r$  would make the derived tensor zero in this degree. Hence  $\text{pd}_A N \geq r$ , as asserted.  $\square$

For a bounded complex  $Z$  with finite-dimensional cohomology,  $\text{pd}_D Z$  is the least integer  $d$  such that  $Z$  is represented by a bounded projective complex zero below degree  $-d$ ; set  $\text{pd}_D 0 = -\infty$ . In particular  $\text{pd}_D(Z[s]) = \text{pd}_D Z + s$ .

**Proposition 6.3.** *Under the hypotheses of Corollary 6.2, if  $F^t N = 0$  for some  $t \geq 1$ , then*

$$\text{pd}_A N = \max_{0 \leq r < t} \{\text{pd}_D(F^r N) + r\}.$$

*Proof.* Put  $S = D/\text{rad } D = A/\text{rad } A$ . A minimal bounded projective complex representing  $Z$  has radical differentials, so

$$\text{pd}_D Z = \sup\{n \in \mathbb{Z} : H^{-n}(S \otimes_D^{\mathbf{L}} Z) \neq 0\}.$$

Such a complex exists by finite global dimension, followed by splitting off contractible projective summands. Reduction modulo the radical kills the remaining differentials and detects every nonzero projective term by Nakayama's lemma. By Corollary 6.2,  $N$  has finite projective dimension. Apply  $S \otimes_D^{\mathbf{L}} -$  to (6.1). Only  $r < t$  survive. The same minimal-resolution argument over  $A$  identifies its projective dimension with the largest detected degree, giving the formula.  $\square$

This is the inflated-module case of [5, Corollary 4.11], with sides reversed.

### 6.3 Tensor iterates and linear bounds

Fix  $R, e, \alpha$  over  $k$  as in Proposition 2.1, construct  $B, P$  by Theorem 4.1, and choose  $[a_*, b_*]$ ,  $C, W, O, T, D, X$  as in Section 5. Put  $A = D \rtimes X$ . For a finite-dimensional  $R$ -module  $V$ , regard  $M(V)$  as supported on the first factor of  $D$ . Put  $E = B \otimes_k C$  and  $Q(V) = O \otimes_B M(V)$ , a left  $E$ -module. Since  $O_B$  is projective, this ordinary tensor computes its derived tensor. Moreover  $Q(V) \neq 0$  for  $V \neq 0$ : the nonzero projective module  $O_B$  is a direct sum of vertex projectives, each of which evaluates to  $V$ . The module  $O$  is nonzero because otherwise  $P = 0$ , whereas (4.1) applied to  $Y_2$  has nonzero right-hand side.

For  $j \geq 0$ , induction from (5.2) and (4.1) gives

$$F^{2j} M(V) \simeq \bigoplus_{q=0}^j M(H^{\circ j} V)[jb_* + 3q]^{\oplus \binom{j}{q}}, \quad (6.4)$$

$$F^{2j+1} M(V) \simeq \bigoplus_{q=0}^j Q(H^{\circ j} V)[jb_* + 3q]^{\oplus \binom{j}{q}}. \quad (6.5)$$

Here the binomial coefficient counts direct summands. Take any  $Y_m$  satisfying (2.2), and inflate  $N_m = M(Y_m)$  along  $A \rightarrow D \rightarrow B$ . Then

$$F^r N_m \neq 0 \quad (0 \leq r < 2m), \quad F^{2m} N_m = 0.$$

For  $0 \leq j < m$ , Proposition 6.3 therefore gives

$$\begin{aligned} \text{pd}_A N_m = \max_{0 \leq j < m} \{ & (b_* + 5)j + \text{pd}_B M(H^{\circ j} Y_m), \\ & (b_* + 5)j + 1 + \text{pd}_E Q(H^{\circ j} Y_m) \}. \end{aligned} \quad (6.6)$$

For each  $j$ , the largest shift occurs at  $q = j$ . The two module dimensions on the right lie respectively in  $[0, 2]$  and  $[0, l + 2]$ . Hence

$$(b_* + 5)(m - 1) + 1 \leq \text{pd}_A N_m \leq (b_* + 5)(m - 1) + l + 3. \quad (6.7)$$

In particular the lower bound is at least  $5m - 4$ .

## 7 Radicals and test modules

**Proposition 7.1.** *The algebra  $A$  above is split basic,  $J_A^7 = 0$ , and  $A/J_A \cong k^{3l+6}$ . Its primitive vertex corners are  $k$ . For every nonzero finite-dimensional  $R$ -module  $Y$ , the inflated module  $M(Y)$  has Loewy length three. For  $R$ -modules  $Y, Z$ ,*

$$\text{Hom}_A(M(Y), M(Z)) \cong \text{Hom}_R(Y, Z).$$

*Proof.* The arrow ideals give  $J_B^3 = 0$ ,  $J_C^2 = 0$ , and split semisimple quotients. Thus

$$J_E = J_B \otimes_k C + B \otimes_k J_C, \quad J_E^4 = 0.$$

Indeed the displayed sum is nilpotent and its quotient is a product of copies of  $k$ ; any product of four terms contains three  $J_B$  factors or two  $J_C$  factors. Also  $J_D = J_B \times J_E$  and

$$J_A = J_D \oplus X, \quad J_A^n = J_D^n + \sum_{i+j=n-1} J_D^i X J_D^j \quad (n \geq 1).$$

A word containing two  $X$  factors vanishes because  $XD X = 0$ . The summand  $O$  has left  $E$ -action and right  $B$ -action, so  $J_D^i O J_D^j = 0$  if  $i \geq 4$  or  $j \geq 3$ . For  $T$  the bounds are reversed. Thus every term for  $n = 7$  vanishes. There are  $3 + 3(l + 1) = 3l + 6$  vertices. Since  $X$  joins distinct factors of  $D$ , it contributes nothing to a primitive vertex corner; each such corner of  $A$  is  $k$ .

The action on  $M(Y)$  factors through  $B$ , and the image of  $J_A$  is  $J_B$ . The path  $s'$  acts as identity from vertex 0 to vertex 2. Therefore  $J_A^2 M(Y) \neq 0$  and  $J_A^3 M(Y) = 0$ . A  $B$ -module map  $M(Y) \rightarrow M(Z)$  has three components. Commutation with  $s, s'$  makes them equal, and commutation with the arrows  $a_h$  makes their common value  $R$ -linear. Inflation preserves these morphisms, proving the Hom identity.  $\square$

### 7.1 Explicit complex representations

For  $k = \mathbb{C}$  use  $p = 2$ . For  $m \geq 1$  set  $S_m = \mathbb{F}_2[t]/(t^m - 1)$ . Each element has a unique representative  $\sum_{r=0}^{m-1} a_r t^r$ . Define  $\lambda_m : S_m \rightarrow \mathbb{F}_2$  by  $\lambda_m(\sum a_r t^r) = a_{m-1}$ . On the complex vector space

$$Y_m^{\text{exp}} = \bigoplus_{x \in S_m^3} \mathbb{C} v_x$$

the vectors  $v_x$  form a basis indexed by  $S_m^3$ . Write  $e_i$  for the  $i$ th coordinate vector in  $S_m^3$ ,  $E_{ij}$  for the  $3 \times 3$  matrix unit, and  $D_\ell$  for the operator multiplying the  $\ell$ th coordinate by  $t$ . Define, for  $i, j, \ell \in \{1, 2, 3\}$ ,  $i \neq j$ , and  $r \in \mathbb{Z}$ ,

$$\begin{aligned} T_\ell v_x &= v_{D_\ell x}, & U_i(r) v_x &= (-1)^{\lambda_m(t^r x_i)} v_x, \\ V_i(r) v_x &= v_{x+t^r e_i}, & W_{ij}(r) v_x &= v_{(I+t^r E_{ij})x}. \end{aligned}$$

The exponent of  $-1$  is read as 0 or 1.

**Proposition 7.2.** *These operators define a simple  $R$ -module of dimension  $2^{3m}$  satisfying (2.2).*

*Proof.* Conjugation by  $T_\ell$  changes the parameter of  $U_i$  by  $-\delta_{\ell i}$ , that of  $V_i$  by  $\delta_{\ell i}$ , and that of  $W_{ij}$  by  $\delta_{\ell i} - \delta_{\ell j}$ . Computing on  $v_x$  gives

$$[U_i(r), W_{ij}(s)] = U_j(r+s), \quad [W_{ij}(r), V_j(s)] = V_i(r+s),$$

and  $[U_i(r), V_i(s)]$  acts as  $(-1)^{\lambda_m(t^{r+s})}$ . The remaining commutations follow from their coordinate supports, and  $U_i(r)^2 = 1$ . Thus all defining relations hold. The elements  $z_0, \dots, z_{m-2}$  act as 1 and  $z_{m-1}$  as  $-1$ , giving the required extinction.

The pairing  $(a, b) \mapsto \lambda_m(ab)$  on  $S_m$  is nondegenerate: in the basis  $1, t, \dots, t^{m-1}$  its matrix has entry 1 exactly when the two exponents sum to  $m-1$  modulo  $m$ , a permutation matrix. For  $a \in S_m^3$ , products of the operators  $U_i(r)$  therefore give the diagonal operator  $U(a)v_y = (-1)^{\lambda_m(\sum_i a_i y_i)} v_y$ . Character cancellation shows that

$$2^{-3m} \sum_{a \in S_m^3} (-1)^{\lambda_m(\sum_i a_i x_i)} U(a)$$

is the projection onto  $\mathbb{C}v_x$ . Any nonzero invariant subspace contains a basis vector; the translations generated by the  $V_i(r)$  then give all basis vectors. This proves simplicity.  $\square$

*Proof of Theorem 1.1.* Use the simple modules from Proposition 2.1, or those from Proposition 7.2 over  $\mathbb{C}$ . Schur's lemma makes their endomorphism rings division rings, and over  $\mathbb{C}$  these rings are  $\mathbb{C}$ . Different extinction times imply that the simple modules are nonisomorphic, so their mutual Hom spaces vanish. Proposition 7.1 transports these conclusions and gives Loewy length three and all assertions about  $A$ . Each encoding has three copies of its input space, giving  $3 \cdot 2^{3m}$  in the explicit case. Finally (6.7) proves the claimed bounds and  $\text{fin. dim } A = \infty$ .  $\square$

## 7.2 Size and other odd shifts

With  $d$  generators in the quadratic presentation of  $R$  and relation basis  $\mathcal{R}$ , counting vertices, arrows, and length-two paths gives

$$\dim_k B = 3 + 2(d+1) + (d+1)^2 - |\mathcal{R}|.$$

Since  $\dim_k C = 2l+1$  and  $\dim_k W = 1$ , the construction gives

$$\dim_k A = (2l+3) \dim_k B + \sum_n \dim_k P^n.$$

The support and the terms of  $P$  depend on the chosen lifts.

**Proposition 7.3.** *For every odd integer  $s \geq 3$ , the realization theorem remains valid with [3] replaced by  $[s]$ , with a complex  $P$  chosen for that  $s$ .*

*Proof.* In the notation of [Lemma 3.2](#), suppose an object  $S_{s-2}$  represents  $U \oplus U[s-2]$ , and  $S_1$  represents  $U \oplus U[1]$ . Take the map  $S_{s-2}[1] \rightarrow S_1$  that is identity on their common  $U[1]$  and zero on the other summands. Fullness of the embedding into the idempotent completion supplies the map in the original category. Its cone represents  $U \oplus U[s]$ . Starting with  $S_1$  proves the assertion for every odd  $s$ . The same finite diagram lifting and rectification apply. The final truncation triangle splits because  $\text{Ext}_B^{s+1}(M(HY), M(HY)) = 0$  for  $s \geq 3$ .  $\square$

For the fixed complex  $P$  with shift three, set  $\Phi = P \otimes_B^{\mathbf{L}} -$ . Its kernels on  $\mathcal{D}^b(B\text{-mod})$  are thick subcategories: they are closed under shifts, cones, and direct summands because  $\Phi$  is an exact additive functor. They form a strictly increasing chain  $\ker \Phi^j \subsetneq \ker \Phi^{j+1}$  for  $j \geq 0$ , where  $\Phi^0$  is the identity. Indeed  $M(Y_{j+1})$  lies in the latter kernel and not in the former, by the iteration formula.

## 8 Left–right asymmetry

The big finitistic dimension  $\text{Fin. dim } A$  is defined as  $\text{fin. dim } A$  with all left modules allowed in the supremum. Thus  $\text{Fin. dim } A \geq \text{fin. dim } A$ . Injective modules generate when the smallest triangulated subcategory of the unbounded derived category containing them and closed under arbitrary coproducts is the whole category. Rickard’s theorem [[8](#), Theorem 4.3], applied to the opposite algebra, implies that injective left modules do not generate for the algebra  $A$  of Theorem [1.1](#).

The following consequence over  $\mathbb{C}$  is retained from [[7](#)].

**Corollary 8.1.** *There is a finite-dimensional complex algebra  $\Lambda$  such that*

$$\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda = \infty, \quad \text{fin. dim } (\Lambda^{\text{op}}) = \text{Fin. dim } (\Lambda^{\text{op}}) = 0.$$

*Injective left  $\Lambda$ -modules do not generate, whereas injective right  $\Lambda$ -modules generate.*

*Proof.* Take  $k = \mathbb{C}$  and put  $C_{\text{aux}} = A^{\text{op}}$ , which is basic by Proposition [7.1](#). Choose its primitive vertex idempotents  $e_1, \dots, e_n$ , and let  $S_i = e_i C_{\text{aux}} / e_i J_{C_{\text{aux}}}$  be the corresponding simple right modules. Set

$$E_{\text{aux}} = \prod_{i=1}^n \mathbb{C}[\epsilon_i] / (\epsilon_i^2), \quad U_{\text{aux}} = \bigoplus_{i=1}^n S_i, \quad \tilde{\Lambda} = \begin{pmatrix} C_{\text{aux}} & 0 \\ U_{\text{aux}} & E_{\text{aux}} \end{pmatrix}.$$

The left  $E_{\text{aux}}$ -action on  $U_{\text{aux}}$  factors through  $E_{\text{aux}} / J_{E_{\text{aux}}} = \mathbb{C}^n$ , with its  $i$ th factor acting on  $S_i$ . This is Cummings’ Construction 2.1. His Propositions 3.2–3.3 [[3](#)] imply that the little and big right finitistic dimensions of  $\tilde{\Lambda}$  are infinite and those of  $\tilde{\Lambda}^{\text{op}}$  are zero. His Proposition 4.4 gives injective generation for right  $\tilde{\Lambda}^{\text{op}}$ -modules. Rickard’s theorem gives failure of injective generation for right  $\tilde{\Lambda}$ -modules. Taking  $\Lambda = \tilde{\Lambda}^{\text{op}}$  proves the assertions.  $\square$

## References

- [1] Paul Balmer and Marco Schlichting. Idempotent completion of triangulated categories. *Journal of Algebra*, 236(2):819–834, 2001. [Publisher DOI](#); [full text](#).
- [2] Hyman Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Transactions of the American Mathematical Society*, 95(3):466–488, 1960. [Publisher DOI](#).
- [3] Charley Cummings. Left–right symmetry of finite finitistic dimension. *Bulletin of the London Mathematical Society*, 56(2):624–633, 2024. [Publisher DOI](#).

- [4] Bernhard Keller. Bimodule complexes via strong homotopy actions. *Algebras and Representation Theory*, 3(4):357–376, 2000. [Publisher DOI](#); [arXiv:math/9910178v1](#).
- [5] Hiroyuki Minamoto and Kota Yamaura. Homological dimension formulas for trivial extension algebras. *Journal of Pure and Applied Algebra*, 224(8):106344, 2020. 30 pp. [Publisher DOI](#); [arXiv:1710.01469v1](#). All result numbers in citations to this paper refer to this arXiv version.
- [6] Amnon Neeman, Andrew Ranicki, and Aidan Schofield. Representations of algebras as universal localizations. *Mathematical Proceedings of the Cambridge Philosophical Society*, 136(1):105–117, 2004. [Publisher DOI](#); [arXiv:math/0205034v2](#).
- [7] OpenAI. An algebra of infinite little finitistic dimension. [OpenAI Math preprint](#), 2026. September 23, 2026. Source version: repository commit `adc7f1241b42e322a6451854ab7e4b4c146bf78a`.
- [8] Jeremy Rickard. Unbounded derived categories and the finitistic dimension conjecture. *Advances in Mathematics*, 354:106735, 2019. [Publisher DOI](#); [arXiv:1804.09801v1](#).
- [9] Yuri Santos Rego. On the finiteness length of some soluble linear groups. *Canadian Journal of Mathematics*, 74(5):1209–1243, 2022. [Publisher DOI](#); [arXiv:1901.06704v3](#).
- [10] The Stacks Project Authors. The Stacks project. <https://stacks.math.columbia.edu>, 2026. Tags [04VI](#), [04VJ](#), [05RG](#), and [05RJ](#); accessed 23 September 2026.
- [11] Birge Zimmermann Huisgen. The finitistic dimension conjectures—a tale of 3.5 decades. In Alberto Facchini and Claudia Menini, editors, *Abelian Groups and Modules*, volume 343 of *Mathematics and Its Applications*, pages 501–517. Kluwer Academic Publishers, Dordrecht, 1995. [Publisher DOI](#); [arXiv:1407.2383v1](#).

Qiyue Tang  
 Zhili College, Tsinghua University  
 Beijing 100084, China  
*Email:* [tangqy24@mails.tsinghua.edu.cn](mailto:tangqy24@mails.tsinghua.edu.cn)