

WEAK CHROMATIC SPLITTING FAILS AT $n = p$ FOR EVERY ODD PRIME p

PROOF PRODUCED BY CLAUDE (ANTHROPIC),
WRITTEN UP AND VERIFIED BY AGNES BEAUDRY AND TOMER SCHLANK

ABSTRACT. We prove that for every odd prime p there is a height n for which the map $L_{n-1}\mathbb{S}_p \rightarrow L_{n-1}L_{K(n)}\mathbb{S}$ is not injective on homotopy groups, hence not the inclusion of a wedge summand. In particular, for such pairs (n, p) the weak chromatic splitting conjecture is false. More precisely, we prove the conjecture fails for $(n, p) = (3, 3)$ and for all pairs (n, p) with $p \leq n < \sqrt{2}(p-1)$. In particular, the conjecture fails for $(n, p) = (p, p)$ for every odd prime p .

This disproof was produced autonomously by Claude in a session working on Freyd's generating hypothesis with no human in the loop. This note is a human write-up of the disproof. We did not contribute any ideas.

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1. INTRODUCTION

The chromatic splitting conjecture is one of the central open problems in chromatic homotopy theory. Chromatic homotopy theory divides the stable homotopy groups of spheres into periodic families, and the chromatic splitting conjecture is the principle which guides *chromatic reassembly*. Namely, by the Hopkins–Ravenel chromatic convergence theorem [Rav92, Thm. 7.5.7], for $\mathbb{S}_p$ the p -complete sphere,

$$\mathbb{S}_p \simeq \varprojlim L_n \mathbb{S}_p$$

where L_n is Bousfield localization at $K(0) \oplus \dots \oplus K(n)$. Furthermore, the chromatic fracture square [Ho95] is a homotopy pullback square

$$\begin{array}{ccc} L_n \mathbb{S}_p & \longrightarrow & L_{K(n)} \mathbb{S} \\ \downarrow & & \downarrow \\ L_{n-1} \mathbb{S}_p & \xrightarrow{\iota} & L_{n-1} L_{K(n)} \mathbb{S} \end{array}$$

so that the chromatic layers can be rebuilt inductively from $L_{K(n)}\mathbb{S}$ and the lower chromatic localizations, according to the *gluing data* ι . The *weak* chromatic splitting conjecture states that ι has a retract, so that chromatic reassembly is simple. The *strong* chromatic splitting conjecture further describes the fiber of ι in terms of lower chromatic localizations. The conjecture is due to Michael Hopkins based on Shimomura and Yabe's computations [SY95] of the homotopy groups $\pi_* L_2 \mathbb{S}$ at the prime $p = 5$. It appears in print in [Ho95] in its strongest form. The terminology *weak* versus *strong* was introduced in [Bea17].

The strong conjecture is known to hold for $n = 1$ and for $n = 2$ at $p \geq 3$ [SY95, Ho95, GHM14]; at $(n, p) = (2, 2)$ the strong form fails [Bea17] but the weak form holds [BGH22]. In this note, we present the proof that for every odd prime p , there is a height n at which the weak chromatic splitting conjecture fails.

Theorem 1.1. *Let (n, p) be such that $p \geq 5$ and $p \leq n < \sqrt{2}(p-1)$, or let $(n, p) = (3, 3)$. The map*

$$L_{n-1}\mathbb{S}_p \longrightarrow L_{n-1}L_{K(n)}\mathbb{S}$$

is not injective on homotopy groups, hence does not admit a retraction.

The proof is short. For each pair (n, p) , we exhibit an explicit element of the kernel. For $p \geq 5$ the element is $\beta_1^{(p-1)^2}$, where β_1 is the first element of the β -family. For $p = 3$, the element is β_1^5 .

We summarize the argument here, noting that for experts, this summary will be sufficient to understand the failure of the conjecture. A class $x \in \pi_*\mathbb{S}_p$ whose image in $L_{K(n)}\mathbb{S}$ is trivial, but which is detected in $\pi_*L_{n-1}\mathbb{S}_p$ must be in the kernel of $L_{n-1}\mathbb{S}_p \rightarrow L_{n-1}L_{K(n)}\mathbb{S}$. If such an x exists, then this map does not admit a retraction, so it suffices to produce such an x . When $(p-1) \nmid n$, the homotopy fixed point spectral sequence computing $\pi_*L_{K(n)}\mathbb{S}$ has a horizontal vanishing line at $s > n^2$, and so any element in the homotopy $\pi_*\mathbb{S}$ whose Adams–Novikov filtration is greater than n^2 must be zero in $\pi_*L_{K(n)}\mathbb{S}$. For $p \geq 5$, β_1^k has nonzero image in π_*EO_{p-1} for $k \leq (p-1)^2$ and hence in $\pi_*L_{p-1}\mathbb{S}_p$. Therefore, for $p \geq 5$, we conclude that $\beta_1^{(p-1)^2}$, whose filtration is $2(p-1)^2 > p^2$, is nonzero in $\pi_*L_{p-1}\mathbb{S}_p$, but zero in $\pi_*L_{K(p)}\mathbb{S}$. For $p = 3$, we instead use that β_1^5 is nonzero in $\pi_*L_{K(2)}\mathbb{S}/3$.

A word on notation. For every homotopy ring R , and $x \in \pi_*\mathbb{S}$, we denote the image of x in π_*R via the unit by the same name. We also abuse notation and use the same name for classes in the Adams–Novikov or homotopy fixed points spectral sequences detecting x . Theorem 1.1 follows from Theorems 2.6 and 2.10 below.

Acknowledgments. The disproof of the weak chromatic splitting conjecture given in this note was produced autonomously by Claude. We did not contribute any ideas and Claude was not prompted on this problem but on Freyd’s generating hypothesis in stable homotopy theory. Although the relationship between Freyd’s generating hypothesis and the chromatic splitting conjecture is well established, for example, in work of Hopkins–Devinatz [Dev90, Dev98] (see also [Ho07, Bar20]), we did not expect Claude to find a counter example to the weak chromatic splitting conjecture, which we both believed to be true. We note recent progress on Freyd’s generating hypothesis recently appeared in Ma–Xu [MX26].

The result arose in the context of paid consulting with Anthropic. Claude was run internally by Anthropic, using its own computing resources. The transcript was shared with us September 8, 2026. The mathematical exposition was drafted by us, Agnès Beaudry and Tomer Schlank, after we verified the details of Claude’s disproof, on September 12, 2026. The result was announced at 2026 Clay Research Workshop on Telescopic Homotopy Theory and Algebraic K-theory, on September 21, 2026. This text is not AI generated. It was produced by us, using Claude only for copyediting.

2. THE DISPROOF

2.1. Vanishing in $\pi_*L_{K(n)}\mathbb{S}$. Let $\iota_{K(n)}: \mathbb{S}_p \rightarrow L_{K(n)}\mathbb{S}$ be the $K(n)$ -localization.

Lemma 2.1. *Let $j < n$ and $x \in \pi_* \mathbb{S}_p$. If $\iota_{K(n)}(x) = 0$ and $\iota_{K(j)}(x) \neq 0$, then $\iota_{K(j)}(x)$ maps to zero under the map*

$$L_{K(j)}\iota_{K(n)}: L_{K(j)}\mathbb{S} \rightarrow L_{K(j)}L_{K(n)}\mathbb{S}.$$

Consequently, this map does not admit a retraction, and neither does

$$L_{n-1}\iota_{K(n)}: L_{n-1}\mathbb{S}_p \rightarrow L_{n-1}L_{K(n)}\mathbb{S}.$$

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{S}_p & \xrightarrow{\iota_{K(n)}} & L_{K(n)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{n-1}\mathbb{S}_p & \xrightarrow{L_{n-1}\iota_{K(n)}} & L_{n-1}L_{K(n)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{K(j)}\mathbb{S} & \xrightarrow{L_{K(j)}\iota_{K(n)}} & L_{K(j)}L_{K(n)}\mathbb{S} \end{array}$$

where the bottom row is the $L_{K(j)}$ -localization of the middle row. The left vertical composite is $\iota_{K(j)}$. By assumption, the image of x is nonzero in $\pi_* L_{K(j)}\mathbb{S}$ and thus in $\pi_* L_{n-1}\mathbb{S}_p$, but is zero in $\pi_* L_{K(n)}\mathbb{S}$ and thus in $\pi_* L_{n-1}L_{K(n)}\mathbb{S}$ and $\pi_* L_{K(j)}L_{K(n)}\mathbb{S}$. Thus the two bottom horizontal maps are not injective on homotopy groups and so do not admit retractions. $\square$

Lemma 2.2. *Suppose $(p-1) \nmid n$. If $x \in \pi_* \mathbb{S}_{(p)}$ in the homotopy of the p -local sphere $\mathbb{S}_{(p)}$ is of Adams–Novikov filtration $s > n^2$, then $\iota_{K(n)}(x) = 0$ in $\pi_* L_{K(n)}\mathbb{S}$.*

Proof. When $(p-1) \nmid n$, the Morava stabilizer group $\mathbb{S}_n$ is a p -adic analytic group of dimension n^2 with no p -torsion, and so has cohomological dimension n^2 (Lazard [La65]; see [BB20, Thm. 5.3.12]). As a consequence, the cohomology $H_c^s(\mathbb{S}_n, \pi_t E_n)$ vanishes for $s > n^2$, and hence so does $H_c^s(\mathbb{G}_n, \pi_t E_n)$, since by [BG18, Lemma 1.32],

$$H_c^s(\mathbb{G}_n, \pi_t E_n) \cong H_c^s(\mathbb{S}_n, \pi_t E_n)^{\text{Gal}}.$$

By [DH04, Thm. 1], the homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_n, \pi_t E_n) \implies \pi_{t-s} L_{K(n)}\mathbb{S}$$

is the $K(n)$ -local E_n -Adams spectral sequence for $\pi_* L_{K(n)}\mathbb{S}$. So there is a map of spectral sequences

$$\begin{array}{ccc} \text{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) & \longrightarrow & H_c^s(\mathbb{G}_n, \pi_t E_n) \\ \Downarrow & & \Downarrow \\ \pi_{t-s} \mathbb{S}_{(p)} & \longrightarrow & \pi_{t-s} L_{K(n)}\mathbb{S} \end{array}$$

where the bottom arrow is the map induced by localization. The right-hand spectral sequence has a horizontal vanishing line at $s > n^2$, which proves the claim. $\square$

In the next result, we let

$$\beta_1 \in \text{Ext}_{BP_*BP}^{2,2(p^2-p)}(BP_*, BP_*)$$

be the first element of the β -family. We also denote by the same name the element of $\pi_{2(p^2-p)-2}\mathbb{S}_{(p)}$ it detects.

Corollary 2.3. *If p is odd and $(p-1) \nmid n$, then $\iota_{K(n)}(\beta_1^k) = 0$ whenever $2k > n^2$.*

Proof. Since β_1 has Adams–Novikov filtration 2, the class β_1^k has Adams–Novikov filtration at least $2k > n^2$. The claim then follows from Lemma 2.2. $\square$

2.2. The case $p \geq 5$. For an odd prime p , let E_{p-1} be Lubin–Tate theory associated to the pair $(\mathbb{F}_{p^{p-1}}, \Gamma_{p-1})$ where Γ_{p-1} is the Honda formal group law of height $p-1$. The Morava stabilizer group $\mathbb{S}_{p-1}$ is the group of $\mathbb{F}_{p^{p-1}}$ -automorphisms of Γ_{p-1} . Let $\Phi \subset \mathbb{S}_{p-1}$ be a maximal finite subgroup containing an element of order p , so that $\Phi \cong \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$ up to conjugation [Na10, §2.2]. Let

$$EO_{p-1} := E_{p-1}^{h\Phi}.$$

This is a $K(p-1)$ -local ring spectrum. Non-triviality of an element in $\pi_* L_{K(p-1)}\mathbb{S}$ can thus be detected in EO_{p-1} by its image under the unit $L_{K(p-1)}\mathbb{S} \rightarrow EO_{p-1}$.

Lemma 2.4. *Let $p \geq 5$ and $1 \leq k \leq (p-1)^2$. Then β_1^k has nonzero image in $\pi_* EO_{p-1}$. Consequently $\iota_{K(p-1)}(\beta_1^k) \neq 0$ in $\pi_* L_{K(p-1)}\mathbb{S}$.*

Proof. The unit $\mathbb{S}_{(p)} \rightarrow EO_{p-1}$ induces a map from the Adams–Novikov spectral sequence computing $\pi_* \mathbb{S}_{(p)}$ to the homotopy fixed point spectral sequence

$$H^*(\Phi, (E_{p-1})_*) \implies \pi_* EO_{p-1}.$$

Modulo the image of the transfer (which are permanent cycles) the spectral sequence is computed in [Na10]. It follows from that computation that β_1^k has nonzero image in $\pi_* EO_{p-1}$ for $k \leq (p-1)^2$. Since EO_{p-1} is $K(p-1)$ -local, the unit factors through $L_{K(p-1)}\mathbb{S}$, and so $\iota_{K(p-1)}(\beta_1^k) \neq 0$ for k in this range. $\square$

Corollary 2.5. *Let $p \geq 5$ and $p \leq n < \sqrt{2}(p-1)$, and let $n^2 < 2k \leq 2(p-1)^2$. Then $\iota_{K(n)}(\beta_1^k) = 0$, while $\iota_{K(p-1)}(\beta_1^k) \neq 0$.*

Proof. Since $p-1 < n < 2(p-1)$, we have $(p-1) \nmid n$. The claim then follows from Lemma 2.2 and Lemma 2.4. $\square$

Theorem 2.6. *Let $p \geq 5$ and $p \leq n < \sqrt{2}(p-1)$. Then*

$$L_{n-1}\mathbb{S}_p \rightarrow L_{n-1}L_{K(n)}\mathbb{S}$$

is not injective on homotopy groups, and so does not admit a retraction.

Proof. Apply Lemma 2.1 with $j = p-1$ and $x = \beta_1^k$ as in Corollary 2.5. $\square$

Remark 2.7. Note that we have shown the stronger result that, under the assumptions of Theorem 2.6,

$$L_{K(p-1)}\mathbb{S} \rightarrow L_{K(p-1)}L_{K(n)}\mathbb{S}$$

does not admit a retraction.

Remark 2.8. The condition $p \geq 5$ is required for the range $p \leq n < \sqrt{2}(p-1)$ to be nonempty. Further, for $p \geq 5$, we can take $n = p$.

2.3. The case $(n, p) = (3, 3)$. At $p = 3$ there is no n with $p \leq n < \sqrt{2}(p-1) = 2\sqrt{2}$. However, since $2 \nmid 3$ and $10 > 9$, Corollary 2.3 gives $\iota_{K(3)}(\beta_1^5) = 0$. So, by Lemma 2.1, it suffices to prove that $\iota_{K(2)}(\beta_1^5) \neq 0$. The following is a consequence of [HKM13, Thm. 1.8].

Theorem 2.9 (Henn–Karamanov–Mahowald). *Let $p = 3$ and let $\mathbb{S}/3$ be the mod 3 Moore spectrum. Then β_1^5 has nonzero image in $\pi_{50} L_{K(2)}\mathbb{S}/3$ under the map*

$$\mathbb{S}_{(p)} \rightarrow L_{K(2)}\mathbb{S} \rightarrow L_{K(2)}\mathbb{S}/3.$$

In particular, $\iota_{K(2)}(\beta_1^5) \neq 0$.

Theorem 2.10. *Let $(n, p) = (3, 3)$. The map*

$$L_{n-1}\mathbb{S}_p \rightarrow L_{n-1}L_{K(n)}\mathbb{S}$$

is not injective on homotopy groups, and so does not admit a retraction.

Proof. Apply Lemma 2.1 with $j = 2$ and $x = \beta_1^5$: Corollary 2.3 gives $\iota_{K(3)}(\beta_1^5) = 0$ and Theorem 2.9 gives $\iota_{K(2)}(\beta_1^5) \neq 0$. $\square$

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