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A POISSON BOUNDARY SOLUTION TO CONNES' BICENTRALIZER CONJECTURE AND KADISON'S PROBLEM ON MASA

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Abstract. We develop Poisson boundary theory for general von Neumann algebras with separable preduals. As an application, we prove that, for every inclusion $M \subset P$ of countably decomposable von Neumann algebras with a normal faithful conditional expectation, $M \subset c(P)$ has the weak Dixmier property, where $c(P)$ is the continuous core of P. As direct consequences, via the characterizations of Haagerup and Marrakchi, we have a completely new proof of Connes' bicentralizer conjecture and prove Marrakchi's relative bicentralizer conjecture. We also give a new proof of the solution to Kadison's problem on maximal abelian subalgebras for inclusions of von Neumann algebras with expectation.

1. Introduction

1.1 Connes' bicentralizer conjecture and the weak Dixmier property. In the theory of von Neumann algebras, as in many areas of mathematics, averaging and averagable objects are a central theme of research. Amenability is the property that describes the possibility of averaging and is a central property in geometric and ergodic group theory. A discrete group Γ is amenable if and only if there exists a Γ-invariant mean on $\ell^\infty(\Gamma)$ [vN29].

For von Neumann algebras, **amenability** is equivalent to **injectivity**. A von Neumann algebra $M \subset B(H)$ is injective if and only if there exists a conditional expectation from $B(H)$ onto M. The weak Dixmier property is an averaging property of inclusions of von Neumann algebras. An inclusion $M \subset \mathcal{A}$ has the **weak Dixmier property** if every $x \in \mathcal{A}$ can be averaged by unitaries of M into the relative commutant $M' \cap \mathcal{A}$, that is,

$$\overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset \qquad (x \in \mathcal{A}).$$

In his celebrated work, Connes [Con76] proved that a von Neumann algebra $M \subset B(H)$ with separable predual is injective if and only if the inclusion $M \subset B(H)$ has the weak Dixmier property. These conditions are further equivalent to M being **hyperfinite**, meaning that it can be approximated by finite-dimensional subalgebras. This connects averaging with finite-dimensional approximation and makes it a tool for the classification of amenable von Neumann algebras.

The classification of von Neumann algebras was a central goal of the work of Murray–von Neumann [MvN36]. Von Neumann algebras with separable predual decompose as direct integrals of **factors**, which are von Neumann algebras M with trivial center, i.e., $\mathcal{Z}(M) = M' \cap M = \mathbb{C}1$. Murray–von Neumann divided factors into types I, II_1, II_∞, and III. A type I factor is isomorphic to $B(H)$ for some Hilbert space H. A type II_1 factor admits a normal faithful finite trace, while a type II_∞ factor admits a normal faithful semifinite trace τ with $\tau(1) = \infty$. Type III factors admit no nonzero normal semifinite trace. Murray–von Neumann [MvN43] proved that, up to isomorphism, there is a unique hyperfinite II_1 factor.

Connes [Con73] refined this classification by dividing type III into types III_λ for $0 \leq \lambda \leq 1$. He subsequently classified the amenable factors with separable predual of types II_1, II_∞, and III_λ for $0 \leq \lambda < 1$ [Con76]. For the remaining type III_1 case, Connes [Con85] introduced the bicentralizer conjecture and showed that a positive solution for amenable factors would imply the uniqueness of the amenable type III_1 factor. Let M be a countably decomposable von Neumann algebra and let ϕ be a normal faithful state on M. The **bicentralizer** $\mathrm{B}(M, \phi)$ consists of the $x \in M$ such that $[x, a_n] \to 0$ strongly for every bounded

sequence $(a_n) \subset M$ satisfying $\|a_n\phi - \phi a_n\| \to 0$; see Subsection 2.2. Connes' bicentralizer conjecture states that $\mathrm{B}(M,\phi) = \mathbb{C}1$ for every type III_1 factor M and every normal faithful state ϕ on M.

The finite trace τ on a type II_1 factor M makes the analogous bicentralizer statement immediate: $\mathrm{B}(M,\tau) = \mathcal{Z}(M) = \mathbb{C}1$. More generally, much of the power of averaging in finite von Neumann algebras comes from the trace: it provides an invariant Hilbert-space geometry in which convexity produces fixed points. Type III algebras have no such trace. For factors with separable predual of type III_λ, $0 < \lambda < 1$, the periodic modular structure gives a discrete decomposition as a crossed product of a type II_∞ factor by $\mathbb{Z}$ [Con73]. This reduces many structural questions to the semifinite setting. Such a periodic reduction is not available in general for type III_1 factors. One must therefore find methods that work without either a trace on the original algebra or such a discrete reduction. This is one reason why the type III_1 case often presents the greatest difficulty.

The same difficulty appears in Popa's problem on the weak Dixmier property (Corollary E) and Kadison's problem on MASA (Theorem G), discussed below. For inclusions $M \subset \mathcal{A}$ with expectation and finite M, the weak Dixmier property follows from classical averaging arguments. The type II_1 case of Kadison's problem was established by Popa [Pop81] in 1981. The type III_1 cases of these problems were resolved only much later, in 2019 and 2026, respectively.

Haagerup [Haa87, Theorem 2.3] solved the bicentralizer problem for amenable factors, thereby proving the uniqueness of the amenable type III_1 factor and completing the classification of amenable factors with separable predual. Haagerup [Haa87, Theorem 3.1] proved that a type III_1 factor M with separable predual has trivial bicentralizer if and only if $M \subset c(M)$ has the weak Dixmier property, where $c(M) = M \rtimes_{\sigma^\phi} \mathbb{R}$ is the **continuous core** of M. Here σ^ϕ is the **modular automorphism group** associated with ϕ; we refer to Subsection 2.3 for background on modular theory and continuous cores. For amenable M, hyperfiniteness allows us to average each element of $c(M)$ by unitaries of M into $M' \cap c(M)$ in the weak topology, so $M \subset c(M)$ has the weak Dixmier property and hence M has trivial bicentralizer.

Houdayer–Marrakchi [HM26, Theorem C] recently solved Connes' bicentralizer conjecture for arbitrary type III_1 factors. Their proof does not follow Haagerup's approach. It relies on the bicentralizer flow [AHHM20], on the ergodicity theorems for this flow proved by Marrakchi [Mar20, Theorem D], [Mar26, Theorem C], and on a resonance phenomenon, discovered in [HM26], for the joint action of the modular flow and the dual bicentralizer flow. In particular, their proof does not use Haagerup's theorem. It is therefore also a new proof that amenable type III_1 factors have trivial bicentralizer, and hence, by [Con85], a new proof of the uniqueness of the amenable type III_1 factor.

In this paper, we follow Haagerup's approach to give a new proof of Connes' bicentralizer conjecture, independent of [HM26]. We prove the weak Dixmier property for $M \subset c(M)$ directly, and more generally for $M \subset c(P)$ whenever $M \subset P$ admits a normal faithful conditional expectation, and then apply Haagerup's characterization. Our proof treats all factors uniformly, with only minor adjustments between the finite and properly infinite cases. In particular, the previously most difficult type III_1 case is treated by the same argument as the other properly infinite cases, without using any recent advances in modular and bicentralizer theory.

Theorem A. *Let $M \subset P$ be an inclusion of countably decomposable von Neumann algebras with a normal faithful conditional expectation. Then $M \subset c(P)$ has the weak Dixmier property. In particular, $M \subset c(M)$ has the weak Dixmier property.*

By the result of Haagerup [Haa87, Theorem 3.1], Theorem A implies Connes' bicentralizer conjecture.

Corollary B (Connes' bicentralizer conjecture [HM26])**.** *Let M be a countably decomposable factor of type III_1. Then*

$$\mathrm{B}(M,\phi) = \mathbb{C}1$$

for every normal faithful state ϕ on M.

By [Mar25, Theorem D], Theorem A also implies Marrakchi's relative bicentralizer conjecture [Mar25, Conjecture C], which generalizes Connes' bicentralizer conjecture. We refer to Subsection 2.2 for the definition of the relative bicentralizer and the statement of the relative bicentralizer conjecture.

Corollary C (The relative bicentralizer conjecture)**.** *Let $M \subset P$ be an inclusion of countably decomposable von Neumann algebras with a normal faithful conditional expectation. Then $M \subset P$ satisfies the relative bicentralizer conjecture.*

The generating inner ucp maps and cutoff arguments used in the proof of Theorem A also apply to crossed products by more general actions. We first obtain the following result for $\mathbb{R}^2$-actions on semifinite von Neumann algebras.

Theorem A.1. *Let $M \subset N$ be an inclusion of von Neumann algebras. Assume that M has separable predual, that τ is a normal faithful semifinite trace on M, and that $E : N \to M$ is a normal faithful conditional expectation. Let $\alpha : \mathbb{R}^2 \curvearrowright N$ be an ultraweakly continuous action such that*

$$\alpha_t(M) = M, \qquad \tau \circ \alpha_t|_M = e^{\ell(t)}\tau \qquad (t \in \mathbb{R}^2),$$

with $\ell : \mathbb{R}^2 \to \mathbb{R}$ linear. Then $M \subset N \rtimes_\alpha \mathbb{R}^2$ has the weak Dixmier property.

Passing to compatible continuous cores and applying Takesaki duality, we then obtain the following result for general flows.

Theorem A.2. *Let $\mathcal{M} \subset \mathcal{N}$ be an inclusion of von Neumann algebras. Assume that $\mathcal{M}$ is properly infinite with separable predual and that $E : \mathcal{N} \to \mathcal{M}$ is a normal faithful conditional expectation. Let $\alpha : \mathbb{R} \curvearrowright \mathcal{N}$ be an ultraweakly continuous action such that*

$$\alpha_t(\mathcal{M}) = \mathcal{M}, \qquad E \circ \alpha_t = \alpha_t|_{\mathcal{M}} \circ E \qquad (t \in \mathbb{R}).$$

Then $\mathcal{M} \subset \mathcal{N} \rtimes_\alpha \mathbb{R}$ has the weak Dixmier property.

We present these extensions in Appendix A to keep the main text focused on the direct proof of Theorem A and its bicentralizer consequences. The appendix explains how the same averaging argument combines with two-dimensional cutoff estimates and passage to compatible continuous cores to treat more general actions.

1.2 The weak Dixmier property via Poisson boundaries. The Poisson boundary of a group is also an object defined through averaging. Let Γ be a countable discrete group, and let $\mu \in \mathrm{Prob}(\Gamma)$ be a generating probability measure, i.e., $\bigcup_{n\geq 1}(\mathrm{supp}\,\mu)^n = \Gamma$. Define the Markov operator P_μ on $\ell^\infty(\Gamma)$ by

$$P_\mu(f)(g) = \sum_{s\in\Gamma} \mu(s) f(gs) \qquad (g \in \Gamma).$$

Hence P_μ can be viewed as a μ-weighted averaging operator. Let $\mathrm{Har}(\Gamma,\mu) = \{f \in \ell^\infty(\Gamma) \mid P_\mu(f) = f\}$ be the space of bounded μ-harmonic functions.

The **Poisson boundary** of (Γ,μ) is the essentially unique nonsingular Γ-probability space (B,ν_B) such that the Poisson transform

$$P_{\nu_B} : L^\infty(B,\nu_B) \longrightarrow \mathrm{Har}(\Gamma,\mu), \qquad P_{\nu_B}(F)(g) = \int_B F(gb)\,\mathrm{d}\nu_B(b)$$

is an isometric isomorphism [Fur63].

For example, let Γ be a non-elementary hyperbolic group with a finite generating set S, and let μ be a generating probability measure supported on S. Then its Poisson boundary can be identified with the Gromov boundary $(\partial_\infty\Gamma, \nu_\infty)$ of the Cayley graph with respect to S, where ν_∞ is the hitting measure of the μ-random walk [Kai00, Theorems 7.6 and 7.7].

Another example comes from lattices in semisimple Lie groups. Furstenberg [Fur67] proved that, for every irreducible lattice $\Gamma < G$ in a connected semisimple real Lie group with finite center and no compact

factors, there exists a fully supported probability measure μ on Γ such that the (Γ,μ)-Poisson boundary is $(G/P,\nu_P)$, where $P<G$ is a minimal parabolic subgroup. A major application of Poisson boundary theory is the proof of Margulis' superrigidity theorem for irreducible lattices in higher rank semisimple Lie groups [Mar91, Chapter VII].

Izumi [Izu02, Izu04] developed a noncommutative Poisson boundary construction for normal ucp maps on von Neumann algebras, motivated by the study of discrete quantum groups. Building on Izumi's construction, Das–Peterson [DP22] initiated Poisson boundary theory for finite von Neumann algebras, which was subsequently studied in [Zho24b, Zho24a, Zho24c, Zho25]. We extend this theory to general von Neumann algebras M with separable preduals. In this general setting, as analogues of generating probability measures on groups, our main objects are **normal, regular, strongly generating (nrsg) hyperstates** on $B(L^2(M))$ (Definition 3.8). More concretely (see Example 3.10), given any normal faithful state ϕ on M with canonical cyclic vector $\xi_\phi\in L^2(M)$, we can always construct a normal faithful state φ on $B(L^2(M))$ with $\varphi|_M=\phi$ and sequences $(z_k),(y_k)\subset M$ such that

$$\overline{\mathbb{C}[z_k]}^{\mathrm{w}}=\overline{\mathbb{C}[y_k]}^{\mathrm{w}}=M,\qquad \sum_k z_k^*z_k=\sum_k y_ky_k^*=1.$$

The corresponding normal inner ucp maps

$$\mathcal{P}_\varphi(T)=\sum_k (Jz_k^*J)T(Jz_kJ),\qquad \mathcal{P}_\varphi^o(T)=\sum_k y_kTy_k^*$$

satisfy

$$\varphi(T)=\langle\mathcal{P}_\varphi(T)\xi_\phi,\xi_\phi\rangle=\langle\mathcal{P}_\varphi^o(T)\xi_\phi,\xi_\phi\rangle\qquad (T\in B(L^2(M))). \tag{1.1}$$

Here J is the modular conjugation on $L^2(M)$, and $\mathbb{C}[z_k]$ and $\mathbb{C}[y_k]$ are the unital subalgebras (rather than the unital $*$-subalgebras) generated by (z_k) and (y_k), respectively.

Let $\mathrm{Har}(\mathcal{P}_\varphi)=\{T\in B(L^2(M))\mid \mathcal{P}_\varphi(T)=T\}$. We define the **Poisson boundary** $\mathcal{B}_\varphi$ to be the von Neumann algebra isomorphic to $\mathrm{Har}(\mathcal{P}_\varphi)$ as operator systems (Definition 3.3). Its existence and uniqueness are proved in Proposition 3.4. Under this identification, M is a von Neumann subalgebra of $\mathcal{B}_\varphi$.

For example, let Γ be a countable discrete group, let $\mu\in\mathrm{Prob}(\Gamma)$ be a generating probability measure, and let (B,ν_B) be its Poisson boundary. For $M=L(\Gamma)$, consider the hyperstate $\varphi\in B(\ell^2(\Gamma))_*$ defined by $\varphi(T)=\sum_{g\in\Gamma}\mu(g)\langle T\delta_g,\delta_g\rangle$. Izumi [Izu04] proved that $M\subset\mathcal{B}_\varphi$ is naturally isomorphic to the inclusion $L(\Gamma)\subset L^\infty(B,\nu_B)\rtimes\Gamma$.

We have the following sufficient condition for the weak Dixmier property in terms of maps into $\mathcal{B}_\varphi$.

Theorem D (Theorem 4.2)**.** *Let M be a von Neumann algebra with separable predual, and let φ be an nrsg hyperstate on $B(L^2(M))$ with Poisson boundary $\mathcal{B}_\varphi$. Assume that one of the following conditions holds:*

(i) *M is properly infinite, i.e., $M=M_{\mathrm{I}_\infty}\oplus M_{\mathrm{II}_\infty}\oplus M_{\mathrm{III}}$;*
(ii) *M is finite with a normal faithful trace τ, and $\varphi=\int_{\mathcal{U}(M)}\langle\cdot\, u\xi_\tau,u\xi_\tau\rangle\,\mathrm{d}\mu(u)$ for some $\mu\in\mathrm{Prob}(\mathcal{U}(M))$.*

Then every W-inclusion $M\subset\mathcal{A}$ equipped with a normal faithful ucp map $\mathcal{P}:\mathcal{A}\to\mathcal{B}_\varphi$ satisfying $\mathcal{P}|_M=\mathrm{id}_M$ has the weak Dixmier property.*

As a direct corollary, we give a new proof of the weak Dixmier property for W*-inclusions with a normal faithful conditional expectation, a problem posed by Popa [Pop00, Remark 1.5] and solved by Marrakchi [Mar21].

Corollary E ([Pop00, Mar21], Corollary 4.3)**.** *Let $M\subset\mathcal{A}$ be an inclusion of countably decomposable von Neumann algebras with a normal faithful conditional expectation. Then $M\subset\mathcal{A}$ has the weak Dixmier property.*

We also prove a semifinite version of Theorem D, replacing the normal faithful ucp map into $\mathcal{B}_\varphi$ by a normal faithful semifinite M-bimodular positive map $\mathcal{T}:\mathcal{A}_+\to\widehat{\mathcal{B}_{\varphi}}_+$, with the averaging conclusion for

positive elements x such that $\mathcal{T}(x)$ is bounded (Theorem 4.7). In particular, we have a new proof of the weak Dixmier result of Isono [Iso25] for inclusions $M \subset \mathcal{A}$ of countably decomposable von Neumann algebras with a normal faithful semifinite operator-valued weight (Corollary 4.8).

The proof of Theorem D is rather simple. We explain the key case where M is properly infinite with separable predual, and there is a normal faithful conditional expectation $E : \mathcal{A} \to M$. Fix a normal faithful state ϕ on M and take an nrsg hyperstate φ extending ϕ as above. Using the coefficients (y_k) of $\mathcal{P}_\varphi^o$, we define the normal M-inner ucp map

$$Q : \mathcal{A} \to \mathcal{A}, \qquad Q(a) = \sum_k y_k a y_k^* \qquad (a \in \mathcal{A}),$$

and let $\mathrm{Har}(Q, \mathcal{A}) = \{a \in \mathcal{A} \mid Q(a) = a\}$. We first prove that $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$.

Define $\varphi_{\mathcal{A}} = \phi \circ E$. For $a \in \mathcal{A}$ with $Q(a) = a$, by Kadison's inequality we have

$$Q(aa^*) - aa^* = Q(aa^*) - Q(a)Q(a)^* \geq 0.$$

By (1.1), $\phi \circ Q|_M = \phi$. Since E is M-bimodular, we also have

$$\varphi_{\mathcal{A}} \circ Q = \phi \circ E \circ Q = \phi \circ Q \circ E = \phi \circ E = \varphi_{\mathcal{A}}. \tag{1.2}$$

Hence $\varphi_{\mathcal{A}}(Q(aa^*) - aa^*) = 0$, and by faithfulness we have $Q(aa^*) = aa^*$. Since $Q(a^*) = a^*$, we have

$$\begin{aligned} &\sum_k (y_k a - a y_k)(y_k a - a y_k)^* \\ &= \sum_k \big(y_k a a^* y_k^* - y_k a y_k^* a^* - a y_k a^* y_k^* + a y_k y_k^* a^* \big) \\ &= Q(aa^*) - Q(a)a^* - aQ(a^*) + aa^* = 0. \end{aligned} \tag{1.3}$$

Hence $y_k a = a y_k$ for every k. Since $\mathbb{C}[y_k]$ is weakly dense in M, we have $a \in M' \cap \mathcal{A}$, proving $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$.

For every $a \in \mathcal{A}$, any weak cluster point of the Cesàro averages $(1/n)\sum_{j=0}^{n-1} Q^j(a)$ lies in $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$. If $y_k = \|y_k\| u_k$ with $u_k \in \mathcal{U}(M)$, these averages would lie in the weakly closed convex hull of the unitary orbit of a, giving the weak Dixmier property for $M \subset \mathcal{A}$ directly. In general, the coefficients (y_k) need not have this form. Nevertheless, the same argument shows that $M \subset \mathcal{A}$ has the inner UCP-Dixmier property defined below.

Definition 1.1. *Let $M \subset \mathcal{A}$ be a W^*-inclusion. A ucp map $\mathcal{P} : \mathcal{A} \to \mathcal{A}$ is called M-**inner** if*

$$\mathcal{P}(a) = \sum_k z_k a z_k^*, \qquad \sum_k z_k z_k^* = 1,$$

*for a sequence $(z_k) \subset M$. We say that $M \subset \mathcal{A}$ has the **inner UCP-Dixmier property** if, for every $a \in \mathcal{A}$,*

$$\overline{\{\mathcal{P}(a) \mid \mathcal{P} \text{ is an } M\text{-inner ucp map}\}}^{\mathrm{w}} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Indeed, each Cesàro averaging map $(1/n)\sum_{j=0}^{n-1} Q^j$ is M-inner, so $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.

We prove that the two properties are equivalent when M is properly infinite, which completes the proof in the case considered above. The proof uses proper infiniteness to pass from inner ucp maps to unitary averages, an idea similar to that used by Haagerup [Haa85, Section 2].

Theorem F (Theorem 4.1)**.** *Let $M \subset \mathcal{A}$ be a W^*-inclusion such that M is properly infinite. Then $M \subset \mathcal{A}$ has the weak Dixmier property if and only if it has the inner UCP-Dixmier property.*

To summarize, the Poisson boundary theory provides inner ucp maps with the properties needed to prove the inner UCP-Dixmier property in Definition 1.1; when M is properly infinite, we then apply Theorem F to obtain the weak Dixmier property.

1.3 Proof strategy for Theorem A. Let $M \subset P$ be a W*-inclusion with a normal faithful conditional expectation $E : P \to M$ as in Theorem A. The proof of the weak Dixmier property for $M \subset c(P)$ is in the same spirit as the preceding argument, with modular-theoretic adaptations to the special case $\mathcal{A} = c(P)$.

We again focus on the key case where M is properly infinite with separable predual. Fix a normal faithful state ϕ on M, and let $\rho = \phi \circ E$. We take a more special nrsg (M, ϕ)-hyperstate φ with symmetry from Example 3.10. Let $\mathcal{A} = c_\rho(P)$, and define $Q(x) = \sum_k y_k x y_k^*$ $(x \in \mathcal{A})$ using the coefficients (y_k) of $\mathcal{P}_\varphi^o$. As in Subsection 1.2, to prove the weak Dixmier property for $M \subset \mathcal{A}$, we only need to prove that $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$.

Let $\mathcal{T} : \mathcal{A}_+ \to \widehat{P}_+$ be the canonical normal faithful semifinite operator-valued weight of the continuous core. Define $\varphi_\mathcal{A} = \rho \circ \mathcal{T}$, the dual weight of ρ; see Subsection 2.3. By (1.3), to prove $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$, we only need to prove that $Q(aa^*) = aa^*$ for every $a \in \mathrm{Har}(Q, \mathcal{A})$. Fix such an a, and let $d := Q(aa^*) - aa^* \geq 0$.

As in (1.2), we have $\varphi_\mathcal{A} \circ Q = \varphi_\mathcal{A}$. Intuitively, this would give $\varphi_\mathcal{A}(d) = \varphi_\mathcal{A}(Q(aa^*)) - \varphi_\mathcal{A}(aa^*) = 0$, but this calculation is justified only when $\varphi_\mathcal{A}(aa^*) < \infty$. Instead, Lemma 2.2 shows that, for every $0 \neq d' \in \mathcal{A}_+$, there exists $x' \in \mathcal{A}_+$ such that $0 \neq x' \leq d'$ and $0 < \varphi_\mathcal{A}(x') < \infty$. Hence we only need to prove that every $x \in \mathcal{A}_+$ satisfying $x \leq d$ and $\varphi_\mathcal{A}(x) < \infty$ must satisfy $x = 0$. Since $x \leq d = Q(aa^*) - aa^*$ and $\sum_{k=0}^{n-1} Q^k(d) = Q^n(aa^*) - aa^*$, we have

$$\sum_{k=0}^{n-1} Q^k(x) \leq C1 \qquad \text{with } C = \|a\|^2 \qquad (n \geq 1).$$

We prove in Lemmas 5.2 and 5.3 that such an x must be 0.

The estimates use a noncommutative analogue of the classical cutoff argument for recurrence of symmetric Markov operators; see [FOT11, Section 1.6, Theorem 1.6.3]. To explain this argument, we present a commutative model in which $L^\infty(\mathbb{R}, \mathrm{d}t)$, equipped with the weight induced by Lebesgue measure, plays the role of $(\mathcal{A}, \varphi_\mathcal{A})$. Our choice of a symmetric measure $\nu \in \mathrm{Prob}(\mathbb{R})$ corresponds to the choice of a symmetric nrsg hyperstate φ.

Proposition 1.2. *Let $\nu \in \mathrm{Prob}(\mathbb{R})$ be symmetric, and assume that $m_2 = \int_\mathbb{R} s^2 \,\mathrm{d}\nu(s) < \infty$. Define the Markov operator Q on $L^\infty(\mathbb{R}, \mathrm{d}t)$ by*

$$Q(b)(t) = \int_\mathbb{R} b(t+s) \,\mathrm{d}\nu(s) \qquad (b \in L^\infty(\mathbb{R}, \mathrm{d}t)).$$

Let $x \in L^\infty(\mathbb{R}, \mathrm{d}t)_+$ with $\int_\mathbb{R} x(t) \,\mathrm{d}t < \infty$. If there exists $0 \leq C < \infty$ such that

$$\sum_{k=0}^{n-1} Q^k(x) \leq C1 \qquad (n \geq 1), \tag{1.4}$$

then $x = 0$.

Proof. We first prove that Q preserves the integral and is a self-adjoint contraction on $L^2(\mathbb{R})$ (cf. Lemma 5.1). Tonelli's theorem and translation invariance give $\int_\mathbb{R} Q(b) \,\mathrm{d}t = \int_\mathbb{R} b \,\mathrm{d}t$ for $b \in L^1(\mathbb{R})_+$. By Kadison's inequality, we then have $\|Q(b)\|_2 \leq \|b\|_2$ for $b \in L^2 \cap L^\infty(\mathbb{R})$, and hence on $L^2(\mathbb{R})$ by density. Since ν is symmetric, Q is self-adjoint, so $1 - Q \geq 0$.

We may assume that $\int_\mathbb{R} x \,\mathrm{d}t > 0$. Take a real $f \in C_c^\infty(\mathbb{R})$ with $0 \leq f \leq 1$, equal to 1 near 0, even and nonincreasing in $|t|$. Let $F_R(t) = f(t/R)$ for $R \geq 1$. Then $F_R \nearrow 1$ pointwise $(R \to \infty)$ and $\int_\mathbb{R} F_R \,\mathrm{d}t < \infty$.

We next prove the partial-sum estimate (cf. Lemma 5.2)

$$\int_\mathbb{R} x \,\mathrm{d}t \leq \liminf_{R \to \infty} C \|(1-Q)^{1/2} F_R\|_2^2. \tag{1.5}$$

Let $a_n = \sum_{k=0}^{n-1} Q^k(x)$. Then $0 \le a_n \le C1$, $\int_{\mathbb{R}} a_n \,\mathrm{d}t = n \int_{\mathbb{R}} x \,\mathrm{d}t$, and $a_n \in L^1 \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Since $(1-Q)a_n = x - Q^n(x)$,

$$\|(1-Q)^{1/2} a_n\|_2^2 = \int_{\mathbb{R}} a_n\big(x - Q^n(x)\big) \,\mathrm{d}t \le C \int_{\mathbb{R}} x \,\mathrm{d}t.$$

For fixed R, by the Cauchy–Schwarz inequality we therefore have

$$\left| \int_{\mathbb{R}} F_R\big(x - Q^n(x)\big) \,\mathrm{d}t \right|^2 \le C \left(\int_{\mathbb{R}} x \,\mathrm{d}t \right) \|(1-Q)^{1/2} F_R\|_2^2. \tag{1.6}$$

By (1.4) and monotone convergence,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} F_R Q^n(x) \,\mathrm{d}t \le C \int_{\mathbb{R}} F_R \,\mathrm{d}t < \infty.$$

Hence $\int_{\mathbb{R}} F_R Q^n(x) \,\mathrm{d}t \to 0$ $(n \to \infty)$, so by (1.6) we have

$$\left(\int_{\mathbb{R}} F_R x \,\mathrm{d}t \right)^2 \le C \left(\int_{\mathbb{R}} x \,\mathrm{d}t \right) \|(1-Q)^{1/2} F_R\|_2^2.$$

Dominated convergence as $R \to \infty$ and division by $\int_{\mathbb{R}} x \,\mathrm{d}t > 0$ give (1.5).

We now prove the cutoff estimate (cf. Lemma 5.3). Expanding the squared difference, for $b \in L^2(\mathbb{R})$ we have

$$\|(1-Q)^{1/2} b\|_2^2 = \frac{1}{2} \int_{\mathbb{R}} \|b(\,\cdot\, + s) - b\|_2^2 \,\mathrm{d}\nu(s). \tag{1.7}$$

The fundamental theorem of calculus and the Cauchy–Schwarz inequality give

$$\|F_R(\,\cdot\, + s) - F_R\|_2 \le |s| \|F_R'\|_2 = \frac{|s|}{\sqrt{R}} \|f'\|_2 \qquad (s \in \mathbb{R}). \tag{1.8}$$

Recall that $m_2 = \int_{\mathbb{R}} s^2 \,\mathrm{d}\nu(s) < \infty$. By (1.7) and (1.8), we have

$$\|(1-Q)^{1/2} F_R\|_2^2 \le \frac{\|f'\|_2^2}{2R} \int_{\mathbb{R}} s^2 \,\mathrm{d}\nu(s) = \frac{m_2 \|f'\|_2^2}{2R}. \tag{1.9}$$

Hence $\|(1-Q)^{1/2} F_R\|_2^2 \to 0$ $(R \to \infty)$, so by (1.5) we have $\int_{\mathbb{R}} x \,\mathrm{d}t = 0$ and $x = 0$. □

Remark 1.3. *Symmetry of Q is essential in Proposition 1.2. Indeed, let $S(f)(t) = f(t-1)$ be the nonsymmetric translation on $L^\infty(\mathbb{R})$, and let $x = \mathbb{1}_{[0,1)}$. Then $x \ne 0$, but*

$$\sum_{n=0}^{\infty} S^n(x) = \mathbb{1}_{[0,\infty)} \le 1.$$

This is why our proof of Theorem A uses the symmetric nrsg hyperstate constructed in Example 3.10, whereas the argument in Subsection 1.2 works with an arbitrary nrsg hyperstate.

1.4 Kadison's problem on MASA. Our Poisson boundary theory has another important application: a new and highly simplified proof of the solution to Kadison's problem on maximal abelian subalgebras (MASA). In 1967, Kadison [Kad67] asked whether, for every inclusion of factors $M \subset P$ with $M' \cap P = \mathbb{C}1$, there is a maximal abelian subalgebra $A \subset M$ which is maximal abelian in P, that is, $A' \cap P = A$. In 1981, Popa [Pop81] answered this question affirmatively when M is of type II and there is a normal faithful conditional expectation from P onto M, by constructing the desired maximal abelian subalgebra as the limit of an increasing sequence of finite-dimensional abelian subalgebras. Ge–Popa [GP98, Corollary 5.1] showed that the answer can be negative without such a conditional expectation, even when M is a type II_1 factor and P is a type II_∞ factor, and asked whether the existence of a normal faithful conditional expectation suffices for factors of arbitrary type; see the discussion after [GP98, Corollary 5.1] and [Pop21, Problem 7.6].

Marrakchi [Mar25, Theorem A] solved this problem when M is not of type III_1, or when M is a type III_1 factor with trivial bicentralizer. Marrakchi [Mar26, Theorem A(2)] completed the solution for arbitrary type III_1 factors, by reducing to the case of approximately finite-dimensional subalgebras [Mar26, Theorem B], which relies on the ergodicity of the relative bicentralizer flow [Mar26, Theorem C].

We give a new proof of the solution to Kadison's problem, different from the proofs of Marrakchi [Mar25, Mar26]. As in our proof of Theorem A, we treat all factors uniformly, with only minor adjustments for finite factors and factors of type I_∞. In particular, the most difficult type III_1 case is treated by the same argument as the other continuous properly infinite cases, without using any recent advances in modular and bicentralizer theory. Moreover, this proof is completely independent of our proof of the bicentralizer conjecture (Theorem A and Corollaries B and C).

Theorem G ([Pop81, Mar25, Mar26])**.** *Let $M \subset P$ be an inclusion of von Neumann algebras with separable preduals. Assume that $M' \cap P = \mathcal{Z}(M)$ and that there is a normal faithful conditional expectation $E : P \to M$. Then there is an abelian von Neumann subalgebra $A \subset M$ such that*

$$A' \cap P = A.$$

We briefly explain the proof of Theorem G in the key case where M is continuous and properly infinite. Let Q be the map of Subsection 1.2 for $\mathcal{A} = P$, that is, $Q(x) = \sum_k y_k x y_k^*$ $(x \in P)$. The argument given there shows that $\mathrm{Har}(Q, P) = M' \cap P = \mathcal{Z}(M)$. Hence the Cesàro means $C_n = (1/n)\sum_{\ell=0}^{n-1} Q^\ell$ satisfy $\|\theta \circ C_n\| \to 0$ for every $\theta \in P_*$ with $\theta|_{\mathcal{Z}(M)} = 0$ (Corollary 6.1). The key point is that we use these Cesàro means to construct matrix algebras in M. Each C_n is a normal M-inner ucp map. Since M is properly infinite, Lemmas 6.2 and 6.3 approximate $\theta \circ C_n$ by $\theta \circ \alpha$, where α is a normal unital injective endomorphism of P with $\alpha(P) = F' \cap P$ for a unital full matrix algebra $F \subset M$. Since $\|\theta|_{F' \cap P}\| = \|\theta \circ \alpha\|$, for such θ and every $\varepsilon > 0$ we have a unital full matrix algebra $F_{\theta,\varepsilon} \subset M$ such that $\|\theta|_{F'_{\theta,\varepsilon} \cap P}\| < \varepsilon$ (Proposition 6.4). This construction replaces the passage used by Marrakchi [Mar26, Theorem B] through an approximately finite-dimensional subalgebra with the same relative commutant, where the ergodicity of the relative bicentralizer flow [Mar26, Theorem C] was used.

Once the matrix algebras $F_{\theta,\varepsilon}$ are constructed, the remaining argument follows the well-established route of Popa [Pop81]. We pass from $F_{\theta,\varepsilon}$ to finite-dimensional abelian subalgebras $D \subset M$ with $\|\theta|_{D' \cap P}\| < \varepsilon$. Along a norm dense sequence (θ_j) in $\{\theta \in P_* \mid \theta|_M = 0\}$, we then construct finite-dimensional abelian subalgebras $D_1 \subset D_2 \subset \cdots$ of M such that $A_0 = \bigvee_n D_n$ satisfies $\theta_j|_{A_0' \cap P} = 0$ for every j. Hence $A_0' \cap P \subset M$, and every maximal abelian subalgebra A of M containing A_0 satisfies $A' \cap P = A$.

Independent and concurrent work. Our solution to the relative bicentralizer conjecture (Theorem A and Corollary C) was obtained in parallel with and independently of the concurrent work [Ope26], using a completely different approach. Their proof builds on [HM26], while ours is based on the Poisson boundary techniques developed here.

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AI-assisted research methodology. It is well known that AI can help solve a given problem by identifying techniques whose relevance had previously gone unnoticed. Our solution to the relative bicentralizer conjecture, however, illustrates the reverse process: starting from a given body of techniques, AI helped identify a solvable problem whose connection with these techniques the author had not previously noticed.

The underlying Poisson boundary theory in Section 3 and the weak Dixmier arguments through Section 4 were developed by the author between February 2025 and May 2026, benefiting from helpful discussions with Cyril Houdayer, Amine Marrakchi, and Soham Chakraborty, and with limited assistance from the $20

version of Gemini 3.1 Pro during April 2026 for language editing only (in the current version). GPT-6 Astra was subsequently used to help identify and repair gaps in this draft.

In September 2026, the author began using GPT-6 Astra to investigate whether every type II_1 factor admits a type III_1 Poisson boundary, a question that had interested the author since the early stages of his doctoral studies. This investigation did not resolve the original question. The author then asked the model to consider the weaker conclusion that every type II_1 factor admits an irreducible embedding into the hyperfinite type III_1 factor. This led to a construction using Poisson boundaries and crossed products. To extend this construction to arbitrary continuous factors, the author supplied the model with his existing manuscript on Poisson boundaries of type III factors. The model then helped develop the corresponding embedding arguments. To keep this paper concise and focused, these irreducible embedding results are not included in the present version. We refer interested readers to a preliminary version of the paper [hex26].

Having recognized Poisson boundaries as an extremely powerful tool, the author then asked GPT-6 Astra on 26 September 2026 which other open problems might be solved using the existing Poisson boundary theory. The model suggested three problems that were still open at the time, before OpenAI's public release on 6 October 2026:

(1) the relative bicentralizer conjecture;
(2) the free group factor problem;
(3) the vanishing cohomology conjecture of Kadison–Ringrose.

The author chose the relative bicentralizer conjecture for further study. The investigation proceeded smoothly and yielded a proof on 27 September 2026, since most of the tools needed for the proof had already been developed in the Poisson boundary manuscript. The author later also asked GPT-6 Astra to investigate the other two problems, but neither attempt was successful. In particular, for the free group factor problem, the author had steered the investigation in the wrong direction by seeking to prove nonisomorphism.

Following the successful treatment of the weak Dixmier property for inclusions with expectation and the relative bicentralizer conjecture, the author recognized the parallels between these two problems: both are formulated for general von Neumann algebras, with classical type II_1 cases but extremely difficult type III_1 cases. The author himself then identified Kadison's problem on MASA as another problem in a similar situation. On 9 October 2026, with AI assistance, the author obtained a substantially simplified proof of the solution to Kadison's problem using inner ucp maps associated with Poisson boundaries.

Lean was also used to check a formalization of a short note consisting of the preliminaries in Subsection 2.3, Example 3.10, Theorem 4.1, and the whole of Section 5. These components form the main line of the proof of Theorem A. Results from the published literature were taken as explicit assumptions.

The author is responsible for the mathematical claims, citations, and final presentation.

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2. Preliminaries

2.1 The standard form of von Neumann algebras. Let M be a countably decomposable von Neumann algebra. Following [Haa75], the standard form of M is a quadruple (M, H, J, P) such that H is a Hilbert space with $M \subset B(H)$; $J : H \to H$ is an anti-unitary with $J^2 = 1$; $P \subset H$ is a self-dual cone, i.e., its dual cone $P^{\mathrm{o}} := \{\xi \in H \mid \langle \xi, \eta \rangle \geq 0,\ \forall \eta \in P\}$ satisfies $P^{\mathrm{o}} = P$; and the following conditions hold:

(1) $JMJ = M' \cap B(H)$;
(2) $JcJ = c^*$ for any $c \in \mathcal{Z}(M)$, the center of M;
(3) $J\xi = \xi$ for any $\xi \in P$;
(4) $a(JaJ)P \subset P$ for any $a \in M$.

The standard form is unique up to unitary equivalence. Assume that $(M, \tilde{H}, \tilde{J}, \tilde{P})$ is another quadruple satisfying the above conditions. Then there exists a unique unitary $u : H \to \tilde{H}$ such that

(a) $\pi_{\tilde{H}}(a) = u\pi_H(a)u^*$ $(a \in M)$, where $\pi_H : M \to B(H)$ and $\pi_{\tilde{H}} : M \to B(\tilde{H})$ are the representations of M on H and $\tilde{H}$, respectively;
(b) $\tilde{J} = uJu^*$;
(c) $\tilde{P} = uP$.

Therefore, the standard form of M depends only on M itself, and we denote it by $(M, L^2(M), J_M, L^2(M)_+)$. In particular, for any normal faithful state φ on M, we can take

$$(M, L^2(M), J_M, L^2(M)_+) = (M, L^2(M, \varphi), J_\varphi, L^2(M, \varphi)_+),$$

where J_φ is the modular conjugation operator determined by the polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ of the Tomita operator, and $L^2(M, \varphi)_+$ is the closure of $\{a(J_\varphi a J_\varphi)\xi_\varphi \mid a \in M\}$ [Haa75, Lemma 2.9].

By [Haa75, Lemma 2.10], any normal positive functional $\omega \in M_*^+$ admits a unique representing vector $\xi_\omega \in L^2(M)_+$, i.e., $\omega = \langle \cdot\, \xi_\omega, \xi_\omega \rangle$. Moreover, $\xi \mapsto \omega_\xi = \langle \cdot\, \xi, \xi \rangle$ is a homeomorphism between $L^2(M)_+$ and M_*^+.

2.2 Centralizers and bicentralizers. Let M be a countably decomposable von Neumann algebra, and let ϕ be a normal faithful state on M. The **centralizer** of ϕ is

$$M_\phi = \{a \in M \mid \phi(ax) = \phi(xa),\ \forall x \in M\} = \{a \in M \mid \sigma_t^\phi(a) = a,\ \forall t \in \mathbb{R}\}.$$

The **asymptotic centralizer** of ϕ is

$$\mathrm{AC}(M, \phi) = \{(a_n) \in \ell^\infty(\mathbb{N}, M) \mid \|a_n\phi - \phi a_n\|_{M_*} \to 0\},$$

where $(a\phi)(x) = \phi(xa)$ and $(\phi a)(x) = \phi(ax)$ for $a, x \in M$. For a W*-inclusion $M \subset P$, the **relative bicentralizer** is defined by

$$\mathrm{B}(M \subset P, \phi) = \{x \in P \mid [x, a_n] \xrightarrow{\mathrm{s}} 0, \quad \forall (a_n) \in \mathrm{AC}(M, \phi)\},$$

where s denotes strong operator convergence; see [AHHM20, Definition 3.1]. For $P = M$, we have the **bicentralizer** of ϕ:

$$\mathrm{B}(M, \phi) = \mathrm{B}(M \subset M, \phi).$$

Marrakchi's **relative bicentralizer conjecture** [Mar25, Conjecture C] asserts that, for every inclusion $M \subset P$ of countably decomposable von Neumann algebras with a normal faithful conditional expectation $E : P \to M$ and every normal faithful state ϕ on M,

$$\mathrm{B}(M \subset c(P), \phi) = \{\phi^{it} \mid t \in \mathbb{R}\}'' \vee (M' \cap c(P)). \tag{2.1}$$

Here we use the compatible realizations $c_\phi(M) \subset c_{\phi \circ E}(P) = c(P)$, and ϕ^{it} denotes the canonical implementing unitary in $c_\phi(M)$. We refer to Subsection 2.3 for the definition of the continuous core.

2.3 Continuous cores and dual weights. For background on modular theory and continuous cores, we refer to [Tak03].

Let P be a countably decomposable von Neumann algebra, and let ρ be a normal faithful state on P. Let $\xi_\rho \in L^2(P)_+$ be the vector representing ρ. The Tomita operator S_ρ is the closure on $L^2(P)$ of the densely defined antilinear map $x\xi_\rho \mapsto x^*\xi_\rho$ $(x \in P)$. Define the modular operator by $\Delta_\rho = S_\rho^* S_\rho$. The modular automorphism group $\sigma^\rho : \mathbb{R} \to \mathrm{Aut}(P)$ is given by

$$\sigma_t^\rho(x) = \Delta_\rho^{it} x \Delta_\rho^{-it} \qquad (x \in P,\ t \in \mathbb{R}).$$

We say that $x \in P$ is **entire analytic** for σ^ρ if $t \mapsto \sigma_t^\rho(x)$ extends to a map $z \mapsto \sigma_z^\rho(x)$ from $\mathbb{C}$ to P such that $z \mapsto \omega(\sigma_z^\rho(x))$ is entire for every $\omega \in P_*$.

Define the continuous core of P by

$$\mathcal{A} = c_\rho(P) = P \rtimes_{\sigma^\rho} \mathbb{R}.$$

We identify P with its canonical image in $\mathcal{A}$ and denote by $(\lambda_t)_{t\in\mathbb{R}}$ the canonical implementing unitaries, satisfying $\lambda_t x \lambda_t^* = \sigma_t^\rho(x)$ $(x \in P,\, t \in \mathbb{R})$. The dual action $\widehat{\sigma} : \mathbb{R} \curvearrowright \mathcal{A}$ is determined by

$$\widehat{\sigma}_r(x) = x, \qquad \widehat{\sigma}_r(\lambda_t) = e^{-irt}\lambda_t \qquad (x \in P,\ r, t \in \mathbb{R}).$$

The canonical operator-valued weight $T : \mathcal{A}_+ \to \widehat{P}_+$ is defined by

$$T(x) = \int_{\mathbb{R}} \widehat{\sigma}_r(x)\, \mathrm{d}r \qquad (x \in \mathcal{A}_+).$$

The dual weight of ρ is $\varphi_{\mathcal{A}} = \rho \circ T$. Write $\tau_{\mathcal{A}}$ for the canonical trace on $\mathcal{A}$, and let

$$h = \frac{\mathrm{d}\varphi_{\mathcal{A}}}{\mathrm{d}\tau_{\mathcal{A}}}, \qquad H = \log h, \qquad \lambda_t = h^{it} = e^{itH}.$$

Products of measurable operators are understood as closed products.

We next prove two lemmas that will be used in Section 5.

Lemma 2.1. *For every positive integrable function g on $\mathbb{R}$,*

$$\varphi_{\mathcal{A}}(g(H)) = \rho(T(g(H))) = \int_{\mathbb{R}} g(s)\, \mathrm{d}s. \tag{2.2}$$

In particular, we have

$$\tau_{\mathcal{A}}(\mathbb{1}_{(s,\infty)}(h)) = \frac{1}{s} \qquad (s > 0). \tag{2.3}$$

Proof. For $r, t \in \mathbb{R}$, by functional calculus we have

$$e^{it\widehat{\sigma}_r(H)} = \widehat{\sigma}_r(e^{itH}) = \widehat{\sigma}_r(\lambda_t) = e^{-irt} e^{itH} = e^{it(H-r)}.$$

By uniqueness of the self-adjoint generator of a strongly continuous unitary group,

$$\widehat{\sigma}_r(H) = H - r. \tag{2.4}$$

Let $\mathcal{E}_H$ be the spectral measure of H. For a positive integrable function g, by (2.4) we have

$$T(g(H)) = \int_{\mathbb{R}} g(H-r)\, \mathrm{d}r = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(s-r)\, \mathrm{d}r \right) \mathrm{d}\mathcal{E}_H(s) = \left(\int_{\mathbb{R}} g(s)\, \mathrm{d}s \right) 1. \tag{2.5}$$

Since $\varphi_{\mathcal{A}} = \rho \circ T$ and $\rho(1) = 1$, by (2.5) we have (2.2).

For $s > 0$, the definition of h and (2.2) give

$$\tau_{\mathcal{A}}(\mathbb{1}_{(s,\infty)}(h)) = \varphi_{\mathcal{A}}\big(h^{-1}\mathbb{1}_{(s,\infty)}(h)\big) = \varphi_{\mathcal{A}}\big(e^{-H}\mathbb{1}_{(\log s,\infty)}(H)\big) = \int_{\log s}^{\infty} e^{-t}\, \mathrm{d}t = \frac{1}{s}.$$

See also [Haa79a, Theorem 1.2(2)]. In particular, h is $\tau_{\mathcal{A}}$-measurable. □

Lemma 2.2. *For every $0 \neq d \in \mathcal{A}_+$, there exists $x \in \mathcal{A}_+$ such that*

$$0 \neq x \leq d, \qquad \varphi_{\mathcal{A}}(x) < \infty.$$

Proof. Take $\delta > 0$ such that $\mathbb{1}_{[\delta,\infty)}(d) \neq 0$. By semifiniteness of $\tau_{\mathcal{A}}$, there is a nonzero projection

$$p \leq \mathbb{1}_{[\delta,\infty)}(d), \qquad 0 < \tau_{\mathcal{A}}(p) < \infty.$$

By (2.3), take $s > 0$ such that $\tau_{\mathcal{A}}(\mathbb{1}_{(s,\infty)}(h)) < \tau_{\mathcal{A}}(p)$, and let $e = \mathbb{1}_{[0,s]}(h)$ and $q = p \wedge e$. Let $(1-e)p = v|(1-e)p|$ be the polar decomposition. The kernel of $(1-e)p$ inside the range of p is the range of $q = p \wedge e$, and the range of $(1-e)p$ is contained in the range of $1-e$. Hence

$$v^*v = p - q, \qquad vv^* \leq 1 - e.$$

Hence $p - q \sim vv^* \leq 1 - e$, so $p - q \precsim 1 - e$ and

$$\tau_{\mathcal{A}}(q) \geq \tau_{\mathcal{A}}(p) - \tau_{\mathcal{A}}(1-e) > 0.$$

Since $q \leq e$, we have

$$\varphi_{\mathcal{A}}(q) = \tau_{\mathcal{A}}(qhq) = \tau_{\mathcal{A}}(q(ehe)q) \leq s\tau_{\mathcal{A}}(q) < \infty.$$

Hence $x = \delta q$ satisfies

$$0 \neq x = \delta q \leq \delta p \leq d, \qquad \varphi_{\mathcal{A}}(x) = \delta\varphi_{\mathcal{A}}(q) < \infty.$$

□

3. Poisson boundaries of general von Neumann algebras

In this section, we develop the Poisson boundary theory for general von Neumann algebras, following the framework of Das–Peterson [DP22] for tracial von Neumann algebras. The proofs of ergodicity, double ergodicity, and rigidity are in the same spirit as those of [DP22]. However, the coefficient families (z_k) and (y_k) for $\mathcal{P}_\varphi$ and $\mathcal{P}_\varphi^o$ need not coincide in the nontracial case, whereas we can take $y_k = z_k$ if $\varphi|_M$ is tracial (Remark 3.7). Thus, in the general setting, we cannot recover all the results obtained in [DP22] for finite von Neumann algebras.

Throughout this section, we fix a von Neumann algebra M with separable predual.

Definition 3.1. *Let $M \subset \mathcal{A}$ be a W^*-inclusion. A normal state φ on $\mathcal{A}$ is called an M-**hyperstate** if its restriction $\varphi|_M$ to M is faithful. Given a fixed normal faithful state ϕ on M, we say that φ is an (M,ϕ)-**hyperstate** if $\varphi|_M = \phi$.*

The framework of Das–Peterson [DP22] concerns (M,τ)-hyperstates for tracial von Neumann algebras (M,τ), whereas we work with general M-hyperstates without fixing a normal faithful state ϕ on M in advance.

The following proposition associates with each normal M-hyperstate on $B(L^2(M))$ an M-bimodular ucp map, which we use to define the Poisson boundary.

Proposition 3.2. *Let $\varphi \in B(L^2(M))_*$ be a normal M-hyperstate. Set $\phi = \varphi|_M$, and let $\xi_\phi \in L^2(M)_+$ be the canonical cyclic and separating vector implementing ϕ. Then there exists a unique normal M-bimodular ucp map $\mathcal{P}_\varphi : B(L^2(M)) \to B(L^2(M))$ such that for all $T \in B(L^2(M))$,*

$$\varphi(T) = \langle \mathcal{P}_\varphi(T)\xi_\phi, \xi_\phi \rangle.$$

Moreover, there exists a (not necessarily unique) sequence $(z_k)_{k=1}^\infty \subset M$ such that

$$\mathcal{P}_\varphi(T) = \sum_{k=1}^\infty (Jz_k^*J)T(Jz_kJ)$$

for all $T \in B(L^2(M))$, where J is the modular conjugation operator on $L^2(M)$.

Proof. We first prove the existence and construct the explicit form of $\mathcal{P}_\varphi$.

Since φ is normal, there exists a sequence $(\xi_k) \subset L^2(M)$ such that

$$\varphi(T) = \sum_k \langle T\xi_k, \xi_k\rangle \qquad (T \in B(L^2(M))).$$

For any $x \in M$, since $\varphi|_M = \phi$, we have:

$$\|x\xi_\phi\|^2 = \phi(x^*x) = \varphi(x^*x) = \sum_k \|x\xi_k\|^2.$$

Hence

$$\|x\xi_n\|^2 \le \|x\xi_\phi\|^2 \qquad (x \in M,\ n \ge 1). \tag{3.1}$$

Define the operator $R_n : M\xi_\phi \to L^2(M)$ by $R_n(x\xi_\phi) = x\xi_n$. By (3.1), $\|R_n(x\xi_\phi)\| \le \|x\xi_\phi\|$, so R_n extends to a contraction on $L^2(M)$. Furthermore, for any $x, y \in M$, we have $R_n(yx\xi_\phi) = yx\xi_n = yR_n(x\xi_\phi)$. Hence $R_n \in M'$. By standard Tomita–Takesaki theory, $M' = JMJ$, so there exists $z_n \in M$ such that $R_n = Jz_nJ$.

Now, for any $x, y \in M$ and $T \in B(L^2(M))$, we calculate:

$$\begin{aligned}\varphi(y^*Tx) &= \sum_k \langle Tx\xi_k, y\xi_k\rangle = \sum_k \langle TR_k(x\xi_\phi), R_k(y\xi_\phi)\rangle \\ &= \sum_k \langle TJz_kJx\xi_\phi, Jz_kJy\xi_\phi\rangle \\ &= \left\langle \left(\sum_k Jz_k^*J\,T\,Jz_kJ\right) x\xi_\phi, y\xi_\phi \right\rangle.\end{aligned} \tag{3.2}$$

Let $\mathcal{P}_\varphi(T) = \sum_k (Jz_k^*J)T(Jz_kJ)$. By (3.2), $\varphi(y^*Tx) = \langle \mathcal{P}_\varphi(T)x\xi_\phi, y\xi_\phi\rangle$. Taking $x = y = 1$, we immediately obtain $\varphi(T) = \langle \mathcal{P}_\varphi(T)\xi_\phi, \xi_\phi\rangle$.

Since $M\xi_\phi$ is dense in $L^2(M)$, the identity

$$\langle \mathcal{P}_\varphi(T)x\xi_\phi, y\xi_\phi\rangle = \varphi(y^*Tx)$$

uniquely determines $\mathcal{P}_\varphi(T)$ for every T. Hence $\mathcal{P}_\varphi$ is unique.

Taking $T = 1$ in (3.2) and using $\varphi|_M = \phi$, we have $\sum_k z_k^*z_k = 1$. Since $Jz_kJ \in M'$, $\mathcal{P}_\varphi$ is a normal M-bimodular ucp map. □

Definition 3.3. *Let φ be a normal M-hyperstate on $B(L^2(M))$. Let* $\mathrm{Har}(\mathcal{P}_\varphi) = \{T \in B(L^2(M)) \mid \mathcal{P}_\varphi(T) = T\}$ *denote the space of* ***φ-harmonic operators****. The* ***Poisson boundary*** $(\mathcal{B}_\varphi, \zeta)$ *is defined as a von Neumann algebra that is isomorphic to* $\mathrm{Har}(\mathcal{P}_\varphi)$ *as an operator system, together with the canonical hyperstate* $\zeta := \varphi|_{\mathrm{Har}(\mathcal{P}_\varphi)}$.

The next proposition shows that the Poisson boundary is well defined and amenable.

Proposition 3.4. *Let φ be a normal M-hyperstate on $B(L^2(M))$. The Poisson boundary $\mathcal{B}_\varphi$ always exists, is unique, and is amenable.*

Proof. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We define a conditional expectation $E : B(L^2(M)) \to \mathrm{Har}(\mathcal{P}_\varphi)$ by

$$E(T) = \operatorname*{wo-lim}_{n\to\omega} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}_\varphi^k(T) \quad \text{for all } T \in B(L^2(M)).$$

By the Choi–Effros theorem [CE77], equipping the operator system $\mathrm{Har}(\mathcal{P}_\varphi)$ with the **Choi–Effros product** defined by

$$x \circ y = E(xy) \quad \text{for } x, y \in \mathrm{Har}(\mathcal{P}_\varphi)$$

makes $(\mathrm{Har}(\mathcal{P}_\varphi), \circ)$ a C*-algebra with its original norm. The harmonic operator system is ultraweakly closed, hence this C*-algebra is a von Neumann algebra, with its canonical ultraweak topology agreeing with the inherited one; see also [Izu02]. The inclusion $M \subset \mathrm{Har}(\mathcal{P}_\varphi)$ is multiplicative for this product, since E is M-bimodular.

The uniqueness of $\mathcal{B}_\varphi$ follows from the fact that any two von Neumann algebras that are completely isometrically isomorphic as operator systems are necessarily *-isomorphic.

Finally, since $\mathcal{B}_\varphi \cong \mathrm{Har}(\mathcal{P}_\varphi)$ as injective operator systems, $\mathcal{B}_\varphi$ is amenable as a von Neumann algebra. □

The following symmetry condition allows us to represent φ also by an M-inner ucp map.

Definition 3.5. *We say that a normal M-hyperstate $\varphi \in B(L^2(M))_*$ is **regular** if $\varphi(Jz^*J) = \varphi(z)$ for all $z \in M$.*

For regular hyperstates, Proposition 3.2 has the following counterpart, with M and JMJ interchanged.

Proposition 3.6. *Let $\varphi \in B(L^2(M))_*$ be a normal regular M-hyperstate, and let $\phi = \varphi|_M$. Then there exists a sequence $(y_k)_{k=1}^\infty \subset M$ with $\sum_k y_k y_k^* = 1$ such that*

$$\mathcal{P}_\varphi^o(T) = \sum_k y_k T y_k^* \qquad (T \in B(L^2(M)))$$

defines a normal JMJ-bimodular ucp map satisfying

$$\varphi(T) = \langle \mathcal{P}_\varphi^o(T)\xi_\phi, \xi_\phi\rangle \qquad (T \in B(L^2(M))).$$

Proof. Let $\phi^o = \varphi|_{JMJ}$. Since φ is regular and $J\xi_\phi = \xi_\phi$, we have

$$\langle Jz^*J\xi_\phi, \xi_\phi\rangle = \phi(z) = \phi^o(Jz^*J) \qquad (z \in M).$$

Hence ξ_ϕ is the canonical cyclic and separating vector implementing ϕ^o. By applying Proposition 3.2 to JMJ, we have the required map and coefficients. □

Remark 3.7. *In general, (y_k) need not coincide with (z_k) from Proposition 3.2. If ϕ is tracial, we can take $y_k = z_k$. This is the main case studied in the Das–Peterson framework [DP22].*

Definition 3.8. *Let φ be a normal regular M-hyperstate. We say that φ is **strongly generating** if there exist sequences $(z_k), (y_k) \subset M$ with*

$$\overline{\mathbb{C}[z_k]}^{\,\mathrm{w}} = \overline{\mathbb{C}[y_k]}^{\,\mathrm{w}} = M$$

such that $\mathcal{P}_\varphi$ and $\mathcal{P}_\varphi^o$ have the standard forms

$$\mathcal{P}_\varphi(T) = \sum_k (Jz_k^*J)T(Jz_kJ), \qquad \mathcal{P}_\varphi^o(T) = \sum_k y_k T y_k^* \qquad (T \in B(L^2(M))).$$

*Here $\mathbb{C}[z_k]$ and $\mathbb{C}[y_k]$ are the unital subalgebras (rather than $*$-algebras) generated by (z_k) and (y_k), respectively. We abbreviate **normal regular strongly generating** hyperstate as **nrsg** hyperstate. In the nontracial setting we assume both density conditions; we do not assert that one follows from the other.*

We next prove some basic properties of nrsg hyperstates.

Proposition 3.9. *Let φ be an nrsg M-hyperstate. For $k \geq 0$, we define the **k-th convolution** φ^{*k} by $\varphi^{*k}(T) = \langle \mathcal{P}_\varphi^k(T)\xi_\phi, \xi_\phi\rangle$. Then the normal state*

$$\overline{\varphi} = \sum_{n=0}^\infty 2^{-n-1}\varphi^{*n}$$

is faithful on $B(L^2(M))$. In particular, $\zeta = \varphi|_{\mathrm{Har}(\mathcal{P}_\varphi)}$ is a normal faithful state on the Poisson boundary $\mathcal{B}_\varphi$. Moreover, M is the largest von Neumann subalgebra of $\mathrm{Har}(\mathcal{P}_\varphi)$ for the original operator product on $B(L^2(M))$.

Proof. Write $\phi = \varphi|_M$. Let $T \in B(L^2(M))_+$ satisfy $\overline{\varphi}(T) = 0$. Since all the coefficients are positive, $\varphi^{*n}(T) = 0$ for every $n \geq 0$. Expanding the iterates of $\mathcal{P}_\varphi$ and using positivity, we have

$$\langle T(JwJ)\xi_\phi, (JwJ)\xi_\phi\rangle = 0$$

for every word w in (z_k), including the empty word. The weak density of $\mathbb{C}[z_k]$ implies that the vectors $(JwJ)\xi_\phi$ have dense linear span in $L^2(M)$. Hence $T = 0$, proving that $\overline{\varphi}$ is faithful. Since $\varphi^{*n}(T) = \varphi(T)$ for $T \in \mathrm{Har}(\mathcal{P}_\varphi)$ and $n \geq 0$, we have $\zeta = \overline{\varphi}|_{\mathrm{Har}(\mathcal{P}_\varphi)}$. Hence ζ is normal and faithful on $\mathcal{B}_\varphi$.

Let $\mathcal{N} \subset \mathrm{Har}(\mathcal{P}_\varphi)$ be a von Neumann subalgebra for the original operator product, and let $v_k = Jz_k^*J$. For $T \in \mathcal{N}$, the operators T, T^*, TT^* are fixed by $\mathcal{P}_\varphi$, so

$$\sum_k (v_kT - Tv_k)(v_kT - Tv_k)^* = \mathcal{P}_\varphi(TT^*) - \mathcal{P}_\varphi(T)T^* - T\mathcal{P}_\varphi(T^*) + TT^* = 0. \tag{3.3}$$

Hence $[T, v_k] = 0$ for every k. By applying (3.3) to T^*, we also have $[T, v_k^*] = 0$ for every k. By the weak density of $\mathbb{C}[\{z_k\}_{k\geq 1}]$, we have $T \in (JMJ)' = M$. Since M itself is contained in the harmonic space, it is the largest such subalgebra. □

In the following example, we construct an nrsg hyperstate for any pair (M, ϕ). The additional conditions (3.4) and (3.5) will be used in Section 5. The generating inner ucp map will also be used in Section 6.

Example 3.10. Let ϕ be a normal faithful state on M, and write $\xi = \xi_\phi$. There exists a faithful nrsg (M, ϕ)-hyperstate φ on $B(L^2(M))$ with inner ucp coefficients $(y_k) \subset M$ for $\mathcal{P}_\varphi^o$ having the following properties:

(i) The series $\sum_k y_k y_k^*$ converges strongly to 1, and the normal ucp map

$$Q(a) = \sum_k y_k a y_k^* \qquad (a \in M),$$

satisfies $\phi \circ Q = \phi$.

(ii) The hyperstate is given by

$$\varphi(T) = \sum_k \langle Ty_k^*\xi, y_k^*\xi\rangle \qquad (T \in B(L^2(M))),$$

and satisfies

$$\begin{gathered} \varphi(JT^*J) = \varphi(T) \quad (T \in B(L^2(M))), \\ \varphi(\Delta_\phi) = \varphi(\Delta_\phi^{-1}) < \infty, \qquad \varphi((\log \Delta_\phi)^2) < \infty. \end{gathered} \tag{3.4}$$

(iii) Each y_k^* admits a bounded modular continuation to $i/2$, and the map

$$y_k \longmapsto \sigma_{i/2}^\phi(y_k^*) \tag{3.5}$$

permutes the family (y_k).

(iv) If ϕ is tracial, we may further take

$$y_k = \lambda_k u_k, \qquad \lambda_k > 0, \quad u_k \in \mathcal{U}(M), \quad \sum_k \lambda_k^2 = 1.$$

Expectations of positive unbounded operators are defined by monotone spectral truncation.

Proof. Take elements $a_n \in M$, entire analytic for σ^ϕ, whose linear span is weakly dense in M and such that $\{a_n\xi \mid n \geq 1\}$ has dense linear span in $L^2(M)$. One obtains such a sequence by applying the Gaussian regularizations

$$a \longmapsto (r/\pi)^{1/2} \int_{\mathbb{R}} e^{-rt^2} \sigma_t^\phi(a)\, \mathrm{d}t \qquad (r \in \mathbb{N}),$$

to a countable strongly dense subset of the unit ball; see [Tak03, Lemma VIII.2.3 and its proof, pp. 98–99].

Let $b_n = \sigma^{\phi}_{-i/2}(a_n^*)$. The standard-form identities give

$$Ja_n\xi = b_n\xi, \qquad a_n\xi = Jb_nJ\xi.$$

Take $c_n > 0$ such that

$$\sum_n c_n(\|a_n\|_\infty^2 + \|b_n\|_\infty^2) < \infty.$$

Rescaling the coefficients, we may assume that

$$C = \sum_n c_n(a_n^*a_n + b_n^*b_n), \qquad \|C\|_\infty = 1, \tag{3.6}$$

where the series converges in norm. In particular, $0 \le C \le 1$. For $\eta \in L^2(M)$, write $\omega_\eta(T) = \langle T\eta, \eta\rangle$, and set

$$\psi = \sum_n c_n(\omega_{a_n\xi} + \omega_{Ja_n\xi})\Big|_M.$$

Since $Ja_nJ, Jb_nJ \in M'$, for $x \in M_+$ we have

$$\psi(x) = \langle JCJx^{1/2}\xi, x^{1/2}\xi\rangle \le \phi(x).$$

By [Haa75, Lemma 2.10(1)], let $\eta \in L^2(M)_+$ with $\phi - \psi = \langle\,\cdot\,\eta, \eta\rangle$, and define

$$\varphi = \sum_n c_n(\omega_{a_n\xi} + \omega_{Ja_n\xi}) + \omega_\eta.$$

Then $\varphi|_M = \phi$. Since $J\eta = \eta$ and $\omega_v(JT^*J) = \omega_{Jv}(T)$, the state φ is J-symmetric and hence regular. The totality of $(a_n\xi)$ makes φ faithful on $B(L^2(M))$.

By the same argument as in the proof of Proposition 3.2, the inequality $\phi - \psi \le \phi$ implies that the map $x\xi \mapsto x\eta$ extends to a contraction $JwJ \in M'$ with $w \in M$. Then

$$w\xi = \eta = JwJ\xi, \qquad \|w\|_\infty \le 1.$$

The two coefficient families associated with φ are

$$(z_k) = \big(\sqrt{c_n}b_n, \sqrt{c_n}a_n\big)_{n\ge1} \cup \{w\}, \qquad (y_k) = \big(\sqrt{c_n}a_n^*, \sqrt{c_n}b_n^*\big)_{n\ge1} \cup \{w^*\}.$$

The two representations are

$$\varphi(T) = \sum_k \langle TJz_kJ\xi, Jz_kJ\xi\rangle = \sum_k \langle Ty_k^*\xi, y_k^*\xi\rangle \qquad (T \in B(L^2(M))). \tag{3.7}$$

Since $\varphi|_M = \phi$ and $\varphi(Ja^*J) = \phi(a)$ for $a \in M$,

$$\sum_k z_k^*z_k = 1, \qquad \sum_k y_ky_k^* = 1$$

strongly. Moreover,

$$a_n \in \mathbb{C}[z_k], \qquad a_n^* \in \mathbb{C}[y_k], \qquad \overline{\mathbb{C}[z_k]}^{\,\mathrm{w}} = \overline{\mathbb{C}[y_k]}^{\,\mathrm{w}} = M,$$

so φ is nrsg. By (3.7),

$$\phi(Q(a)) = \sum_k \langle ay_k^*\xi, y_k^*\xi\rangle = \varphi(a) = \phi(a) \qquad (a \in M).$$

This proves (i) and the first assertion of (ii).

To prove (iii), note that

$$\sigma^{\phi}_{i/2}(a_n) = b_n^*, \qquad \sigma^{\phi}_{i/2}(b_n) = a_n^*.$$

It remains to treat w^*. For $r > 0$, define

$$w_r = (r/\pi)^{1/2}\int_{\mathbb{R}} e^{-rt^2}\sigma_t^{\phi}(w)\,\mathrm{d}t.$$

The identity $Jw\xi = w\xi$ and the commutation relation $J\Delta_\phi^{it} = \Delta_\phi^{it}J$ give $Jw_r\xi = w_r\xi$. The Tomita identity and the separating property of ξ imply

$$\sigma^\phi_{-i/2}(w_r^*) = w_r, \qquad \sigma^\phi_{i/2}(w_r) = w_r^*.$$

The boundary values on the half-strip are bounded by $\|w\|_\infty$. The three-lines theorem and the ultraweak closedness of the analytic generator give $\sigma^\phi_{i/2}(w) = w^*$; see [VD14, p. 5]. Hence (3.5) exchanges the paired coefficients and fixes w^*.

We next check (3.4). The vectors $a_n\xi$, $b_n\xi$ and $w\xi$ belong to the domain of the Tomita operator $S_\phi = J\Delta_\phi^{1/2}$. Hence

$$\varphi(\Delta_\phi) = \sum_n c_n\big(\|a_n^*\xi\|^2 + \|b_n^*\xi\|^2\big) + \|w^*\xi\|^2 \le \sum_n c_n(\|a_n\|_\infty^2 + \|b_n\|_\infty^2) + 1 < \infty.$$

Here the equality follows by monotone spectral truncation. Since $J\Delta_\phi J = \Delta_\phi^{-1}$, by J-symmetry we have $\varphi(\Delta_\phi^{-1}) = \varphi(\Delta_\phi)$. The inequality $(\log t)^2 \le t + t^{-1} - 2$ $(t > 0)$ proves $\varphi((\log\Delta_\phi)^2) < \infty$. In particular,

$$\varphi(|\log\Delta_\phi|) < \infty, \qquad \varphi(\log\Delta_\phi) = 0.$$

Indeed, $J(\log\Delta_\phi)J = -\log\Delta_\phi$, so by J-symmetry we have $\varphi(\log\Delta_\phi) = -\varphi(\log\Delta_\phi)$.

(iv): Assume that ϕ is tracial. The modular action σ^ϕ is trivial, so every element of M is entire analytic. Since M is linearly spanned by its unitaries, we may take $a_n \in \mathcal{U}(M)$ in the preceding construction with the same density properties. For this choice, $b_n = a_n^*$, and hence

$$C = 2\left(\sum_n c_n\right)1.$$

By (3.6), $2\sum_n c_n = 1$. Since each $\omega_{a_n\xi}$ restricts to ϕ on M, we have $\psi = \phi$. Hence $\eta = 0$ and $w = 0$. Omitting the zero coefficient, the inner ucp coefficients of Q are

$$y_{2n-1} = \sqrt{c_n}a_n^*, \qquad y_{2n} = \sqrt{c_n}a_n \qquad (n \ge 1).$$

This has the required form $y_k = \lambda_k u_k$, with $\sum_k \lambda_k^2 = 2\sum_n c_n = 1$, which finishes the proof. □

Remark 3.11. *The construction in Example 3.10 is natural. Although regularity only concerns the restrictions to M and JMJ, it is easier to impose J-symmetry on all of $B(L^2(M))$. Indeed, for any normal faithful state φ_0 on $B(L^2(M))$, the state $\varphi(T) = \frac{1}{2}\big(\varphi_0(T) + \varphi_0(JT^*J)\big)$ $(T \in B(L^2(M)))$ is a normal faithful J-symmetric M-hyperstate, and hence regular.*

Let $\xi = \xi_\phi$. To construct a strongly generating symmetric hyperstate, we want the coefficient families (z_k) and (y_k) in (3.7) to generate M and satisfy

$$\sum_k \langle Ty_k^*\xi, y_k^*\xi\rangle = \sum_k \langle TJy_k^*\xi, Jy_k^*\xi\rangle \qquad (T \in B(L^2(M))).$$

This equality holds if $(Jy_k^*\xi)$ is a permutation of $(y_k^*\xi)$. For analytic coefficients, we have

$$Jy_k^*\xi = \sigma^\phi_{-i/2}(y_k)\xi = \big(\sigma^\phi_{i/2}(y_k^*)\big)^*\xi.$$

Hence we ask that $y_k \mapsto \sigma^\phi_{i/2}(y_k^*)$ permute the coefficient family, as in (3.5). On the z_k side, the corresponding identity is $Jz_k\xi = \sigma^\phi_{-i/2}(z_k^*)\xi$. We therefore take an analytic family (a_n) with weakly dense linear span and pair each a_n with $b_n = \sigma^\phi_{-i/2}(a_n^*)$, using the same weight c_n for the two vector states $\omega_{a_n\xi}$ and $\omega_{b_n\xi}$.

For a prescribed restriction $\varphi|_M = \phi$, we take these weights small enough that the restriction ψ of the paired sum satisfies $\psi \le \phi$. Let $\eta \in L^2(M)_+$ represent $\phi - \psi$. Since $J\eta = \eta$, the sum of the paired vector states and ω_η is still J-symmetric and restricts to ϕ on M.

Finally, the finiteness $\varphi(\Delta_\phi) = \varphi(\Delta_\phi^{-1}) < \infty$ *follows from simply taking* c_n *such that* $\sum_n c_n(\|a_n\|_\infty^2 + \|b_n\|_\infty^2) < \infty$.

From now on, we fix a normal faithful state ϕ **on** M **and an nrsg** (M,ϕ)**-hyperstate** φ **in the sense of Definition 3.8, and let** $(\mathcal{B}_\varphi, \zeta)$ **denote its Poisson boundary.**

We prove ergodicity, double ergodicity, and rigidity for Poisson boundaries. The proofs are in the same spirit as Das–Peterson [DP22].

The following is a well-known result [DP22, Lemma 2.6], which also appears in earlier literature such as [FNW94], [BJKW00, Lemma 3.4], and [CD20, Lemma 3.1].

Lemma 3.12. *Let* $\mathcal{A}$ *be a unital* C^**-algebra, and let* $\mathcal{P} : \mathcal{A} \to \mathcal{A}$ *be a ucp map preserving a faithful state on* $\mathcal{A}$*. Then*

$$\mathrm{Har}(\mathcal{A}, \mathcal{P}) = \{x \in \mathcal{A} \mid \mathcal{P}(x) = x\}$$

is a C^**-subalgebra of* $\mathcal{A}$*. If* $\mathcal{A}$ *is a von Neumann algebra and* $\mathcal{P}$ *is normal, then* $\mathrm{Har}(\mathcal{A}, \mathcal{P})$ *is a von Neumann subalgebra of* $\mathcal{A}$*.*

Proposition 3.13. *We have the* ***ergodicity*** *of the Poisson boundary, that is,* $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$.

Proof. By identifying $\mathcal{B}_\varphi$ with $\mathrm{Har}(\mathcal{P}_\varphi)$, we consider elements in $M' \cap \mathrm{Har}(\mathcal{P}_\varphi)$. Note that $M' \cap B(L^2(M)) = JMJ$. Hence, we only need to compute $JMJ \cap \mathrm{Har}(\mathcal{P}_\varphi)$.

By the standard form of $\mathcal{P}_\varphi$, we have $\mathcal{P}_\varphi(JMJ) \subset JMJ$. Since φ is regular, for any $T \in JMJ$, we have:

$$\phi^o(\mathcal{P}_\varphi(T)) = \langle \mathcal{P}_\varphi(T)\xi_\phi, \xi_\phi\rangle = \varphi(T) = \phi^o(T).$$

Here the first equality uses $\mathcal{P}_\varphi(T) \in JMJ$ and the fact that ξ_ϕ implements ϕ^o on JMJ. Hence, the restriction $\mathcal{P}_\varphi|_{JMJ}$ preserves the normal faithful state ϕ^o on JMJ. By Lemma 3.12, the fixed-point space $\mathrm{Har}(JMJ, \mathcal{P}_\varphi|_{JMJ})$ is a von Neumann subalgebra of JMJ.

Because φ is strongly generating, M is the largest von Neumann subalgebra contained in $\mathrm{Har}(\mathcal{P}_\varphi)$. Therefore, $\mathrm{Har}(\mathcal{P}_\varphi) \cap JMJ \subset M$, which implies $\mathrm{Har}(\mathcal{P}_\varphi) \cap JMJ \subset M \cap JMJ = \mathcal{Z}(M)$. Since $\mathcal{Z}(M) \subset \mathrm{Har}(\mathcal{P}_\varphi)$ is trivial, we conclude $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$. □

We next prove the double ergodicity of the Poisson boundary.

Proposition 3.14. *Let* $M \subset \mathcal{A}$ *be a* W^**-inclusion, then* $\mathcal{P}_\varphi^o : T \mapsto \sum_k y_k T y_k^*$ *can be naturally defined on* $\mathcal{A}$*. Let*

$$\mathrm{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = \{T \in \mathcal{A} \mid \mathcal{P}_\varphi^o(T) = T\}.$$

Assume there exists a normal faithful ucp map $\mathcal{P} : \mathcal{A} \to \mathcal{B}_\varphi$ *such that* $\mathcal{P}|_M = \mathrm{id}_M$*. Then we have* $\mathrm{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = M' \cap \mathcal{A}$.

In particular, we have the ***double ergodicity*** *of the Poisson boundary:*

$$\mathrm{Har}(\mathcal{P}_\varphi) \cap \mathrm{Har}(\mathcal{P}_\varphi^o) = \mathcal{Z}(M).$$

Proof. Let ζ be the normal faithful stationary state on the Poisson boundary $\mathcal{B}_\varphi$. We define a normal faithful state $\varphi_{\mathcal{A}}$ on $\mathcal{A}$ by $\varphi_{\mathcal{A}} = \zeta \circ \mathcal{P}$.

Because $\mathcal{P}|_M = \mathrm{id}_M$, $\mathcal{P}$ is an M-bimodular map. Hence

$$\mathcal{P}(\mathcal{P}_\varphi^o(T)) = \sum_{k=1}^{\infty} \mathcal{P}(y_k T y_k^*) = \sum_{k=1}^{\infty} y_k \mathcal{P}(T) y_k^* = \mathcal{P}_\varphi^o(\mathcal{P}(T)). \tag{3.8}$$

We now prove that ζ is $\mathcal{P}_\varphi^o$-invariant on $\mathcal{B}_\varphi$. Identifying $\mathcal{B}_\varphi$ with $\mathrm{Har}(\mathcal{P}_\varphi)$, the state is implemented by the vector state $\zeta(\,\cdot\,) = \langle \mathcal{P}_\varphi(\,\cdot\,)\xi_\phi, \xi_\phi\rangle = \langle\,\cdot\,\xi_\phi, \xi_\phi\rangle$, which coincides with the original hyperstate φ on this space. By the defining construction of $\mathcal{P}_\varphi^o$, we have $\langle \mathcal{P}_\varphi^o(T)\xi_\phi, \xi_\phi\rangle = \varphi(T)$ for any $T \in B(L^2(M))$. Hence, for any $S \in \mathcal{B}_\varphi$, we have:

$$\zeta(\mathcal{P}_\varphi^o(S)) = \langle \mathcal{P}_\varphi^o(S)\xi_\phi, \xi_\phi\rangle = \varphi(S) = \zeta(S). \tag{3.9}$$

By (3.8) and (3.9), for $T \in \mathcal{A}$,

$$\varphi_{\mathcal{A}}(\mathcal{P}_\varphi^o(T)) = \zeta(\mathcal{P}(\mathcal{P}_\varphi^o(T))) = \zeta(\mathcal{P}_\varphi^o(\mathcal{P}(T))) = \zeta(\mathcal{P}(T)) = \varphi_{\mathcal{A}}(T).$$

This shows that $\mathcal{P}_\varphi^o$ preserves the normal faithful state $\varphi_{\mathcal{A}}$. By Lemma 3.12, the fixed-point space $\mathcal{A}_0 = \mathrm{Har}(\mathcal{P}_\varphi^o, \mathcal{A})$ is a von Neumann subalgebra of $\mathcal{A}$.

Clearly, if $T \in M' \cap \mathcal{A}$, then $\mathcal{P}_\varphi^o(T) = \sum_{k=1}^\infty y_k T y_k^* = \sum_{k=1}^\infty y_k y_k^* T = T$, so $M' \cap \mathcal{A} \subset \mathcal{A}_0$.

Conversely, suppose $T \in \mathcal{A}_0$. Because $\mathcal{A}_0$ is a von Neumann subalgebra, we also have $TT^* \in \mathcal{A}_0$, meaning $\mathcal{P}_\varphi^o(T) = T$ and $\mathcal{P}_\varphi^o(TT^*) = TT^*$. We compute the following sum of positive elements:

$$\begin{aligned}
&\sum_{k=1}^\infty (y_k T - T y_k)(y_k T - T y_k)^* \\
=&\sum_{k=1}^\infty (y_k T T^* y_k^* - y_k T y_k^* T^* - T y_k T^* y_k^* + T y_k y_k^* T^*) \\
=&\mathcal{P}_\varphi^o(TT^*) - \mathcal{P}_\varphi^o(T) T^* - T \mathcal{P}_\varphi^o(T^*) + T \left(\sum_{k=1}^\infty y_k y_k^* \right) T^* \\
=&TT^* - TT^* - TT^* + TT^* \\
=&0.
\end{aligned} \tag{3.10}$$

By (3.10), $[y_k, T] = 0$ for every k. Since $\overline{\mathbb{C}[y_k]}^{\mathrm{w}} = M$, $T \in M' \cap \mathcal{A}$. Hence $\mathrm{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = M' \cap \mathcal{A}$.

For the double ergodicity of $\mathcal{B}_\varphi$, we set $\mathcal{A} = \mathcal{B}_\varphi \cong \mathrm{Har}(\mathcal{P}_\varphi)$ and let $\mathcal{P} = \mathrm{id}_{\mathcal{B}_\varphi}$. The hypothesis is trivially satisfied, yielding:

$$\mathrm{Har}(\mathcal{P}_\varphi) \cap \mathrm{Har}(\mathcal{P}_\varphi^o) = \mathrm{Har}(\mathcal{P}_\varphi^o, \mathcal{B}_\varphi) = M' \cap \mathcal{B}_\varphi.$$

By Proposition 3.13, $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$, which concludes the proof. □

As a consequence, the Cesàro means of $\mathcal{P}_\varphi^o$ give the inner UCP-Dixmier property.

Corollary 3.15. *Let $M \subset \mathcal{A}$ be a W*-inclusion, and assume there exists a normal faithful ucp map $\mathcal{P} : \mathcal{A} \to \mathcal{B}_\varphi$ such that $\mathcal{P}|_M = \mathrm{id}_M$. Then $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.*

Proof. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. By Proposition 3.14, we can define a conditional expectation $E : \mathcal{A} \to M' \cap \mathcal{A}$ by

$$E(x) := \operatorname*{wo-lim}_{n \to \omega} \frac{1}{n} \sum_{k=1}^n (\mathcal{P}_\varphi^o)^k(x).$$

Recall that $\mathcal{P}_\varphi^o$ is an M-inner ucp map. Therefore, $E(x)$ lies in the weak closure of the inner UCP orbit of x:

$$E(x) \in \overline{\{\mathcal{P}(x) \mid \mathcal{P} \text{ is an } M\text{-inner ucp map}\}}^{\mathrm{w}} \cap (M' \cap \mathcal{A}).$$

Hence $M \subset \mathcal{A}$ has the inner UCP-Dixmier property. □

We now prove the rigidity of the Poisson boundary.

Theorem 3.16. *Let $\mathcal{C}$ be a weakly closed operator system such that $M \subset \mathcal{C} \subset \mathcal{B}_\varphi$ and $M\mathcal{C} + \mathcal{C}M \subset \mathcal{C}$. If $\Psi : \mathcal{C} \to \mathcal{B}_\varphi$ is a normal ucp map such that $\Psi|_M = \mathrm{id}_M$, then $\Psi = \mathrm{id}_{\mathcal{C}}$.*

Proof. We identify $\mathcal{C}$ with its image under the isomorphism $\mathcal{B}_\varphi \cong \mathrm{Har}(\mathcal{P}_\varphi)$. Then Ψ is a normal ucp map from $\mathcal{C}$ to $\mathrm{Har}(\mathcal{P}_\varphi)$. Since $\Psi|_M = \mathrm{id}_M$, the map Ψ is M-bimodular. Using the explicit formula $\mathcal{P}_\varphi^o(T) = \sum_{k=1}^\infty y_k T y_k^*$, for any $T \in \mathcal{C}$ we have:

$$\langle \Psi(\mathcal{P}_\varphi^o(T))\xi_\phi, \xi_\phi \rangle = \sum_{k=1}^\infty \langle \Psi(y_k T y_k^*)\xi_\phi, \xi_\phi \rangle = \sum_{k=1}^\infty \langle y_k \Psi(T) y_k^* \xi_\phi, \xi_\phi \rangle = \langle \mathcal{P}_\varphi^o(\Psi(T))\xi_\phi, \xi_\phi \rangle.$$

Since $\Psi(T) \in \mathrm{Har}(\mathcal{P}_\varphi)$ and φ is regular, the term on the right evaluates to:

$$\langle \mathcal{P}_\varphi(\Psi(T))\xi_\phi, \xi_\phi\rangle = \langle \Psi(T)\xi_\phi, \xi_\phi\rangle.$$

Hence, $\langle \Psi(\mathcal{P}_\varphi^o(T))\xi_\phi, \xi_\phi\rangle = \langle \Psi(T)\xi_\phi, \xi_\phi\rangle$ for all $T \in \mathcal{C}$. By linearity, this invariance extends to Cesàro means:

$$\left\langle \Psi\left(\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\varphi^o)^n(T)\right)\xi_\phi, \xi_\phi\right\rangle = \langle \Psi(T)\xi_\phi, \xi_\phi\rangle.$$

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. By normality and Proposition 3.14, we have

$$z := \operatorname*{wo-lim}_{N\to\omega} \frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\varphi^o)^n(T) \in \mathrm{Har}(\mathcal{P}_\varphi) \cap \mathrm{Har}(\mathcal{P}_\varphi^o) = \mathcal{Z}(M).$$

Since $z \in M$, we have $\Psi(z) = z$. By normality of Ψ, we have the first equality below. The last equality follows from (3.9) and normality of ζ:

$$\langle \Psi(T)\xi_\phi, \xi_\phi\rangle = \langle \Psi(z)\xi_\phi, \xi_\phi\rangle = \langle z\xi_\phi, \xi_\phi\rangle = \langle T\xi_\phi, \xi_\phi\rangle.$$

Now, for any $a, b \in M$ and $T \in \mathcal{C}$, the element b^*Ta is still in $\mathcal{C}$. Using the M-bimodularity of Ψ, we have:

$$\langle \Psi(T)a\xi_\phi, b\xi_\phi\rangle = \langle \Psi(b^*Ta)\xi_\phi, \xi_\phi\rangle = \langle b^*Ta\xi_\phi, \xi_\phi\rangle = \langle Ta\xi_\phi, b\xi_\phi\rangle.$$

Because $M\xi_\phi$ is dense in $L^2(M)$, this forces $\Psi(T) = T$ for all $T \in \mathcal{C}$. Hence, $\Psi = \mathrm{id}_{\mathcal{C}}$. □

4. Weak Dixmier properties via Poisson boundaries

In this section, we use Poisson boundaries to give a sufficient condition for a W*-inclusion $M \subset \mathcal{A}$ to have the weak Dixmier property. We begin by proving Theorem F, which shows that, for properly infinite M, inner ucp averages can be replaced by unitary averages.

Theorem 4.1 (Thm. F)**.** *Let $M \subset \mathcal{A}$ be a W*-inclusion such that M is properly infinite. Then $M \subset \mathcal{A}$ has the weak Dixmier property if and only if it has the inner UCP-Dixmier property. Here $\mathcal{A}$ need not have separable predual.*

Proof. We use a dilation argument similar to that of Haagerup [Haa85, Proposition 2.1 and the proof of Theorem 2.2], passing from completely positive maps to unitary conjugations by using proper infiniteness. We only need to prove the if direction, since every finite convex combination of unitary conjugations is an M-inner ucp map. Assume that $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.

Because M is properly infinite, it contains an infinite sequence of mutually orthogonal projections $(p_k)_{k=1}^\infty$ such that $p_k \sim 1_M$ for all k and $\sum_{k=1}^\infty p_k = 1_M$. Take isometries $v_k \in M$ with $v_k v_k^* = p_k$. By this partition of unity, we have a spatial isomorphism

$$\Theta : \mathcal{A} \xrightarrow{\sim} \mathcal{A}\bar{\otimes}B(H), \qquad \Theta(a) = (v_i^* a v_j)_{i,j\geq 1},$$

where $H = \ell^2(\mathbb{N})$. Its restriction maps M onto $M\bar{\otimes}B(H)$ and p_k to $1_M \otimes e_{kk}$, where $(e_{ij})_{i,j\geq 1}$ are the standard matrix units of $B(H)$. Indeed, on a Hilbert space on which $\mathcal{A}$ acts, the unitary $\xi \otimes e_k \mapsto v_k\xi$ implements this isomorphism. Hence we identify the inclusion $M \subset \mathcal{A}$ with

$$M\bar{\otimes}B(H) \subset \mathcal{A}\bar{\otimes}B(H).$$

We only need to prove the weak Dixmier property for an arbitrary element $x \in \mathcal{A}\bar{\otimes}B(H)$. Write

$$K_x = \overline{\mathrm{conv}}^{\mathrm{w}}\{UxU^* \mid U \in \mathcal{U}(M\bar{\otimes}B(H))\}.$$

Since $B(H)$ is hyperfinite, take increasing unital full matrix algebras $F_n \cong \mathrm{M}_{2^n}(\mathbb{C})$ in $B(H)$ with $(\bigcup_n F_n)'' = B(H)$. For a free ultrafilter ω on $\mathbb{N}$, by normalized Haar averaging we have

$$a \otimes 1_{B(H)} := \operatorname*{wo-lim}_{n\to\omega} \int_{\mathcal{U}(F_n)} (1_M \otimes u)x(1_M \otimes u^*)\,\mathrm{d}u \in K_x \qquad (a \in \mathcal{A}).$$

Since K_x is weakly closed, convex and invariant under these unitary conjugations, it also contains the weakly closed convex hull of the unitary orbit of $a \otimes 1_{B(H)}$.

We only need to prove that for any M-inner ucp map

$$\mathcal{P}(a) = \sum_{k=1}^{\infty} z_k a z_k^*, \qquad z_k \in M, \qquad \sum_{k=1}^{\infty} z_k z_k^* = 1_M,$$

the element $\mathcal{P}(a) \otimes 1_{B(H)}$ is contained in the weak closure of the convex hull of the unitary orbit of $a \otimes 1_{B(H)}$.

Fix an integer $n \geq 1$. We construct a partial isometry $Z_n \in M \bar{\otimes} B(H)$ structured as a block matrix where each block is of size $n \times n$. The first block-row consists of a zero block followed by scaled identity matrices $z_k \mathrm{I}_n$, and all subsequent block-rows are identically zero:

$$Z_n = \begin{pmatrix} 0_{n\times n} & z_1 \mathrm{I}_n & z_2 \mathrm{I}_n & z_3 \mathrm{I}_n & \cdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The strong convergence of $\sum_{k=1}^{\infty} z_k z_k^* = 1_M$ makes this row a bounded operator. Let $E_n = \sum_{j=1}^{n} e_{jj}$. Then

$$Z_n Z_n^* = 1_M \otimes E_n.$$

Hence, Z_n^* is a partial isometry with initial projection $1_M \otimes E_n$. Because the first block-column of Z_n is identically zero, its final projection $Z_n^* Z_n$ is orthogonal to $1_M \otimes E_n$. Since M is properly infinite, each $1_M \otimes e_{jj}$ is equivalent to $1_{M \bar{\otimes} B(H)}$. Hence

$$1 - Z_n^* Z_n \geq 1_M \otimes E_n \geq 1_M \otimes e_{11} \sim 1_{M \bar{\otimes} B(H)}.$$

We also have

$$1 - (1_M \otimes E_n) \geq 1_M \otimes e_{n+1,n+1} \sim 1_{M \bar{\otimes} B(H)}.$$

Therefore, the orthogonal complements of both projections are Murray–von Neumann equivalent to the identity:

$$1 - (1_M \otimes E_n) \sim 1 - Z_n^* Z_n \sim 1_{M \bar{\otimes} B(H)}.$$

Hence there exists a partial isometry $W_n \in M \bar{\otimes} B(H)$ with

$$W_n^* W_n = 1 - Z_n^* Z_n, \qquad W_n W_n^* = 1 - (1_M \otimes E_n).$$

Define $U_n = Z_n + W_n$. Since the initial and final projections of Z_n and W_n are respectively orthogonal, all mixed terms vanish. Therefore,

$$U_n^* U_n = Z_n^* Z_n + W_n^* W_n = 1, \qquad U_n U_n^* = Z_n Z_n^* + W_n W_n^* = 1.$$

Hence we complete the partial isometry into a unitary $U_n \in \mathcal{U}(M \bar{\otimes} B(H))$. Since $(1_M \otimes E_n) W_n = 0$, the first block-row of U_n remains the first block-row of Z_n. In block-row notation, we write

$$U_n = \begin{pmatrix} Z_n \\ W_n \end{pmatrix}.$$

We conjugate $a \otimes 1_{B(H)}$ by U_n. Since

$$Z_n (a \otimes 1_{B(H)}) Z_n^* = \mathcal{P}(a) \otimes E_n,$$

we have, with respect to the first n coordinates and their orthogonal complement,

$$U_n (a \otimes 1_{B(H)}) U_n^* = \begin{pmatrix} \mathcal{P}(a)\mathrm{I}_n & * \\ * & * \end{pmatrix}. \tag{4.1}$$

Let $\mathcal{D} \subset B(H)$ be the abelian von Neumann algebra of diagonal operators, and apply the conditional expectation onto the relative commutant of $1_{\mathcal{A}} \otimes \mathcal{D}$:

$$E_{\mathcal{D}'} : \mathcal{A}\bar{\otimes}B(H) \longrightarrow \mathcal{A}\bar{\otimes}\mathcal{D}.$$

This expectation is approximated weakly by convex combinations of diagonal unitary conjugations. Indeed, for $t \in [0,1]$, let

$$U_t = 1_M \otimes \sum_{k=1}^{\infty} e^{2k\pi it} e_{kk} \in \mathcal{U}(1_M \otimes \mathcal{D}).$$

Then

$$E_{\mathcal{D}'}(T) = \int_0^1 U_t T U_t^* \, \mathrm{d}t, \tag{4.2}$$

where the integral is taken ultraweakly. By (4.2) and (4.1),

$$E_{\mathcal{D}'}\big(U_n(a \otimes 1_{B(H)})U_n^*\big) = \mathcal{P}(a) \otimes E_n + b_n,$$

where

$$b_n = (1 - 1_{\mathcal{A}} \otimes E_n) b_n (1 - 1_{\mathcal{A}} \otimes E_n), \qquad \|b_n\| \leq \|a\|.$$

As $n \to \infty$, $E_n \nearrow 1_{B(H)}$ strongly. Hence, the support and uniform norm bound of b_n give $b_n \to 0$ strongly, and hence weakly. Moreover, $\mathcal{P}(a) \otimes E_n \to \mathcal{P}(a) \otimes 1_{B(H)}$ weakly. Therefore,

$$\operatorname*{wo-lim}_{n\to\infty} E_{\mathcal{D}'}\big(U_n(a \otimes 1_{B(H)})U_n^*\big) = \mathcal{P}(a) \otimes 1_{B(H)}.$$

Because this holds for any M-inner ucp map $\mathcal{P}$, the entire set

$$\{\mathcal{P}(a) \otimes 1_{B(H)} \mid \mathcal{P} \text{ is an } M\text{-inner ucp map}\}$$

is contained in K_x.

By the assumption that the original inclusion $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, there exists an element

$$y \in \overline{\{\mathcal{P}(a) \mid \mathcal{P} \text{ is an } M\text{-inner ucp map}\}}^{\mathrm{w}} \cap (M' \cap \mathcal{A}).$$

Because the mapping $T \mapsto T \otimes 1_{B(H)}$ is normal, the element $y \otimes 1_{B(H)}$ lies in the weak closure of $\{\mathcal{P}(a) \otimes 1_{B(H)}\}$. Since K_x is already weakly closed, it contains $y \otimes 1_{B(H)}$.

Hence, $y \otimes 1_{B(H)}$ belongs to

$$(M\bar{\otimes}B(H))' \cap (\mathcal{A}\bar{\otimes}B(H)) = (M' \cap \mathcal{A})\bar{\otimes}\mathbb{C}1_{B(H)}$$

and lies in the weakly closed convex hull of the unitary orbit of x, which finishes the proof. □

Combining Theorem 4.1 with Corollary 3.15, we have the weak Dixmier property for inclusions that map into a Poisson boundary.

Theorem 4.2 (Thm. D)**.** *Let M be a von Neumann algebra with separable predual and φ be an nrsg hyperstate on $B(L^2(M))$ with the Poisson boundary $\mathcal{B}_\varphi$. Assume that one of the following conditions holds:*

(i) *M is properly infinite, i.e., $M = M_{\mathrm{I}_\infty} \oplus M_{\mathrm{II}_\infty} \oplus M_{\mathrm{III}}$;*
(ii) *M is finite with normal faithful trace τ and $\varphi = \int_{\mathcal{U}(M)} \langle \cdot\, u\xi_\tau, u\xi_\tau\rangle \mathrm{d}\mu(u)$ for some $\mu \in \mathrm{Prob}(\mathcal{U}(M))$.*

Then any W^*-inclusion $M \subset \mathcal{A}$ equipped with a normal faithful ucp map $\mathcal{P} : \mathcal{A} \to \mathcal{B}_\varphi$ with $\mathcal{P}|_M = \mathrm{id}_M$ has the weak Dixmier property.

Proof. Let $M \subset \mathcal{A}$ be a W*-inclusion equipped with a normal faithful ucp map $\mathcal{P} : \mathcal{A} \to \mathcal{B}_\varphi$ satisfying $\mathcal{P}|_M = \mathrm{id}_M$. By Corollary 3.15, $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.

We consider the two cases separately.

Case (i): Assume M is properly infinite. By Theorem 4.1, for properly infinite von Neumann algebras, the weak Dixmier property is equivalent to the inner UCP-Dixmier property. Since $M \subset \mathcal{A}$ already has the inner UCP-Dixmier property by Corollary 3.15, it has the weak Dixmier property.

Case (ii): Assume M is finite with trace τ, and $\varphi = \int_{\mathcal{U}(M)} \langle \cdot\, u\xi_\tau, u\xi_\tau\rangle \mathrm{d}\mu(u)$ for some probability measure $\mu \in \mathrm{Prob}(\mathcal{U}(M))$. In this setting, the associated ucp map $\mathcal{P}_\varphi^o : \mathcal{A} \to \mathcal{A}$ is given by $\mathcal{P}_\varphi^o(x) = \int_{\mathcal{U}(M)} u^* x u \,\mathrm{d}\mu(u)$ for all $x \in \mathcal{A}$. The adjoint direction follows from the vector-state convention $\varphi = \int \omega_{u\xi_\tau} \,\mathrm{d}\mu(u)$. Because $\mathcal{P}_\varphi^o$ is an integral of unitary conjugations, its iterates $(\mathcal{P}_\varphi^o)^k$ and their Cesàro means $C_N(x) = (1/N)\sum_{k=1}^N (\mathcal{P}_\varphi^o)^k(x)$ lie in the weakly closed convex hull of the unitary orbit of x:

$$C_N(x) \in \overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\}.$$

By the proof of Corollary 3.15, any weak limit point z of the sequence $(C_N(x))_{N=1}^\infty$ is an element of $M' \cap \mathcal{A}$. Since the weakly closed convex hull of the unitary orbit is weakly closed, the limit point z remains within it. Hence:

$$z \in \overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

This proves that $M \subset \mathcal{A}$ has the weak Dixmier property.

□

As a first application, we recover the weak Dixmier property for inclusions with expectation.

Corollary 4.3 (Cor. E, [Pop00, Mar21])**.** *Let $M \subset \mathcal{A}$ be an inclusion of countably decomposable von Neumann algebras with a normal faithful conditional expectation. Then $M \subset \mathcal{A}$ has the weak Dixmier property.*

Proof. Let $E : \mathcal{A} \to M$ be the given normal faithful conditional expectation. As in [Mar21, p. 122, proof of the main theorem], take an increasing net $(M_i)_{i\in I}$ of von Neumann subalgebras with separable preduals and normal faithful conditional expectations $E_{M_i} : M \to M_i$, such that $(\bigcup_i M_i)'' = M$. For $x \in \mathcal{A}$, let

$$K_x^i = \overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M_i)\}.$$

We only need to prove that $K_x^i \cap M_i' \neq \varnothing$ for every i: by weak compactness, any choice $y_i \in K_x^i \cap M_i'$ has a weak cluster point in

$$\overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}).$$

Since $E_{M_i} \circ E : \mathcal{A} \to M_i$ is a normal faithful conditional expectation, we may therefore assume that M has separable predual. No separability assumption on $\mathcal{A}$ is needed in the argument below or in the auxiliary averaging results used here. Decompose $M = M_{\mathrm{fin}} \oplus M_\infty$ into its finite and properly infinite parts, omitting zero summands, and write p_1, p_∞ for their units. Averaging by $p_1 + e^{2\pi i t} p_\infty$ removes the off-diagonal terms of every $x \in \mathcal{A}$:

$$\int_0^1 (p_1 + e^{2\pi i t} p_\infty) x (p_1 + e^{2\pi i t} p_\infty)^* \,\mathrm{d}t = p_1 x p_1 + p_\infty x p_\infty. \tag{4.3}$$

The restrictions $E_i : p_i \mathcal{A} p_i \to p_i M$ are normal faithful conditional expectations.

Take a normal faithful trace τ on M_{fin}, a countable adjoint-closed family of unitaries (u_k) generating M_{fin}, and $c_k > 0$ with $\sum_k c_k = 1$. The state

$$\varphi_1 = \sum_k c_k \langle \cdot\, u_k \xi_\tau, u_k \xi_\tau\rangle$$

is normal and regular. Both strong generation conditions follow from the weak density of $\mathbb{C}[u_k]$, since (u_k) is adjoint-closed. Composing E_1 with $M_{\mathrm{fin}} \subset \mathcal{B}_{\varphi_1}$, by Theorem 4.2(ii), we have that $M_{\mathrm{fin}} \subset p_1 \mathcal{A} p_1$ has the weak Dixmier property.

For M_∞, take an nrsg hyperstate by Example 3.10. Composing E_∞ with the boundary inclusion and applying Theorem 4.2(i), we have the same conclusion for this corner. Product nets of the two unitary averaging nets, after applying (4.3), give an element of $M' \cap \mathcal{A}$ in the closed convex unitary orbit of x. □

We now turn to the partial weak Dixmier property.

Definition 4.4. *Let $M \subset \mathcal{A}$ and $M \subset \mathcal{B}$ be W*-inclusions. A map $\mathcal{T} : \mathcal{A}_+ \to \widehat{\mathcal{B}}_+$ is called an M-**bimodular semifinite positive map** if it is additive, positively homogeneous and satisfies*

$$\mathcal{T}(a^* x a) = a^* \mathcal{T}(x) a \qquad (x \in \mathcal{A}_+,\ a \in M).$$

We require normality, meaning preservation of increasing bounded nets, faithfulness, and ultraweak density of

$$\mathfrak{m}_{\mathcal{T}}^+ = \{x \in \mathcal{A}_+ \mid \mathcal{T}(x) \in \mathcal{B}_+\}.$$

Here $\widehat{\mathcal{B}}_+$ is the extended positive cone, and $\mathcal{T}(x) < \infty$ will always mean that $\mathcal{T}(x)$ is bounded.

For every $\rho \in \mathcal{B}_*^+$, the function $\rho \circ \mathcal{T}$ is a normal weight. In particular, its sublevel sets are ultraweakly closed. It follows that, for $C < \infty$, the set

$$\{x \in \mathcal{A}_+ \mid \mathcal{T}(x) \le C1\} \tag{4.4}$$

is ultraweakly closed. We use this scalar interpretation of lower semicontinuity throughout; see [Haa79b, Sections 1–2].

When the invariant state is replaced by an invariant semifinite weight, Cesàro limits still satisfy the following multiplicativity.

Lemma 4.5. *Let $\varphi_{\mathcal{A}}$ be a normal faithful semifinite weight on a von Neumann algebra $\mathcal{A}$, and let $\mathcal{P} : \mathcal{A} \to \mathcal{A}$ be a normal ucp map satisfying $\varphi_{\mathcal{A}} \circ \mathcal{P} = \varphi_{\mathcal{A}}$. Fix a free ultrafilter ω and define*

$$E(x) = \operatorname*{wo-lim}_{n \to \omega} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}^k(x).$$

If $x \in \mathcal{A}_+$ and $\varphi_{\mathcal{A}}(x) < \infty$, then $\mathcal{P}(E(x)) = E(x)$ and $\mathcal{P}(E(x)^2) = E(x)^2$.

Proof. Write $C_n(x) = (1/n) \sum_{k=0}^{n-1} \mathcal{P}^k(x)$ and $y = E(x)$. Since $\|\mathcal{P}(C_n(x)) - C_n(x)\| \le 2\|x\|/n$, by normality of $\mathcal{P}$, we have $\mathcal{P}(y) = y$. Lower semicontinuity and invariance of the weight give

$$\varphi_{\mathcal{A}}(y) \le \varphi_{\mathcal{A}}(x) < \infty, \qquad \varphi_{\mathcal{A}}(y^2) \le \|y\| \varphi_{\mathcal{A}}(y) < \infty.$$

By Kadison's inequality, $\mathcal{P}(y^2) - y^2 \ge 0$. Since $\varphi_{\mathcal{A}} \circ \mathcal{P} = \varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{A}}(y^2) < \infty$,

$$\varphi_{\mathcal{A}}(\mathcal{P}(y^2) - y^2) = \varphi_{\mathcal{A}}(\mathcal{P}(y^2)) - \varphi_{\mathcal{A}}(y^2) = 0.$$

By faithfulness, we have $\mathcal{P}(y^2) = y^2$. □

By this lemma, we have the following semifinite version of Proposition 3.14.

Theorem 4.6. *Let $M \subset \mathcal{A}$ be a W*-inclusion. Assume that M has separable predual, and let φ be an nrsg M-hyperstate. Assume that there exists a normal faithful semifinite M-bimodular positive map $\mathcal{T} : \mathcal{A}_+ \to \widehat{\mathcal{B}_{\varphi}}_+$ in the preceding sense. Set $Q = \mathcal{P}_\varphi^o$, acting on $\mathcal{A}$ by $Q(x) = \sum_k y_k x y_k^*$. For any $x \in \mathcal{A}_+$ with $\mathcal{T}(x) < \infty$, every ultraweak cluster point of $(1/n) \sum_{k=0}^{n-1} Q^k(x)$ belongs to $M' \cap \mathcal{A}$.*

Proof. Let ζ be the canonical normal faithful state on $\mathcal{B}_\varphi$ and let $\varphi_{\mathcal{A}} = \zeta \circ \mathcal{T}$. This is a normal faithful weight. It is semifinite because its finite positive domain contains the ultraweakly dense cone $\mathfrak{m}_{\mathcal{T}}^+$.

Normality and M-bimodularity give, in the extended positive cone,

$$\mathcal{T}(Q(x)) = \sum_k y_k \mathcal{T}(x) y_k^* = Q(\mathcal{T}(x)).$$

By (3.9), $\zeta \circ Q = \zeta$ on $(\mathcal{B}_\varphi)_+$. This identity extends to the extended positive cone by monotone approximation. Hence $\varphi_{\mathcal{A}} \circ Q = \varphi_{\mathcal{A}}$.

Let T be an ultraweak cluster point of $(1/n)\sum_{k=0}^{n-1} Q^k(x)$. By the proof of Lemma 4.5, applied along a convergent subnet, we have $Q(T) = T$ and $Q(T^2) = T^2$. Hence,

$$\sum_k (y_k T - T y_k)(y_k T - T y_k)^* = Q(T^2) - Q(T)T - TQ(T) + T^2 = 0.$$

Hence $[T, y_k] = 0$ for every k. Since $\overline{\mathbb{C}[y_k]}^{\mathrm{w}} = M$, $T \in M' \cap \mathcal{A}$. □

As in Theorem 4.2, we can convert these Cesàro limits into unitary averages.

Theorem 4.7. *Let M have separable predual and let φ be an nrsg hyperstate. Assume that one of the following conditions holds:*

(i) *M is properly infinite;*
(ii) *M is finite with a normal faithful tracial state τ and*

$$\varphi(T) = \int_{\mathcal{U}(M)} \langle Tu\xi_\tau, u\xi_\tau \rangle \, \mathrm{d}\mu(u) \tag{4.5}$$

for some $\mu \in \mathrm{Prob}(\mathcal{U}(M))$.

Let $M \subset \mathcal{A}$ be a W^*-inclusion and let $\mathcal{T} : \mathcal{A}_+ \to \widehat{\mathcal{B}_{\varphi}}_+$ be a normal faithful semifinite M-bimodular positive map. Then, for $x \in \mathcal{A}_+$ with $\mathcal{T}(x) < \infty$,

$$\overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Proof. Let $C = \|\mathcal{T}(x)\|$ and $K_x = \overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\}$. For $u \in \mathcal{U}(M)$, $\mathcal{T}(uxu^*) = u\mathcal{T}(x)u^* \leq C1$. By the ultraweak closedness of (4.4),

$$\mathcal{T}(b) \leq C1 \qquad (b \in K_x).$$

Suppose first that M is properly infinite. Take isometries $(v_i) \subset M$ with $v_i^* v_j = \delta_{ij} 1$ and $\sum_i v_i v_i^* = 1$, and use the isomorphism

$$\Theta : \mathcal{A} \xrightarrow{\sim} \mathcal{A} \bar{\otimes} B(\ell^2(\mathbb{N})), \qquad \Theta(b) = \sum_{i,j} v_i^* b v_j \otimes e_{ij}.$$

It maps M onto $M \bar{\otimes} B(\ell^2(\mathbb{N}))$. As in the proof of Theorem 4.1, by unitary averaging over the type I tensor factor, we have $a \otimes 1 \in \Theta(K_x)$ for some $a \in \mathcal{A}_+$. Let $b = \Theta^{-1}(a \otimes 1)$. Since $b \in K_x$ and $a = v_1^* b v_1$,

$$\mathcal{T}(a) = v_1^* \mathcal{T}(b) v_1 \leq C1.$$

By Theorem 4.6, an ultraweak cluster point c of the Cesàro means of $(\mathcal{P}_\varphi^o)^k(a)$ belongs to $M' \cap \mathcal{A}$.

The matrix-dilation argument in the proof of Theorem 4.1 applies to each M-inner ucp map. Hence every Cesàro mean of $(\mathcal{P}_\varphi^o)^k(a)$, tensored with 1, belongs to the weakly closed unitary convex hull of $a \otimes 1$. That hull is contained in $\Theta(K_x)$, so $c \otimes 1 \in \Theta(K_x)$. Since $c \otimes 1 \in (M \bar{\otimes} B(\ell^2(\mathbb{N})))'$,

$$\Theta^{-1}(c \otimes 1) \in K_x \cap (M' \cap \mathcal{A}).$$

Suppose now that M is finite and φ satisfies (4.5). With the present vector-state convention,

$$\mathcal{P}_\varphi^o(b) = \int_{\mathcal{U}(M)} u^* b u \, \mathrm{d}\mu(u).$$

Its iterates and Cesàro means preserve K_x. Their ultraweak cluster points belong to $M' \cap \mathcal{A}$ by Theorem 4.6, which proves the assertion. □

Composing an operator-valued weight onto M with the boundary inclusion recovers Isono's result.

Corollary 4.8 ([Iso25]). *Let $M \subset \mathcal{A}$ be an inclusion of countably decomposable von Neumann algebras, and let $\mathcal{T} : \mathcal{A}_+ \to \widehat{M}_+$ be a normal faithful semifinite operator-valued weight. For every $x \in \mathcal{A}_+$ such that $\mathcal{T}(x)$ is bounded,*

$$\overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Proof. As in the beginning of the proof of Corollary 4.3, we may assume that M has separable predual. Write $M = M_{\mathrm{fin}} \oplus M_\infty$ for its finite and properly infinite parts, with units p_1, p_∞. Averaging over the unitaries $p_1 + e^{2\pi i t} p_\infty$ sends x to $p_1 x p_1 + p_\infty x p_\infty$ within its unitary convex hull.

The restrictions $\mathcal{T}_i : (p_i \mathcal{A} p_i)_+ \to \widehat{p_i M}_+$ are normal, faithful and semifinite. Indeed, compression of the ultraweakly dense finite positive domain of $\mathcal{T}$ is ultraweakly dense in $(p_i \mathcal{A} p_i)_+$, and

$$\mathcal{T}_i(p_i a p_i) = p_i \mathcal{T}(a) p_i.$$

Moreover, $\mathcal{T}_i(p_i x p_i)$ is bounded.

On M_{fin} take an atomic measure on a countable symmetric family of unitaries generating M_{fin}, with every atom having positive mass. Then we have an nrsg hyperstate with respect to a normal faithful tracial state. On M_∞ use Example 3.10. Compose each $\mathcal{T}_i$ with the extension to extended positive cones of the canonical embedding $p_i M \subset \mathcal{B}_{\varphi_i}$. These composites satisfy the hypotheses of Theorem 4.7; no normalization on $p_i M$ is required. Apply that theorem separately to the two positive diagonal blocks.

Finally, block-diagonal unitaries in $\mathcal{U}(M_{\mathrm{fin}}) \oplus \mathcal{U}(M_\infty)$ belong to $\mathcal{U}(M)$. Taking product nets of the two averaging nets, we have a limit in $M' \cap \mathcal{A}$, still in the original unitary convex hull. □

5. The relative bicentralizer conjecture

In this section, we prove Theorem A. Fix an inclusion $M \subset P$ of countably decomposable von Neumann algebras with a normal faithful conditional expectation $E : P \to M$. We only need to prove that $M \subset c(P)$ has the weak Dixmier property.

By the same argument as in the beginning of the proof of Corollary 4.3, we may assume that M has separable predual; see also [Mar25, Theorems D and E(6)]. For the remainder of this section, we therefore assume that M has separable predual, while P need only be countably decomposable. Fix a normal faithful state ϕ on M, and let $\rho = \phi \circ E$. We use the compatible realizations

$$c_\phi(M) \subset c_\rho(P) = P \rtimes_{\sigma^\rho} \mathbb{R}.$$

We write ϕ^{it} for the canonical implementing unitaries in $c_\phi(M)$. Let $\mathcal{A} = c_\rho(P)$, and let $\varphi_{\mathcal{A}} = \rho \circ T$ be the dual weight of ρ, where $T : \mathcal{A}_+ \to \widehat{P}_+$ is the canonical operator-valued weight. We use the notation $h = \frac{\mathrm{d}\varphi_{\mathcal{A}}}{\mathrm{d}\tau_{\mathcal{A}}}$ and $H = \log h$ from Subsection 2.3, where $\tau_{\mathcal{A}}$ is the canonical trace on $\mathcal{A}$. Here $\lambda_t = h^{it} = \phi^{it}$. The modular action $\sigma_t^{\varphi_{\mathcal{A}}} = \mathrm{Ad}(\lambda_t)$ restricts to σ_t^ϕ on M. Take the (M, ϕ)-hyperstate φ and the inner ucp coefficients (y_k) from Example 3.10, and define

$$Q(x) = \sum_k y_k x y_k^* \qquad (x \in \mathcal{A}). \tag{5.1}$$

To prove that $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, we only need to prove that $d := Q(aa^*) - aa^* = 0$ for every $a \in \mathrm{Har}(Q, \mathcal{A})$. As in the proof of Proposition 3.14 and the argument showing $E(x) \in \mathrm{Har}(\mathcal{P})$ in Lemma 4.5, followed by Cesàro averaging, we have

$$\begin{aligned} & Q(aa^*) = aa^* \quad (a \in \mathrm{Har}(Q, \mathcal{A})) \\ \Longrightarrow \quad & \mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A} \\ \Longrightarrow \quad & M \subset \mathcal{A} \text{ has the inner UCP-Dixmier property.} \end{aligned}$$

Fix $a \in \mathrm{Har}(Q, \mathcal{A})$. By Kadison's inequality, $d \geq 0$. Lemma 2.2 reduces $d = 0$ to proving that every $x \in \mathcal{A}_+$ with $0 \leq x \leq d$ and $\varphi_{\mathcal{A}}(x) < \infty$ must be $x = 0$. Such an x satisfies

$$\sum_{k=0}^{n-1} Q^k(x) \leq \sum_{k=0}^{n-1} Q^k(d) = Q^n(aa^*) - aa^* \leq C1 \qquad \text{with } C = \|a\|^2.$$

After constructing the pairing and the self-adjoint contraction S in Lemma 5.1, Lemmas 5.2 and 5.3 prove $\varphi_{\mathcal{A}}(x) = 0$ for such x. By faithfulness, we then have $x = 0$.

Theorem 4.1 converts the inner UCP-Dixmier property into the weak Dixmier property when M is properly infinite. For finite M, we take ϕ tracial so that the Cesàro means are already unitary averages. The general case follows by decomposing M into its finite and properly infinite parts.

Since $\phi \circ Q = \phi$, normality and M-bimodularity of E and T give

$$\varphi_{\mathcal{A}} \circ Q = \phi \circ E \circ T \circ Q = \phi \circ Q \circ E \circ T = \phi \circ E \circ T = \varphi_{\mathcal{A}}, \tag{5.2}$$

where the compositions are understood on the extended positive cone.

Define

$$\mathfrak{m}_{\varphi_{\mathcal{A}}} = \mathrm{span}\{a \in \mathcal{A}_+ \mid \varphi_{\mathcal{A}}(a) < \infty\}, \qquad \xi(a) = h^{1/4} a h^{1/4} \quad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}).$$

On $L^2(\mathcal{A}, \tau_{\mathcal{A}})$ we use the inner product $\langle X, Y \rangle_2 = \tau_{\mathcal{A}}(Y^* X)$, linear in the first variable.

Define $S : D \to D$, where $D = \{\xi(a) \mid a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}\}$, by

$$S\xi(a) = \xi(Q(a)) \qquad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}). \tag{5.3}$$

This is well-defined since ξ is injective and $Q(\mathfrak{m}_{\varphi_{\mathcal{A}}}) \subset \mathfrak{m}_{\varphi_{\mathcal{A}}}$ by (5.2).

Lemma 5.1. *For every $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, we have $\xi(a) \in L^2(\mathcal{A}, \tau_{\mathcal{A}})$. Let $x \in \mathcal{A}_+$ satisfy $\varphi_{\mathcal{A}}(x) < \infty$. Then*

$$\omega_x(a) = \tau_{\mathcal{A}}(h^{1/2} x h^{1/2} a) \qquad (a \in \mathcal{A})$$

defines a finite normal positive functional on $\mathcal{A}$, with $\omega_x(1) = \varphi_{\mathcal{A}}(x)$, and

$$\langle \xi(a), \xi(x) \rangle_2 = \omega_x(a) \qquad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}). \tag{5.4}$$

Moreover, S extends to a self-adjoint contraction on $L^2(\mathcal{A}, \tau_{\mathcal{A}})$.

Proof. For $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, let

$$k(a) = h^{1/2} a h^{1/2} \in L^1(\mathcal{A}, \tau_{\mathcal{A}}).$$

We first prove the assertions about ξ and ω_x. Both $ah^{1/2}$ and $h^{1/2}a$ belong to $L^2(\mathcal{A}, \tau_{\mathcal{A}})$. Traciality and the Cauchy–Schwarz inequality give

$$\begin{aligned} \|\xi(a)\|_2^2 &= \tau_{\mathcal{A}}(h^{1/4} a^* h^{1/2} a h^{1/4}) \\ &= \tau_{\mathcal{A}}\big((ah^{1/2})^* (h^{1/2} a)\big) \\ &\leq \|ah^{1/2}\|_2 \|h^{1/2} a\|_2 \\ &= \tau_{\mathcal{A}}(h^{1/2} a^* a h^{1/2})^{1/2} \tau_{\mathcal{A}}(h^{1/2} a a^* h^{1/2})^{1/2} \\ &= \varphi_{\mathcal{A}}(a^* a)^{1/2} \varphi_{\mathcal{A}}(a a^*)^{1/2}. \end{aligned} \tag{5.5}$$

The calculation is first made for $a_n = e_n a e_n$, where $e_n = \mathbb{1}_{[1/n, n]}(h)$. The convergences $a_n h^{1/2} \to a h^{1/2}$ and $h^{1/2} a_n \to h^{1/2} a$ in L^2 justify passage to the limit.

Since $k(x) \in L^1(\mathcal{A}, \tau_{\mathcal{A}})_+$ and $\tau_{\mathcal{A}}(k(x)) = \varphi_{\mathcal{A}}(x)$, ω_x is a finite normal positive functional with $\omega_x(1) = \varphi_{\mathcal{A}}(x)$. To prove (5.4), let

$$e_n = \mathbb{1}_{[\frac{1}{n}, n]}(h), \qquad x_n = e_n x e_n, \qquad a_n = e_n a e_n.$$

By (2.3), the projections e_n have finite trace. By traciality in $e_n\mathcal{A}e_n$, we have

$$\langle \xi(a_n), \xi(x_n)\rangle_2 = \tau_{\mathcal{A}}(h^{1/2}x_n h^{1/2}a_n).$$

Since $e_n h = h e_n$ and $e_n \nearrow 1$ strongly,

$$k(x_n) = e_n k(x) e_n \longrightarrow k(x) \quad \text{in } L^1, \qquad \xi(x_n) = e_n \xi(x) e_n \longrightarrow \xi(x) \quad \text{in } L^2, \qquad \xi(a_n) \longrightarrow \xi(a) \quad \text{in } L^2.$$

Moreover,

$$\sup_n \|a_n\| \le \|a\|, \qquad a_n \longrightarrow a \quad *\text{-strongly}.$$

Passing to the limit proves (5.4).

We next prove that S extends to a self-adjoint contraction. Identifying $\mathcal{A}_*$ with $L^1(\mathcal{A}, \tau_{\mathcal{A}})$, we first show that the preadjoint of Q satisfies

$$Q_* k(a) = k(Q(a)) \qquad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}). \tag{5.6}$$

Let $v_k = \sigma^{\phi}_{i/2}(y_k^*)$. By Example 3.10, (v_k) is a permutation of (y_k). We have

$$y_k^* h^{1/2} = h^{1/2} v_k, \qquad h^{1/2} y_k = v_k^* h^{1/2}. \tag{5.7}$$

Formula (5.7) follows first on compact spectral vectors for H, and then for the closed measurable products. For $a \in \mathcal{A}_+$ with $\varphi_{\mathcal{A}}(a) < \infty$, we have

$$y_k^* k(a) y_k = k(v_k a v_k^*).$$

By (5.7), the positive partial sums $\sum_{k\le n} y_k^* k(a) y_k$ converge in L^1, since

$$\tau_{\mathcal{A}}\left(\sum_{k\le n} y_k^* k(a) y_k\right) = \tau_{\mathcal{A}}\left(k(a)\sum_{k\le n} y_k y_k^*\right) \nearrow \varphi_{\mathcal{A}}(a).$$

By monotone convergence of the corresponding quadratic forms, we have

$$Q_* k(a) = \sum_k y_k^* k(a) y_k = k\left(\sum_k v_k a v_k^*\right) = k(Q(a)).$$

This proves (5.6) for positive a, and the general case follows by linearity.

By (5.6) and (5.4), for $a, b \in \mathfrak{m}_{\varphi_{\mathcal{A}}} \cap \mathcal{A}_+$,

$$\langle S\xi(a), \xi(b)\rangle_2 = \tau_{\mathcal{A}}(k(b)Q(a)) = \tau_{\mathcal{A}}(Q_* k(b) a) = \tau_{\mathcal{A}}(k(Q(b))a) = \langle \xi(a), S\xi(b)\rangle_2.$$

By linearity, we have $\langle Sv, w\rangle_2 = \langle v, Sw\rangle_2$ for $v, w \in D$. Also $S(D) \subset D$ by (5.2). For $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, (5.3), (5.5), Kadison's inequality for Q^n, and $\varphi_{\mathcal{A}} \circ Q^n = \varphi_{\mathcal{A}}$ give

$$\begin{aligned}
\|S^n \xi(a)\|_2^2 &= \|\xi(Q^n(a))\|_2^2 \\
&= \|h^{1/4} Q^n(a) h^{1/4}\|_2^2 \\
&= \tau_{\mathcal{A}}(h^{1/2} Q^n(a)^* h^{1/2} Q^n(a)) \\
&\le \varphi_{\mathcal{A}}((Q^n(a))^* Q^n(a))^{1/2} \varphi_{\mathcal{A}}(Q^n(a)(Q^n(a))^*)^{1/2} \\
&\le \varphi_{\mathcal{A}}(Q^n(a^*a))^{1/2} \varphi_{\mathcal{A}}(Q^n(aa^*))^{1/2} \\
&= \varphi_{\mathcal{A}}(a^*a)^{1/2} \varphi_{\mathcal{A}}(aa^*)^{1/2}.
\end{aligned}$$

Hence

$$C_v := \sup_{m\ge 0} \|S^m v\|_2 < \infty \qquad (v \in D).$$

For $v \in D$, by symmetry, we have

$$\|Sv\|_2^2 = \langle S^2 v, v\rangle_2 \le \|S^2 v\|_2 \|v\|_2. \tag{5.8}$$

Iterating (5.8), we have

$$\|Sv\|_2 \le \|S^{2^n}v\|_2^{1/2^n}\|v\|_2^{1-1/2^n} \le C_v^{1/2^n}\|v\|_2^{1-1/2^n} \longrightarrow \|v\|_2.$$

It remains to check that D is dense. With $e_n = \mathbb{1}_{[1/n,n]}(h)$, both h and h^{-1} are bounded on e_n, and $\tau_{\mathcal{A}}(e_n) < \infty$. Every bounded $b \in e_n\mathcal{A}e_n$ has the form

$$b = \xi(h^{-1/4}bh^{-1/4}), \qquad h^{-1/4}bh^{-1/4} \in e_n\mathcal{A}e_n \subset \mathfrak{m}_{\varphi_{\mathcal{A}}}.$$

These finite-corner elements are dense in $L^2(\mathcal{A},\tau_{\mathcal{A}})$. Therefore S extends to a self-adjoint contraction. □

Let $f \in C_c^\infty(\mathbb{R})$ be real with $0 \le f \le 1$, equal to 1 near 0, even and nonincreasing in $|s|$. Define $H_R = f(H/R)$, $R \ge 1$. Then $H_R \nearrow 1$ strongly as $R \to \infty$, and by (2.2), we have

$$\varphi_{\mathcal{A}}(H_R) = R\int_{\mathbb{R}} f(s)\,\mathrm{d}s < \infty.$$

Lemma 5.2. *Let $x \in \mathcal{A}_+$ satisfy $\varphi_{\mathcal{A}}(x) < \infty$. Assume that there is a constant $0 \le C < \infty$ such that*

$$\sum_{k=0}^{n-1} Q^k(x) \le C1 \qquad (n \ge 1). \tag{5.9}$$

Then

$$\varphi_{\mathcal{A}}(x) \le \liminf_{R\to\infty} C\|(1-S)^{1/2}\xi(H_R)\|_2^2. \tag{5.10}$$

Proof. The case $\varphi_{\mathcal{A}}(x) = 0$ is immediate. Assume that $\varphi_{\mathcal{A}}(x) > 0$. Let $a_n = \sum_{k=0}^{n-1} Q^k(x) \le C1$. By (5.2) and (5.3),

$$\varphi_{\mathcal{A}}(a_n) = n\varphi_{\mathcal{A}}(x) < \infty, \qquad (1-S)\xi(a_n) = \xi(x - Q^n(x)).$$

By (5.4) and positivity,

$$\|(1-S)^{1/2}\xi(a_n)\|_2^2 = \langle \xi(x-Q^n(x)), \xi(a_n)\rangle_2 = \omega_x(a_n) - \omega_{a_n}(Q^n(x)) \le \omega_x(a_n) \le C\omega_x(1) = C\varphi_{\mathcal{A}}(x). \tag{5.11}$$

Fix $R \ge 1$. By (5.4) and positivity, $\omega_x(H_R) = \overline{\omega_{H_R}(x)} = \omega_{H_R}(x)$. Since $a_n - Q(a_n) = x - Q^n(x)$ and $1 - S \ge 0$, (5.3) and (5.11) give

$$\begin{aligned}
&|\omega_x(H_R) - \omega_{H_R}(Q^n(x))|^2 \\
=&\big|\omega_{H_R}(x - Q^n(x))\big|^2 \\
=&\big|\langle \xi(x-Q^n(x)), \xi(H_R)\rangle_2\big|^2 \\
=&\big|\langle (1-S)\xi(a_n), \xi(H_R)\rangle_2\big|^2 \\
=&\big|\langle (1-S)^{1/2}\xi(a_n), (1-S)^{1/2}\xi(H_R)\rangle_2\big|^2 \\
\le&\|(1-S)^{1/2}\xi(a_n)\|_2^2\|(1-S)^{1/2}\xi(H_R)\|_2^2 \\
\le&C\varphi_{\mathcal{A}}(x)\|(1-S)^{1/2}\xi(H_R)\|_2^2.
\end{aligned} \tag{5.12}$$

By (5.9), for every $N \ge 1$,

$$\sum_{n=0}^{N-1} \omega_{H_R}(Q^n(x)) = \omega_{H_R}\left(\sum_{n=0}^{N-1} Q^n(x)\right) \le C\omega_{H_R}(1) = C\varphi_{\mathcal{A}}(H_R) < \infty.$$

Letting $N \to \infty$, we have

$$\sum_{n=0}^{\infty} \omega_{H_R}(Q^n(x)) \le C\varphi_{\mathcal{A}}(H_R) < \infty,$$

so $\omega_{H_R}(Q^n(x)) \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (5.12), we have

$$|\omega_x(H_R)|^2 \le C\varphi_{\mathcal{A}}(x)\|(1-S)^{1/2}\xi(H_R)\|_2^2 \qquad (R \ge 1). \tag{5.13}$$

Since $H_R \nearrow 1$ strongly as $R \to \infty$, normality of ω_x and (5.13) give

$$\varphi_{\mathcal{A}}(x)^2 = |\omega_x(1)|^2 = \lim_{R\to\infty} |\omega_x(H_R)|^2 \le \liminf_{R\to\infty} C\varphi_{\mathcal{A}}(x)\|(1-S)^{1/2}\xi(H_R)\|_2^2.$$

Dividing by $\varphi_{\mathcal{A}}(x)$ proves (5.10). □

Let ν be the spectral distribution of $\log \Delta_\phi$ under φ. By (3.4),

$$m_2 := \int_{\mathbb{R}} s^2 \,\mathrm{d}\nu(s) = \varphi((\log \Delta_\phi)^2) < \infty.$$

Lemma 5.3. *For every $R \ge 1$,*

$$\|(1-S)^{1/2}\xi(H_R)\|_2^2 \le \frac{m_2\|f'\|_2^2}{2R}. \tag{5.14}$$

Moreover, if $x \in \mathcal{A}_+$ satisfies $\varphi_{\mathcal{A}}(x) < \infty$ and $\sum_{k=0}^{n-1} Q^k(x) \le C1$ for some $0 \le C < \infty$ and every $n \ge 1$, then $x = 0$.

Proof. Let $g \in C_c^\infty(\mathbb{R})$ be real. By (2.2), $g(H) \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, and $g(H) \in \mathcal{A}_{\varphi_{\mathcal{A}}}$. Expanding the commutators, we have

$$\sum_k [y_k, g(H)][y_k, g(H)]^* = Q(g(H)^2) - Q(g(H))g(H) - g(H)Q(g(H)) + g(H)^2.$$

By Kadison's inequality, (5.2), and (2.2),

$$\varphi_{\mathcal{A}}(Q(g(H))^2) \le \varphi_{\mathcal{A}}(Q(g(H)^2)) = \varphi_{\mathcal{A}}(g(H)^2) = \int_{\mathbb{R}} |g(s)|^2 \,\mathrm{d}s < \infty.$$

Hence all four terms on the right belong to $\mathfrak{m}_{\varphi_{\mathcal{A}}}$. Applying $\varphi_{\mathcal{A}}$ and using the $\varphi_{\mathcal{A}}$-centralizer property of $g(H)$, the linear extension of (5.4) to self-adjoint elements, and (5.3), we have

$$\begin{aligned}\sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) &= 2\big(\varphi_{\mathcal{A}}(g(H)^2) - \varphi_{\mathcal{A}}(g(H)Q(g(H)))\big)\\ &= 2\big(\langle \xi(g(H)), \xi(g(H))\rangle_2 - \langle S\xi(g(H)), \xi(g(H))\rangle_2\big)\\ &= 2\|(1-S)^{1/2}\xi(g(H))\|_2^2.\end{aligned} \tag{5.15}$$

We compute the left-hand side by proving

$$\sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) = \int_{\mathbb{R}}\int_{\mathbb{R}} |g(r+s) - g(r)|^2 \,\mathrm{d}r \,\mathrm{d}\nu(s). \tag{5.16}$$

We use the unitary Fourier transform

$$\widehat{g}(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-itr} g(r) \,\mathrm{d}r.$$

By the crossed-product relation, we have

$$[y_k, g(H)] = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{g}(t)(y_k - \sigma_t^\phi(y_k))\lambda_t \,\mathrm{d}t. \tag{5.17}$$

With the normalization in (2.2), for $X = (2\pi)^{-1/2}\int_{\mathbb{R}} a_t\lambda_t \,\mathrm{d}t = (2\pi)^{-1/2}\int_{\mathbb{R}} a_t e^{itH} \,\mathrm{d}t$, we have

$$\varphi_{\mathcal{A}}(X^*X) = \int_{\mathbb{R}} \rho(a_t^* a_t) \,\mathrm{d}t, \qquad \varphi_{\mathcal{A}}(XX^*) = \int_{\mathbb{R}} \rho(a_t a_t^*) \,\mathrm{d}t, \tag{5.18}$$

The identities in (5.18) hold first on the crossed-product Hilbert algebra and extend by its L^2 isometry; see [Haa78a]. In the first identity in (5.18), the zeroth coefficient is $\int_{\mathbb{R}} \sigma^{\rho}_{-t}(a_t^* a_t)\,\mathrm{d}t$, and the modular invariance of ρ removes this twist.

By $\varphi = \sum_k \langle \cdot\, y_k^* \xi_\phi, y_k^* \xi_\phi \rangle$ and the definition of ν as the spectral distribution of $\log \Delta_\phi$ under φ, we have

$$\begin{aligned}
\sum_k \phi\big((y_k - \sigma_t^\phi(y_k))(y_k - \sigma_t^\phi(y_k))^*\big) &= \sum_k \|(1-\Delta_\phi^{it}) y_k^* \xi_\phi\|^2 \\
&= \sum_k \big\langle (1-\Delta_\phi^{-it})(1-\Delta_\phi^{it}) y_k^*\xi_\phi, y_k^*\xi_\phi \big\rangle \\
&= \varphi\big((1-\Delta_\phi^{-it})(1-\Delta_\phi^{it})\big) \\
&= \int_{\mathbb{R}} (1-e^{-its})(1-e^{its})\,\mathrm{d}\nu(s) \\
&= \int_{\mathbb{R}} |1-e^{its}|^2\,\mathrm{d}\nu(s).
\end{aligned}$$

By (5.17), (5.18), and scalar Plancherel,

$$\begin{aligned}
\sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) &= \int_{\mathbb{R}} |\widehat{g}(t)|^2 \int_{\mathbb{R}} |1-e^{its}|^2\,\mathrm{d}\nu(s)\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{g}(t)|^2 |1-e^{its}|^2\,\mathrm{d}t\,\mathrm{d}\nu(s) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |g(r+s)-g(r)|^2\,\mathrm{d}r\,\mathrm{d}\nu(s).
\end{aligned}$$

This proves (5.16). Now take $g(r) = f(r/R)$, so $g(H) = H_R$. Since $\|g(\cdot + s) - g\|_2 \le |s|\|g'\|_2 = \frac{|s|}{\sqrt{R}}\|f'\|_2$, by (5.16), we have

$$\sum_k \varphi_{\mathcal{A}}([y_k, H_R][y_k, H_R]^*) \le \frac{m_2 \|f'\|_2^2}{R}. \tag{5.19}$$

Combining (5.15) with $g(H) = H_R$ and (5.19) proves (5.14).

Now let x satisfy the hypotheses of Lemma 5.2. By (5.10) and (5.14),

$$0 \le \varphi_{\mathcal{A}}(x) \le \liminf_{R\to\infty} \frac{C m_2 \|f'\|_2^2}{2R} = 0.$$

Hence $\varphi_{\mathcal{A}}(x) = 0$, and by faithfulness of $\varphi_{\mathcal{A}}$, we have $x = 0$. □

We can now identify $\mathrm{Har}(Q, \mathcal{A})$.

Proposition 5.4. *The fixed-point space of Q is*

$$\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}.$$

Proof. We only need to prove that $\mathrm{Har}(Q, \mathcal{A}) \subset M' \cap \mathcal{A}$. Let $a \in \mathrm{Har}(Q, \mathcal{A})$. By Kadison's inequality,

$$d := Q(aa^*) - aa^* = Q(aa^*) - Q(a)Q(a)^* \ge 0.$$

We have

$$\sum_{k=0}^{n-1} Q^k(d) = Q^n(aa^*) - aa^* \le \|a\|^2 1 \qquad (n \ge 1).$$

Suppose that $d \neq 0$. By Lemma 2.2, take $0 \neq x \leq d$ with $\varphi_{\mathcal{A}}(x) < \infty$. Then

$$\sum_{k=0}^{n-1} Q^k(x) \leq \sum_{k=0}^{n-1} Q^k(d) \leq \|a\|^2 1 \qquad (n \geq 1).$$

By Lemma 5.3, we have $x = 0$, a contradiction. Hence $Q(aa^*) = aa^*$. Applying the same argument to $a^* \in \mathrm{Har}(Q, \mathcal{A})$, we have $Q(a^*a) = a^*a$, so a belongs to the multiplicative domain of Q. Expanding as in (3.10), we have

$$\sum_k (y_k a - a y_k)(y_k a - a y_k)^* = 0.$$

Hence $[a, y_k] = 0$ for every k. Since $\overline{\mathbb{C}[y_k]}^{\mathrm{w}} = M$, $a \in M' \cap \mathcal{A}$. □

Marrakchi proved the relative bicentralizer conjecture, in particular, when M is amenable, has no type III_1 direct summand, is a tensor product of two type III_1 von Neumann algebras, or admits a Cartan subalgebra [Mar25, Theorem E(1)–(4)]. For completeness, we give a new proof covering these cases as well. We now finish the proof by applying Theorem 4.1.

Proof of Theorem A. As in the beginning of the proof of Corollary 4.3, we may assume that M has separable predual.

Suppose first that M is either finite or properly infinite. Fix a normal faithful state ϕ on M, taking ϕ tracial when M is finite. Let Q be the map in (5.1) on $\mathcal{A} = c_{\phi \circ E}(P)$, with coefficients from Example 3.10. For $x \in \mathcal{A}$, define

$$C_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} Q^k(x), \qquad K_x = \overline{\mathrm{conv}}^{\mathrm{w}}\{uxu^* \mid u \in \mathcal{U}(M)\}.$$

Since

$$\|Q(C_n(x)) - C_n(x)\| \leq \frac{2\|x\|}{n},$$

normality of Q and Proposition 5.4 imply that every ultraweak cluster point of $(C_n(x))$ belongs to $M' \cap \mathcal{A}$.

If M is properly infinite, each C_n is an M-inner ucp map. Hence $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, and by Theorem 4.1, we have the weak Dixmier property.

If M is finite, by the tracial choice in Example 3.10, we have

$$y_k = \lambda_k u_k, \qquad Q(x) = \sum_k \lambda_k^2 u_k x u_k^*, \qquad \sum_k \lambda_k^2 = 1.$$

Hence $Q(K_x) \subset K_x$ and $C_n(x) \in K_x$ for every n. Every ultraweak cluster point belongs to $K_x \cap (M' \cap \mathcal{A})$, which proves the finite case.

For general M, write

$$M = M_{\mathrm{fin}} \oplus M_\infty$$

for its finite and properly infinite parts. We may assume that both summands are nonzero. Let p_1, p_2 be their units and let $P_i = p_i P p_i$. There is a normal faithful conditional expectation

$$F : P \to P_1 \oplus P_2, \qquad F(a) = p_1 a p_1 + p_2 a p_2.$$

The restrictions

$$E_1 = E|_{P_1} : P_1 \to M_{\mathrm{fin}}, \qquad E_2 = E|_{P_2} : P_2 \to M_\infty$$

are normal faithful conditional expectations.

Take a normal faithful state ϕ on M whose restriction to M_{fin} is tracial, and let $\rho = \phi \circ E$, $\mathcal{A} = c_\rho(P)$, and $\rho_i = \rho|_{P_i}/\rho(p_i)$. Since $p_i \in M_\phi$, by bimodularity, we have $p_i \in P_\rho$. Hence F preserves ρ and extends to a normal faithful conditional expectation

$$\widetilde{F} : \mathcal{A} \to p_1 \mathcal{A} p_1 \oplus p_2 \mathcal{A} p_2, \qquad p_i \mathcal{A} p_i \cong c_{\rho_i}(P_i).$$

For $x \in \mathcal{A}$, since $p_1 - p_2 \in \mathcal{U}(M)$,

$$\widetilde{F}(x) = p_1 x p_1 + p_2 x p_2 = \frac{1}{2}\big(x + (p_1 - p_2)x(p_1 - p_2)\big) \in K_x.$$

The finite and properly infinite cases give the weak Dixmier property for $M_{\mathrm{fin}} \subset p_1 \mathcal{A} p_1$ and $M_\infty \subset p_2 \mathcal{A} p_2$, respectively. As in the proof of Corollary 4.3, taking product nets of the two unitary averaging nets, we have

$$K_x \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

This proves that $M \subset c(P)$ has the weak Dixmier property. Taking $P = M$, we have the last assertion. □

Proof of Corollary B. By Theorem A, $M \subset c(M)$ has the weak Dixmier property. Since M is of type III_1, by [Haa87, Theorem 3.1], we have $\mathrm{B}(M, \phi) = \mathbb{C}1$ for every normal faithful state ϕ on M. □

Proof of Corollary C. By Theorem A, we have the weak Dixmier property for $M \subset c(P)$. Hence by [Mar25, Theorem D], we have (2.1) for every normal faithful state ϕ on M. □

6. Kadison's problem on MASA

In this section, we use the generating inner ucp map associated with an nrsg hyperstate to give a new proof of the solution to Kadison's problem (Theorem G).

Fix an inclusion $M \subset P$ of von Neumann algebras with separable preduals and a normal faithful conditional expectation $E : P \to M$ as in Theorem G. Fix a normal faithful state ϕ on M, and let $\rho = \phi \circ E$. Since ρ is faithful, the only $x \in P$ with $\rho(ax) = 0$ for all $a \in P$ is $x = 0$. By the Hahn–Banach theorem,

$$\overline{\{\rho(a\,\cdot\,) \mid a \in P\}}^{\|\cdot\|} = P_*. \tag{6.1}$$

For a unital finite-dimensional subalgebra $F \subset P$, define

$$\Psi_F(x) = \int_{\mathcal{U}(F)} uxu^*\,\mathrm{d}u \qquad (x \in P),$$

where $\mathrm{d}u$ is the normalized Haar measure on $\mathcal{U}(F)$. Then Ψ_F is the normal conditional expectation onto $F' \cap P$. For a finite-dimensional unital abelian subalgebra $D \subset P$, let $\mathrm{At}(D)$ denote the set of minimal projections of D, and define

$$\Phi_D(x) = \sum_{p \in \mathrm{At}(D)} pxp \qquad (x \in P).$$

Then $\Phi_D = \Psi_D$ is the normal conditional expectation onto $D' \cap P$. In particular,

$$\|\theta \circ \Phi_D\| = \|\theta|_{D' \cap P}\| \qquad (\theta \in P_*). \tag{6.2}$$

We use the same notation for a corner ePe in place of P.

Take an nrsg (M, ϕ)-hyperstate φ with inner ucp coefficients $(y_k) \subset M$. For example, we can take the hyperstate constructed in Example 3.10. In particular,

$$\sum_k y_k y_k^* = 1, \qquad \overline{\mathbb{C}[y_k]}^{\mathrm{w}} = M.$$

Define the normal M-inner ucp map

$$Q(x) = \sum_k y_k x y_k^* \qquad (x \in P). \tag{6.3}$$

The restriction of Q to M preserves ϕ, and Q is the map $\mathcal{P}_\varphi^o$ of Proposition 3.14 for $\mathcal{A} = P$. We will also use this construction for the corners and central summands in Lemma 6.12 and Proposition 6.11.

We first outline the proof. Let $C_n = (1/n)\sum_{\ell=0}^{n-1} Q^\ell$ be the Cesàro means of Q, and let

$$M_\perp = \{\theta \in P_* \mid \theta|_M = 0\}.$$

Since P_* is separable, we can take a sequence $(\theta_n)_{n\geq 1}$ which is norm dense in $M_\perp$. Let $\theta \in M_\perp$ and $\varepsilon > 0$; finitely many functionals, such as $\theta_1, \ldots, \theta_n$, are treated simultaneously. When M is **continuous** (i.e., M has no type I direct summand) and properly infinite, the proof follows the chain

$$
\begin{aligned}
& \mathrm{Har}(Q,P) = \mathcal{Z}(M) && \text{(Proposition 3.14)}\\
(6.4)\quad \Longrightarrow\ & \|\theta \circ C_n\| \to 0 \quad (n\to\infty) && \text{(Corollary 6.1)}\\
(6.5)\quad \Longrightarrow\ & \exists \text{ matrix algebra } F_{\theta,\varepsilon} \subset M : && \\
& \quad \|\theta|_{F'_{\theta,\varepsilon}\cap P}\| < \varepsilon && \text{(Proposition 6.4)}\\
(6.6)\quad \Longrightarrow\ & \exists \text{ finite-dimensional abelian } D_{\theta,\varepsilon} \subset M : && \\
& \quad \|\theta \circ \Phi_{D_{\theta,\varepsilon}}\| = \|\theta|_{D'_{\theta,\varepsilon}\cap P}\| < \varepsilon && \text{(Corollary 6.8)}\\
(6.7)\quad \Longrightarrow\ & \exists \text{ finite-dimensional abelian } D_1 \subset D_2 \subset \cdots \subset M : && \\
& \quad \|\theta_j|_{D'_n\cap P}\| < 2^{-n} \quad (j \leq n) && \text{(Lemma 6.12)}\\
(6.8)\quad \Longrightarrow\ & \theta_j|_{A'_0\cap P} = 0 \text{ for } A_0 = \textstyle\bigvee_n D_n \quad (j\geq 1) && \text{(proof of Theorem G)}\\
(6.9)\quad \Longrightarrow\ & A'_0 \cap P \subset M && \text{(proof of Theorem G)}\\
(6.10)\quad \Longrightarrow\ & \text{every MASA } A \text{ of } M \text{ containing } A_0 \text{ satisfies} && \\
& \quad A' \cap P = A && \text{(proof of Theorem G).}
\end{aligned}
$$

The Poisson boundary method gives the generating inner ucp map Q and, by Proposition 3.14 and $M' \cap P = \mathcal{Z}(M)$, the fixed-point equality $\mathrm{Har}(Q,P) = \mathcal{Z}(M)$. The passage to (6.4) is the standard predual Cesàro argument. The main simplification is the passage from (6.4) to (6.5): we obtain the matrix approximation directly from the inner ucp maps, without passing through an AFD subalgebra with the same relative commutant as in [Mar26, Theorem B]. The steps (6.6)–(6.10) are well-established steps originating in Popa's work [Pop81] and also used by Marrakchi [Mar25, Mar26]. For the passage to (6.6), we give a separate argument in Proposition 6.7, which has the same purpose as [Mar26, Lemma 1.1] but uses a different proof: we enlarge F inside the centralizer of a suitable state and use (6.13). By (6.2), (6.6) is the formulation of local quantization for normal functionals vanishing on $M' \cap P = \mathcal{Z}(M)$; see [Mar25, Definition 10.2]. The recursive refinement (6.7) follows Popa's construction [Pop81] and is essentially the same as [Mar26, Lemma 1.3], while the passage through (6.8) and (6.9) to (6.10) follows Popa's construction and the predual argument in [Mar25, Proposition 10.5] and [Mar26, Theorem 1.4]. We include these steps for completeness. For finite M, Lemma 6.10 is essentially Popa's averaging and refinement argument [Pop81], using the tracial case of Example 3.10 to express unitary averaging through an inner ucp map.

The following corollary follows from Proposition 3.14 and $M' \cap P = \mathcal{Z}(M)$.

Corollary 6.1. *Let $M \subset P$ be as in Theorem G, and let Q be defined by (6.3). Then $\mathrm{Har}(Q,P) = \mathcal{Z}(M)$. Moreover, the Cesàro means $C_n = (1/n)\sum_{j=0}^{n-1} Q^j$ satisfy*

$$\|\theta \circ C_n\| \longrightarrow 0 \quad (n\to\infty)$$

for every $\theta \in P_*$ with $\theta|_{\mathcal{Z}(M)} = 0$.

Proof. The composition $P \xrightarrow{E} M \subset \mathcal{B}_\varphi$ is a normal faithful ucp map whose restriction to M is the identity. By Proposition 3.14, $\mathrm{Har}(Q,P) = M' \cap P = \mathcal{Z}(M)$.

Let $T = Q_*$ be the predual map on P_*, so that $T\theta = \theta \circ Q$. The annihilator of $\mathrm{Ran}(\mathrm{id} - T)$ in $P = (P_*)^*$ is $\mathrm{Har}(Q,P) = \mathcal{Z}(M)$. Hence by the Hahn–Banach theorem, we have

$$\overline{\mathrm{Ran}(\mathrm{id}-T)}^{\|\cdot\|} = \{\theta \in P_* \mid \theta|_{\mathcal{Z}(M)} = 0\}.$$

If $\theta = (\mathrm{id} - T)\eta$ for some $\eta \in P_*$, then we have

$$\|\theta \circ C_n\| = \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j \theta \right\| = \frac{1}{n} \|\eta - T^n \eta\| \leq \frac{2\|\eta\|}{n}.$$

Since $\|\theta \circ C_n\| \leq \|\theta\|$ for all $\theta \in P_*$ and $n \geq 1$, the convergence extends to the norm closure of $\mathrm{Ran}(\mathrm{id} - T)$, which finishes the proof. □

We next approximate normal M-inner ucp maps by M-inner ucp maps with finitely many inner ucp coefficients.

Lemma 6.2. *Let $M \subset P$ be a W^*-inclusion, and let $R(x) = \sum_{i \geq 1} a_i x a_i^*$ be a normal M-inner ucp map on P. Then there are M-inner ucp maps R_m on P, each with finitely many inner ucp coefficients, such that*

$$\|\theta \circ R_m - \theta \circ R\| \longrightarrow 0 \quad (m \to \infty)$$

for every $\theta \in P_*$.

Proof. For $m \geq 1$, define

$$b_m = \left(1 - \sum_{i=1}^{m} a_i a_i^*\right)^{1/2}, \qquad R_m(x) = \sum_{i=1}^{m} a_i x a_i^* + b_m x b_m.$$

Since $b_m \in M$, each R_m is an M-inner ucp map with $m+1$ inner ucp coefficients. For $\theta \in P_*^+$, the functional $\theta \circ R_m - \theta \circ R$ is the difference of the positive functionals $\theta(b_m \cdot b_m)$ and $\sum_{i>m} \theta(a_i \cdot a_i^*)$. Since $\sum_{i>m} a_i a_i^* = b_m^2$, we have

$$\|\theta \circ R_m - \theta \circ R\| \leq 2\theta(b_m^2) \longrightarrow 0 \quad (m \to \infty).$$

Every normal functional is a linear combination of positive normal functionals, which proves the lemma. □

When M is properly infinite, an M-inner ucp map with finitely many inner ucp coefficients is a compression of an endomorphism of P whose range is the relative commutant of a full matrix algebra in M. The following lemma removes the compression asymptotically.

Lemma 6.3. *Let $M \subset P$ be a W^*-inclusion such that M is a properly infinite von Neumann algebra with separable predual. Let $R(x) = \sum_{i=1}^{r} a_i x a_i^*$ be an M-inner ucp map on P. Then there are normal unital injective endomorphisms $\alpha_n : P \to P$ and unital full matrix algebras $F_n \subset M$ such that*

$$\alpha_n(P) = F_n' \cap P, \qquad \|\theta \circ \alpha_n - \theta \circ R\| \longrightarrow 0 \quad (n \to \infty,\ \theta \in P_*).$$

Proof. We use the same matrix construction as in the proof of Theorem 4.1. Let $d = r + 1$, and fix a projection $e \in M$ with $e \sim 1 - e \sim 1$. Take matrix units $(e_{ij})_{1 \leq i,j \leq d}$ in $\mathrm{M}_d(\mathbb{C}) \otimes 1$. Take an isometry $v_e \in M$ with $v_e v_e^* = e$ and an isomorphism

$$\Theta_e : M \xrightarrow{\sim} \mathrm{M}_d(\mathbb{C}) \bar{\otimes} M, \qquad \Theta_e(e) = e_{11},$$

which extends to an isomorphism $P \xrightarrow{\sim} \mathrm{M}_d(\mathbb{C}) \bar{\otimes} P$, still denoted by Θ_e, such that

$$\Theta_e^{-1} \begin{pmatrix} x & 0_{1\times r} \\ 0_{r\times 1} & 0_{r\times r} \end{pmatrix} = v_e x v_e^* \qquad (x \in P).$$

In $\mathrm{M}_d(\mathbb{C}) \bar{\otimes} M$, let

$$Z = \begin{pmatrix} 0 & a_1 & \cdots & a_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $\sum_{i=1}^r a_i a_i^* = 1$ and the first column of Z is 0,

$$ZZ^* = e_{11}, \qquad 1 - Z^*Z \geq e_{11} \sim 1, \qquad 1 - e_{11} \sim 1.$$

Take a partial isometry $W \in \mathrm{M}_d(\mathbb{C}) \bar{\otimes} M$ with

$$W^*W = 1 - Z^*Z, \qquad WW^* = 1 - e_{11}.$$

Then $U = Z + W$ is unitary and its first row is $(0, a_1, \ldots, a_r)$. For $x \in P$, by matrix multiplication, we have

$$U \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} U^* = \begin{pmatrix} R(x) & * \\ * & * \end{pmatrix}.$$

Define

$$\alpha_e(x) = \Theta_e^{-1}\big(U(1_d \otimes x)U^*\big), \qquad F_e = \Theta_e^{-1}\big(U(\mathrm{M}_d(\mathbb{C}) \otimes 1)U^*\big).$$

Then α_e is a normal unital injective endomorphism, $F_e \subset M$ is a unital full matrix algebra, and

$$\alpha_e(P) = F_e' \cap P.$$

Since $\Theta_e(e) = e_{11}$, we have

$$e\alpha_e(x)e = \Theta_e^{-1} \begin{pmatrix} R(x) & 0 \\ 0 & 0 \end{pmatrix} = v_e R(x) v_e^* \qquad (x \in P). \tag{6.11}$$

Finally, take projections $e_n \nearrow 1$ in M with

$$e_n - e_{n-1} \sim 1 - e_n \sim 1 \qquad (n \geq 1).$$

Take partial isometries $v_n \in M$ with

$$v_n^* v_n = 1 - e_{n-1}, \qquad v_n v_n^* = e_n - e_{n-1},$$

and use $v_{e_n} = e_{n-1} + v_n$ in the construction above. Then

$$v_{e_n}^* v_{e_n} = 1, \qquad v_{e_n} v_{e_n}^* = e_n, \qquad v_{e_n} e_{n-1} = v_{e_n}^* e_{n-1} = e_{n-1},$$

so $v_{e_n} \to 1$ $*$-strongly. Let $\alpha_n = \alpha_{e_n}$ and $F_n = F_{e_n}$. For $\theta \in P_*$, by (6.11), we have

$$\theta \circ \alpha_n - \theta \circ R = (\theta - \theta(e_n \cdot e_n)) \circ \alpha_n + (\theta(v_{e_n} \cdot v_{e_n}^*) - \theta) \circ R,$$

and hence

$$\|\theta \circ \alpha_n - \theta \circ R\| \leq \|\theta - \theta(e_n \cdot e_n)\| + \|\theta(v_{e_n} \cdot v_{e_n}^*) - \theta\| \longrightarrow 0 \qquad (n \to \infty).$$

The convergence follows from $e_n \to 1$ strongly and $v_{e_n} \to 1$ $*$-strongly, first for vector functionals and then for every $\theta \in P_*$ by norm approximation. This finishes the proof. □

We now combine Corollary 6.1 with Lemmas 6.2 and 6.3.

Proposition 6.4. *Let $M \subset P$ be as in Theorem G, and assume that M is properly infinite. Let $\theta_1, \ldots, \theta_s \in P_*$ satisfy $\theta_j|_{Z(M)} = 0$, and let $\varepsilon > 0$. Then there is a unital full matrix algebra $F \subset M$ such that*

$$\|\theta_j\big|_{F' \cap P}\| < \varepsilon \qquad (1 \leq j \leq s).$$

Proof. For $n \geq 1$, let $C_n = (1/n) \sum_{\ell=0}^{n-1} Q^\ell$ as in Corollary 6.1. Each C_n is a normal M-inner ucp map. By Corollary 6.1 and Lemmas 6.2 and 6.3, we can successively take $n \geq 1$, an M-inner ucp map R with finitely many inner ucp coefficients, and a normal unital injective endomorphism $\alpha : P \to P$ with $\alpha(P) = F' \cap P$ for a unital full matrix algebra $F \subset M$, such that

$$\|\theta_j \circ C_n\| < \frac{\varepsilon}{3}, \quad \|\theta_j \circ R - \theta_j \circ C_n\| < \frac{\varepsilon}{3}, \quad \|\theta_j \circ \alpha - \theta_j \circ R\| < \frac{\varepsilon}{3} \quad (1 \leq j \leq s).$$

Since α is a $*$-isomorphism of P onto $F' \cap P$, it maps the unit ball of P onto that of $F' \cap P$. Therefore

$$\|\theta_j\big|_{F'\cap P}\| = \|\theta_j \circ \alpha\| \leq \|\theta_j \circ \alpha - \theta_j \circ R\| + \|\theta_j \circ R - \theta_j \circ C_n\| + \|\theta_j \circ C_n\| < \varepsilon$$

for $1 \leq j \leq s$, which proves the proposition. □

The next lemma computes the average of the maps Φ_D over the diagonal subalgebras of a matrix algebra.

Lemma 6.5. *Let $F \subset P$ be a unital full matrix algebra with $F \cong \mathrm{M}_d(\mathbb{C})$, and let $D \subset F$ be a maximal abelian subalgebra. Then*

$$\int_{\mathcal{U}(F)} \Phi_{uDu^*}\,\mathrm{d}u = \frac{\mathrm{id} + d\Psi_F}{d+1}. \tag{6.12}$$

If ρ is a normal faithful state on P with $F \subset P_\rho$, then

$$\int_{\mathcal{U}(F)} \|\Phi_{uDu^*}(b)\|_{2,\rho}^2\,\mathrm{d}u = \frac{\|b\|_{2,\rho}^2}{d+1} \tag{6.13}$$

for every $b \in P$ with $\Psi_F(b) = 0$.

Proof. Throughout this section, tr_n denotes the normalized trace on $\mathrm{M}_n(\mathbb{C})$. Under the identification $P = F\bar{\otimes}(F' \cap P)$, both sides of (6.12) act only on the first tensor factor. Hence we only need to treat $P = F = \mathrm{M}_d(\mathbb{C})$, where $\Psi_F(x) = \mathrm{tr}_d(x)1$. The case $d = 1$ is immediate, so assume that $d \geq 2$. By [AP17, Theorem 3.1.4], we may take matrix units (e_{ij}) such that $D = \bigoplus_{i=1}^d \mathbb{C}e_{ii}$. Hence

$$\Phi_D(x) = \sum_{i=1}^d e_{ii}xe_{ii} = \mathrm{diag}(x_{11}, \ldots, x_{dd}).$$

Let Ψ denote the left-hand side of (6.12). For $v \in \mathcal{U}(F)$, by the invariance of Haar measure under $u \mapsto vu$, we have

$$\mathrm{Ad}(v) \circ \Psi \circ \mathrm{Ad}(v^*) = \int_{\mathcal{U}(F)} \mathrm{Ad}(v) \circ \Phi_{uDu^*} \circ \mathrm{Ad}(v^*)\,\mathrm{d}u = \int_{\mathcal{U}(F)} \Phi_{(vu)D(vu)^*}\,\mathrm{d}u = \Psi.$$

Hence $\Psi(vxv^*) = v\Psi(x)v^*$ for $x \in F$ and $v \in \mathcal{U}(F)$. Fix $i \neq j$, and write $\Psi(e_{ij}) = \sum_{p,q} c_{pq}e_{pq}$. For $w = \mathrm{diag}(z_1, \ldots, z_d)$ with $|z_\ell| = 1$, we have

$$w\Psi(e_{ij})w^* = \Psi(we_{ij}w^*) = z_i\overline{z_j}\Psi(e_{ij}),$$

so, by comparing the (p, q) entries, we have

$$c_{pq}\big(z_p\overline{z_q} - z_i\overline{z_j}\big) = 0 \qquad (|z_1| = \cdots = |z_d| = 1).$$

Varying $z_1, \ldots, z_d$ independently shows that $c_{pq} = 0$ unless $(p, q) = (i, j)$. Hence $\Psi(e_{ij}) = t_{ij}e_{ij}$. For a permutation σ of $\{1, \ldots, d\}$, let u_σ be its permutation matrix. Then

$$\Psi(e_{\sigma(i)\sigma(j)}) = \Psi(u_\sigma e_{ij}u_\sigma^*) = u_\sigma\Psi(e_{ij})u_\sigma^* = t_{ij}e_{\sigma(i)\sigma(j)}.$$

Hence $t_{\sigma(i)\sigma(j)} = t_{ij}$. Since permutations act transitively on ordered pairs of distinct indices, there is $t \in \mathbb{C}$ such that

$$\Psi(e_{ij}) = te_{ij} \qquad (i \neq j).$$

Let

$$v = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus 1_{d-2}.$$

Since $v(e_{12} + e_{21})v^* = e_{11} - e_{22}$, we have

$$\Psi(e_{11} - e_{22}) = t(e_{11} - e_{22}).$$

By conjugating by permutation matrices, we have the same identity for $e_{ii} - e_{jj}$. Since $\Psi(1) = 1$, it follows that

$$\Psi(x) = \mathrm{tr}_d(x)1 + t\big(x - \mathrm{tr}_d(x)1\big) \qquad (x \in \mathrm{M}_d(\mathbb{C})).$$

Each Φ_{uDu^*} is an idempotent of rank d. Taking traces as linear operators on the d^2-dimensional space $\mathrm{M}_d(\mathbb{C})$, we have

$$1 + t(d^2 - 1) = d, \qquad t = \frac{1}{d+1}.$$

Hence

$$\Psi(x) = \frac{x + d\,\mathrm{tr}_d(x)1}{d+1},$$

which proves (6.12).

Assume now that $F \subset P_\rho$. Then each Φ_{uDu^*} is the ρ-preserving conditional expectation onto $(uDu^*)' \cap P$, and hence acts as an orthogonal projection on $L^2(P, \rho)$. For $b \in P$ with $\Psi_F(b) = 0$, by (6.12) we have

$$\int_{\mathcal{U}(F)} \|\Phi_{uDu^*}(b)\|_{2,\rho}^2 \,\mathrm{d}u = \int_{\mathcal{U}(F)} \langle \Phi_{uDu^*}(b), b\rangle_{2,\rho} \,\mathrm{d}u = \frac{\langle b + d\Psi_F(b), b\rangle_{2,\rho}}{d+1} = \frac{\|b\|_{2,\rho}^2}{d+1}.$$

This proves (6.13). □

We next construct matrix algebras in the centralizer of a suitable state.

Lemma 6.6. *Let N be a continuous von Neumann algebra with separable predual. Then there is a normal faithful state χ on N such that, for every integer $k \geq 1$, the centralizer N_χ contains a unital subalgebra isomorphic to $\mathrm{M}_k(\mathbb{C})$.*

Proof. By [HS90, Theorem 11.1], there is a normal faithful state χ on N such that N_χ is of type II_1. Fix $k \geq 1$. Since N_χ is of type II_1, we can take mutually orthogonal projections $p_1, \ldots, p_k \in N_\chi$ such that

$$\sum_{i=1}^k p_i = 1, \qquad \tau_{N_\chi}(p_i) = \frac{1}{k}1,$$

where τ_{N_χ} is its normalized center-valued trace. By [AP17, Proposition 9.1.8], the p_i are equivalent. Take partial isometries $v_i \in N_\chi$ with $v_i^* v_i = p_1$ and $v_i v_i^* = p_i$. Then $e_{ij} = v_i v_j^*$ form matrix units, and

$$K = \mathrm{span}\{e_{ij} \mid 1 \leq i, j \leq k\} \cong \mathrm{M}_k(\mathbb{C}), \qquad \sum_{i=1}^k e_{ii} = 1.$$

□

We can now replace the matrix algebra in Proposition 6.4 by a finite-dimensional abelian subalgebra, at the cost of an arbitrarily small error.

Proposition 6.7. *Let $M \subset P$ be a W^*-inclusion with a normal faithful conditional expectation $E : P \to M$. Assume that M is a continuous von Neumann algebra with separable predual. Let $F \subset M$ be a unital full matrix algebra. For every finite family $\theta_1, \ldots, \theta_s \in P_*$ and every $\varepsilon > 0$, there is a finite-dimensional unital abelian subalgebra $D \subset M$ with*

$$\|\theta_j \circ \Phi_D\| < \|\theta_j|_{F' \cap P}\| + \varepsilon \qquad (1 \leq j \leq s).$$

Proof. Let $F \cong \mathrm{M}_r(\mathbb{C})$ and $N = F' \cap M$. Then $M = F \bar{\otimes} N$, and N is a continuous von Neumann algebra. By Lemma 6.6, there is a normal faithful state χ on N such that, for every $k \geq 1$, the centralizer N_χ contains a unital subalgebra isomorphic to $\mathrm{M}_k(\mathbb{C})$. Let

$$\psi = \mathrm{tr}_r \otimes \chi \in M_*, \qquad \rho = \psi \circ E \in P_*.$$

By (6.1), we can take $a_1, \dots, a_s \in P$ and $k \geq 1$ such that

$$\|\theta_j - \rho(a_j\,\cdot\,)\| < \frac{\varepsilon}{4}, \qquad \left(\frac{4\sum_{j=1}^s \|a_j\|^2}{rk+1}\right)^{1/2} < \frac{\varepsilon}{2}.$$

Let $K \subset N_\chi$ be a unital subalgebra with $K \cong \mathrm{M}_k(\mathbb{C})$, and let $F_1 = F \vee K$. Then

$$F_1 \cong \mathrm{M}_{rk}(\mathbb{C}), \qquad F_1 \subset M_\psi \subset P_\rho.$$

Let $c_j = a_j - \Psi_F(a_j)$ for $1 \leq j \leq s$. Since $\mathcal{U}(F) \subset \mathcal{U}(F_1)$, by the invariance of Haar measure we have $\Psi_{F_1}\Psi_F = \Psi_{F_1}$, and hence $\Psi_{F_1}(c_j^*) = 0$. Fix a maximal abelian subalgebra $A_1 \subset F_1$. Since $\rho(1) = 1$ and $\|\Psi_F\| = 1$, we have

$$\|c_j^*\|_{2,\rho} \leq \|c_j\| \leq \|a_j\| + \|\Psi_F(a_j)\| \leq 2\|a_j\|.$$

Using $F_1 \cong \mathrm{M}_{rk}(\mathbb{C})$, $F_1 \subset P_\rho$ and $\Psi_{F_1}(c_j^*) = 0$, by (6.13) we have

$$\int_{\mathcal{U}(F_1)} \sum_{j=1}^s \|\Phi_{uA_1u^*}(c_j^*)\|_{2,\rho}^2 \,\mathrm{d}u = \sum_{j=1}^s \int_{\mathcal{U}(F_1)} \|\Phi_{uA_1u^*}(c_j^*)\|_{2,\rho}^2 \,\mathrm{d}u = \frac{\sum_{j=1}^s \|c_j^*\|_{2,\rho}^2}{rk+1} \leq \frac{4\sum_{j=1}^s \|a_j\|^2}{rk+1} < \frac{\varepsilon^2}{4}.$$

Since $\mathrm{d}u$ has total mass 1, there is $u \in \mathcal{U}(F_1)$ for which the sum is no greater than its average. Hence the finite-dimensional unital abelian subalgebra $D = uA_1u^* \subset M$ satisfies

$$\sum_{j=1}^s \|\Phi_D(c_j^*)\|_{2,\rho}^2 \leq \frac{\sum_{j=1}^s \|c_j^*\|_{2,\rho}^2}{rk+1} < \frac{\varepsilon^2}{4}.$$

Since $F, D \subset P_\rho$, the conditional expectations Ψ_F and Φ_D preserve ρ, and by bimodularity we have

$$\rho(c_j\,\cdot\,) = \rho(a_j\,\cdot\,) \circ (\mathrm{id} - \Psi_F), \qquad \rho(c_j\,\cdot\,) \circ \Phi_D = \rho(\Phi_D(c_j)\,\cdot\,).$$

Hence

$$\theta_j \circ \Phi_D = \theta_j \circ \Psi_F \circ \Phi_D + \rho(\Phi_D(c_j)\,\cdot\,) + (\theta_j - \rho(a_j\,\cdot\,)) \circ (\mathrm{id} - \Psi_F) \circ \Phi_D. \tag{6.14}$$

By (6.14), the triangle inequality, and the Cauchy–Schwarz inequality,

$$\|\theta_j \circ \Phi_D\| \leq \|\theta_j \circ \Psi_F\| + \|\Phi_D(c_j^*)\|_{2,\rho} + 2\|\theta_j - \rho(a_j\,\cdot\,)\| < \|\theta_j\big|_{F'\cap P}\| + \varepsilon \qquad (1 \leq j \leq s).$$

□

Combining Propositions 6.4 and 6.7, we have the continuous properly infinite case.

Corollary 6.8. *Let $M \subset P$ be as in Theorem G, and assume that M is a properly infinite continuous von Neumann algebra. For any $\theta_1, \dots, \theta_s \in P_*$ with $\theta_j|_{\mathcal{Z}(M)} = 0$ and any $\varepsilon > 0$, there is a finite-dimensional unital abelian subalgebra $D \subset M$ such that*

$$\|\theta_j \circ \Phi_D\| < \varepsilon \qquad (1 \leq j \leq s).$$

Proof. By Proposition 6.4, there is a unital full matrix algebra $F \subset M$ such that $\|\theta_j\big|_{F'\cap P}\| < \varepsilon/2$ for $1 \leq j \leq s$. By Proposition 6.7, applied with $\varepsilon/2$, we have a finite-dimensional unital abelian subalgebra $D \subset M$ with

$$\|\theta_j \circ \Phi_D\| < \|\theta_j\big|_{F'\cap P}\| + \frac{\varepsilon}{2} < \varepsilon \qquad (1 \leq j \leq s).$$

This proves the corollary. □

The following lemma collects the facts on corners used below.

Lemma 6.9. *Let $M \subset P$ be an inclusion of von Neumann algebras with a normal faithful conditional expectation $E : P \to M$, and assume that $M' \cap P \subset M$. Then $M' \cap P = \mathcal{Z}(M)$, and the following hold.*

(i) *For every nonzero projection $e \in M$, the restriction $E|_{ePe} : ePe \to eMe$ is a normal faithful conditional expectation, and*

$$(eMe)' \cap ePe = e(M' \cap P) = e\mathcal{Z}(M) = \mathcal{Z}(eMe). \tag{6.15}$$

In particular, $(eMe)' \cap ePe \subset eMe$.

(ii) *For every finite-dimensional unital abelian subalgebra $D \subset M$,*

$$(D' \cap M)' \cap P = D \vee \mathcal{Z}(M) \subset M.$$

(iii) *Let $e_1, \dots, e_m \in M$ be mutually orthogonal nonzero projections with $\sum_i e_i = 1$, and let $D_i \subset e_i M e_i$ be finite-dimensional unital abelian subalgebras. Then $D = \bigoplus_{i=1}^m D_i$ is a finite-dimensional unital abelian subalgebra of M, and*

$$\|\theta \circ \Phi_D\| \le \sum_{i=1}^m \|\theta|_{e_i P e_i} \circ \Phi_{D_i}\| \qquad (\theta \in P_*).$$

Proof. Since $\mathcal{Z}(M) \subset M' \cap P$, by the assumption $M' \cap P \subset M$ we have $M' \cap P = \mathcal{Z}(M)$.

(i) Since E is M-bimodular, $E(exe) = eE(x)e$ for $x \in P$. Hence $E|_{ePe}$ is a normal faithful conditional expectation onto eMe. By [AP17, Propositions 4.2.1 and 4.2.2] (see also [Jon83]) and $M' \cap P = \mathcal{Z}(M)$, we have (6.15).

(ii) Let $e_1, \dots, e_r$ be the minimal projections of D, and let $N = D' \cap M$, so that

$$D = \bigoplus_{i=1}^r \mathbb{C} e_i, \qquad N = \bigoplus_{i=1}^r e_i M e_i.$$

If $x \in N' \cap P$, then x commutes with every $e_i \in N$, and by (6.15) we have

$$x = \sum_{i=1}^r e_i x e_i, \qquad e_i x e_i \in (e_i M e_i)' \cap e_i P e_i = e_i \mathcal{Z}(M).$$

Hence $N' \cap P \subset \bigoplus_i e_i \mathcal{Z}(M) = D \vee \mathcal{Z}(M)$. The reverse inclusion follows from $D \vee \mathcal{Z}(M) \subset \mathcal{Z}(N)$.

(iii) Since $\mathrm{At}(D)$ is the disjoint union of the sets $\mathrm{At}(D_i)$, and $p \le e_i$ for $p \in \mathrm{At}(D_i)$, we have

$$\Phi_D(x) = \sum_{i=1}^m \Phi_{D_i}(e_i x e_i) \qquad (x \in P).$$

Since the maps $x \mapsto e_i x e_i$ are contractive, the estimate follows. □

For finite M, the proof of the following lemma is essentially Popa's averaging and refinement argument [Pop81]. We include it for completeness, using the tracial case of Example 3.10 to express the unitary averaging through an inner ucp map.

Lemma 6.10. *Let $M \subset P$ be as in Theorem G, and assume that M is finite. Let $\theta_1, \dots, \theta_s \in P_*$ vanish on M, and let $\varepsilon > 0$. Then there is a finite-dimensional unital abelian subalgebra $D \subset M$ such that*

$$\|\theta_j \circ \Phi_D\| < \varepsilon \qquad (1 \le j \le s).$$

Proof. Let τ be a normal faithful tracial state on M, and let $\rho = \tau \circ E$. Since $M_\tau = M$, we have $M \subset P_\rho$.

We first reduce the lemma to an L^2 estimate. By (6.1), we can take $a_1, \dots, a_s \in P$ such that

$$\|\theta_j - \rho(a_j^* \cdot)\| < \frac{\varepsilon}{8}.$$

Let $b_j = a_j - E(a_j)$ and $\eta_j = \rho(b_j^* \cdot)$. Since $\rho \circ E = \rho$ and E is M-bimodular,

$$\rho(a_j^* E(x)) = \rho(E(a_j)^* x) \qquad (x \in P).$$

Using $\theta_j \circ E = 0$ and $\|\mathrm{id} - E\| \le 2$, we have

$$E(b_j) = 0, \qquad \theta_j - \eta_j = (\theta_j - \rho(a_j^* \cdot)) \circ (\mathrm{id} - E), \qquad \|\theta_j - \eta_j\| < \frac{\varepsilon}{4}.$$

Let $D \subset M$ be a finite-dimensional unital abelian subalgebra. Since $D \subset M \subset P_\rho$, we have

$$\eta_j(\Phi_D(x)) = \sum_{p \in \mathrm{At}(D)} \rho(b_j^* p x p) = \rho(\Phi_D(b_j)^* x) \qquad (x \in P).$$

By the Cauchy–Schwarz inequality, we have

$$\|\rho(\Phi_D(b_j)^* \cdot)\| \le \|\Phi_D(b_j)\|_{2,\rho}.$$

Hence, by the triangle inequality and $\|\Phi_D\| = 1$,

$$\|\theta_j \circ \Phi_D\| \le \|\theta_j - \eta_j\| + \|\rho(\Phi_D(b_j)^* \cdot)\| < \frac{\varepsilon}{4} + \|\Phi_D(b_j)\|_{2,\rho}.$$

Let $\mathcal{D}$ be the family of finite-dimensional unital abelian subalgebras of M, and let

$$t = \inf_{D \in \mathcal{D}} \sum_{j=1}^{s} \|\Phi_D(b_j)\|_{2,\rho}^2.$$

We only need to prove that $t = 0$, since then some $D \in \mathcal{D}$ satisfies $\|\Phi_D(b_j)\|_{2,\rho} < \varepsilon/2$ for $1 \le j \le s$.

Fix $\delta > 0$, and take $D \in \mathcal{D}$ such that

$$\sum_{j=1}^{s} \|\Phi_D(b_j)\|_{2,\rho}^2 < t + \delta.$$

Let $c_j = \Phi_D(b_j)$ and $N = D' \cap M$. By Lemma 6.9(ii), $N' \cap P = D \vee \mathcal{Z}(M) \subset M$. Moreover, by the M-bimodularity of E, we have

$$E(c_j) = \Phi_D(E(b_j)) = 0.$$

The map $\Phi_D \circ E : P \to N$ is a normal faithful conditional expectation. Applying Example 3.10 to $(N, \tau|_N)$ and then Proposition 3.14, we have

$$Q_N(x) = \sum_k \lambda_k^2 u_k x u_k^*, \qquad u_k \in \mathcal{U}(N), \qquad \sum_k \lambda_k^2 = 1, \qquad \mathrm{Har}(Q_N, P) = N' \cap P = D \vee \mathcal{Z}(M).$$

Let $C_n^N = (1/n) \sum_{\ell=0}^{n-1} Q_N^\ell$. Since $EQ_N = Q_N E$,

$$E(C_n^N(c_j)) = 0, \qquad \|C_n^N(c_j)\| \le \|c_j\|, \qquad \|Q_N(C_n^N(c_j)) - C_n^N(c_j)\| \le \frac{2\|c_j\|}{n}.$$

Every ultraweak cluster point of $(C_n^N(c_j))_n$ belongs to $(D \vee \mathcal{Z}(M)) \cap \ker E = \{0\}$, so $C_n^N(c_j) \to 0$ ultraweakly. In particular,

$$\sum_{j=1}^{s} \mathrm{Re}\, \rho(c_j^* C_n^N(c_j)) \longrightarrow 0 \qquad (n \to \infty).$$

Each C_n^N is a countable convex combination of conjugations by unitary words in the u_k. Hence there is $u \in \mathcal{U}(N)$ such that

$$\sum_{j=1}^{s} \mathrm{Re}\, \rho(c_j^* u c_j u^*) < \delta.$$

Approximating u in norm by a finite-spectrum unitary $v \in W^*(u) \subset N$, we may also arrange

$$\sum_{j=1}^{s} \mathrm{Re}\, \rho(c_j^* v c_j v^*) < \delta.$$

Let $B = D \vee W^*(v)$. Since D is finite-dimensional and $v \in D' \cap M$ has finite spectrum, B is a finite-dimensional unital abelian subalgebra of M containing D, so $B \in \mathcal{D}$. Since $M \subset P_\rho$, the map Φ_B is a contraction on $L^2(P,\rho)$ and $\|vc_jv^*\|_{2,\rho} = \|c_j\|_{2,\rho}$. Using $\Phi_B\Phi_D = \Phi_B$ and $\Phi_B \circ \mathrm{Ad}(v) = \Phi_B$, we have

$$t \le \sum_j \|\Phi_B(b_j)\|_{2,\rho}^2 = \sum_j \left\|\Phi_B\left(\frac{c_j + vc_jv^*}{2}\right)\right\|_{2,\rho}^2 \le \frac{1}{2}\sum_j \|c_j\|_{2,\rho}^2 + \frac{1}{2}\sum_j \mathrm{Re}\,\rho(c_j^* vc_jv^*) < \frac{t}{2} + \delta.$$

Letting $\delta \to 0$, we have $t = 0$, which proves the lemma. □

We now combine the three cases.

Proposition 6.11. *Let $M \subset P$ be as in Theorem G. Let $\theta_1, \dots, \theta_s \in P_*$ vanish on M, and let $\varepsilon > 0$. Then there is a finite-dimensional unital abelian subalgebra $D \subset M$ such that*

$$\|\theta_j \circ \Phi_D\| < \varepsilon \qquad (1 \le j \le s).$$

Proof. Take mutually orthogonal central projections $z_1, z_2, z_3 \in \mathcal{Z}(M)$ with $z_1 + z_2 + z_3 = 1$ such that

$$M = M_{\mathrm{fin}} \oplus M_{\mathrm{I}_\infty} \oplus M_{\mathrm{c},\infty},$$

where $M_{\mathrm{fin}} = z_1M$ is finite, $M_{\mathrm{I}_\infty} = z_2M$ is of type I_∞, and $M_{\mathrm{c},\infty} = z_3M$ is continuous and properly infinite. By Lemma 6.9(i), each inclusion $z_iM \subset z_iPz_i$ with $z_i \neq 0$ again satisfies the hypotheses of Theorem G, and the restrictions $\theta_j|_{z_iPz_i}$ vanish on z_iM. By Lemma 6.9(iii), we only need to find, for each i with $z_i \neq 0$, a finite-dimensional unital abelian subalgebra $D_i \subset z_iM$ such that

$$\|\theta_j|_{z_iPz_i} \circ \Phi_{D_i}\| < \frac{\varepsilon}{3} \qquad (1 \le j \le s).$$

We may therefore assume that M is finite, of type I_∞, or continuous and properly infinite.

If M is finite, the assertion is Lemma 6.10. If M is continuous and properly infinite, it follows from Corollary 6.8, since each θ_j vanishes on $\mathcal{Z}(M) \subset M$.

Finally, assume that M is of type I_∞. Write $M = B(H)\bar{\otimes}\mathcal{Z}(M)$ for some separable Hilbert space H, identify $B(H)$ with $B(H)\bar{\otimes}\mathbb{C}1$, and let $P_0 = \mathcal{Z}(M)' \cap P$. Then

$$B(H)' \cap P_0 = (B(H) \vee \mathcal{Z}(M))' \cap P = M' \cap P = \mathcal{Z}(M).$$

Since $B(H)$ is a type I factor,

$$P_0 = B(H)\bar{\otimes}(B(H)' \cap P_0) = B(H)\bar{\otimes}\mathcal{Z}(M) = M.$$

Let $A \subset M$ be a maximal abelian subalgebra. Since $\mathcal{Z}(M) \subset A$,

$$A' \cap P \subset \mathcal{Z}(M)' \cap P = M, \qquad A' \cap P = A.$$

Take increasing finite-dimensional unital abelian subalgebras $A_n \subset A$ with $(\bigcup_n A_n)'' = A$. Then $\bigcap_n (A_n' \cap P) = A$. By (6.2), ultraweak compactness of the unit ball of P, and normality of θ_j,

$$\|\theta_j \circ \Phi_{A_n}\| = \|\theta_j|_{A_n' \cap P}\| \longrightarrow \|\theta_j|_A\| = 0 \qquad (n \to \infty).$$

We can therefore take $D = A_n$ for n sufficiently large. □

The cornerwise refinement below is essentially the same as in [Mar26, Lemma 1.3], following Popa's construction [Pop81]. We use Proposition 6.11 as the approximation estimate in the corners and include the refinement argument for completeness.

Lemma 6.12. *Let $M \subset P$ be as in Theorem G. Let $D_0 \subset M$ be a finite-dimensional unital abelian subalgebra, let $\theta_1, \dots, \theta_s \in P_*$ vanish on M, and let $\varepsilon > 0$. Then there is a finite-dimensional unital abelian subalgebra $D_1 \subset M$ such that*

$$D_0 \subset D_1, \qquad \|\theta_j \circ \Phi_{D_1}\| < \varepsilon \quad (1 \le j \le s).$$

Proof. Let $e_1, \dots, e_m$ be the minimal projections of D_0. By Lemma 6.9(i), each inclusion $e_i M e_i \subset e_i P e_i$ satisfies the hypotheses of Theorem G, and the restrictions $\theta_j|_{e_i P e_i}$ vanish on $e_i M e_i$. By Proposition 6.11, there are finite-dimensional unital abelian subalgebras $D_{1,i} \subset e_i M e_i$ such that

$$\|\theta_j|_{e_i P e_i} \circ \Phi_{D_{1,i}}\| < \frac{\varepsilon}{m} \qquad (1 \le j \le s).$$

Let $D_1 = \bigoplus_{i=1}^m D_{1,i}$. Then $D_0 \subset D_1 \subset M$, and by Lemma 6.9(iii) we have

$$\|\theta_j \circ \Phi_{D_1}\| \le \sum_{i=1}^m \|\theta_j|_{e_i P e_i} \circ \Phi_{D_{1,i}}\| < \varepsilon \qquad (1 \le j \le s).$$

□

The final construction follows Popa [Pop81]. The increasing sequence of abelian subalgebras and the predual argument are essentially the same as in [Mar25, Proposition 10.5] and [Mar26, Theorem 1.4]. We include the argument for completeness.

Proof of Theorem G. Let $M_\perp = \{\theta \in P_* \mid \theta\big|_M = 0\}$. Since P_* is separable, take a norm-dense sequence $(\theta_j)_{j \ge 1}$ in $M_\perp$. Starting with $D_0 = \mathbb{C}1$ and applying Lemma 6.12 to $\theta_1, \dots, \theta_n$ with $\varepsilon = 2^{-n}$ at the nth step, we have finite-dimensional unital abelian subalgebras $D_n \subset M$ such that

$$D_{n-1} \subset D_n, \qquad \|\theta_j \circ \Phi_{D_n}\| < 2^{-n} \qquad (1 \le j \le n).$$

Let $A_0 = (\bigcup_n D_n)''$, and let $x \in A_0' \cap P$. Then $\Phi_{D_n}(x) = x$ for every n, and hence

$$|\theta_j(x)| \le \|\theta_j \circ \Phi_{D_n}\| \, \|x\| \le 2^{-n}\|x\| \qquad (n \ge j).$$

Letting $n \to \infty$, we have $\theta_j(x) = 0$ for every j. By density, $\theta(x) = 0$ for every $\theta \in M_\perp$. Since M is ultraweakly closed, by the Hahn–Banach theorem we have $x \in M$. Hence $A_0' \cap P \subset M$. Let A be a maximal abelian subalgebra of M containing A_0. Then

$$A' \cap P \subset A_0' \cap P \subset M,$$

and hence $A' \cap P = A' \cap M = A$, which finishes the proof. □

A. Weak Dixmier properties for actions of $\mathbb{R}$ and $\mathbb{R}^2$

We extend the averaging argument to crossed products by actions of $\mathbb{R}^2$ on semifinite von Neumann algebras. We then deduce the weak Dixmier property for crossed products by equivariant actions of $\mathbb{R}$ on inclusions with expectation and properly infinite source algebras.

Theorem A.1. *Let $M \subset N$ be an inclusion of von Neumann algebras. Assume that M has separable predual, that τ is a normal faithful semifinite trace on M, and that $E : N \to M$ is a normal faithful conditional expectation. Let $\alpha : \mathbb{R}^2 \curvearrowright N$ be an ultraweakly continuous action such that*

$$\alpha_t(M) = M, \qquad \tau \circ \alpha_t|_M = e^{\ell(t)}\tau \qquad (t \in \mathbb{R}^2),$$

with $\ell : \mathbb{R}^2 \to \mathbb{R}$ linear. Then $M \subset N \rtimes_\alpha \mathbb{R}^2$ has the weak Dixmier property.

In particular, the trace condition holds with $\ell = 0$ for every action on a II_1 factor. The following theorem follows by passing to compatible continuous cores and applying Takesaki duality.

Theorem A.2. *Let $\mathcal{M} \subset \mathcal{N}$ be an inclusion of von Neumann algebras. Assume that $\mathcal{M}$ is properly infinite with separable predual and that $E : \mathcal{N} \to \mathcal{M}$ is a normal faithful conditional expectation. Let $\alpha : \mathbb{R} \curvearrowright \mathcal{N}$ be an ultraweakly continuous action such that*

$$\alpha_t(\mathcal{M}) = \mathcal{M}, \qquad E \circ \alpha_t = \alpha_t|_{\mathcal{M}} \circ E \qquad (t \in \mathbb{R}). \tag{A.1}$$

Then $\mathcal{M} \subset \mathcal{N} \rtimes_\alpha \mathbb{R}$ has the weak Dixmier property.

No α-invariant state is assumed, and $\mathcal{N}$ need not have separable predual.

A.1 A commutative model. The estimates use a noncommutative analogue of the classical cutoff argument for recurrence of symmetric Markov operators; see [FOT11, Section 1.6, Theorem 1.6.3]. To explain the partial-sum argument and the logarithmic cutoffs, we give a two-dimensional commutative model. Integration against Lebesgue measure plays the role of the semifinite trace in Lemma A.7, and the symmetric measure ν corresponds to the symmetric unitary coefficients of Q.

Proposition A.3. *Let $\nu \in \mathrm{Prob}(\mathbb{R}^2)$ be symmetric, and assume that $m_2 = \int_{\mathbb{R}^2} |s|^2 \, \mathrm{d}\nu(s) < \infty$. Define the Markov operator Q on $L^\infty(\mathbb{R}^2, \mathrm{d}t)$ by*

$$Q(b)(t) = \int_{\mathbb{R}^2} b(t+s) \, \mathrm{d}\nu(s) \qquad (b \in L^\infty(\mathbb{R}^2, \mathrm{d}t)).$$

Let $x \in L^\infty(\mathbb{R}^2, \mathrm{d}t)_+$ with $\int_{\mathbb{R}^2} x(t) \, \mathrm{d}t < \infty$. If there exists $0 \le C < \infty$ such that

$$\sum_{k=0}^{n-1} Q^k(x) \le C1 \qquad (n \ge 1), \tag{A.2}$$

then $x = 0$.

Proof. We first prove that Q preserves the integral and is a self-adjoint contraction on L^2 (cf. the discussion before Lemma A.7). Tonelli's theorem and translation invariance give $\int_{\mathbb{R}^2} Q(b) \, \mathrm{d}t = \int_{\mathbb{R}^2} b \, \mathrm{d}t$ for $b \in L^1_+$. By Kadison's inequality, Q is contractive on L^2, and by symmetry of ν it is self-adjoint. Hence $1 - Q \ge 0$.

We may assume that $\int_{\mathbb{R}^2} x \, \mathrm{d}t > 0$. Take $f \in C_c^\infty(\mathbb{R})$ with $0 \le f \le 1$, equal to 1 near 0, and nonincreasing on $[0, \infty)$. For $R > 1$, define

$$F_R(t) = f\left(\frac{\log(1 + |t|^2)}{\log R}\right) \qquad (t \in \mathbb{R}^2).$$

Then $F_R \in C_c^\infty(\mathbb{R}^2)$, $0 \le F_R \le 1$, and $F_R \nearrow 1$ pointwise ($R \to \infty$).

We next prove the partial-sum estimate (cf. Lemma A.7)

$$\int_{\mathbb{R}^2} x \, \mathrm{d}t \le \liminf_{R \to \infty} C \|(1 - Q)^{1/2} F_R\|_2^2. \tag{A.3}$$

Let $a_n = \sum_{k=0}^{n-1} Q^k(x)$. Then $0 \le a_n \le C1$, $\int_{\mathbb{R}^2} a_n \,\mathrm{d}t = n \int_{\mathbb{R}^2} x \,\mathrm{d}t$, and $a_n \in L^1 \cap L^\infty \subset L^2$. Since $(1-Q)a_n = x - Q^n(x)$,

$$\|(1-Q)^{1/2} a_n\|_2^2 = \int_{\mathbb{R}^2} a_n\big(x - Q^n(x)\big) \,\mathrm{d}t \le C \int_{\mathbb{R}^2} x \,\mathrm{d}t.$$

For fixed R, by the Cauchy–Schwarz inequality we have

$$\left| \int_{\mathbb{R}^2} F_R\big(x - Q^n(x)\big) \,\mathrm{d}t \right|^2 \le C \left(\int_{\mathbb{R}^2} x \,\mathrm{d}t \right) \|(1-Q)^{1/2} F_R\|_2^2. \tag{A.4}$$

By (A.2) and monotone convergence,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}^2} F_R Q^n(x) \,\mathrm{d}t \le C \int_{\mathbb{R}^2} F_R \,\mathrm{d}t < \infty.$$

Hence $\int_{\mathbb{R}^2} F_R Q^n(x) \,\mathrm{d}t \to 0$ $(n \to \infty)$, so by (A.4) we have

$$\left(\int_{\mathbb{R}^2} F_R x \,\mathrm{d}t \right)^2 \le C \left(\int_{\mathbb{R}^2} x \,\mathrm{d}t \right) \|(1-Q)^{1/2} F_R\|_2^2.$$

Dominated convergence as $R \to \infty$ and division by $\int_{\mathbb{R}^2} x \,\mathrm{d}t > 0$ give (A.3).

We now prove the cutoff estimate (cf. Lemma A.8). Expanding the squared difference, we have

$$\|(1-Q)^{1/2} b\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^2} \|b(\,\cdot\, + s) - b\|_2^2 \,\mathrm{d}\nu(s) \qquad (b \in L^2). \tag{A.5}$$

Polar coordinates give

$$\|\nabla F_R\|_2^2 = \frac{4\pi}{\log R} \int_0^\infty (1 - R^{-u}) |f'(u)|^2 \,\mathrm{d}u \le \frac{4\pi}{\log R} \int_0^\infty |f'(u)|^2 \,\mathrm{d}u. \tag{A.6}$$

By the translation estimate for $W^{1,2}(\mathbb{R}^2)$, we have

$$\|F_R(\,\cdot\, + s) - F_R\|_2 \le |s| \|\nabla F_R\|_2 \qquad (s \in \mathbb{R}^2). \tag{A.7}$$

By (A.5)–(A.7),

$$\|(1-Q)^{1/2} F_R\|_2^2 \le \frac{2\pi m_2}{\log R} \int_0^\infty |f'(u)|^2 \,\mathrm{d}u \longrightarrow 0 \qquad (R \to \infty). \tag{A.8}$$

Combining this with (A.3), we have $\int_{\mathbb{R}^2} x \,\mathrm{d}t = 0$, and hence $x = 0$. □

Remark A.4. *We do not extend Theorem A.1 to actions of $\mathbb{R}^3$ here because the present cutoff and partial-sum method no longer applies in general. The cutoff argument used in Theorem A.1 does not extend to $\mathbb{R}^3$ in this form. In fact, the conclusion of Proposition A.3 fails in dimension 3. Let ν be the standard Gaussian probability measure on $\mathbb{R}^3$, so that ν is symmetric and $\int |s|^2 \,\mathrm{d}\nu(s) = 3$. Let Q be its convolution operator, and let $x = \mathbb{1}_B$, where $B \subset \mathbb{R}^3$ is the unit ball. For every $n \ge 1$,*

$$Q^n(x)(t) = \int_B \frac{e^{-|s-t|^2/(2n)}}{(2\pi n)^{3/2}} \,\mathrm{d}s \le \frac{|B|}{(2\pi n)^{3/2}} \qquad (t \in \mathbb{R}^3),$$

with $|B|$ denoting Lebesgue measure. Hence,

$$\sum_{n=0}^{\infty} Q^n(x) \le C1 \qquad \text{with } C = 1 + \frac{|B|}{(2\pi)^{3/2}} \sum_{n=1}^{\infty} n^{-3/2} < \infty,$$

although $x \neq 0$ and $\int_{\mathbb{R}^3} x \,\mathrm{d}t < \infty$.

A.2 Proof of Theorem A.1. Recall the assumptions of Theorem A.1. We fix an inclusion $M \subset N$ of von Neumann algebras with M having separable predual, a normal faithful semifinite trace τ on M, a normal faithful conditional expectation $E : N \to M$, and an ultraweakly continuous action $\alpha : \mathbb{R}^2 \curvearrowright N$ such that

$$\alpha_t(M) = M, \qquad \tau \circ \alpha_t|_M = e^{\ell(t)}\tau \qquad (t \in \mathbb{R}^2),$$

with $\ell : \mathbb{R}^2 \to \mathbb{R}$ linear. Let $\mathcal{A} = N \rtimes_\alpha \mathbb{R}^2$, let $\rho = \tau \circ E$, and let $\varphi_{\mathcal{A}}$ be the dual weight of ρ on $\mathcal{A}$. Since E is M-bimodular and τ is tracial, $M \subset N_\rho$. The modular group of $\varphi_{\mathcal{A}}$ restricts to that of ρ on N, so $M \subset \mathcal{A}_{\varphi_{\mathcal{A}}}$; see [Haa78b]. We write $\|\cdot\|_2$ for the norm of $L^2(M,\tau)$.

We first construct the coefficients of the unitary averaging map Q.

Lemma A.5. *There are unitaries $u_n \in M$ $(n \geq 1)$ which generate M and satisfy $u_n - 1 \in L^2(M,\tau)$. For these unitaries, there exist constants $0 \leq D_n < \infty$ such that*

$$\max\{\|\alpha_t(u_n) - u_n\|_2^2, \|\alpha_t(u_n^*) - u_n^*\|_2^2\} \leq D_n^2 |t|^2 (1 + e^{\ell(t)}) \qquad (t \in \mathbb{R}^2). \tag{A.9}$$

Proof. Since $\tau \circ \alpha_t|_M = e^{\ell(t)}\tau$, for $b \in M \cap L^2(M,\tau)$ and $t \in \mathbb{R}^2$ we have

$$\|\alpha_t(b)\|_2 = e^{\ell(t)/2}\|b\|_2. \tag{A.10}$$

The action on $L^2(M,\tau)$ is strongly continuous: for bounded b with finite-trace support, the identity

$$\|\alpha_t(b) - b\|_2^2 = (e^{\ell(t)} + 1)\|b\|_2^2 - 2\operatorname{Re}\tau(b^*\alpha_t(b))$$

proves continuity, and the general case follows by density.

Take finite-trace projections $p_k \in M$ increasing strongly to 1. Let $(a_j)_{j\geq 1}$ be a strong*-dense sequence in the self-adjoint unit ball of M. Take $\kappa_n \in C_c^\infty(\mathbb{R}^2)_+$ such that

$$\int_{\mathbb{R}^2} \kappa_n(t)\,\mathrm{d}t = 1, \qquad \operatorname{supp}\kappa_n \subset \{t \in \mathbb{R}^2 \mid |t| < 1/n\},$$

and define

$$b_{j,k,n} = \int_{\mathbb{R}^2} \kappa_n(t)\alpha_t(p_k a_j p_k)\,\mathrm{d}t \qquad (j,k,n \geq 1).$$

Since $\|p_k a_j p_k\|_2^2 \leq \tau(p_k) < \infty$, each $b_{j,k,n}$ is a self-adjoint contraction in $M \cap L^2(M,\tau)$.

For $\kappa \in C_c^\infty(\mathbb{R}^2)$ and $b \in M \cap L^2(M,\tau)$, the convolution $b_\kappa = \int_{\mathbb{R}^2} \kappa(t)\alpha_t(b)\,\mathrm{d}t$ satisfies

$$\partial_j b_\kappa = -\int_{\mathbb{R}^2} (\partial_j\kappa)(t)\alpha_t(b)\,\mathrm{d}t \qquad (j = 1,2),$$

where ∂_j denotes the derivative of $t \mapsto \alpha_t(b_\kappa)$ at $t = 0$ in the jth coordinate. All these derivatives belong to $L^2(M,\tau)$, since $\|\alpha_t(b)\|_2 = e^{\ell(t)/2}\|b\|_2$ and κ has compact support. Moreover,

$$b_{j,k,n} \xrightarrow[n\to\infty]{\text{strong}^*} p_k a_j p_k, \qquad p_k a_j p_k \xrightarrow[k\to\infty]{\text{strong}^*} a_j.$$

Hence the family $(b_{j,k,n})_{j,k,n\geq 1}$ generates M. Enumerating this family, we have self-adjoint contractions $b_n \in M \cap L^2(M,\tau)$ which generate M and are smooth in $L^2(M,\tau)$ in the sense that the orbit maps $t \mapsto \alpha_t(b_n)$ are infinitely differentiable in the $L^2(M,\tau)$ norm.

Let $u_n = e^{ib_n}$ and $D_n = \left(\sum_{j=1}^2 \|\partial_j b_n\|_2^2\right)^{1/2}$. The unitaries u_n generate M. By the exponential difference formula, we have $\|e^{ix} - e^{iy}\|_2 \leq \|x - y\|_2$ for self-adjoint $x, y \in M$ with $x - y \in L^2(M,\tau)$. Taking $(x,y) = (b_n, 0)$, we have $\|u_n - 1\|_2 \leq \|b_n\|_2 < \infty$, so $u_n - 1 \in L^2(M,\tau)$. Fix $t = (t_1,t_2) \in \mathbb{R}^2$. By integration along $s \mapsto st$ in $L^2(M,\tau)$, we have

$$\alpha_t(b_n) - b_n = \int_0^1 \alpha_{st}\left(\sum_{j=1}^2 t_j \partial_j b_n\right)\,\mathrm{d}s.$$

By (A.10), the Cauchy–Schwarz inequality, and the definition of D_n, we have

$$\begin{aligned}\|\alpha_t(b_n) - b_n\|_2 &\le \int_0^1 e^{s\ell(t)/2} \left\| \sum_{j=1}^2 t_j \partial_j b_n \right\|_2 \mathrm{d}s \\ &\le |t| D_n \int_0^1 e^{s\ell(t)/2}\,\mathrm{d}s \le |t| D_n \max\{1, e^{\ell(t)/2}\}.\end{aligned}$$

Applying the exponential difference estimate with $(x, y) = (\pm\alpha_t(b_n), \pm b_n)$ proves (A.9). □

Let $d_n = \|u_n - 1\|_2$, and take

$$\lambda_n = \frac{2^{-n}}{Z(1 + d_n^2 + D_n^2)} \qquad \text{with } Z = \sum_n \frac{2^{-n}}{1 + d_n^2 + D_n^2}.$$

Then $\sum_n \lambda_n = 1$ and $\sum_n \lambda_n(d_n^2 + D_n^2) < \infty$. Define the normal M-inner ucp map

$$Q(a) = \frac{1}{2} \sum_n \lambda_n (u_n a u_n^* + u_n^* a u_n) \qquad (a \in \mathcal{A}). \tag{A.11}$$

Since $u_n \in M \subset \mathcal{A}_{\varphi_\mathcal{A}}$, the map Q preserves $\varphi_\mathcal{A}$, and its implementation on $L^2(\mathcal{A}, \varphi_\mathcal{A})$ is a self-adjoint contraction.

We first outline the proof. Since Q is a countable convex combination of unitary conjugations by elements of M, Cesàro averaging reduces the weak Dixmier property to the identity $\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}$. Let $a \in \mathrm{Har}(Q, \mathcal{A})$, and let $d = Q(aa^*) - aa^*$. By Kadison's inequality, $d \ge 0$, and the partial sums telescope:

$$\sum_{j=0}^{r-1} Q^j(d) = Q^r(aa^*) - aa^* \le C1 \qquad \text{with } C = \|a\|^2.$$

We therefore only need to prove that every positive element with bounded partial sums is 0; then (1.3) shows that a commutes with every u_n. We first prove Lemma A.6, and then prove the partial-sum estimate in Lemma A.7. The cutoffs constructed in Lemma A.8 make this estimate tend to 0. Proposition A.9 then identifies $\mathrm{Har}(Q, \mathcal{A})$.

Since $\varphi_\mathcal{A}$ need not be a trace, we use the continuous core $\mathcal{C} = c_{\varphi_\mathcal{A}}(\mathcal{A})$, with canonical trace $\tau_\mathcal{C}$. Let $h = \mathrm{d}\widehat{\varphi_\mathcal{A}}/\mathrm{d}\tau_\mathcal{C}$, where $\widehat{\varphi_\mathcal{A}}$ is the dual weight of $\varphi_\mathcal{A}$. Fix $p = \mathbb{1}_{[1/2,1]}(h)$. The following identity relates the trace pairing in $\mathcal{C}$ to the weight on $\mathcal{A}$.

Lemma A.6. *We have*

$$\tau_\mathcal{C}(pxp) = \varphi_\mathcal{A}(x) \qquad (x \in \mathcal{A}_+), \tag{A.12}$$

with equality in $[0, \infty]$. *Hence,*

$$\|pXp\|_{2,\tau_\mathcal{C}}^2 \le \varphi_\mathcal{A}(X^*X) \qquad (X \in \mathcal{A}). \tag{A.13}$$

Proof. Let θ and T be the dual action and the canonical operator-valued weight of $\mathcal{C}$. The operator-valued calculation in (2.5) remains valid for normal faithful semifinite weights, so $T(p) = (\log 2)1$. Define $\Phi_p(x) = \tau_\mathcal{C}(pxp)$ for $x \in \mathcal{A}_+$. Since $\int_\mathbb{R} e^{-s}\mathbb{1}_{[1/2,1]}(e^s h)\,\mathrm{d}s = h$, by (2.4) and the trace-scaling identity $\tau_\mathcal{C} \circ \theta_s = e^{-s}\tau_\mathcal{C}$ [Haa78b], for $z \in \mathcal{C}_+$ we have

$$\Phi_p(T(z)) = \tau_\mathcal{C}\left(z^{1/2} \left[\int_\mathbb{R} e^{-s}\mathbb{1}_{[1/2,1]}(e^s h)\,\mathrm{d}s \right] z^{1/2} \right) = \tau_\mathcal{C}(z^{1/2} h z^{1/2}) = \varphi_\mathcal{A}(T(z)).$$

Taking $z = x^{1/2} p x^{1/2}$, we have $T(z) = (\log 2)x$ by $\mathcal{A}$-bimodularity, which proves (A.12). Finally, since $(pXp)^*(pXp) \le pX^*Xp$, we have (A.13). □

Extend Q to $\mathcal{C}$ by the same coefficient formula (A.11). Since $u_n \in \mathcal{A}_{\varphi_{\mathcal{A}}}$, each u_n commutes with h and with p. The restriction $Q_p = Q|_{p\mathcal{C}p}$ is therefore unital, with unit p, and preserves $\tau_{\mathcal{C}}$. Its implementation on $L^2(p\mathcal{C}p, \tau_{\mathcal{C}})$ is a self-adjoint contraction. Expanding the commutators, we have

$$\|(1-Q_p)^{1/2}b\|_{2,\tau_{\mathcal{C}}}^2 = \frac{1}{4}\sum_n \lambda_n\big(\|[u_n,b]\|_{2,\tau_{\mathcal{C}}}^2 + \|[u_n^*,b]\|_{2,\tau_{\mathcal{C}}}^2\big) \tag{A.14}$$

for $b \in L^2(p\mathcal{C}p, \tau_{\mathcal{C}})$.

We next apply the partial-sum argument of Lemma 5.2 to the tracial algebra $p\mathcal{C}p$.

Lemma A.7. *Let $x \in (p\mathcal{C}p)_+$ satisfy $\tau_{\mathcal{C}}(x) < \infty$, and assume that there is a constant $0 \le C < \infty$ such that*

$$\sum_{k=0}^{n-1} Q_p^k(x) \le Cp \qquad (n \ge 1). \tag{A.15}$$

Then, for every positive contraction $z \in p\mathcal{C}p$ with $\tau_{\mathcal{C}}(z) < \infty$,

$$|\tau_{\mathcal{C}}(xz)|^2 \le C\tau_{\mathcal{C}}(x)\|(1-Q_p)^{1/2}z\|_{2,\tau_{\mathcal{C}}}^2. \tag{A.16}$$

Proof. The argument leading to (5.13) in the proof of Lemma 5.2 applies to $(p\mathcal{C}p, \tau_{\mathcal{C}})$, with $S = Q_p$, $\xi(b) = b$, $\omega_x(b) = \tau_{\mathcal{C}}(xb)$, and H_R replaced by z. Indeed, Q_p preserves $\tau_{\mathcal{C}}$ and is self-adjoint on $L^2(p\mathcal{C}p, \tau_{\mathcal{C}})$, while ω_x is a finite normal positive functional. Hence we have (A.16). □

We now construct the cutoffs for the partial-sum estimate. We use the logarithmic cutoffs of Proposition A.3, with Gaussian smoothing to handle the trace-scaling factor.

Lemma A.8. *There are positive contractions $a_n \in \mathcal{A}$ such that*

$$\begin{aligned} &a_n \to 1 \text{ strongly}, \qquad \varphi_{\mathcal{A}}(a_n) < \infty, \\ &\varepsilon_n := \frac{1}{2}\sum_j \lambda_j\big(\|[u_j,a_n]\|_{2,\varphi_{\mathcal{A}}}^2 + \|[u_j^*,a_n]\|_{2,\varphi_{\mathcal{A}}}^2\big) \longrightarrow 0. \end{aligned} \tag{A.17}$$

The positive contractions $z_n = pa_np$ satisfy $z_n \to p$ strongly, $\tau_{\mathcal{C}}(z_n) < \infty$, and

$$\|(1-Q_p)^{1/2}z_n\|_{2,\tau_{\mathcal{C}}}^2 \le \frac{\varepsilon_n}{2} \longrightarrow 0. \tag{A.18}$$

Proof. As in the proof of Proposition A.3, define

$$F_n(s) = f\left(\frac{\log(1+|s|^2)}{n}\right) \qquad (s \in \mathbb{R}^2).$$

Then $F_n \in C_c^\infty(\mathbb{R}^2)$, $0 \le F_n \le 1$, and $F_n \nearrow 1$ pointwise $(n \to \infty)$. By (A.6),

$$\|\nabla F_n\|_2^2 \le \frac{4\pi}{n}\int_0^\infty |f'(u)|^2\,\mathrm{d}u \longrightarrow 0 \qquad (n \to \infty).$$

We use the unitary Fourier transform $\widehat{g}(t) = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-it\cdot s}g(s)\,\mathrm{d}s$ for $g \in \mathcal{S}(\mathbb{R}^2)$, extended to $L^2(\mathbb{R}^2)$ by density. Let $\eta(s) = (4\pi)^{-1}e^{-|s|^2/4}$ and define $f_n = F_n * \eta$. Then $\widehat{f_n}(t) = e^{-|t|^2}\widehat{F_n}(t)$, f_n is a Schwartz function, $0 \le f_n \le 1$, and $f_n \nearrow 1$ pointwise $(n \to \infty)$. Define

$$\begin{aligned} L_n &= \int_{\mathbb{R}^2} (1+e^{-\ell(t)})|\widehat{f_n}(t)|^2\,\mathrm{d}t < \infty, \\ W_n &= \int_{\mathbb{R}^2} |t|^2(1+e^{-\ell(t)})|\widehat{f_n}(t)|^2\,\mathrm{d}t. \end{aligned} \tag{A.19}$$

Let $C_\ell = \sup_{t\in\mathbb{R}^2}(1+e^{-\ell(t)})e^{-2|t|^2} < \infty$. By Plancherel and (A.6),

$$\begin{aligned} W_n &= \int_{\mathbb{R}^2} |t|^2(1+e^{-\ell(t)})e^{-2|t|^2}|\widehat{F_n}(t)|^2\,\mathrm{d}t \le C_\ell\|\nabla F_n\|_2^2 \\ &\le \frac{4\pi C_\ell}{n}\int_0^\infty |f'(u)|^2\,\mathrm{d}u \longrightarrow 0 \qquad (n\to\infty). \end{aligned}$$

We next compute the relevant weight norms. Write $v_t = e^{it\cdot H}$ $(t\in\mathbb{R}^2)$ for the implementing unitaries of $\mathcal{A} = N\rtimes_\alpha \mathbb{R}^2$. We take the scalar normalization of the dual weight so that the crossed-product L^2 formula reads

$$\varphi_{\mathcal{A}}(X^*X) = \int_{\mathbb{R}^2}\rho\big(\alpha_{-t}(a_t^*a_t)\big)\,\mathrm{d}t \qquad \text{with } X = (2\pi)^{-1}\int_{\mathbb{R}^2} a_t v_t\,\mathrm{d}t.$$

For coefficients $a_t\in M$, we have $\rho|_M = \tau$, and hence

$$\varphi_{\mathcal{A}}(X^*X) = \int_{\mathbb{R}^2} e^{-\ell(t)}\tau(a_t^*a_t)\,\mathrm{d}t. \tag{A.20}$$

Only coefficients in M are used below. Let $y\in M\cap L^2(M,\tau)$ and $g\in L^2\cap L^\infty(\mathbb{R}^2)$ satisfy

$$\int_{\mathbb{R}^2}(1+|t|^2)(1+e^{-\ell(t)})|\widehat{g}(t)|^2\,\mathrm{d}t < \infty.$$

First assume that $\widehat{g}\in C_c^\infty(\mathbb{R}^2)$. Fourier inversion and $v_t y = \alpha_t(y)v_t$ give

$$\begin{aligned} yg(H) &= (2\pi)^{-1}\int_{\mathbb{R}^2}\widehat{g}(t)yv_t\,\mathrm{d}t, \\ g(H)y &= (2\pi)^{-1}\int_{\mathbb{R}^2}\widehat{g}(t)\alpha_t(y)v_t\,\mathrm{d}t, \\ [u_j,g(H)] &= (2\pi)^{-1}\int_{\mathbb{R}^2}\widehat{g}(t)\big(u_j-\alpha_t(u_j)\big)v_t\,\mathrm{d}t. \end{aligned}$$

By (A.20) and $\|\alpha_t(y)\|_2^2 = e^{\ell(t)}\|y\|_2^2$, we have

$$\begin{aligned} \|yg(H)\|_{2,\varphi_{\mathcal{A}}}^2 &= \|y\|_2^2\int_{\mathbb{R}^2} e^{-\ell(t)}|\widehat{g}(t)|^2\,\mathrm{d}t, \\ \|g(H)y\|_{2,\varphi_{\mathcal{A}}}^2 &= \|y\|_2^2\int_{\mathbb{R}^2}|\widehat{g}(t)|^2\,\mathrm{d}t, \\ \|[u_j,g(H)]\|_{2,\varphi_{\mathcal{A}}}^2 &= \int_{\mathbb{R}^2} e^{-\ell(t)}\|u_j-\alpha_t(u_j)\|_2^2|\widehat{g}(t)|^2\,\mathrm{d}t. \end{aligned} \tag{A.21}$$

The crossed-product L^2 isometry extends these identities to every g satisfying the above conditions; see [Haa78b]. By (A.9), the last expression with $g = f_n$ is at most $D_j^2 W_n$, and the same bound holds for u_j^*.

Finally, we compress the cutoffs by finite-trace projections, since $\tau(1)$ may be infinite. Take finite-trace projections $p_k\in M$ increasing strongly to 1, and let $\delta_k = \sum_j \lambda_j\|[u_j,p_k]\|_2^2$. Then $\delta_k\to 0$: use $u_j - 1\in L^2(M,\tau)$, $p_k\to 1$ strongly, and the dominating bound $4d_j^2$. Take $k(n)\ge n$ with $L_n\delta_{k(n)}\le 1/n$, and let $a_n = p_{k(n)}f_n(H)p_{k(n)}$. Then $0\le a_n\le 1$ and $a_n - 1\to 0$ strongly $(n\to\infty)$. Since $f_n\ge 0$ is a Schwartz function, $\sqrt{f_n}\in L^2(\mathbb{R}^2)$. Applying the second identity in (A.21) with $y = p_{k(n)}$ and $g = \sqrt{f_n}$, using its extension to $L^2\cap L^\infty(\mathbb{R}^2)$, we have

$$\varphi_{\mathcal{A}}(a_n) = \|f_n(H)^{1/2}p_{k(n)}\|_{2,\varphi_{\mathcal{A}}}^2 = \tau(p_{k(n)})\int_{\mathbb{R}^2} f_n(s)\,\mathrm{d}s < \infty.$$

For $e = p_{k(n)}$ and $u\in\{u_j,u_j^*\}$, we have

$$[u,ef_n(H)e] = [u,e]f_n(H)e + e[u,f_n(H)]e + ef_n(H)[u,e].$$

Since $e \in \mathcal{A}_{\varphi_{\mathcal{A}}}$, we have $\|eX\|_{2,\varphi_{\mathcal{A}}}, \|Xe\|_{2,\varphi_{\mathcal{A}}} \le \|X\|_{2,\varphi_{\mathcal{A}}}$. By (A.21) and (A.9),

$$\begin{aligned}
\|[u,e]f_n(H)e\|_{2,\varphi_{\mathcal{A}}}^2 &\le \|[u,e]\|_2^2 \int_{\mathbb{R}^2} e^{-\ell(t)} |\widehat{f_n}(t)|^2 \,\mathrm{d}t, \\
\|e[u,f_n(H)]e\|_{2,\varphi_{\mathcal{A}}}^2 &\le D_j^2 W_n, \\
\|ef_n(H)[u,e]\|_{2,\varphi_{\mathcal{A}}}^2 &\le \|[u,e]\|_2^2 \int_{\mathbb{R}^2} |\widehat{f_n}(t)|^2 \,\mathrm{d}t.
\end{aligned}$$

Hence, by the Cauchy–Schwarz inequality,

$$\|[u,a_n]\|_{2,\varphi_{\mathcal{A}}}^2 \le 3\big(L_n \|[u,e]\|_2^2 + D_j^2 W_n\big).$$

Since $W_n \to 0$ and $\sum_j \lambda_j D_j^2 < \infty$, using $\|[u_j^*, e]\|_2 = \|[u_j, e]\|_2$ and $L_n \delta_{k(n)} \le 1/n$ and summing with weights $\lambda_j/2$, we have

$$\varepsilon_n \le 3\left(L_n \delta_{k(n)} + W_n \sum_j \lambda_j D_j^2\right) \le \frac{3}{n} + 3W_n \sum_j \lambda_j D_j^2 \longrightarrow 0 \qquad (n \to \infty).$$

This proves (A.17).

Let $z_n = p a_n p$. By (A.12),

$$\tau_{\mathcal{C}}(z_n) = \varphi_{\mathcal{A}}(a_n) < \infty.$$

Moreover, $0 \le z_n \le p$ and $z_n \to p$ strongly. Since $[u_j, z_n] = p[u_j, a_n]p$, equations (A.14) and (A.13) give

$$\|(1-Q_p)^{1/2} z_n\|_{2,\tau_{\mathcal{C}}}^2 \le \frac{1}{4} \sum_j \lambda_j \big(\|[u_j,a_n]\|_{2,\varphi_{\mathcal{A}}}^2 + \|[u_j^*,a_n]\|_{2,\varphi_{\mathcal{A}}}^2\big) = \frac{\varepsilon_n}{2}.$$

This proves (A.18). □

We can now identify the harmonic space.

Proposition A.9. *The fixed-point space of the map Q in (A.11) is*

$$\mathrm{Har}(Q, \mathcal{A}) = M' \cap \mathcal{A}.$$

Proof. As in the proof of Proposition 5.4, we only need to show that $d = Q(aa^*) - aa^* = 0$ for $a \in \mathrm{Har}(Q, \mathcal{A})$; Kadison's inequality and telescoping give

$$d \ge 0, \qquad \sum_{k=0}^{r-1} Q^k(d) \le \|a\|^2 1 \qquad (r \ge 1).$$

Suppose that $d \ne 0$. By (A.12) and faithfulness, $\tau_{\mathcal{C}}(pdp) = \varphi_{\mathcal{A}}(d) > 0$, so $pdp \ne 0$. By semifiniteness of $\tau_{\mathcal{C}}$, there exists $0 \ne x \le pdp$ with $\tau_{\mathcal{C}}(x) < \infty$. Since $Q_p^k(pdp) = pQ^k(d)p$, we have

$$\sum_{k=0}^{r-1} Q_p^k(x) \le p\left(\sum_{k=0}^{r-1} Q^k(d)\right) p \le \|a\|^2 p.$$

Let $z_n = p a_n p$, with a_n as in Lemma A.8. Applying (A.16) and (A.18), we have

$$|\tau_{\mathcal{C}}(x z_n)|^2 \le \frac{\|a\|^2}{2} \tau_{\mathcal{C}}(x) \varepsilon_n \longrightarrow 0.$$

Since $z_n \to p$ strongly and $\tau_{\mathcal{C}}(x) < \infty$, the left-hand side tends to $\tau_{\mathcal{C}}(x)^2$, a contradiction. Hence $d = 0$. By the commutator argument at the end of the proof of Proposition 5.4, we have $a \in M' \cap \mathcal{A}$, since the u_n generate M. The reverse inclusion follows from (A.11). □

Proof of Theorem A.1. By Proposition A.9, $\mathrm{Har}(Q,\mathcal{A}) = M'\cap\mathcal{A}$. Since Q is a countable convex combination of unitary conjugations from M, by the Cesàro averaging argument in the proof of Theorem A we have the weak Dixmier property for $M \subset \mathcal{A}$. □

A.3 Proof of Theorem A.2.

Proof. Fix a normal faithful state ϕ on $\mathcal{M}$ and let $\rho = \phi\circ E$. Let $M = c_\phi(\mathcal{M}) \subset N = c_\rho(\mathcal{N})$, with the normal faithful expectation $\widehat{E} : N \to M$ induced by E and the canonical trace τ on M. Write w_r for the common implementing unitaries, θ for the dual action, and $\mathcal{A} = \mathcal{N}\rtimes_\alpha\mathbb{R}$.

By equivariance of E and [Con73, Lemmas 1.4.3–1.4.4],

$$c_t(r) := (D(\rho\circ\alpha_{-t}) : D\rho)_r = (D(\phi\circ\alpha_{-t}|_{\mathcal{M}}) : D\phi)_r \in \mathcal{M}.$$

The canonical extension $\widetilde{\alpha}$ to N satisfies $\widetilde{\alpha}_t|_{\mathcal{N}} = \alpha_t$ and $\widetilde{\alpha}_t(w_r) = c_t(r)w_r$. It commutes with θ, and its restriction to M is the canonical extension of $\alpha|_{\mathcal{M}}$, so it preserves τ; see [HS90, Propositions 12.1–12.2]. It is ultraweakly continuous, as follows from the standard implementation V_t of α and the implementation $(U_t\xi)(r) = c_t(-r)^*V_t\xi(r)$, using continuity of $c_t(r)$.

Define $\beta_{(t,s)} = \widetilde{\alpha}_t\theta_s$. Then

$$\beta_{(t,s)}(M) = M, \qquad \tau\circ\beta_{(t,s)}|_M = e^{-s}\tau \qquad ((t,s)\in\mathbb{R}^2).$$

Since M has separable predual, Theorem A.1 applies to $M\subset N$, τ, $\widehat{E}$, and β, with $\ell(t,s) = -s$. Hence $M \subset N\rtimes_\beta\mathbb{R}^2$ has the weak Dixmier property. Let $\mu_s \in N\rtimes_\beta\mathbb{R}^2$ implement θ. Since $\mu_s M\mu_s^* = M$, by further Cesàro averaging under $(\mathrm{Ad}\,\mu_s)_{s\in\mathbb{R}}$ we have the weak Dixmier property for $M\rtimes_\theta\mathbb{R}\subset N\rtimes_\beta\mathbb{R}^2$.

By Takesaki duality [Tak73, Theorem 4.5] and proper infiniteness of $\mathcal{M}$, we have

$$M\rtimes_\theta\mathbb{R} \cong \mathcal{M}\bar{\otimes}B(L^2\mathbb{R}) \cong \mathcal{M},$$
$$N\rtimes_\beta\mathbb{R}^2 \cong \mathcal{A}\bar{\otimes}B(L^2\mathbb{R}) \cong \mathcal{A}.$$

Since $c_t(r)\in\mathcal{M}$, these identifications can be taken compatibly with the inclusions. Hence $\mathcal{M}\subset\mathcal{A}$ has the weak Dixmier property, which finishes the proof. □

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