

POISSON BOUNDARIES OF TYPE III FACTORS AND THE RELATIVE BICENTRALIZER CONJECTURE

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ABSTRACT. We study Poisson boundaries associated with normal regular strongly generating hyperstates on von Neumann algebras with separable preduals. We establish their rigidity and double ergodicity, characterize the Choquet–Deny property, and obtain criteria for the weak Dixmier property. We prove that every continuous factor with separable predual admits irreducible embeddings with the weak Dixmier property into the hyperfinite factors of type III_λ , $0 < \lambda \leq 1$, and into some hyperfinite factor of type III_0 . Using techniques developed for these Poisson boundaries, we prove that, for every inclusion with a faithful normal conditional expectation $M \subset N$, $M \subset c(N)$ has the weak Dixmier property. By Marrakchi’s characterization, this gives the relative bicentralizer conjecture.

1. INTRODUCTION

Throughout this paper, we assume that all von Neumann algebras have separable preduals. The main application of the Poisson boundary techniques developed here is a proof of the relative bicentralizer conjecture (Theorem G). The proof adapts the arguments for double ergodicity and the weak Dixmier property to the continuous core.

We construct Poisson boundaries from normal regular strongly generating hyperstates, where $\overline{\mathbb{C}[z_k]}^w = \overline{\mathbb{C}[y_k]}^w = M$; see Definition 3.7.

Example 3.9 supplies such a hyperstate with any prescribed faithful normal marginal, together with a modular symmetry and a finite second logarithmic moment. We use these properties to construct type III Poisson boundaries and, in Section 7, to identify the harmonic elements of an inner ucp map on the continuous core. Weak closures of bounded sets are understood in the ultraweak topology.

Theorem A (Theorem 5.2). *Let M be a von Neumann algebra with separable predual and φ be a normal regular strongly generating hyperstate on $B(L^2(M))$ with the Poisson boundary $\mathcal{B}_\varphi$. Assume that one of the following conditions holds:*

- (i) *M is properly infinite, i.e., $M = M_{\text{I}_\infty} \oplus M_{\text{II}_\infty} \oplus M_{\text{III}}$;*
- (ii) *M is finite with normal faithful trace τ and $\varphi = \int_{\mathcal{U}(M)} \langle \cdot, u\xi_\tau, u\xi_\tau \rangle d\mu(u)$ for some $\mu \in \text{Prob}(\mathcal{U}(M))$.*

Then any W^ -inclusion $M \subset \mathcal{A}$ equipped with a normal faithful ucp map $\mathcal{P} : \mathcal{A} \rightarrow \mathcal{B}_\varphi$ with $\mathcal{P}|_M = \text{id}_M$ has the weak Dixmier property.*

As a corollary of Theorem A, we recover the classical weak Dixmier property for W^* -inclusions with expectation [Pop00, Mar21]. The argument uses the existence

of nrsg hyperstates, double ergodicity, and the passage from inner ucp averaging to unitary averaging.

Corollary B ([Pop00, Mar21] Corollary 5.3). *Any W^* -inclusion $M \subset \mathcal{A}$ with normal faithful conditional expectation has the weak Dixmier property.*

Definition 1.1. *Let $M \subset \mathcal{A}$ be a W^* -inclusion. A ucp map $P : \mathcal{A} \rightarrow \mathcal{A}$ is called **M -inner** if*

$$P(a) = \sum_k z_k a z_k^*, \quad \sum_k z_k z_k^* = 1,$$

for a sequence $(z_k) \subset M$, where the second sum converges strongly and the first converges ultraweakly. We say that $M \subset \mathcal{A}$ has the **inner UCP-Dixmier property** if, for every $a \in \mathcal{A}$,

$$\overline{\{P(a) : P \text{ is an } M\text{-inner ucp map}\}}^{\text{uw}} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

To establish Theorem A(i), our strategy is to first utilize the noncommutative Poisson boundary theory to deduce that the inclusion $M \subset \mathcal{A}$ automatically satisfies the inner UCP-Dixmier property (Corollary 3.12). To bridge the gap from this ucp-averaging condition to the classical weak Dixmier property, which relies strictly on unitary averaging, we must prove that these two properties are in fact equivalent. The following theorem establishes this crucial equivalence for the properly infinite case.

Theorem C (Theorem 5.1). *Let $M \subset \mathcal{A}$ be a W^* -inclusion such that M is properly infinite. Then $M \subset \mathcal{A}$ has the weak Dixmier property if and only if it has the inner UCP-Dixmier property.*

Theorem D (Theorem 6.3). *Let M and φ satisfy either condition (i) or condition (ii) of Theorem A. Let $M \subset \mathcal{A}$ be a W^* -inclusion equipped with a normal faithful semifinite M -bimodular positive map*

$$\mathcal{P} : \mathcal{A}_+ \longrightarrow \widehat{\mathcal{B}}_{\varphi+}$$

in the extended-positive-cone sense specified before Lemma 6.1. For every $x \in \mathcal{A}_+$ for which $\mathcal{P}(x)$ is bounded, we have

$$\overline{\text{conv}}^w \{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Remark 1.2. A bounded M -bimodular positive map $F : \mathcal{A} \rightarrow B(L^2(M))$ is completely positive. Indeed, for $[x_{ij}] \in M_n(\mathcal{A})_+$ and $a_1, \dots, a_n \in M$,

$$\sum_{i,j} \langle F(x_{ij})a_j \xi_\phi, a_i \xi_\phi \rangle = \left\langle F \left(\sum_{i,j} a_i^* x_{ij} a_j \right) \xi_\phi, \xi_\phi \right\rangle \geq 0.$$

Since $M\xi_\phi$ is dense, $[F(x_{ij})] \geq 0$. This observation concerns bounded linear maps; it is not a claim of ordinary complete positivity for maps defined only on the positive cone with extended-positive values.

Corollary E ([Iso25] Corollary 6.4). *Let $M \subset \mathcal{A}$ be a W^* -inclusion with a normal faithful semi-finite operator-valued weight $T : \mathcal{A}_+ \rightarrow \widehat{M}_+$. Then for any $x \in \mathcal{A}_+$ with $T(x) < +\infty$, we have*

$$\overline{\text{conv}}^w \{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

We now apply the existence and rigidity of type III Poisson boundaries to irreducible embeddings. A factor is called *continuous* if it is of type II or type III. For $0 < \lambda \leq 1$, let R_λ denote the unique hyperfinite type III_λ factor with separable predual [Con76, Haa87].

Theorem F (Theorem 8.8). *Let M be a continuous factor with separable predual. For every $0 < \lambda \leq 1$, there exists a normal unital embedding $\iota_\lambda : M \rightarrow R_\lambda$ such that*

$$\iota_\lambda(M)' \cap R_\lambda = \mathbb{C}1.$$

Moreover, M admits a normal unital irreducible embedding into some hyperfinite type III_0 factor with separable predual. The type III_0 target is not prescribed in advance. Moreover, these irreducible inclusions can be chosen to have the weak Dixmier property.

The following table records selected earlier results and the cases covered by Theorem F. Here R is the hyperfinite II_1 factor and $R_{\text{II}_\infty} = R \bar{\otimes} B(\ell^2)$.

Source M	Target R_{II_∞}	Some hyperfinite III_0	R_λ , $0 < \lambda < 1$	Target R_1
II_1	[Pop81]	Thm F	Thm F	Thm F
II_∞	[Pop81]	Thm F	Thm F	Thm F
III_0	[Pop21]	[Pop85]	Thm F	Thm F
III_μ $0 < \mu < 1$	[Pop81]	Thm F	[AHHM20] if $\lambda = \mu$; Thm F for all λ	Thm F
III_1	[Pop21]	Thm F	[Mar20]	[Mar20]

For target R_{II_∞} , the cases of type II_1 and type III_μ , $0 < \mu < 1$, already follow from [Pop81, Theorem 3.2 and Corollary 4.1], as recalled after [Pop21, Corollary 1.3]. The type II_∞ case follows by amplifying the type II_1 embedding. The remaining cited results are [Pop21, Corollaries 1.3 and 6.6], [Pop85, Corollary 3.4], [AHHM20, Corollary 5.7], and [Mar20, Theorem D(i)]. The latter three results are stated as the existence of irreducible hyperfinite subfactors. Applying them to M' in a standard representation and taking commutants gives the displayed embeddings. Indeed, $P \subset M'$ and $P' \cap M' = \mathbb{C}1$ imply $M \subset P'$ and $M' \cap P' = \mathbb{C}1$.

Our construction starts with a rigid type III boundary and uses at most two crossed products by countable amenable groups. The resulting ambient factors are not asserted to be Poisson boundaries, and the resulting embeddings are not asserted to admit normal conditional expectations onto M .

We now apply the Poisson boundary techniques developed above to the relative bicentralizer conjecture, using the symmetric hyperstate of Example 3.9. For an inclusion with expectation $M \subset N$, let $c(N)$ denote the continuous core of N . Marrakchi's characterization [Mar25, Theorem D] identifies the relative bicentralizer conjecture with the weak Dixmier property for $M \subset c(N)$. We prove the following result.

Theorem G (Theorem 7.1). *Let $M \subset N$ be a W^* -inclusion with a faithful normal conditional expectation. Then $M \subset c(N)$ has the weak Dixmier property. Consequently, for every faithful normal state ϕ on M ,*

$$B(M \subset c(N), \phi) = \{\phi^{it} : t \in \mathbb{R}\}'' \vee (M' \cap c(N)).$$

Here $B(M \subset c(N), \phi)$ is the relative bicentralizer recalled in Section 7, and $(\phi^{it})_{t \in \mathbb{R}}$ denotes the canonical group of unitaries in $c_\phi(M) \subset c(N)$ implementing σ^ϕ . The proof comes from the Poisson boundary arguments for double ergodicity (Theorem 3.11) and for $E(x) \in \text{Har}(P)$ in Lemma 6.1. We apply the symmetric inner ucp map supplied by Example 3.9 to $c(N)$. Estimates for smooth spectral cutoffs then identify its harmonic elements with $M' \cap c(N)$. Cesàro averaging and Theorem 5.1 then give the weak Dixmier property when M is properly infinite; in the finite case the inner ucp coefficients can be chosen as scalar multiples of unitaries, so the same Cesàro means are already unitary averages.

Status and relation to the OpenAI Math Release. This preliminary version is prepared for submission to Hexagon immediately after the author learned of the OpenAI Math Release [OAI26a, OAI26b], in order to record the present independently developed proof. The exposition has not yet been fully polished. The proof of the relative bicentralizer conjecture was developed independently of those manuscripts and does not use their results. Its conclusion overlaps with that of [OAI26a], but its method comes from the Poisson boundary techniques developed in this paper. The approach in [OAI26a] uses bicentralizer flows and a spectral comparison of multiplication maps; our argument uses a symmetric inner ucp map on the continuous core, estimates for spectral cutoffs, and the passage from harmonic elements to the weak Dixmier property.

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CONTENTS

1	Introduction	1
2	Preliminaries	4
2.1	Continuous cores and dual weights	4
3	Poisson boundaries of type III factors	6
4	The Choquet-Deny property	16
5	Proofs of main results on the weak Dixmier property	19
6	A new proof of Isono's theorem	23
7	The relative bicentralizer conjecture	27
8	Irreducible embeddings into hyperfinite factors	35

2. PRELIMINARIES

2.1 Continuous cores and dual weights Let N be a von Neumann algebra with separable predual, and let ρ be a faithful normal state on N . Define

$$\mathcal{A} = c_\rho(N) = N \rtimes_{\sigma^\rho} \mathbb{R}.$$

Let $T : \mathcal{A}_+ \rightarrow \widehat{N}_+$ be the canonical operator-valued weight, and let $\varphi_{\mathcal{A}} = \rho \circ T$ be the dual weight of ρ . Write τ for the canonical trace on $\mathcal{A}$, and let

$$h = \frac{d\varphi_{\mathcal{A}}}{d\tau}, \quad H = \log h, \quad \lambda_t = h^{it} = e^{itH}.$$

Products of measurable operators are understood as closed products. We normalize the dual action $\widehat{\sigma}$ and the dual operator-valued weight by

$$\widehat{\sigma}_r(\lambda_t) = e^{-irt} \lambda_t, \quad T(x) = \int_{\mathbb{R}} \widehat{\sigma}_r(x) dr.$$

Proposition 2.1. *For every positive integrable function g on $\mathbb{R}$,*

$$(2.1) \quad \varphi_{\mathcal{A}}(g(H)) = \rho(T(g(H))) = \int_{\mathbb{R}} g(s) ds.$$

In particular, we have

$$(2.2) \quad \tau(\mathbf{1}_{(s,\infty)}(h)) = \frac{1}{s} \quad (s > 0).$$

Proof. For $r, t \in \mathbb{R}$, functional calculus gives

$$e^{it\widehat{\sigma}_r(H)} = \widehat{\sigma}_r(e^{itH}) = \widehat{\sigma}_r(\lambda_t) = e^{-irt} e^{itH} = e^{it(H-r)}.$$

By uniqueness of the self-adjoint generator of a strongly continuous unitary group,

$$(2.3) \quad \widehat{\sigma}_r(H) = H - r.$$

Let $\mathcal{E}_H$ be the spectral measure of H . For a positive integrable function g , (2.3) gives

$$(2.4) \quad \begin{aligned} T(g(H)) &= \int_{\mathbb{R}} g(H - r) dr \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(s - r) dr \right) d\mathcal{E}_H(s) \\ &= \left(\int_{\mathbb{R}} g(s) ds \right) 1. \end{aligned}$$

Since $\varphi_{\mathcal{A}} = \rho \circ T$ and $\rho(1) = 1$, (2.4) gives (2.1).

For $s > 0$, the definition of h and (2.1) give

$$\begin{aligned} \tau(\mathbf{1}_{(s,\infty)}(h)) &= \varphi_{\mathcal{A}}(h^{-1} \mathbf{1}_{(s,\infty)}(h)) \\ &= \varphi_{\mathcal{A}}(e^{-H} \mathbf{1}_{(\log s, \infty)}(H)) \\ &= \int_{\log s}^{\infty} e^{-t} dt = \frac{1}{s}. \end{aligned}$$

See also [Haa79a, Theorem 1.2(2)]. In particular, h is τ -measurable. $\square$

Lemma 2.2. *For every $0 \neq d \in \mathcal{A}_+$, there exists $x \in \mathcal{A}_+$ such that*

$$0 \neq x \leq d, \quad \varphi_{\mathcal{A}}(x) < \infty.$$

Proof. Choose $\delta > 0$ such that $\mathbf{1}_{[\delta,\infty)}(d) \neq 0$. By semifiniteness of τ , there is a nonzero projection

$$p \leq \mathbf{1}_{[\delta,\infty)}(d), \quad 0 < \tau(p) < \infty.$$

By (2.2), choose $s > 0$ such that $\tau(\mathbf{1}_{(s,\infty)}(h)) < \tau(p)$, and let $e = \mathbf{1}_{[0,s]}(h)$ and $q = p \wedge e$. Let $(1 - e)p = v|(1 - e)p|$ be the polar decomposition. The kernel of

$(1-e)p$ inside the range of p is the range of $q = p \wedge e$, and the range of $(1-e)p$ is contained in the range of $1-e$. Hence

$$v^*v = p - q, \quad vv^* \leq 1 - e.$$

Thus $p - q \sim vv^* \leq 1 - e$, so $p - q \lesssim 1 - e$ and

$$\tau(q) \geq \tau(p) - \tau(1 - e) > 0.$$

Since $q \leq e$, we have

$$\varphi_{\mathcal{A}}(q) = \tau(qhq) = \tau(qeheq) \leq s\tau(q) < \infty.$$

Thus $x = \delta q$ satisfies

$$0 \neq x = \delta q \leq \delta p \leq d, \quad \varphi_{\mathcal{A}}(x) = \delta\varphi_{\mathcal{A}}(q) < \infty.$$

□

3. POISSON BOUNDARIES OF TYPE III FACTORS

Let M be a von Neumann algebra with separable predual.

Definition 3.1. Let $M \subset \mathcal{A}$ be a W^* -inclusion. A normal state φ on $\mathcal{A}$ is called an M -**hyperstate** if its restriction $\varphi|_M$ to M is faithful. Given a fixed normal faithful state ϕ on M , we say that φ is an (M, ϕ) -**hyperstate** if $\varphi|_M = \phi$.

Proposition 3.2. Let $\varphi \in B(L^2(M))_*$ be a normal M -hyperstate. Set $\phi = \varphi|_M$, and let $\xi_\phi \in L^2(M)_+$ be the canonical cyclic and separating vector implementing ϕ . Then there exists a unique normal M -bimodular ucp map $\mathcal{P}_\varphi : B(L^2(M)) \rightarrow B(L^2(M))$ such that for all $T \in B(L^2(M))$,

$$\varphi(T) = \langle \mathcal{P}_\varphi(T)\xi_\phi, \xi_\phi \rangle.$$

Moreover, there exists a (not necessarily unique) sequence $(z_k)_{k=1}^\infty \subset M$ such that

$$(3.1) \quad \mathcal{P}_\varphi(T) = \sum_{k=1}^{\infty} (Jz_k^*J)T(Jz_kJ)$$

for all $T \in B(L^2(M))$, where J is the modular conjugation operator on $L^2(M)$.

Proof. We first prove the existence and construct the explicit form of $\mathcal{P}_\varphi$.

Since $\varphi \in B(L^2(M))_*$ is a normal state, there exists a positive trace-class operator ρ on $L^2(M)$ with $\text{Tr}(\rho) = 1$ such that $\varphi(T) = \text{Tr}(\rho T)$. We can decompose $\rho = \sum_k \lambda_k |\xi_k\rangle\langle\xi_k|$, where (ξ_k) is an orthonormal system in $L^2(M)$, $\lambda_k > 0$, and $\sum_k \lambda_k = 1$.

For any $x \in M$, since $\varphi|_M = \phi$, we have:

$$\|x\xi_\phi\|^2 = \phi(x^*x) = \varphi(x^*x) = \text{Tr}(\rho x^*x) = \sum_k \lambda_k \langle x^*x\xi_k, \xi_k \rangle = \sum_k \lambda_k \|x\xi_k\|^2.$$

Hence

$$(3.2) \quad \lambda_m \|x\xi_m\|^2 \leq \|x\xi_\phi\|^2 \quad (x \in M, m \geq 1).$$

Define the operator $R_m : M\xi_\phi \rightarrow L^2(M)$ by $R_m(x\xi_\phi) = x\xi_m$. By (3.2), $\|R_m(x\xi_\phi)\| \leq \frac{1}{\sqrt{\lambda_m}} \|x\xi_\phi\|$, so R_m extends to a bounded operator on $L^2(M)$. Furthermore, for any $x, y \in M$, we have $R_m(yx\xi_\phi) = yx\xi_m = yR_m(x\xi_\phi)$. Hence $R_m \in M'$.

By standard Tomita-Takesaki theory, $M' = JMJ$, so there exists $y_m \in M$ such that $R_m = Jy_mJ$.

Now, for any $x, y \in M$ and $T \in B(L^2(M))$, we calculate:

$$\begin{aligned}
 \varphi(y^*Tx) &= \text{Tr}(\rho y^*Tx) = \sum_k \lambda_k \langle y^*Tx\xi_k, \xi_k \rangle = \sum_k \lambda_k \langle Tx\xi_k, y\xi_k \rangle \\
 &= \sum_k \lambda_k \langle TR_k(x\xi_\phi), R_k(y\xi_\phi) \rangle \\
 (3.3) \quad &= \sum_k \lambda_k \langle TJy_kJx\xi_\phi, Jy_kJy\xi_\phi \rangle \\
 &= \left\langle \left(\sum_k \lambda_k Jy_k^*JTJy_kJ \right) x\xi_\phi, y\xi_\phi \right\rangle.
 \end{aligned}$$

Define $z_k = \sqrt{\lambda_k}y_k \in M$. Let $\mathcal{P}_\varphi(T) = \sum_k (Jz_k^*J)T(Jz_kJ)$.

By (3.3), $\varphi(y^*Tx) = \langle \mathcal{P}_\varphi(T)x\xi_\phi, y\xi_\phi \rangle$.

Taking $x = y = 1$, we immediately obtain $\varphi(T) = \langle \mathcal{P}_\varphi(T)\xi_\phi, \xi_\phi \rangle$.

Since $M\xi_\phi$ is dense in $L^2(M)$, the identity

$$\langle \mathcal{P}_\varphi(T)x\xi_\phi, y\xi_\phi \rangle = \varphi(y^*Tx)$$

uniquely determines $\mathcal{P}_\varphi(T)$ for every T . Thus $\mathcal{P}_\varphi$ is unique.

Finally, we verify the properties of $\mathcal{P}_\varphi$. Since $Jz_kJ \in M'$, $\mathcal{P}_\varphi$ is structurally completely positive and M -bimodular. Setting $T = 1$ gives:

$$\langle \mathcal{P}_\varphi(1)x\xi_\phi, y\xi_\phi \rangle = \varphi(y^*x) = \phi(y^*x) = \langle x\xi_\phi, y\xi_\phi \rangle.$$

This forces $\mathcal{P}_\varphi(1) = 1$. Thus $\sum_k Jz_k^*z_kJ = 1$, which implies $J(\sum_k z_k^*z_k)J = 1$, concluding that $\sum_k z_k^*z_k = 1$. Therefore, $\mathcal{P}_\varphi$ is a ucp map. It is normal: for $T \geq 0$, the partial sums in its defining series increase strongly, and interchanging the two increasing suprema shows that it preserves bounded increasing nets of positive operators. $\square$

Definition 3.3. Let $\text{Har}(\mathcal{P}_\varphi) = \{T \in B(L^2(M)) \mid \mathcal{P}_\varphi(T) = T\}$ denote the space of φ -harmonic operators. The **Poisson boundary** $(\mathcal{B}_\varphi, \zeta)$ is defined as a von Neumann algebra that is isomorphic to $\text{Har}(\mathcal{P}_\varphi)$ as an operator system, together with the canonical hyperstate $\zeta := \varphi|_{\text{Har}(\mathcal{P}_\varphi)}$.

Proposition 3.4. The Poisson boundary $\mathcal{B}_\varphi$ always exists, is unique, and is amenable.

Proof. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We define a map $E : B(L^2(M)) \rightarrow \text{Har}(\mathcal{P}_\varphi)$ by taking the ultraweak limit of the Cesàro means:

$$E(T) = \text{w-}\lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}_\varphi^k(T) \quad \text{for all } T \in B(L^2(M)).$$

Since $\mathcal{P}_\varphi$ is normal, E is a unital completely positive idempotent with range $\text{Har}(\mathcal{P}_\varphi)$. It need not be normal, and its range need not be a subalgebra for the original product on $B(L^2(M))$. Thus we do not call it a conditional expectation onto an ordinary von Neumann subalgebra. Its range is injective as an operator system.

By the Choi-Effros theorem, equipping the operator system $\text{Har}(\mathcal{P}_\varphi)$ with the Choi-Effros product defined by

$$x \circ y = E(xy) \quad \text{for } x, y \in \text{Har}(\mathcal{P}_\varphi)$$

makes $(\text{Har}(\mathcal{P}_\varphi), \circ)$ a C^* -algebra with its original norm. The harmonic operator system is ultraweakly closed, hence a dual Banach space with the quotient of $B(L^2(M))_*$ as predual. Sakai's characterization therefore makes this C^* -algebra a von Neumann algebra, with its canonical ultraweak topology agreeing with the inherited one; see also [Izu02]. The inclusion $M \subset \text{Har}(\mathcal{P}_\varphi)$ is multiplicative for this product, since E is M -bimodular. The canonical Poisson transform is a normal unital complete order isomorphism onto the harmonic operator system. We use this transform whenever we identify boundary elements with operators on $L^2(M)$; products of two boundary elements must otherwise be taken in the Choi-Effros algebra.

The uniqueness of $\mathcal{B}_\varphi$ follows from the fact that any two von Neumann algebras that are completely isometrically isomorphic as operator systems are necessarily $*$ -isomorphic.

Finally, since $\mathcal{B}_\varphi \cong \text{Har}(\mathcal{P}_\varphi)$ as injective operator systems, $\mathcal{B}_\varphi$ is amenable as a von Neumann algebra. $\square$

Definition 3.5. We say that an M -hyperstate $\varphi \in B(L^2(M))_*$ is **regular** if $\varphi(Jz^*J) = \varphi(z)$ for all $z \in M$.

Proposition 3.6. Assume that $\varphi \in B(L^2(M))_*$ is a regular M -hyperstate. Let $\phi = \varphi|_M$ and $\phi^\circ = \varphi|_{JMJ}$. Since $\xi_\phi \in L^2(M)_+$ is fixed by J , for $z \in M$, we have $\langle Jz^*J\xi_\phi, \xi_\phi \rangle = \phi(z) = \phi^\circ(Jz^*J)$. Hence we can naturally identify $L^2(M, \phi) = L^2(JMJ, \phi^\circ)$ sharing the same canonical cyclic vector ξ_ϕ . The same argument as in Proposition 3.2 guarantees the existence of a JMJ -bimodular ucp map $\mathcal{P}_\varphi^\circ : B(L^2(M)) \rightarrow B(L^2(M))$ such that

$$\varphi(T) = \langle \mathcal{P}_\varphi^\circ(T)\xi_\phi, \xi_\phi \rangle \quad \text{for all } T \in B(L^2(M)).$$

Moreover, there exists a sequence $(y_k)_{k=1}^\infty \subset M$ satisfying $\sum_{k=1}^\infty y_k y_k^* = 1$ such that $\mathcal{P}_\varphi^\circ(T) = \sum_{k=1}^\infty y_k T y_k^*$. Note that we do not necessarily have $(y_k) = (z_k)$ unless (M, ϕ) is tracial.

For $k \geq 0$, we define the k -th convolution φ^{*k} by $\varphi^{*k}(T) = \langle \mathcal{P}_\varphi^k(T)\xi_\phi, \xi_\phi \rangle$.

Definition 3.7. Let φ be a normal regular M -hyperstate, and write

$$\mathcal{P}_\varphi(T) = \sum_k (Jz_k^*J)T(Jz_kJ), \quad \mathcal{P}_\varphi^\circ(T) = \sum_k y_k T y_k^*.$$

We say that φ is **strongly generating** if

$$\overline{\mathbb{C}[z_k]}^w = \overline{\mathbb{C}[y_k]}^w = M.$$

Here no adjoints are adjoined in forming either algebra. We abbreviate normal, regular, strongly generating to **nrsg**. In the nontracial setting we impose both density conditions; we do not assert that one follows from the other.

These conditions do not depend on the choice of inner ucp coefficients. Indeed, comparison with a minimal Stinespring representation expresses each coefficient of

either representation as a square-summable scalar combination of the coefficients of the other. These combinations converge in norm: for example,

$$\left\| \sum_k c_k z_k \right\| \leq \|(c_k)\|_2 \quad \text{when} \quad \sum_k z_k^* z_k = 1,$$

and the analogous inequality holds for (y_k) because $\sum_k y_k y_k^* = 1$. Thus the norm-closed linear spans, and hence the weakly closed unital algebras, agree. When $\phi = \varphi|_M$ is tracial, the same coefficient family can be used for z_k and y_k , recovering the tracial definition.

Proposition 3.8. *Let φ be a normal regular M -hyperstate. Assume that $\mathbb{C}[\{z_k\}_{k \geq 1}]$ is weakly dense in M . Then the normal state*

$$\bar{\varphi} = \sum_{n=0}^{\infty} 2^{-n-1} \varphi^{*n}$$

is faithful on $B(L^2(M))$. Moreover, M is the largest von Neumann subalgebra of $\text{Har}(\mathcal{P}_\varphi)$ for the original operator product on $B(L^2(M))$.

Proof. Write $\phi = \varphi|_M$. Let $T \in B(L^2(M))_+$ satisfy $\bar{\varphi}(T) = 0$. Since all the coefficients are positive, $\varphi^{*n}(T) = 0$ for every $n \geq 0$. Expanding the iterates of $\mathcal{P}_\varphi$ and using positivity, we obtain

$$\langle T(JwJ)\xi_\phi, (JwJ)\xi_\phi \rangle = 0$$

for every word w in (z_k) , including the empty word. The weak density of $\mathbb{C}[z_k]$ implies that the vectors $(JwJ)\xi_\phi$ have dense linear span in $L^2(M)$. Thus $T = 0$, proving that $\bar{\varphi}$ is faithful.

Let $\mathcal{N} \subset \text{Har}(\mathcal{P}_\varphi)$ be a von Neumann subalgebra for the original operator product, and let $v_k = Jz_k^*J$. For $T \in \mathcal{N}$, the operators T, T^*, TT^* are fixed by $\mathcal{P}_\varphi$, so

$$(3.4) \quad \sum_k (v_k T - T v_k)(v_k T - T v_k)^* = \mathcal{P}_\varphi(TT^*) - \mathcal{P}_\varphi(T)T^* - T\mathcal{P}_\varphi(T^*) + TT^* = 0.$$

Thus $[T, v_k] = 0$ for every k . Applying (3.4) to T^* also gives $[T, v_k^*] = 0$ for every k .

The weak density of $\mathbb{C}[\{z_k\}_{k \geq 1}]$ gives $T \in (JMJ)' = M$. Since M itself is contained in the harmonic space, it is the largest such subalgebra. $\square$

The following example gives an nrsg hyperstate with any prescribed faithful normal marginal. The additional modular symmetry and moment bounds will be used in Proposition 8.4 and in Section 7.

Example 3.9. Let M have separable predual, let ϕ be a faithful normal state on M , and write $\xi = \xi_\phi$. There exists a faithful nrsg (M, ϕ) -hyperstate φ on $B(L^2(M))$ with inner ucp coefficients $(y_k) \subset M$ for $\mathcal{P}_\varphi^o$ having the following properties:

- (i) The series $\sum_k y_k y_k^*$ converges strongly to 1, and the normal ucp map

$$Q(a) = \sum_k y_k a y_k^*, \quad a \in M,$$

satisfies $\phi \circ Q = \phi$.

(ii) The hyperstate is given by

$$\varphi(T) = \sum_k \langle T y_k^* \xi, y_k^* \xi \rangle, \quad T \in B(L^2(M)),$$

and satisfies

$$(3.5) \quad \begin{aligned} \varphi(JT^*J) &= \varphi(T) \quad (T \in B(L^2(M))), \\ \varphi(\Delta_\phi) &= \varphi(\Delta_\phi^{-1}) < \infty, \quad \varphi((\log \Delta_\phi)^2) < \infty. \end{aligned}$$

(iii) Each y_k^* admits a bounded modular continuation to $i/2$, and the map

$$(3.6) \quad y_k \longmapsto \sigma_{i/2}^\phi(y_k^*)$$

permutes the family (y_k) .

(iv) If ϕ is tracial, we may further choose

$$y_k = \lambda_k u_k, \quad \lambda_k > 0, \quad u_k \in \mathcal{U}(M), \quad \sum_k \lambda_k^2 = 1.$$

Expectations of positive unbounded operators are defined by monotone spectral truncation.

Proof. Choose elements $a_n \in M$, entire analytic for σ^ϕ , whose linear span is weakly dense in M and such that $\{a_n \xi : n \geq 1\}$ has dense linear span in $L^2(M)$. One obtains such a sequence by applying the Gaussian regularizations

$$a \longmapsto (r/\pi)^{1/2} \int_{\mathbb{R}} e^{-rt^2} \sigma_t^\phi(a) dt, \quad r \in \mathbb{N},$$

to a countable strongly dense subset of the unit ball; see [Tak03].

Let $b_n = \sigma_{-i/2}^\phi(a_n^*)$. The standard-form identities give

$$Ja_n \xi = b_n \xi, \quad a_n \xi = Jb_n J \xi.$$

Choose $c_n > 0$ such that

$$\sum_n c_n (\|a_n\|_\infty^2 + \|b_n\|_\infty^2) < \infty.$$

Rescaling the coefficients, we may assume that

$$(3.7) \quad C = \sum_n c_n (a_n^* a_n + b_n^* b_n), \quad \|C\|_\infty = 1,$$

where the series converges in norm. In particular, $0 \leq C \leq 1$. For $\eta \in L^2(M)$, write $\omega_\eta(T) = \langle T\eta, \eta \rangle$, and set

$$\psi = \sum_n c_n (\omega_{a_n \xi} + \omega_{Ja_n \xi}) \Big|_M.$$

Since $Ja_n J, Jb_n J \in M'$, for $x \in M_+$ we have

$$\psi(x) = \langle JCJx^{1/2}\xi, x^{1/2}\xi \rangle \leq \phi(x).$$

Let $\eta \in L^2(M)_+$ implement $\phi - \psi$, and define

$$\varphi = \sum_n c_n (\omega_{a_n \xi} + \omega_{Ja_n \xi}) + \omega_\eta.$$

Then $\varphi|_M = \phi$. Since $J\eta = \eta$ and $\omega_v(JT^*J) = \omega_{Jv}(T)$, the state φ is J -symmetric and hence regular. The totality of $(a_n \xi)$ makes φ faithful on $B(L^2(M))$.

The inequality $\phi - \psi \leq \phi$ implies that the map $x\xi \mapsto x\eta$ extends to a contraction $r \in M'$. Let $p = JrJ \in M$. Then

$$p\xi = \eta = JpJ\xi, \quad \|p\|_\infty \leq 1.$$

The two coefficient families associated with φ are

$$\begin{aligned} (z_k) &= (\sqrt{c_n}b_n, \sqrt{c_n}a_n)_{n \geq 1} \cup \{p\}, \\ (y_k) &= (\sqrt{c_n}a_n^*, \sqrt{c_n}b_n^*)_{n \geq 1} \cup \{p^*\}. \end{aligned}$$

The two representations are

$$\begin{aligned} \varphi(T) &= \sum_k \langle TJz_kJ\xi, Jz_kJ\xi \rangle \\ (3.8) \quad &= \sum_k \langle Ty_k^*\xi, y_k^*\xi \rangle \quad (T \in B(L^2(M))). \end{aligned}$$

Since $\varphi|_M = \phi$ and $\varphi(Ja^*J) = \phi(a)$ for $a \in M$,

$$\sum_k z_k^* z_k = 1, \quad \sum_k y_k y_k^* = 1$$

strongly. Moreover,

$$a_n \in \mathbb{C}[z_k], \quad a_n^* \in \mathbb{C}[y_k], \quad \overline{\mathbb{C}[z_k]}^w = \overline{\mathbb{C}[y_k]}^w = M,$$

so φ is nrsg. By (3.8),

$$\phi(Q(a)) = \sum_k \langle ay_k^*\xi, y_k^*\xi \rangle = \varphi(a) = \phi(a), \quad a \in M.$$

This proves (i) and the first assertion of (ii).

To verify (iii), note that

$$\sigma_{i/2}^\phi(a_n) = b_n^*, \quad \sigma_{i/2}^\phi(b_n) = a_n^*.$$

It remains to treat p^* . For $r > 0$, define

$$p_r = (r/\pi)^{1/2} \int_{\mathbb{R}} e^{-rt^2} \sigma_t^\phi(p) dt.$$

The identity $Jp\xi = p\xi$ and the commutation relation $J\Delta_\phi^{it} = \Delta_\phi^{it}J$ give $Jp_r\xi = p_r\xi$. The Tomita identity and the separating property of ξ imply

$$\sigma_{-i/2}^\phi(p_r^*) = p_r, \quad \sigma_{i/2}^\phi(p_r) = p_r^*.$$

The boundary values on the half-strip are bounded by $\|p\|_\infty$. The three-lines theorem and the ultraweak closedness of the analytic generator give $\sigma_{i/2}^\phi(p) = p^*$; see [VD14, p. 5]. Thus (3.6) exchanges the paired coefficients and fixes p^* .

We next check (3.5). The vectors $a_n\xi$, $b_n\xi$ and $p\xi$ belong to the domain of the Tomita operator $S_\phi = J\Delta_\phi^{1/2}$. Hence

$$\begin{aligned} \varphi(\Delta_\phi) &= \sum_n c_n (\|a_n^*\xi\|^2 + \|b_n^*\xi\|^2) + \|p^*\xi\|^2 \\ &\leq \sum_n c_n (\|a_n\|_\infty^2 + \|b_n\|_\infty^2) + 1 < \infty. \end{aligned}$$

Here the equality follows by monotone spectral truncation. Since $J\Delta_\phi J = \Delta_\phi^{-1}$, J -symmetry gives $\varphi(\Delta_\phi^{-1}) = \varphi(\Delta_\phi)$. The inequality $(\log t)^2 \leq t + t^{-1} - 2$, $t > 0$, proves $\varphi((\log \Delta_\phi)^2) < \infty$. In particular,

$$(3.9) \quad \varphi(|\log \Delta_\phi|) < \infty, \quad \varphi(\log \Delta_\phi) = 0.$$

(iv): The modular action σ^ϕ is trivial, so every element of M is entire analytic. Since M is linearly spanned by its unitaries, we may choose the a_n in the preceding construction to be unitaries with the same density properties. For this choice, $b_n = a_n^*$, and hence

$$C = 2 \left(\sum_n c_n \right) 1.$$

By (3.7), $2 \sum_n c_n = 1$. Since each $\omega_{a_n \xi}$ restricts to ϕ on M , we have $\psi = \phi$. Thus $\eta = 0$ and $p = 0$. Omitting the zero coefficient, the inner ucp coefficients of Q are

$$y_{2n-1} = \sqrt{c_n} a_n^*, \quad y_{2n} = \sqrt{c_n} a_n, \quad n \geq 1.$$

This has the required form $y_k = \lambda_k u_k$, with $\sum_k \lambda_k^2 = 2 \sum_n c_n = 1$, which finishes the proof. $\square$

From now on, we fix a normal faithful state ϕ on M and an nrsg (M, ϕ) -hyperstate φ in the sense of Definition 3.7, and let $(\mathcal{B}_\varphi, \zeta)$ denote its Poisson boundary. By Proposition 3.8, the normal state $\bar{\varphi} = \sum_{n=0}^{\infty} 2^{-n-1} \varphi^{*n}$ is faithful on $B(L^2(M))$. All the states φ^{*n} agree with φ on $\text{Har}(\mathcal{P}_\varphi)$, so the canonical boundary state ζ is normal and faithful.

Proposition 3.10. *We have $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$.*

Proof. By identifying $\mathcal{B}_\varphi$ with $\text{Har}(\mathcal{P}_\varphi)$, we consider elements in $M' \cap \text{Har}(\mathcal{P}_\varphi)$. Note that $M' \cap B(L^2(M)) = JMJ$. Thus, it suffices to compute $JMJ \cap \text{Har}(\mathcal{P}_\varphi)$.

By the standard form of $\mathcal{P}_\varphi$, we have $\mathcal{P}_\varphi(JMJ) \subset JMJ$. Since φ is regular, for any $T \in JMJ$, we have:

$$\phi^o(\mathcal{P}_\varphi(T)) = \langle \mathcal{P}_\varphi(T) \xi_\phi, \xi_\phi \rangle = \varphi(T) = \phi^o(T).$$

Here the first equality uses $\mathcal{P}_\varphi(T) \in JMJ$ and the fact that ξ_ϕ implements ϕ^o on JMJ . Hence, the restriction $\mathcal{P}_\varphi|_{JMJ}$ preserves the normal faithful state ϕ^o on JMJ . It follows that the fixed-point space $\text{Har}(JMJ, \mathcal{P}_\varphi|_{JMJ})$ is a von Neumann subalgebra of JMJ .

Because φ is strongly generating, M is the largest von Neumann subalgebra contained in $\text{Har}(\mathcal{P}_\varphi)$. Therefore, $\text{Har}(\mathcal{P}_\varphi) \cap JMJ \subset M$, which implies $\text{Har}(\mathcal{P}_\varphi) \cap JMJ \subset M \cap JMJ = \mathcal{Z}(M)$. Since $\mathcal{Z}(M) \subset \text{Har}(\mathcal{P}_\varphi)$ is trivial, we conclude $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$. $\square$

Theorem 3.11. *Let $M \subset \mathcal{A}$ be a W^* -inclusion, then $\mathcal{P}_\varphi^o : T \mapsto \sum_k y_k T y_k^*$ can be naturally defined on $\mathcal{A}$. Let*

$$\text{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = \{T \in \mathcal{A} \mid \mathcal{P}_\varphi^o(T) = T\}.$$

Assume there exists a normal faithful ucp map $\mathcal{P} : \mathcal{A} \rightarrow \mathcal{B}_\varphi$ such that $\mathcal{P}|_M = \text{id}_M$. Then we have $\text{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = M' \cap \mathcal{A}$.

In particular, we have the double ergodicity of the Poisson boundary:

$$\text{Har}(\mathcal{P}_\varphi) \cap \text{Har}(\mathcal{P}_\varphi^o) = \mathcal{Z}(M).$$

Proof. Let ζ be the normal faithful stationary state on the Poisson boundary $\mathcal{B}_\varphi$. We define a normal faithful state $\varphi_{\mathcal{A}}$ on $\mathcal{A}$ by $\varphi_{\mathcal{A}} = \zeta \circ \mathcal{P}$.

Recall that the ucp map $\mathcal{P}_\varphi^o$ acting on $\mathcal{A}$ is given by $\mathcal{P}_\varphi^o(T) = \sum_{k=1}^{\infty} y_k T y_k^*$ for $T \in \mathcal{A}$, where $(y_k)_{k=1}^{\infty} \subset M$ and $\sum_{k=1}^{\infty} y_k y_k^* = 1$. Because $\mathcal{P}|_M = \text{id}_M$, $\mathcal{P}$ is an M -bimodular map. Hence

$$(3.10) \quad \mathcal{P}(\mathcal{P}_\varphi^o(T)) = \sum_{k=1}^{\infty} \mathcal{P}(y_k T y_k^*) = \sum_{k=1}^{\infty} y_k \mathcal{P}(T) y_k^* = \mathcal{P}_\varphi^o(\mathcal{P}(T)).$$

We now verify that ζ is $\mathcal{P}_\varphi^o$ -invariant on $\mathcal{B}_\varphi$. Identifying $\mathcal{B}_\varphi$ with $\text{Har}(\mathcal{P}_\varphi)$, the state is implemented by the vector state $\zeta(\cdot) = \langle \mathcal{P}_\varphi(\cdot) \xi_\phi, \xi_\phi \rangle = \langle \cdot \xi_\phi, \xi_\phi \rangle$, which coincides with the original hyperstate φ on this space. By the defining construction of $\mathcal{P}_\varphi^o$, we have $\langle \mathcal{P}_\varphi^o(T) \xi_\phi, \xi_\phi \rangle = \varphi(T)$ for any $T \in B(L^2(M))$. Thus, for any $S \in \mathcal{B}_\varphi$, we have:

$$(3.11) \quad \zeta(\mathcal{P}_\varphi^o(S)) = \langle \mathcal{P}_\varphi^o(S) \xi_\phi, \xi_\phi \rangle = \varphi(S) = \zeta(S).$$

By (3.10) and (3.11), for $T \in \mathcal{A}$,

$$\varphi_{\mathcal{A}}(\mathcal{P}_\varphi^o(T)) = \zeta(\mathcal{P}(\mathcal{P}_\varphi^o(T))) = \zeta(\mathcal{P}_\varphi^o(\mathcal{P}(T))) = \zeta(\mathcal{P}(T)) = \varphi_{\mathcal{A}}(T).$$

This shows that $\mathcal{P}_\varphi^o$ preserves the normal faithful state $\varphi_{\mathcal{A}}$. By [DP22, Lemma 2.6], the fixed-point space $\mathcal{A}_0 = \text{Har}(\mathcal{P}_\varphi^o, \mathcal{A})$ is a von Neumann subalgebra of $\mathcal{A}$.

Clearly, if $T \in M' \cap \mathcal{A}$, then $\mathcal{P}_\varphi^o(T) = \sum_{k=1}^{\infty} y_k T y_k^* = \sum_{k=1}^{\infty} y_k y_k^* T = T$, so $M' \cap \mathcal{A} \subset \mathcal{A}_0$.

Conversely, suppose $T \in \mathcal{A}_0$. Because $\mathcal{A}_0$ is an algebra, we also have $TT^* \in \mathcal{A}_0$, meaning $\mathcal{P}_\varphi^o(T) = T$ and $\mathcal{P}_\varphi^o(TT^*) = TT^*$. We compute the following sum of positive elements:

$$(3.12) \quad \begin{aligned} & \sum_{k=1}^{\infty} (y_k T - T y_k)(y_k T - T y_k)^* \\ &= \sum_{k=1}^{\infty} (y_k T T^* y_k^* - y_k T y_k^* T^* - T y_k T^* y_k^* + T y_k y_k^* T^*) \\ &= \mathcal{P}_\varphi^o(TT^*) - \mathcal{P}_\varphi^o(T)T^* - T\mathcal{P}_\varphi^o(T^*) + T \left(\sum_{k=1}^{\infty} y_k y_k^* \right) T^* \\ &= TT^* - TT^* - TT^* + TT^* \\ &= 0. \end{aligned}$$

By (3.12), $[y_k, T] = 0$ for every k . Since $\overline{\mathbb{C}[y_k]}^w = M$, $T \in M' \cap \mathcal{A}$. Thus $\text{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = M' \cap \mathcal{A}$.

For the "in particular" statement, we set $\mathcal{A} = \mathcal{B}_\varphi \cong \text{Har}(\mathcal{P}_\varphi)$ and let $\mathcal{P} = \text{id}_{\mathcal{B}_\varphi}$. The hypothesis is trivially satisfied, yielding:

$$\text{Har}(\mathcal{P}_\varphi) \cap \text{Har}(\mathcal{P}_\varphi^o) = \text{Har}(\mathcal{P}_\varphi^o, \mathcal{B}_\varphi) = M' \cap \mathcal{B}_\varphi.$$

By Proposition 3.10, $M' \cap \mathcal{B}_\varphi = \mathcal{Z}(M)$, which concludes the proof. $\square$

Corollary 3.12. *Let $M \subset \mathcal{A}$ be a W^* -inclusion, and assume there exists a normal faithful ucp map $\mathcal{P} : \mathcal{A} \rightarrow \mathcal{B}_\varphi$ such that $\mathcal{P}|_M = \text{id}_M$. Then $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.*

Proof. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. By Theorem 3.11, the fixed space of $\mathcal{P}_\varphi^o$ is the ordinary von Neumann subalgebra $M' \cap \mathcal{A}$. The following ucp idempotent is therefore a conditional expectation onto $M' \cap \mathcal{A}$, without any assertion of normality:

$$E(x) := \text{w-}\lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=1}^n (\mathcal{P}_\varphi^o)^k(x).$$

Recall that $\mathcal{P}_\varphi^o$ is an M -inner ucp map. Therefore, $E(x)$ lies in the weak closure of the inner UCP orbit of x :

$$E(x) \in \overline{\{P(x) \mid P \text{ is an } M\text{-inner ucp map}\}}^w.$$

Normality of $\mathcal{P}_\varphi^o$ gives $\mathcal{P}_\varphi^o(E(x)) = E(x)$. By Theorem 3.11, $E(x) \in \text{Har}(\mathcal{P}_\varphi^o, \mathcal{A}) = M' \cap \mathcal{A}$. Hence

$$E(x) \in \overline{\{P(x) : P \text{ is an } M\text{-inner ucp map}\}}^w \cap (M' \cap \mathcal{A}).$$

□

Theorem 3.13. *Let $\mathcal{C}$ be a weakly closed operator system such that $M \subset \mathcal{C} \subset \mathcal{B}_\varphi$ and $M\mathcal{C} + \mathcal{C}M \subset \mathcal{C}$. If $\Psi : \mathcal{C} \rightarrow \mathcal{B}_\varphi$ is a normal ucp map such that $\Psi|_M = \text{id}_M$, then $\Psi = \text{id}_\mathcal{C}$.*

Proof. We identify $\mathcal{C}$ with its image in $\text{Har}(\mathcal{P}_\varphi)$ under the Poisson transform. Then Ψ is a normal ucp map from $\mathcal{C}$ to $\text{Har}(\mathcal{P}_\varphi)$. Since $\Psi|_M = \text{id}_M$, the map Ψ is M -bimodular. Using the explicit formula $\mathcal{P}_\varphi^o(T) = \sum_{k=1}^\infty y_k T y_k^*$, for any $T \in \mathcal{C}$ we have:

$$\begin{aligned} \langle \Psi(\mathcal{P}_\varphi^o(T)) \xi_\phi, \xi_\phi \rangle &= \sum_{k=1}^\infty \langle \Psi(y_k T y_k^*) \xi_\phi, \xi_\phi \rangle \\ &= \sum_{k=1}^\infty \langle y_k \Psi(T) y_k^* \xi_\phi, \xi_\phi \rangle \\ &= \langle \mathcal{P}_\varphi^o(\Psi(T)) \xi_\phi, \xi_\phi \rangle. \end{aligned}$$

Since $\Psi(T) \in \text{Har}(\mathcal{P}_\varphi)$ and φ is regular, the term on the right evaluates to:

$$\langle \mathcal{P}_\varphi(\Psi(T)) \xi_\phi, \xi_\phi \rangle = \langle \Psi(T) \xi_\phi, \xi_\phi \rangle.$$

Thus, $\langle \Psi(\mathcal{P}_\varphi^o(T)) \xi_\phi, \xi_\phi \rangle = \langle \Psi(T) \xi_\phi, \xi_\phi \rangle$ for all $T \in \mathcal{C}$. By linearity, this invariance extends to Cesàro means:

$$\left\langle \Psi \left(\frac{1}{N} \sum_{n=1}^N (\mathcal{P}_\varphi^o)^n(T) \right) \xi_\phi, \xi_\phi \right\rangle = \langle \Psi(T) \xi_\phi, \xi_\phi \rangle.$$

Define $A_N(T) = (1/N) \sum_{n=1}^N (\mathcal{P}_\varphi^o)^n(T)$. Since $\mathcal{P}_\varphi \mathcal{P}_\varphi^o = \mathcal{P}_\varphi^o \mathcal{P}_\varphi$,

$$\mathcal{P}_\varphi(A_N(T)) = A_N(T), \quad \|\mathcal{P}_\varphi^o(A_N(T)) - A_N(T)\| \leq \frac{2\|T\|}{N} \rightarrow 0.$$

Choose an ultraweakly convergent subnet with limit $z \in \mathcal{C}$. Normality and Theorem 3.11 give

$$z \in \text{Har}(\mathcal{P}_\varphi) \cap \text{Har}(\mathcal{P}_\varphi^o) = \mathcal{Z}(M).$$

Since $z \in M$, we have $\Psi(z) = z$. Normality of Ψ gives the first equality below. The last equality follows from (3.11) and normality of ζ :

$$\langle \Psi(T)\xi_\phi, \xi_\phi \rangle = \langle \Psi(z)\xi_\phi, \xi_\phi \rangle = \langle z\xi_\phi, \xi_\phi \rangle = \langle T\xi_\phi, \xi_\phi \rangle.$$

Now, for any $a, b \in M$ and $T \in \mathcal{C}$, the element b^*Ta is still in $\mathcal{C}$. Using the M -bimodularity of Ψ , we obtain:

$$\langle \Psi(T)a\xi_\phi, b\xi_\phi \rangle = \langle \Psi(b^*Ta)\xi_\phi, \xi_\phi \rangle = \langle b^*Ta\xi_\phi, \xi_\phi \rangle = \langle Ta\xi_\phi, b\xi_\phi \rangle.$$

Because $M\xi_\phi$ is dense in $L^2(M)$, this forces $\Psi(T) = T$ for all $T \in \mathcal{C}$. Hence, $\Psi = \text{id}_{\mathcal{C}}$. $\square$

Lemma 3.14. *Let φ be a normal regular strongly generating hyperstate with $\phi = \varphi|_M$, and let $0 < t < 1$. Set $\varphi^{(t)} = (1-t)\varphi + t\langle \cdot, \xi_\phi, \xi_\phi \rangle$. Then $\varphi^{(t)}$ is normal, regular and strongly generating, and its Poisson boundary agrees with that of φ , as an inclusion containing M . Moreover, if φ_i is such a hyperstate on M_i , then $\varphi_1^{(t_1)} \otimes \varphi_2^{(t_2)}$ is strongly generating in the sense of Definition 3.7.*

Proof. The two coefficient families for $\varphi^{(t)}$ are obtained by multiplying the original families by $\sqrt{1-t}$ and adjoining $\sqrt{t}1$. In particular, both nonselfadjoint unital algebras remain weakly dense. Since $\mathcal{P}_{\varphi^{(t)}} = (1-t)\mathcal{P}_\varphi + t\text{id}$, the two harmonic operator systems coincide, and hence so do their Poisson boundaries. The coefficient families for the product hyperstate contain nonzero scalar multiples of $z_{1,k} \otimes 1$, $1 \otimes z_{2,l}$, and, respectively, $y_{1,k} \otimes 1$, $1 \otimes y_{2,l}$. Their unital algebras are therefore weakly dense in $M_1 \bar{\otimes} M_2$. $\square$

Theorem 3.15. *Suppose for each $i \in \{1, 2\}$, M_i is a von Neumann algebra. Let φ_i and $\varphi_1 \otimes \varphi_2$ be normal regular strongly generating hyperstates for M_i and $M_1 \bar{\otimes} M_2$ on $B(L^2(M_i))$ and $B(L^2(M_1 \bar{\otimes} M_2))$, respectively. Then*

$$\text{Har}(\mathcal{P}_{\varphi_1 \otimes \varphi_2}) = \text{Har}(\mathcal{P}_{\varphi_1}) \bar{\otimes} \text{Har}(\mathcal{P}_{\varphi_2}),$$

and

$$\mathcal{B}_{\varphi_1 \otimes \varphi_2} \cong \mathcal{B}_{\varphi_1} \bar{\otimes} \mathcal{B}_{\varphi_2}.$$

Proof. Write $P_i = \mathcal{P}_{\varphi_i}$ and $\mathcal{H} = \text{Har}(P_1 \otimes P_2)$. The normal maps $P_1 \otimes \text{id}$ and $\text{id} \otimes P_2$ preserve $\mathcal{H}$ and fix $M_1 \bar{\otimes} M_2$ pointwise. Theorem 3.13, applied to the product hyperstate, shows that both maps act identically on $\mathcal{H}$. Thus, for $x \in \mathcal{H}$,

$$(\text{id} \otimes \omega_2)(x) \in \text{Har}(P_1), \quad (\omega_1 \otimes \text{id})(x) \in \text{Har}(P_2) \quad (\omega_i \in B(L^2(M_i))_*).$$

By Proposition 3.4, these harmonic operator systems are normally completely order isomorphic to injective von Neumann algebras. The slice-map property for an injective von Neumann algebra, as used in [DP22, Theorem 4.4], now gives

$$\mathcal{H} \subset \text{Har}(P_1) \bar{\otimes} \text{Har}(P_2).$$

The reverse inclusion follows from normality and the identity $(P_1 \otimes P_2)(S \otimes T) = S \otimes T$ for harmonic S, T . The tensor product of the Poisson transforms is therefore a normal complete order isomorphism onto $\mathcal{H}$. Uniqueness of the Choi–Effros von Neumann algebra structure gives the stated $*$ -isomorphism, which fixes $M_1 \bar{\otimes} M_2$. $\square$

4. THE CHOQUET-DENY PROPERTY

Proposition 4.1. *Any type I factor $M = B(H)$ is Choquet-Deny.*

Proof. Let $M = B(H)$. Since M is a type I factor, its standard representation $B(L^2(M))$ canonically decomposes as the von Neumann algebra tensor product $M \bar{\otimes} M'$. Under this spatial identification, any $a \in M$ acts as $a \otimes 1$, and any $b \in M'$ acts as $1 \otimes b$.

Let φ be a normal regular strongly generating M -hyperstate. The associated Poisson transform $\mathcal{P}_\varphi : B(L^2(M)) \rightarrow B(L^2(M))$ takes the standard form:

$$\mathcal{P}_\varphi(T) = \sum_{k=1}^{\infty} (Jz_k^* J) T (Jz_k J),$$

where $(z_k)_{k=1}^{\infty} \subset M$ and $\sum_{k=1}^{\infty} z_k^* z_k = 1$.

Because the inner ucp coefficients $Jz_k J$ belong to M' , they correspond to elements of the form $1 \otimes y_k$ in $M \bar{\otimes} M'$. Consequently, the map $\mathcal{P}_\varphi$ splits as a tensor product $\mathcal{P}_\varphi = \text{id}_M \otimes \Phi$, where $\Phi : M' \rightarrow M'$ is the normal ucp map defined by $\Phi(x) = \sum_{k=1}^{\infty} y_k^* x y_k$.

For a normal ucp map of the form $\text{id}_M \otimes \Phi$ on a tensor product, its space of harmonic operators is precisely the tensor product of the respective harmonic spaces:

$$\text{Har}(\mathcal{P}_\varphi) = M \bar{\otimes} \text{Har}(\Phi).$$

To determine $\text{Har}(\Phi)$, note that Φ is simply the restriction of $\mathcal{P}_\varphi$ to M' . Because φ is a regular hyperstate, Φ preserves the normal faithful state ϕ^o on M' . By [DP22, Lemma 2.6], preserving a faithful state ensures that the fixed-point space $\text{Har}(\Phi)$ is a von Neumann subalgebra of M' .

For $x \in \text{Har}(\Phi)$, the argument in (3.4) and (3.12) gives $[x, y_k] = 0$ for every k , or equivalently $[x, Jz_k J] = 0$ in $B(L^2(M))$. Since $\overline{\mathbb{C}[z_k]}^w = M$,

$$x \in (M')' \cap M' = M \cap M' = \mathcal{Z}(M) = \mathbb{C}1.$$

Substituting this back into the tensor product yields:

$$\text{Har}(\mathcal{P}_\varphi) = M \bar{\otimes} \mathbb{C}1 = M.$$

This proves that the Poisson boundary $\mathcal{B}_\varphi \cong \text{Har}(\mathcal{P}_\varphi)$ coincides entirely with M , meaning it is trivial. Thus, $M = B(H)$ is Choquet-Deny. $\square$

Proposition 4.2. *Let M be a von Neumann algebra, and let φ be a normal regular strongly generating hyperstate on $B(L^2(M))$. Then $\text{Har}(\mathcal{P}_\varphi) \subset \mathcal{Z}(M)'$.*

Proof. Fix a unitary $u \in \mathcal{U}(\mathcal{Z}(M))$, and define $\Psi_u : \mathcal{B}_\varphi \rightarrow \mathcal{B}_\varphi$ by

$$\Psi_u(x) = u x u^*, \quad x \in \mathcal{B}_\varphi.$$

Then Ψ_u is a normal ucp map and $\Psi_u|_M = \text{id}_M$. By the rigidity of the Poisson boundary (Theorem 3.13), we have $\Psi_u = \text{id}_{\mathcal{B}_\varphi}$.

Therefore, for any $u \in \mathcal{U}(\mathcal{Z}(M))$ and $x \in \mathcal{B}_\varphi$, we have $u x u^* = x$. By the isomorphism $\mathcal{B}_\varphi \cong \text{Har}(\mathcal{P}_\varphi)$, which is M -bimodular, we know that for any $u \in \mathcal{U}(\mathcal{Z}(M))$ and $T \in \text{Har}(\mathcal{P}_\varphi)$, $u T u^* = T$. Therefore, $\text{Har}(\mathcal{P}_\varphi) \subset \mathcal{Z}(M)'$. $\square$

For the next theorem, we refer to [Tak02, Chapter IV] for details on direct integral theory.

Theorem 4.3. *Let M be a von Neumann algebra, and let φ be a normal regular strongly generating hyperstate on $B(L^2(M))$. Let $\phi = \varphi|_M$ be the corresponding normal faithful state on M .*

Assume that (M, ϕ) admits a direct integral decomposition on a standard probability measure space (X, μ_X) :

$$(M, \phi) = \int_X^\oplus (M_x, \phi_x) d\mu_X(x).$$

By direct integral theory, φ admits the following decomposition when restricted to the algebra of decomposable operators:

$$\varphi|_{\int_X^\oplus B(L^2(M_x)) d\mu_X(x)} = \int_X^\oplus \varphi_x d\mu_X(x).$$

Then we have that for μ_X -a.e. $x \in X$, φ_x is a normal regular strongly generating hyperstate on $B(L^2(M_x))$. Moreover, we have

$$\text{Har}(\mathcal{P}_\varphi) = \int_X^\oplus \text{Har}(\mathcal{P}_{\varphi_x}) d\mu_X(x)$$

and

$$\mathcal{B}_\varphi = \int_X^\oplus \mathcal{B}_{\varphi_x} d\mu_X(x).$$

Proof. We use a measurable field of standard forms, so that $J = \int_X^\oplus J_x d\mu_X(x)$ and $\xi_\phi = \int_X^\oplus \xi_{\phi_x} d\mu_X(x)$. Choose the two coefficient families (z_k) and (y_k) in the definition of strong generation. Outside one null set their decompositions satisfy

$$\sum_k z_k(x)^* z_k(x) = 1, \quad \sum_k y_k(x) y_k(x)^* = 1,$$

with strong convergence. The representation using (z_k) gives

$$(4.1) \quad \varphi_x(T) = \sum_k \langle T J_x z_k(x) J_x \xi_{\phi_x}, J_x z_k(x) J_x \xi_{\phi_x} \rangle, \quad T \in B(L^2(M_x)).$$

Indeed, integrating (4.1) against arbitrary central projections

recovers the disintegration of φ . In particular, the fiber functionals are normal states. The representation using (y_k) disintegrates in the same way. Testing the two marginal identities against central projections and a countable fundamental family of sections shows that

$$\varphi_x|_{M_x} = \phi_x, \quad \varphi_x(J_x a^* J_x) = \phi_x(a), \quad a \in M_x,$$

almost everywhere. Hence the fiber hyperstates are regular.

We next check nonselfadjoint generation. Enumerate the words in (z_k) , including the empty word. If the fiber words failed to have ultraweakly dense linear span on a set of positive measure, Hahn–Banach and measurable selection in the measurable field of preduals would give a nonzero integrable measurable field $x \mapsto f_x \in (M_x)_*$ annihilating every fiber word. One may choose $\|f_x\| \leq 1$ and support this field on that set. Choose a positive-measure subset and one bounded fundamental section $a_j(x)$ on which $|f_x(a_j(x))| > 1/m$ for some m . Multiplying the functionals by measurable scalar phases makes these values positive real. Their direct integral is then a nonzero normal functional on M annihilating every global word, contradicting

strong generation. Here measurable selection applies to the countably many equations $f_x(w(x)) = 0$; nonzeroness can be tested on a countable fundamental family of sections. Applying the same argument to (y_k) proves both required density statements almost everywhere.

By Proposition 4.2, every harmonic operator is decomposable.

For $T = \int_X^\oplus T_x \, d\mu_X(x)$, (3.1) gives

$$\mathcal{P}_\varphi(T) = \int_X^\oplus \mathcal{P}_{\varphi_x}(T_x) \, d\mu_X(x).$$

Consequently, T is harmonic if and only if T_x is harmonic almost everywhere, proving the asserted equality of operator systems. This identification also preserves the boundary products. Indeed, for a harmonic T , the increasing bounded sequence $\mathcal{P}_\varphi^n(T^*T)$ converges strongly to its Choi–Effros square; the same construction applies in every fiber and commutes with the direct integral. Polarization gives the assertion for all products. Thus the resulting complete order identification is the stated normal $*$ -isomorphism of Poisson boundaries. $\square$

Theorem 4.4. *A von Neumann algebra M is Choquet-Deny if and only if it is of type I.*

Proof. Suppose first that M is Choquet-Deny. Example 3.9 supplies an nrsg hyperstate; its boundary is M by assumption. Proposition 3.4 therefore implies that M is injective. Write $M = M_I \oplus N$, where N is the continuous part, and suppose $N \neq 0$.

Every continuous injective factor with separable predual absorbs the hyperfinite II_1 factor R . For semifinite factors this follows from the uniqueness of the injective II_1 factor. In type III_λ , $0 \leq \lambda < 1$, tensoring by R does not change the flow of weights, so the classification of injective factors applies; in type III_1 one uses uniqueness of the injective factor of that type. See [Con76, Haa87]. Disintegrating N over its center gives this absorption fiberwise. The fiber isomorphisms can be chosen measurably: in standard form their unitary implementations are selected from the measurable isomorphism relation, specified by countably many equations on fundamental operator sections. Integrating them gives

$$N \cong N \bar{\otimes} R.$$

By [Zho24, Proposition 5.3], R admits a normal regular strongly generating hyperstate φ_R with nontrivial boundary. The cited construction uses a semigroup-generating measure on group unitaries of S_∞ ; its two coefficient families agree in the tracial setting, and $\mathbb{C}[z_k] = \mathbb{C}[y_k]$ is weakly dense. Choose an nrsg hyperstate φ_0 on N using Example 3.9. Replace both hyperstates by their lazy versions from Lemma 3.14. Their boundaries do not change, and both coefficient families of their product now generate $N \bar{\otimes} R$. Theorem 3.15 gives

$$\mathcal{B}_{\varphi_0 \otimes \varphi_R} \cong \mathcal{B}_{\varphi_0} \bar{\otimes} \mathcal{B}_{\varphi_R}.$$

This inclusion is nontrivial: take a harmonic operator $T \notin R$ for φ_R . Then $1 \otimes T$ is product-harmonic and does not belong to $N \bar{\otimes} R$, as is seen by slicing with a normal state on N . Transport through a standard-form implementation of $N \cong N \bar{\otimes} R$ gives an nrsg hyperstate φ_N on $B(L^2(N))$ with nontrivial boundary. This transport need not preserve a previously chosen marginal, which is immaterial to the definition of Choquet-Deny.

If $M_I = 0$, this already contradicts the hypothesis. Otherwise choose an nrsg hyperstate φ_I on M_I and lazify both summand hyperstates. With respect to $L^2(M) = L^2(M_I) \oplus L^2(N)$ define

$$(4.2) \quad \varphi(T) = \frac{1}{2}\varphi_I(T_{11}) + \frac{1}{2}\varphi_N(T_{22}).$$

Its marginal is $\frac{1}{2}\phi_I \oplus \frac{1}{2}\phi_N$. Its two coefficient families (z_k) and (y_k) are the corresponding summand families extended by zero. Because the summands were lazified, these families contain nonzero scalar multiples of both central summand projections. Consequently both $\mathbb{C}[z_k]$ and $\mathbb{C}[y_k]$ are weakly dense in M , and φ is nrsg. The maps $\mathcal{P}_\varphi$ and $\mathcal{P}_\varphi^o$ kill off-diagonal operators, and on the diagonal they are the direct sums of the summand maps. Hence

$$\mathcal{B}_\varphi \cong \mathcal{B}_{\varphi_I} \oplus \mathcal{B}_{\varphi_N} \neq M,$$

a contradiction. Thus $N = 0$ and M is of type I.

Conversely, let M be of type I and let φ be any nrsg hyperstate. Decompose the standard form over the center:

$$(M, \phi) = \int_X^\oplus (M_x, \phi_x) d\mu_X(x),$$

where almost every M_x is a type I factor. Theorem 4.3 gives nrsg fiber hyperstates φ_x almost everywhere. Proposition 4.1 gives $\text{Har}(\mathcal{P}_{\varphi_x}) = M_x$. Integrating and using Theorem 4.3 again yields

$$\text{Har}(\mathcal{P}_\varphi) = \int_X^\oplus M_x d\mu_X(x) = M.$$

Therefore M is Choquet-Deny. $\square$

5. PROOFS OF MAIN RESULTS ON THE WEAK DIXMIER PROPERTY

Theorem 5.1. *Let $M \subset \mathcal{A}$ be a W^* -inclusion such that M is properly infinite. Then $M \subset \mathcal{A}$ has the weak Dixmier property if and only if it has the inner UCP-Dixmier property.*

Proof. We only need to prove the if direction, since every finite convex combination of unitary conjugations is an M -inner ucp map. Assume that $M \subset \mathcal{A}$ has the inner UCP-Dixmier property.

Because M is properly infinite, it contains an infinite sequence of mutually orthogonal projections $(p_k)_{k=1}^\infty$ such that $p_k \sim 1_M$ for all k and $\sum_{k=1}^\infty p_k = 1_M$. Choose isometries $s_k \in M$ with $s_k s_k^* = p_k$. This partition of unity gives a spatial isomorphism

$$\Theta : \mathcal{A} \xrightarrow{\sim} \mathcal{A} \bar{\otimes} B(H), \quad \Theta(a) = (s_i^* a s_j)_{i,j \geq 1},$$

where $H = \ell^2(\mathbb{N})$. Its restriction maps M onto $M \bar{\otimes} B(H)$ and p_k to $1_M \otimes e_{kk}$, where $(e_{ij})_{i,j \geq 1}$ are the standard matrix units of $B(H)$. Indeed, on a Hilbert space on which $\mathcal{A}$ acts, the unitary $\xi \otimes e_k \mapsto s_k \xi$ implements this isomorphism. Thus we identify the inclusion $M \subset \mathcal{A}$ with

$$M \bar{\otimes} B(H) \subset \mathcal{A} \bar{\otimes} B(H).$$

It suffices to prove the weak Dixmier property for an arbitrary element $x \in \mathcal{A} \bar{\otimes} B(H)$. Write

$$K_x = \overline{\text{conv}}^w \{U x U^* : U \in \mathcal{U}(M \bar{\otimes} B(H))\}.$$

First, we can take a weak limit of convex combinations of unitary conjugates of the form $(1_M \otimes u)x(1_M \otimes u^*)$ to average x into the relative commutant $\mathcal{A} \otimes 1_{B(H)}$ of $1_{\mathcal{A}} \otimes B(H)$. To see this explicitly, average over the compact group $\mathcal{U}(M_m(\mathbb{C}))$ acting on the first m basis vectors of H and identically on their orthogonal complement. These averages lie in K_x and are uniformly bounded. Any weak cluster point commutes with each of these nested unitary groups, and hence with $1_{\mathcal{A}} \otimes B(H)$. Thus, for some $a \in \mathcal{A}$, we have

$$a \otimes 1_{B(H)} \in K_x.$$

Since K_x is weakly closed, convex and invariant under these unitary conjugations, it also contains the weakly closed convex hull of the unitary orbit of $a \otimes 1_{B(H)}$.

To proceed, it is sufficient to prove that for any M -inner ucp map

$$P(a) = \sum_{k=1}^{\infty} z_k a z_k^*, \quad z_k \in M, \quad \sum_{k=1}^{\infty} z_k z_k^* = 1_M,$$

the element $P(a) \otimes 1_{B(H)}$ is contained in the weak closure of the convex hull of the unitary orbit of $a \otimes 1_{B(H)}$.

Fix an integer $n \geq 1$. We construct a partial isometry $Z_n \in M \bar{\otimes} B(H)$ structured as a block matrix where each block is of size $n \times n$. The first block-row consists of a zero block followed by scaled identity matrices $z_k I_n$, and all subsequent block-rows are identically zero:

$$Z_n = \begin{pmatrix} 0_{n \times n} & z_1 I_n & z_2 I_n & z_3 I_n & \cdots \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The strong convergence of $\sum_{k=1}^{\infty} z_k z_k^* = 1_M$ makes this row a bounded operator. Multiplying this block matrix by its adjoint yields the identity 1_M on the first $n \times n$ diagonal block and zero elsewhere. Letting E_n denote the projection onto the first n basis vectors of H , we have

$$Z_n Z_n^* = 1_M \otimes E_n.$$

Thus, Z_n^* is a partial isometry with initial projection $1_M \otimes E_n$. Because the first block-column of Z_n is identically zero, its final projection $Z_n^* Z_n$ is orthogonal to $1_M \otimes E_n$. Since M is properly infinite, each $1_M \otimes e_{jj}$ is equivalent to $1_{M \bar{\otimes} B(H)}$. For $j = 1$, this is implemented by the strong sum $\sum_k s_k \otimes e_{1k}$, and the other cases follow by matrix-unit equivalence. Hence

$$1 - Z_n^* Z_n \geq 1_M \otimes E_n \geq 1_M \otimes e_{11} \sim 1_{M \bar{\otimes} B(H)}.$$

We also have

$$1 - (1_M \otimes E_n) \geq 1_M \otimes e_{n+1, n+1} \sim 1_{M \bar{\otimes} B(H)}.$$

Therefore, the orthogonal complements of both projections are Murray-von Neumann equivalent to the identity:

$$1 - (1_M \otimes E_n) \sim 1 - Z_n^* Z_n \sim 1_{M \bar{\otimes} B(H)}.$$

This guarantees the existence of a partial isometry $W_n \in M \bar{\otimes} B(H)$ with

$$W_n^* W_n = 1 - Z_n^* Z_n, \quad W_n W_n^* = 1 - (1_M \otimes E_n).$$

Define $U_n = Z_n + W_n$. Since the initial and final projections of Z_n and W_n are respectively orthogonal, all mixed terms vanish. Therefore,

$$U_n^* U_n = Z_n^* Z_n + W_n^* W_n = 1, \quad U_n U_n^* = Z_n Z_n^* + W_n W_n^* = 1.$$

Hence we complete the partial isometry into a unitary $U_n \in \mathcal{U}(M \bar{\otimes} B(H))$. Since $(1_M \otimes E_n)W_n = 0$, the first block-row of U_n remains the first block-row of Z_n . In block-row notation, we write

$$U_n = \begin{pmatrix} Z_n \\ W_n \end{pmatrix}.$$

We conjugate $a \otimes 1_{B(H)}$ by U_n . Since

$$Z_n(a \otimes 1_{B(H)})Z_n^* = P(a) \otimes E_n,$$

we have, with respect to the first n coordinates and their orthogonal complement,

$$(5.1) \quad U_n(a \otimes 1_{B(H)})U_n^* = \begin{pmatrix} P(a)I_n & * \\ * & * \end{pmatrix}.$$

To ensure a diagonal output, let $\mathcal{D} \subset B(H)$ be the abelian von Neumann algebra of diagonal operators, and apply the conditional expectation onto the relative commutant of $1_{\mathcal{A}} \otimes \mathcal{D}$:

$$E_{\mathcal{D}'} : \mathcal{A} \bar{\otimes} B(H) \longrightarrow \mathcal{A} \bar{\otimes} \mathcal{D}.$$

This expectation is approximated weakly by convex combinations of diagonal unitary conjugations. Indeed, for $t \in [0, 1]$, let

$$U_t = 1_M \otimes \sum_{k=1}^{\infty} e^{2k\pi it} e_{kk} \in \mathcal{U}(1_M \otimes \mathcal{D}).$$

Then

$$(5.2) \quad E_{\mathcal{D}'}(T) = \int_0^1 U_t T U_t^* dt,$$

where the integral is taken ultraweakly. By (5.2) and (5.1),

$$E_{\mathcal{D}'}(U_n(a \otimes 1_{B(H)})U_n^*) = P(a) \otimes E_n + b_n,$$

where

$$b_n = (1 - 1_{\mathcal{A}} \otimes E_n)b_n(1 - 1_{\mathcal{A}} \otimes E_n), \quad \|b_n\| \leq \|a\|.$$

As $n \rightarrow \infty$, $E_n \nearrow 1_{B(H)}$ strongly. Thus, the support and uniform norm bound of b_n give $b_n \rightarrow 0$ strongly, and hence weakly. Concurrently, $P(a) \otimes E_n \rightarrow P(a) \otimes 1_{B(H)}$ weakly. Therefore,

$$\text{w-} \lim_{n \rightarrow \infty} E_{\mathcal{D}'}(U_n(a \otimes 1_{B(H)})U_n^*) = P(a) \otimes 1_{B(H)}.$$

Because this holds for any M -inner ucp map P , the entire set

$$\{P(a) \otimes 1_{B(H)} \mid P \text{ is an } M\text{-inner ucp map}\}$$

is contained in K_x .

By the assumption that the original inclusion $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, there exists an element

$$y \in \overline{\{P(a) \mid P \text{ is an } M\text{-inner ucp map}\}}^w \cap (M' \cap \mathcal{A}).$$

Because the mapping $T \mapsto T \otimes 1_{B(H)}$ is normal, the element $y \otimes 1_{B(H)}$ lies in the weak closure of $\{P(a) \otimes 1_{B(H)}\}$. Since K_x is already weakly closed, it contains $y \otimes 1_{B(H)}$.

Thus, $y \otimes 1_{B(H)}$ belongs to

$$(M \bar{\otimes} B(H))' \cap (\mathcal{A} \bar{\otimes} B(H)) = (M' \cap \mathcal{A}) \bar{\otimes} \mathbb{C}1_{B(H)}$$

and lies in the weakly closed convex hull of the unitary orbit of x , which finishes the proof. $\square$

Theorem 5.2. *Let M be a von Neumann algebra with separable predual and φ be a normal regular strongly generating hyperstate on $B(L^2(M))$ with the Poisson boundary $\mathcal{B}_\varphi$. Assume that one of the following conditions holds:*

- (i) *M is properly infinite, i.e., $M = M_{I_\infty} \oplus M_{II_\infty} \oplus M_{III}$;*
- (ii) *M is finite with normal faithful trace τ and $\varphi = \int_{\mathcal{U}(M)} \langle \cdot, u\xi_\tau, u\xi_\tau \rangle d\mu(u)$ for some $\mu \in \text{Prob}(\mathcal{U}(M))$.*

Then any W^ -inclusion $M \subset \mathcal{A}$ equipped with a normal faithful ucp map $\mathcal{P} : \mathcal{A} \rightarrow \mathcal{B}_\varphi$ with $\mathcal{P}|_M = \text{id}_M$ has the weak Dixmier property.*

Proof. Let $M \subset \mathcal{A}$ be a W^* -inclusion equipped with a normal faithful ucp map $\mathcal{P} : \mathcal{A} \rightarrow \mathcal{B}_\varphi$ satisfying $\mathcal{P}|_M = \text{id}_M$. By Corollary 3.12, the existence of such a map $\mathcal{P}$ guarantees that the inclusion $M \subset \mathcal{A}$ automatically satisfies the inner UCP-Dixmier property.

We now verify that the weak Dixmier property holds under either of the assumed conditions:

Case (i): Assume M is properly infinite. By Theorem 5.1, for properly infinite von Neumann algebras, the weak Dixmier property is equivalent to the inner UCP-Dixmier property. Since $M \subset \mathcal{A}$ already has the inner UCP-Dixmier property by Corollary 3.12, it inherently possesses the weak Dixmier property.

Case (ii): Assume M is finite with trace τ , and $\varphi = \int_{\mathcal{U}(M)} \langle \cdot, u\xi_\tau, u\xi_\tau \rangle d\mu(u)$ for some probability measure $\mu \in \text{Prob}(\mathcal{U}(M))$. In this setting, the associated ucp map $\mathcal{P}_\varphi^o : \mathcal{A} \rightarrow \mathcal{A}$ is explicitly given by integrating over unitary conjugates: $\mathcal{P}_\varphi^o(x) = \int_{\mathcal{U}(M)} u^*xu d\mu(u)$ for all $x \in \mathcal{A}$. The adjoint direction follows from the vector-state convention $\varphi = \int \omega_{u\xi_\tau} d\mu(u)$. Because $\mathcal{P}_\varphi^o$ is constructed solely from integrals of unitary conjugations, its iterates $(\mathcal{P}_\varphi^o)^k$ and their Cesàro means $\Psi_N(x) = (1/N) \sum_{k=1}^N (\mathcal{P}_\varphi^o)^k(x)$ must lie inside the weakly closed convex hull of the unitary orbit of x :

$$\Psi_N(x) \in \overline{\text{conv}}^w \{uxu^* \mid u \in \mathcal{U}(M)\}.$$

As established in the proof of Corollary 3.12, any weak limit point z of the sequence $(\Psi_N(x))_{N=1}^\infty$ is an element of $M' \cap \mathcal{A}$. Since the weakly closed convex hull of the unitary orbit is weakly closed, the limit point z remains within it. Thus:

$$z \in \overline{\text{conv}}^w \{uxu^* \mid u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset,$$

which satisfies the exact definition of the weak Dixmier property for $M \subset \mathcal{A}$. $\square$

Corollary 5.3 ([Pop00, Mar21]). *Any W^* -inclusion $M \subset \mathcal{A}$ with normal faithful conditional expectation has the weak Dixmier property.*

Proof. Let $E : \mathcal{A} \rightarrow M$ be the given normal faithful conditional expectation. Decompose $M = M_{\text{fin}} \oplus M_{\infty}$ into its finite and properly infinite parts, omitting zero summands, and write p_1, p_{∞} for their units. Averaging by $p_1 + e^{2\pi it} p_{\infty}$ removes the off-diagonal terms of every $x \in \mathcal{A}$:

$$(5.3) \quad \int_0^1 (p_1 + e^{2\pi it} p_{\infty}) x (p_1 + e^{2\pi it} p_{\infty})^* dt = p_1 x p_1 + p_{\infty} x p_{\infty}.$$

The restrictions $E_i : p_i \mathcal{A} p_i \rightarrow p_i M$ are normal faithful conditional expectations.

Choose a faithful normal trace τ on M_{fin} , a countable adjoint-closed family of unitaries (u_k) generating M_{fin} , and $c_k > 0$ with $\sum_k c_k = 1$. The state

$$\varphi_1 = \sum_k c_k \langle \cdot, u_k \xi_{\tau}, u_k \xi_{\tau} \rangle$$

is normal and regular. Both strong generation conditions follow from the weak density of $\mathbb{C}[u_k]$, since (u_k) is adjoint-closed. Composing E_1 with $M_{\text{fin}} \subset \mathcal{B}_{\varphi_1}$, Theorem 5.2(ii) gives the weak Dixmier property for $M_{\text{fin}} \subset p_1 \mathcal{A} p_1$.

For M_{∞} , choose an nrsg hyperstate by Example 3.9. Composing E_{∞} with the boundary inclusion and applying Theorem 5.2(i) gives the same conclusion for this corner. Product nets of the two unitary averaging nets, after applying (5.3), give an element of $M' \cap \mathcal{A}$ in the closed convex unitary orbit of x . $\square$

6. A NEW PROOF OF ISONO'S THEOREM

In this section, an M -bimodular semifinite positive map with values in a von Neumann algebra $\mathcal{B} \supset M$ means an additive, positively homogeneous map $\mathcal{P} : \mathcal{A}_+ \rightarrow \widehat{\mathcal{B}}_+$ satisfying

$$\mathcal{P}(a^* x a) = a^* \mathcal{P}(x) a \quad (x \in \mathcal{A}_+, a \in M).$$

We require normality, meaning preservation of increasing bounded nets, faithfulness, and ultraweak density of

$$\mathfrak{m}_{\mathcal{P}}^+ = \{x \in \mathcal{A}_+ : \mathcal{P}(x) \in \mathcal{B}_+\}.$$

Here $\widehat{\mathcal{B}}_+$ is the extended positive cone, and $\mathcal{P}(x) < \infty$ will always mean that $\mathcal{P}(x)$ is bounded. For every $\rho \in \mathcal{B}_*^+$, the function $\rho \circ \mathcal{P}$ is a normal weight. In particular, its sublevel sets are ultraweakly closed. It follows that, for $C < \infty$, the set

$$(6.1) \quad \{x \in \mathcal{A}_+ : \mathcal{P}(x) \leq C1\}$$

is ultraweakly closed. We use this scalar interpretation of lower semicontinuity throughout; see [Haa79b, Sections 1–2].

Lemma 6.1. *Let φ_A be a normal faithful semifinite weight on a von Neumann algebra A , and let $P : A \rightarrow A$ be a normal ucp map satisfying $\varphi_A \circ P = \varphi_A$. Fix a free ultrafilter ω and define*

$$E(x) = \text{uw-}\lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x).$$

If $x \in A_+$ and $\varphi_A(x) < \infty$, then $P(E(x)) = E(x)$ and $P(E(x)^2) = E(x)^2$.

Proof. Write $A_n(x) = (1/n) \sum_{k=0}^{n-1} P^k(x)$ and $y = E(x)$. Since $\|P(A_n(x)) - A_n(x)\| \leq 2\|x\|/n$, normality of P gives $P(y) = y$. Lower semicontinuity and invariance of the weight give

$$\varphi_A(y) \leq \varphi_A(x) < \infty, \quad \varphi_A(y^2) \leq \|y\| \varphi_A(y) < \infty.$$

By Kadison's inequality, $P(y^2) - y^2 \geq 0$. Since $\varphi_A \circ P = \varphi_A$ and $\varphi_A(y^2) < \infty$,

$$\varphi_A(P(y^2) - y^2) = \varphi_A(P(y^2)) - \varphi_A(y^2) = 0.$$

Faithfulness gives $P(y^2) = y^2$. □

Theorem 6.2. *Let $M \subset \mathcal{A}$ be a W^* -inclusion and let φ be a normal regular strongly generating hyperstate. Assume that there exists a normal faithful semifinite M -bimodular positive map $\mathcal{P} : \mathcal{A}_+ \rightarrow \widehat{\mathcal{B}}_{\varphi+}$ in the preceding sense. Set $Q = \mathcal{P}^\circ$, acting on $\mathcal{A}$ by $Q(x) = \sum_k y_k x y_k^*$. For any $x \in \mathcal{A}_+$ with $\mathcal{P}(x) < \infty$, every ultraweak cluster point of*

$$(6.2) \quad \frac{1}{n} \sum_{k=0}^{n-1} Q^k(x)$$

belongs to $M' \cap \mathcal{A}$.

Proof. Let ζ be the canonical faithful normal state on $\mathcal{B}_\varphi$ and let $\varphi_{\mathcal{A}} = \zeta \circ \mathcal{P}$. This is a normal faithful weight. It is semifinite because its finite positive domain contains the ultraweakly dense cone $\mathfrak{m}_{\mathcal{P}}^+$.

Normality and M -bimodularity give, in the extended positive cone,

$$\mathcal{P}(Q(x)) = \sum_k y_k \mathcal{P}(x) y_k^* = Q(\mathcal{P}(x)).$$

By (3.11), $\zeta \circ Q = \zeta$ on $(\mathcal{B}_\varphi)_+$. This identity extends

to the extended positive cone by monotone approximation. Hence $\varphi_{\mathcal{A}} \circ Q = \varphi_{\mathcal{A}}$.

Let T be an ultraweak cluster point of (6.2). The proof of Lemma 6.1, applied along a convergent subnet, gives $Q(T) = T$ and $Q(T^2) = T^2$. Consequently,

$$\sum_k (y_k T - T y_k) (y_k T - T y_k)^* = Q(T^2) - Q(T)T - TQ(T) + T^2 = 0.$$

Thus $[T, y_k] = 0$ for every k . Since $\overline{\mathbb{C}[y_k]}^w = M$, $T \in M' \cap \mathcal{A}$. □

Theorem 6.3. *Let M have separable predual and let φ be a normal regular strongly generating hyperstate. Assume either that M is properly infinite, or that M is finite with a faithful normal tracial state τ and*

$$(6.3) \quad \varphi(T) = \int_{\mathcal{U}(M)} \langle T u \xi_\tau, u \xi_\tau \rangle d\mu(u)$$

for a probability measure μ . Let $M \subset \mathcal{A}$ be a W^ -inclusion and let $\mathcal{P} : \mathcal{A}_+ \rightarrow \widehat{\mathcal{B}}_{\varphi+}$ be a normal faithful semifinite M -bimodular positive map. Then, for $x \in \mathcal{A}_+$ with $\mathcal{P}(x) < \infty$,*

$$\overline{\text{conv}}^w \{u x u^* : u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Proof. Let $C = \|\mathcal{P}(x)\|$ and $K_x = \overline{\text{conv}}^w\{uxu^* : u \in \mathcal{U}(M)\}$.

For $u \in \mathcal{U}(M)$, $\mathcal{P}(uxu^*) \leq C1$. By the ultraweak closedness of (6.1),

$$\mathcal{P}(b) \leq C1 \quad (b \in K_x).$$

Suppose first that M is properly infinite. Choose isometries $(v_i) \subset M$ with $v_i^*v_j = \delta_{ij}1$ and $\sum_i v_i v_i^* = 1$, and use the isomorphism

$$\pi : \mathcal{A} \xrightarrow{\sim} \mathcal{A} \bar{\otimes} B(\ell^2(\mathbb{N})), \quad \pi(b) = \sum_{i,j} v_i^* b v_j \otimes e_{ij}.$$

It maps M onto $M \bar{\otimes} B(\ell^2(\mathbb{N}))$. As in the proof of Theorem 5.1, unitary averaging over the type I tensor factor gives $a \otimes 1 \in \pi(K_x)$ for some $a \in \mathcal{A}_+$. Let $b = \pi^{-1}(a \otimes 1)$. Since $b \in K_x$ and $a = v_1^* b v_1$,

$$\mathcal{P}(a) = v_1^* \mathcal{P}(b) v_1 \leq C1.$$

By Theorem 6.2, an ultraweak cluster point c of the Cesàro means of $(\mathcal{P}_\varphi^a)^k(a)$ belongs to $M' \cap \mathcal{A}$.

The matrix-dilation argument in the proof of Theorem 5.1 applies to each M -inner ucp map. Thus every Cesàro mean of $(\mathcal{P}_\varphi^a)^k(a)$, tensored with 1, belongs to the weakly closed unitary convex hull of $a \otimes 1$. That hull is contained in $\pi(K_x)$, so $c \otimes 1 \in \pi(K_x)$.

Since $c \otimes 1 \in (M \bar{\otimes} B(\ell^2(\mathbb{N})))'$,

$$\pi^{-1}(c \otimes 1) \in K_x \cap (M' \cap \mathcal{A}).$$

Suppose now that M is finite and φ satisfies (6.3).

With the present vector-state convention,

$$\mathcal{P}_\varphi^a(b) = \int_{\mathcal{U}(M)} u^* b u \, d\mu(u).$$

Its iterates and Cesàro means preserve K_x . Their ultraweak cluster points belong to $M' \cap \mathcal{A}$ by Theorem 6.2, which proves the assertion. $\square$

Corollary 6.4 ([Iso25]). *Let $M \subset \mathcal{A}$ be a W^* -inclusion with M having separable predual, and let $T : \mathcal{A}_+ \rightarrow \widehat{M}_+$ be a normal faithful semifinite operator-valued weight. For every $x \in \mathcal{A}_+$ such that $T(x)$ is bounded,*

$$\overline{\text{conv}}^w\{uxu^* : u \in \mathcal{U}(M)\} \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

Proof. Write $M = M_{\text{fin}} \oplus M_\infty$ for its finite and properly infinite parts, with units p_1, p_∞ . Averaging over the unitaries $p_1 + e^{2\pi i t} p_\infty$ sends x to $p_1 x p_1 + p_\infty x p_\infty$ within its unitary convex hull.

The restrictions $T_i : (p_i \mathcal{A} p_i)_+ \rightarrow \widehat{p_i M}_+$ are normal, faithful and semifinite. Indeed, compression of the ultraweakly dense finite positive domain of T is ultraweakly dense in $(p_i \mathcal{A} p_i)_+$, and

$$T_i(p_i a p_i) = p_i T(a) p_i.$$

Moreover, $T_i(p_i x p_i)$ is bounded.

On M_{fin} choose an atomic measure on a countable symmetric family of unitaries generating M_{fin} , with every atom having positive mass. This gives a normal regular strongly generating hyperstate with respect to a faithful normal tracial state. On M_∞ use Example 3.9. Compose each T_i with the extension to extended positive

cones of the canonical embedding $p_i M \subset \mathcal{B}_{\varphi_i}$. These composites satisfy the hypotheses of Theorem 6.3; no normalization on $p_i M$ is required. Apply that theorem separately to the two positive diagonal blocks.

Finally, block-diagonal unitaries in $\mathcal{U}(M_{\text{fin}}) \oplus \mathcal{U}(M_{\infty})$ belong to $\mathcal{U}(M)$. Taking product nets of the two averaging nets gives a limit in $M' \cap \mathcal{A}$, still in the original unitary convex hull. $\square$

We next show that a nontrivial Poisson boundary admits no normal faithful semifinite operator-valued weight onto the original algebra.

Theorem 6.5. *Let M be a von Neumann algebra and $\mathcal{B}_{\varphi}$ be the Poisson boundary associated with a normal regular strongly generating hyperstate. If the Poisson boundary is non-trivial (i.e., $\mathcal{B}_{\varphi} \neq M$), then there exists no normal faithful semifinite operator-valued weight $T : (\mathcal{B}_{\varphi})_+ \rightarrow \widehat{M}_+$.*

Proof. Write $\mathcal{B} = \mathcal{B}_{\varphi}$ and $Q = \mathcal{P}_{\varphi}^o$, and suppose that such an operator-valued weight $T : \mathcal{B}_+ \rightarrow \widehat{M}_+$ exists. By Proposition 3.10,

$$M' \cap \mathcal{B} = \mathcal{Z}(M) = \mathcal{Z}(\mathcal{B}).$$

To justify the second equality, let $u \in \mathcal{U}(\mathcal{Z}(M))$. The normal automorphism $\text{Ad}(u)$ of $\mathcal{B}$ fixes M pointwise, so Theorem 3.13 gives $\text{Ad}(u) = \text{id}_{\mathcal{B}}$. Hence $\mathcal{Z}(M) \subset \mathcal{Z}(\mathcal{B})$; the reverse inclusion follows from $\mathcal{Z}(\mathcal{B}) \subset M' \cap \mathcal{B} = \mathcal{Z}(M)$.

Let $h = T(1)$. By M -bimodularity, $uhu^* = h$ for every $u \in \mathcal{U}(M)$; hence h belongs to the extended positive cone of $\mathcal{Z}(M)$. Let e denote the infinite spectral projection of h . We first prove that $e = 0$.

If $e \neq 0$, semifiniteness and M -bimodularity of T give a nonzero $x \in (e\mathcal{B})_+$ with $T(x) \leq C1$ for some $C < \infty$. Every Cesàro mean $A_n(x) = (1/n) \sum_{k=0}^{n-1} Q^k(x)$ satisfies $T(A_n(x)) \leq C1$. Choose an ultraweak cluster point z of these means. Normality of Q and Theorem 3.11 give

$$z \in \mathcal{Z}(M)_+, \quad z = ez, \quad \zeta(z) = \zeta(x) > 0.$$

For every $\rho \in M_*^+$, the normal weight $\rho \circ T$ is lower semicontinuous. Thus $\rho(T(z)) \leq C\rho(1)$, and consequently $T(z) \leq C1$. Choose $\varepsilon > 0$ such that $q = \mathbb{1}_{[\varepsilon, \infty)}(z) \neq 0$. Then $q \leq e$ and

$$T(q) \leq \varepsilon^{-1}T(z) \leq \varepsilon^{-1}C1.$$

On the other hand, M -bimodularity gives $T(q) = qhq$, which has infinite value on the nonzero projection $q \leq e$. This contradiction proves $e = 0$.

Faithfulness of T also shows that the zero spectral projection of h vanishes. Therefore

$$q_n = \mathbb{1}_{[\frac{1}{n}, \infty)}(h) \in \mathcal{Z}(M), \quad q_n \nearrow 1.$$

On each nonzero corner, $h_n = q_n h$ is bounded and invertible in $\mathcal{Z}(q_n M)$. Since $T(q_n) = h_n$, the formula

$$E_n(a) = h_n^{-1/2} T(a) h_n^{-1/2}, \quad a \in (q_n \mathcal{B})_+,$$

extends to a normal faithful conditional expectation $q_n \mathcal{B} \rightarrow q_n M$. Here boundedness follows from $T(a) \leq \|a\| h_n$. Extend E_n to the normal ucp map

$$\Psi_n : \mathcal{B} \rightarrow \mathcal{B}, \quad \Psi_n(b) = E_n(q_n b) + (1 - q_n)b.$$

The centrality of q_n makes this a direct-sum map, and $\Psi_n|_M = \text{id}_M$. Theorem 3.13 gives $\Psi_n = \text{id}_B$. Hence $q_n B = q_n M$ for every n . Passing to the strong limit $q_n \nearrow 1$ yields $B = M$, contrary to the hypothesis. $\square$

Corollary 6.6. *Every non-type I von Neumann algebra M admits an irreducible W^* -inclusion $M \subset \mathcal{A}$ (i.e., $M' \cap \mathcal{A} = \mathcal{Z}(M)$) that satisfies the weak Dixmier property but admits no normal faithful semi-finite operator-valued weight.*

Proof. Write $M = M_{\text{fin}} \oplus M_{\infty}$ for its finite and properly infinite parts, omitting zero summands. On each nonzero finite part fix a faithful normal tracial state. If M_{fin} is not of type I, [Zho24, Theorem 5.6, condition (ii)] supplies an atomic unitary-measure nrsg hyperstate with nontrivial boundary. Reflect the measure if necessary to use the convention $\varphi_{\mu} = \int \omega_{u\xi_{\tau}} d\mu(u)$. This is a relabelling of the same displayed hyperstate, not a change of its boundary. If M_{fin} is of type I, use positive weights on a countable adjoint-closed generating family of unitaries.

On the properly infinite part, if it is not of type I, Theorem 4.4 supplies an nrsg hyperstate with nontrivial boundary. Otherwise use Example 3.9. Lazify the chosen states by Lemma 3.14; this preserves their boundaries, and in the finite case merely adds an atom at the identity.

If both summands are present, form the diagonal hyperstate as in (4.2). The two identity coefficients ensure both strong generation conditions, as checked in the proof of Theorem 4.4. If only one summand is present, retain its state. The resulting boundary is

$$\mathcal{A} = \mathcal{B}_{\varphi} = \mathcal{B}_{\varphi_1} \oplus \mathcal{B}_{\varphi_{\infty}},$$

with absent summands omitted. At least one summand is nontrivial, since M is not of type I; thus $\mathcal{A} \neq M$. Proposition 3.10 gives $M' \cap \mathcal{A} = \mathcal{Z}(M)$.

Theorem 5.2(ii) applies to the finite boundary inclusion, and part (i) applies to the properly infinite one. Combining their unitary averaging nets gives the weak Dixmier property for $M \subset \mathcal{A}$. Finally, Theorem 6.5 rules out a normal faithful semifinite operator-valued weight from $\mathcal{A}$ to M . $\square$

7. THE RELATIVE BICENTRALIZER CONJECTURE

Let $M \subset N$ be an inclusion of von Neumann algebras with separable preduals and a faithful normal conditional expectation $E : N \rightarrow M$. Fix a faithful normal state ϕ on M and let $\rho = \phi \circ E$. We use the compatible realizations

$$c_{\phi}(M) \subset c_{\rho}(N) = N \rtimes_{\sigma^{\rho}} \mathbb{R}.$$

Following [Mar25, Definition 7.1], the **relative bicentralizer** is

$$B(M \subset c(N), \phi) = (M_{\phi^{\omega}}^{\omega})' \cap c(N),$$

where ω is a free ultrafilter on $\mathbb{N}$ and the commutant is taken in the compatible ultrapower realization. This algebra is independent of ω . We write ϕ^{it} for the canonical implementing unitaries in $c_{\phi}(M)$.

We prove the following theorem.

Theorem 7.1. *Let $M \subset N$ be a W^* -inclusion with a faithful normal conditional expectation. Then $M \subset c(N)$ has the weak Dixmier property:*

$$(7.1) \quad \overline{\text{conv}}^{\text{uw}} \{uxu^* : u \in \mathcal{U}(M)\} \cap (M' \cap c(N)) \neq \emptyset \quad (x \in c(N)).$$

Consequently, for every faithful normal state ϕ on M ,

$$(7.2) \quad B(M \subset c(N), \phi) = \{\phi^{it} : t \in \mathbb{R}\}'' \vee (M' \cap c(N)).$$

Fix a normal faithful state ϕ on M , and let $\rho = \phi \circ E$ and $\mathcal{A} = c_\rho(N)$. Let $\varphi_{\mathcal{A}} = \rho \circ T$ be the dual weight of ρ , where $T : \mathcal{A}_+ \rightarrow \widehat{N}_+$ is the canonical operator-valued weight. We use the notation $h = \frac{d\varphi_{\mathcal{A}}}{d\tau}$ and $H = \log h$ from Subsection 2.1, where τ is the canonical trace on $\mathcal{A}$. Here $\lambda_t = h^{it} = \phi^{it}$. The modular action $\sigma_t^{\varphi_{\mathcal{A}}} = \text{Ad}(\lambda_t)$ restricts to σ_t^ϕ on M . Choose the (M, ϕ) -hyperstate φ and the inner ucp coefficients (y_k) from Example 3.9, and define

$$(7.3) \quad Q(x) = \sum_k y_k x y_k^*, \quad x \in \mathcal{A}.$$

By [Mar25, Theorem D], it suffices to prove (7.1). To prove that $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, it suffices to show that $d := Q(aa^*) - aa^* = 0$ for every $a \in \text{Har}(Q)$. As in the proof of Theorem 3.11 and the argument showing $E(x) \in \text{Har}(P)$ in Lemma 6.1, followed by Cesàro averaging, we have

$$\begin{aligned} Q(aa^*) &= aa^* \quad (a \in \text{Har}(Q)) \\ \implies \text{Har}(Q) &= M' \cap \mathcal{A} \\ \implies M \subset \mathcal{A} &\text{ has the inner UCP-Dixmier property.} \end{aligned}$$

Fix $a \in \text{Har}(Q)$. By Kadison's inequality, $d \geq 0$. Lemma 2.2 reduces $d = 0$ to proving that every $x \in \mathcal{A}_+$ with $0 \leq x \leq d$ and $\varphi_{\mathcal{A}}(x) < \infty$ must be $x = 0$. Such an x satisfies

$$\sum_{k=0}^{n-1} Q^k(x) \leq \sum_{k=0}^{n-1} Q^k(d) = Q^n(aa^*) - aa^* \leq C1, \quad C = \|a\|^2.$$

After constructing the pairing and the self-adjoint contraction S in Lemma 7.2, Lemmas 7.3 and 7.4 prove $\varphi_{\mathcal{A}}(x) = 0$ for such x . Faithfulness then gives $x = 0$.

Theorem 5.1 converts the inner UCP-Dixmier property into (7.1) when M is properly infinite. For finite M , we choose ϕ tracial so that the Cesàro means are already unitary averages. The general case follows by decomposing M into its finite and properly infinite parts.

Since $\phi \circ Q = \phi$, normality and M -bimodularity of E and T give

$$(7.4) \quad \varphi_{\mathcal{A}} \circ Q = \phi \circ E \circ T \circ Q = \phi \circ Q \circ E \circ T = \phi \circ E \circ T = \varphi_{\mathcal{A}},$$

where the compositions are understood on the extended positive cone.

Define

$$\mathfrak{m}_{\varphi_{\mathcal{A}}} = \text{span}\{a \in \mathcal{A}_+ : \varphi_{\mathcal{A}}(a) < \infty\}, \quad \xi(a) = h^{1/4} a h^{1/4} \quad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}).$$

On $L^2(\mathcal{A}, \tau)$ we use the inner product $\langle X, Y \rangle_2 = \tau(Y^* X)$, linear in the first variable.

Define $S : D \rightarrow D$, where $D = \{\xi(a) : a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}\}$, by

$$(7.5) \quad S\xi(a) = \xi(Q(a)) \quad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}).$$

This is well-defined since ξ is injective and $Q(\mathfrak{m}_{\varphi_{\mathcal{A}}}) \subset \mathfrak{m}_{\varphi_{\mathcal{A}}}$ by (7.4).

Lemma 7.2. *For every $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, we have $\xi(a) \in L^2(\mathcal{A}, \tau)$. Let $x \in \mathcal{A}_+$ satisfy $\varphi_{\mathcal{A}}(x) < \infty$. Then*

$$\omega_x(a) = \tau(h^{1/2} x h^{1/2} a) \quad (a \in \mathcal{A})$$

defines a finite normal positive functional on $\mathcal{A}$, with $\omega_x(1) = \varphi_{\mathcal{A}}(x)$, and

$$(7.6) \quad \langle \xi(a), \xi(x) \rangle_2 = \omega_x(a) \quad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}).$$

Moreover, S extends to a self-adjoint contraction on $L^2(\mathcal{A}, \tau)$.

Proof. For $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, let

$$k(a) = h^{1/2} a h^{1/2} \in L^1(\mathcal{A}, \tau).$$

We first verify the assertions about ξ and ω_x . Both $ah^{1/2}$ and $h^{1/2}a$ belong to $L^2(\mathcal{A}, \tau)$. Traciality and the Cauchy–Schwarz inequality give

$$(7.7) \quad \begin{aligned} \|\xi(a)\|_2^2 &= \tau(h^{1/4} a^* h^{1/2} a h^{1/4}) \\ &= \tau((ah^{1/2})^* (h^{1/2}a)) \\ &\leq \|ah^{1/2}\|_2 \|h^{1/2}a\|_2 \\ &= \tau(h^{1/2} a^* ah^{1/2})^{1/2} \tau(h^{1/2} aa^* h^{1/2})^{1/2} \\ &= \varphi_{\mathcal{A}}(a^* a)^{1/2} \varphi_{\mathcal{A}}(aa^*)^{1/2}. \end{aligned}$$

The calculation is first made for $a_n = e_n a e_n$, where $e_n = \mathbf{1}_{[1/n, n]}(h)$. The convergences $a_n h^{1/2} \rightarrow ah^{1/2}$ and $h^{1/2} a_n \rightarrow h^{1/2} a$ in L^2 justify passage to the limit.

Since $k(x) \in L^1(\mathcal{A}, \tau)_+$ and $\tau(k(x)) = \varphi_{\mathcal{A}}(x)$, ω_x is a finite normal positive functional with $\omega_x(1) = \varphi_{\mathcal{A}}(x)$. To prove (7.6), let

$$e_n = \mathbf{1}_{[1/n, n]}(h), \quad x_n = e_n x e_n, \quad a_n = e_n a e_n.$$

By (2.2), the projections e_n have finite trace. Traciality in $e_n \mathcal{A} e_n$ gives

$$\langle \xi(a_n), \xi(x_n) \rangle_2 = \tau(h^{1/2} x_n h^{1/2} a_n).$$

Since $e_n h = h e_n$ and $e_n \nearrow 1$ strongly,

$$\begin{aligned} k(x_n) &= e_n k(x) e_n \longrightarrow k(x) \quad \text{in } L^1, \\ \xi(x_n) &= e_n \xi(x) e_n \longrightarrow \xi(x) \quad \text{in } L^2, \\ \xi(a_n) &\longrightarrow \xi(a) \quad \text{in } L^2. \end{aligned}$$

Moreover,

$$\sup_n \|a_n\| \leq \|a\|, \quad a_n \longrightarrow a \quad \text{*strongly}.$$

Passing to the limit proves (7.6).

We next prove that S extends to a self-adjoint contraction. Identifying $\mathcal{A}_*$ with $L^1(\mathcal{A}, \tau)$, we first show that the preadjoint of Q satisfies

$$(7.8) \quad Q_* k(a) = k(Q(a)) \quad (a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}).$$

Let $v_k = \sigma_{i/2}^\phi(y_k^*)$. By Example 3.9, (v_k) is a permutation of (y_k) . We have

$$(7.9) \quad y_k^* h^{1/2} = h^{1/2} v_k, \quad h^{1/2} y_k = v_k^* h^{1/2}.$$

Formula (7.9) follows first on compact spectral vectors for H , and then for the closed measurable products. For $a \in \mathcal{A}_+$ with $\varphi_{\mathcal{A}}(a) < \infty$, we obtain

$$y_k^* k(a) y_k = k(v_k a v_k^*).$$

By (7.9), the positive partial sums $\sum_{k \leq n} y_k^* k(a) y_k$ converge in L^1 , since

$$\tau \left(\sum_{k \leq n} y_k^* k(a) y_k \right) = \tau \left(k(a) \sum_{k \leq n} y_k y_k^* \right) \nearrow \varphi_{\mathcal{A}}(a).$$

Monotone convergence of the corresponding quadratic forms gives

$$Q_* k(a) = \sum_k y_k^* k(a) y_k = k \left(\sum_k v_k a v_k^* \right) = k(Q(a)).$$

This proves (7.8) for positive a , and the general case follows by linearity.

By (7.8) and (7.6), for $a, b \in \mathfrak{m}_{\varphi_{\mathcal{A}}} \cap \mathcal{A}_+$,

$$\begin{aligned} \langle S\xi(a), \xi(b) \rangle_2 &= \tau(k(b)Q(a)) = \tau(Q_* k(b)a) \\ &= \tau(k(Q(b))a) = \langle \xi(a), S\xi(b) \rangle_2. \end{aligned}$$

Linearity gives $\langle Sv, w \rangle_2 = \langle v, Sw \rangle_2$ for $v, w \in D$. Also $S(D) \subset D$ by (7.4). For $a \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, (7.5), (7.7), Kadison's inequality for Q^n , and $\varphi_{\mathcal{A}} \circ Q^n = \varphi_{\mathcal{A}}$ give

$$\begin{aligned} \|S^n \xi(a)\|_2^2 &= \|\xi(Q^n(a))\|_2^2 \\ &= \|h^{1/4} Q^n(a) h^{1/4}\|_2^2 \\ &= \tau(h^{1/2} Q^n(a)^* h^{1/2} Q^n(a)) \\ &\leq \varphi_{\mathcal{A}}((Q^n(a))^* Q^n(a))^{1/2} \varphi_{\mathcal{A}}(Q^n(a) (Q^n(a))^*)^{1/2} \\ &\leq \varphi_{\mathcal{A}}(Q^n(a^* a))^{1/2} \varphi_{\mathcal{A}}(Q^n(a a^*))^{1/2} \\ &= \varphi_{\mathcal{A}}(a^* a)^{1/2} \varphi_{\mathcal{A}}(a a^*)^{1/2}. \end{aligned}$$

Thus

$$C_v := \sup_{m \geq 0} \|S^m v\|_2 < \infty \quad (v \in D).$$

For $v \in D$, symmetry gives

$$(7.10) \quad \|Sv\|_2^2 = \langle S^2 v, v \rangle_2 \leq \|S^2 v\|_2 \|v\|_2.$$

Iterating (7.10), we obtain

$$\|Sv\|_2 \leq \|S^{2^n} v\|_2^{1/2^n} \|v\|_2^{1-1/2^n} \leq C_v^{1/2^n} \|v\|_2^{1-1/2^n} \longrightarrow \|v\|_2.$$

It remains to check that D is dense. With $e_n = \mathbf{1}_{[1/n, n]}(h)$, both h and h^{-1} are bounded on e_n , and $\tau(e_n) < \infty$. Every bounded $b \in e_n \mathcal{A} e_n$ has the form

$$b = \xi(h^{-1/4} b h^{-1/4}), \quad h^{-1/4} b h^{-1/4} \in e_n \mathcal{A} e_n \subset \mathfrak{m}_{\varphi_{\mathcal{A}}}.$$

These finite-corner elements are dense in $L^2(\mathcal{A}, \tau)$. Therefore S extends to a self-adjoint contraction. $\square$

Let $f \in C_c^\infty(\mathbb{R})$ be real with $0 \leq f \leq 1$, equal to 1 near 0, even and nonincreasing in $|s|$. Define $H_R = f(H/R)$, $R \geq 1$. Then $H_R \nearrow 1$ strongly as $R \rightarrow \infty$, and (2.1) gives

$$\varphi_{\mathcal{A}}(H_R) = R \int_{\mathbb{R}} f(s) \, ds < \infty.$$

Lemma 7.3. *Let $x \in \mathcal{A}_+$ satisfy $\varphi_{\mathcal{A}}(x) < \infty$. Assume that there is a constant $0 \leq C < \infty$ such that*

$$(7.11) \quad \sum_{k=0}^{n-1} Q^k(x) \leq C1 \quad (n \geq 1).$$

Then

$$(7.12) \quad \varphi_{\mathcal{A}}(x) \leq \liminf_{R \rightarrow \infty} C \|(1-S)^{1/2} \xi(H_R)\|_2^2.$$

Proof. The case $\varphi_{\mathcal{A}}(x) = 0$ is immediate. Assume that $\varphi_{\mathcal{A}}(x) > 0$. Let $a_n = \sum_{k=0}^{n-1} Q^k(x) \leq C1$. By (7.4) and (7.5),

$$\varphi_{\mathcal{A}}(a_n) = n\varphi_{\mathcal{A}}(x) < \infty, \quad (1-S)\xi(a_n) = \xi(x - Q^n(x)).$$

By (7.6) and positivity,

$$(7.13) \quad \begin{aligned} \|(1-S)^{1/2} \xi(a_n)\|_2^2 &= \langle \xi(x - Q^n(x)), \xi(a_n) \rangle_2 \\ &= \omega_x(a_n) - \omega_{a_n}(Q^n(x)) \\ &\leq \omega_x(a_n) \\ &\leq C\omega_x(1) = C\varphi_{\mathcal{A}}(x). \end{aligned}$$

Fix $R \geq 1$. By (7.6) and positivity, $\omega_x(H_R) = \overline{\omega_{H_R}(x)} = \omega_{H_R}(x)$. Since $a_n - Q(a_n) = x - Q^n(x)$ and $1-S \geq 0$, (7.5) and (7.13) give

$$(7.14) \quad \begin{aligned} &|\omega_x(H_R) - \omega_{H_R}(Q^n(x))|^2 \\ &= |\omega_{H_R}(x - Q^n(x))|^2 \\ &= |\langle \xi(x - Q^n(x)), \xi(H_R) \rangle_2|^2 \\ &= |\langle (1-S)\xi(a_n), \xi(H_R) \rangle_2|^2 \\ &= |\langle (1-S)^{1/2} \xi(a_n), (1-S)^{1/2} \xi(H_R) \rangle_2|^2 \\ &\leq \|(1-S)^{1/2} \xi(a_n)\|_2^2 \|(1-S)^{1/2} \xi(H_R)\|_2^2 \\ &\leq C\varphi_{\mathcal{A}}(x) \|(1-S)^{1/2} \xi(H_R)\|_2^2. \end{aligned}$$

By (7.11), for every $N \geq 1$,

$$\begin{aligned} \sum_{n=0}^{N-1} \omega_{H_R}(Q^n(x)) &= \omega_{H_R}\left(\sum_{n=0}^{N-1} Q^n(x)\right) \\ &\leq C\omega_{H_R}(1) = C\varphi_{\mathcal{A}}(H_R) < \infty. \end{aligned}$$

Letting $N \rightarrow \infty$, we obtain

$$\sum_{n=0}^{\infty} \omega_{H_R}(Q^n(x)) \leq C\varphi_{\mathcal{A}}(H_R) < \infty,$$

so $\omega_{H_R}(Q^n(x)) \rightarrow 0$ as $n \rightarrow \infty$. Letting $n \rightarrow \infty$ in (7.14) gives

$$(7.15) \quad |\omega_x(H_R)|^2 \leq C\varphi_{\mathcal{A}}(x) \|(1-S)^{1/2} \xi(H_R)\|_2^2 \quad (R \geq 1).$$

Since $H_R \nearrow 1$ strongly as $R \rightarrow \infty$, normality of ω_x and (7.15) give

$$\begin{aligned}\varphi_{\mathcal{A}}(x)^2 &= |\omega_x(1)|^2 \\ &= \lim_{R \rightarrow \infty} |\omega_x(H_R)|^2 \\ &\leq \liminf_{R \rightarrow \infty} C\varphi_{\mathcal{A}}(x) \|(1-S)^{1/2}\xi(H_R)\|_2^2.\end{aligned}$$

Dividing by $\varphi_{\mathcal{A}}(x)$ proves (7.12). $\square$

Let ν be the spectral distribution of $\log \Delta_\phi$ under φ . By (3.5),

$$m_2 := \int_{\mathbb{R}} s^2 d\nu(s) = \varphi((\log \Delta_\phi)^2) < \infty.$$

Lemma 7.4. *For every $R \geq 1$,*

$$(7.16) \quad \|(1-S)^{1/2}\xi(H_R)\|_2^2 \leq \frac{m_2 \|f'\|_2^2}{2R}.$$

Moreover, if $x \in \mathcal{A}_+$ satisfies $\varphi_{\mathcal{A}}(x) < \infty$ and (7.11) for some $0 \leq C < \infty$, then $\varphi_{\mathcal{A}}(x) = 0$ and $x = 0$.

Proof. Let $g \in C_c^\infty(\mathbb{R})$ be real. By (2.1), $g(H) \in \mathfrak{m}_{\varphi_{\mathcal{A}}}$, and $g(H) \in \mathcal{A}_{\varphi_{\mathcal{A}}}$. Expanding the commutators gives

$$\sum_k [y_k, g(H)][y_k, g(H)]^* = Q(g(H)^2) - Q(g(H))g(H) - g(H)Q(g(H)) + g(H)^2.$$

By Kadison's inequality, (7.4), and (2.1),

$$\varphi_{\mathcal{A}}(Q(g(H))^2) \leq \varphi_{\mathcal{A}}(Q(g(H)^2)) = \varphi_{\mathcal{A}}(g(H)^2) = \int_{\mathbb{R}} |g(s)|^2 ds < \infty.$$

Thus all four terms on the right belong to $\mathfrak{m}_{\varphi_{\mathcal{A}}}$. Applying $\varphi_{\mathcal{A}}$ and using the $\varphi_{\mathcal{A}}$ -centralizer property of $g(H)$, the linear extension of (7.6) to self-adjoint elements, and (7.5), we obtain

$$\begin{aligned}(7.17) \quad \sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) &= 2(\varphi_{\mathcal{A}}(g(H)^2) - \varphi_{\mathcal{A}}(g(H)Q(g(H)))) \\ &= 2(\langle \xi(g(H)), \xi(g(H)) \rangle_2 - \langle S\xi(g(H)), \xi(g(H)) \rangle_2) \\ &= 2\|(1-S)^{1/2}\xi(g(H))\|_2^2.\end{aligned}$$

We compute the left-hand side by proving

$$(7.18) \quad \sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) = \int_{\mathbb{R}} \int_{\mathbb{R}} |g(r+s) - g(r)|^2 dr d\nu(s).$$

We use the unitary Fourier transform

$$\widehat{g}(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-itr} g(r) dr.$$

The crossed-product relation gives

$$(7.19) \quad [y_k, g(H)] = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{g}(t)(y_k - \sigma_t^\phi(y_k))\lambda_t dt.$$

By (2.1), for $X = (2\pi)^{-1/2} \int_{\mathbb{R}} a_t \lambda_t dt = (2\pi)^{-1/2} \int_{\mathbb{R}} a_t e^{itH} dt$, we have

$$(7.20) \quad \varphi_{\mathcal{A}}(X^*X) = \int_{\mathbb{R}} \rho(a_t^* a_t) dt, \quad \varphi_{\mathcal{A}}(XX^*) = \int_{\mathbb{R}} \rho(a_t a_t^*) dt,$$

The identities in (7.20) hold first on the crossed-product Hilbert algebra and extend by its L^2 isometry; see [Haa78]. In the first identity in (7.20), the zeroth coefficient is $\int_{\mathbb{R}} \sigma_{-t}^\rho(a_t^* a_t) dt$, and the modular invariance of ρ removes this twist.

By $\varphi = \sum_k \langle \cdot, y_k^* \xi_\phi, y_k^* \xi_\phi \rangle$ and the definition of ν as the spectral distribution of $\log \Delta_\phi$ under φ , we have

$$\begin{aligned} \sum_k \phi((y_k - \sigma_t^\phi(y_k))(y_k - \sigma_t^\phi(y_k))^*) &= \sum_k \|(1 - \Delta_\phi^{it}) y_k^* \xi_\phi\|^2 \\ &= \sum_k \langle (1 - \Delta_\phi^{-it})(1 - \Delta_\phi^{it}) y_k^* \xi_\phi, y_k^* \xi_\phi \rangle \\ &= \varphi((1 - \Delta_\phi^{-it})(1 - \Delta_\phi^{it})) \\ &= \int_{\mathbb{R}} (1 - e^{-its})(1 - e^{its}) d\nu(s) \\ &= \int_{\mathbb{R}} |1 - e^{its}|^2 d\nu(s). \end{aligned}$$

By (7.19), (7.20), and scalar Plancherel,

$$\begin{aligned} \sum_k \varphi_{\mathcal{A}}([y_k, g(H)][y_k, g(H)]^*) &= \int_{\mathbb{R}} |\hat{g}(t)|^2 \int_{\mathbb{R}} |1 - e^{its}|^2 d\nu(s) dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{g}(t)|^2 |1 - e^{its}|^2 dt d\nu(s) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |g(r+s) - g(r)|^2 dr d\nu(s). \end{aligned}$$

This proves (7.18). Now take $g(r) = f(r/R)$, so $g(H) = H_R$. Since $\|g(\cdot + s) - g\|_2 \leq |s| \|g'\|_2 = \frac{|s|}{\sqrt{R}} \|f'\|_2$, (7.18) gives

$$(7.21) \quad \sum_k \varphi_{\mathcal{A}}([y_k, H_R][y_k, H_R]^*) \leq \frac{m_2 \|f'\|_2^2}{R}.$$

Combining (7.17) with $g(H) = H_R$ and (7.21) proves (7.16).

Now let x satisfy the hypotheses of Lemma 7.3. By (7.12) and (7.16),

$$0 \leq \varphi_{\mathcal{A}}(x) \leq \liminf_{R \rightarrow \infty} \frac{C m_2 \|f'\|_2^2}{2R} = 0.$$

Thus $\varphi_{\mathcal{A}}(x) = 0$, and faithfulness of $\varphi_{\mathcal{A}}$ gives $x = 0$. $\square$

We can now identify $\text{Har}(Q)$.

Proposition 7.5. *The fixed-point space of Q is*

$$\text{Har}(Q) = M' \cap \mathcal{A}.$$

Proof. We only need to prove that $\text{Har}(Q) \subset M' \cap \mathcal{A}$. Let $a \in \text{Har}(Q)$. By Kadison's inequality,

$$d := Q(aa^*) - aa^* = Q(aa^*) - Q(a)Q(a)^* \geq 0.$$

We have

$$\sum_{k=0}^{n-1} Q^k(d) = Q^n(aa^*) - aa^* \leq \|a\|^2 1 \quad (n \geq 1).$$

Suppose that $d \neq 0$. By Lemma 2.2, choose $0 \neq x \leq d$ with $\varphi_{\mathcal{A}}(x) < \infty$. Then

$$\sum_{k=0}^{n-1} Q^k(x) \leq \sum_{k=0}^{n-1} Q^k(d) \leq \|a\|^2 1 \quad (n \geq 1).$$

Lemma 7.4 gives $x = 0$, a contradiction. Hence $Q(aa^*) = aa^*$. Applying the same argument to $a^* \in \text{Har}(Q)$ gives $Q(a^*a) = a^*a$, so a belongs to the multiplicative domain of Q . Expanding as in (3.12), we obtain

$$\sum_k (y_k a - a y_k)(y_k a - a y_k)^* = 0.$$

Thus $[a, y_k] = 0$ for every k . Since $\overline{\mathbb{C}[y_k]}^w = M$, $a \in M' \cap \mathcal{A}$. $\square$

We can now finish the proof by applying the equivalence established in Theorem 5.1.

Proof of Theorem 7.1. Suppose first that M is either finite or properly infinite. Choose a faithful normal state ϕ on M , taking ϕ tracial when M is finite. Let Q be the map in (7.3) on $\mathcal{A} = c_{\phi \circ E}(N)$, with coefficients from Example 3.9. For $x \in \mathcal{A}$, define

$$Q_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} Q^k(x), \quad K_x = \overline{\text{conv}}^{uw} \{uxu^* : u \in \mathcal{U}(M)\}.$$

Since

$$\|Q(Q_n(x)) - Q_n(x)\| \leq \frac{2\|x\|}{n},$$

normality of Q and Proposition 7.5 imply that every ultraweak cluster point of $(Q_n(x))$ belongs to $M' \cap \mathcal{A}$.

If M is properly infinite, each Q_n is an M -inner ucp map. Thus $M \subset \mathcal{A}$ has the inner UCP-Dixmier property, and Theorem 5.1 gives the weak Dixmier property.

If M is finite, the tracial choice in Example 3.9 gives

$$y_k = \lambda_k u_k, \quad Q(x) = \sum_k \lambda_k^2 u_k x u_k^*, \quad \sum_k \lambda_k^2 = 1.$$

Thus $Q(K_x) \subset K_x$ and $Q_n(x) \in K_x$ for every n . Every ultraweak cluster point belongs to $K_x \cap (M' \cap \mathcal{A})$, which proves the finite case.

For general M , write

$$M = M_{\text{fin}} \oplus M_{\infty}$$

for its finite and properly infinite parts. We may assume that both summands are nonzero. Let p_1, p_2 be their units and let $N_i = p_i N p_i$. There is a faithful normal conditional expectation

$$F : N \rightarrow N_1 \oplus N_2, \quad F(a) = p_1 a p_1 + p_2 a p_2.$$

The restrictions

$$E_1 = E|_{N_1} : N_1 \rightarrow M_{\text{fin}}, \quad E_2 = E|_{N_2} : N_2 \rightarrow M_{\infty}$$

are faithful normal conditional expectations.

Choose a faithful normal state ϕ on M whose restriction to M_{fin} is tracial, and let $\rho = \phi \circ E$, $\mathcal{A} = c_{\rho}(N)$, and $\rho_i = \rho|_{N_i}/\rho(p_i)$. Since $p_i \in M_{\phi}$, bimodularity gives $p_i \in N_{\rho}$. Thus F preserves ρ and extends to a faithful normal conditional expectation

$$\tilde{F} : \mathcal{A} \rightarrow p_1 \mathcal{A} p_1 \oplus p_2 \mathcal{A} p_2, \quad p_i \mathcal{A} p_i \cong c_{\rho_i}(N_i).$$

For $x \in \mathcal{A}$, since $p_1 - p_2 \in \mathcal{U}(M)$,

$$\tilde{F}(x) = p_1 x p_1 + p_2 x p_2 = \frac{1}{2}(x + (p_1 - p_2)x(p_1 - p_2)) \in K_x.$$

The finite and properly infinite cases give the weak Dixmier property for $M_{\text{fin}} \subset p_1 \mathcal{A} p_1$ and $M_\infty \subset p_2 \mathcal{A} p_2$, respectively. As in the proof of Corollary 5.3, taking product nets of the two unitary averaging nets gives

$$K_x \cap (M' \cap \mathcal{A}) \neq \emptyset.$$

This proves (7.1).

Finally, [Mar25, Theorem D] gives (7.2) for every faithful normal state ϕ on M , which finishes the proof. $\square$

8. IRREDUCIBLE EMBEDDINGS INTO HYPERFINITE FACTORS

To construct the desired irreducible inclusion $M \subset R_\lambda$, we first choose a Poisson boundary $\mathcal{B}_\varphi$ of M and then take crossed products. When a single crossed product suffices, the construction has the form

$$M \subset \mathcal{B}_\varphi \subset \mathcal{B}_\varphi \rtimes_\alpha G \cong R_\lambda.$$

Here $M \subset \mathcal{B}_\varphi$ is irreducible by Proposition 3.10 and rigid by Theorem 3.13. We choose a suitable outer action α of a countable discrete abelian group G to obtain the hyperfinite factor of the desired type. The following Lemma 8.1 then shows that $M \subset R_\lambda$ is irreducible. The cases requiring a second crossed product are treated in Lemma 8.7.

Lemma 8.1. *Let $M \subset \mathcal{B}$ be an irreducible inclusion of factors. Assume that every normal automorphism of $\mathcal{B}$ which fixes M pointwise is the identity. Let G be a countable discrete group and let $\alpha : G \curvearrowright \mathcal{B}$ be an action such that α_g is outer for every $g \neq e$. Then*

$$M' \cap (\mathcal{B} \rtimes_\alpha G) = \mathbb{C}1.$$

Proof. We first observe that, for $\beta \in \text{Aut}(\mathcal{B})$,

$$(8.1) \quad \{x \in \mathcal{B} : mx = x\beta(m) \text{ for all } m \in M\} \neq \{0\} \implies \beta \text{ is inner.}$$

Indeed, for a nonzero such x ,

$$xx^* \in M' \cap \mathcal{B} = \mathbb{C}1, \quad x^*x \in \beta(M)' \cap \mathcal{B} = \mathbb{C}1.$$

Thus $x = cv$ for some $c > 0$ and $v \in \mathcal{U}(\mathcal{B})$. The automorphism $\text{Ad}(v) \circ \beta$ fixes M pointwise, so it is the identity on $\mathcal{B}$.

Let $E_{\mathcal{B}} : \mathcal{B} \rtimes_\alpha G \rightarrow \mathcal{B}$ be the canonical faithful normal conditional expectation and let u_g be the implementing unitaries. For $x \in M' \cap (\mathcal{B} \rtimes_\alpha G)$, define $x_g = E_{\mathcal{B}}(xu_g^*)$. Bimodularity gives

$$mx_g = x_g \alpha_g(m) \quad (m \in M).$$

By (8.1), $x_g = 0$ for $g \neq e$, while $x_e \in M' \cap \mathcal{B} = \mathbb{C}1$. Uniqueness of Fourier coefficients gives $x = x_e \in \mathbb{C}1$. $\square$

We first establish the type and existence results for Poisson boundaries needed below.

Theorem 8.2. *Let M be a semifinite factor and let φ be a normal regular strongly generating hyperstate. If $\mathcal{B}_\varphi \neq M$, then $\mathcal{B}_\varphi$ is a type III factor.*

Proof. By Proposition 3.10, $\mathcal{B}_\varphi$ is a factor. If it were semifinite, Haagerup's theorem [Haa79b, Theorem 2.7], applied to the inclusion $M \subset \mathcal{B}_\varphi$ and faithful normal semifinite traces on its two algebras, would give a normal faithful semifinite operator-valued weight

$$T : (\mathcal{B}_\varphi)_+ \longrightarrow \widehat{M}_+.$$

This contradicts Theorem 6.5. Hence $\mathcal{B}_\varphi$ is of type III. $\square$

We next prove an existence result for type III source factors. The argument uses the symmetric hyperstates constructed in Example 3.9, rather than trace estimates for unitary averages.

Lemma 8.3. *Let $\mathcal{B}$ be a semifinite von Neumann algebra, let ζ be a faithful normal state on $\mathcal{B}$, and let $J_\mathcal{B}$ and Δ_ζ be its modular conjugation and modular operator. Suppose that Ω is a normal state on $B(L^2(\mathcal{B}, \zeta))$ satisfying*

$$\Omega(b) = \zeta(b) = \Omega(J_\mathcal{B} b J_\mathcal{B}) \quad (b \in \mathcal{B}_+).$$

If $\Omega(\Delta_\zeta) < \infty$, then

$$\Omega(|\log \Delta_\zeta|) < \infty, \quad \Omega(\log \Delta_\zeta) = 0.$$

Proof. Choose a faithful normal semifinite trace τ on $\mathcal{B}$ and write $\zeta = \tau(h \cdot)$, where $h \in L^1(\mathcal{B}, \tau)_+$ has support 1. In the tracial standard form, $\xi_\zeta = h^{1/2}$ and

$$\Delta_\zeta = L_h R_h^{-1}, \quad \log \Delta_\zeta = L_{\log h} - R_{\log h}.$$

The two operators on the right strongly commute. Their joint spectral measure, evaluated against Ω , gives a probability measure on $\mathbb{R}^2$. Let X and Y denote its coordinate functions. The two marginal distributions agree by the hypothesis on Ω , applied to the spectral projections of $\log h$.

Since X and Y have the same distribution, for $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(Y \leq t < X) &= \mathbb{P}(Y \leq t) - \mathbb{P}(X \leq t, Y \leq t) \\ &= \mathbb{P}(X \leq t) - \mathbb{P}(X \leq t, Y \leq t) = \mathbb{P}(X \leq t < Y). \end{aligned}$$

Integrating in t gives

$$\begin{aligned} \mathbb{E}((X - Y)_+) &= \int_{\mathbb{R}} \mathbb{P}(Y \leq t < X) dt \\ &= \int_{\mathbb{R}} \mathbb{P}(X \leq t < Y) dt = \mathbb{E}((Y - X)_+), \end{aligned}$$

initially allowing infinite values. Since $(\log s)_+ \leq s$ for $s > 0$,

$$\mathbb{E}((X - Y)_+) = \Omega((\log \Delta_\zeta)_+) \leq \Omega(\Delta_\zeta) < \infty.$$

Hence

$$\Omega(|\log \Delta_\zeta|) = 2\mathbb{E}((X - Y)_+) < \infty, \quad \Omega(\log \Delta_\zeta) = 0.$$

No integrability of $h \log h$ is required. $\square$

Proposition 8.4. *Let M have separable predual, and let φ be a faithful normal regular strongly generating hyperstate on $B(L^2(M))$. Let $\phi = \varphi|_M$. Assume that*

$$\varphi(JT^*J) = \varphi(T) \quad (T \in B(L^2(M))), \quad \varphi(\Delta_\phi) < \infty.$$

If $\mathcal{B}_\varphi$ is semifinite, then $\mathcal{B}_\varphi = M$. Consequently, if M is a type III factor, then $\mathcal{B}_\varphi$ is a type III factor.

Proof. Write $\mathcal{B} = \mathcal{B}_\varphi$, and suppose that $\mathcal{B}$ is semifinite. Let ζ be its canonical faithful normal state. Denote the modular objects for (M, ϕ) and $(\mathcal{B}, \zeta)$ by S_M, J_M, Δ_M and $S_{\mathcal{B}}, J_{\mathcal{B}}, \Delta_{\mathcal{B}}$, respectively. Since $\zeta|_M = \phi$, there is an isometry

$$V : L^2(M, \phi) \longrightarrow L^2(\mathcal{B}, \zeta), \quad V(x\xi_\phi) = x\xi_\zeta.$$

Define the normal state $\Omega(T) = \varphi(V^*TV)$ on $B(L^2(\mathcal{B}, \zeta))$. Writing $Q = \mathcal{P}_\varphi^o$, we have

$$Q(b) = \sum_k y_k b y_k^*, \quad \varphi = \sum_k \omega_{y_k^* \xi_\phi}, \quad \Omega = \sum_k \omega_{y_k^* \xi_\zeta}.$$

By (3.11), $\zeta \circ Q = \zeta$.

Thus, for $b \in \mathcal{B}_+$,

$$\Omega(b) = \sum_k \zeta(y_k b y_k^*) = \zeta(b).$$

Since $J_{\mathcal{B}} b J_{\mathcal{B}}$ commutes with every y_k , the identity $\sum_k y_k y_k^* = 1$ also gives

$$\Omega(J_{\mathcal{B}} b J_{\mathcal{B}}) = \langle J_{\mathcal{B}} b J_{\mathcal{B}} \xi_\zeta, \xi_\zeta \rangle = \zeta(b).$$

We next prove (8.2) and (8.3). The Tomita identities on the individual vectors give

$$\begin{aligned} \Omega(\Delta_{\mathcal{B}}) &= \sum_k \|S_{\mathcal{B}}(y_k^* \xi_\zeta)\|^2 = \sum_k \zeta(y_k^* y_k) \\ &= \sum_k \phi(y_k^* y_k) = \sum_k \|S_M(y_k^* \xi_\phi)\|^2 = \varphi(\Delta_M) < \infty. \end{aligned}$$

All sums here are understood through monotone spectral truncation. Lemma 8.3 gives

$$(8.2) \quad \Omega(|\log \Delta_{\mathcal{B}}|) < \infty, \quad \Omega(\log \Delta_{\mathcal{B}}) = 0.$$

Similarly, J_M -symmetry and $J_M \Delta_M J_M = \Delta_M^{-1}$ imply

$$(8.3) \quad \varphi(|\log \Delta_M|) < \infty, \quad \varphi(\log \Delta_M) = 0.$$

We now compare the modular resolvents. On $M\xi_\phi$,

$$S_{\mathcal{B}} V(x\xi_\phi) = x^* \xi_\zeta = V S_M(x\xi_\phi).$$

Because $M\xi_\phi$ is a core for S_M and $S_{\mathcal{B}}$ is closed, this identity extends to $\text{Dom}(S_M)$. In particular,

$$(8.4) \quad \|\Delta_{\mathcal{B}}^{1/2} V \eta\| = \|\Delta_M^{1/2} \eta\| \quad (\eta \in \text{Dom}(\Delta_M^{1/2})).$$

For a nonnegative self-adjoint operator A and $s > 0$,

$$(8.5) \quad \langle (A + s)^{-1} \xi, \xi \rangle = \sup_{\eta \in \text{Dom}(A^{1/2})} (2 \operatorname{Re} \langle \xi, \eta \rangle - \|A^{1/2} \eta\|^2 - s \|\eta\|^2).$$

By (8.4), restricting the supremum in (8.5) for $A = \Delta_{\mathcal{B}}$ to $V \text{Dom}(\Delta_M^{1/2})$ gives

$$(8.6) \quad D_s := V^*(\Delta_{\mathcal{B}} + s)^{-1} V - (\Delta_M + s)^{-1} \geq 0 \quad (s > 0).$$

This argument does not assert equality of the domains of the compressed modular operators.

For $0 < a < b < \infty$, define

$$g_{a,b}(r) = \int_a^b \left(\frac{1}{1+s} - \frac{1}{r+s} \right) ds \quad (r > 0).$$

The integrand has the sign of $r - 1$, so $|g_{a,b}(r)| \leq |\log r|$. Moreover, $g_{1/n,n}(r) \rightarrow \log r$. By (8.2), (8.3), and dominated convergence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\frac{1}{n}}^n \varphi(D_s) ds &= \lim_{n \rightarrow \infty} (\varphi(g_{\frac{1}{n},n}(\Delta_M)) - \Omega(g_{\frac{1}{n},n}(\Delta_{\mathcal{B}}))) \\ &= \varphi(\log \Delta_M) - \Omega(\log \Delta_{\mathcal{B}}) = 0. \end{aligned}$$

The integrand is nonnegative and norm continuous in s . Hence $\varphi(D_s) = 0$ for every $s > 0$. Faithfulness of φ on $B(L^2(M))$ implies $D_s = 0$ for every $s > 0$.

Let $R_s = (\Delta_{\mathcal{B}} + s)^{-1}$ and $e = VV^*$. Since $D_s = 0$, (8.6) gives $V^*R_sV = (\Delta_M + s)^{-1}$. Differentiating in operator norm yields

$$V^*R_s^2V = (\Delta_M + s)^{-2} = (V^*R_sV)^2.$$

Consequently $V^*R_s(1-e)R_sV = 0$, and hence $(1-e)R_sV = 0$. It follows that

$$R_sV = V(\Delta_M + s)^{-1}, \quad \Delta_{\mathcal{B}}^{it}V = V\Delta_M^{it} \quad (t \in \mathbb{R}).$$

For $x \in M$, we obtain

$$\sigma_t^\zeta(x)\xi_\zeta = \Delta_{\mathcal{B}}^{it}V(x\xi_\phi) = V(\sigma_t^\phi(x)\xi_\phi) = \sigma_t^\phi(x)\xi_\zeta.$$

Since ξ_ζ is separating for $\mathcal{B}$, this proves $\sigma_t^\zeta(x) = \sigma_t^\phi(x)$. Thus $\sigma_t^\zeta(M) = M$ for every $t \in \mathbb{R}$. Takesaki's theorem [Tak03, Theorem IX.4.2] gives a ζ -preserving faithful normal conditional expectation $E : \mathcal{B} \rightarrow M$. Composing E with $M \subset \mathcal{B}$ and applying Theorem 3.13 gives $E = \text{id}_{\mathcal{B}}$, so $\mathcal{B} = M$.

Finally, if M is a type III factor, a semifinite $\mathcal{B}$ would contradict this equality. Proposition 3.10 says that $\mathcal{B}$ is a factor, so it must have type III. $\square$

Corollary 8.5. *Every type III factor M with separable predual admits a normal regular strongly generating hyperstate whose Poisson boundary is a nontrivial type III factor. Moreover, for every prescribed faithful normal state ϕ on M , there exists a faithful nrsg (M, ϕ) -hyperstate with a type III Poisson boundary, without an assertion of nontriviality for this prescribed-marginal choice.*

Proof. For a prescribed ϕ , choose the hyperstate of Example 3.9. It satisfies the hypotheses of Proposition 8.4, so its boundary has type III. This proves the last assertion.

If M is not injective, this boundary is nontrivial, since Poisson boundaries are injective by Proposition 3.4. Suppose therefore that M is injective. Let R be the hyperfinite II_1 factor. The classification of injective factors gives

$$M \cong M \bar{\otimes} R;$$

the absorption statement and the relevant type III_0 case are recalled in the proof of Theorem 4.4. Choose a hyperstate φ_0 on M as above, with type III boundary $\mathcal{B}_0$. By [Zho24, Proposition 5.3], choose an nrsg hyperstate ψ on R with $\mathcal{B}_\psi \neq R$. Replace both hyperstates by their lazy versions from Lemma 3.14. Their boundaries

do not change, and their product is nrsg. Theorem 3.15 gives an isomorphism of inclusions

$$M \bar{\otimes} R \subset \mathcal{B}_{\varphi_0 \otimes \psi} \cong M \bar{\otimes} R \subset \mathcal{B}_0 \bar{\otimes} \mathcal{B}_\psi.$$

The tensor product is a type III factor, since $\mathcal{B}_0$ is a type III factor. It strictly contains $M \bar{\otimes} R$: if $b \in \mathcal{B}_\psi \setminus R$, then $1 \otimes b \notin M \bar{\otimes} R$, as follows by slicing with a normal state on $\mathcal{B}_0$. Transporting this hyperstate through a standard-form implementation of $M \cong M \bar{\otimes} R$ proves the first assertion. This transport need not preserve a previously prescribed marginal. $\square$

Remark 8.6. Corollary 8.5 proves existence of a nontrivial type III Poisson boundary for every type III factor. It does not prove that the boundary of every nrsg hyperstate has type III. Proposition 8.4 uses faithfulness of the hyperstate on $B(L^2(M))$, its J -symmetry, and its finite modular moment. These are additional properties of the chosen hyperstate, not consequences of regularity alone.

For an arbitrary nrsg hyperstate on a type III factor, the boundary is a properly infinite factor and is not of type I. Indeed, if $\mathcal{B}_\varphi \cong B(H)$, irreducibility and the bicommutant theorem would force $M = B(H)$. Thus the remaining possibilities in the arbitrary case are type II_∞ and type III. The weak Dixmier property does not identify its scalar unitary-average limit with $\zeta(x)1$, and it is not used to exclude semifiniteness in the preceding modular argument.

The next lemma provides the required changes of type. We allow an iterated crossed product: after the first enlargement, rigidity need not persist, so the second step requires an additional intertwiner argument.

Lemma 8.7. *Let $\mathcal{B}$ be a hyperfinite type III factor with separable predual. For every $0 \leq \lambda \leq 1$, there exists a hyperfinite type III_λ factor N_λ with separable predual containing $\mathcal{B}$, obtained by at most two crossed products by countable discrete amenable groups. Each nontrivial crossed-product action is outer, and $\mathcal{B} \subset N_\lambda$ has a faithful normal conditional expectation.*

Moreover, if $M \subset \mathcal{B}$ is irreducible and every normal automorphism of $\mathcal{B}$ fixing M pointwise is the identity, the construction satisfies

$$M' \cap N_\lambda = \mathbb{C}1.$$

For $\lambda = 0$, only the existence of some type III_0 target is asserted.

Proof. We first fix the modular conventions. For a faithful normal state ω on $\mathcal{B}$, write

$$D = \mathcal{B} \rtimes_{\sigma^\omega} \mathbb{R}, \quad C = \mathcal{Z}(D),$$

and denote the canonical implementing unitaries by $\ell_\omega(r)$. The dual action is normalized by

$$\widehat{\sigma}_s(b) = b, \quad \widehat{\sigma}_s(\ell_\omega(r)) = e^{-isr} \ell_\omega(r).$$

Let $\theta = \widehat{\sigma}|_C$. The flow (C, θ) is ergodic. It is the one-point flow in type III_1 , the circle flow of period $|\log \mu|$ in type III_μ , $0 < \mu < 1$, and a properly ergodic flow in type III_0 [CT77].

Write

$$\text{T}(\mathcal{B}) = \{r \in \mathbb{R} : \sigma_r^\omega \text{ is inner}\}.$$

We use its point-spectrum description

$$(8.7) \quad \text{T}(\mathcal{B}) = \{r \in \mathbb{R} : \exists z \in \mathcal{U}(C), \theta_s(z) = e^{-isr} z \text{ for all } s \in \mathbb{R}\}.$$

To check (8.7), suppose $\sigma_r^\omega = \text{Ad}(v)$. Since $\omega \circ \text{Ad}(v) = \omega$, the unitary v belongs to the centralizer of ω . Hence $v^* \ell_\omega(r)$ is central in D and has the stated eigenfrequency. Conversely, for such an eigenunitary z , the unitary $z^* \ell_\omega(r)$ is fixed by the dual action, hence belongs to $\mathcal{B}$, and implements σ_r^ω . See also [GS85, p. 210]. In particular,

$$\text{T}(\mathcal{B}) = \{0\} \quad \text{in type III}_1, \quad \text{T}(\mathcal{B}) = \frac{2\pi}{|\log \mu|} \mathbb{Z} \quad \text{in type III}_\mu.$$

A type III $_\mu$ factor, $0 < \mu < 1$, admits a faithful normal periodic state with this fundamental modular period, by the discrete decomposition [Tak03].

By the splitting theorem of Sutherland–Takesaki [ST98, Theorem 1.1 and Corollary 1.3], the module has a group-homomorphic section

$$s : \text{Aut}_\theta(C) \longrightarrow \text{Aut}(\mathcal{B}), \quad \text{mod}(s(g)) = g.$$

Here $\text{Aut}_\theta(C)$ consists of the automorphisms of C commuting with θ . This is a genuine group section, including on finite cyclic subgroups.

Lifting time-maps. Let G be a countable subgroup of the group of actual time-maps $\{\theta_h : h \in \mathbb{R}\}$, with the discrete topology, and define

$$\alpha_g = s(g), \quad A = \mathcal{B} \rtimes_\alpha G.$$

If $g \neq e$, then $\text{mod}(\alpha_g) = g \neq \text{id}_C$, so α_g is outer. Hence A is a factor. If M is as in the statement, Lemma 8.1 gives $M' \cap A = \mathbb{C}1$.

The core of A is

$$\text{c}(A) \cong D \rtimes_{\tilde{\alpha}} G, \quad \tilde{\alpha}_g|_C = g,$$

with the compatible dual action [Sek90, Proposition 2.1]. We claim that

$$(8.8) \quad \mathcal{Z}(\text{c}(A)) = C^G, \quad \text{the weight flow of } A \text{ is } \theta|_{C^G}.$$

A nonidentity time-map acts essentially freely on C : in a standard measurable realization of the flow, its fixed-point set is invariant under the whole flow, hence null or conull; the conull case would make the time-map the identity. If $z \in \mathcal{Z}(D \rtimes_{\tilde{\alpha}} G)$ and $z_g \in D$ is a Fourier coefficient, then

$$cz_g = z_g g(c) \quad (c \in C).$$

Testing on a countable family of measurable projections separating points shows that the central support of z_g is contained in the fixed-point set of g . Thus $z_g = 0$ for $g \neq e$.

The remaining coefficient belongs to C^G , and conversely each element of C^G is central. This proves (8.8).

We also need the following relative statement, for every faithful normal state ψ on A :

$$(8.9) \quad r \notin \text{T}(\mathcal{B}) \implies \{y \in A : my = y\sigma_r^\psi(m) \text{ for all } m \in M\} = \{0\}.$$

First take $\psi_0 = \omega \circ E_{\mathcal{B}}$, so that $\sigma_r^{\psi_0}|_{\mathcal{B}} = \sigma_r^\omega$. The Fourier coefficients of such a y satisfy

$$my_g = y_g \alpha_g(\sigma_r^\omega(m)) \quad (m \in M).$$

If $y_g \neq 0$, (8.1) implies that $\alpha_g \circ \sigma_r^\omega$ is inner. A modular automorphism has trivial module, since its canonical extension is $\text{Ad}(\ell_\omega(r))$. Therefore $g = e$, and then $r \in \text{T}(\mathcal{B})$, a contradiction. For an arbitrary ψ , let $w_r = [D\psi : D\psi_0]_r$.

Since $\sigma_r^\psi = \text{Ad}(w_r) \circ \sigma_r^{\psi_0}$,

$$m(yw_r) = y\sigma_r^\psi(m)w_r = (yw_r)\sigma_r^{\psi_0}(m) \quad (m \in M).$$

The case ψ_0 gives $yw_r = 0$, hence $y = 0$.

This proves (8.9). The same argument is available with $M = \mathcal{B}$.

The target type III₁. Take $G = \{\theta_q : q \in \mathbb{Q}\}$ and set $N_1 = \mathcal{B} \rtimes_{s|_G} G$.

By (8.8), continuity, and ergodicity,

$$\mathcal{Z}(\mathbf{c}(N_1)) = C^{\theta(\mathbb{Q})} = C^\theta = \mathbb{C}1.$$

Thus N_1 is of type III₁, and Lemma 8.1 gives $M' \cap N_1 = \mathbb{C}1$. If the original flow is trivial, this simply gives $N_1 = \mathcal{B}$.

The target type III₀. If $\mathcal{B}$ is already of type III₀, take $N_0 = \mathcal{B}$. Otherwise let $H \subset \mathbb{R}$ be a countable additive subgroup with $H \cap \mathbf{T}(\mathcal{B}) = \{0\}$ and let $N = \mathcal{B} \rtimes_{\sigma^\omega} H$. The action is outer. We verify that

$$(8.10) \quad \mathbf{T}(N) = H + \mathbf{T}(\mathcal{B}).$$

In type III _{μ} , choose ω periodic with fundamental period $T_\mu = 2\pi/|\log \mu|$. Changing the state only cocycle conjugates the modular action and gives an isomorphism of crossed products fixing $\mathcal{B}$. For $\tilde{\omega} = \omega \circ E_{\mathcal{B}}$,

$$\sigma_r^{\tilde{\omega}}(b) = \sigma_r^\omega(b), \quad \sigma_r^{\tilde{\omega}}(u_h) = u_h.$$

If $r = h + q$ with $h \in H$ and $q \in \mathbf{T}(\mathcal{B})$, then $\sigma_r^{\tilde{\omega}} = \text{Ad}(u_h)$; in type III₁ one has $q = 0$. Conversely, if $\sigma_r^{\tilde{\omega}} = \text{Ad}(v)$, a nonzero Fourier coefficient v_h satisfies

$$v_h \sigma_h^\omega(b) = \sigma_r^\omega(b) v_h \quad (b \in \mathcal{B}).$$

Factoriality makes v_h a nonzero scalar multiple of a unitary, so $r - h \in \mathbf{T}(\mathcal{B})$. This proves (8.10).

For type III₁, choose $H = \mathbb{Q}$. For type III _{μ} , choose $H = c\mathbb{Q}$ with $c/T_\mu \notin \mathbb{Q}$. In both cases $\mathbf{T}(N)$ is a countable dense proper subgroup of $\mathbb{R}$. It is neither $\mathbb{R}$, nor $\{0\}$, nor a nonzero discrete cyclic subgroup. Thus N is neither semifinite, nor of type III₁, nor of type III _{ν} with $0 < \nu < 1$. It has type III₀. Lemma 8.1 preserves irreducibility over M .

A prescribed target type III _{λ} , $0 < \lambda < 1$. Let

$$a = |\log \lambda|, \quad t = \frac{2\pi}{a}, \quad K = \mathbf{T}(\mathcal{B}) \cap t\mathbb{Z}.$$

Either $K = \{0\}$ or $K = dt\mathbb{Z}$ for an integer $d \geq 1$. Take the actual cyclic group $G = \langle \theta_a \rangle$ and set $A = \mathcal{B} \rtimes_{s|_G} G$. Using actual time-maps means that the kernel of $n \mapsto \theta_{na}$ has already been divided out. By (8.8), the flow of A is the restriction of θ to C^{θ_a} .

This restricted flow is ergodic and factors through $\mathbb{R}/a\mathbb{Z}$. Its eigenfrequency group is K , by (8.7). To identify the whole flow, average a faithful normal state on C^{θ_a} over this compact group. Its L^2 -space decomposes into character spaces. Each nonzero character space is one-dimensional: the modulus of an eigenfunction is invariant and hence constant, and the quotient of two eigenunitaries of the same frequency is invariant. If $K = \{0\}$, only constants occur, and A has type III₁. If $K = dt\mathbb{Z}$, an eigenunitary of frequency dt generates C^{θ_a} . The flow has smallest positive period a/d , so A has type III _{$\lambda^{1/d}$} .

Suppose first that $K = \{0\}$. Choose a faithful normal state ψ on A and let $N_\lambda = A \rtimes_{\sigma^\psi} t\mathbb{Z}$. This action is outer because $T(A) = \{0\}$. We compute the flow rather than infer the type only from $T(N_\lambda)$. Let $D_A = c(A)$ and $v = \ell_\psi(t)$. The canonical extension of the generator is $\text{Ad}(v)$. If U is the implementing unitary in $D_A \rtimes_{\text{Ad}(v)} \mathbb{Z}$, then $w = v^*U$ is central and untwists the inner action. Since D_A is a factor,

$$c(N_\lambda) \cong D_A \bar{\otimes} L(\mathbb{Z}), \quad \mathcal{Z}(c(N_\lambda)) = W^*(w), \quad \theta_s(w) = e^{ist}w.$$

The weight flow is the circle flow of period $2\pi/t = a$, so N_λ has type III_λ . If $x \in M' \cap N_\lambda$, its coefficient $x_k \in A$ at kt satisfies

$$mx_k = x_k \sigma_{kt}^\psi(m) \quad (m \in M).$$

For $k \neq 0$, $kt \notin T(\mathcal{B})$, so (8.9) gives $x_k = 0$. The zeroth coefficient is scalar.

Suppose next that $K = dt\mathbb{Z}$. For $d = 1$, take $N_\lambda = A$. For $d \geq 2$, choose a faithful normal state χ on A with modular period exactly dt , and set

$$N_\lambda = A \rtimes_{\sigma_t^\chi} \mathbb{Z}/d\mathbb{Z}.$$

This is a genuine outer action, since $T(A) = dt\mathbb{Z}$. Let $D_A = c(A)$ and $v = \ell_\chi(t)$. Then $v^d = \ell_\chi(dt)$ is central and generates $\mathcal{Z}(D_A)$: it is an eigenunitary at the fundamental frequency dt of the circle flow of period a/d . In $D_A \rtimes_{\text{Ad}(v)} \mathbb{Z}/d\mathbb{Z}$, let $U^d = 1$ be the implementing unitary. The unitary $w = v^*U$ is central and $w^d = v^{-d}$. For a central element with Fourier coefficients $z_j \in D_A$, commutation with D_A implies $z_j v^j \in \mathcal{Z}(D_A)$. Since $\mathcal{Z}(D_A) = W^*(v^d) = W^*(w^d)$, this proves

$$\mathcal{Z}(c(N_\lambda)) = W^*(w).$$

The dual action fixes U and satisfies $\theta_s(w) = e^{ist}w$. The spectrum of w is nonempty and invariant under all rotations, hence is the full circle. The weight flow has smallest positive period a , so N_λ has type III_λ . Finally, if $x \in M' \cap N_\lambda$, its coefficients satisfy

$$mx_j = x_j \sigma_{jt}^\chi(m) \quad (m \in M).$$

For $1 \leq j < d$, $jt \notin T(\mathcal{B})$, so (8.9) makes $x_j = 0$. Again the zeroth coefficient is scalar.

All groups used above are countable and abelian, hence amenable. Crossed products therefore preserve injectivity and separability; the resulting factors are hyperfinite [Con76]. Composing their canonical faithful normal expectations gives an expectation onto $\mathcal{B}$. Taking $M = \mathcal{B}$ in the arguments also proves the assertions without an intermediate subfactor. $\square$

We are now ready to prove the embedding theorem.

Theorem 8.8. *Let M be a continuous factor with separable predual. For every $0 < \lambda \leq 1$, there exists a normal unital irreducible embedding of M into the hyperfinite type III_λ factor R_λ . Moreover, M admits such an embedding into some hyperfinite type III_0 factor with separable predual. Moreover, these irreducible inclusions can be chosen to have the weak Dixmier property.*

Proof. Suppose first that M is of type II_1 . By [Zho24, Theorem 5.6], there exists an atomic unitary-measure nrsg hyperstate with nontrivial Poisson boundary $\mathcal{B} = \mathcal{B}_\varphi$. In the tracial setting both $\mathbb{C}[z_k]$ and $\mathbb{C}[y_k]$ in Definition 3.7 are weakly dense; taking adjoints of the coefficient algebra verifies the second condition. The boundary is a type III factor by [DP22, Corollary 4.3].

Suppose next that M is of type II_∞ . By Theorem 4.4, choose an nrsg hyperstate with nontrivial Poisson boundary $\mathcal{B} = \mathcal{B}_\varphi$. Since M is semifinite, Theorem 8.2 shows that $\mathcal{B}$ is a type III factor.

Finally, if M is of type III, choose the nontrivial type III boundary supplied by Corollary 8.5.

In all three cases, $\mathcal{B}$ is injective by Proposition 3.4 and $M' \cap \mathcal{B} = \mathbb{C}1$ by Proposition 3.10. It has separable predual: its normal Poisson transform identifies it with an ultraweakly closed operator system in $B(L^2(M))$, whose predual is a quotient of the separable trace class. Theorem 3.13 says that every normal automorphism of $\mathcal{B}$ fixing M pointwise is the identity. Lemma 8.7 therefore gives all the required irreducible embeddings.

Let N be any of the ambient factors just constructed. By Lemma 8.7, there is a faithful normal conditional expectation

$E : N \rightarrow \mathcal{B} = \mathcal{B}_\varphi$, which fixes M pointwise. If M is of type II_1 , the chosen hyperstate comes from an atomic probability measure on $\mathcal{U}(M)$, so Theorem 5.2(ii) applies. If M is of type II_∞ or III, it is properly infinite, so Theorem 5.2(i) applies. Thus, in all cases, $M \subset N$ has the weak Dixmier property.

For $0 < \lambda < 1$, uniqueness follows from [Con76]; for $\lambda = 1$, it follows from [Haa87]. For $\lambda = 0$, no uniqueness or prescribed target is needed. This finishes the proof. $\square$

Remark 8.9. *The modular automorphisms used in the second crossed product have trivial Connes–Takesaki module. They must not be confused with the lifts of time-maps used in the first crossed product. Also, a nontrivial last crossed product by an abelian group has nontrivial dual automorphisms fixing the preceding algebra, and hence fixing M pointwise. By Theorem 3.13, such a final inclusion is not itself a Poisson-boundary inclusion over M . The expectations in Lemma 8.7 are onto the original boundary $\mathcal{B}$, not onto M .*

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