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Strongly positive bases and quantum greedy expansions

Qiyue Tang

Abstract

We prove that every element of a strongly positive basis of a coefficient-free rank-two quantum cluster algebra has nonnegative Laurent-polynomial coefficients in the quantum greedy basis. This establishes the strongly positive basis assertion of Conjecture 16 of Lee, Li, Rupel, and Zelevinsky. We also give a counterexample to their Conjecture 13(b).

1 Introduction

The quantum greedy basis and the triangular basis provide two distinguished bases of rank-two quantum cluster algebras. Lee, Li, Rupel, and Zelevinsky introduced the quantum greedy basis and asked how these bases compare with bases having positive multiplication [3, 4].

We prove the assertion about arbitrary strongly positive bases in [3, Conjecture 16]. We also disprove the identification of the triangular basis with the proposed upper strongly positive basis in [3, Conjecture 13(b)].

Fix positive integers b, c and an indeterminate v. Set

$$R = \mathbb{Z}[v^{\pm 1}], \qquad R_+ = \mathbb{Z}_{\geq 0}[v^{\pm 1}].$$

The quantum torus has normalized monomials

$$X^{(r,s)} = v^{rs} X_1^r X_2^s \quad (r, s \in \mathbb{Z}), \qquad X_2 X_1 = v^2 X_1 X_2.$$

For vectors $m = (m_1, m_2)$ and $n = (n_1, n_2)$, write $\det(m, n) = m_1 n_2 - m_2 n_1$. Then

$$X^m X^n = v^{-\det(m,n)} X^{m+n}. \tag{1}$$

Inside its skew field, define X_m, $m \in \mathbb{Z}$, by

$$X_{m+1} X_{m-1} = \begin{cases} v^b X_m^b + 1 & m \text{ odd}, \\ v^c X_m^c + 1 & m \text{ even}. \end{cases}$$

The R-subalgebra generated by these variables is $\mathcal{A}_v(b, c)$. Its clusters are the adjacent pairs (X_m, X_{m+1}), and its normalized cluster monomials are $v^{ij} X_m^i X_{m+1}^j$ for $i, j \in \mathbb{Z}_{\geq 0}$. The bar operation is the additive anti-involution fixing the X_m and sending v to v^{-1}; it fixes each $X^{(r,s)}$.

Definition 1.1. A strongly positive basis is an R-basis $\mathcal{B}$ of $\mathcal{A}_v(b, c)$ whose members are bar invariant, which is independent of the initial cluster, contains every normalized cluster monomial, and satisfies

$$YY' \in \bigoplus_{Z \in \mathcal{B}} R_+ Z \qquad (Y, Y' \in \mathcal{B}).$$

All basis expansions in this paper are finite.

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Write $G[a] = G[a_1, a_2]$ for the quantum greedy element with label $a = (a_1, a_2) \in \mathbb{Z}^2$; its defining recurrence is given in Section 2. For any basis element L in a specified basis, $[L]f$ denotes its coefficient in the finite expansion of f in that basis. In particular, $[X^m]f$ denotes a Laurent coefficient in the normalized torus basis. For $h(v) = \sum_k h_k v^k \in R$, we set $[v^k]h = h_k$, with absent coefficients equal to zero.

Theorem 1.2. *Let b, c be positive integers and let $\mathcal{B}$ be a strongly positive basis of $\mathcal{A}_v(b, c)$. For every $Y \in \mathcal{B}$ and $t \in \mathbb{Z}^2$,*

$$[G[t]]Y \in R_+.$$

The proof uses a pointed parametrization $\mathcal{B} = \{B[a] : a \in \mathbb{Z}^2\}$, established in [5] and proved again below. A standard-monomial bound and bar invariance force finite transition blocks from B to G to become the identity along a generic large direction. A finite matrix factorization then extracts a prescribed greedy coordinate as a multiplication coefficient of $\mathcal{B}$.

Let $C[a] = C[a_1, a_2]$ denote the triangular basis in the normalization of [3, Section 4]; Section 6 recalls its defining congruence.

Theorem 1.3. *For $(b, c) = (2, 3)$,*

$$[C[0, 0]](X_3^2 X_0^3) = v^7 + v^3 + v + v^{-1} + v^{-3} - v^{-5}.$$

Consequently the triangular basis is not strongly positive for these parameters, and Conjecture 13(b) of [3] is false.

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2 Pointed expansions

For an integer n, put $[n]_+ = \max(n, 0)$. For $n, k \in \mathbb{Z}_{\geq 0}$ and an invertible indeterminate w, define

$$[n]_w = \frac{w^n - w^{-n}}{w - w^{-1}}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_w = \frac{[n]_w [n-1]_w \cdots [n-k+1]_w}{[k]_w [k-1]_w \cdots [1]_w} \quad (0 \leq k \leq n).$$

The empty product is one, and $\begin{bmatrix} n \\ k \end{bmatrix}_w = 0$ if $k > n$ or $k < 0$. These symmetric Gaussian polynomials belong to $\mathbb{Z}_{\geq 0}[w^{\pm 1}]$. Their generating polynomial is

$$K_{n,d}(z) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{v^d} z^k = \prod_{j=1}^{n} (1 + v^{d(n+1-2j)} z), \qquad n \in \mathbb{Z}_{\geq 0}, d \in \mathbb{Z}_{>0}. \tag{2}$$

Here z commutes with v. Expanding the product proves the formula by induction on n. Its inverse in $R[[z]]$ is

$$K_{n,d}(z)^{-1} = \sum_{k \geq 0} (-1)^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{v^d} z^k \quad (n > 0); \qquad K_{0,d}(z)^{-1} = 1. \tag{3}$$

This follows by multiplying the two series, or by the same induction using the product in (2).

An element is *pointed at* $a = (a_1, a_2) \in \mathbb{Z}^2$ if its normalized Laurent expansion is

$$f = \sum_{p,q \geq 0} e(p,q) X^{(-a_1+bp, -a_2+cq)}, \qquad e(0,0) = 1. \tag{4}$$

Its corner is X^{-a}; p, q are nonnegative integer indices. The sum is finite. We use coordinate-wise order on labels, writing $t \leq a$ if $t_i \leq a_i$ for $i = 1, 2$.

The standard monomials are

$$M[a] = M[a_1, a_2] = v^{a_1 a_2} X_3^{[a_1]_+} X_1^{[-a_1]_+} X_2^{[-a_2]_+} X_0^{[a_2]_+}. \tag{5}$$

They form an R-basis by the acyclic standard-basis theorem [1, Theorems 7.3 and 7.5]. The compatible matrices in that convention are

$$\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad q^{1/2} = v;$$

the product of the transpose of the first with the second is $\mathrm{diag}(c, b)$. There are no frozen variables. The displayed order in (5) differs from the standard order only by powers of v from adjacent quasi-commutation. The quantum Laurent phenomenon [1, Corollary 5.2] gives a finite Laurent expansion of every element in every cluster.

The exchange relations give $X_3 = X^{(-1,0)} + X^{(-1,c)}$ and $X_0 = X^{(0,-1)} + X^{(b,-1)}$. The quantum binomial formula and (1) therefore give

$$M[a] = \sum_{p,q \geq 0} v^{bcpq} \begin{bmatrix} [a_2]_+ \\ p \end{bmatrix}_{v^b} \begin{bmatrix} [a_1]_+ \\ q \end{bmatrix}_{v^c} X^{(-a_1+bp, -a_2+cq)}. \tag{6}$$

In particular $M[a]$ is pointed. If either coordinate of a is nonpositive, it is a normalized cluster monomial.

The quantum greedy basis theorem [4, Theorems 1.7 and 1.9] gives a bar-invariant, mutation-invariant R-basis $G[a]$, containing the normalized cluster monomials. Its pointed coefficients $e_a(p,q)$ satisfy $e_a(0,0) = 1$ and, for $(p,q) \neq (0,0)$,

$$e_a(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{p} (-1)^{k-1} e_a(p-k, q) \begin{bmatrix} [a_2 - cq]_+ + k - 1 \\ k \end{bmatrix}_{v^b}, & ca_1 q \leq ba_2 p, \\ \displaystyle\sum_{k=1}^{q} (-1)^{k-1} e_a(p, q-k) \begin{bmatrix} [a_1 - bp]_+ + k - 1 \\ k \end{bmatrix}_{v^c}, & ca_1 q \geq ba_2 p. \end{cases} \tag{7}$$

Both equations hold at equality. In particular its two axes have coefficients $\begin{bmatrix} [a_2]_+ \\ p \end{bmatrix}_{v^b}$ and $\begin{bmatrix} [a_1]_+ \\ q \end{bmatrix}_{v^c}$.

We use the support theorem in [4, Definition 3.1, Proposition 3.7, and Theorem 5.8]. In the open region

$$\mathcal{W} = \{a \in \mathbb{R}^2_{>0} : a_1 < ba_2,\ a_2 < ca_1\}, \tag{8}$$

the pointed support lies in

$$\begin{aligned} &\{(p,q) \in \mathbb{R}^2_{\geq 0} : p < a_1/b, q + (b - ba_2/(ca_1))p < a_1\} \\ \cup\, &\{(p,q) \in \mathbb{R}^2_{\geq 0} : q < a_2/c, p + (c - ca_1/(ba_2))q < a_2\} \cup \{(0, a_1), (a_2, 0)\}. \end{aligned} \tag{9}$$

Outside $\mathcal{W}$, the support is respectively a point, an axis segment, or one of

$$\{p \leq a_2,\ q \leq a_1 - bp\} \quad (0 < ba_2 \leq a_1), \qquad \{q \leq a_1,\ p \leq a_2 - cq\} \quad (0 < ca_1 \leq a_2),$$

always with $p, q \geq 0$. In each of these outside cases $G[a]$ is a normalized cluster monomial [4, Corollary 3.6].

Lemma 2.1. *Let $\{L[s] : s \in \mathbb{Z}^2\}$ be a pointed basis. If the Laurent support of f lies in $\{(x, y) : x \geq -a_1, y \geq -a_2\}$, then $[L[s]]f \neq 0$ implies $s \leq a$. Its coefficient at $L[a]$ equals $[X^{-a}]f$. If f is supported in one coset of $b\mathbb{Z} \times c\mathbb{Z}$, its basis expansion uses that coset alone.*

Proof. Choose a coordinatewise maximal label s in the finite nonzero basis expansion. Only a label at least s can contribute its corner; hence that corner cannot cancel and $s \leq a$. Every label is below a maximal one. Only $L[a]$ can then contribute at X^{-a}. Finally the torus is the direct sum of its lattice-coset components; pointed basis elements are homogeneous for this decomposition. Linear independence excludes a nonzero expansion in any other coset. □

Proposition 2.2. *Every strongly positive basis has a unique member $B[a]$ pointed at each $a \in \mathbb{Z}^2$, and these are all its members. In every cluster,*

$$0 \leq B[a] \leq M[a] \tag{10}$$

coefficientwise in normalized Laurent monomials, using its label in that cluster. Its two axis polynomials equal those of $M[a]$.

Proof. For $Y \in \mathcal{B}$, right multiplication by a sufficiently large initial cluster monomial clears its negative Laurent exponents. Its Laurent expansion then consists of distinct members of $\mathcal{B}$. Uniqueness of basis expansion and positive multiplication imply nonnegative Laurent coefficients. The monomial phases in (1) preserve this property when the denominator-clearing monomial is removed. This applies in every cluster.

Each $M[a]$ is a power of v times a product of cluster variables, so $M[a] = \sum_Y h_Y Y$ with $h_Y \in R_+$. No Laurent terms cancel; every summand is supported inside the rectangle in (6). At the corner, $1 = \sum_Y h_Y [X^{-a}]Y$. Exactly one product is nonzero and it equals one. Its factors are inverse powers of v; bar invariance of Y makes both equal to one. Denote this pointed member by $B[a]$.

Lemma 2.1 gives $B[a] = M[a] + \sum_{t \leq a, t \neq a} d_t M[t]$. On the boundary $a_1 \leq 0$ or $a_2 \leq 0$, $B[a] = M[a]$ because the latter is already a member of $\mathcal{B}$. For positive coordinates, induction on $a_1 + a_2$ expresses every $M[a]$ in the span of the selected members: each lower term either lies on the boundary or has a smaller positive coordinate sum. Thus the selected subset spans and equals $\mathcal{B}$. Corner minima distinguish its labels and give uniqueness. Its coefficient in $M[a]$ is one, proving (10).

For the horizontal axis, Laurentness in the adjacent cluster gives divisibility by $K_{[a_2]_+, b}(z)$, by the quantum exchange relation, equivalently [4, Lemma 2.2]. The axis has degree at most $[a_2]_+$ by (10), and constant coefficient one. The quotient is therefore one. The vertical argument uses $K_{[a_1]_+, c}$. Here degree means the largest power of z with nonzero coefficient. □

Lemma 2.3. *In the reflected adjacent charts (X_1, X_0) and (X_3, X_2), the labels of $B[a]$ are respectively*

$$(a_1, c[a_1]_+ - a_2), \qquad (b[a_2]_+ - a_1, a_2). \tag{11}$$

The same formulas hold for greedy elements. Reversed chart orientation uses v^{-1} in place of v.

Proof. Consider the first chart. Replacing X_2 by its exchange expression in X_1, X_0 reverses its exponent. Powers of the exchange polynomial have constant term one as series in X_1^b, and introduce only nonnegative powers of X_1. For negative powers this is the formal inverse series. Thus the transformed expansion has first exponent at least $-a_1$ and second exponent at least $a_2 - c[a_1]_+$, by the rectangle bound. The vertex with initial exponent $(-a_1, c[a_1]_+ - a_2)$ has coefficient one by the axis formula; its constant exchange term is the unique contribution at the transformed corner. The formal expansion equals the Laurent expansion since the element is Laurent in that chart. This proves its pointed label; normalization of the corner is also

fixed by bar invariance. The second chart follows by exchanging the coordinates and b, c. For $G[a]$ these are the mutation formulas of [4, Proposition 4.5 and Theorem 5.8]. □

By Lemma 2.1, there is a finite expansion

$$B[a] = G[a] + \sum_{\substack{d \in b\mathbb{Z}_{\geq 0} \times c\mathbb{Z}_{\geq 0} \\ d \neq 0}} T_a(d) G[a-d], \qquad T_a(d) \in R. \tag{12}$$

Here d is a label difference. Set $T_a(0) = 1$, and set $T_a(d) = 0$ for absent terms. Bar invariance makes every $T_a(d)$ bar invariant.

3 Rigidity along large labels

Lemma 3.1. *If $T_a(e) \neq 0$, then*

$$e_2 \leq ce_1 \quad \text{if } a_1 > e_1, \qquad e_1 \leq be_2 \quad \text{if } a_2 > e_2.$$

Proof. Rewrite (12) in the first reflected chart. The reflected greedy labels are distinct, and their coefficients remain nonzero even if v is inverted. Lemmas 2.1 and 2.3 give

$$c[a_1]_+ - a_2 \geq c[a_1 - e_1]_+ - (a_2 - e_2).$$

For $a_1 > e_1 \geq 0$, this is $e_2 \leq ce_1$. The second reflected chart gives the other inequality. □

For $a_1, a_2 > 0$, the coefficient of $M[a]$ at step (p, q) has powers of v in the interval with center $bcpq$ and half-width $cq(a_1 - q) + bp(a_2 - p)$, when $0 \leq p \leq a_2$ and $0 \leq q \leq a_1$. This follows from the largest and smallest powers in the Gaussian polynomials. Define

$$\Delta_a(p, q) = ca_1q + ba_2p - bp^2 - bcpq - cq^2. \tag{13}$$

By (10) and bar invariance, the coefficient of $B[a]$ is zero if $\Delta_a(p, q) < 0$: the standard coefficient then has only strictly positive powers of v, while the coefficient of $B[a]$ is bar invariant. Outside the standard rectangle it is zero as well.

Lemma 3.2. *Fix $\nu \in \mathcal{W} \cap \mathbb{Z}^2$, a finite set $H \subset \mathbb{Z}^2$, and nonnegative integers D_1, D_2. There exists N_0, depending only on these data and on b, c, such that for every strongly positive basis, every integer $N \geq N_0$, and every $h \in H$,*

$$0 \leq d_i \leq D_i \ (i = 1, 2), \quad \det(\nu, d) \neq 0 \quad \Longrightarrow \quad T_{N\nu+h}(d) = 0.$$

Proof. Write $a = N\nu + h$. First consider $\det(\nu, d) > 0$. Lemma 3.1, once $a_1 > D_1$, confines all relevant terms to

$$E_+ = \{e \in b\mathbb{Z}_{\geq 0} \times c\mathbb{Z}_{\geq 0} : e_1 \leq D_1, \ e_2 \leq ce_1\}.$$

This set is finite. Increase N so that every $a - e$, for $e \in E_+$ and $h \in H$, lies in $\mathcal{W}$.

The upper axis vertex of $G[a - d]$ is

$$m_d = (-a_1 + d_1, ca_1 - a_2 - cd_1 + d_2),$$

with coefficient one. Relative to the a grid it has $p = d_1/b$ and $q = a_1 - d_1 + d_2/c$. Direct substitution in (13) gives

$$\Delta_a(d_1/b, a_1 - d_1 + d_2/c) = -\frac{d_1^2}{b} + d_1d_2 - \frac{d_2^2}{c} - \det(a, d). \tag{14}$$

Since $\det(a,d) = N\det(\nu,d) + \det(h,d)$, this is negative for all sufficiently large N. Thus $[X^{m_d}]B[a] = 0$.

If $T_a(e) \neq 0$ and $G[a-e]$ contributes, its first exponent gives $e_1 \leq d_1$; confinement gives $e \in E_+$. Its relative step indices are

$$p_e = (d_1 - e_1)/b, \qquad q_e = a_1 - d_1 + (d_2 - e_2)/c.$$

For a label $s = (s_1, s_2) \in \mathcal{W}$, (9) implies

$$q \leq s_1 - \left(b - \frac{bs_2}{cs_1}\right) p \quad \text{whenever } 0 \leq p \leq s_1/b. \tag{15}$$

Indeed the first region gives this directly; in the second, $q < s_2/c$, whereas the right side is at least s_2/c on the stated interval. The upper vertex satisfies equality, and the right vertex has $p = s_2 > s_1/b$. For N large, $a_1 \geq d_1$, so (15) applies to p_e. Substitution gives the necessary condition

$$\begin{aligned} 0 &\geq \det(a-e, d-e) \\ &= N\big(\det(\nu,d) - \det(\nu,e)\big) + \det(h, d-e) - \det(e,d). \end{aligned} \tag{16}$$

There are finitely many pairs $d, e \in E_+$ and $h \in H$. A common threshold therefore excludes every competitor with $\det(\nu,e) < \det(\nu,d)$.

Order the positive-determinant members of E_+ by increasing first coordinate and, within a fixed first coordinate, decreasing second coordinate. Induct in this order. A surviving competitor with $e_1 < d_1$ has already been eliminated. If $e_1 = d_1$, its axis bound $q_e \leq a_1 - e_1$ gives $e_2 \geq d_2$, and strict inequality again puts it earlier. The only remaining contributor is $e = d$, whose vertex coefficient is one. Therefore $0 = [X^{m_d}]B[a] = T_a(d)$.

For $\det(\nu,d) < 0$, use

$$E_- = \{e \in b\mathbb{Z}_{\geq 0} \times c\mathbb{Z}_{\geq 0} : e_2 \leq D_2,\ e_1 \leq be_2\}$$

and the right vertex $(ba_2 - a_1 + d_1 - bd_2, -a_2 + d_2)$. Its degree expression is

$$-d_1^2/b + d_1d_2 - d_2^2/c + \det(a,d),$$

which is negative for large N. A contributor satisfies $e_2 \leq d_2$ and $\det(a-e, d-e) \geq 0$, by the coordinate-reversed version of (15). Thus only $\det(\nu,e) \leq \det(\nu,d) < 0$ survive. Induct by increasing e_2, then decreasing e_1. At equal e_2, the horizontal axis forces $e_1 \geq d_1$. This leaves the desired coefficient alone and sets it to zero.

Every threshold used is a maximum over the two finite sets and H of inequalities affine in N with a strictly positive leading coefficient. These thresholds depend only on b, c, the fixed rectangle, H, and ν. □

Corollary 3.3. *Fix a finite rectangle $0 \leq d_i \leq D_i$, finite offsets H, and an open interval of directions in $\mathcal{W}$. One can choose an integral ν in that interval such that, for all sufficiently large N, every transition (12) at $N\nu + h$, $h \in H$, is the identity throughout the rectangle.*

Proof. Only finitely many nonzero integer vectors occur in the rectangle. Choose a rational direction in the open interval different from all their directions, and represent it by a positive integral vector ν. Lemma 3.2 eliminates every nonzero deficit in the rectangle. □

4 Finite matrices and angular coordinates

Work over $K = \mathbb{Q}(v)$ in this section. Define the step map $D(p, q) = (bp, cq)$ for integer pairs. A finite positive label interval consists of all labels in one lattice coset between two fixed, strictly positive endpoints. A downward matrix on this interval has entry zero unless its column label is $a - D(p, q)$ for some $p, q \in \mathbb{Z}_{\geq 0}$, where a is its row label. Such matrices multiply by summing over intermediate labels. Unitriangular means that all diagonal entries are one. Restriction to an interval preserves downward matrix products and inverses: every intervening label lies between the endpoints.

For a positive vector a, let $\theta(a) \in (0, \pi/2)$ be its ordinary polar angle. Nonzero vectors on the coordinate axes have angles 0 and $\pi/2$. Define on a finite positive interval

$$E_{a,t} = [X^{-t}]G[a],$$
$$J_{a,a-(bp,0)} = \begin{bmatrix} a_2 \\ p \end{bmatrix}_{v^b}, \qquad L_{a,a-(0,cq)} = \begin{bmatrix} a_1 \\ q \end{bmatrix}_{v^c}.$$

All unspecified entries of J, L are zero. These are unitriangular matrices.

Lemma 4.1. *Off the diagonal, EJ^{-1} has entries only at $\theta(t) < \theta(a)$, and EL^{-1} only at $\theta(t) > \theta(a)$. Consequently*

$$LJ^{-1} = (EL^{-1})^{-1}(EJ^{-1}) \tag{17}$$

is the unique factorization into a unitriangular strictly increasing-angle matrix followed by a unitriangular strictly decreasing-angle matrix.

Proof. For $t = a - D(p, q)$, the entry of EJ^{-1} is

$$\sum_{k=0}^{p} e_a(p-k, q)(-1)^k \begin{bmatrix} a_2 - cq + k - 1 \\ k \end{bmatrix}_{v^b}.$$

The parameter $a_2 - cq = t_2$ is positive throughout this interval. By (7) it vanishes for $ba_2p - ca_1q \geq 0$ off the diagonal. This expression equals $\det(a, t)$, so the asserted angle direction follows. The calculation for EL^{-1} uses the other recurrence and $t_1 > 0$. At equal angles both vanish.

Strict angle increase and decrease are transitive and thus each defines a group of these unitriangular matrices. Their intersection is the identity. If $U_1V_1 = U_2V_2$ are two factorizations in the stated order, then $U_2^{-1}U_1 = V_2V_1^{-1}$ lies in that intersection. Equation (17) follows by multiplication. □

Let $\widehat{\mathcal{T}}_K$ be the algebra of series $\sum_{m \in \mathbb{Z}^2} f_m X^m$ over K whose exponent support is bounded below in each coordinate. Multiplication is (1); each coefficient is a finite sum. A series with constant term one supported on $D(\mathbb{Z}^2_{\geq 0})$ has an inverse of the same kind. Conjugation by such a series defines an automorphism of $\widehat{\mathcal{T}}_K$.

Introduce an auxiliary completed algebra with monomials Z_d, $d \in \mathbb{Z}^2_{\geq 0}$, and product

$$Z_iZ_j = v^{bc\det(i,j)}Z_{i+j}.$$

The map $\iota(Z_d) = X^{Dd}$ reverses multiplication. For each $d = (p, q)$ define downward matrices on integer labels by

$$(T_d^{\pm})_{a,a-Dd} = v^{\pm(ba_2p - ca_1q)}.$$

Finite interval restrictions make every sum and product below finite. Direct calculation gives

$$T_i^+T_j^+ = v^{bc\det(i,j)}T_{i+j}^+, \qquad T_i^-T_j^- = v^{-bc\det(i,j)}T_{i+j}^-, \qquad T_i^+T_j^- = T_j^-T_i^+. \tag{18}$$

Thus $\rho_+(Z_d) = T_d^+$ preserves multiplication and $\rho_-(Z_d) = T_d^-$ reverses it. Extend these maps coefficientwise to series. For a series F with constant term one, put

$$\rho(F) = \rho_+(F)\rho_-(F^{-1}). \tag{19}$$

Equations (18) imply $\rho(FG) = \rho(F)\rho(G)$. On a row monomial X^{-a}, T_d^+ is left multiplication by X^{Dd} and T_d^- is right multiplication. Therefore the row action $f\rho(F)$ is $\iota(F)f\iota(F)^{-1}$. Row products apply their maps from left to right. This convention also applies to all Ψ below.

Define the formal Euler series, for $d > 0$, by

$$\mathsf{E}_d(z) = \sum_{n\geq 0} \frac{v^{dn^2}}{\prod_{j=1}^{n}(1 - v^{2dj})} z^n \in K[[z]],$$

and put $\mathsf{A} = \mathsf{E}_b(Z_{1,0})$, $\mathsf{B} = \mathsf{E}_c(Z_{0,1})$. The scalar identity $\mathsf{E}_d(v^{2d}z) = \mathsf{E}_d(z)/(1 + v^d z)$ follows by comparing coefficients. Iterating it yields

$$\frac{\mathsf{E}_d(v^{dn}z)}{\mathsf{E}_d(v^{-dn}z)} = K_{n,d}(z)^{-1} \quad (n \in \mathbb{Z}_{\geq 0}).$$

Along horizontal strings a_2 is fixed, and along vertical strings a_1 is fixed. The signs in $T_d^{\pm}$ therefore give

$$\rho(\mathsf{A}) = J^{-1}, \qquad \rho(\mathsf{B}) = L, \qquad LJ^{-1} = \rho(\mathsf{BA}). \tag{20}$$

Lemma 4.2. *There is a unique ordered factorization*

$$\mathsf{BA} = \prod_{\alpha \text{ increasing}} W_\alpha, \tag{21}$$

where W_α has constant term one and all its other monomials have Dd on the ray of angle α. The product is defined coefficientwise. In each fixed step rectangle only finitely many factors act nontrivially.

Proof. Induct on total step degree $p + q$. Once all coefficients below degree n have been assigned, products of those coefficients have determined the degree-n contribution except for the linear contribution of the unknown degree-n terms of the factors. Assign the discrepancy at each $Z_{p,q}$ to its unique ray. This defines the factorization, proves its uniqueness, and is compatible with increasing truncation. Each bounded rectangle contains finitely many nonzero step directions, so the final assertion follows. □

For an angle $\theta \in (0, \pi/2)$ let

$$F_\theta = \prod_{\alpha<\theta}^{\alpha \text{ increasing}} W_\alpha, \qquad \Psi_\theta = \rho(F_\theta^{-1}\mathsf{B}). \tag{22}$$

Each Ψ_θ is an automorphism of the formal algebra, and both it and its inverse add only nonnegative steps to monomial exponents.

Proposition 4.3. *For a positive integer label a, the coefficients of $G[a]$ at labels $t \leq a$ with $t > 0$ agree with those of $X^{-a}\Psi_{\theta(a)}$. If a fixed finite rectangle of deficits and finite offsets h are prescribed, and a positive integer vector ν is parallel to none of the nonzero deficits in that rectangle, then for all sufficiently large N these coefficients satisfy*

$$G[N\nu + h] = X^{-N\nu-h}\Psi_{\theta(\nu)} \quad \textit{throughout the prescribed rectangle.} \tag{23}$$

Proof. Fix a finite positive interval. The action of a ray factor $\rho(W_\alpha)$ joins only labels on strings parallel to that ray. Split it as $U_\alpha V_\alpha$, where U_α is the full action on strings above the ray and the identity on the others, while V_α is the full action below the ray and the identity elsewhere. On parallel strings the action is the identity, since the conjugating series commutes with the monomial. The sign of the determinant with the ray is constant along each string. Thus the two factors commute; U_α strictly increases angle and V_α strictly decreases it.

If $\alpha < \beta$, then

$$(V_\alpha - I)(U_\beta - I) = 0 = (U_\beta - I)(V_\alpha - I).$$

Indeed an angle-decreasing move below α cannot reach a source above β, and an angle-increasing move above β cannot reach a source below α. All intermediate labels remain positive. Moving the earlier V factors past later U factors gives

$$LJ^{-1} = \left(\prod_\alpha U_\alpha\right)\left(\prod_\alpha V_\alpha\right).$$

Lemma 4.1 now gives

$$E = \left(\prod_\alpha U_\alpha\right)^{-1} L. \tag{24}$$

Act on row a. The inverse product acts in decreasing ray angle. Every factor above $\theta(a)$, and the factor at $\theta(a)$, fixes the initial row. After a lower ray acts, subtracting a positive multiple of that direction increases the angle of the remaining positive label. Every later, still lower ray therefore acts by its full inverse. The row in (24) is exactly the row of $\rho(F_{\theta(a)}^{-1}\mathsf{B})$, proving the first assertion.

Only finitely many ray directions can act in a fixed deficit rectangle. If none is parallel to ν, all the labels $N\nu + h$ eventually lie in the same chamber between these ray directions as ν. In detail, for every relevant $d \neq 0$ require

$$N|\det(\nu, Dd)| > |\det(h, Dd)|$$

and require every label in the window to be positive. The prefixes in (22) then coincide on the rectangle, proving (23). □

Lemma 4.4. *Fix $t \in \mathbb{Z}^2_{>0}$ and a finite set $S \subset \mathbb{Z}^2$. There is an open interval I containing $\theta(t)$ such that*

$$[X^{-t}](G[s]\Psi_\theta^{-1}) = \delta_{s,t} \qquad (s \in S,\ \theta \in I). \tag{25}$$

Proof. Only s with $s - t \in D(\mathbb{Z}^2_{\geq 0})$ can contribute. For these s, every intermediate label between s and t is positive, so Proposition 4.3 gives the transition

$$\Psi_{\theta(s)}\Psi_\theta^{-1} = \rho(F_{\theta(s)}^{-1}\mathsf{B})\rho(\mathsf{B}^{-1}F_\theta) = \rho(F_{\theta(s)}^{-1}F_\theta).$$

The two prefixes cancel in the written order, leaving only factors with ray angles between $\theta(s)$ and θ.

Suppose $s \neq t$ and they are not parallel. Since $s = t + (s-t)$, $\theta(s)$ lies strictly between $\theta(t)$ and $\theta(s-t)$. Hence $\theta(s-t)$ is outside the closed interval between $\theta(s)$ and $\theta(t)$, and remains outside when θ is sufficiently close to $\theta(t)$. Every shift produced by the remaining factors is a sum of nonnegative multiples of rays in that interval. It cannot equal $s - t$.

If s, t are parallel and different, their difference has the same angle. All remaining non-parallel rays lie strictly on one side, so a sum using any of them cannot have that angle. The parallel factor, if present, is first in the row action under the strict-prefix convention and fixes X^{-s}. It thus contributes no nonzero parallel shift either. Finally $s = t$ gives coefficient one by unitriangularity. Intersect the finitely many neighborhoods required in the nonparallel cases. □

5 Positive greedy coefficients

Proposition 5.1. *Let f be pointed at $a \in \mathbb{Z}^2$ and let $t \in \mathbb{Z}^2_{>0}$ satisfy $a - t \in b\mathbb{Z}_{\geq 0} \times c\mathbb{Z}_{\geq 0}$. There exists a positive integer vector ν in any sufficiently small open interval about $\theta(t)$, avoiding any prescribed finite set of directions, such that for every sufficiently large integer N,*

$$[G[N\nu + t]](fG[N\nu]) = v^{-\det(t,N\nu)}[G[t]]f. \tag{26}$$

Proof. By Lemma 2.1, all labels of f are at most a. Only labels in the finite interval $[t, a]$ and the coset of a can contribute. Apply Lemma 4.4 to this finite set and choose ν in its interval, avoiding all nonzero step directions with deficits bounded by $a - t$ and the additional prescribed directions.

For all sufficiently large N, Proposition 4.3 gives the same chart $\Psi_{\theta(\nu)}$ for $G[N\nu]$ and every $G[N\nu + h]$, $t \leq h \leq a$, through the deficit rectangle bounded by $a - t$. Apply its inverse to $fG[N\nu]$. The multiplier becomes $X^{-N\nu}$ and every relevant output row becomes its corner. Since it is an algebra automorphism,

$$\begin{aligned}
[G[N\nu + t]](fG[N\nu]) &= [X^{-N\nu - t}]\big((f\Psi^{-1}_{\theta(\nu)})X^{-N\nu}\big) \\
&= v^{-\det(t,N\nu)}[X^{-t}](f\Psi^{-1}_{\theta(\nu)}) \\
&= v^{-\det(t,N\nu)}[G[t]]f.
\end{aligned}$$

All omitted terms have a nonnegative deficit exceeding the chosen bound in at least one coordinate. Multiplication adds such deficits, as do both chart maps, so these terms cannot return to the coefficient under consideration. Output labels outside $N\nu + [t, a]$ likewise cannot contribute under downward elimination. □

Proof of Theorem 1.2. Write $Y = B[a]$ by Proposition 2.2. Labels outside $\{a - (bp, cq) : p, q \in \mathbb{Z}_{\geq 0}\}$ have coefficient zero by Lemma 2.1. Fix a remaining target t.

First suppose $t \in \mathcal{W}$. Choose the interval in Proposition 5.1 inside $\mathcal{W}$, and choose ν avoiding every nonzero deficit in the rectangle bounded by $a - t$. Apply Lemma 3.2 to the offsets 0 and all integer $h \in [t, a]$. For large N, the input transition from $B[N\nu]$ to $G[N\nu]$ is the identity in that rectangle. Any correction with deficit d could affect the target only if $d \leq a - t$, by pointedness and Lemma 2.1. Thus no input correction contributes. The output transition block on $N\nu + [t, a]$ is also the identity, and so is its inverse. All other output labels are excluded by downward order. Consequently

$$\begin{aligned}
[B[N\nu + t]](B[a]B[N\nu]) &= [G[N\nu + t]](B[a]G[N\nu]) \\
&= v^{-\det(t,N\nu)}[G[t]]B[a].
\end{aligned} \tag{27}$$

The left side belongs to R_+ by strong positivity. Removing a power of v preserves R_+.

For $t \notin \mathcal{W}$, $G[t]$ is a cluster monomial by the five outside cases in Section 2. Choose a cluster in which it is $X^{(i,j)}$, $i, j \in \mathbb{Z}_{\geq 0}$. No other greedy element contains this normalized monomial. Indeed an element with nonpositive label has just its own corner; in either mixed-sign case one Laurent exponent is strictly negative; in the first trapezoid case $bp - a_1 \leq 0$, and equality forces $q = 0$, giving second exponent $-a_2 < 0$. The other trapezoid is symmetric. In (9), each of the two regions has a strictly negative coordinate, and each axis vertex has one too. These cases exhaust all labels in the chosen cluster. Thus $[G[t]]B[a]$ is a single Laurent coefficient in that cluster and belongs to R_+ by Proposition 2.2. □

6 A negative triangular structure constant

The triangular basis is characterized by

$$\overline{C[a]} = C[a], \qquad C[a] - M[a] \in \bigoplus_{s\in\mathbb{Z}^2} v\mathbb{Z}[v]M[s]. \tag{28}$$

Existence and uniqueness follow from [2, Theorems 1.1 and 1.4]; this is the normalization in [3, Section 4]. On the boundary with one label coordinate nonpositive, $C[a] = M[a]$ since those standard monomials are bar invariant and satisfy (28). In particular $1 = C[0,0]$, $X_3^2 = C[2,0]$ and $X_0^3 = C[0,3]$.

Proof of Theorem 1.3. Set $b = 2, c = 3$. The following finite identity determines $C[2,3]$:

$$\begin{aligned} M[2,3] - C[2,3] = {} & (v^{13} + v^9 + v^7 + v^5 + v^3 - v)M[0,0] \\ & + (v^{19} + v^{15} + v^{13} + v^{11} + v^9 + v^5)M[-2,0] \\ & + (v^{16} + v^{12} + v^8)M[0,-3] \\ & + (v^{21} + v^{15})M[-4,0] \\ & + (v^{28} + v^{24} + v^{20})M[-2,-3] + v^{36}M[-4,-3]. \end{aligned} \tag{29}$$

Subtracting the right side of (29) from $M[2,3]$ using (6) gives

$$\begin{aligned} & X^{(-2,-3)} + (v^3 + v^{-3})X^{(-2,0)} + X^{(-2,3)} \\ & \quad + (v^4 + 1 + v^{-4})(X^{(0,-3)} + X^{(2,-3)}) + X^{(4,-3)} + (v + v^{-1})X^{(0,0)}. \end{aligned} \tag{30}$$

Every coefficient is bar invariant. Every coefficient on the right of (29) belongs to $v\mathbb{Z}[v]$, so uniqueness in (28) identifies (30) with $C[2,3]$. All remaining standard monomials in (29) are boundary triangular elements. Hence the coefficient of $C[0,0]$ in $M[2,3]$ is exactly $v^{13} + v^9 + v^7 + v^5 + v^3 - v$. Finally $M[2,3] = v^6 X_3^2 X_0^3$, giving

$$[C[0,0]](X_3^2 X_0^3) = v^7 + v^3 + v + v^{-1} + v^{-3} - v^{-5}.$$

Since X_3^2, X_0^3 are triangular basis elements, the coefficient -1 at v^{-5} disproves strong positivity and hence Conjecture 13(b). □

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Zhili College, Tsinghua University, Beijing 100084, China
Email: tangqy24@mails.tsinghua.edu.cn