An explicit nonideal rooted cluster morphism

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Abstract

We construct an explicit, inducible rooted cluster morphism that is not ideal, answering a question of Chang and Zhu. Its image is a polynomial ring, whereas its image-seed algebra is an integrally closed, non-Noetherian subring. We also give a coefficient-free example in which specializing a mutable variable to one does not map into the cluster algebra of the deleted seed.

1 Introduction

A rooted cluster algebra is a cluster algebra with a specified initial seed. Assem, Dupont and Schiffler introduced rooted cluster morphisms to study maps that respect initial variables and mutation along biadmissible sequences [1]. Such a morphism determines an image seed, whose cluster algebra is contained in its ring-theoretic image. The morphism is called ideal when these two algebras agree. Chang and Zhu constructed nonideal morphisms whose values on the initial variables do not determine the morphism. They then asked whether every explicit rooted cluster morphism is ideal [2, §2.2, Problem 2]. Here explicit means uniquely determined, for fixed source and target, by its values on the initial variables.

We answer this question negatively with three mutable variables in both source and target and one frozen variable in the source. Throughout the paper, the ground ring is $\mathbb{Z}$, and frozen variables are not inverted. This is the coefficient convention of [1, 2].

Here is the construction. Let $C = \mathcal{A}(\Sigma')$ have mutable variables (a, b, z), no frozen variables, and exchange matrix B. Let $A = \mathcal{A}(\Sigma)$ have mutable variables (a, b, u), frozen variable t, and extended exchange matrix $\widetilde{B}$, where

$$B = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}. \tag{1.1}$$

Rows are ordered by the displayed mutable variables followed by the frozen variable; columns are ordered by mutable variables. The source and target have independent ambient coordinates; the repeated names a, b specify which coordinates the map identifies. In $\mathbb{Q}(a, b, z)$ put

$$s = \frac{a^2 + b^2 + z}{ab}, \qquad P = \mathbb{Z}[a, b, s]. \tag{1.2}$$

Since $z = abs - a^2 - b^2$, the elements a, b, s are algebraically independent. Thus P is a polynomial ring in three variables. The notation $(a, b)P$ denotes its ideal generated by a, b.

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Theorem 1.1. *The assignment*

$$a \longmapsto a, \qquad b \longmapsto b, \qquad u \longmapsto 1, \qquad t \longmapsto z$$

extends to an explicit, inducible rooted cluster morphism $f : A \to C$. Its image seed has mutable variables a, b, frozen variable z, and matrix

$$B_R = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ 1 & -1 \end{pmatrix}.$$

Writing $R = \mathcal{A}(f(\Sigma))$, one has

$$R = \mathbb{Z} + (a, b)P \subsetneq f(A) = P \subsetneq C.$$

In particular, f is not ideal.

The global definition of f is the main issue: an assignment on initial variables need not send all source cluster variables into the target cluster algebra. We prove this containment by specializing Laurent expansions in three source clusters and computing the intersection of the resulting rings. The image-seed algebra is then determined by a rank-two recurrence.

The invariant s has an established annulus interpretation. At $q = 1$, the loop formulas of Muller [7, §12.2, pp. 490–491] give

$$\ell x_i = x_{i+1} + x_{i-1}, \qquad \ell = \frac{x_i^2 + x_{i+1}^2 + r_0 r_1}{x_i x_{i+1}},$$

where r_0, r_1 are the two boundary variables. Setting $r_0 = z, r_1 = 1$ recovers s. Muller's cluster algebra inverts these boundary variables. Here the polynomial coefficient ring permits the strict inclusion in Theorem 1.1. We compute its conductor and integral closure in Section 4.

Section 5 gives a separate four-variable example for which freezing fails to give a cluster localization. The same example answers the specialization-at-one question of Assem–Dupont–Schiffler [1, Problem 6.16 in arXiv v1] and the question following Theorem 6.7 of Huang–Li–Yang [5].

2 Mutation and polynomial coefficients

We use finite seeds $\Sigma = (E, F, B)$. The disjoint sets E and F of mutable and frozen variables are algebraically independent over $\mathbb{Q}$. The integer matrix B has rows indexed by $X = E \sqcup F$ and columns by E. Its principal part is skew-symmetrizable: there are positive integers d_i for $i \in E$ with $d_i b_{ij} = -d_j b_{ji}$. The ambient field is $\mathbb{Q}(X)$. For an integer r, write $[r]_+ = \max(r, 0)$. Mutation at $k \in E$ replaces x_k by x_k' according to

$$x_k x_k' = \prod_{i \in X} x_i^{[b_{ik}]_+} + \prod_{i \in X} x_i^{[-b_{ik}]_+}, \tag{2.1}$$

and replaces the matrix by

$$b_{ij}' = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & \text{otherwise.} \end{cases} \tag{2.2}$$

An empty product is 1. The algebra $\mathcal{A}(\Sigma)$ is generated over $\mathbb{Z}$ by F and all mutable variables obtained by finite mutation sequences. For a set of independent variables E, the notation $\mathbb{Z}[E^{\pm 1}, F]$ means that only the variables in E are inverted.

We use the Laurent phenomenon of Fomin–Zelevinsky [3, Theorem 3.1] and its polynomial-coefficient form [4, Proposition 11.2]. The latter occurs in a section explicitly allowing infinite cluster type. The following argument explains why no frozen inverses are needed.

Lemma 2.1. *For every seed (E_*, F, B_*) mutation equivalent to Σ,*

$$\mathcal{A}(\Sigma) \subseteq \mathbb{Z}[E_*^{\pm 1}, F].$$

Proof. The ordinary Laurent phenomenon gives this inclusion with $F^{\pm 1}$ in place of F. Fix a frozen variable $t \in F$ and use E_* as base coordinates. Regard rational functions as elements of $K(t)$, where $K = \mathbb{Q}(E_*, F \setminus \{t\})$. The order at $t = 0$ of a nonzero rational function is the exponent of its first nonzero term in t; its leading coefficient belongs to K.

Initially every mutable variable has order zero and a leading coefficient given by a subtraction-free rational expression in the remaining coordinates. Suppose these properties hold at a current seed. If the entry of the t-row in the mutated column is c, the two terms on the right of (2.1) have orders $[c]_+$ and $[-c]_+$. Their minimum is zero. Their leading coefficients, when both contribute, cannot cancel: they are subtraction-free rational expressions and are positive upon any positive specialization of the remaining coordinates. Dividing by the old mutable variable preserves order zero and this property of the leading coefficient. Induction along any mutation word therefore gives order zero for every mutable cluster variable.

In the Laurent expansion already supplied by the ordinary theorem, order zero excludes negative powers of t. Applying this to each frozen variable proves the assertion for the generators and hence for the algebra. □

For clarity, we recall the morphism convention of [2, Definition 2.5]. A unital ring map $f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')$ is a rooted cluster morphism if $f(E) \subseteq E' \cup \mathbb{Z}$, $f(F) \subseteq E' \cup F' \cup \mathbb{Z}$, and it commutes with mutation on every biadmissible sequence, on all initial variables. A sequence is admissible if each term is mutable when used; it is biadmissible if its termwise image is also admissible in the target. Integer images remain unchanged. In particular, a frozen source variable may map to a mutable target variable.

The image seed $f(\Sigma)$ retains from the target the mutable variables $f(E) \cap E'$, all variables $f(X) \cap X'$, and the corresponding matrix rows and mutable columns. Its other retained variables are frozen. The morphism is ideal when $f(\mathcal{A}(\Sigma)) = \mathcal{A}(f(\Sigma))$. It is explicit when no other rooted cluster morphism with the same source, target, and initial values differs from it. It is inducible when it extends to a homomorphism between the ambient Laurent rings $\mathbb{Q}[E^{\pm 1}, F]$ and $\mathbb{Q}[(E')^{\pm 1}, F']$ [2, Definition 2.8].

3 The morphism and its image

We now use the two seeds in (1.1) and the polynomial ring P in (1.2). Denote the target mutations at a and b, performed separately from the initial seed, by

$$a' = \frac{b^2 + z}{a} = bs - a, \qquad b' = \frac{a^2 + z}{b} = as - b. \tag{3.1}$$

Lemma 3.1. *Inside $\mathbb{Q}(a, b, z)$ one has*

$$\mathbb{Z}[a^{\pm 1}, b^{\pm 1}, z] \cap \mathbb{Z}[(a')^{\pm 1}, b^{\pm 1}, z] \cap \mathbb{Z}[a^{\pm 1}, (b')^{\pm 1}, z] = P.$$

Proof. The three rings are, respectively,

$$P[(ab)^{-1}], \qquad P[(a'b)^{-1}], \qquad P[(ab')^{-1}]. \tag{3.2}$$

For the middle equality, use

$$s = \frac{(a')^2 + b^2 + z}{a'b}, \qquad a = bs - a'.$$

The last equality follows by exchanging a, b, and the first follows directly from (1.2).

The ring P is a unique factorization domain. Any irreducible factor remaining in a reduced denominator of an element of all three rings in (3.2) must divide ab, $a'b$, and ab'. The only possible factors of ab are a, b, up to units. But modulo a the element $a'b$ is $b^2 s \neq 0$, and modulo b the element ab' is $a^2 s \neq 0$. Thus no denominator factor remains. Each ring contains P, proving equality. □

Lemma 3.2. *The Laurent-ring homomorphism*

$$\phi : \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, u^{\pm 1}, t] \longrightarrow \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, z], \qquad (a, b, u, t) \longmapsto (a, b, 1, z)$$

satisfies $\phi(A) \subseteq P \subseteq C$.

Proof. The map is defined because $a, b, 1$ are units in its codomain. The source one-step mutations at a, b are

$$\widetilde{a}' = \frac{b^2 + ut}{a}, \qquad \widetilde{b}' = \frac{a^2 + ut}{b}.$$

Their images are a', b'. By Lemma 2.1, any $h \in A$ has a Laurent expansion in each of the source clusters (a, b, u), $(\widetilde{a}', b, u)$, and $(a, \widetilde{b}', u)$, with coefficients in $\mathbb{Z}[t]$. To specialize a neighboring expansion, multiply it by its mutable monomial denominator. The resulting identity lies in the initial Laurent ring, so ϕ applies to it. The image of that denominator is nonzero, since a, b, a', b' are nonzero. Dividing in $\mathbb{Q}(a, b, z)$ gives $\phi(h)$ in the corresponding target ring of Lemma 3.1. Hence $\phi(h) \in P$.

For the other inclusion, mutate the target in positions $(3, 1, 2)$, in that order. The first two new matrices are

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The three new variables, denoted p, q, v, satisfy

$$p = \frac{a + b}{z}, \qquad q = \frac{1 + bp}{a}, \qquad v = \frac{q + p}{b} = \frac{(a + b)^2 + z}{abz}. \tag{3.3}$$

Thus $s = zv - 2 \in C$, and $a, b \in C$ imply $P \subseteq C$. □

Proposition 3.3. *The restriction* $f = \phi|_A : A \to C$ *is an explicit, inducible rooted cluster morphism, and* $f(A) = P$.

Proof. Lemma 3.2 supplies the ring homomorphism. The initial assignments satisfy the two conditions on mutable and frozen images. A biadmissible sequence cannot have a first mutation in the third source position: before that step the variable is still u, with image 1. Hence every biadmissible sequence uses only positions $1, 2$.

Conversely, consider any word using those two positions. In the source, the u- and t-rows remain equal by (2.2), because neither row is mutated. Restricted to columns $1, 2$, they agree with the target z-row; the two mutable principal parts agree as well. In each exchange monomial their contribution has the form $u^r t^r$, which specializes to z^r. Induction proves that the current variables in positions $1, 2$ map to the corresponding target variables. In particular the word is biadmissible, and mutation commutes with f. The unchanged variables u, t and their images $1, z$ also satisfy the required identities. This proves the mutation condition for every biadmissible sequence and every initial variable.

Extending ϕ to rational coefficients and including its codomain in $\mathbb{Q}[a^{\pm 1}, b^{\pm 1}, z^{\pm 1}]$ proves inducibility. To prove explicitness directly, let $g : A \to C$ be any unital ring map with the same initial values. For $h \in A$, Lemma 2.1 gives nonnegative integers i, j, k and a polynomial

$Q \in \mathbb{Z}[a,b,u,t]$ with $a^i b^j u^k h = Q$. Applying f, g and cancelling their common nonzero factor $a^i b^j$ in the domain C gives $f(h) = g(h)$.

Finally, the source word $(3,1,2)$ produces

$$\widetilde{p} = \frac{a+b}{u}, \qquad \widetilde{q} = \frac{t+b\widetilde{p}}{a}, \qquad \widetilde{v} = \frac{\widetilde{q}+\widetilde{p}}{b} = \frac{(a+b)^2+ut}{abu}.$$

The frozen row is $(1,-1,0)$ after the first step and $(-1,0,1)$ after the second, verifying the coefficients in these exchanges. Consequently $f(\widetilde{v}) = s+2$. Since a, b also belong to $f(A)$, Lemma 3.2 now gives $f(A) = P$. □

Proposition 3.4. *The image-seed algebra is $R = \mathbb{Z} + (a,b)P$.*

Proof. Its mutable variables are a, b, and z is frozen, so its matrix is B_R in Theorem 1.1. Mutation at either mutable position negates this entire matrix. Cancelling consecutive repetitions of a position reduces every word to an alternating word. Thus all mutable variables are the elements x_i, $i \in \mathbb{Z}$, defined in both directions by

$$x_0 = a, \quad x_1 = b, \qquad x_{i-1}x_{i+1} = x_i^2 + z. \tag{3.4}$$

All clusters are adjacent pairs, up to their ordering. The identity

$$\frac{x^2+y^2+z}{xy} = \frac{x+x'}{y} = \frac{(x')^2+y^2+z}{x'y}, \qquad xx' = y^2+z,$$

shows that the expression defining s is invariant under mutation. Therefore

$$s = \frac{x_{i-1}^2 + x_i^2 + z}{x_{i-1}x_i}, \qquad x_{i+1} = sx_i - x_{i-1} \quad (i \in \mathbb{Z}). \tag{3.5}$$

Every x_i is a $\mathbb{Z}[s]$-linear combination of a, b. Also $z = abs - a^2 - b^2 \in (a,b)P$. This proves $R \subseteq \mathbb{Z} + (a,b)P$.

To prove the reverse inclusion, (3.5) gives, for every $n \geq 0$,

$$x_{n+1} - bs^n,\ x_{-n} - as^n \ \in \ \sum_{j=0}^{n-1} (\mathbb{Z}as^j + \mathbb{Z}bs^j).$$

For $n = 0$ the sum is zero; the general formulas follow by induction from the two directions of (3.5). Simultaneous induction on n now proves $as^n, bs^n \in R$: subtract the lower-degree terms, already in R, from the corresponding cluster variable. Multiplication by nonnegative powers of a, b then gives every monomial in $(a,b)P$. Thus $(a,b)P \subseteq R$. □

Proof of Theorem 1.1. Propositions 3.3 and 3.4 give the morphism and the two image descriptions. Modulo $(a,b)P$, the image of R is $\mathbb{Z}$, whereas $P/(a,b)P = \mathbb{Z}[s]$. Hence $s \notin R$ and $R \subsetneq P$. Finally $P \subseteq \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, z]$, but the target variable $(a+b)/z$ in (3.3) does not lie in this ring. Therefore $P \subsetneq C$. □

4 The image-seed ring

The exact image description permits a comparison with the upper algebra. In the same polynomial coefficient convention, define

$$U^{\mathrm{pol}}(R) = \bigcap_{i \in \mathbb{Z}} \mathbb{Z}[z, x_i^{\pm 1}, x_{i+1}^{\pm 1}] \subseteq \mathbb{Q}(a,b,z),$$

where the x_i are defined in (3.4).

Proposition 4.1. *One has*

$$U^{\mathrm{pol}}(R) = P, \qquad R[z^{-1}] = P[z^{-1}] = C[z^{-1}].$$

Proof. The invariant expression in (3.5) puts s in every ring defining $U^{\mathrm{pol}}(R)$. Reversing the linear recurrence expresses a, b as $\mathbb{Z}[s]$-linear combinations of any adjacent pair x_i, x_{i+1}. Thus $P \subseteq U^{\mathrm{pol}}(R)$. The initial pair and its two neighbors have intersection P by Lemma 3.1, giving equality.

Using (3.1) and $z = abs - a^2 - b^2$, we obtain

$$a'b' - ab = zs. \tag{4.1}$$

All four variables on the left belong to R, so $s \in R[z^{-1}]$. Since $R \subseteq P$ and $a, b \in R$, this proves $R[z^{-1}] = P[z^{-1}]$. The target Laurent expansions in its initial cluster and its separate neighbors at a, b yield

$$C \subseteq P[(abz)^{-1}] \cap P[(a'bz)^{-1}] \cap P[(ab'z)^{-1}] = P[z^{-1}].$$

Indeed, the denominator argument of Lemma 3.1 allows only prime factors dividing z after these localizations. Together with $P \subseteq C$, this proves $C[z^{-1}] = P[z^{-1}]$. □

Let $I = (a, b)P$, now also viewed as an ideal of $R = \mathbb{Z} + I$. The *conductor* of R in P is the ideal $\{r \in R : rP \subseteq R\}$.

Proposition 4.2. *The conductor of R in P equals I. The ring R is integrally closed in its fraction field and is not Noetherian. The element s is not integral over R; in particular, P is neither integral nor module-finite over R.*

Proof. Since I is a P-ideal contained in R, it lies in the conductor. For $r = c + i$, with $c \in \mathbb{Z}$, $i \in I$, the image of rs in $P/I = \mathbb{Z}[s]$ is cs. This belongs to $R/I = \mathbb{Z}$ only if $c = 0$. The conductor is therefore exactly I.

The square of I as an R-ideal is $(a^2, ab, b^2)P$: for example, $a^2 p = a(ap)$ with both factors in I, and the other generators are handled identically. Consequently

$$I/I^2 \cong \mathbb{Z}[s]\overline{a} \oplus \mathbb{Z}[s]\overline{b}.$$

Here $\overline{a}, \overline{b}$ denote the classes modulo I^2. This is an infinite-rank free $\mathbb{Z}$-module, whereas the R-action factors through $R/I = \mathbb{Z}$. A finite set of R-generators for I would give a finite set of $\mathbb{Z}$-generators for I/I^2, a contradiction. Hence R is not Noetherian, and in particular is not a finitely generated $\mathbb{Z}$-algebra.

The relation $as \in R$ shows that $\mathrm{Frac}(R) = \mathrm{Frac}(P) = \mathbb{Q}(a, b, s)$. If w in this field is integral over R, it is integral over the polynomial UFD P, and thus belongs to P. Reducing a monic equation for w modulo I shows that its class in $\mathbb{Z}[s]$ is integral over $\mathbb{Z}$. A nonconstant polynomial of degree $d > 0$ cannot satisfy a monic equation of degree m over $\mathbb{Z}$: its leading term of degree md cannot cancel with any lower power. The class of w is therefore an integer, so $w \in \mathbb{Z} + I = R$.

The same reduction shows that s is not integral over R. A module-finite ring extension is integral, so P is not module-finite over R either. □

5 Freezing and specialization at one

We give a coefficient-free example where the difference persists after the frozen variable is inverted. Let C_4 have independent mutable variables (a, b, c, z), no frozen variables, and matrix

$$B_4 = \begin{pmatrix} 0 & 2 & -2 & -1 \\ -2 & 0 & 2 & 1 \\ 2 & -2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

Let R_4 be obtained by freezing z: its mutable variables are a, b, c, its frozen variable is z, and its matrix consists of the first three columns of B_4. Let M be the coefficient-free Markov cluster algebra obtained by deleting the row and column of z, with initial variables a, b, c. Finally let

$$\sigma : \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, z^{\pm 1}] \longrightarrow \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$$

fix a, b, c and send z to 1. The freezing is called a cluster localization when $R_4[z^{-1}] = C_4[z^{-1}]$.

Theorem 5.1. *The inclusions*

$$R_4[z^{-1}] \subsetneq C_4[z^{-1}], \qquad M \subsetneq \sigma(C_4)$$

are strict. No unital ring homomorphism $C_4 \to M$ sends (a, b, c, z) to $(a, b, c, 1)$. The same assertions hold after extending scalars from $\mathbb{Z}$ to $\mathbb{Q}$.

Proof. Mutate B_4 in positions $(4, 1, 2)$. The first two matrices are

$$\begin{pmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 2 & -1 \\ 2 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 0 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}.$$

Denoting the three new variables by p_4, q_4, v_4, we obtain

$$p_4 = \frac{a+b}{z}, \qquad q_4 = \frac{c^2 + bp_4}{a}, \qquad v_4 = \frac{q_4 + p_4}{b} = \frac{(a+b)^2 + zc^2}{abz} \in C_4.$$

Give the initial Laurent ring the integer grading determined by $\deg a = \deg b = \deg c = 1$ and $\deg z = 0$; the degree of a Laurent monomial is the sum of its exponents of a, b, c. Mutation of the Markov principal matrix at any of its three positions negates that matrix. Every exchange relation in R_4 consequently has the form

$$y_k y_k' = z^{r_+} y_i^2 + z^{r_-} y_j^2, \qquad r_+, r_- \geq 0,$$

where i, j, k are the three distinct mutable positions. Induction gives degree one for every mutable cluster variable. Thus R_4 is nonnegatively graded with degree-zero part $\mathbb{Z}[z]$, and $(R_4[z^{-1}])_0 = \mathbb{Z}[z^{\pm 1}]$. But v_4 has degree zero and its Laurent term $c^2/(ab)$ prevents it from belonging to $\mathbb{Z}[z^{\pm 1}]$. It is therefore outside $R_4[z^{-1}]$. Mutations avoiding z identify R_4 with a subalgebra of C_4, proving the first strict inclusion.

Lemma 2.1 defines σ on C_4. Every mutation word of the deleted seed lifts to a word avoiding z, and setting $z = 1$ in its exchanges gives the corresponding Markov variables. Therefore $M \subseteq \sigma(C_4)$. The same grading argument gives $M_0 = \mathbb{Z}$, whereas

$$\sigma(v_4) = \frac{(a+b)^2 + c^2}{ab}$$

has degree zero and is not constant. This proves strictness. Moreover, any ring homomorphism h with the prescribed initial values would send the identity $abzv_4 = (a+b)^2 + zc^2$ to $ab\,h(v_4) = (a+b)^2 + c^2$. Cancellation in $\mathbb{Q}(a,b,c)$ forces the same forbidden value for $h(v_4)$.

Over $\mathbb{Q}$, the degree-zero parts become $\mathbb{Q}[z^{\pm 1}]$ and $\mathbb{Q}$, respectively. The identical witnesses prove all assertions there. □

This failure concerns the ordinary deleted algebra. For example, the element $(a^2 + b^2 + c^2)/(abc)$ is invariant under Markov mutation, hence Laurent in every Markov cluster. Its degree is -1, so it belongs to the upper algebra but not to M. Thus the deletion in Theorem 5.1 does not satisfy the equality of ordinary and upper cluster algebras required in the specialization theorem of Keller–Plamondon–Qin [6, Corollary 4.3].

References

[1] I. Assem, G. Dupont and R. Schiffler, *On a category of cluster algebras*, J. Pure Appl. Algebra **218** (2014), 553–582. arXiv:1201.5986v1.

[2] W. Chang and B. Zhu, *On rooted cluster morphisms and cluster structures in 2-Calabi–Yau triangulated categories*, J. Algebra **458** (2016), 387–421. arXiv:1410.5702v2.

[3] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529. arXiv:math/0104151.

[4] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121. arXiv:math/0208229v2.

[5] M. Huang, F. Li and Y. Yang, *On structure of cluster algebras of geometric type I: In view of sub-seeds and seed homomorphisms*, Sci. China Math. **61** (2018), 831–854. arXiv:1509.01050v4. The question cited here follows Theorem 6.7 in this arXiv version.

[6] B. Keller, P.-G. Plamondon and F. Qin, *A refined multiplication formula for cluster characters*, arXiv:2301.01059v2.

[7] G. Muller, *Skein and cluster algebras of marked surfaces*, Quantum Topol. **7** (2016), 435–503. doi:10.4171/QT/79.

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