Algebraic dependence of cluster X-variables

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Abstract

We prove that, over any field and with algebraically independent initial seed coordinates, two cluster X-variables are algebraically dependent if and only if they are equal or reciprocal. In characteristic zero, this is equivalent to the vanishing of the wedge product of their differentials.

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1 Introduction

Cluster X-mutations identify the function fields of seed tori. Each seed gives a rational coordinate system, and coordinates from different seeds can satisfy algebraic relations. This paper determines when a relation involves only two coordinates: algebraic dependence forces equality or reciprocity. The result holds over an arbitrary field, with any chosen subset of mutation directions, provided the initial coordinates are independent indeterminates.

The differential version of this question was posed by Elbabak on MathOverflow [2]. For the standard mutation formula of Fock–Goncharov [3, §1.2.4, (13)], we prove that, in characteristic zero, two cluster X-coordinates have vanishing wedge product of differentials precisely when they are equal or reciprocal. Section 5 records the distinction between the standard formula and the formula displayed in that question, together with examples concerning specialization and positive characteristic.

The equality of cluster X-variables has also been studied through exchange pairs. Sherman-Bennett [4, Theorem 1.1 and Corollary 1.2] gives a combinatorial description in finite type: the classical types are treated using quadrilateral models, and the exceptional types are checked computationally. Cao–Keller–Qin [1, Theorems 7.5–7.6] give equality criteria and a parametrization by ordered exchange pairs in their general setting. Combined with our theorem, this parametrization identifies the one-variable subfields generated by X-coordinates with unordered exchange pairs. The matrix convention and coefficient ring for this application are specified in Section 4.2.

The proof has two steps. Algebraic dependence between two rational coordinates gives equality of their generated subfields and hence a fractional linear relation. For each monomial valuation arising from an ordered exponent lattice, the values of the coordinates in every seed form an integral basis and their leading coefficients are one. Two choices of order force the fractional linear map to preserve the pair consisting of zero and infinity; the leading coefficients determine its scalar factor.

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2 Seeds and mutations

Let k be a field, let $n \geq 1$, and let $x_1, \ldots, x_n$ be algebraically independent indeterminates over k. Put $K = k(x_1, \ldots, x_n)$ and $I = \{1, \ldots, n\}$. Fix a subset $I_{\mathrm{mut}} \subseteq I$ of mutable indices; the other indices are frozen. The initial exchange matrix $B = (b_{ij})_{i,j \in I}$ is an integer skew-symmetrizable matrix. Thus there are positive integers $d_1, \ldots, d_n$ such that $d_i b_{ij} = -d_j b_{ji}$ for all i, j, and in particular $b_{ii} = 0$.

A rational coordinate system for K/k is an ordered algebraically independent n-tuple generating K over k. A seed consists of an exchange matrix and such a coordinate system. For a current seed $(B, (z_i)_{i \in I})$ and $j \in I_{\mathrm{mut}}$, write $[a]_+ = \max(a, 0)$ for an integer a. Mutation in direction j gives

$$z_j' = z_j^{-1}, \qquad z_i' = z_i z_j^{[b_{ij}]_+} (1 + z_j)^{-b_{ij}} \quad (i \neq j), \tag{1}$$

and the matrix changes by

$$b_{i\ell}' = \begin{cases} -b_{i\ell}, & i = j \text{ or } \ell = j, \\ b_{i\ell} + [b_{ij}]_+ [b_{j\ell}]_+ - [-b_{ij}]_+ [-b_{j\ell}]_+, & i, \ell \neq j. \end{cases} \tag{2}$$

All powers in (1) are taken in K. If $b_{ij} = 0$, the formula gives $z_i' = z_i$. If $b_{ij} \neq 0$, it is equivalent to

$$z_i' = z_i \bigl(1 + z_j^{-\operatorname{sgn}(b_{ij})}\bigr)^{-b_{ij}},$$

where $\operatorname{sgn}(a)$ is 1 for $a > 0$ and -1 for $a < 0$. This is the convention in [3, §1.2.4, (13)]. There, the pullback of a target coordinate means its composition with the mutation map; here these pullbacks are written directly as elements of K.

Lemma 2.1. *Mutation is a birational involution. Every seed obtained by a finite sequence of mutations has a rational coordinate system for K/k.*

Proof. For $i \neq j$, a second mutation in direction j gives

$$z_i'' = z_i z_j^{[b_{ij}]_+} (1 + z_j)^{-b_{ij}} (z_j^{-1})^{[-b_{ij}]_+} (1 + z_j^{-1})^{b_{ij}} = z_i,$$

using $[b_{ij}]_+ - [-b_{ij}]_+ = b_{ij}$ and $1 + z_j^{-1} = (1 + z_j)/z_j$. The pivot coordinate returns to z_j. In (2), the second increment for $i, \ell \neq j$ is the negative of the first, and the entries in row or column j change sign twice. Thus the matrix also returns to its original value.

The matrix remains skew-symmetrizable by the same integers d_i. For $i, \ell \neq j$, use

$$[b_{\ell j}]_+ = \frac{d_j}{d_\ell} [-b_{j\ell}]_+, \qquad [b_{ji}]_+ = \frac{d_i}{d_j} [-b_{ij}]_+,$$

and the analogous identities with both signs reversed. Substitution in (2) shows that the two increments satisfy the required skew-symmetry after multiplication by d_i and d_ℓ.

Since z_j is transcendental over k, both z_j and $1 + z_j$ are nonzero. The inverse formulas therefore give

$$k(z_1', \ldots, z_n') = k(z_1, \ldots, z_n) = K.$$

The transcendence degree of K/k is n, so these n generators are algebraically independent. Induction on the length of the mutation sequence proves the assertion. □

Let $\mathscr{X} \subset K^\times$, where $K^\times = K \setminus \{0\}$, be the set of all coordinates occurring in seeds reachable from the initial seed. Frozen indices restrict the permitted mutation directions; their coordinates still transform by (1) when another direction is mutated. In particular, all initial coordinates, including the frozen ones, remain independent indeterminates in this definition. Simultaneously relabelling coordinates, matrix indices, and mutable indices preserves $\mathscr{X}$.

3 Algebraic dependence

Elements $X, Y \in K$ are algebraically dependent over k if $P(X, Y) = 0$ for some nonzero polynomial $P(U, V) \in k[U, V]$. Write $k(X)$ for the subfield of K generated by k and X.

Theorem 3.1. *For $X, Y \in \mathscr{X}$, the following conditions are equivalent:*

1. *X and Y are algebraically dependent over k;*
2. *$k(X) = k(Y)$;*
3. *$Y = X$ or $Y = X^{-1}$.*

We first reduce dependence between rational coordinates to a fractional linear relation.

Lemma 3.2. *Let F be a field, let $m \geq 0$, and let $t_1, \ldots, t_m$ be algebraically independent over F. Every element of $F(t_1, \ldots, t_m)$ that is algebraic over F belongs to F.*

Proof. The case $m = 0$ is immediate. Write an algebraic element as $h = P/Q$, with $P, Q \in F[t_1, \ldots, t_m]$ and $Q \neq 0$. Choose a nonzero polynomial $f \in F[T]$ with $f(h) = 0$. It has positive degree. In an algebraic closure $\overline{F}$ of F, factor

$$f(T) = c \prod_{r=1}^{d} (T - \alpha_r), \qquad c \neq 0,$$

listing each root with its multiplicity. Clearing denominators gives

$$c \prod_{r=1}^{d} (P - \alpha_r Q) = 0 \quad \text{in } \overline{F}[t_1, \ldots, t_m].$$

This polynomial ring is an integral domain, so $P = \alpha_r Q$ for some r. Comparing a nonzero coefficient of Q with the corresponding coefficient of P gives $\alpha_r \in F$. Hence $h \in F$. □

Lemma 3.3. *Suppose that each of $X, Y \in K$ belongs to a rational coordinate system for K/k. If X and Y are algebraically dependent over k, then*

$$Y = \frac{aX + b}{cX + d} \quad \textit{for some } a, b, c, d \in k \textit{ with } ad - bc \neq 0.$$

Proof. Since X is transcendental over k, a nonzero relation $P(X, Y) = 0$ gives a nonzero polynomial over $k(X)$ having Y as a root. Complete X to a rational coordinate system, so that $K = k(X)(z_2, \ldots, z_n)$. Lemma 3.2 gives $Y \in k(X)$. Interchanging X and Y gives $X \in k(Y)$, and therefore $k(X) = k(Y)$.

There are nonconstant rational functions $R, S \in k(T)$ such that $Y = R(X)$ and $X = S(Y)$. Transcendence of X implies $S \circ R = T$. For relatively prime polynomials P, Q with $Q \neq 0$, the degree of the nonconstant rational function P/Q is $\max(\deg P, \deg Q)$, where polynomial degree is the largest exponent with nonzero coefficient. Nonconstant rational functions satisfy

$$\deg(S \circ R) = \deg S \, \deg R. \tag{3}$$

Indeed, homogenize the numerator and denominator of each function to its degree. Each resulting pair of homogeneous polynomials has no common zero on the projective line $\mathbb{P}^1(\overline{k}) = \overline{k} \cup \{\infty\}$, where $\overline{k}$ is an algebraic closure of k. Substituting the pair for R into the pair for S produces homogeneous polynomials of degree $\deg S \deg R$. A common zero would map to a common zero of the pair for S, which is impossible. Thus the composed pair has no common factor, proving (3). It follows that $\deg R = 1$, which gives the stated expression. Its nonconstancy is equivalent to $ad - bc \neq 0$. □

Fix the coordinates $z_1, \dots, z_n$ of a seed and a translation-invariant total order on the additive group $\mathbb{Z}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, where $\mathbb{N} = \{0, 1, \dots\}$, put $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$. The support of a nonzero polynomial $P = \sum_\alpha p_\alpha z^\alpha$ is the finite set of exponents with $p_\alpha \neq 0$. Define $\nu(P)$ to be its least exponent and $\operatorname{lc}(P)$ to be the coefficient of that exponent. For $P, Q \neq 0$, set

$$\nu(P/Q) = \nu(P) - \nu(Q), \qquad \operatorname{lc}(P/Q) = \operatorname{lc}(P)/\operatorname{lc}(Q). \tag{4}$$

Thus ν takes values in the ordered group $\mathbb{Z}^n$, and lc takes values in $k^\times$.

Lemma 3.4. *The definitions in (4) are independent of the chosen fraction. For $f, g \in K^\times$,*

$$\nu(fg) = \nu(f) + \nu(g), \qquad \operatorname{lc}(fg) = \operatorname{lc}(f)\operatorname{lc}(g).$$

If $\nu(f) \neq \nu(g)$, then $f + g \neq 0$, and its value and leading coefficient equal those of the summand with smaller value.

Proof. Let α_0 and β_0 be the least support exponents of two nonzero polynomials. Every other pair of support exponents satisfies $\alpha + \beta > \alpha_0 + \beta_0$: both exponents are at least their respective minima, and at least one is larger. The least term of the product is therefore the product of the least terms, with nonzero coefficient. Applying this fact to $PQ' = P'Q$ proves that (4) is well defined and proves the multiplication rules. For addition, pass to a common denominator. When the two numerator values differ, the smaller term survives with its coefficient unchanged. □

Lemma 3.5. *For every seed $(w_1, \dots, w_n)$ reachable from $(z_1, \dots, z_n)$, the vectors $v_i = \nu(w_i)$ form a basis of $\mathbb{Z}^n$, and $\operatorname{lc}(w_i) = 1$ for every i.*

Proof. In the initial seed, the values are the standard basis vectors $e_1, \dots, e_n$, and all leading coefficients are one. Suppose the assertion holds for the current seed, with exchange matrix $(b_{i\ell})$. Its basis property gives $v_j \neq 0$ for every mutation direction j. Lemma 3.4 then gives

$$\nu(1 + w_j) = \min(0, v_j), \qquad \operatorname{lc}(1 + w_j) = 1.$$

Applying (1) yields

$$v_j' = -v_j, \qquad v_i' = v_i + [b_{ij}]_+ v_j - b_{ij} \min(0, v_j) \quad (i \neq j). \tag{5}$$

The minimum is either 0 or v_j. Thus (5) adds integer multiples of v_j to the other basis vectors and changes the sign of v_j. This transformation has determinant -1 and preserves the integral basis property. The multiplication rules in Lemma 3.4, applied also to inverses and integer powers, show that every new leading coefficient is one. Induction proves the assertion. □

Proof of Theorem 3.1. Suppose X and Y are algebraically dependent. Choose a seed containing X and relabel its coordinates so that $z_1 = X$. By Lemma 2.1, reversing a mutation path and then following another path reaches a seed containing Y. Lemma 3.3 gives

$$Y = R(X) = \frac{aX + b}{cX + d}, \qquad ad - bc \neq 0.$$

Use the lexicographic order on $\mathbb{Z}^n$, comparing two vectors at their first unequal component. Then $\nu(X) = e_1 > 0$. If $R(0)$ is finite and nonzero, both b and d are nonzero. The numerator and denominator have value zero, so $\nu(Y) = 0$, contrary to Lemma 3.5. Consequently $R(0) \in \{0, \infty\}$.

Now reverse the entire lexicographic order and apply the construction (4) with this new order. Each finite support has a least element, and $\nu(X) = e_1 < 0$. If $R(\infty)$ is finite and nonzero, then $a, c \neq 0$. The numerator and denominator both have value e_1, again giving $\nu(Y) = 0$. Hence $R(\infty) \in \{0, \infty\}$.

The condition $ad - bc \neq 0$ makes R an automorphism of the projective line, so $R(0)$ and $R(\infty)$ are distinct. Therefore $R(T) = \lambda T$ or $R(T) = \lambda / T$ for some $\lambda \in k^\times$. In either valuation, X, X^{-1}, and Y have leading coefficient one. Comparing leading coefficients gives $\lambda = 1$, proving (1)⇒(3). Condition (3) implies (2). Under (2), expressing Y as a rational function of X and clearing its nonzero denominator gives a nonzero polynomial relation, proving (1). □

4 Differentials and exchange pairs

4.1 Differentials

Let $d : K \to \Omega^1_{K/k}$ be the universal k-derivation. In seed coordinates $z_1, \dots, z_n$, the space $\Omega^1_{K/k}$ has K-basis $dz_1, \dots, dz_n$; the derivation is zero on k and satisfies the Leibniz rule. Wedge products below belong to $\Omega^2_{K/k} = \bigwedge^2_K \Omega^1_{K/k}$.

Corollary 4.1. *If* $\operatorname{char} k = 0$*, then for* $X, Y \in \mathscr{X}$*,*

$$dX \wedge dY = 0 \quad \Longleftrightarrow \quad Y = X \text{ or } Y = X^{-1}.$$

Proof. Suppose first that X, Y are algebraically independent. Extend them to a transcendence basis of K/k, and let F be the field generated by that basis over k. The extension K/F is finite: K is finitely generated over F, and its generators are algebraic. It is separable because the characteristic is zero.

The partial derivations with respect to X and Y on F extend to K. To see this, adjoin the finitely many algebraic generators one at a time. If an adjoined element α has minimal polynomial $f(T) = \sum_i c_i T^i$, a derivation D of the preceding field extends by

$$D(\alpha) = -\frac{\sum_i D(c_i)\alpha^i}{f'(\alpha)}.$$

Separability gives $f'(\alpha) \neq 0$. This formula makes the ideal generated by f stable under the extended derivation on the polynomial ring, so it defines a derivation on the quotient field. The two extended partial derivations take the pairs (X, Y) to $(1, 0)$ and $(0, 1)$. Applying them to any K-linear relation between dX and dY makes both coefficients zero. Thus $dX \wedge dY \neq 0$.

It follows that a vanishing wedge product forces algebraic dependence, and Theorem 3.1 gives $Y = X$ or $Y = X^{-1}$. Conversely, these identities give $dY = dX$ and $dY = -X^{-2}dX$, respectively, and hence a zero wedge product. □

4.2 Exchange pairs

Assume in this subsection that $\operatorname{char} k = 0$ and $I_{\text{mut}} = I$. Label seed positions along mutation paths by t, and denote our exchange matrix and coordinates at t by B_t and $X_{i;t}$, respectively, for $1 \leq i \leq n$. Consider the upper cluster algebra $\mathcal{U}$ over $\mathbb{Z}$ with initial exchange matrix B^T and trivial coefficient semifield $\{1\}$, whose multiplication and auxiliary addition both send $(1, 1)$ to 1. Its initial cluster variables $A_1, \dots, A_n$ are independent indeterminates, with ambient field $\mathbb{Q}(A_1, \dots, A_n)$. Write $A_{i;t}$ for the cluster variables along the same mutation paths. Explicitly, $\mathcal{U}$ is the intersection, inside this ambient field, of the Laurent polynomial rings

$$\mathbb{Z}[A_{1;t}^{\pm 1}, \dots, A_{n;t}^{\pm 1}]$$

over all seed positions t.

Across an edge $t \leftrightarrow t'$ in direction j, all A-coordinates except the jth remain unchanged, and

$$A_{j;t}A_{j;t'} = \prod_{i=1}^{n} A_{i;t}^{[(B_t^T)_{ij}]_+} + \prod_{i=1}^{n} A_{i;t}^{[-(B_t^T)_{ij}]_+}. \tag{6}$$

An empty product equals one. The ordered pair $(A_{j;t}, A_{j;t'})$ is an ordered exchange pair; its unordered version is the corresponding unordered exchange pair.

In [1, Definition 2.5], mutation of X_i in direction j uses the (j, i) matrix entry. Our formula uses the (i, j) entry, which accounts for the transpose above. Formula (2) commutes with transposition, so the X-variables in the two conventions agree along each mutation path. Moreover, B^T is skew-symmetrizable: reciprocal symmetrizing factors $1/d_i$, multiplied by a common denominator, give positive integer factors for B^T.

The standing assumptions of [1, §1.4] require a factorial base domain of characteristic zero and exchange binomials that are nonunits over the coefficient ring. The choice $\mathbb{Z}$ satisfies these conditions. Each nonconstant exchange binomial in (6) is a nonunit; a zero column gives the binomial 2, also a nonunit over $\mathbb{Z}$. Quotients of nonzero polynomials in independent initial X-variables with nonnegative integer coefficients form the universal semifield used in [1], with ordinary multiplication and addition of rational functions. Our X-mutation pattern over this semifield has initial matrix B^T in that paper's convention. By [1, Theorem 7.6(i)], there is a bijection

$$\psi : X_{j;t} \longmapsto (A_{j;t}, A_{j;t'}). \tag{7}$$

Mutation at j inverts $X_{j;t}$, so

$$\psi(X_{j;t}^{-1}) = (A_{j;t'}, A_{j;t}).$$

Finally, the embedding $\mathbb{Q}(x_1, \ldots, x_n) \hookrightarrow k(x_1, \ldots, x_n)$ preserves all equalities among these rational functions. Thus (7) defines a bijection on the set $\mathscr{X}$ in the present characteristic-zero setting.

Corollary 4.2. *Under the assumptions of this subsection, $k(X) = k(Y)$ for $X, Y \in \mathscr{X}$ if and only if $\psi(X)$ and $\psi(Y)$ have the same unordered exchange pair.*

Proof. By Theorem 3.1, equality of the fields is equivalent to $Y = X$ or $Y = X^{-1}$. Under ψ, these alternatives give the same ordered pair or the reversed pair. Conversely, two orderings of the same unordered pair are equal or reversed, and injectivity of ψ gives the corresponding identity between X and Y. □

4.3 Parallel classes

Return to the arbitrary field and mutable index set of Section 2. Algebraic independence defines an algebraic matroid on $\mathscr{X}$: a finite subset is independent when its elements are algebraically independent over k, and an arbitrary subset is independent when each finite subset is independent. Every coordinate is transcendental over k, so this matroid has no loops. The parallel class of X consists of X and the distinct elements that form a dependent pair with X.

Corollary 4.3. *The parallel class of $X \in \mathscr{X}$ is*

$$\mathscr{X} \cap \{X, X^{-1}\}.$$

If X occurs in a mutable position of a seed, this class has exactly two elements.

Proof. The description follows from Theorem 3.1. Mutation in a mutable position containing X produces X^{-1}. These elements are distinct because X is transcendental over k. □

5 Examples

5.1 A noninvolutive mutation rule

The question [2] displays the rule

$$x_i' = x_i\bigl(1 + x_j^{-[b_{ij}]_+}\bigr)^{-b_{ij}}, \qquad i \neq j,$$

with $[a]_+ = \max(a, 0)$. Over $\mathbb{C}(x_1, x_2)$, take

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad j = 2.$$

This displayed rule gives $x_1' = 2x_1$, which is algebraically dependent on x_1 and distinct from both x_1 and x_1^{-1}. Applying the same rule a second time in direction 2 gives $2x_1/(1 + x_2)$. Thus this rule differs from the involutive mutation (1), which is the formula used in Theorem 3.1.

5.2 Specializing a frozen coordinate

Take $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $I_{\mathrm{mut}} = \{1\}$. Mutation gives

$$x_1' = x_1^{-1}, \qquad x_2' = x_2(1 + x_1).$$

These expressions are regular along $x_2 = 1$, and substitution gives $x_2' = 1 + x_1$. The specialized coordinate is algebraically dependent on x_1 and distinct from x_1 and x_1^{-1}. The independent initial frozen coordinate in Theorem 3.1 therefore has an essential role when the set $\mathscr{X}$ includes frozen coordinates.

5.3 Differentials in positive characteristic

Let $\operatorname{char} k = p > 0$ and take

$$B = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}.$$

Mutation in direction 2 produces the pair $X = x_1$ and $Y = x_1(1 + x_2)^p = x_1(1 + x_2^p)$. Hence

$$dY = (1 + x_2^p)\, dx_1, \qquad dX \wedge dY = 0.$$

On the other hand, $k(X, Y) = k(x_1, x_2^p)$ has transcendence degree two over k, so X, Y are algebraically independent. This gives a positive-characteristic failure of the differential criterion in Corollary 4.1.

5.4 A dependent triple

In $k(u, v)$, mutation in direction 1 for $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ produces $w = v(1 + u)$. The three coordinates u, v, w satisfy $w - v - uv = 0$. Each pair is algebraically independent: u, v are the initial coordinates, and

$$k(u, w) = k(u, v), \qquad k(v, w) = k(u, v),$$

as follows from $v = w/(1 + u)$ and $u = w/v - 1$. This example works over every field.

A An asymptotic proof over the complex numbers

Proof of Theorem 3.1 over $\mathbb{C}$. Suppose $X, Y \in \mathscr{X}$ are algebraically dependent, and choose seed coordinates $z_1, \dots, z_n$ with $z_1 = X$. Lemma 3.3 gives $Y = R(X)$ for a fractional linear map R. Choose positive real numbers $r_1, \dots, r_n$ that are linearly independent over $\mathbb{Q}$, and substitute $z_i = t^{r_i}$, where $t > 0$ tends to zero. The mutation formulas preserve positive coordinates, so all expressions along a finite mutation path are defined.

We prove inductively that every current coordinate has an asymptotic expression

$$w_i(t) = t^{s_i}(1 + o(1)),$$

where $o(1)$ denotes a real function tending to zero, and $s_1, \dots, s_n$ are linearly independent over $\mathbb{Q}$. Initially $s_i = r_i$. At a mutation in direction j, the exponent s_j is nonzero. Thus $1 + w_j$ has leading term 1 if $s_j > 0$ and t^{s_j} if $s_j < 0$, with coefficient one in both cases. The new exponents are

$$s_j' = -s_j, \qquad s_i' = s_i + [b_{ij}]_+ s_j - b_{ij} \min(0, s_j) \quad (i \neq j).$$

This is an invertible integer linear transformation of the exponent tuple. It preserves rational linear independence, and the leading coefficients remain one.

Along this path, X tends to zero, and Y tends to zero or infinity. Hence $R(0) \in \{0, \infty\}$. Repeating the argument with $z_i = t^{-r_i}$ makes X tend to infinity and gives $R(\infty) \in \{0, \infty\}$. Since

R is a projective automorphism, it permutes these two points. Therefore $Y = \lambda X$ or $Y = \lambda X^{-1}$ for some $\lambda \in \mathbb{C}^{\times}$. Along the first path, comparison of exponents and leading coefficients gives $\lambda = 1$. The remaining implications follow by taking generated fields and clearing denominators, as in the proof of Theorem 3.1. □

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