

CRYSTALLINE AND INFINITESIMAL POISSON STACKS

ALEX KARAPETYAN

ABSTRACT. A systematic way to construct Poisson brackets on moduli stacks in algebraic geometry is through shifted Poisson structures on derived stacks. The standard construction, due to Calaque–Pantev–Toën–Vaquié–Vezzosi, imports Gelfand–Kazhdan formal geometry to derived stacks in order to establish formal descent for polyvector fields. The result is a graded Lie algebra which we call the “crystalline” polyvector fields on a derived stack. Using instead the formal derived geometry machinery of Gaitsgory–Rozenblyum, we construct a graded Lie algebra of “infinitesimal” polyvector fields which has a direct geometric interpretation: it governs deformations of the formal shifted cotangent stack, together with its canonical exact symplectic form. Our main result is an equivalence between crystalline and infinitesimal polyvector fields on a derived stack as graded Lie algebra objects, providing a deformation-theoretic perspective on the graded Lie algebra constructed by Calaque–Pantev–Toën–Vaquié–Vezzosi.

1. INTRODUCTION

This article is part of the subject of derived symplectic and derived Poisson geometry, as pioneered by [Pan+13; Cal+17]. Specifically, in [Cal+17], the authors construct a Lie algebra (actually $\mathfrak{p}_n$ -algebra) of *polyvectors* on a derived stack. It enables them to define Poisson structures on derived stacks as something resembling Maurer–Cartan elements, initiating the study of Poisson structures in derived geometry. The nature of this construction is in some sense algebraic: the authors set up a suitable kind of descent which allows them to reduce to derived algebra, where they make the definition after a detailed study of certain Hodge-filtered de Rham complexes. Following the classical Maurer–Cartan philosophy, one expects the resulting Lie algebra to govern some sort of formal deformations of a geometric structure. In this article, we try to address the question: what is a deformation-theoretic interpretation of the construction of [Cal+17]?

Motivation from differential geometry. To explain the substance of our result, let us first orient ourselves by analogy with classical differential geometry. A Poisson structure on a smooth manifold M is a Lie bracket

$$\{-, -\}: C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M)$$

on the algebra of functions which is a derivation in each argument. A key feature of a Poisson manifold is the collection of Hamiltonian vector fields $\{f, -\}$ sweeps out a singular foliation of M , called the symplectic foliation. This foliation can be very complicated and pathological.

It is well-known that the data of a Poisson bracket is equivalent to the data of a bivector field

$$\pi \in \Gamma(M, \Lambda^2 TM)$$

satisfying the Maurer–Cartan equation $[\pi, \pi]_S = 0$. Here,

$$[-, -]_S: \Gamma(M, \Lambda^* TM) \otimes \Gamma(M, \Lambda^* TM) \longrightarrow \Gamma(M, \Lambda^* TM)[-1]$$

is the Schouten bracket of polyvector fields, which is the unique Lie bracket of degree -1 which restricts to the Lie bracket of vector fields in degree 1 and satisfies the (graded) Leibniz rule. That is, $[-, -]_S$ is itself a type of Poisson bracket, specifically a Gerstenhaber bracket. Modern (differential) Poisson geometry has the following interpretation for this phenomenon.

The philosophy is that Poisson manifolds should arise as Lagrangian submanifolds of symplectic manifolds which are treated as “desingularizations” of the symplectic foliation. Consider first the zero Poisson structure, $\pi = 0$. Here, the symplectic foliation is simply by the points of M . It naturally embeds as a Lagrangian into the cotangent bundle

$$M \rightarrow T^*M,$$

which is symplectic. The (rather trivial) symplectic foliation on M is encoded by the submersion

$$T^*M \rightarrow M,$$

together with the *additive* structure on the cotangent bundle which preserves the fibers. From the stacky perspective, this data encodes the Lagrangian zero section in the *shifted* cotangent bundle

$$M \rightarrow T^*[1]M,$$

which is a graded symplectic manifold forming a “desingularization” of (leaf space of) the symplectic foliation for the zero Poisson structure on M . The bracket $[-, -]_S$ now has a clear interpretation: it is the Poisson bracket of formal functions on the graded symplectic manifold $T^*[1]M$. Nonzero Poisson bivectors, as Maurer–Cartan elements in the resulting nilpotent dg Lie algebra, are formal deformations of the zero Poisson structure. It is natural to therefore attempt to reinterpret their deformation theory in terms of $T^*[1]M$.

The key is the following observation. The graded symplectic form ω on $T^*[1]M$ defines a Hamiltonian vector field map

$$v_-: C^\infty(T^*[1]M) = \Gamma(M, \Lambda^* TM) \rightarrow \text{VF}(T^*[1]M),$$

where we are for now suppressing the grading shifts for clarity. This map is a Lie algebra morphism and is an injection modulo locally constant functions. Thus, the Schouten bracket is entirely determined by the Lie bracket of vector fields on $T^*[1]M$, as long as we can define the subspace of Hamiltonian vector fields.

To do so, recall that $v \in \text{VF}(T^*[1]M)$ is Hamiltonian if and only if there is a function $f \in \Gamma(M, \Lambda^* TM)$ such that

$$\iota_v \omega = -df,$$

where ι_v denotes interior product. Using the fact that $\omega = d\lambda$ is *exact* and applying the Cartan homotopy formula $[d, \iota_v] = L_v$, we get

$$L_v \lambda = d(\iota_v \lambda - f).$$

Thus, a vector field v is Hamiltonian if and only if its flow preserves λ *up to an exact 1-form*, i.e. as a class in the two-term complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M).$$

Maurer–Cartan elements of the graded Lie algebra of Hamiltonian vector fields on M are, up to locally constant functions, precisely the Poisson structures on M . The upshot is:

Poisson bivectors are formal deformations of the 1-shifted cotangent bundle as an exact graded symplectic manifold.

Derived algebraic geometry. Now, let us return to the world of algebraic geometry. Suppose X is a smooth scheme over a field k of characteristic 0. There is a well-studied subject of algebraic symplectic structures: Hodge 2-forms

$$\omega \in H^0(X; \Lambda^2 \Omega_X^1)$$

such that $d\omega = 0$ in $\Lambda^3 \Omega_X^1$, and the induced map between the tangent and cotangent sheaves is an isomorphism. On the other hand, a Poisson structure on X can be defined as either a Poisson bracket on the sheaf of functions

$$\{-, -\}: \mathcal{O}_X \otimes_k \mathcal{O}_X \longrightarrow \mathcal{O}_X,$$

or a global section

$$P \in H^0(X; \Lambda^2 T_X)$$

such that $[P, P]_S = 0$, where the Schouten bracket has the same definition as in differential geometry. There is also a rich theory of algebraic Poisson structures on schemes.

Passing to derived stacks, we have a good notion of de Rham theory, namely complete Hodge-filtered derived de Rham cohomology,

$$F_H^* dR_X,$$

defined by smooth descent from affines. As pioneered by [Pan+13], it is the basis of a theory of symplectic structures in derived algebraic geometry. A 2-form on a derived stack X is a section of the quasicoherent sheaf $\Lambda^2 \mathbb{L}_X$; the condition of having vanishing de Rham differential becomes the structure of admitting a lift along the canonical map of complexes

$$F_H^2 dR_X \rightarrow R\Gamma(X; \Lambda^2 \mathbb{L}_X).$$

Finally, the nondegeneracy condition is that the induced map $T_X \rightarrow \mathbb{L}_X$ is an equivalence in the ∞ -category of quasi-coherent sheaves on X .

In contrast, a serious problem is immediately encountered in an attempt to study *Poisson* structures on derived stacks. Namely, as soon as we leave the relatively restrictive world of Deligne–Mumford stacks, the Lie bracket of vector fields not only doesn’t descend in the naive way, but isn’t even functorial. The work [Cal+17] solves this problem by instead using descent of the Lie bracket of vector fields over the space of jets, or more precisely the sheaf of principal parts together with its canonical flat connection. In practice, this means studying in detail the geometry of the map

$$p: X \rightarrow X_{\text{dR}}$$

to the de Rham stack of X , in particular the extent to which formal neighborhoods of points descend along p .

One of the accomplishments of [Cal+17] is to produce a sheaf of graded Lie algebras on X_{dR} , which we denote

$$\text{Pol}_X^{\text{crys}} \in \text{Alg}_{\text{Lie}}^{\text{gr}}(\mathcal{QCoh}(X_{\text{dR}})),$$

such that its underlying graded sheaf is $p_*(\Lambda^* T_X)$. On affines this Lie algebra reduces to the standard definition of polyvectors as multiderivations, justifying its definition. The superscript stands for “crystalline” — we will explain the reasoning for this later in the

introduction. Then, Poisson structures on X are defined to be “graded Maurer–Cartan elements.” More precisely, the space of Poisson structures on X is the mapping space

$$\mathrm{Pois}^{\mathrm{crys}}(X, 0) = \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(X_{\mathrm{dR}})}(\mathrm{ins}^2 \mathcal{O}_{X_{\mathrm{dR}}}[-1], \mathrm{Pol}_X^{\mathrm{crys}}),$$

where $\mathrm{ins}^2 \mathcal{O}_{X_{\mathrm{dR}}}[-1]$ is the trivial Lie algebra object $\mathcal{O}_{X_{\mathrm{dR}}}[-1] \in \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{QCoh}(X_{\mathrm{dR}}))$ placed in grading 2. This turns out to be the right invariant definition which recovers ordinary Poisson bivectors on smooth schemes — see [Cal+17; MS18]. We will try to provide a bit of additional intuition on this point in Theorem 4.6.

The construction. We will borrow the ideas from differential geometry outlined previously to provide an alternative construction of a graded Lie algebra structure on $p_*(\Lambda^* \mathbb{T}_X)$. As a feature, it will have a direct deformation-theoretic interpretation. Specifically, given a derived stack X with perfect cotangent complex, we define a graded Lie algebra

$$\mathrm{Pol}_X^{\mathrm{inf}} \in \mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathcal{QCoh}(X_{\mathrm{dR}}))$$

governing the deformations of the formal symplectic stack $\widehat{\mathbb{T}}^*[1]X$ over X and prove its underlying graded sheaf is equivalent to $p_*(\Lambda^* \mathbb{T}_X)$. The superscript stands for “infinitesimal” to indicate the nature of this Lie algebra as arising from the realm of D-modules.

The conceptual idea of the definition is as follows. Let X be a derived stack with perfect cotangent complex and write $F_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}} \in \mathcal{QCoh}(X_{\mathrm{dR}})$ for the complete Hodge-filtered de Rham cohomology of the canonical morphism $X \rightarrow X_{\mathrm{dR}}$. Since X_{dR} has vanishing cotangent complex, $F_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}}$ is a lift of the absolute Hodge-filtered de Rham cohomology of X to a sheaf over X_{dR} . By functoriality of $F_H^* \mathrm{dR}$, there is an action

$$\widehat{\mathrm{Aut}}(X) \curvearrowright F_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}}$$

over X_{dR} , where $\widehat{\mathrm{Aut}}(X)$ is the group of *formal automorphisms* of X over X_{dR} . The group of formal automorphisms is the object encoding deformation theory of X . Replacing X by $\widehat{\mathbb{T}}^*[1]X$ and recalling that $(\widehat{\mathbb{T}}^*[1]X)_{\mathrm{dR}} \simeq X_{\mathrm{dR}}$, we also have an action

$$\widehat{\mathrm{Aut}}(\widehat{\mathbb{T}}^*[1]X) \curvearrowright F_H^* \mathrm{dR}_{\widehat{\mathbb{T}}^*[1]X/X_{\mathrm{dR}}}.$$

The symplectic structure on $\widehat{\mathbb{T}}^*[1]X$ is *exact*, which means it defines a class in the cofiber

$$(1) \quad \mathrm{coFib} \left(F_H^0 \mathrm{dR}_{\widehat{\mathbb{T}}^*[1]X/X_{\mathrm{dR}}} \rightarrow F_H^2 \mathrm{dR}_{\widehat{\mathbb{T}}^*[1]X/X_{\mathrm{dR}}} \right),$$

on which there is an induced action of $\widehat{\mathrm{Aut}}(\widehat{\mathbb{T}}^*[1]X)$. Moreover, the whole picture is compatible with the fiberwise grading on $\widehat{\mathbb{T}}^*[1]X$ since the symplectic form is fiberwise *linear*. The object $\mathrm{Pol}_X^{\mathrm{inf}}$ is defined to be the Lie algebra of the formal group given by an appropriately defined *stabilizer* of the symplectic form as a class in (1) under the action of $\widehat{\mathrm{Aut}}(\widehat{\mathbb{T}}^*[1]X)$.

We can now give an alternative definition of Poisson structures as the mapping space

$$\mathrm{Pois}^{\mathrm{inf}}(X, 0) = \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(X_{\mathrm{dR}})}(\mathrm{ins}^2 \mathcal{O}_{X_{\mathrm{dR}}}[-1], \mathrm{Pol}_X^{\mathrm{inf}}).$$

A priori, this space is unrelated to the previously defined $\mathrm{Pois}^{\mathrm{crys}}(X, 0)$. However, our main theorem will provide an equivalence of graded Lie algebra objects

$$\mathrm{Pol}_X^{\mathrm{crys}} \simeq \mathrm{Pol}_X^{\mathrm{inf}}$$

in $\mathcal{QCoh}(X_{\mathrm{dR}})$, and therefore an equivalence of spaces

$$\mathrm{Pois}^{\mathrm{inf}}(X, 0) \simeq \mathrm{Pois}^{\mathrm{crys}}(X, 0).$$

Put differently, we prove that the deformation theory of the Poisson structures of [Cal+17] recovers the deformation theory of the exact symplectic stack $\widehat{T}^*[1]X$.

Cartan homotopy formula. To prove that the resulting object has the right underlying graded sheaf requires the knowledge that the de Rham differential is dual (in a Koszul sense) to the Lie bracket of vector fields. In our generality, that means determining the relationship between the $\widehat{\text{Aut}}(X)$ -action on $F_H^* dR_X$ over X_{dR} and the duality $\mathbb{L}_X \simeq T_X^\vee$ over X itself — a version of the Cartan homotopy formula. For that purpose, we take inspiration from [KS04] and re-interpret the Cartan formula as an action on a mapping stack of a semi-direct product of groups naturally defined in terms of the source and target.

Returning momentarily to differential geometry, let M be a manifold and let $\mathbf{R}[-1]$ be the odd line, i.e. the graded manifold over a point whose graded ring of functions is $\mathbf{R}[\eta]$ with $\deg \eta = -1$ (in particular $\eta^2 = 0$). Then we have an expression for differential forms on M in terms of (smooth) mapping spaces:

$$C^\infty(\text{Map}(\mathbf{R}[-1], M)) \simeq C^\infty(T[-1]M) \simeq \Omega^\bullet(M)$$

(with reversed grading), and the de Rham differential corresponds to a vector field of degree 1 on the mapping space. Following [KS04], we describe d in terms of group actions.

Specifically, on the mapping space

$$\text{Map}(\mathbf{R}[-1], M)$$

there are actions of two natural groups:

$$\text{Diff}(\mathbf{R}[-1]) \text{ and } \text{Map}(\mathbf{R}[-1], \text{Diff}(M)).$$

On the level of Lie algebras, it is easy to see that the latter acts by Lie derivatives L_v (corresponding to the base) and interior products ι_v (corresponding to the nilpotent fibers) by vector fields $v \in \text{VF}(M)$. On the other hand, the Lie algebra of vector fields on $\mathbf{R}[-1]$ is spanned by two elements:

$$\frac{\partial}{\partial \eta} \text{ and } \eta \frac{\partial}{\partial \eta}.$$

One can see that $\eta \frac{\partial}{\partial \eta}$ governs the *degree* of differential forms, in that it acts by multiplication by p on p -forms. On the other hand, the derivation of degree 1 on $\Omega^\bullet(M)$ induced by $\frac{\partial}{\partial \eta}$ is precisely the de Rham differential. Finally, a computation reveals that the Cartan homotopy formula

$$[d, \iota_v] = L_v$$

on $\Omega^\bullet(M)$ is equivalent to the statement that the actions of the two groups above extend to an action of the *semidirect product*

$$\text{Diff}(\mathbf{R}[-1]) \ltimes \text{Map}(\mathbf{R}[-1], \text{Diff}(M)).$$

This idea lends itself to import into derived geometry, and indeed our statement which helps identify the underlying sheaf of the Lie algebra $\text{Pol}_X^{\text{inf}}$ parallels the above discussion. Working in formal moduli problems, M becomes a derived stack X , diffeomorphism groups are replaced by groups of formal automorphisms, and the role of $\mathbf{R}[-1]$ is played by the derived stack referred to as the *graded circle*:

$$S_{\text{gr}}^1 = [\mathbf{B}\mathbf{G}_a/\mathbf{G}_m] \in \text{dSt}/\mathbf{B}\mathbf{G}_m,$$

so named because $\mathbf{B}\mathbf{G}_a$ is the affinization of the constant derived stack S^1 .

What is infinitesimal here? We choose to distinguish the two a priori different definitions of polyvectors with the words “crystalline” and “infinitesimal”, which deserves explanation. The construction of Pol^{crys} fundamentally involves studying the derived algebra of modules over the complete Hodge-filtered de Rham algebra

$$F_H^* dR_{R/\mathbb{k}}$$

for a map of derived rings $\mathbb{k} \rightarrow R$. It is well-known that in characteristic p , the underlying object $F_H^0 dR_{R/\mathbb{k}}$ computes crystalline cohomology of $R/\mathbb{k}$; thus, the algebra of $F_H^* dR_{R/\mathbb{k}}$ has a “crystalline” nature.

The de Rham stack X_{dR} of a smooth scheme X over $\mathbf{Q}$ governs Grothendieck’s infinitesimal site of X : $(X_{\text{dR}})_{\text{ét}} \simeq X_{\text{inf}}$. Our construction of Pol^{inf} fundamentally relies on sheaf theory on X_{dR} , as studied in [GR17b], without involving modules over a de Rham algebra. Thus we chose the name “infinitesimal” to emphasize that distinguishing feature of the definition. Our main theorem states simply that $\text{Pol}^{\text{inf}} \simeq \text{Pol}^{\text{crys}}$, so that one can immediately return to the standard notation without superscripts.

We hasten to point out that [Cal+17] do also work over X_{dR} . However, as pointed out in the original work, the objects they define over X_{dR} are in fact *not* sheaves. In fact, as we will see, these objects are a certain kind of *ind-coherent sheaves* on X_{dR} .

Of course, we work over $\mathbf{Q}$, where there is *no difference* between the crystalline and infinitesimal sites. This is one reason to believe that Pol^{inf} and Pol^{crys} represent the same object. On the other hand, in general characteristics there may appear a difference. We have set up the exposition in a way which hopefully facilitates studying generalizations in this direction. For example, there are several options away from characteristic 0 for what replaces the de Rham stack.

Main results. Here, we give somewhat more precise statements of the results we prove. Recall that we write $p: X \rightarrow X_{\text{dR}}$ for the canonical map.

The first result is independently proven (in greater generality) in a paper in preparation by N. Rozenblyum, to whom we are grateful for many discussions on the topic.

Theorem 1.1 (Theorem 6.4). *Let X be a derived stack admitting a perfect cotangent complex. Let $n \in \mathbf{Z}$. There exists a formal group $\mathcal{GP}_X(n)$ over $X_{\text{dR}} \times \mathbf{BG}_m$ such that*

$$\text{Pol}_X^{\text{inf}}(n) = \text{Lie}(\mathcal{GP}_X(n)) \in \text{Alg}_{\text{Lie}}^{\text{gr}}(\text{QCoh}(X_{\text{dR}}))$$

governs the deformation theory of $\widehat{\mathbf{T}}^[n+1]X$ as an exact $(n+1)$ -symplectic derived stack. Moreover, the underlying graded sheaf satisfies*

$$\text{Forget}(\text{Pol}_X^{\text{inf}}(n)) \simeq p_* \text{Sym}_X^*(\mathbb{T}_X[-n-1])[n+1]$$

in $\text{GrQCoh}(X_{\text{dR}})$.

As we have already mentioned, the last point of Theorem 6.4 requires a version of Cartan calculus adapted to the objects $\widehat{\text{Aut}}(X)$ and $F_H^* dR_X$. A general statement of this form is proved in an upcoming paper of Brav–Rozenblyum, from which ours can be deduced directly. We will provide an independent proof which will be sufficient for our setup. It for the most part does not depend on the underlying ∞ -category, and so we have chosen to place it in an appendix where it can be read on its own.

In parallel with the diffeo-geometric discussion and following [TV11], the HKR theorem becomes the statement that the de Rham differential on $p_* \text{Sym}_X^*(\mathbb{L}_X[1])$ corresponds to the

action of S_{gr}^1 on

$$\widehat{T}[-1]X \simeq \widehat{\text{Map}}_{X_{\text{dR}} \times \text{BG}_m}(X_{\text{dR}} \times S_{\text{gr}}^1, X \times \text{BG}_m)$$

by “rotation of formal loops”. This action evidently commutes with the natural action of $\widehat{\text{Aut}}(X)$. The result we require is then the following extension of the HKR theorem.

Theorem 1.2 (Theorem 5.7). *Let X be a derived stack admitting a perfect cotangent complex. The $S_{\text{gr}}^1 \times \widehat{\text{Aut}}(X)$ -action on $\widehat{T}[-1]X$ lifts to an action of*

$$S_{\text{gr}}^1 \ltimes \widehat{T}[-1]\widehat{\text{Aut}}(X)$$

such that its restriction to $\widehat{T}[-1]\widehat{\text{Aut}}(X)$ is equivalent to the canonical action on $\widehat{T}[-1]X$.

The *existence* of the action in Theorem 5.7 is essentially a statement about internal mapping objects in any ∞ -category. Thus we have chosen to place that part of the proof in the Appendix, which does not make reference to the particulars of the categories and objects involved in the theorem. The geometric significance of Theorem 5.7 for us arises from the identification of the restricted action of the constituent group objects. In particular, the (-1) -shift of the formal automorphisms of X is identified with the *translation* action on the shifted tangent stack. On the algebra of global functions, the translation action is given by the duality $\mathbb{T}_X \simeq \mathbb{L}_X^\vee$ — the interior product.

Finally, the main theorem. We let

$$\text{Pol}_X^{\text{crys}}(n)$$

denote the graded Lie algebra object of $\mathcal{QCoh}(X_{\text{dR}})$ constructed in [Cal+17] which governs n -Poisson structures on X^1 . Write

$$\text{Pois}^{\text{crys}}(X, n) = \text{Map}_{\text{Alg}_{\text{Lie}}^{\text{gr}}(X_{\text{dR}})}(\text{ins}^2 \mathcal{O}_{X_{\text{dR}}}[-1], \text{Pol}_X^{\text{crys}}(n)),$$

$$\text{Pois}^{\text{inf}}(X, n) = \text{Map}_{\text{Alg}_{\text{Lie}}^{\text{gr}}(X_{\text{dR}})}(\text{ins}^2 \mathcal{O}_{X_{\text{dR}}}[-1], \text{Pol}_X^{\text{inf}}(n))$$

for the spaces of *crystalline n -Poisson structures* and *infinitesimal n -Poisson structures* on X respectively.

Theorem 1.3 (Theorem 8.1). *Let X be a derived stack with perfect cotangent complex. Let $n \in \mathbf{Z}$. There is an equivalence of Lie algebra objects*

$$\text{Pol}_X^{\text{crys}}(n) \simeq \text{Pol}_X^{\text{inf}}(n)$$

in $\mathcal{GrQCoh}(X_{\text{dR}})$. In particular, there is an equivalence of spaces

$$\text{Pois}^{\text{crys}}(X, n) \simeq \text{Pois}^{\text{inf}}(X, n).$$

The proof proceeds by comparing both sides to a “hybrid” construction: a graded Lie algebra of crystalline *vector fields* on $\widehat{T}^*[n+1]X$. The idea is to construct morphisms of Lie algebras

$$(2) \quad \text{Pol}_X^{\text{crys}}(n) \rightarrow \text{gr}^1 \text{Pol}_{\widehat{T}^*[n+1]X}^{\text{crys}}(0) \leftarrow \text{Pol}_X^{\text{inf}}(n)$$

and another Lie algebra

$$\mathfrak{h}_X(n) \rightarrow \text{gr}^1 \text{Pol}_{\widehat{T}^*[n+1]X}^{\text{crys}}(0)$$

¹To avoid confusion, we note that this is the graded sheaf on X_{dR} which [Cal+17] denote $\text{Pol}(\mathcal{B}_X/\mathcal{O}_X, n+1)[n+1]$.

such that both maps in (2) lift to $\mathfrak{h}_X(n)$, and moreover

$$\mathrm{Pol}_X^{\mathrm{crys}}(n) \simeq \mathfrak{h}_X(n) \simeq \mathrm{Pol}_X^{\mathrm{inf}}(n).$$

The Lie algebra $\mathfrak{h}_X(n)$ is defined analogously to $\mathrm{Pol}_X^{\mathrm{inf}}(n)$ but using the crystalline framework. As such, it is constructed using cofibrant models, and the comparison involves choosing differential graded Lie algebroid models for the formal moduli problems involved.

Outline. In Section 2, we begin with algebraic preliminaries. The material here is largely standard and serves as a convenient reference for what is to come. Section 3 goes on to outline the necessary background on derived algebraic geometry. Aside from material studied in detail in [GR17b], it also summarizes the key features of the perspective on formal moduli problems from [Nui19b]. In Section 4, we introduce the point of view on derived de Rham theory and symplectic structures [Pan+13] that we will take. Section 5 goes on to establish a version of the Cartan homotopy formula for derived stacks from this perspective. It crucially relies on a basic category-theoretic result, which we deduce from existing results of [Lur] in Section A. We give the definition of infinitesimal polyvectors in Section 6 and use the material on Cartan calculus to justify it. In the short Section 7, we remind the reader of the main definitions of [Cal+17] and orient it within the world of ind-coherent sheaves. Finally, Section 8 is dedicated to the statement and proof of the main theorem. It contains a detailed outline of the proof itself.

Relationship to other work. The foundational work [Cal+17] is, of course, what allowed for the ideas of this article to be developed. The construction of what we call infinitesimal polyvectors was discovered by and will appear in an upcoming work of Nick Rozenblyum. It is moreover based on a paper in preparation by Brav–Rozenblyum on a general approach to Cartan calculus, of which we develop a special case here. We would also like to draw the attention of the reader to the two preprints [Tom25; Tom26], in which the author proves a complementary theorem: crystalline Poisson structures on a derived stack X are equivalent to Lagrangian thickenings of X . Lagrangian thickenings of X can be interpreted as formal symplectic groupoids exhibiting the deformations classified by infinitesimal Poisson structures — indeed, they are the correct analogue in formal derived geometry of the “desingularizations” of symplectic foliations discussed in the introduction.

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2. PRELIMINARIES: ALGEBRA

2.1. Assumptions and notation. Fix a field k of characteristic 0 throughout (one can safely assume $k = \mathbf{C}$). Gradings are cohomological unless otherwise indicated; in particular,

$\pi_n = H^{-n}$ for linear objects. All categorical notions will be implicitly ∞ -categorical, unless otherwise indicated. We use calligraphic letters to denote ∞ -categories, and any 1-categories will be denoted in Roman letters. Set-theoretical issues will not exceed those of [Lur], so we leave sizes of categories implicit.

We mostly use the conventions of [GR17a; GR17b] for derived (pre)stacks and sheaves on them. In particular, all (derived) schemes are assumed to be separated, and all derived prestacks are assumed to be locally almost of finite type (laft), convergent and infinitesimally cohesive. We write

- $\mathcal{Pr}_k^{\mathbf{L}}$ for the symmetric monoidal category of stable presentable k -linear categories;
- $\mathcal{Mod}_k$ for the category of k -modules (cochain complexes);
- $\mathcal{CAlg}_k$ for the category of E_∞ -algebras (alias: derived commutative algebras) over k ;
- $\mathbf{aCAlg}_k$ for the category of connective E_∞ -algebras (alias: animated commutative algebras) over k ;
- $\mathcal{Aff}_k = (\mathbf{aCAlg}_k^{\text{fp}})^{\text{op}}$ for the category of (finitely presented) affine schemes;
- $\mathcal{GrC}$ for the category of graded objects in $\mathcal{C}$;
- $\mathcal{FilC}$ for the category of (decreasing) filtered objects in $\mathcal{C}$;
- $\mathbf{cFilC}$ for the category of complete (decreasing) filtered objects in $\mathcal{C}$;

When it is relevant, we refer to increasing filtrations as “cofiltrations” and treat them as filtrations in the opposite category. We generally use the notation $\mathcal{Alg}, \mathcal{CAlg}$ for algebra and commutative algebra objects in a (symmetric) monoidal category respectively. Similarly, we use the notation $\mathbf{cAlg}, \mathbf{cCAlg}$ for coalgebra and cocommutative algebra objects. We will tacitly assume coalgebras are *conilpotent*. We use the notation

$$\mathbf{cCAlg}^{\text{aug}}(\mathcal{C}) = (\mathbf{cCAlg})_{1_{\mathcal{C}}}/$$

for coaugmented cocommutative coalgebras.

2.2. Derived rings. Let $\mathcal{C} \in \mathcal{CAlg}(\mathcal{Pr}_k^{\mathbf{L}})$ be a presentable k -linear symmetric monoidal category. As mentioned in the previous section, we use the notation

$$\mathcal{CAlg}_k$$

for the category of E_∞ -algebra objects in $\mathcal{Mod}_k$ and refer to them as derived commutative algebras. Given $R \in \mathcal{CAlg}_k$, we will similarly write $\mathcal{CAlg}_R$ for the category of R -algebras, and $\mathcal{Mod}_R$ for the category of R -modules in $\mathcal{Mod}_k$.

We refer to the full subcategory $\mathbf{aCAlg}_k \subset \mathcal{CAlg}_k$ spanned by connective objects as the category of animated commutative rings. It is well-known that this category has several convenient models in terms of 1-categorical data; here, we will find the model category of (cohomologically) nonpositively graded cdgas useful later. We refer to classical (non-derived) commutative rings, viewed in $\mathbf{aCAlg}_k$, as *static*.

Definition 2.1. We say that an animated commutative ring $R \in \mathbf{aCAlg}_k$ is *finitely generated* (*finitely presented*) if

- $\pi_0 R$ is a finitely generated (finitely presented) static k -algebra,
- $\pi_i R$ is a finitely generated (finitely presented) $\pi_0 R$ -module for each i , and
- $\pi_i R = 0$ for $i \gg 0$.

We say an affine $\text{Spec } A$ is of *finite type* (*finite presentation*) if A is finitely generated (finitely presented). Let

$$\mathbf{aCAlg}_k^{\text{fg}}, \mathbf{aCAlg}_k^{\text{fp}}$$

denote the full subcategory spanned by finitely generated (finitely presented) animated commutative rings. The category of *affine (derived) schemes* is

$$\mathcal{A}\mathrm{ff}_k = (\mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}})^{\mathrm{op}}.$$

Thus, we only deal with finitely presented affine schemes. Our stacks will by definition be defined on such affines. A piece of notation: given a morphism $f: A \rightarrow B$ of animated commutative rings, we write $N^\bullet(f) \in \mathrm{Fun}(\Delta, \mathfrak{a}\mathcal{C}\mathrm{Alg}_k)$ for its conerve (aka Amitsur complex).

Definition 2.2. The category of *derived prestacks* (or simply prestacks) is by definition the presheaf category

$$\mathrm{d}\mathcal{P}\mathrm{St}_k = \mathrm{Fun}(\mathcal{A}\mathrm{ff}_k^{\mathrm{op}}, \mathrm{Spc}) \simeq \mathrm{Fun}(\mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}}, \mathrm{Spc}).$$

A derived prestack $X: \mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}} \rightarrow \mathrm{Spc}$ is a *derived stack* if it satisfies étale descent, i.e. for every étale cover $f: A \rightarrow B$ in $\mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}}$, the natural map of spaces

$$X(A) \rightarrow \lim_{\Delta} X(N^\bullet(f))$$

is an equivalence. Let $\mathrm{dSt}_k \subset \mathrm{d}\mathcal{P}\mathrm{St}_k$ be the full subcategory spanned by derived stacks.

There is a fully faithful inclusion

$$\mathcal{A}\mathrm{ff}_k \hookrightarrow \mathrm{dSt}_k.$$

Remark 2.3. (Pre)stacks in the sense used here are related to presheaves defined on all of $\mathfrak{a}\mathcal{C}\mathrm{Alg}_k$ via left Kan extension.

Remark 2.4. We first restrict to finitely generated animated commutative rings and assume all derived stacks are left Kan extended from $\mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fg}}$ because that is required for the formalism of [GR17a; GR17b] to work. To apply some of the results of [Nui19b], we require our rings to be *coherent*. We thus impose the more severe restriction to finite presentation out of desire for simplicity, and the reader is free to replace finitely presented animated commutative rings with coherent ones where it is appropriate.

2.3. Gradings and filtrations. Here, we give a rapid recall of definitions and statements on filtered objects; see [Rak20, Sec. 3.1] for a detailed discussion. Recall that the graded category of $\mathcal{C}$ is

$$\mathrm{Gr}\mathcal{C} = \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathcal{C}),$$

where $\mathbf{Z}^{\mathrm{ds}}$ is $\mathbf{Z}$ viewed as a discrete category, i.e. a set. The filtered category is

$$\mathrm{Fil}\mathcal{C} = \mathrm{Fun}(\mathbf{Z}^{\geq *}, \mathcal{C})$$

where we now consider the integers as a poset (viewed as a 1-category). Note that our filtrations are assumed to be decreasing; at some points we work with *increasing* filtrations, and refer to them as *cofiltrations*. An important role is played by the *nonnegatively filtered category*,

$$\mathrm{Fil}^+\mathcal{C} = \mathrm{Fun}(\mathbf{N}^{\geq *}, \mathcal{C}),$$

which is in fact a full subcategory of $\mathrm{Fil}\mathcal{C}$, which can be seen by considering left Kan extension along $\mathbf{N}^{\geq *} \rightarrow \mathbf{Z}^{\geq *}$. The category $\mathrm{Fil}^+\mathcal{C}$ is equivalently described as the full subcategory spanned by objects X^* for which $X^{i+1} \rightarrow X^i$ is an equivalence for all $i < 0$. In general, we use the term *weight* to denote the filtration (grading) degree in a filtered (graded) object. We will write $X^*(j) = X^{*+j}$ for the weight shift.

There are *evaluation* functors

$$\mathrm{ev}^n: \mathrm{Gr}\mathcal{C} \rightarrow \mathcal{C}, \quad \mathrm{ev}^n: \mathrm{Fil}\mathcal{C} \rightarrow \mathcal{C}$$

given by restriction to $\{n\} \hookrightarrow \mathbf{Z}$. They both admit fully faithful left adjoints

$$\mathrm{ins}^n: \mathcal{C} \rightarrow \mathrm{Gr}\mathcal{C}, \quad \mathrm{ins}^n: \mathcal{C} \rightarrow \mathrm{Fil}\mathcal{C},$$

called *insertion* functors. Write also

$$\begin{aligned} F^{-\infty}: \mathrm{Fil}\mathcal{C} &\longrightarrow \mathcal{C} \\ X^* &\longmapsto \operatorname{colim}_{n \rightarrow -\infty} X^n \end{aligned}$$

for the underlying object functor. The functor $F^{-\infty}$ admits a right adjoint which sends an object $A \in \mathcal{C}$ to the constant filtration

$$\cdots = A = A = A = A = \cdots.$$

If $X^* \in \mathrm{Fil}^+\mathcal{C}$ then

$$F^{-\infty}X^* \simeq X^0 = \mathrm{ev}^0 X^*.$$

The associated graded functor is denoted by

$$\mathrm{gr}: \mathrm{Fil}\mathcal{C} \rightarrow \mathrm{Gr}\mathcal{C}$$

and is given by the formula

$$\mathrm{gr}^n X = \operatorname{coFib}(X^{n+1} \rightarrow X^n).$$

An object $X^* \in \mathrm{Fil}\mathcal{C}$ is *complete* if

$$\lim_{n \rightarrow \infty} X^n \simeq 0.$$

The full subcategory of $\mathrm{Fil}\mathcal{C}$ (respectively $\mathrm{Fil}^+\mathcal{C}$) spanned by all complete objects is denoted by $\widehat{\mathrm{Fil}}\mathcal{C}$ (respectively $\widehat{\mathrm{Fil}}^+\mathcal{C}$). The inclusion admits a left adjoint, called the completion functor

$$\widehat{(-)}: \mathrm{Fil}\mathcal{C} \rightarrow \widehat{\mathrm{Fil}}\mathcal{C},$$

which restricts to

$$\widehat{(-)}: \mathrm{Fil}^+\mathcal{C} \rightarrow \widehat{\mathrm{Fil}}^+\mathcal{C}.$$

Moreover, the restriction

$$\mathrm{gr}: \widehat{\mathrm{Fil}}\mathcal{C} \rightarrow \mathrm{Gr}\mathcal{C}$$

is conservative.

Definition 2.5. Write

$$\mathbb{D}^\pm = \operatorname{Sym}_k^*(k[\mp 1](1)) \in \mathrm{Gr}\mathcal{C}\mathrm{Alg}_k$$

for the free graded derived commutative k -algebra on a generator of cohomological degree ± 1 and weight 1.

The following theorem is a manifestation of Koszul duality, and in particular the fact that $(\mathbb{D}^+)^{\vee}$ is the bar construction of the Rees algebra. It can be interpreted as a result on convergence of spectral sequences — see [Ant24] for precise statements.

Theorem 2.6 (Ariotta, Raksit). *There is a symmetric monoidal equivalence*

$$\widehat{\mathcal{F}\mathrm{ilMod}}_k \rightarrow \mathrm{Mod}_{\mathbb{D}^+}(\mathcal{G}\mathrm{rMod}_k),$$

making the diagram

$$\begin{array}{ccc} \widehat{\mathcal{F}\mathrm{ilMod}}_k & \xrightarrow{\simeq} & \mathrm{Mod}_{\mathbb{D}^+}(\mathcal{G}\mathrm{rMod}_k) \\ & \searrow \mathrm{gr}^* & \downarrow \\ & & \mathcal{G}\mathrm{rMod}_k \end{array}$$

commute. There is a shear equivalence

$$\mathrm{Mod}_{\mathbb{D}^+}(\mathcal{G}\mathrm{rMod}_k) \simeq \mathrm{Mod}_{\mathbb{D}^-}(\mathcal{G}\mathrm{rMod}_k).$$

Proof. See [Ari21] and [Rak20]. □

The above theorem is a precise statement regarding the additional structure present on the associated graded object of a complete filtered object. Given $X^* \in \widehat{\mathcal{F}\mathrm{ilMod}}_k$, the composition

$$\mathrm{gr}^i X \rightarrow X^{i+1}[-1] \rightarrow \mathrm{gr}^{i+1} X[-1],$$

where the maps are the natural ones coming from the fiber sequences defining $\mathrm{gr}^* X$, defines a sequence of degree 1 maps

$$\cdots \mathrm{gr}^{i-1} X \rightarrow \mathrm{gr}^i X \rightarrow \mathrm{gr}^{i+1} X \rightarrow \cdots,$$

with all compositions being nullhomotopic — a kind of cochain complex. Theorem 2.6 states that the coherence data for these nullhomotopies, in the form of a $\mathbb{D}^+$ -module structure, precisely encodes the original complete filtration.

There is an explicit cochain model for $\mathbb{D}^+$ given simply by

$$\mathbb{D}^+ = k[\epsilon],$$

where $\deg \epsilon = 1$ and $\mathrm{wt} \epsilon = 1$. The symmetric monoidal model category of $k[\epsilon]$ -modules in $\mathrm{Gr} C^\bullet(k)$ is described in detail in [Cal+17, Sec. 1.1]; see also [TV20, Sec. 1.2–1.3]. In the cited references, these are called *graded mixed modules*². We also call them *(strict) dg $\mathbb{D}^+$ -modules*. Write

$$\mathrm{Mod}_{\mathbb{D}^+}$$

for the symmetric monoidal model category of dg $\mathbb{D}^+$ -modules over k ; write also

$$\mathrm{CAlg}_{\mathbb{D}^+}$$

for the model category of commutative algebra objects in $\mathrm{Mod}_{\mathbb{D}^+}$.

Part of the philosophy underlying this article is:

*The more fundamental objects are complete filtered modules/algebras, and
graded mixed/dg $\mathbb{D}^+$ -objects are avatars of filtrations.*

Of course, this statement is strictly about interpretations, since Theorem 2.6 states that the two categories are equivalent.

²It is also worth noting that the graded mixed objects of [Pan+13] correspond to $\mathbb{D}^-$ -modules, and are thus related by a shear.

2.4. de Rham cohomology. We will mostly follow Raksit [Rak20] and Antieau [Ant25a; Ant25b] in our presentation in this section. Let $\mathbb{k}$ be any derived commutative ring. Recall that the zeroth associated graded functor

$$\mathrm{gr}^0: \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}_{\mathbb{k}} \rightarrow \mathcal{C}\mathrm{Alg}_{\mathbb{k}}$$

admits a left adjoint.

Definition 2.7. We write $F_{\mathrm{H}}^* \mathrm{dR}_{(-)/\mathbb{k}}$ for the left adjoint of gr^0 . For R a derived commutative $\mathbb{k}$ -algebra, we refer to

$$F_{\mathrm{H}}^* \mathrm{dR}_{R/\mathbb{k}}$$

as the *(Hodge-filtered) de Rham cohomology of R over $\mathbb{k}$* . We emphasize that we always take the *completed* Hodge filtration.

Proposition 2.8 (Raksit). *The associated graded of the Hodge filtration is computed by*

$$\mathrm{gr}_{\mathrm{H}}^p \mathrm{dR}_{R/\mathbb{k}} \simeq \mathrm{Sym}_R^p(\mathbb{L}_{R/\mathbb{k}}[-1]).$$

Moreover, in the case that $\mathbb{k}$ is static and R is an animated commutative $\mathbb{k}$ -algebra, $F_{\mathrm{H}}^ \mathrm{dR}_{R/\mathbb{k}}$ agrees with the Hodge-complete derived de Rham cohomology.*

Proof. See [Ant25a; Rak20]. □

As discussed in [Rak20], Theorem 2.8 goes through with the derived ring $\mathbb{k}$ replaced by any $K^* \in \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_k$. That is, there is an adjunction

$$\mathrm{gr}^0: \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}_{K^*} \rightleftarrows \mathcal{C}\mathrm{Alg}_{K^*}: F_{\mathrm{H}}^* \mathrm{dR}_{(-)/K^*},$$

and for any K^* -algebra R ,

$$\mathrm{gr}_{\mathrm{H}}^* \mathrm{dR}_{R/K^*} \simeq \mathrm{Sym}_R^*(\mathbb{L}_{R/K^*}[-1])$$

in the category $\mathrm{Gr} \mathcal{C}\mathrm{Alg}_{K^*}$.

Definition 2.9 (Filtered underlying objects). Let $K^* \in \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_k$. The underlying object functor

$$F^{-\infty}: \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{K^*} \rightarrow \mathcal{C}\mathrm{Alg}_{F^{-\infty}K}$$

induces a symmetric monoidal functor

$$F_K^{-\infty}: \mathrm{cFil}(\widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{K^*}) \rightarrow \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{F^{-\infty}K}.$$

Informally, an object $X^{*,\star} \in \mathrm{cFil}(\widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{K^*})$ is equipped with two filtrations: one from the K^* -module structure, and an “external” one which we label by $\star$. Then

$$F_K^{-\infty} X^* \in \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{F^{-\infty}K}$$

extracts the “inner” underlying object and retains the “external” filtration.

In particular, suppose $R^* \in \mathcal{C}\mathrm{Alg}_{K^*}$. Then we get an underlying “absolute” de Rham cohomology

$$F_K^{-\infty} F_{\mathrm{H}}^* \mathrm{dR}_{R^*/K^*} \in \widehat{\mathrm{Fil}} \mathcal{C}\mathrm{Alg}_{F^{-\infty}K}.$$

For clarity, we will drop the superscripts on R^*, K^* and write

$$F_K^{-\infty} F_{\mathrm{H}}^* \mathrm{dR}_{R/K} = F_K^{-\infty} F_{\mathrm{H}}^* \mathrm{dR}_{R^*/K^*}.$$

This emphasizes which filtration “remains” after taking underlying objects.

Remark 2.10. In [Cal+17], the analogue of the “inner underlying object” functor $F_K^{-\infty}$ goes by the name Tate realization, denoted $|-|^t$. We will elaborate on this comparison in a later section.

The de Rham cohomology of more general stacks will feature heavily. As such, it is expected that we will need a symmetric monoidality statement.

Theorem 2.11 (Base change for Hodge-filtered de Rham cohomology). *Suppose given a commutative diagram*

$$\begin{array}{ccccc} \mathbb{k}_1 & \longleftarrow & \mathbb{k}_0 & \longrightarrow & \mathbb{k}_2 \\ \downarrow & & \downarrow & & \downarrow \\ R_1 & \longleftarrow & R_0 & \longrightarrow & R_2 \end{array}$$

in $\mathcal{CAlg}_k$. The natural morphism

$$F_H^* dR_{R_1/\mathbb{k}_1} \otimes_{F_H^* dR_{R_0/\mathbb{k}_0}} F_H^* dR_{R_2/\mathbb{k}_2} \longrightarrow F_H^* dR_{R_1 \otimes_{R_0} R_2 / \mathbb{k}_1 \otimes_{\mathbb{k}_0} \mathbb{k}_2}$$

in $\widehat{\mathcal{F}il}^+ \mathcal{CAlg}_k$ is an equivalence.

Proof. In this form, the statement is proved in [Ant25b, Thm. 2.8]. See also [Cal+17, Lemma 1.3.18] and [CS24, Cor. 2.8]. $\square$

We will need a particular type of descent statement for de Rham cohomology in the proof of the main theorem, which we isolate here. The key player is the reduced locus functor.

Definition 2.12. Let R be an animated commutative ring. The *associated reduced ring* is

$$R^{\text{red}} = \pi_0 R / \text{Nil}(\pi_0 R),$$

where Nil is the functor associating to a static ring its ideal of nilpotents. In particular note that R^{red} is a static commutative ring.

The following statement is well-known.

Lemma 2.13. *The functor*

$$G_a^{\text{dR}} : R \mapsto R^{\text{red}}$$

defines a ring stack on $\mathbf{aCAlg}_k$ valued in 0-truncated objects.

Observe that there is a natural map

$$R \rightarrow R^{\text{red}}$$

which is connected, i.e. induces a surjection on π_0 . Connectedness is the property underlying the following.

Proposition 2.14. *Let $R \rightarrow S_i$ be an étale covering family in $\mathbf{aCAlg}_k$ and let $S^\bullet$ denote its conerve. Then*

$$F_H^* dR_{R^{\text{red}}/R} \simeq \text{Tot } F_H^* dR_{(S^\bullet)^{\text{red}}/S^\bullet}.$$

Proof. By [Ant25b, Theorem 6.1], it suffices to show that $(R^{\text{red}}/R) \rightarrow (S_i^{\text{red}}/S_i)$ is a cover for the canonical topology on the category of pairs. This follows from Theorem 2.13. $\square$

The fundamental importance of the object $F_H^* dR_{R^{\text{red}}/R}$ arises from the following.

Theorem 2.15. *Let $R \in \mathbf{aCAlg}_k^{\text{fp}}$. Then the canonical map*

$$R \rightarrow F_H^{-\infty} dR_{R^{\text{red}}/R} \simeq F_H^0 dR_{R^{\text{red}}/R}$$

in $\mathbf{CAlg}_k$ is an equivalence.

Proof. This follows from [Bha12, Cor. 4.14]. It is also proven directly in [Cal+17, Lemma 2.2.4]. See [Ant25b, Sec. 7] for another perspective. $\square$

2.5. Lie algebras. In this section we gather definitions and structural results to do with Lie algebra objects which we will be using. See [Hei25] and [GR17b, Ch. 6] for the relevant details; we use the formalism for (symmetric) operads described in [BCN25]. Let

$$\mathbf{Com}, \mathbf{Lie}$$

denote the commutative and Lie operads in $\mathbf{Mod}_k$ respectively. Let $\mathcal{C} \in \mathbf{CAlg}(\mathcal{P}\mathbf{r}_k^{\text{L}})$. For any operad $\mathbf{a}$ in $\mathbf{Mod}_k$, the symmetric monoidal unit map

$$i: \mathbf{Mod}_k^{\otimes} \rightarrow \mathcal{C}^{\otimes}$$

defines an operad $i(\mathbf{a})$ in $\mathcal{C}$. We will drop the unit map from the notation and denote the induced operad by the same symbol. Recall that $\mathbf{Lie}$ and $\mathbf{Com}$ are Koszul dual, so that

$$\mathbf{Lie} \simeq \Lambda(\mathbf{cCom})^{\perp},$$

as operads in $\mathbf{Mod}_k$. Here, Λ denotes shift of symmetric sequences, $(-)^{\perp}$ denotes the Koszul dual operad of a cooperad, and $\mathbf{cCom}$ is the operad $\mathbf{Com}$ in $\mathcal{C}^{\text{op}}$, treated as a cooperad in $\mathcal{C}$. We will also use the n -Poisson operad

$$\mathbf{p}_n = \mathbf{Com} \circ \Lambda^n \mathbf{Lie},$$

whose algebras have Lie brackets of cohomological degree $1 - n$.

Write

$$C_*: \mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C}) \rightarrow \mathbf{cCAlg}(\mathcal{C})$$

for the cobar construction on algebras over the Koszul operad $\mathbf{Lie}$. When $\mathcal{C} = \mathbf{Mod}_k$, C_* admits an explicit model given by the restriction of the Chevalley–Eilenberg coalgebra functor to fibrant-cofibrant objects.

We will consider $\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})$ together with its Cartesian symmetric monoidal structure. In particular

$$\mathbf{Alg}(\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})) \simeq \mathbf{Mon}(\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})).$$

Let

$$\begin{aligned} \Omega^{\mathbf{Lie}}: \mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C}) &\longrightarrow \mathbf{Alg}(\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})) \\ \mathfrak{g} &\longmapsto 0 \times 0. \end{aligned}$$

Proposition 2.16. *The functor $\Omega^{\mathbf{Lie}}$ is an equivalence. Moreover, the composition*

$$\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C}) \xrightarrow{\Omega^{\mathbf{Lie}}} \mathbf{Alg}(\mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})) \xrightarrow{\text{Forget}} \mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C})$$

factors through the trivial Lie algebra functor

$$\text{triv}_{\mathbf{Lie}}: \mathcal{C} \rightarrow \mathbf{Alg}_{\mathbf{Lie}}(\mathcal{C}).$$

Proof. See [Toë13, Lemma 5.3]. $\square$

We use B^{Lie} to denote an inverse of Ω^{Lie} . Write

$$\text{bAlg}(\mathcal{C}) = \text{Alg}(\text{cCAlg}(\mathcal{C}))$$

for the category of cocommutative bialgebras in $\mathcal{C}$. Note that the monoidal structure on coalgebras is the tensor product of the underlying modules and coincides with the categorical product in $\text{cCAlg}(\mathcal{C})$.

Remark 2.17. It is conceptually helpful to keep in mind that if $\mathcal{C} \in \text{CAlg}(\text{Pr}_k^{\text{L}})$ then the inclusion

$$\text{bAlg}(\mathcal{C}) \rightarrow \text{Grp}(\text{cCAlg}(\mathcal{C}))$$

is an equivalence, so that all cocommutative bialgebras are in fact Hopf algebras. See [Hei25, Cor. 4.11] for more on this point.

We give a model-independent definition of the universal enveloping algebra, [GR17b, Ch. 6] and [Hei25]. We remark that the functor C_* admits a symmetric monoidal structure.

Definition 2.18. Let $\mathcal{C} \in \text{CAlg}(\text{Pr}_k^{\text{L}})$. The *universal enveloping algebra* functor, denoted by U , is the composition

$$\text{Alg}_{\text{Lie}}(\mathcal{C}) \xrightarrow{\Omega^{\text{Lie}}} \text{Alg}(\text{Alg}_{\text{Lie}}(\mathcal{C})) \xrightarrow{C_*} \text{bAlg}(\mathcal{C}).$$

We collect some of the main results of [GR17b, Ch. 6] and [Hei25] in one theorem for convenience of reference. Note that the former uses a slightly different (but equivalent) formalism for operads.

Theorem 2.19 (Gaitsgory–Rozenblyum; Heine). *Fix $\mathcal{C} \in \text{CAlg}(\text{Pr}_k^{\text{L}})$.*

1. (*Primitives*) *The functor U admits a right adjoint, denoted*

$$\text{prim}: \text{bAlg}(\mathcal{C}) \rightarrow \text{Alg}_{\text{Lie}}(\mathcal{C}).$$

2. (*PBW theorem*) *There is a canonical equivalence*

$$U \circ \text{triv}_{\text{Lie}} \simeq \text{Sym}$$

of functors $\mathcal{C} \rightarrow \text{Alg}(\mathcal{C})$.

3. (*Milnor–Moore theorem*) *The functor*

$$U: \text{Alg}_{\text{Lie}} \rightarrow \text{bAlg}(\mathcal{C})$$

is fully faithful.

Proof. (1) [GR17b, Ch. 6, 4.4.2], [Hei25, p. 4.14]; (2) [GR17b, Ch. 6, Thm. 5.2.4]; (3) [GR17b, Ch. 6, Thm. 4.4.6], [Hei25, Thm. 4.28]. $\square$

We record the following convention, matching that of [Cal+17; MS18].

Convention 2.20 (Graded Lie algebras). Let $\mathcal{C}$ be a stable presentable k -linear symmetric monoidal category and consider the Lie operad Lie in $\mathcal{C}$. Recall the unit map

$$i: \text{Mod}_k \rightarrow \text{GrMod}_k.$$

The operad of *graded* Lie algebras in $\text{Gr}\mathcal{C}$ is by definition the twist

$$\text{Lie}^{\text{gr}} = i(\text{Lie})(-1)$$

obtained from $i(\text{Lie})$ by assigning weight -1 to the Lie bracket. Informally, a Lie^{gr} -algebra is a graded module $\mathfrak{g}^*$ together with an antisymmetric map

$$[-, -]: \mathfrak{g}^* \otimes \mathfrak{g}^* \longrightarrow \mathfrak{g}^{*-1},$$

which satisfies the Jacobi identity up to coherent higher homotopy³. The key example arising in nature is the graded Lie algebra of polyvector fields on a smooth scheme. We write

$$\mathcal{A}lg_{\text{Lie}}^{\text{gr}}(\mathcal{C}) \simeq \mathcal{A}lg_{\text{Lie}}(\text{Gr}\mathcal{C})$$

for the category of Lie^{gr} -algebras in $\mathcal{C}$.

Remark 2.21. A conceptual reason weight -1 Lie brackets arise in the subject is the fact automorphisms of a smooth scheme should be thought of as having weight 1. This is clear on the level of neighborhoods of points: module automorphisms are linear.

Finally, we introduce derivations and multiderivations. As they are defined by operadic formulas, the definition must pass through cofibrant models, which are summarized before giving the definition. For more details, see [Cal+17, p. 1.4].

Let $\mathcal{K}$ be a cofibrant commutative dg algebra over k , and let $\mathcal{R} \rightarrow \mathcal{A}$ be a cofibration of cofibrant commutative dg $\mathcal{K}$ -algebras. Then the tangent space

$$T_{\mathcal{A}/\mathcal{R}} = \text{Hom}_{\mathcal{A}}(\Omega_{\mathcal{A}/\mathcal{R}}^1, \mathcal{A})$$

is equipped with the structure of a dg Lie algebra over $\mathcal{R}$, given by the restriction of the commutator in $\text{End}_{\mathcal{R}}(\mathcal{A})$. Moreover, this Lie structure is invariant under equivalences of the triple $(\mathcal{K}, \mathcal{R}, \mathcal{A})$. That is, given any other cofibrant commutative dg algebra $\mathcal{K}'$ and cofibration of commutative dg $\mathcal{K}'$ -algebras $\mathcal{R}' \rightarrow \mathcal{A}'$ such that $\mathcal{K}' \simeq \mathcal{K}$ and $(\mathcal{R}', \mathcal{A}') \simeq (\mathcal{R}, \mathcal{A})$ as pairs of $\mathcal{K}$ -algebras, we have an equivalence of dg Lie algebras $T_{\mathcal{A}'/\mathcal{R}'} \simeq T_{\mathcal{A}/\mathcal{R}}$.

For $\mathcal{K}, \mathcal{R}, \mathcal{A}$ as above and $n \in \mathbf{Z}$, the symmetric algebra

$$\text{Sym}_{\mathcal{A}}^*(T_{\mathcal{A}/\mathcal{R}}[-n-1])$$

acquires the structure of a graded $\mathfrak{p}_{n+2}$ -algebra by extending the Lie bracket of derivations via the Leibniz rule; the grading is the symmetric algebra grading. This graded $\mathfrak{p}_{n+2}$ -algebra is also invariant under equivalences of cofibrant triples $(\mathcal{K}, \mathcal{R}, \mathcal{A})$, since the Lie bracket of derivations is.

Finally, observe that the same considerations apply when $\mathcal{K}$ (and therefore $\mathcal{R}, \mathcal{A}$) is equipped with an additional grading, complete filtration or dg $\mathbb{D}^+$ -module structure.

Definition 2.22. Let $\mathcal{C}$ be one of $\text{Mod}_k, \text{GrMod}_k, \widehat{\text{FilMod}}_k, \text{Mod}_{\mathbb{D}^+}$ and fix a base commutative algebra $\mathbb{k} \in \mathcal{C}\mathcal{A}lg(\mathcal{C})$. Let $R \rightarrow A$ be a morphism in $\mathcal{C}\mathcal{A}lg(\mathcal{C})$. We define the *Lie algebra of derivations of A/R* by choosing any cofibrant representative $(\mathcal{K}, \mathcal{R}, \mathcal{A})$ of the triple $(\mathbb{k}, R, A)$ and letting

$$\text{Der}(A/R) = (T_{\mathcal{A}/\mathcal{R}}, [-, -]),$$

where $[-, -]$ is the dg Lie structure defined in the paragraph above. Similarly, we define the *graded $\mathfrak{p}_{n+2}$ -algebra of $(n+1)$ -derivations of A/R* , $\text{MDer}_{\mathfrak{p}}(A/R, n+1)$, to be

$$\text{MDer}_{\mathfrak{p}}(A/R, n+1) = \text{Sym}_{\mathcal{A}}^*(\text{Der}(\mathcal{A}/\mathcal{R})[-n-1]) \in \mathcal{A}lg_{\mathfrak{p}_{n+2}}^{\text{gr}}(\text{Mod}_R).$$

Finally, the *graded Lie algebra of $(n+1)$ -derivations of A/R* is by definition

$$\text{MDer}(A/R, n+1) = \text{MDer}_{\mathfrak{p}}(A/R, n+1) \in \mathcal{A}lg_{\text{Lie}}^{\text{gr}}(\text{Mod}_R).$$

In [Cal+17], the graded $\mathfrak{p}_{n+2}$ -algebra $\text{MDer}(\mathcal{A}/\mathcal{R}, n+1)$ is denoted by $\text{Pol}(\mathcal{A}/\mathcal{R}, n+1)$, for “polyvectors”. We choose to reserve the notation Pol for when we pass to the geometric setting in an attempt to keep a clear distinction between several different objects, any of which may be called “polyvectors”.

³This operad is Koszul dual to the commutative operad with multiplication of weight 1.

3. PRELIMINARIES: GEOMETRY

Here we gather key facts about the geometry of stacks which we will use. See [GR17a; GR17b] for more details. Recall all our stacks are defined on the category of finitely presented animated commutative rings.

3.1. The de Rham stack and formal moduli problems. Let $X \in \mathbf{dPSt}_k$.

Definition 3.1. The *de Rham stack* of X is the prestack given by

$$X_{\mathrm{dR}}(R) = X(\mathbf{G}_a^{\mathrm{dR}}(R)) = X(R^{\mathrm{red}}).$$

The prestack X_{dR} is a stack for the étale topology if X is, which follows from Theorem 2.13⁴. The quotient map $\mathbf{G}_a \rightarrow \mathbf{G}_a^{\mathrm{dR}}$ induces a canonical projection

$$p: X \rightarrow X_{\mathrm{dR}}.$$

By convention, we will always denote this morphism by p .

The functor

$$(-)_{\mathrm{dR}}: \mathbf{dSt}_k \rightarrow \mathbf{dSt}_k$$

admits a left adjoint, called the *reduced stack functor* and denoted

$$(-)_{\mathrm{red}}: \mathbf{dSt}_k \rightarrow \mathbf{dSt}_k.$$

It satisfies $(\mathrm{Spec} R)_{\mathrm{red}} \simeq \mathrm{Spec}(R^{\mathrm{red}})$.

Definition 3.2. Let $f: X \rightarrow Y$ be a morphism of stacks. Then f is a *nil-isomorphism* if the induced map

$$f_{\mathrm{dR}}: X_{\mathrm{dR}} \rightarrow Y_{\mathrm{dR}}$$

is an equivalence. We note that f_{dR} is an equivalence if and only if f_{red} is an equivalence.

Definition 3.3. Let $f: X \rightarrow Y$ be a morphism of stacks admitting cotangent complex. We say that f is a *formal moduli problem under X* if f is a nil-isomorphism. By the standard abuse of terminology, we will say that Y is a formal moduli problem under X , with the morphism f understood. Write

$$\mathcal{FMP}_{X/} \subset \mathbf{dSt}_{X/}$$

for the full subcategory of the under-category of X spanned by formal moduli problems $f: X \rightarrow Y$ under X .

Definition 3.4. Let $\pi: \mathcal{X} \rightarrow X$ be a morphism of stacks with cotangent complex. We say that π is a *formal moduli problem over X* if the following two conditions hold:

- π is a nil-isomorphism;
- for any affine scheme S , the base change

$$\mathcal{X}_S^\wedge = \mathcal{X} \times_X S$$

has a cotangent complex and has the property that $(\mathcal{X}_S^\wedge)_{\mathrm{red}}$ is equivalent to a quasi-compact scheme.

⁴We also have that X_{dR} is laft if X is [GR17b, Ch. 4, 1.1.4].

Write

$$\mathcal{FMP}_{/X} \subset \mathrm{dSt}_{/X}$$

for the full subcategory of the over-category of X spanned by the formal moduli problems over X . Write also

$$\mathcal{FMP}_{/X,*} = (\mathcal{FMP}_{/X})_{X/}$$

for the category of *pointed* formal moduli problems over X .

Given an object $Y \in \mathcal{FMP}_{X/}$, we have automatically that X is a formal moduli problem over Y . We will often have a fixed stack X and work with the categories $\mathcal{FMP}_{X/}$ and $\mathcal{FMP}_{/X_{\mathrm{dR}}}$.

3.2. Sheaf theory. We give a brief review of the basic structure of ind-coherent sheaves that we will use from [GR17a; GR17b]. For an animated commutative ring $R \in \mathbf{aCAlg}_k^{\mathrm{fp}}$, the category of quasi-coherent sheaves on the affine scheme $\mathrm{Spec} R$ is by definition the symmetric monoidal category

$$\mathcal{QCoh}(\mathrm{Spec} R) = \mathrm{Mod}_R.$$

It is contravariantly functorial in $\mathrm{Spec} R$: for any map $f: \mathrm{Spec} R_0 \rightarrow \mathrm{Spec} R_1$, there is a colimit-preserving symmetric monoidal functor

$$f^*: \mathcal{QCoh}(\mathrm{Spec} R_1) \rightarrow \mathcal{QCoh}(\mathrm{Spec} R_0),$$

namely $f^* = - \otimes_{R_1} R_0$. We also have the quasi-coherent pushforward

$$f_*: \mathcal{QCoh}(\mathrm{Spec} R_0) \rightarrow \mathcal{QCoh}(\mathrm{Spec} R_1),$$

which is right adjoint to f^* .

As is well-known, $\mathcal{QCoh}(\mathrm{Spec} R)$ has the property that compact objects and dualizable objects coincide. The full subcategory of compact objects in $\mathcal{QCoh}(\mathrm{Spec} R)$ is denoted

$$\mathrm{Perf}(\mathrm{Spec} R) = \mathrm{Perf}_R,$$

and its objects are called *perfect complexes over R* . Perfect complexes are preserved by base change, and we have

$$\mathcal{QCoh}(\mathrm{Spec} R) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathrm{Spec} R)).$$

A quasi-coherent sheaf $\mathcal{F} \in \mathcal{QCoh}(\mathrm{Spec} R)$ is *coherent* if $\pi_i \mathcal{F}$ is a finitely generated module over the static ring $\pi_0 R$, and $\pi_i \mathcal{F} = 0$ for all but finitely many $i \in \mathbf{Z}$. The full subcategory of $\mathcal{QCoh}(\mathrm{Spec} R)$ spanned by coherent sheaves is denoted $\mathrm{Coh}(\mathrm{Spec} R)$ or sometimes $\mathrm{Coh}(R)$. If R is static, $\mathrm{Coh}(R)$ coincides with the derived category of ordinary coherent modules over R . There is a natural inclusion functor

$$(3) \quad \mathrm{Perf}(\mathrm{Spec} R) \rightarrow \mathrm{Coh}(\mathrm{Spec} R)$$

which is an equivalence when R is a smooth static ring. The category of *ind-coherent sheaves* on $\mathrm{Spec} R$ is by definition

$$\mathrm{IndCoh}(\mathrm{Spec} R) = \mathrm{Ind}(\mathrm{Coh}(\mathrm{Spec} R)).$$

It is contravariantly functorial in the affine scheme: for any map $f: \mathrm{Spec} R_0 \rightarrow \mathrm{Spec} R_1$, there is a colimit-preserving functor

$$f^!: \mathrm{IndCoh}(\mathrm{Spec} R_1) \rightarrow \mathrm{IndCoh}(\mathrm{Spec} R_0).$$

There is a symmetric monoidal structure on $\mathrm{IndCoh}(\mathrm{Spec} R)$, which we denote simply by $\otimes$ (in [GR17a], this is denoted $\otimes^!$). The *dualizing sheaf* of $S = \mathrm{Spec} R$

$$\omega_S = \omega_R \in \mathrm{IndCoh}(S)$$

is by definition the unit of the symmetric monoidal structure. Equivalently, $\omega_R = \pi^!(k)$, where $\pi: \mathrm{Spec} R \rightarrow \mathrm{Spec} k$ is the structure map (note that k is coherent over itself). For S a smooth classical scheme of dimension d , ω_S is represented by the usual dualizing sheaf $\Omega_S^d[-d]$.

There is a symmetric monoidal functor

$$\Upsilon: \mathrm{QCoh}(\mathrm{Spec} R) \rightarrow \mathrm{IndCoh}(\mathrm{Spec} R),$$

given by the ind-extension of (3). It is fully faithful and preserves compact objects (recall R is finitely presented). For any map f in $\mathcal{A}\mathrm{ff}_k$, there is a natural equivalence

$$\Upsilon \circ f^* \simeq f^! \circ \Upsilon.$$

Let $X \in \mathrm{dSt}_k$. By definition, the (symmetric monoidal) category of *quasicoherent sheaves on X* is

$$\mathrm{QCoh}(X) = \lim_{S \in \mathcal{A}\mathrm{ff}/X} \mathrm{QCoh}(S),$$

where the limit is taken with respect to the base change functors. For $f: X \rightarrow Y$ a morphism of derived stacks, the colimit-preserving pullback functor

$$f^*: \mathrm{QCoh}(Y) \rightarrow \mathrm{QCoh}(X)$$

admits a right adjoint which may not preserve colimits. We denote it by

$$f_* = \Gamma_Y: \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y).$$

One of the main results of [GR17a; GR17b] is the construction of the (symmetric monoidal) category of ind-coherent sheaves on X :

$$\mathrm{IndCoh}(X) = \lim_{S \in \mathcal{A}\mathrm{ff}/X} \mathrm{Ind}(\mathrm{Coh}(S)),$$

where the limit is with respect to $!$ -pullback, together with a plethora of structure on it. Denote by ω_X the symmetric monoidal unit. There is a symmetric monoidal functor

$$\Upsilon: \mathrm{QCoh}(X) \rightarrow \mathrm{IndCoh}(X)$$

taking $\mathcal{O}_X$ to ω_X , which is fully faithful and intertwines $*$ -pullback of quasicoherent sheaves with $!$ -pullback of ind-coherent sheaves.

An advantage of IndCoh is it is better suited to dealing with *right* adjoints of pushforward functors. For us, this will take the following guise. Suppose $\pi: X \rightarrow Y$ is a formal moduli problem under X . Then by [GR17b, Ch. 3, Thm. 5.4.3], there is an adjunction

$$\pi_!: \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(Y): \pi^!$$

in stable presentable k -linear categories, where the right adjoint $\pi^!$ is symmetric monoidal, hence $\pi_!$ is oplax symmetric monoidal. In particular, the oplax symmetric monoidal structure on $\pi_!$ induces a functor

$$\pi_!: \mathrm{cCAlg}(\mathrm{IndCoh}(X)) \rightarrow \mathrm{cCAlg}(\mathrm{IndCoh}(Y)).$$

Warning: in loc. cit., the functor $\pi_!$ is denoted π_*^{IndCoh} . We reserve the notation π_* for the (in general discontinuous) *right* adjoint of $\pi^!$ to keep “ind-coherent notation” distinct from “quasicoherent notation”.

Definition 3.5. Let $\pi: X \rightarrow Y$ be a formal moduli problem under X . The *coalgebra of distributions on X/Y* is

$$\mathcal{D}_{X/Y} = \pi_! \omega_X \in \mathrm{c}\mathcal{C}\mathrm{Alg}(\mathrm{IndCoh}(Y)).$$

When $\pi = p: X \rightarrow X_{\mathrm{dR}}$, write

$$\mathcal{D}_X = \mathcal{D}_{X/X_{\mathrm{dR}}}.$$

Suppose moreover that π is pointed, i.e. is equipped with a section $Y \rightarrow X$. Then $\mathcal{D}_{X/Y}$ is naturally equipped with an augmentation

$$\eta: \omega_Y \rightarrow \mathcal{D}_{X/Y},$$

lifting it to an object $\mathcal{D}_{X/Y}^a \in \mathrm{c}\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}(\mathrm{IndCoh}(Y))$, and we write

$$\mathcal{D}_{X/Y}^+ = \mathrm{coFib}(\eta).$$

Definition 3.6. Let $X \in \mathrm{dSt}_k$ and suppose X has a cotangent complex. The *inf-spectrum* functor

$$\mathrm{Spec}^{\mathrm{inf}}: \mathrm{c}\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \rightarrow \mathcal{F}\mathcal{M}\mathcal{P}_{/X,*}$$

is by definition the right adjoint to

$$\mathcal{D}_{X/Y}^a: \mathcal{F}\mathcal{M}\mathcal{P}_{/X,*} \rightarrow \mathrm{c}\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

We will be dealing with Lie algebra objects in ind-coherent sheaves. These are by definition objects of the category

$$\mathcal{A}\mathrm{lg}_{\mathrm{Lie}}(\mathrm{IndCoh}(X)).$$

Categories of the type $\mathrm{IndCoh}(X)$ are the prime example to which the convention on graded Lie algebras will be applied.

3.3. Gradings and filtrations, geometrized. Graded and filtered objects of sheaves over a stack will be integral to the results of this paper. Here, we record the well-known geometric formulation of graded and filtered objects, as well as verifying certain properties which we will require.

We begin with graded objects, which we view as living over the smooth stack BG_m .

Proposition 3.7. *Let $X \in \mathrm{dSt}_k$. There are equivalences of stable presentable k -linear symmetric monoidal categories*

$$\mathrm{Gr}\mathcal{Q}\mathrm{Coh}(X) \simeq \mathcal{Q}\mathrm{Coh}(X \times \mathrm{BG}_m)$$

and

$$\mathrm{Gr}\mathrm{IndCoh}(X) \simeq \mathrm{IndCoh}(X \times \mathrm{BG}_m).$$

Proof. See, for instance, [Mou21, Thm. 4.1] for the first equivalence. The second follows from loc. cit. due to

$$\mathrm{IndCoh}(X \times \mathrm{BG}_m) \simeq \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(\mathrm{BG}_m) \simeq \mathrm{IndCoh}(X) \otimes \mathcal{Q}\mathrm{Coh}(\mathrm{BG}_m),$$

using the fact that BG_m is a smooth stack. $\square$

The analogous statement for filtered sheaves requires replacing the classifying stack BG_m with the Artin stack $\mathbf{A}^1/\mathbf{G}_m$. This is the main theorem of [Mou21], going back to Simpson.

Theorem 3.8. *Let $X \in \mathrm{dSt}_k$. There is an equivalence of stable presentable k -linear symmetric monoidal categories*

$$\mathrm{FilQcoh}(X) \simeq \mathrm{Qcoh}(X \times \mathbf{A}^1/\mathbf{G}_m).$$

Moreover, it is t -exact for the neutral t -structure on $\mathrm{FilQcoh}(X)$ and the standard t -structure on the right side.

There is an extension of this theorem to *complete* filtered objects, obtained by introducing the formal substack

$$j: \widehat{\mathbf{A}}^1/\mathbf{G}_m \hookrightarrow \mathbf{A}^1/\mathbf{G}_m,$$

where $\widehat{\mathbf{A}}^1$ denotes completion at the origin. It is surely well-known to experts, however the author was unable to locate a written reference. We will make this result explicit for completeness.

Theorem 3.9 (Folklore). *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathbf{A}^1/\mathbf{G}_m) & \xrightarrow{\simeq} & \mathrm{FilMod}_k \\ j^* \downarrow & & \downarrow (-)^\wedge \\ \mathrm{Qcoh}(\widehat{\mathbf{A}}^1/\mathbf{G}_m) & \xrightarrow{\gamma} & \widehat{\mathrm{FilMod}}_k, \end{array}$$

where the right vertical arrow is the completion functor. Moreover, γ is an equivalence of symmetric monoidal categories.

To prove the theorem, we need a little bit of homological algebra — a simple case of Greenlees–May duality. Write

$$\mathrm{GrMod}_{k[t]}^{\mathrm{tors}}$$

for the category of t -torsion graded $k[t]$ -modules, and

$$\mathrm{GrMod}_{k[t]}^{\mathrm{comp}}$$

for the category of t -complete graded $k[t]$ -modules.

Lemma 3.10. *The completion functor*

$$(-)^\wedge: \mathrm{GrMod}_{k[t]} \rightarrow \mathrm{GrMod}_{k[t]}^{\mathrm{comp}}$$

induces an equivalence of categories

$$\mathrm{GrMod}_{k[t]}^{\mathrm{tors}} \simeq \mathrm{GrMod}_{k[t]}^{\mathrm{comp}}.$$

Proof. This is a special case of the main theorem of [PSY14]. However, the proof is considerably simplified for $k[t]$; see [GM92, Lemma 2.6] and [Stacks, Tag 0A6V]. The core of the proof is to approximate the (derived) completion of a module using Koszul complexes, and deduce the statement from the Milnor exact sequence for $\lim^1$. $\square$

Proof of Theorem 3.9. Let

$$\mathrm{Qcoh}(\mathbf{A}^1/\mathbf{G}_m)_{\mathbf{BG}_m} = \mathrm{Fib}(i_1^*: \mathrm{Qcoh}(\mathbf{A}^1/\mathbf{G}_m) \rightarrow \mathrm{Qcoh}(\mathbf{G}_m/\mathbf{G}_m))$$

denote the category of quasicoherent sheaves set-theoretically supported at the special point $\mathbf{BG}_m$. By the graded version of [GR23, Prop. 7.1.3], we have that

$$\mathrm{Qcoh}(\mathbf{A}^1/\mathbf{G}_m)_{\mathbf{BG}_m} \simeq \mathrm{Qcoh}(\widehat{\mathbf{A}}^1/\mathbf{G}_m),$$

induced by the pullback j^* . By Theorem 3.8 and the Rees equivalence,

$$\mathcal{QCoh}(\mathbf{A}^1/\mathbf{G}_m)_{\mathbf{BG}_m} \simeq \mathcal{GrMod}_{k[t]}^{\text{tors}}.$$

On the other hand, the Rees equivalence for complete filtered objects states

$$\widehat{\mathcal{FilMod}}_k \simeq \mathcal{GrMod}_{k[t]}^{\text{comp}}.$$

By Theorem 3.10, we deduce that γ is an equivalence. Commutativity of the diagram is deduced from the fact that the functor in Theorem 3.10 is completion. Finally, to upgrade to a symmetric monoidal equivalence we note that the symmetric monoidal structures on both sides are obtained by localization from the equivalence of Theorem 3.8. $\square$

Remark 3.11. In fact, the functor

$$\Upsilon: \mathcal{QCoh}(\widehat{\mathbf{A}}^1/\mathbf{G}_m) \rightarrow \mathcal{IndCoh}(\widehat{\mathbf{A}}^1/\mathbf{G}_m)$$

is a t-exact equivalence of categories. One can verify this by a support argument, together with the fact that $\mathbf{A}^1/\mathbf{G}_m$ is a smooth stack of dimension 0. A different way to deduce it comes from Theorem 3.8 and Theorem 3.9: both equivalences are compatible with the neutral t-structure, and then it suffices to verify the neutral structure is left complete. The latter follows immediately from the left completeness of the standard t-structure on $\mathcal{Mod}_k$.

3.4. Formal groups and automorphisms. The main reference for this subsection is [GR17b, Ch. 7].

Definition 3.12. Let $X \in \mathbf{dSt}_k$ have a cotangent complex. The category of *formal groups over X* is by definition

$$\mathbf{Grp}(\mathcal{FMP}_{/X}).$$

By [GR17b, Ch. 5, Lemma 1.6.2], the inclusion of this category into $\mathbf{Mon}(\mathcal{FMP}_{/X})$ is an equivalence — compare this with Theorem 2.17.

We will be crucially using the analogue of Lie’s Third Theorem for formal groups, namely [GR17b, Ch. 7, Thm. 3.1.4]. We collect the most relevant aspects below.

There is a naturally defined loop functor

$$\begin{aligned} \Omega_X: \mathcal{FMP}_{/X,*} &\longrightarrow \mathbf{Grp}(\mathcal{FMP}_{/X}) \\ Y &\longmapsto Y \times_X Y. \end{aligned}$$

By [GR17b, Ch. 5, Thm. 1.6.4], it is an equivalence; denote the inverse by B_X .

Definition 3.13. The *Lie differentiation functor* is defined to be

$$\text{Lie}_X = B^{\text{Lie}} \circ \text{prim} \circ \mathcal{D}_X^a: \mathbf{Grp}(\mathcal{FMP}_{/X}) \rightarrow \mathcal{Alg}_{\text{Lie}}(\mathcal{IndCoh}(X)).$$

The *Lie integration functor* is

$$\exp_X = \text{Spec}^{\text{inf}} \circ C_* \circ \Omega^{\text{Lie}}: \mathcal{Alg}_{\text{Lie}}(\mathcal{IndCoh}(X)) \rightarrow \mathbf{Grp}(\mathcal{FMP}_{/X}).$$

Note that this definition uses the fact that Spec^{inf} is limit-preserving.

Theorem 3.14 (Gaitsgory–Rozenblyum). *The functors Lie_X and $\exp_X$ are mutually inverse equivalences of categories. In particular,*

$$\mathbf{Grp}(\mathcal{FMP}_{/X}) \simeq \mathcal{Alg}_{\text{Lie}}(\mathcal{IndCoh}(X)).$$

Proof. See [GR17b, Ch. 7]. $\square$

The most important example of a formal group for us is the following.

Example 3.15. Fix a stack $\mathcal{Z}$ with cotangent complex, and let $X \rightarrow \mathcal{Z}$ be a formal moduli problem over $\mathcal{Z}$. Let

$$\widehat{\mathrm{Aut}}(X/\mathcal{Z}): (\mathrm{Aff}_k)_{/\mathcal{Z}} \rightarrow \mathrm{Spc}$$

be the prestack defined by

$$(S \rightarrow \mathcal{Z}) \mapsto \mathrm{End}_S(X_S^\wedge) \times_{\mathrm{End}_{S_{\mathrm{red}}}((X_S^\wedge)_{\mathrm{red}})} \mathrm{pt},$$

where the point maps to the identity. Recall that $X_S^\wedge = X \times_{\mathcal{Z}} S$. In the special case that $X \rightarrow \mathcal{Z}$ is the canonical map $p: X \rightarrow X_{\mathrm{dR}}$, we write

$$\widehat{\mathrm{Aut}}(X) = \widehat{\mathrm{Aut}}(X/X_{\mathrm{dR}}).$$

Write

$$\mathrm{VF}(X/\mathcal{Z}) = \mathrm{Lie}_X(\widehat{\mathrm{Aut}}(X/\mathcal{Z})) \in \mathcal{A}\mathrm{lg}_{\mathrm{Lie}}(\mathrm{IndCoh}(\mathcal{Z}))$$

for the Lie algebra. Once again in the “absolute” case, write

$$\mathrm{VF}(X) = \mathrm{VF}(X/X_{\mathrm{dR}}).$$

It is proved in [GR17b] that, for $\pi: X \rightarrow \mathcal{Z}$ a formal moduli problem over $\mathcal{Z}$, the underlying object of $\mathrm{VF}(X/\mathcal{Z})$ satisfies

$$\mathrm{Forget} \mathrm{VF}(X/\mathcal{Z}) \simeq \pi_* \mathbb{T}_{X/\mathcal{Z}} \in \mathrm{IndCoh}(\mathcal{Z}).$$

Here, π_* is the right adjoint of $\pi^!$, which is not a morphism in $\mathrm{Pr}_k^{\mathrm{L}}$.

3.5. Linear formal stacks. Let $\mathcal{F} \in \mathcal{QCoh}(X)$. There are a priori two ways of associating a “formally complete total space” of $\mathcal{F}$ as a formal moduli problem over X . In the following, we use the convention that, given

$$u: \mathrm{Spec} R \rightarrow X,$$

we denote by u_{red} the composite morphism

$$\mathrm{Spec} R^{\mathrm{red}} \rightarrow \mathrm{Spec} R \rightarrow X.$$

Definition 3.16. Let $\mathcal{F} \in \mathcal{QCoh}(X)$. We define the *complete total space of $\mathcal{F}$* as the presheaf

$$\widehat{\mathrm{Tot}}_X(\mathcal{F}): (\mathrm{Aff}_{/X})^{\mathrm{op}} \rightarrow \mathrm{Spc}$$

given by

$$(u: \mathrm{Spec} R \rightarrow X) \mapsto |u^* \mathcal{F}| \times_{|u_{\mathrm{red}}^* \mathcal{F}|} \mathrm{pt}.$$

Note that this is precisely the formal completion along the zero section of [CS24, Def. 1.19].

Definition 3.17. Assume $X \in \mathrm{Aff}_k$ and let $\mathcal{G} \in \mathrm{IndCoh}(X)$. We define the *vector prestack associated to $\mathcal{G}$* as the formal moduli problem over X

$$\mathbb{V}_X(\mathcal{G}): (\mathrm{Aff}_{/X}^{\mathrm{sm}})^{\mathrm{op}} \rightarrow \mathrm{Spc}$$

given by

$$\mathbb{V}_X(\mathcal{G}) = \mathrm{Spec}^{\mathrm{inf}}(\mathrm{cSym}(\mathcal{G})).$$

As in [GR17b, Ch. 7, 1.4], the definition extends to stacks X by descent of formal moduli problems.

Lemma 3.18. *The functor*

$$\mathrm{Lie}_X \circ \Omega_X^2 : \mathcal{FMP}_{/X,*} \rightarrow \mathcal{Alg}_{\mathrm{Lie}}(\mathrm{IndCoh}(X))$$

factors through the trivial Lie algebra functor.

Proof. Let $Y \in \mathcal{FMP}_{/X,*}$. Recall that $\Omega_X^2 Y$ is naturally an object of $\mathrm{Grp}(\mathrm{Grp}(\mathcal{FMP}_{/X}))$. We claim that there is a functorial equivalence

$$(4) \quad \mathrm{Grp}(\mathrm{Lie}_X)(\Omega_X^2 Y) \simeq \Omega^{\mathrm{Lie}} \mathrm{Lie}_X(\Omega_X Y)$$

in $\mathcal{Alg}(\mathcal{Alg}_{\mathrm{Lie}}(\mathrm{IndCoh}(X)))$. The lemma then follows from the fact that $\mathrm{Forget}_{\mathrm{Lie}} \circ \Omega^{\mathrm{Lie}}$ factors through the trivial Lie algebra functor [GR17b, Ch. 6, Prop. 1.7.2].

To prove (4), note that by [GR17b, Ch. 7, Cor. 3.2.3] and [GR17b, Ch. 7, Cor. 3.3.2],

$$\begin{aligned} \mathrm{Grp}(\mathbb{D}_X) \circ \exp_X \circ \mathrm{B}^{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{Lie}_X)(\Omega_X^2 Y) &\simeq \mathrm{Grp}(C_* \circ \Omega_X^{\mathrm{Lie}})(\Omega_X^2 Y) \\ &\simeq \mathrm{Grp}(\mathcal{D}_X^a)(\mathrm{B}_X \Omega_X^2 Y). \end{aligned}$$

Since $\mathrm{Grp}(\mathcal{D}_X^a)$ is fully faithful, we get that

$$\exp_X \mathrm{B}^{\mathrm{Lie}} \mathrm{Grp}(\mathrm{Lie}_X)(\Omega_X^2 Y) \simeq \Omega_X Y,$$

which implies (4) by applying $\Omega^{\mathrm{Lie}} \circ \mathrm{Lie}_X$. $\square$

Proposition 3.19. *Let $X \in \mathrm{dSt}_k$ have deformation theory, and let $\mathcal{E} \in \mathrm{Perf}(X)$. Then*

$$\widehat{\mathrm{Tot}}_X(\mathcal{E}) \simeq \mathbb{V}_X(\Upsilon \mathcal{E})$$

as pointed formal moduli problems over X .

Proof. We observe that $\widehat{\mathrm{Tot}}_X(\mathcal{F})$ defines an object of $\mathcal{FMP}_{/X,*}$. Indeed, the projection

$$\pi : \widehat{\mathrm{Tot}}_X(\mathcal{F}) \rightarrow X$$

is an inf-schematic nil-isomorphism by construction, and the distinguished X -point is given by $\widehat{\mathrm{Tot}}_X(0 \rightarrow \mathcal{F})$. Observe moreover that

$$\widehat{\mathrm{Tot}}_X(\mathcal{F}) \simeq \Omega_X \widehat{\mathrm{Tot}}_X(\mathcal{F}[1]).$$

Thus, by Theorem 3.18, the Lie algebra structure corresponding to $\widehat{\mathrm{Tot}}_X(\mathcal{F})$ is trivial, and we have

$$\widehat{\mathrm{Tot}}_X(\mathcal{F}) \simeq \exp_X(\mathrm{triv}_{\mathrm{Lie}}(\Upsilon \mathcal{F})).$$

Now the right hand side is easily seen to be equivalent to $\mathbb{V}_X(\Upsilon \mathcal{F})$ from the construction of [GR17b, Ch. 7, S.3], since $C_*(\mathrm{triv}_{\mathrm{Lie}}(\mathcal{F})) \simeq \mathrm{cSym}_X(\Upsilon \mathcal{F})$. $\square$

3.6. de Rham cohomology of stacks. In this short subsection, we record the definition of de Rham cohomology of arbitrary stacks with cotangent complex.

Definition 3.20 (Hodge-filtered de Rham cohomology of stacks). Let $X \rightarrow Y$ be a morphism in dSt_k such that both X, Y admit a cotangent complex. Given an affine $\mathrm{Spec} R \rightarrow Y$, write

$$X_R^Y = X \times_Y \mathrm{Spec} R.$$

We define the *Hodge-filtered de Rham cohomology of X/Y* to be

$$\mathrm{F}_H^* \mathrm{dR}_{X/Y} = \lim_{\mathrm{Spec} R \rightarrow Y} \lim_{\mathrm{Spec} A \rightarrow X_R^Y} \mathrm{F}_H^* \mathrm{dR}_{A/R} \in \mathrm{cFil}^+ \mathrm{QCoh}(Y).$$

Theorem 3.21 (Calaque–Safronov). *Let $X \rightarrow Y$ be a morphism in dSt_k such that both X, Y admit a perfect cotangent complex. The natural map*

$$\Gamma_Y(X, \mathrm{Sym}_X^*(\mathbb{L}_X[-1])) \rightarrow \mathrm{gr}_H^* \mathrm{dR}_{X/Y}$$

in $\mathrm{GrQCoh}(Y)$ is an equivalence.

Proof. In our generality, this is [CS24, Thm. 2.6]. See also [Pan+13]. $\square$

Proposition 3.22 (Base change for the de Rham algebra). *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a Cartesian square in dSt_k such that X, Y have perfect cotangent complex. Then the natural morphism

$$g^* \mathrm{F}_H \mathrm{dR}_{X/Y} \rightarrow \mathrm{F}_H \mathrm{dR}_{X'/Y'}$$

is an equivalence in $\widehat{\mathrm{FilQCoh}}(Y')$.

Proof. This is [CS24, Cor. 2.8]; it also follows from the affine statement. $\square$

3.7. Lie algebroids. Let $X \in \mathrm{dSt}_k$. We combine the philosophies of [GR17b, Ch. 8] and [Nui19b] and give a “tautological” definition of Lie algebroids. Namely, by definition a *Lie algebroid over X* is an object

$$L \in \mathcal{FMP}_{X/}.$$

Before we give a more detailed discussion in terms of derived algebra, we briefly address the relationship to *formal groupoids*.

Remark 3.23. In [GR17b], the category of Lie algebroids over X is defined to be the category of Segal groupoid objects in $\mathcal{FMP}_{X/}$, which we denote by $\mathcal{LieAbd}_X$. This is motivated by Theorem 3.14, together with the fundamental idea of differential geometry that Lie algebroids over smooth manifolds have the same relationship to Lie groupoids that Lie algebras do to Lie groups. There is a natural (relative) based loops functor,

$$\begin{aligned} \Omega_X : \mathcal{FMP}_{X/} &\longrightarrow \mathcal{LieAbd}_X \\ \mathcal{X} &\longmapsto \mathcal{X} \times_X \mathcal{X} \end{aligned}$$

For instance, if X is a smooth scheme then $\Omega_X X_{\mathrm{dR}} \simeq (X \times X)_{\Delta}^{\wedge}$, the *infinitesimal groupoid*. The result [GR17b, Ch. 5, Thm. 2.3.2] is:

The functor Ω_X is an equivalence of categories.

We will not be using the category of formal groupoids in the rest of the article, except for brief mentions in Theorem 3.36 and Theorem 4.9. See [GR17b, Ch. 5] for details.

We will now explain the key aspects of the theory which we will use. Fix $A \in \mathbf{a}\mathcal{CAlg}_k^{\text{fp}}$ and write $X = \text{Spec } A$. A summary of this subsection is given by the following diagram:

$$(5) \quad \begin{array}{ccc} & \mathcal{FMP}_{X/} & \\ \begin{array}{c} \nearrow B_A \\ \searrow \Omega_A \end{array} & & \begin{array}{c} \nwarrow F_H^* dR_{X/-} \\ \swarrow B_A \circ (\text{gr}^1)^\vee \end{array} \\ \mathcal{LA}_A & \xrightleftharpoons[(\text{gr}^1)^\vee]{C^*} & \widehat{\text{Fil}}^+ \mathcal{CAlg}_{F_H^* dR_{A/k}}^{\text{fol}} \end{array}$$

The only aspects which will be required for the main theorems and constructions are the two functors between $\mathcal{FMP}_{X/}$ and $\widehat{\text{Fil}}^+ \mathcal{CAlg}_{F_H^* dR_{A/k}}^{\text{fol}}$. However, the rest of the diagram is required to justify the properties these two functors have, hence its inclusion in this subsection. We first address the functor $F_H^* dR_{X/-}$.

Definition 3.24 ([TV25]). Let $A \in \mathbf{a}\mathcal{CAlg}_k^{\text{fp}}$ and let $C^* \in \widehat{\text{Fil}}^+ \mathcal{CAlg}_k$. We say that C^* is an *affine derived foliation over A* if

- (1) $\text{gr}^0 C \simeq A$,
- (2) $\text{gr}^1 C$ is a perfect $\text{gr}^0 C$ -module, and
- (3) the natural map

$$\text{Sym}_{\text{gr}^0 C}^*(\text{gr}^1 C) \rightarrow C^*$$

is an equivalence in $\text{GrCalg}_{\text{gr}^0 C}$.

Let K^* be an affine derived foliation over A . Write

$$\widehat{\text{Fil}}^+ \mathcal{CAlg}_{K^*}^{\text{fol}} \subset (\widehat{\text{Fil}}^+ \mathcal{CAlg}_k)_{K^* /}$$

for the full subcategory spanned by the set of objects $K^* \rightarrow C^*$ such that C^* is an affine derived foliation over A . In particular, set $\widehat{\text{Fil}}^+ \mathcal{CAlg}_A^{\text{fol}} = \widehat{\text{Fil}}^+ \mathcal{CAlg}_{\text{ins}^0 A}^{\text{fol}}$.

Proposition 3.25. *There is a functor*

$$\begin{aligned} \widehat{\text{Fil}}^+ \mathcal{CAlg}_k^{\text{fol}} : \mathbf{a}\mathcal{CAlg}_k &\longrightarrow \text{Cat} \\ A &\longmapsto \widehat{\text{Fil}}^+ \mathcal{CAlg}_A^{\text{fol}}. \end{aligned}$$

Moreover, $\widehat{\text{Fil}}^+ \mathcal{CAlg}_k^{\text{fol}}$ satisfies descent for the étale topology.

Proof. This is [TV25, Prop. 2.1.3.1]. See also [TV20, Prop. 1.2.3]. $\square$

By Theorem 3.21, we see that for any $Z \in \mathcal{FMP}_{X/}$ with $\mathbb{T}_{X/Z} \in \text{Perf}(X)$,

$$F_H^* dR_{X/Z} \in \widehat{\text{Fil}}^+ \mathcal{CAlg}_{F_H^* dR_{X/k}}^{\text{fol}}.$$

We move on to the category $\mathcal{LA}$. As mentioned, its details will not be necessary outside of this subsection, so we will not give the precise construction. Indeed, this is the main result of [Nui19a] and requires the use of semi-model categories. However, we will provide the definition of the 1-category which underlies the semi-model category of [Nui19a] in order to give a sufficient description of the functor C^* of (5).

Definition 3.26. Assume that A is represented by a cofibrant commutative dg algebra of finite presentation. A *dg Lie algebroid over A* is an A -module equipped with the structure of a dg Lie algebra L over k and a dg Lie algebra morphism

$$\rho: L \rightarrow \operatorname{Der}(A/k)$$

(the *anchor map*) such that ρ is an A -module map, and the equality

$$[e_1, ae_2] = (-1)^{\deg e_1 \cdot \deg a} a[e_1, e_2] + \rho(e_1)(a)e_2$$

holds for all $e_1, e_2 \in L$, $a \in A$. A (1-)*morphism* $\varphi: (L_0, \rho_0, [-, -]_0) \rightarrow (L_1, \rho_1, [-, -]_1)$ of dg Lie algebroids over A is a dg A -module morphism $\phi: L_0 \rightarrow L_1$ such that $\phi\rho_0 = \rho_1\phi$, and for all $x, y \in L_0$,

$$[\phi(x), \phi(y)]_1 = \phi([x, y]_0).$$

We say φ is an *equivalence* if ϕ is a quasi-isomorphism of dg A -modules. Write

$$\operatorname{LA}_A \subset (\operatorname{Alg}_{\operatorname{Lie}/k}) / \operatorname{Der}(A/k)$$

for the 1-category of dg Lie algebroids over A and their morphisms.

With A as above, the A -module $\operatorname{Der}(A/k)$ has the structure of a dg Lie algebroid over A , with the anchor map being the identity. More generally, for a map $B \rightarrow A$ of commutative dg algebras, $\operatorname{Der}(A/B)$ acquires the structure of a Lie algebroid, the *tangent Lie algebroid of A/B* . We will also denote this Lie algebroid by $T_{A/B}$.

Informally, $\mathcal{LA}_A$ is the ∞ -categorical localization of LA_A by the equivalences of dg Lie algebroids. It is equipped with forgetful functors $\mathcal{LA}_A \rightarrow \operatorname{Mod}_A$ and $\mathcal{LA}_A \rightarrow \operatorname{Alg}_{\operatorname{Lie}}(\operatorname{Mod}_k)$. The key for the present discussion is the following result.

Theorem 3.27 (Nuiten). *Let $A \in \mathbf{aCAlg}_k^{\operatorname{fp}}$. The tangent complex functor*

$$\operatorname{T}_{\operatorname{Spec} A/-} : \mathcal{FMP}_{\operatorname{Spec} A/-} \rightarrow \operatorname{Mod}_A$$

lifts to a functor

$$\Omega_A : \mathcal{FMP}_{\operatorname{Spec} A/-} \rightarrow \mathcal{LA}_A.$$

Moreover, Ω_A is an equivalence.

Proof. See [Nui19b, Thm. 5.1]. Note that in loc. cit. the theorem is valid more generally for coherent animated commutative rings. $\square$

Remark 3.28. Under the equivalence Ω_A , the tangent Lie algebroid $T_{A/B}$ corresponds to the formal moduli problem $\operatorname{Spec} A \rightarrow \mathcal{Z}_B$, where $\mathcal{Z}_B = (\operatorname{Spec} B)_A^\wedge$. In particular, $\mathcal{Z}_B \simeq \operatorname{Spec} B$ if $B \rightarrow A$ is a nil-isomorphism.

In the diagram (5), B_A denotes an inverse for Ω_A . In [Nui19b], this is written as MC_A , by analogy with the Maurer–Cartan set of dg Lie algebras.

We finally address the functor C^* . On cofibrant objects L of LA_A , it is familiar. Namely, it is explicitly computed as the dg $\mathbb{D}^+$ -algebra

$$(6) \quad C^*(L) \simeq (\operatorname{Sym}_A^*(L^\vee[-1]), \partial_{\operatorname{CE}}),$$

where

$$\partial_{\operatorname{CE}}(l_1 \cdots l_k) = \sum (-1)^i \rho(l_i) \cdot l_1 \cdots \hat{l}_i \cdots l_k + \sum (-1)^{i+j} [l_i, l_j] l_1 \cdots \hat{l}_i \cdots \hat{l}_j \cdots l_k.$$

In particular, for the tangent Lie algebroid $L = T_{A/B}$ of a formal moduli problem of the form $\operatorname{Spec} A \rightarrow \operatorname{Spec} B$, we have

$$(7) \quad C^*(T_{A/B}) \simeq F_H^* \operatorname{dR}_{A/B}.$$

Proposition 3.29. *There is a functor*

$$C^*: \mathcal{LA}_A \rightarrow \widehat{\mathcal{F}\mathrm{il}}^+ \mathcal{CAlg}_A,$$

factoring through $\widehat{\mathcal{F}\mathrm{il}}^+ \mathcal{CAlg}_A^{\mathrm{fol}}$ on perfect objects, such that for a cofibrant perfect dg Lie algebroid L , the affine derived foliation $C^(L)$ is given by (6).*

Proof. Several attributions could be given. The definition in the discussion above is precisely written down in [CG19, A.2], expanding on [Nui19b]. See also [Buc25, Sec. 4.1]. $\square$

The data of the anchor map, together with the equivalence (7), now yields a factorization of C^* through the full subcategory of complete filtered commutative algebras under $F_H^* dR_{A/k}$. We call the resulting functor

$$C^*: \mathcal{LA} \rightarrow \widehat{\mathcal{F}\mathrm{il}}^+ \mathcal{CAlg}_{F_H^* dR_{A/k}}$$

the *Hodge-filtered Chevalley–Eilenberg* functor.

Theorem 3.30 (Fu). *Let $A \in \mathfrak{aCAlg}_k$ be finitely presented. The Hodge-filtered Chevalley–Eilenberg functor C^* is fully faithful.*

Proof. See [Fu24, Thm. 4.26]. The definition of Lie algebroids in [Fu24] is (at least a priori) different, however the same proof applies once we verify that 1) C^* has a left adjoint, 2) every Lie algebroid over A is a sifted colimit of Lie algebras over A (this is [Nui19a, Cor. 3.8]), and 3) the complete filtered Chevalley–Eilenberg algebra of Lie *algebras* is a fully faithful functor. Point (3) can be deduced from the derived Milnor–Moore theorem Theorem 2.19 by identifying $C^*(\mathfrak{g})$ for $\mathfrak{g} \in \mathcal{Alg}_{\mathrm{Lie}/A}$ with a filtered cobar construction over $U(\mathfrak{g})$ as in [Buc25, Rmk. 4.1.7]. $\square$

Finally, we state the results of [Buc25] on sheaves over formal moduli problems, which will be a key ingredient in the main theorem. First, we define the relevant category of sheaves after [Ber21].

Definition 3.31 ([Ber21]). Let $A \in \mathfrak{aCAlg}_k^{\mathrm{fp}}$, $X = \mathrm{Spec} A$, and let $X \rightarrow \mathcal{Z}$ be a formal moduli problem under X . Write

$$\mathrm{IndCoh}_0^X(\mathcal{Z}) = \mathrm{IndCoh}(\mathcal{Z}) \times_{\mathrm{IndCoh}(X)} \mathcal{QCoh}(X).$$

Note that $\mathrm{IndCoh}_0^X(\mathcal{Z}) \rightarrow \mathrm{IndCoh}(\mathcal{Z})$ is fully faithful.

For our application, we will need a categorical property of IndCoh_0 , which we record here.

Proposition 3.32 (Beraldo). *Let $A \in \mathfrak{aCAlg}_k^{\mathrm{fp}}$, $X = \mathrm{Spec} A$. Let $f: X \rightarrow \mathcal{Z}$ be an object of $\mathcal{FMP}_X$. Then the category $\mathrm{IndCoh}_0^X(\mathcal{Z})$ is compactly generated by the image of the quasi-coherent pushforward functor*

$$f_*: \mathrm{Perf}(X) \rightarrow \mathrm{IndCoh}_0^X(\mathcal{Z}).$$

Proof. See [Ber21, Cor. 3.1.7]. $\square$

Definition 3.33. Let $A \in \mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}}$. Suppose C^* is an affine derived foliation over A . The category of *induced C^* -modules* is defined to be the pullback

$$\begin{array}{ccc} \mathrm{Mod}_{C^*}^{\mathrm{in}} & \xrightarrow{\quad} & \mathrm{Mod}_{C^*} \\ \downarrow & \lrcorner & \downarrow \mathrm{gr}^* \\ \mathrm{Mod}_A & \xrightarrow{-\otimes_A \mathrm{gr}^* C} & \mathrm{Mod}_{\mathrm{gr}^* C}. \end{array}$$

Note that induced modules are what [Buc25] refers to as *constant*. By [Buc25, Rmk. 2.2.2], the functor $\mathrm{Mod}_{C^*}^{\mathrm{in}} \rightarrow \mathrm{Mod}_{C^*}$ is fully faithful.

Theorem 3.34 (Buccisano). *Let $A \in \mathfrak{a}\mathcal{C}\mathrm{Alg}_k^{\mathrm{fp}}$ and let $L \in \mathcal{L}A$ be perfect. Let $\mathcal{Z} = B_A L$. There is a symmetric monoidal equivalence of categories*

$$\nu_L : \mathrm{IndCoh}_0^X(B_A L) \xrightarrow{\simeq} \mathrm{Mod}_{C^*(L)}^{\mathrm{in}}$$

such that there is a natural equivalence of functors

$$\mathrm{gr}^* \circ \nu_L(-) \simeq \mathrm{pr}_2(-) \otimes_A \mathrm{gr}^* C(L),$$

where $\mathrm{pr}_2 : \mathrm{IndCoh}_0^X(B_A L) \rightarrow \mathcal{Q}\mathrm{Coh}(X)$ is the projection.

Proof. This is the main result of [Buc25]. In particular, it follows by combining [Buc25, Sec. 3.3] with [Buc25, Thm. 4.1]. $\square$

3.8. Actions and representations. In this section, we state some preparatory definitions about actions of formal groups. First, we remind the reader of the definition of action objects.

Definition 3.35. ([GR17b, pp. VII, 5.1.1]). Let X be a derived stack admitting a cotangent complex and let $G \in \mathrm{Grp}(\mathcal{FMP}_X)$, represented by a simplicial object $B_\bullet G \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{FMP}_X)$. The *category of (left) G -modules* is the limit

$$\mathrm{IndCoh}^G(X) = \mathrm{Tot}(\mathrm{IndCoh}(B_\bullet G)),$$

taken with respect to the $!$ -pullback functors.

The following Lemma is essentially a definition and is easy for readers of [GR17b].

Lemma 3.36. *Let X be a derived stack admitting a cotangent complex and let $G \in \mathrm{Grp}(\mathcal{FMP}_X)$. There is a limit-preserving functor*

$$\mathbb{V}_X^G : \mathrm{IndCoh}^G(X) \rightarrow \mathcal{L}\mathrm{Mod}_G(\mathcal{FMP}_X)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^G(X) & \xrightarrow{\mathbb{V}_X^G} & \mathcal{L}\mathrm{Mod}_G(\mathcal{FMP}_X) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}(X) & \xrightarrow{\mathbb{V}_X} & \mathcal{FMP}_X \end{array}$$

commutes.

Proof. The functor $\mathbb{V}_X$ preserves limits. It follows that $\mathcal{F} \mapsto (\text{pr}: \mathbb{V}_X(\mathcal{F}) \rightarrow X)$ upgrades to a functor

$$\tilde{\mathbb{V}}_X^G: \text{TotJndCoh}(\mathbf{B}_\bullet G) \rightarrow \text{Fun}(\Delta^{\text{op}} \times \Delta^1, \mathcal{FMP}_{/X}),$$

whose essential image is contained in the full subcategory of Segal groupoid objects. Given a Segal groupoid object $\mathcal{G}$ in $\mathcal{FMP}_{/X}$, [GR17b, Ch. 5, 2.4.1] gives a functorial construction of $\mathbf{B}_X \mathcal{G} \in \mathcal{FMP}_{X/}$, the “formal quotient stack”. The functor $\mathbf{B}_X$ is proved to be an equivalence of categories [GR17b, Ch. 5, Thm. 2.3.2]. The functor $\mathbb{V}_X^G$ is given by $\mathbf{B}_X \circ \tilde{\mathbb{V}}_X^G$. $\square$

Fix a derived stack X with cotangent complex. In the next section, we will need to consider semidirect products of formal groups. Here, we record a convenient definition. It is applicable in some generality, but we focus on the situation at hand. Thus:

For the rest of this subsection, we denote $\mathcal{C} = \mathcal{FMP}_{/X}$.

Definition 3.37. Recall that the category $\text{Mon}(\mathcal{C}) \simeq \text{Grp}(\mathcal{C})$ of monoid objects in $\mathcal{C}$ itself has the Cartesian monoidal structure, and there is a functor

$$t = \text{Mon}(\text{triv}_{\mathcal{C}}): \text{Mon}(\mathcal{C}) \rightarrow \text{Mon}(\text{Mon}(\mathcal{C}))$$

which takes the trivial “inner” monoid structure. We write

$$\text{Act}_{\text{Mon}}(\mathcal{C}) = \mathcal{L}\text{Mod}(\text{Mon}(\mathcal{C})) \times_{\text{Mon}(\text{Mon}(\mathcal{C}))} \text{Mon}(\mathcal{C}),$$

where the fiber product is taken with respect to t . There is an evident forgetful functor

$$U_{\text{ActMon}}: \text{ActMon}(\mathcal{C}) \rightarrow \text{Mon}(\mathcal{C}) \times \text{Mon}(\mathcal{C})$$

extracting an underlying pair of monoids. We write

$$\text{Act}_{\text{Grp}}(\mathcal{C}) = U_{\text{ActMon}}^{-1}(\text{Grp}(\mathcal{C}) \times \text{Grp}(\mathcal{C})) \subset \text{ActMon}(\mathcal{C})$$

for the full subcategory on those objects both of whose underlying monoids are group objects.

Lemma 3.38. *There is a functor*

$$\mathcal{G}: \text{Arr}(\text{Grp}(\mathcal{C})) \rightarrow \text{Act}_{\text{Grp}}(\mathcal{C})$$

commuting with the forgetful functors to $\text{Grp}(\mathcal{C})^{\times 2}$.

Proof. There is an “adjoint action” functor

$$(-)^{S^1}: \text{Grp}(\mathcal{C}) \rightarrow \mathcal{L}\text{Mod}(\mathcal{C}),$$

namely the cotensoring with the circle $S^1 \in \text{sSet}$. Cotensoring with the circle commutes with limits, so $(-)^{S^1}$ lifts to an endofunctor on $\text{Grp}(\mathcal{C})$. The functor $\mathcal{G}$ can now be defined as

$$\mathcal{G}: (G \rightarrow H) \longmapsto (G \times_H H^{S^1} \rightarrow G) \in \text{Act}_{\text{Grp}}(\mathcal{C}).$$

Commutation with forgetful functors is manifest. $\square$

Lemma 3.39. *The functor $\mathcal{G}$ admits a left adjoint*

$$\mathcal{F}: \text{Act}_{\text{Grp}}(\mathcal{C}) \rightarrow \text{Arr}(\text{Grp}(\mathcal{C}))$$

such that the composition of $\mathcal{F}$ with the forgetful functor $\text{Arr}(\text{Grp}(\mathcal{C})) \rightarrow \text{Arr}(\mathcal{C})$ is naturally equivalent to

$$(G \curvearrowright H) \longmapsto (\text{pr}: G \times H \rightarrow G),$$

where pr denotes the canonical projection.

Proof. We apply the adjoint functor theorem after recalling that $(-)^{S^1}$ preserves filtered colimits because the circle is a compact object, and fiber products preserve colimits in an ∞ -topos. $\square$

Definition 3.40. The *semidirect product functor* $G \ltimes H \in \mathrm{Grp}(\mathcal{C})$ is the composition

$$\mathrm{Act}_{\mathrm{Grp}}(\mathcal{C}) \xrightarrow{\mathcal{F}} \mathrm{Arr}(\mathrm{Grp}(\mathcal{C})) \xrightarrow{s} \mathrm{Grp}(\mathcal{C}),$$

where s is the source map. We denote the image of an object $G \curvearrowright H$ under this functor by $G \ltimes H$.

4. PRELIMINARIES: SYMPLECTIC GEOMETRY

In this section we set up the de Rham theory and symplectic geometry of stacks. Nothing here is new, but we attempt to give definitions best suited for our purposes. We will define the graded circle and explain its relationship to de Rham cohomology and deformations. In the second subsection, we briefly outline the theory of derived symplectic stacks following [Pan+13; CS24].

4.1. Formal loops and the HKR theorem.

Definition 4.1. Let X be a derived stack with perfect cotangent complex. Define the *formal loop space* of X (from now on referred to simply as the *loop space*) to be the formal stack

$$\mathrm{LX} = X \times_{(X \times X)_{\Delta}^{\wedge}} X \in \mathcal{FMP}/_X.$$

It acquires the structure of a formal group over X ; we do not distinguish the group object from the underlying stack notationally.

Lemma 4.2. *Let X be a derived stack with perfect cotangent complex. There is an equivalence of derived stacks over X*

$$\mathrm{LX} \simeq \widehat{\mathrm{T}}[-1]X \simeq \widehat{\mathrm{Map}}(\mathrm{BG}_a, X).$$

Proof. We have $\mathrm{Lie}_X(\mathrm{LX}) \simeq \mathrm{T}_X[-1]$. But

$$\mathrm{Forget}_{\mathrm{Grp}}(\exp_X(\mathrm{T}_X[-1])) \simeq \widehat{\mathrm{T}}[-1]X,$$

and we conclude by Theorem 3.14. The second equivalence is proved similarly. $\square$

Definition 4.3 (Graded circle and its shear). We adopt the convention that $\mathbf{G}_m$ act on $\mathbf{G}_a$ with geometric weight 1, i.e. the coordinate t on $\mathbf{G}_a$ has weight -1 . Let

$$\mathbf{T} = [\mathrm{BG}_a/\mathbf{G}_m] \in \mathrm{Grp}(\mathcal{FMP}/_{\mathrm{BG}_m}),$$

and

$$\mathbf{T}^{\vee} = [\Omega\mathbf{G}_a/\mathbf{G}_m] \in \mathrm{Grp}(\mathcal{FMP}/_{\mathrm{BG}_m}),$$

where $\Omega\mathbf{G}_a$ denotes the based loop group of $\mathbf{G}_a$. We refer to $\mathbf{T}$ as the *graded circle*. We will also write

$$\mathrm{B}^{\mathrm{gr}}\mathbf{T} = \mathrm{B}_{\mathrm{BG}_m}\mathbf{T}; \quad \mathrm{B}^{\mathrm{gr}}\mathbf{T}^{\vee} = \mathrm{B}_{\mathrm{BG}_m}\mathbf{T}^{\vee}$$

for the corresponding objects under the equivalence

$$\mathrm{B}_{\mathrm{BG}_m} : \mathrm{Grp}(\mathcal{FMP}/_{\mathrm{BG}_m}) \simeq \mathrm{Ptd}(\mathcal{FMP}/_{\mathrm{BG}_m}).$$

Remark 4.4. The graded circle has in addition the structure of a *cogroup* object in derived stacks, which may be obtained by taking the conerve of

$$\mathrm{Spec}(k[t, s]/(t^2 - s^2)) \rightarrow \mathrm{Spec} k.$$

See [Mou24] for more on this point.

The following justifies the name of $\mathbf{T}$.

Lemma 4.5. *There are equivalences*

$$\Gamma(\mathbf{T}, \mathcal{O}_{\mathbf{T}}) \simeq \mathbb{D}^+$$

and

$$\Gamma(\mathbf{T}^\vee, \mathcal{O}_{\mathbf{T}^\vee}) \simeq \mathbb{D}^-$$

in $\mathrm{bAlg}(\mathrm{GrMod}_k)$. It follows that there are equivalences of symmetric monoidal categories

$$(8) \quad \mathcal{QCoh}(\mathbf{BT}) \simeq \mathrm{Mod}_{\mathbb{D}^+} \simeq \widehat{\mathrm{FilMod}}_k \simeq \mathrm{Mod}_{\mathbb{D}^-} \simeq \mathcal{QCoh}(\mathbf{BT}^\vee).$$

Proof. The computation of the bialgebras of functions is immediate since we are in characteristic 0. The equivalence (8) follows from descent for comodules and Theorem 2.6. $\square$

In particular, the category of representations of $\mathbf{T}$ is equivalent to the category of complete filtered k -modules. A geometric perspective on this statement is the fact that

$$(9) \quad \mathrm{B}^{\mathrm{gr}} \mathbf{T}^\vee \simeq \widehat{\mathbf{A}}^1 / \mathbf{G}_m,$$

which can be seen by direct computation. Compare also to [Cal+17, Rmk. 1.1.3].

Let

$$\mathfrak{t}^+ = \mathrm{Lie}_{\mathbf{BG}_m}(\mathbf{T}) \in \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{JndCoh}(\mathbf{BG}_m)) \simeq \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{GrMod}_k)$$

be the (graded) Lie algebra of $\mathbf{T}$. It is readily seen that

$$(10) \quad \mathfrak{t}^+ \simeq \mathrm{triv}_{\mathrm{Lie}}(\omega_{\mathbf{BG}_m}),$$

and $\mathrm{U}(\mathfrak{t}^+) \simeq \Upsilon(\mathbb{D}^+)^\vee$ by Theorem 4.5.

Remark 4.6. Let us relate the appearance of the graded circle to the use of the graded Lie algebra $\mathrm{ins}^2 k[-1]$ in defining Poisson structures. Let $\mathcal{Z}$ be a derived stack and let

$$\mathcal{H} \in \mathrm{Grp}(\mathcal{FMP}_{/\mathbf{BG}_m \times \mathcal{Z}})$$

be such that $\mathrm{Lie}(\mathcal{H}) \simeq \Upsilon \mathfrak{h}$ for some $\mathfrak{h} \in \mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathcal{QCoh}(\mathcal{Z}))$. We think of $\mathcal{H}$ as classifying formal deformations of some geometric structure over $\mathcal{Z}$, for example closed relative differential forms on a morphism $X \rightarrow \mathcal{Z}$. Accounting for our convention on graded Lie algebra structures, $\mathrm{ins}^2 k[-1]$ corresponds to the weight 0 Lie algebra object $\mathrm{ins}^1 k[-1]$ in $\mathcal{QCoh}(\mathbf{BG}_m)$, and so

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathcal{QCoh}(\mathcal{Z}))}(\mathrm{ins}^2 k[-1] \otimes_k \mathcal{O}_{\mathcal{Z}}, \mathfrak{h}) &\simeq \mathrm{Map}_{\mathrm{Grp}(\mathcal{FMP}_{/\mathbf{BG}_m \times \mathcal{Z}})}(\mathbf{T}^\vee \times \mathcal{Z}, \mathcal{H}) \\ &\simeq \mathrm{Map}_{\mathcal{FMP}_{/\mathbf{BG}_m \times \mathcal{Z}}}(\mathrm{B}^{\mathrm{gr}} \mathbf{T}^\vee \times \mathcal{Z}, \mathrm{B}_{\mathbf{BG}_m} \mathcal{H}). \end{aligned}$$

By (9), we see that

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathcal{QCoh}(\mathcal{Z}))}(\mathrm{ins}^2 k[-1] \otimes_k \mathcal{O}_{\mathcal{Z}}, \mathfrak{h}) \simeq \mathrm{Map}_{\mathcal{FMP}_{/\mathbf{BG}_m \times \mathcal{Z}}}(\widehat{\mathbf{A}}^1 / \mathbf{G}_m \times \mathcal{Z}, \mathrm{B}_{\mathbf{BG}_m} \mathcal{H}).$$

That is, the graded Lie algebra $\mathrm{ins}^2 k[-1]$ corepresents graded formal deformations of objects classified by $\mathrm{B}_{\mathbf{BG}_m} \mathcal{H}$.

We will now discuss the graded circle action on the formal cotangent stack in some more detail.

Definition 4.7. We write

$$\Gamma^{\mathbf{T}}: \mathrm{dSt}/_{\mathrm{B}^{\mathrm{gr}}\mathbf{T}} \rightarrow \mathrm{Mod}_{\mathbb{D}^+}$$

for the composition of the pushforward $\Gamma_{\mathrm{B}^{\mathrm{gr}}\mathbf{T}}$ with the equivalence (8). For a derived stack X , write $\Gamma_X^{\mathbf{T}}$ for the composition

$$\begin{aligned} \mathrm{dSt}/_{X \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}} &\longrightarrow \mathcal{Q}\mathrm{Coh}(X) \otimes \mathcal{Q}\mathrm{Coh}(\mathrm{B}\mathbf{T}) \xrightarrow{\mathrm{id} \otimes \Gamma^{\mathbf{T}}} \mathcal{Q}\mathrm{Coh}(X) \otimes \mathrm{Mod}_{\mathbb{D}^+} \\ &\simeq \mathrm{Mod}_{\mathbb{D}^+}(\mathcal{Q}\mathrm{Coh}(X)). \end{aligned}$$

Now equip $\widehat{\mathrm{T}}[-1]X$ with its fiberwise grading; that is, consider

$$\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}} = \mathbb{V}_{X \times \mathrm{B}\mathbf{G}_m}(\mathrm{ins}^1 \mathrm{T}_X[-1]).$$

It is easy to see that there is an equivalence of formal moduli problems

$$(11) \quad \widehat{\mathrm{T}}[-1]X^{\mathrm{gr}} \simeq \widehat{\mathcal{M}\mathrm{ap}}_{\mathrm{B}\mathbf{G}_m}(\mathbf{T}, X \times \mathrm{B}\mathbf{G}_m).$$

In particular, we get a $\mathbf{T}$ -action on $\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}$ induced by left multiplication on $\mathbf{T}$. More precisely, the left multiplication action is the result of applying the endomorphism $-\star[0]$ of Δ^{op} to the group object $\mathbf{T}$, which then acts on the functor of maps out of $\mathbf{T}$.

Definition 4.8. Let

$$\widehat{\mathcal{L}}X \rightarrow X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}$$

be the object of $\mathcal{F}\mathcal{M}\mathcal{P}/_{X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}}$ given by formal moduli problem corresponding to the action object $X_{\mathrm{dR}} \times \mathbf{T} \curvearrowright \widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}$. We call it the *formal loop stack* of X .

Before we state the version of the HKR theorem that we will need, let us record a descent statement for the formal loop stack.

Lemma 4.9. *Let X be a derived stack with perfect cotangent complex. Then the natural map*

$$\eta: \operatorname{colim}_{\mathrm{Spec} R \rightarrow X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}} \operatorname{colim}_{\mathrm{Spec} A \rightarrow X_R^\wedge} \widehat{\mathcal{L}}(A/R) \rightarrow \widehat{\mathcal{L}}X,$$

where $X_R^\wedge = (X \times \mathrm{B}\mathbf{G}_m) \times_{(X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T})} \mathrm{Spec} R$, is an equivalence in the category $\mathrm{dSt}_{X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}}$.

Proof. First, we verify the graded version, with $\widehat{\mathcal{L}}X \rightarrow X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}$ replaced by $\mathrm{T}[-1]X^{\mathrm{gr}} \rightarrow X_{\mathrm{dR}} \times \mathrm{B}\mathbf{G}_m$. This is (up to shear) precisely [CS24, Cor. 1.29], whose proof we summarize briefly. By (11), it suffices to replace tangent stacks by formal mapping stacks. In fact, we may drop the completion at the zero section, as colimits in stacks commute with pullbacks. But then the statement reduces to the fact that colimits of presheaves are computed pointwise.

To deduce the lemma from the graded version, we observe that $\widehat{\mathcal{L}}X$ is the formal moduli problem associated to the formal groupoid $\mathcal{L}_\bullet X \in (\mathcal{F}\mathcal{M}\mathcal{P}/_{X_{\mathrm{dR}} \times \mathrm{B}\mathbf{G}_m})^{\Delta^{\mathrm{op}}}$. The category of formal moduli problems by construction satisfies base change, and geometric realizations are preserved under pullback, so the formation of this formal groupoid is compatible with base change. We get that the natural map

$$\operatorname{colim}_{\mathrm{Spec} R \rightarrow X_{\mathrm{dR}} \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}} \widehat{\mathcal{L}}(X_R^\wedge/R) \rightarrow \widehat{\mathcal{L}}X$$

is an equivalence. The lemma then follows because $\widehat{\mathcal{L}}(X_R^\wedge/R)$ is a stack. $\square$

Remark 4.10. We would like to draw the reader's attention to a subtlety in the constructions of this section. The formal loop stack $\widehat{\mathcal{L}}X$ is constructed as a sifted colimit in the category

$$\mathcal{FMP}/_{X_{\mathrm{dR}} \times \mathrm{BG}_m}.$$

This is *not the same* as the corresponding sifted colimit in $\mathrm{dSt}/_{X_{\mathrm{dR}} \times \mathrm{BG}_m}$, which is the ambient category for the colimits featuring in the statement of Theorem 4.9. In particular, the second step in the proof of Theorem 4.9 involves both types of colimits, and therefore is not quite tautological.

Lemma 4.11 (Quasicoherent base change for $\mathbf{T}$ -equivariant functions). *Let $Y \in \mathrm{dSt}_k$ have cotangent complex and let $\pi: X \rightarrow \mathrm{B}^{\mathrm{gr}}\mathbf{T} \times Y$ be a formal moduli problem. Let*

$$\begin{array}{ccc} X' & \xrightarrow{\bar{i}} & X \\ \bar{\pi} \downarrow & & \downarrow \pi \\ Y \times \mathrm{BG}_m & \xrightarrow{i} & Y \times \mathrm{B}^{\mathrm{gr}}\mathbf{T} \end{array}$$

be a Cartesian square and let $\mathcal{F} \in \mathrm{QCoh}(X)$. Then the base change morphism

$$i^* \pi_* \mathcal{F} \rightarrow \bar{\pi}_* \bar{i}^* \mathcal{F}$$

is an equivalence.

Proof. The proof is similar to the proof of [GR17b, Ch. 3, Prop. 2.1.2]. By construction of the QCoh package, it suffices to base change to an affine

$$S = \mathrm{Spec} R \rightarrow Y \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}.$$

Let $S' = S \times_{Y \times \mathrm{B}^{\mathrm{gr}}\mathbf{T}} Y \times \mathrm{BG}_m$ and let

$$\begin{array}{ccc} X'_S & \xrightarrow{\bar{i}_S} & X_S \\ \bar{\pi}^S \downarrow & & \downarrow \pi^S \\ S' & \xrightarrow{i_S} & S \end{array}$$

denote the resulting Cartesian square over S . Because π is inf-schematic, we may write

$$X_S \simeq \mathrm{colim}_{\alpha} X_{\alpha}$$

in dSt_k , where for every α X_{α} is a finite type scheme, and $\pi^{\alpha}: X_{\alpha} \rightarrow S$ is a nil-isomorphism. In particular, each π^{α} is schematic and quasi-compact, and therefore satisfies the base change property for quasicoherent sheaves [GR17a, Ch. 3, Prop. 2.2.2]. The sheaf $\mathcal{F}$ is given by a compatible family $\mathcal{F}_{\alpha} \in \mathrm{QCoh}(X_{\alpha})$. Letting $X'_{\alpha} = X' \times_{S'} X_{\alpha}$, we have the resulting Cartesian diagram

$$\begin{array}{ccc} X'_{\alpha} & \xrightarrow{\bar{i}_{\alpha}} & X_{\alpha} \\ \bar{\pi}^{\alpha} \downarrow & & \downarrow \pi^{\alpha} \\ S' & \xrightarrow{i_S} & S. \end{array}$$

Finally, we observe that the morphism i_S is affine such that $(i_S)_* \mathcal{O}_{S'} \in \mathrm{Perf}(S)$; indeed, this follows from the explicit presentation

$$\mathrm{B}^{\mathrm{gr}}\mathbf{T} \simeq \mathrm{Spec}_{\mathrm{BG}_m}(\mathrm{Sym}(\mathcal{O}_{\mathrm{BG}_m}[2])).$$

In particular, the natural morphism

$$i_S^* \lim_{\alpha} \pi_*^\alpha \rightarrow \lim_{\alpha} i_S^* \pi_*^\alpha$$

is an equivalence. Combining the above, we have that the following natural maps are equivalences:

$$\begin{aligned} i_S^* \pi_*^S \mathcal{F} &\rightarrow i_S^* \lim_{\alpha} \pi_*^\alpha \mathcal{F}_\alpha \\ &\rightarrow \lim_{\alpha} i_S^* \pi_*^\alpha \mathcal{F}_\alpha \\ &\rightarrow \lim_{\alpha} \bar{\pi}_*^\alpha i_\alpha^* \mathcal{F}_\alpha \\ &\rightarrow \bar{\pi}_*^S i_S^* \mathcal{F}, \end{aligned}$$

from which the lemma follows. $\square$

The following is essentially a restatement of the equivariant HKR equivalence [TV11], except that we assume fairly little about our stacks. See also [Mou24], where the loop stacks are formally complete like in our setting.

Proposition 4.12. *Suppose X is a derived stack with perfect cotangent complex. There is an equivalence of $\mathbb{D}^+$ -algebras*

$$\Gamma_{X_{\mathrm{dR}}}^{\mathbf{T}}(\widehat{\mathcal{L}}X; \mathcal{O}_{\widehat{\mathcal{L}}X}) \simeq (\mathrm{gr}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}}[-2*], \mathrm{d}).$$

In particular, the $\mathbf{T}$ -action on $\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}$ induces the de Rham differential on graded functions.

Proof. By definition of and descent for the de Rham algebra,

$$\mathrm{F}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}} \simeq \lim_{\mathrm{Spec} R \rightarrow X_{\mathrm{dR}}} \lim_{\mathrm{Spec} A \rightarrow X_R^\wedge} \mathrm{F}_H^* \mathrm{dR}_{A/R},$$

where $X_R^\wedge = (X \times \mathbf{B}\mathbf{G}_m) \times_{(X_{\mathrm{dR}} \times \mathbf{B}^{\mathrm{gr}}\mathbf{T})} \mathrm{Spec} R$. On the other hand, by Theorem 4.9,

$$\widehat{\mathcal{L}}X \simeq \lim_{\mathrm{Spec} R \rightarrow X_{\mathrm{dR}} \times \mathbf{B}^{\mathrm{gr}}\mathbf{T}} \lim_{\mathrm{Spec} A \rightarrow X_R^\wedge} \widehat{\mathcal{L}}(A/R)$$

in $\mathrm{dSt}_{\mathbf{B}^{\mathrm{gr}}\mathbf{T}}$. By applying Theorem 4.11, we may replace $X \rightarrow X_{\mathrm{dR}}$ by

$$X_R^\wedge \rightarrow \mathrm{Spec} R,$$

and we can furthermore assume that $X_R^\wedge \simeq \mathrm{Spec} A$ is affine. Performing base change along $X_{\mathrm{dR}} \times \mathbf{B}\mathbf{G}_m \rightarrow X_{\mathrm{dR}} \times \mathbf{B}^{\mathrm{gr}}\mathbf{T}$, we get a natural equivalence of underlying graded objects

$$(12) \quad \mathrm{gr}^* \Gamma_{\mathrm{Spec} R}^{\mathbf{T}}(\mathcal{O}_{\widehat{\mathcal{L}}(A/R)}) \simeq \mathrm{gr}_H^* \mathrm{dR}_{A/R}[-2*].$$

The universal property of the de Rham algebra then implies there is a natural map of $\mathbb{D}^+$ -algebras

$$\sigma: (\mathrm{gr}_H^* \mathrm{dR}_{A/R}, \mathrm{d}) \rightarrow \Gamma_{\mathrm{Spec} R}^{\mathbf{T}}(\mathcal{O}_{\widehat{\mathcal{L}}(A/R)}).$$

It suffices to show σ is an equivalence; but it induces the map (12) on graded objects, and the associated graded functor is conservative. $\square$

Remark 4.13. The geometric formulation of the de Rham differential described above, as well as the approach to Cartan calculus in the next section, is partly inspired by [KS04]. In particular, we can re-express the groups involved as

$$\mathrm{Aut}(\mathbf{A}^1[-1]) \simeq \mathbf{G}_m \ltimes \mathbf{B}\mathbf{G}_a,$$

and

$$\widehat{\text{Aut}}([\mathbf{A}^1[-1]/\mathbf{G}_m]/\mathbf{BG}_m) \simeq [\mathbf{BG}_a/\mathbf{G}_m].$$

Then, as in loc. cit., the grading and de Rham differential on differential forms are re-expressed as group actions on a mapping space, and the Cartan formula becomes a statement about interaction of this group action with the action of diffeomorphisms of the codomain.

4.2. Closed and exact forms.

Definition 4.14. Let $p \in \mathbf{N}$ and $X \rightarrow S$ be a morphism of stacks. The *sheaf of closed p -forms on X/S* is

$$F_H^p dR_{X/S} \in \mathcal{QCoh}(S).$$

The *sheaf of exact p -forms on X* is

$$\Omega_{X/S}^{p,\text{ex}} = \text{coFib}(F_H^p dR_{X/S} \rightarrow F_H^0 dR_{X/S}) \in \mathcal{QCoh}(S).$$

Observe that the definition of exact p -forms immediately lifts to the filtered setting, endowing $\Omega_X^{p,\text{ex}}$ with a p -step complete filtration, which we also refer to as the Hodge filtration. More precisely, we write

$$F_H^* \Omega_{X/S}^{p,\text{ex}} = \text{coFib}(F_H^{*+p} dR_{X/S} \rightarrow F_H^* dR_{X/S}) \in \widehat{\text{Fil}}^+ \mathcal{QCoh}(S),$$

observing that

$$\text{gr}_H^* \Omega_{X/S}^{p,\text{ex}} \simeq 0 \text{ when } * \geq p.$$

There is a natural map

$$(13) \quad F_H^* \Omega_{X/S}^{p,\text{ex}} \rightarrow F_H^{p+*} dR_{X/S}1,$$

namely the connecting map of the defining cofiber sequence.

Lemma 4.15. *There is a commutative square*

$$\begin{array}{ccc} \text{gr}_H^{p-1} \Omega^{p,\text{ex}} & \longrightarrow & \text{gr}_H^p dR_{X/S} \\ \simeq \downarrow & & \downarrow \simeq \\ \wedge^{p-1} \mathbb{L}_{X/S} & \xrightarrow{d} & \wedge^p \mathbb{L}_{X/S}[1], \end{array}$$

where the top horizontal arrow is the image of (13) under the functor gr^{p-1} .

Definition 4.16 (Stacks of differential forms). Let $n \in \mathbf{Z}$, $p \in \mathbf{N}$. The *stack of n -shifted closed p -forms on X* is the complete linear stack

$$\Omega_X^{p,\text{cl}}(n) = \widehat{\mathbf{V}}_{X_{\text{dR}}} (F_H^p dR_{X/X_{\text{dR}}}[n+p]) \in \mathcal{FMP}_{X_{\text{dR}}}.$$

Similarly, the *prestack of n -shifted exact p -forms on X* is

$$\Omega_X^{p,\text{ex}}(n) = \widehat{\mathbf{V}}_{X_{\text{dR}}} (\Omega_{X/X_{\text{dR}}}^{p,\text{ex}}[n+p-1]) \in \mathcal{FMP}_{X_{\text{dR}}}.$$

For every n, p there is a natural map

$$\Omega_X^{p,\text{ex}}(n) \rightarrow \Omega_X^{p,\text{cl}}(n),$$

thought of as the de Rham differential.

Definition 4.17. Let

$$\omega \in \pi_0 F_H^2 dR_X[2+n]$$

be an n -shifted closed 2-form on X . We say that ω is an n -symplectic form on X if the morphism of sheaves

$$\omega_2: \mathbb{T}_X \rightarrow \mathbb{L}_X[n]$$

is an equivalence.

Definition 4.18. Let $f: L \rightarrow X$ and let ω be an n -symplectic form on X . An *isotropic structure on f* is a nullhomotopy for $f^*\omega$, i.e. a path

$$h \in \pi_0 P_{f^*\omega,0}(\tau_{\geq 0} F_H^2 dR_L).$$

An isotropic structure h on f is a *Lagrangian structure* if the induced morphism of sheaves

$$h_2: \mathbb{T}_L \rightarrow \mathbb{L}_{L/X}[n-1]$$

is an equivalence. An isotropic (resp. Lagrangian) structure h is said to be an *exact isotropic* (resp. *exact Lagrangian*) structure if ω is exact, and h is in the image of $\Omega_{L/X}^{2,\text{ex}}$.

Remark 4.19. In the above definition, the map h_2 is obtained as follows. By composing with the underlying 2-form map, h induces a nullhomotopy of

$$f^*\omega_2: \mathbb{T}_L \rightarrow f^*\mathbb{T}_X \rightarrow f^*\mathbb{L}_X[n] \rightarrow \mathbb{L}_L[n],$$

and therefore we get a nullhomotopy for the composite

$$\mathbb{T}_{L/X} \xrightarrow{\nu} \mathbb{T}_L \xrightarrow{f^*\omega_2} \mathbb{L}_L[n],$$

where ν is the natural map. On the other hand, the normal fiber sequence for f yields a canonical nullhomotopy for ν , and composing the two nullhomotopies yields an element

$$h_2 \in \pi_0 \Omega_0 \text{Map}_{\mathcal{QCoh}(L)}(\mathbb{T}_{L/X}, \mathbb{L}_L[n]) \simeq \pi_0 \text{Map}_{\mathcal{QCoh}(L)}(\mathbb{T}_{L/X}, \mathbb{L}_L[n-1]).$$

Definition 4.20 (Stack of isotropic structures). Let $f: L \rightarrow X$ be a nil-isomorphism (in particular $f_{dR}: X_{dR} \simeq L_{dR}$) and write

$$\mathcal{I}_f = \text{Fib}(F_H^2 dR_{X/X_{dR}} \rightarrow F_H^2 dR_{L/L_{dR}}) \in \mathcal{QCoh}(X_{dR}).$$

The *stack of n -isotropic structures* on $f: L \rightarrow X$ is the formal stack

$$\mathbf{Isot}_f(n) = \widehat{\mathbb{V}}_{X_{dR}}(\mathcal{I}_f[n]) \in \mathcal{FMP}_{/X_{dR}}.$$

By construction, there is an “underlying closed 2-form” map of stacks

$$\mathbf{Isot}_f(n) \rightarrow \Omega_X^{2,\text{cl}}(n).$$

Similarly, letting

$$\mathcal{I}_f^{\text{ex}} = \text{Fib}(\Omega_{X/X_{dR}}^{2,\text{ex}} \rightarrow \Omega_{L/L_{dR}}^{2,\text{ex}}),$$

the *stack of exact n -isotropic structures* on $f: L \rightarrow X$ is the formal stack

$$\mathbf{Isot}_f^{\text{ex}}(n) = \widehat{\mathbb{V}}_{X_{dR}}(\mathcal{I}_f^{\text{ex}}[n]) \in \mathcal{FMP}_{/X_{dR}}.$$

It comes with an underlying exact 2-form map

$$\mathbf{Isot}_f^{\text{ex}}(n) \rightarrow \Omega_X^{2,\text{ex}}(n).$$

For a morphism $f: L \rightarrow X$ which is not necessarily a nil-isomorphism, we define n -isotropic structures on f to be n -isotropic structures on the formal completion

$$L \rightarrow X_L^\wedge = X \times_{X_{\text{dR}}} L_{\text{dR}}.$$

The following is a combination of results of [Pan+13; Cal19; CS24].

Theorem 4.21 (Pantev–Toën–Vaquié–Vezzosi; Calaque; Calaque–Safronov). *Let X be a derived stack with perfect cotangent complex. Then $\widehat{\mathbf{T}}^*[n]X$ is equipped with a canonical exact n -shifted symplectic structure*

$$\omega \in \pi_0 \Omega_{\widehat{\mathbf{T}}^*[n]X/X_{\text{dR}}}^{2,\text{ex}}[n+1].$$

Moreover, the zero section

$$\iota: X \rightarrow \widehat{\mathbf{T}}^*[n]X$$

is equipped with a canonical exact Lagrangian structure.

5. DIFFERENTIAL CALCULUS

It will be crucial for our goals to understand the interaction of formal automorphisms of a derived stack with the Hodge filtration on de Rham cohomology. This section is therefore dedicated to studying this interaction, culminating in a version of the Cartan homotopy formula for derived stacks. The perspective will be through the lens of the HKR theorem: the associated graded of the Hodge filtration is governed geometrically by the formal tangent stack, and the de Rham differential corresponds to the graded circle action.

We first treat the easier case of the associated graded, which corresponds to the notion of precalculus due to Tamarkin–Tsygan. Here most features come down to functoriality of the formal tangent stack. The most notable feature is that the identity fiber of the tangent of the automorphism group acts by fiberwise translation, which is the correct geometric notion of “interior derivative”. In the second subsection, we re-introduce the circle action, which acts on both the tangent stack and the tangent of the automorphism group. The main theorem will be an appropriate compatibility between them.

We fix a derived stack X with perfect cotangent complex; recall all of our stacks are left Kan extended from finitely presented animated commutative rings. We will write

$$p: X \rightarrow X_{\text{dR}}$$

for the canonical projection. For a point $S \rightarrow X_{\text{dR}}$, we write

$$X_S^\wedge = S \times_{X_{\text{dR}}} X.$$

5.1. Precalculus. Consider the abelian formal group

$$\mathbb{V}_{X_{\text{dR}}}(p_* \mathbb{T}_X[-1]) = \exp_{X_{\text{dR}}}(\text{triv}_{\text{Lie}}(p_* \mathbb{T}_X[-1])) \in \text{Grp}(\mathcal{FMP}_{/X_{\text{dR}}}).$$

The pullback

$$\mathbb{V}_{X_{\text{dR}}}(p_* \mathbb{T}_X[-1]) \times_{X_{\text{dR}}} X$$

is an object of $\text{Grp}(\mathcal{FMP}_{/X})$ with trivial multiplication. By perfection of $\mathbb{T}_X$ and Theorem 3.19,

$$\mathbb{V}_{X_{\text{dR}}}(p_* \mathbb{T}_X[-1]) \times_{X_{\text{dR}}} X \simeq \widehat{\text{Tot}}_X(p^* p_* \mathbb{T}_X[-1]),$$

where we've dropped Υ from the notation. In particular, the canonical map

$$p^*p_*\mathbb{T}_X[-1] \rightarrow \mathbb{T}_X[-1]$$

in $\mathcal{QCoh}(X)$ induces a morphism

$$\mathbb{V}_{X_{\mathrm{dR}}}(p_*\mathbb{T}_X[-1]) \times_{X_{\mathrm{dR}}} X \rightarrow \widehat{\mathbb{T}}[-1]X$$

over X . By taking the left translation action of $\widehat{\mathbb{T}}[-1]X$ on itself and pulling back, we thus get an action

$$(14) \quad \mathbb{V}_{X_{\mathrm{dR}}}(p_*\mathbb{T}_X[-1]) \times_{X_{\mathrm{dR}}} X \curvearrowright \widehat{\mathbb{T}}[-1]X$$

in $\mathcal{FMP}/_X$. The resulting group morphism

$$\mathbb{V}_{X_{\mathrm{dR}}}(p_*\mathbb{T}_X[-1]) \longrightarrow \mathrm{Res}_{X_{\mathrm{dR}}}^X \widehat{\mathrm{Aut}}(\widehat{\mathbb{T}}[-1]X/X)$$

over X_{dR} is the Lie exponential of the map

$$p_*\mathbb{T}_X[-1] \rightarrow p_*\pi_*\pi^!\mathbb{T}_X[-1]$$

arising from the unit of the (discontinuous) adjunction $\pi^! \dashv \pi_*$

Definition 5.1. We refer to the action (14) as the *global fiberwise translation action*.

The following statement is the global analogue of the action of

$$\mathrm{VF}(\mathbf{A}^n) \oplus \mathrm{VF}(\mathbf{A}^n)[-1]$$

on the graded module $\Omega_{\mathbf{A}^n/k}^*$ given by Lie derivative and interior product.

Proposition 5.2. *There is a canonically defined action*

$$\widehat{\mathbb{T}}[-1]\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \curvearrowright \widehat{\mathbb{T}}[-1]X^{\mathrm{gr}},$$

which lifts the natural action

$$\widehat{\mathrm{Aut}}(X) \curvearrowright \widehat{\mathbb{T}}[-1]X^{\mathrm{gr}}.$$

Moreover, the restriction to the identity fiber

$$\Omega_{X_{\mathrm{dR}}}\widehat{\mathrm{Aut}}(X) \simeq \mathbb{V}_{X_{\mathrm{dR}}}(\mathrm{triv}_{\mathrm{Lie}}(p_*\mathbb{T}_X[-1]))$$

is identified with the global fiberwise translation action.

Proof. The functor

$$\widehat{\mathbb{T}}[-1](-/\mathbb{Z})^{\mathrm{gr}} \simeq \widehat{\mathcal{M}\mathrm{ap}}_{\mathbb{Z}}(\mathbf{T} \times \mathbb{Z}, -): \mathcal{FMP}/_{\mathbb{Z}} \rightarrow \mathcal{FMP}/_{\mathbb{Z} \times \mathbf{BG}_m}$$

preserves limits, and therefore automatically has the structure of a monoidal functor for the Cartesian product. The action we are after is obtained by applying $\widehat{\mathbb{T}}[-1](-/X_{\mathrm{dR}})^{\mathrm{gr}}$ to the tautological action of $\widehat{\mathrm{Aut}}(X)$ on X .

It remains to prove the last claim. By (4),

$$\Omega_{X_{\mathrm{dR}}}\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \simeq \exp_{X_{\mathrm{dR}} \times \mathbf{BG}_m}(\mathrm{ins}^1 \Omega^{\mathrm{Lie}} \mathrm{VF}(X)).$$

So, as formal groups, $\Omega_{X_{\mathrm{dR}}}\widehat{\mathrm{Aut}}(X) \simeq \mathbb{V}_{X_{\mathrm{dR}}}(\mathrm{triv}_{\mathrm{Lie}}(p_*\mathbb{T}_X[-1]))$, compatibly with the grading. The action

$$\Omega_{X_{\mathrm{dR}}}\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \times_{X_{\mathrm{dR}}} X \curvearrowright \widehat{\mathbb{T}}[-1]X^{\mathrm{gr}}$$

is given by the action of

$$\mathrm{Fib}_{\mathrm{id}} \left(\widehat{\mathrm{Map}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(\mathbf{T}, \widehat{\mathrm{Aut}}(X)) \rightarrow \widehat{\mathrm{Map}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}}, \widehat{\mathrm{Aut}}(X)) \right)$$

on $\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}$ over X . Given a point

$$u: \mathrm{Spec} R \rightarrow X,$$

this is the tangent map at (id, u) of the canonical action

$$\widehat{\mathrm{Aut}}(X_R^\wedge) \curvearrowright X_R^\wedge.$$

Thus the action of $\Omega_{X_{\mathrm{dR}}} \widehat{\mathrm{Aut}}(X)^{\mathrm{gr}}$ on $\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}$ is linear over X , and the computation of $\mathrm{Lie}(\widehat{\mathrm{Aut}}(X))$ shows that it is equivalent to the fiberwise translation action. $\square$

Lemma 5.3. *The $\Omega_{X_{\mathrm{dR}}} \widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \times_{X_{\mathrm{dR}}} X$ -module structure on the graded object*

$$\Upsilon \Gamma_{X \times \mathrm{BG}_m}(\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}, \mathcal{O}_{\widehat{\mathrm{T}}[-1]X^{\mathrm{gr}}}) \simeq \Upsilon \mathrm{Sym}_{X \times \mathrm{BG}_m}^*(\mathrm{ins}^{-1} \mathbb{L}_X[1]) \in \mathrm{GrCAlg}(\mathrm{IndCoh}(X))$$

induced by the fiberwise translation action is equivalent to the extension of the duality pairing

$$\mathrm{T}_X[-1] \otimes_X \mathbb{L}_X[1] \longrightarrow \mathcal{O}_X.$$

Proof. Since the group structure here is abelian, by Theorem 4.9 we may assume X is a derived *scheme*. Moreover, we may reduce to $X = \mathbf{A}^1$, since every derived scheme can be written as a cosifted limit of products of $\mathbf{A}^1$. On the affine line, the statement can be checked explicitly:

$$\widehat{\mathrm{T}}[-1]\mathbf{A}^1 \simeq \mathbf{A}^1 \times \mathbf{A}^1[-1],$$

and fiberwise translation corresponds to infinitesimal translation in $\mathbf{A}^1[-1]$, which on the sheaf of functions is identified with the duality pairing. $\square$

Let us clarify this subsection using the classical example.

Example 5.4. Set $X = \mathbf{A}^n$ with coordinates $x^1, \dots, x^n$. We may write

$$\widehat{\mathrm{T}}[-1]\mathbf{A}^n = \mathrm{Spec} k[x^1, \dots, x^n, \zeta^1, \dots, \zeta^n],$$

where ζ^i has cohomological degree -1 and weight 1 . This defines $\mathrm{gr}_H^* \mathrm{dR}_{X/k}$. Now let $v = v^i \frac{\partial}{\partial x^i}$ be a derivation of $\mathcal{O}(X)$. Interior product by v is identified with the derivation

$$\iota_v = v^i \frac{\partial}{\partial \zeta^i}.$$

Thus, the flow of ι_v fixes the zero section and translates the fibers proportionally to the components, explaining the name “fiberwise translation”.

5.2. Interaction with the circle action. There is a natural $\mathbf{T}$ -action on $\widehat{\mathrm{T}}[-1]\widehat{\mathrm{Aut}}(X)$ as a stack over X_{dR} , defining a morphism of (graded) group objects:

$$\mathrm{D}_C: \mathbf{T} \rightarrow \mathrm{Aut}(\widehat{\mathrm{T}}[-1]\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}}).$$

Lemma 5.5. *The map D_C lifts to*

$$\mathbf{T} \rightarrow \mathrm{Aut}_{\mathrm{Grp}}(\widehat{\mathrm{T}}[-1]\widehat{\mathrm{Aut}}(X)).$$

Proof. Follows from the fact that the action of $\mathbf{T}$ on $\widehat{\mathrm{T}}[-1]\widehat{\mathrm{Aut}}(X)$ arises from its action on the monoidal functor $\widehat{\mathrm{T}}[-1](-)$. $\square$

Definition 5.6. We define the formal group

$$\mathrm{Calc}(X) = \mathbf{T} \ltimes \widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \in \mathrm{Grp}(\mathcal{FMP}_{/X_{\mathrm{dR}} \times \mathrm{BG}_m})$$

using the group action D_C . We also write $\mathfrak{c}(X) \in \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{GrIndCoh}(X_{\mathrm{dR}}))$ for its Lie algebra.

We arrive at the main theorem of this section. The author is grateful to Nick Rozenblyum for communicating that this theorem independently follows from much more general results of an upcoming paper of Brav–Rozenblyum.

Theorem 5.7. *There is an action*

$$\mathrm{Calc}(X) \curvearrowright \widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}$$

in the category $\mathcal{FMP}_{/X_{\mathrm{dR}} \times \mathrm{BG}_m}$ which restricts to

- (1) the $X_{\mathrm{dR}} \times \mathbf{T}$ -action defining the formal loop stack $\widehat{\mathcal{L}}X$, and
- (2) the $\widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X)$ -action of Theorem 5.2.

Proof. By construction,

$$\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}} = \widehat{\mathrm{Map}}_{\mathcal{FMP}_{/X_{\mathrm{dR}} \times \mathrm{BG}_m}}(X_{\mathrm{dR}} \times \mathbf{T}, X \times \mathrm{BG}_m)$$

is an internal mapping object in $\mathcal{FMP}_{/X_{\mathrm{dR}} \times \mathrm{BG}_m}$. Similarly,

$$(15) \quad \widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \simeq \widehat{\mathrm{Map}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}} \times \mathbf{T}, \underline{\mathrm{end}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X \times \mathrm{BG}_m)).$$

There is a natural group morphism

$$X_{\mathrm{dR}} \times \mathbf{T} \rightarrow \widehat{\mathrm{Aut}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}} \times \mathbf{T}),$$

which under (15) intertwines the action defining $\widehat{\mathcal{L}}(\widehat{\mathrm{Aut}}(X))$ with the standard precomposition action of the right side.

By Theorem A.1, $\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}$ acquires an action of

$$\begin{aligned} & \widehat{\mathrm{Aut}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}} \times \mathbf{T}) \ltimes \widehat{\mathrm{Map}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}} \times \mathbf{T}, \underline{\mathrm{end}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X \times \mathrm{BG}_m)) \\ & \simeq \widehat{\mathrm{Aut}}_{X_{\mathrm{dR}} \times \mathrm{BG}_m}(X_{\mathrm{dR}} \times \mathbf{T}) \ltimes \widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X)^{\mathrm{gr}} \end{aligned}$$

which restricts to the action of Theorem 5.2 and the natural action of $\widehat{\mathrm{Aut}}(X_{\mathrm{dR}} \times \mathbf{T})$. To verify the hypotheses of Theorem A.1, recall that the inclusion $\mathrm{Mon}(\mathcal{FMP}_{/X}) \rightarrow \mathrm{Grp}(\mathcal{FMP}_{/X})$ is an equivalence. Now we pull back the action by

$$\mathrm{Calc}(X) \rightarrow \widehat{\mathrm{Aut}}(\mathbf{T}) \ltimes \widehat{\mathrm{Map}}(\mathbf{T}, \underline{\mathrm{end}}(X \times \mathrm{BG}_m))$$

and observe that resulting action lifts the $X_{\mathrm{dR}} \times \mathbf{T}$ -action defining $\widehat{\mathcal{L}}X$ by design. $\square$

Corollary 5.8 (Cartan homotopy formula). *The canonical graded $\mathrm{VF}(X)$ -module structure on $\mathrm{gr}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}}$ lifts to $\mathrm{Mod}_c(\mathrm{IndCoh}(X_{\mathrm{dR}}))$. In particular, for any class*

$$v \in \pi_p \mathrm{VF}(X) \cong \pi_{p+1} \Gamma(X, \mathbb{T}_X[-1]),$$

the following equality holds in $\pi_p \mathrm{End}_{\mathrm{GrIndCoh}(X_{\mathrm{dR}})}(\mathrm{gr}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}})$:

$$[d, \iota_v] = L_v.$$

Proof. By definition,

$$\mathfrak{c}(X) \simeq \mathfrak{t}^+ \ltimes \mathrm{Lie}(\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}).$$

The action of Theorem 5.7 yields a $\mathrm{Calc}(X)$ -module structure, and therefore a $\mathfrak{c}(X)$ -module structure, on

$$\Gamma_{\mathrm{BG}_m}(\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}; \mathcal{O}_{\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}}}) \simeq \mathrm{gr}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}}.$$

Write

$$\rho: \pi_p \mathfrak{c}(X) \rightarrow \pi_p \mathrm{End}_{\mathrm{grJndCoh}(X_{\mathrm{dR}})}(\mathrm{gr}_H^* \mathrm{dR}_{X/X_{\mathrm{dR}}})$$

for the action morphism on homotopy groups.

Let

$$\xi: \omega_{\mathrm{BG}_m} \rightarrow \mathfrak{t}^+$$

represent a generator. For $v \in \pi_p \mathrm{VF}(X)$, write

$$v^l \in \pi_p \mathrm{Lie}(\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}})$$

for the class induced by the inclusion of the zero section, and

$$v^i \in \pi_{p+1} \mathrm{Lie}(\widehat{\mathbf{T}}[-1]X^{\mathrm{gr}})$$

for the corresponding class induced by the map from the identity fiber. By construction of D_C ,

$$(16) \quad [\xi, v^i] = v^l.$$

We now identify the image of these three classes under ρ . Theorem 4.12 implies

$$\rho(\xi) = \mathrm{d}.$$

Restricting the action of $\widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X)$ to the zero section is precisely the canonical automorphism action, whence

$$\rho(v^l) = L_v.$$

Finally, Theorem 5.3 yields

$$\rho(v^i) = \iota_v,$$

since Theorem 5.2 shows the identity fiber acts by fiberwise translation. Applying ρ to (16) lets us conclude. $\square$

Remark 5.9. The above corollary can be thought of as generalizing Gelfand–Kazhdan descent for the classical Cartan homotopy formula on manifolds.

No part of the proof required the base we work over to be of the form X_{dR} . In particular, all statements go through for the formal moduli problem $X \rightarrow X_{\mathrm{dR}}$ replaced by any formal moduli problem $X \rightarrow \mathbb{Z}$, where we keep the standing assumption that cotangent complexes are perfect. More precisely, we define

$$\mathrm{Calc}(X/\mathbb{Z}) = \mathbf{T} \ltimes \widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X/\mathbb{Z})^{\mathrm{gr}} \in \mathrm{Grp}(\mathcal{FMP}_{/\mathbb{Z} \times \mathrm{BG}_m}),$$

and have the following.

Corollary 5.10. *There is a natural action*

$$\mathrm{Calc}(X/\mathbb{Z}) \curvearrowright \widehat{\mathbf{T}}[-1](X/\mathbb{Z})^{\mathrm{gr}}$$

such that the restriction to $\mathbf{T}$ is the de Rham action, and the restriction to $\widehat{\mathbf{T}}[-1]\widehat{\mathrm{Aut}}(X/\mathbb{Z})^{\mathrm{gr}}$ recovers the action given by functoriality.

6. POISSON STRUCTURES

We arrive at the definition of infinitesimal Poisson structures. In the first subsection, we will construct a formal group of “Hamiltonian” deformations of the cotangent stack, and use Theorem 5.7 to prove that its tangent space at the identity coincides with the graded module underlying the polyvectors algebra. In the second section, we will introduce the relative case, which requires a little bit of care.

6.1. A formal group of Poisson deformations. Let $X \rightarrow \mathbb{Z}$ be a formal moduli problem over $\mathbb{Z}$. An integral part of the Cartan package is the action

$$\widehat{\mathrm{Aut}}(X/\mathbb{Z}) \curvearrowright \Omega_{X/\mathbb{Z}}^{p,\mathrm{ex}},$$

for any $p \in \mathbb{N}$, which is already well-defined by functoriality of the filtered de Rham algebra. One readily sees that it equips $\Omega_{X/\mathbb{Z}}^{p,\mathrm{ex}}$ with the structure of an $\widehat{\mathrm{Aut}}(X/\mathbb{Z})$ -module.

Definition 6.1. Let $\mathbb{Z} \in \mathrm{dSt}_k$, let $X \in \mathcal{FMP}/\mathbb{Z}$ and assume both X and $\mathbb{Z}$ admit a perfect cotangent complex. Fix $p \in \mathbb{N}, n \in \mathbb{Z}$. We define

$$\tilde{\Omega}^{p,\mathrm{ex}}(X/\mathbb{Z}, n) = \mathbb{V}_{\mathbb{Z}}^{\widehat{\mathrm{Aut}}(X/\mathbb{Z})}(\Omega^{p,\mathrm{ex}}(X/\mathbb{Z}, n)),$$

where we’ve used the notation of Theorem 3.36. As an action object, it is equipped with a projection morphism

$$\tilde{\Omega}^{p,\mathrm{ex}}(X/\mathbb{Z}, n) \rightarrow \mathrm{B}_{\mathbb{Z}} \widehat{\mathrm{Aut}}(X/\mathbb{Z}).$$

We adopt the notational convention

$$\tilde{\Omega}^{p,\mathrm{ex}}(X, n) = \tilde{\Omega}^{p,\mathrm{ex}}(X/X_{\mathrm{dR}}, n).$$

We finally introduce the most important definition in the article.

Definition 6.2. We define the *group of n -Poisson deformations of X* to be the based loop stack

$$\mathcal{GP}_X(n) = \Omega_{\omega} \tilde{\Omega}^{2,\mathrm{ex}}(\widehat{\mathrm{T}}^*[n+1]X, n+1)$$

at the point

$$\omega: X_{\mathrm{dR}} \rightarrow \tilde{\Omega}^{2,\mathrm{ex}}(\widehat{\mathrm{T}}^*[n+1]X, n+1)$$

defined by the standard exact symplectic form. As a based loop construction, it canonically acquires the structure of a group in $\mathcal{FMP}/X_{\mathrm{dR}}$; we abuse notation and do not distinguish between the group object and its underlying stack.

Points $\alpha \in \mathcal{GP}_X(n)$ are to be thought of as Hamiltonian vector fields on $\widehat{\mathrm{T}}^*[n+1]X$. There is a canonical morphism of group objects

$$v_X: \mathcal{GP}_X(n) \rightarrow \widehat{\mathrm{Aut}}(\widehat{\mathrm{T}}^*[n+1]X).$$

We think of $v_X(\alpha)$ as the “underlying vector field” of the Hamiltonian vector field α . Moreover, we have the following.

Lemma 6.3. *The underlying formal stack of $\mathcal{GP}_X(n)$ is given by the fiber at ω of the morphism*

$$(17) \quad \widehat{\mathrm{Aut}}(\widehat{\mathrm{T}}^*[n+1]X) \rightarrow \Omega^{2,\mathrm{ex}}(\widehat{\mathrm{T}}^*[n+1]X, n+1)$$

induced by the action of formal automorphisms of ω .

Proof. Elementary from the definition of based loop stacks of a quotient stack. $\square$

Theorem 6.4. *Let $e: X_{\text{dR}} \rightarrow \mathcal{GP}_X(n)$ denote the identity section. The underlying sheaf of $\text{Lie}(\mathcal{GP}_X(n))$ satisfies*

$$e^* \mathbb{T}_{\mathcal{GP}_X(n)/X_{\text{dR}}} \simeq \text{gr}_H^0 \Omega_{\widehat{T}^*[n+1]X/X_{\text{dR}}}^{2,\text{ex}}[n+1] \simeq \Gamma_{X_{\text{dR}}}(\widehat{T}^*[n+1]X, \mathcal{O})[n+1],$$

which moreover lifts to an equivalence of graded objects

$$e^* \mathbb{T}_{\mathcal{GP}_X(n)^{\text{gr}}/X_{\text{dR}}} \simeq \Gamma_{X_{\text{dR}}}^{\text{gr}}(\widehat{T}^*[n+1]X, \mathcal{O})[n+1]$$

in $\text{GrQCoh}(X_{\text{dR}})$.

Proof. Let

$$\mathfrak{h} = e^* \mathbb{T}_{\mathcal{GP}_X(n)/X_{\text{dR}}}, \quad \mathfrak{v} = e^* \mathbb{T}_{\widehat{\text{Aut}}(\widehat{T}^*[n+1]X)/X_{\text{dR}}} \in \text{GrQCoh}(X_{\text{dR}}).$$

Write also

$$\mathfrak{f} = \Omega_{\widehat{T}^*[n+1]X/X_{\text{dR}}}^{2,\text{ex}}[n+2] \in \text{GrQCoh}(X_{\text{dR}}),$$

where we will suppress the distinction between the Hodge-filtered object and the underlying object.

We have by construction that

$$(18) \quad \mathfrak{h} \rightarrow \mathfrak{v} \rightarrow \mathfrak{f}$$

is a fiber sequence of sheaves on X_{dR} . Let w_0 denote the composition

$$w_0: \text{gr}_H^0 \mathfrak{f} \xrightarrow{d} \text{gr}_H^1 \mathfrak{f}[1] \xrightarrow{\rho} \mathfrak{v}[1]$$

where ρ is an inverse for $\omega_2^\sharp$.

Lemma 6.5. *There is a factorization*

$$\begin{array}{ccc} & & \mathfrak{h}[1] \\ & \nearrow w & \downarrow \\ \text{gr}_H^0 \mathfrak{f} & \xrightarrow{w_0} & \mathfrak{v}[1]. \end{array}$$

Proof. It suffices to verify that the composition

$$\text{gr}_H^0 \mathfrak{f} \xrightarrow{w_0} \mathfrak{v}[1] \longrightarrow \mathfrak{f}[1]$$

is nullhomotopic. For that it's enough to prove that

$$\text{gr}_H^0 \mathfrak{f} \xrightarrow{w_0} \mathfrak{v}[1] \longrightarrow \text{gr}_H^1 \mathfrak{f}[1]$$

factors through the connecting map

$$\text{gr}_H^0 \mathfrak{f} \rightarrow \text{gr}_H^1 \mathfrak{f}[1]$$

which is identified with the first differential in the homotopy coherent complex

$$(\text{gr}_H^* dR_{\widehat{T}^*[n+1]X/X_{\text{dR}}}[n+1], d).$$

By Theorem 5.8, this composite is equivalent to $\iota_{\rho(d(-))} d\lambda$ modulo d . But by definition of ρ and Theorem 5.2, $\iota_{\rho(d(-))} \simeq d(-)$, and thus the lemma follows. $\square$

Now, using Theorem 5.2, we have

$$h: \mathfrak{h}[1] \longrightarrow \mathfrak{v}[1] \xrightarrow{\iota\lambda} \mathfrak{f} \longrightarrow \mathrm{gr}_H^0 \mathfrak{f}.$$

We claim that h is an inverse for w . Indeed, $h \circ w \simeq \mathrm{id}$ follows from the lemma and an application of Theorem 5.8. On the other hand, applying Theorem 5.8 to

$$\mathfrak{v}[1] \xrightarrow{\iota\lambda} \mathrm{gr}_H^0 \mathfrak{f} \xrightarrow{d} \mathrm{gr}_H^1 \mathfrak{f}[1] \xrightarrow{\rho} \mathfrak{v}[1],$$

we see that

$$\begin{array}{ccc} \mathfrak{h}[1] & \xrightarrow{\mathrm{id}} & \mathfrak{h}[1] \\ \downarrow & & \downarrow \\ \mathfrak{v}[1] & \xrightarrow{w_0 \circ \iota\lambda} & \mathfrak{v}[1] \end{array}$$

commutes. By the universal property of the fiber in (18), we must have $w \circ h \simeq \mathrm{id}$, finishing the proof. $\square$

Remark 6.6. The equivalence

$$h: \mathfrak{h}[1] \simeq \mathrm{gr}_H^0 \mathfrak{f}$$

in the proof above can informally be thought of as follows. An element of $\mathfrak{h}$ is a vector field v together with a function f such that $L_v \lambda = df$. The morphism h takes (v, f) to the function $f - \iota_v \lambda$, which is a Hamiltonian for v by the Cartan homotopy formula.

Definition 6.7 (Infinitesimal polyvectors). Let

$$\mathrm{Pol}_X^{\mathrm{inf}}(n) = \mathrm{Lie}(\mathcal{GP}_X(n)^{\mathrm{gr}}) \in \mathcal{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathrm{QCoh}(X_{\mathrm{dR}}))$$

be the graded Lie algebra of the group of n -Poisson deformations of X . We refer to $\mathrm{Pol}_X^{\mathrm{inf}}(n)$ as the *Lie algebra of infinitesimal n -polyvectors*.

Now we can define a notion of Poisson bivectors.

Definition 6.8 (Infinitesimal Poisson structures). Let X be a derived stack with perfect cotangent complex. The space of *infinitesimal n -Poisson structures on X* is the mapping space

$$\mathrm{Pois}^{\mathrm{inf}}(X, n) = \mathrm{Map}_{\mathcal{Alg}_{\mathrm{Lie}}^{\mathrm{gr}}(\mathrm{QCoh}(X_{\mathrm{dR}}))}(\mathrm{ins}^2 \mathcal{O}_{X_{\mathrm{dR}}}[-1], \mathrm{Pol}_X^{\mathrm{inf}}(n)).$$

Following Theorem 4.6, this definition can be rephrased as

$$\mathrm{Pois}^{\mathrm{inf}}(X, n) \simeq \mathrm{Map}_{\mathcal{FMP}/\mathrm{BG}_m \times X_{\mathrm{dR}}}(\widehat{\mathbf{A}}^1/\mathbf{G}_m \times X_{\mathrm{dR}}, \mathrm{B}_{\mathrm{BG}_m} \mathcal{GP}_X(n)^{\mathrm{gr}}).$$

Example 6.9 (Classifying stacks). Let G be a reductive group with Lie algebra $\mathfrak{g}$ and consider $X = \mathrm{BG}$. As studied in [Saf21], the space of 1-Poisson structures on BG is closely related to Poisson Lie group structures on G . To illustrate the definition of infinitesimal 1-Poisson structures, we will outline how it works for BG and relate it to Poisson Lie group structures in the next subsection.

Consider

$$\widehat{\mathrm{T}}^*[2]\mathrm{BG} \simeq \widehat{\mathfrak{g}}^{\vee}[1]/G.$$

It is helpful to keep in mind that

$$\mathrm{T}_{\widehat{\mathrm{T}}^*[2]\mathrm{BG}^{\mathrm{gr}}}|_{\mathrm{BG}_m} \simeq \mathfrak{g}[1] \oplus \mathrm{ins}^{-1} \mathfrak{g}^{\vee}[1].$$

Graded functions on $\widehat{\mathfrak{g}}^\vee[1]/G$ can be computed as

$$\Gamma_{\mathbf{BG}_m}^*(\widehat{\mathfrak{g}}^\vee[1]/G; \mathcal{O}_{\widehat{\mathfrak{g}}^\vee[1]/G}) \simeq \lim_{\Delta} \left(\mathcal{O}(G^{\times \bullet}) \otimes_k \mathrm{Sym}_k^*(\mathrm{ins}^1 \mathfrak{g}[-1]) \right).$$

By base change, we have

$$(19) \quad \Gamma_{\mathbf{BG}_m}^*(\widehat{\mathfrak{g}}^\vee[1]/G; \mathbb{T}_{\widehat{\mathfrak{g}}^\vee[1]/G}) \simeq \lim_{\Delta} \left(\mathcal{O}(G^{\times \bullet}) \otimes_k \mathrm{Sym}_k^*(\mathrm{ins}^1 \mathfrak{g}[-1]) \right) \otimes_k (\mathfrak{g}[1] \oplus \mathrm{ins}^{-1} \mathfrak{g}^\vee[1]).$$

The graded Lie algebra structure on (19) defining

$$\mathrm{VF}(\widehat{\mathfrak{g}}^\vee[1]/G)^{\mathrm{gr}}$$

can be checked to be the one induced by the pairing

$$\mathrm{ins}^1 \mathfrak{g}[-1] \otimes_k \mathrm{ins}^{-1} \mathfrak{g}^\vee[1] \longrightarrow k$$

and the action of $\mathfrak{g}[1]$ on $G^{\times \bullet}$ by (usual) invariant vector fields. Observe that this graded Lie algebra vanishes in weight i for all $i < -1$.

Infinitesimal 1-Poisson structures are given by Lie algebra morphisms

$$\mathrm{ins}^2 k[-1] \longrightarrow \mathrm{VF}(\widehat{\mathfrak{g}}^\vee[1]/G) \times_{\mathrm{VF}(\widehat{\mathfrak{g}}^\vee[1]/G) \ltimes \Omega_{\widehat{\mathfrak{g}}^\vee[1]/G}^{2, \mathrm{ex}}} \mathrm{VF}(\widehat{\mathfrak{g}}^\vee[1]/G),$$

where the right side is the fiber product of the section λ with the zero section — this is the Lie algebra of $\mathcal{GP}_{\mathbf{BG}}(1)$, where we have omitted the grading from the notation. Using the explicit resolution of $\mathrm{ins}^2 k[-1]$ given by [Mel16, Sec. 4] and leveraging the fact that the weights of $\mathrm{VF}(\widehat{\mathfrak{g}}^\vee[1]/G)$ are bounded below by -1 , one finds that points of

$$\mathrm{Pois}^{\mathrm{inf}}(\mathbf{BG}, 1)$$

correspond to pairs

$$\pi \in \mathcal{O}(G) \otimes_k \mathrm{Sym}_k^2(\mathfrak{g}[-1]), \quad \phi \in \mathrm{Sym}_k^3(\mathfrak{g}[-1]),$$

which satisfy the equations for a quasi-Poisson Lie group structure on G , the most important of which is $[\pi, \pi] = 2\phi$.

6.2. Relative setting and coisotropics. In this section, we introduce a formal group controlling deformations of coisotropic structures. To do so, we borrow an idea from [Cal19; CS24]. Let

$$j: C \rightarrow X$$

be a nil-isomorphism of derived stacks with perfect cotangent complex; in general we may replace any morphism between such stacks by its completion without loss of generality. By [Cal19, Thm. 2.8] and its extension [CS24, Thm. 3.6], the diagram

$$\begin{array}{ccc} \widehat{\mathrm{T}}^*[n]C/X & \xrightarrow{i} & \widehat{\mathrm{T}}^*[n+1]X \\ \pi_C \downarrow & & \downarrow \pi_X \\ C & \xrightarrow{j} & X \end{array}$$

is equipped with the following structure⁵, all of which is compatible with the fiberwise grading:

- an exact $(n+1)$ -symplectic form $\lambda \in \pi_0 \Omega_{\widehat{\mathrm{T}}^*[n+1]X}^{2, \mathrm{ex}}(n+1)$;

⁵In [Cal19], such a package is referred to as a nondegenerate isotropic square.

- an exact $(n+1)$ -Lagrangian structure $l \in \pi_0 \mathbf{Isot}_i^{\text{ex}}(n+1)$;
- a nondegenerate isotropic fibration structure on π_X , i.e. a path

$$\eta \in P_{\lambda/X,0}(\Omega_{\widehat{T}^*[n+1]X/X}^{2,\text{ex}}(n+1)),$$

such that the induced isotropic structure on $\mathbb{T}_{\pi_X} \rightarrow \mathbb{T}_{\widehat{T}^*[n+1]X}$ is nondegenerate; and

- a nullhomotopy ξ for the composite path $l \circ (i^* \eta)^{-1}$ in $\Omega_{\pi_C}^{2,\text{ex}}(n)$ whose leading term is nondegenerate.

The existence of l is essentially due to the factorization of i through

$$i_C: \widehat{T}^*[n]C/X \rightarrow \widehat{T}^*[n+1]X \times_X C,$$

namely it is the pullback of the canonical Lagrangian structure on $C \rightarrow \widehat{T}^*[n+1]C$ through i_C . Moreover (see [Cal19, p. 2.3.2]), the paths l, η compose to a formal loop β in

$$\Omega_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n+1)$$

based at 0, which is natural. That is, β defines a morphism

$$\beta: X \rightarrow \Omega_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n)$$

in $\mathcal{FMP}_{/X \times \mathbf{BG}_m}$.

Lemma 6.10. *There exists a morphism of formal group objects*

$$c_j: \mathcal{GP}_X(n) \rightarrow \text{Res}_{X_{\text{dR}}}^X \left(\Omega_{\beta} \widetilde{\Omega}_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n) \right)$$

over X_{dR} .

Proof. We may replace

$$\Omega_{\widehat{T}^*[n+1]X}^{2,\text{ex}}(n+1), \Omega_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n)$$

by

$$(\Omega_{\widehat{T}^*[n+1]X}^{2,\text{ex}}(n+1))_{\lambda}^{\wedge}, (\Omega_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n))_{\beta}^{\wedge},$$

since both based loop spaces are taken in formal moduli problems. Then the construction of β implies there is a linear map

$$(\Omega_{\widehat{T}^*[n+1]X}^{2,\text{ex}}(n+1))_{\lambda}^{\wedge} \times_{X_{\text{dR}}} X \rightarrow (\Omega_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n))_{\beta}^{\wedge},$$

taking the difference of homotopies. Since $\beta = \gamma \eta^{-1}$, it lifts to $\widehat{\text{Aut}}(\widehat{T}^*[n+1]X)$ -invariants. So, we get an induced morphism

$$(\widetilde{\Omega}_{\widehat{T}^*[n+1]X}^{2,\text{ex}}(n+1))_{\lambda}^{\wedge} \times_{X_{\text{dR}}} X \rightarrow \widetilde{\Omega}_{(\widehat{T}^*[n]C/X)/X}^{2,\text{ex}}(n),$$

which yields the desired group map upon taking loops. $\square$

Definition 6.11. Let $j: C \rightarrow X$ be a map of derived stacks with perfect cotangent complex. The group of n -coisotropic deformations of j , denoted by $\mathcal{GC}_j(n)$, is defined as the fiber

$$\mathcal{GC}_j(n) = \text{Fib}(c_j) \in \text{Grp}(\mathcal{FMP}_{/X_{\text{dR}}}),$$

where c_j is the morphism of Theorem 6.10. We will often omit the morphism from the notation and write

$$\mathcal{GC}_{C/X}(n) = \mathcal{GC}_j(n).$$

Theorem 6.12. *Let $j: C \rightarrow X$ be a morphism of stacks with perfect cotangent complex, and let $e: X_{\text{dR}} \rightarrow \mathcal{GC}_j(n)$ denote the identity section. The underlying graded object of $\text{Lie}(\mathcal{GC}_j(n))$ satisfies*

$$\begin{aligned} e^* \mathbb{T}_{\mathcal{GC}_j(n)/X_{\text{dR}}} &\simeq \text{gr}_H^0 \text{Fib} \left(\Omega_{X/X_{\text{dR}}}^{2,ex} [n+1] \rightarrow p_* \Omega_{C/X}^{2,ex} [n+1] \right) \\ &\simeq \text{Fib} \left(\Gamma_{X_{\text{dR}}}^{\text{gr}} (\widehat{T}^*[n+1]X, \mathcal{O}[n+1]) \rightarrow \Gamma_{X_{\text{dR}}}^{\text{gr}} (\widehat{T}^*[n](C/X), \mathcal{O}[n+1]) \right) \end{aligned}$$

in $\text{GrIndCoh}(X_{\text{dR}})$.

Proof. It will suffice to compute the relative tangent complex of

$$\mathcal{GC}_{C/X}(n) \rightarrow \mathcal{GP}_X(n)$$

at the identity. To do this, we employ the same strategy as in Theorem 6.4, replacing the structure map $\widehat{T}^*[n+1]X \rightarrow X_{\text{dR}}$ with $\widehat{T}^*[n]C/X \rightarrow X$. The nondegeneracy of ω is replaced by the nondegeneracy of β [Cal19, Lemma 2.18], and otherwise the proof proceeds the same way.

The final thing to verify is that the map

$$e^* \mathbb{T}_{\mathcal{GC}_{C/X}(n)} \rightarrow e^* \mathbb{T}_{\mathcal{GP}_X(n)}$$

is the one claimed. But this follows from the fact that c_j is induced by a linear map of sheaves, hence the tangent maps are identified. $\square$

Definition 6.13 (Relative infinitesimal polyvectors). Let

$$\text{Pol}_{C/X}^{\text{inf}}(n) = \text{Lie}(\mathcal{GC}_{C/X}(n)^{\text{gr}}) \in \text{Alg}_{\text{Lie}}^{\text{gr}}(\mathcal{QCoh}(X_{\text{dR}}))$$

be the graded Lie algebra of the group of n -coisotropic deformations of $C \rightarrow X$. We refer to $\text{Pol}_{C/X}^{\text{inf}}(n)$ as the *Lie algebra of infinitesimal relative n -polyvectors*.

Definition 6.14 (Infinitesimal coisotropic structures). Let $j: C \rightarrow X$ be a morphism of derived stacks with perfect cotangent complex. The space of *infinitesimal n -coisson structures on j* is the mapping space

$$\text{Cois}^{\text{inf}}(C/X, n) = \text{Map}_{\text{Alg}_{\text{Lie}}^{\text{gr}}(\mathcal{QCoh}(X_{\text{dR}}))}(\text{ins}^2 \mathcal{O}_{X_{\text{dR}}}[-1], \text{Pol}_{C/X}^{\text{inf}}(n)).$$

Observe that there is a map

$$\text{Cois}^{\text{inf}}(C/X, n) \rightarrow \text{Pois}^{\text{inf}}(X, n).$$

Suppose moreover that X is equipped with an infinitesimal n -Poisson structure $\pi \in \pi_0 \text{Pois}^{\text{inf}}(X, n)$. The space of *infinitesimal n -coisotropic structures on j* is

$$\text{Cois}_{\pi}^{\text{inf}}(C/X, n) = \text{Fib}_{\pi} \left(\text{Cois}^{\text{inf}}(C/X, n) \rightarrow \text{Pois}^{\text{inf}}(X, n) \right).$$

Example 6.15 (Classifying stacks). We revisit Theorem 6.9 and consider infinitesimal 1-coisson structures on the canonical morphism

$$\text{pt} \rightarrow BG.$$

What follows is a conceptual explanation; the precise calculations justifying it are omitted for the sake of space. We have

$$\widehat{T}^*[1](\text{pt}/BG) \simeq \widehat{\mathfrak{g}}^{\vee}[1],$$

and the conormal square thus takes the form

$$\begin{array}{ccc} \widehat{\mathfrak{g}}^\vee[1] & \xrightarrow{i} & \widehat{\mathfrak{g}}^\vee[1]/G \\ \downarrow & & \downarrow \pi \\ \mathrm{pt} & \longrightarrow & BG. \end{array}$$

Now an infinitesimal 1-Poisson structure on $\widehat{\mathfrak{g}}^\vee[1]/G$ yields in particular a deformation of the Lie algebra

$$\mathbb{T}_{\widehat{\mathfrak{g}}^\vee[1]/G}[-1] \simeq \mathfrak{g} \oplus \mathfrak{g}^\vee,$$

where the bracket on $\mathfrak{g}$ is the original bracket and restricts to zero on $\mathfrak{g}^\vee$. This is the deformation of the stack $\widehat{\mathfrak{g}}^\vee[1]/G$ underlying the Poisson structure. The deformation of the symplectic form yields a deformation of the *metric* on $\mathfrak{g} \oplus \mathfrak{g}^\vee$ given by the duality pairing, and exactness states that the deformation is compatible with the (G -equivariant) map of Lie algebras

$$\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}^\vee.$$

The complex

$$\Omega_{\widehat{T}^*[1](\mathrm{pt}/BG)^{\mathrm{gr}}/BG}^{2,\mathrm{ex}}$$

is represented by the graded complex

$$\mathrm{Sym}_k^*(\mathrm{ins}^1 \mathfrak{g}[-1]) \otimes \mathcal{O}_{BG} \xrightarrow{d} \mathrm{Sym}_k^*(\mathrm{ins}^1 \mathfrak{g}[-1]) \otimes (\mathrm{ins}^1 \mathfrak{g}[-1] \oplus \mathfrak{g}^\vee) \otimes \mathcal{O}_{BG}$$

of sheaves on BG . The element ξ is represented by the (G -equivariant) duality pairing between $\mathfrak{g}[-1]$ and $\mathbb{L}_{BG}$. The additional freedom to deform ξ in Hodge weight 0 corresponds to a deformation of the Lie algebra structure on $\mathfrak{g}^\vee$, compatibly with the G -action.

Thus 1-coisson structures on $\mathrm{pt} \rightarrow BG$ yield deformations of the canonical Manin triple

$$(\mathfrak{g} \oplus \mathfrak{g}^\vee, \mathfrak{g}, \mathfrak{g}^\vee).$$

By Drinfeld's theorem, these correspond to deformations of G as an algebraic Poisson Lie group. This can be upgraded to an equivalence of spaces, recovering [Saf21, Cor. 2.14].

Remark 6.16. By the discussion in the previous example and results of [Saf21], Theorem 8.1 can be understood as a generalization of Drinfeld's theorem on the correspondence between Poisson Lie groups and Manin triples. It is conceivable that this idea can be generalized to Manin triples of Lie *algebroids*, which would yield a definition of Lie bialgebroids over derived stacks based on shifted symplectic geometry. We hope to investigate this problem in the future.

7. CRYSTALLINE AND INFINITESIMAL

We will now begin setting the stage for the comparison between infinitesimal polyvectors and the theory of [Cal+17]. We will first summarize the construction of polyvectors from loc. cit., which we call *crystalline* polyvectors. It fundamentally uses the module theory of certain Hodge-filtered de Rham algebras, i.e. crystals. The construction is traditionally explained in terms of model categories of graded mixed algebras (or dg $\mathbb{D}^+$ -algebras), so we will attempt to spell out exactly how to translate between that language and the complete filtered world. In particular [Cal+17] define a certain ind-object of presheaves of graded mixed algebras which is considered the “true receptacle” of Poisson brackets of functions

on a derived stack. We will explain what happens to it upon passage to complete filtered categories.

In the second subsection, we explain how to correctly pass between the sheaf theories underlying crystalline and infinitesimal polyvectors. The upshot, which is essentially due to [Buc25], is that the “extra” complete filtrations appearing behind the scenes of crystalline polyvectors play the role of passing to ind-coherent sheaves. Thus the necessity of the passage to crystals reflects the idea that IndCoh is the natural category on which to do Lie theory on a stack.

7.1. Crystalline Poisson structures. We review the construction of [Cal+17]. Let X be a derived stack with perfect cotangent complex. Given a geometric point $u \in X(k)$, the shifted tangent complex at u , $\mathrm{T}_{X,u}[-1] \in \mathrm{Mod}_k$, acquires the structure of a Lie algebra. The philosophy of loc. cit. is to repackage the formal geometry of a derived stack X in terms of derived algebra by studying descent over X_{dR} for the Hodge-filtered Chevalley–Eilenberg cohomology of this Lie algebra — a vast generalization of Gelfand–Kazhdan descent. Specifically, the shifted tangent Lie algebra localized over X_{dR} encodes the Grothendieck flat connection on the sheaf of principal parts $p_*\mathcal{O}_X$. The difference between this approach and that of [GR17b] is that [Cal+17] takes the crystalline perspective: flat connections are encoded by crystals in the sense of Grothendieck, which in our setting translates to induced modules over the Hodge-filtered de Rham algebra.

First, we introduce the analogue of the structure sheaf of $\mathcal{O}_{X_{\mathrm{dR}}}$ in this picture.

Definition 7.1. The *filtered crystalline structure sheaf* of X is the presheaf of complete filtered algebras on X_{dR} defined by

$$\mathcal{D}_X^*(\mathrm{Spec} R \rightarrow X_{\mathrm{dR}}) = F_H^* \mathrm{dR}_{R^{\mathrm{red}}/R}.$$

By Theorem 2.15, $F^{-\infty}\mathcal{D}_X^* \simeq \mathcal{O}_{X_{\mathrm{dR}}}$, explaining the name.

To encode $p_*\mathcal{O}_{X_{\mathrm{dR}}}$, [Cal+17] use the idea that the shifted tangent Lie algebra $\mathrm{T}[-1]$ controls formal neighborhoods of points.

Lemma 7.2. *Let Y be a derived stack with perfect cotangent complex. We have*

$$F_H^* \mathrm{dR}_{Y_{\mathrm{red}}/Y} \simeq \lim_{\mathrm{Spec} R \rightarrow Y} F_H^* \mathrm{dR}_{R^{\mathrm{red}}/R}.$$

Proof. By descent for Y_{red} ,

$$(Y_{\mathrm{red}} \rightarrow Y) \simeq \lim_{\mathrm{Spec} R \rightarrow Y} (\mathrm{Spec} R^{\mathrm{red}} \rightarrow \mathrm{Spec} R)$$

in the category of pairs. The lemma now follows from Theorem 2.14. $\square$

Definition 7.3. The *filtered sheaf of principal parts* of X is the presheaf of $\mathcal{D}_X^*$ -algebras on X_{dR} given by

$$\mathcal{B}_X^*(\mathrm{Spec} R \rightarrow X_{\mathrm{dR}}) = F_H^* \mathrm{dR}_{R^{\mathrm{red}}/X_R^\wedge}.$$

By Theorem 2.15, $F^{-\infty}\mathcal{B}_X^* \simeq p_*\mathcal{O}_X$. There is a natural morphism of presheaves of complete filtered algebras $\mathcal{D}_X^* \rightarrow \mathcal{B}_X^*$.

The following is immediate from Theorem 7.2.

Corollary 7.4. *Let X be a derived stack with perfect cotangent complex. Then there is an equivalence*

$$\mathrm{gr}^*\mathcal{B} \simeq \Gamma_{X_{\mathrm{dR}}}(X, \mathrm{Sym}_X^*(\mathbb{L}_{X/X_{\mathrm{dR}}}[-1]))$$

in $\mathrm{GrQCoh}(X_{\mathrm{dR}})$.

We remark that, in [Cal+17], the above is stated only under the further assumption that X is an Artin derived stack. That assumption, at least for the results which we use, is removed by Theorem 7.2.

Recall our notation $\mathrm{MDer}(-, n)$ for the Lie algebra of n -shifted multiderivations.

Definition 7.5. Let X be a derived stack with perfect cotangent complex and let $n \in \mathbf{Z}$. Let

$$F_{\mathcal{D}}^* \mathrm{Pol}_X^{\mathrm{crys}}(n) = \mathrm{Forget}_{\mathrm{Lie}^{\mathrm{gr}}}^{\mathbf{p}_{n+2}^{\mathrm{gr}}}(\mathrm{MDer}(\mathcal{B}_X^*/\mathcal{D}_X^*, n+1)) \in \mathrm{Alg}_{\mathrm{Lie}^{\mathrm{gr}}}(\widehat{\mathrm{Fil}} \mathrm{QCoh}(X_{\mathrm{dR}})).$$

Note that the grading, coming from the graded Poisson operad, is distinct from the complete filtration degree arising from the crystalline nature of $\mathcal{B}_X^*$. We refer to the underlying object

$$\mathrm{Pol}_X^{\mathrm{crys}}(n) = F_{\mathcal{D}}^{-\infty} \mathrm{Pol}_X^{\mathrm{crys}}(n)$$

as the *Lie algebra of crystalline n -polyvectors on X* . Most of the time we will opt to retain the $F_{\mathcal{D}}^{-\infty}$ notation in order to emphasize the fact that we must take underlying objects.

The fact that the presheaf $\mathrm{Pol}_X^{\mathrm{crys}}(n)$ defines a quasi-coherent sheaf is not a priori immediate. See [Cal+17, Rmk. 2.4.10]. We prefer not to take sections and always work over X_{dR} . A careful comparison of $\mathrm{Pol}_X^{\mathrm{crys}}(n)$ to the objects of [Cal+17] will be given below.

Before stating what we will use about $\mathrm{Pol}_X^{\mathrm{crys}}(n)$, we elaborate on the somewhat confusing filtration $F_{\mathcal{D}}^*$. We have

$$(20) \quad F_{\mathrm{H}}^* \mathrm{dR}_{\mathcal{B}_X^*/\mathcal{D}_X^*} \in \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}_{\mathcal{D}_X^*},$$

where there are two complete filtrations: the “internal” filtration coming from $\mathcal{B}_X^*$ and the Hodge filtration F_{H}^* . The underlying object functor

$$F^{-\infty}: \mathcal{C}\mathrm{Alg}_{\mathcal{D}_X^*} \rightarrow \mathcal{C}\mathrm{Alg}_{F^{-\infty} \mathcal{D}_X^*} \simeq \mathcal{C}\mathrm{Alg}(\mathrm{QCoh}(X_{\mathrm{dR}}))$$

lifts to complete filtered objects:

$$F_{\mathcal{D}}^{-\infty}: \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}_{\mathcal{D}_X^*} \rightarrow \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}(\mathrm{QCoh}(X_{\mathrm{dR}})).$$

The notation emphasizes that underlying objects are only taken with respect to the “internal” filtration. In particular, applying it to (20) yields

$$F_{\mathcal{D}}^{-\infty} F_{\mathrm{H}}^* \mathrm{dR}_{\mathcal{B}_X^*/\mathcal{D}_X^*} \in \widehat{\mathrm{Fil}}^+ \mathcal{C}\mathrm{Alg}(\mathrm{QCoh}(X_{\mathrm{dR}})).$$

We use the same notation for the analogous construction on graded Lie algebra objects, as in Theorem 7.5.

The following result makes precise the statement that the filtered sheaf of principal parts governs differential calculus on the stack X .

Theorem 7.6 (Calaque–Pantev–Toën–Vaquié–Vezzosi). *Let X be a derived stack with perfect cotangent complex and $n \in \mathbf{Z}$.*

1. *There is a natural equivalence of complete filtered commutative algebra objects of $\mathrm{QCoh}(X_{\mathrm{dR}})$*

$$F_{\mathcal{D}}^{-\infty} F_{\mathrm{H}}^* \mathrm{dR}_{\mathcal{B}_X^*/\mathcal{D}_X^*} \simeq F_{\mathrm{H}}^* \mathrm{dR}_{X/X_{\mathrm{dR}}}.$$

2. *There is a natural equivalence of graded quasi-coherent sheaves over X_{dR}*

$$F_{\mathcal{D}}^{-\infty} \mathrm{Pol}_X^{\mathrm{crys}}(n) \simeq \Gamma_{X_{\mathrm{dR}}}(X, \mathrm{Sym}_X^*(\mathbb{T}_{X/X_{\mathrm{dR}}}[-n-1])).$$

Proof. See [Cal+17, Thm. 2.3.3, Thm. 2.3.8]. □

7.2. Explicit comparison to the graded mixed picture. In this extended remark, we make explicit the correspondence of objects and notation between ours and that of [Cal+17]. A summary is given by the following table:

Object	Counterpart in [Cal+17]
$\widehat{\text{Fil}}(\text{Mod}_k)$	$\epsilon - \text{Mod}_k^{\text{gr}}$
$F^{-\infty}$	$ - ^t$
$\text{ins}^2 \mathcal{O}_{X_{\text{dR}}}[-1]$	$k(2)[-1]$
$\mathcal{D}_X^*, \mathcal{B}_X^*$	$\mathbb{D}_{X_{\text{dR}}}, \mathcal{B}_X$
$F_{\mathcal{D}}^* \text{Pol}_X^{\text{crys}}(n)$	$\mathbf{Pol}^{\text{int}}(\mathcal{B}_X/\mathbb{D}_{X_{\text{dR}}}, n+1)[n+1]$
$F_{\mathcal{D}}^{-\infty} \text{Pol}_X^{\text{crys}}(n)$	$\mathbf{Pol}^t(X, n)[n+1]$

The objects $\mathbb{D}_{X_{\text{dR}}}$ and $\mathcal{B}_X$ are graded mixed algebra models for the presheaves $\mathcal{D}_X^*$ and $\mathcal{B}_X^*$, which follows from the fact that the de Rham graded mixed algebra models the Hodge-filtered de Rham algebra. The object $\text{Pol}_X^{\text{crys}}(n)$ is a graded Lie algebra object in the category of presheaves of complete filtered commutative algebras, so it corresponds to what [Cal+17] denote

$$\mathbf{Pol}^{\text{int}}(\mathcal{B}_X/\mathbb{D}_{X_{\text{dR}}}, n+1)[n+1].$$

Readers of [Cal+17] will notice that we do not consider an analogue of the ind-object $\mathcal{B}_X(\infty) \in \text{Ind}(\epsilon - \text{Mod}_k^{\text{gr}})$. Its use is only necessary when one considers the correct notion of $\mathfrak{p}_n^{\text{gr}}$ -algebra structure associated to a given polyvector. In fact it seems more natural not to introduce ind-objects at all and instead do the following.

For brevity, write $\mathcal{P}\text{Sh}(X_{\text{dR}}, k)$ for the category of Mod_k -valued presheaves on X_{dR} , and

$$\mathcal{C} = \widehat{\text{grFil}}(\mathcal{P}\text{Sh}(X_{\text{dR}}, k)).$$

Let $A \in \mathcal{C}\text{Alg}(\mathcal{C})$ and write

$$\mathfrak{p}_n^{\text{gr}}(A) = \text{Fib} \left(\text{Map}_{\text{Op}(\mathcal{C})}(\mathfrak{p}_n^{\text{gr}}, \text{end}(A)) \longrightarrow \text{Map}_{\text{Op}(\mathcal{C})}(\text{Com}, \text{end}(A)) \right)$$

for the space of lifts of the commutative algebra structure on A to a $\mathfrak{p}_n^{\text{gr}}$ -algebra structure, where $\text{end}(A)$ is the endomorphism operad of A . Since objects of $\mathcal{C}$ are in particular complete filtered objects in another category $\mathcal{C}'$, the mapping space

$$\text{Map}_{\mathcal{C}}(c, d) \in \mathcal{S}\text{pc}$$

admits a lift to *complete filtered spaces*

$$F^* \text{Map}_{\mathcal{C}}(c, d) \in \widehat{\text{Fil}}\mathcal{S}\text{pc},$$

such that

$$F^0 \text{Map}_{\mathcal{C}}(c, d) \simeq \text{Map}_{\mathcal{C}}(c, d).$$

Now if we replace F^0 by $F^{-\infty}$, the latter of which is the correct notion of underlying object, we get a modified definition of $\mathfrak{p}_n^{\text{gr}}(A)$:

$$\text{Fib} \left(F^{-\infty} \text{Map}_{\text{Op}(\mathcal{C})}(\mathfrak{p}_n^{\text{gr}}, \text{end}(A)) \longrightarrow F^{-\infty} \text{Map}_{\text{Op}(\mathcal{C})}(\text{Com}, \text{end}(A)) \right).$$

Letting $A = \mathcal{B}_X^*$ and working relative to $\mathcal{D}_X^*$ yields a modified space of $\mathfrak{p}_n^{\text{gr}}$ -algebra structures, which is the counterpart to $\mathfrak{p}_n^{\text{gr}}$ -algebra structures on $\mathcal{B}_X(\infty)$. Indeed, all uses of the (∞) -twist in [Cal+17] can be replaced by the analogous redefinition of algebras over operads in graded mixed objects. Namely, given a Koszul dg operad $\mathfrak{p}$ and graded mixed complex A , consider the convolution Lie algebra model

$$\text{Conv}(\mathfrak{p}, A) = \prod_{p \geq 0} \text{Hom}_{\Sigma_p}(\Omega \mathfrak{p}^\perp(p) \otimes A^{\otimes p}, A)$$

for the space of $\mathfrak{p}$ -algebra structures on A . In all definitions, replace the convolution Lie algebra by

$$\text{Conv}'(\mathfrak{p}, A) = \prod_{p \geq 0} \left| \text{Hom}_{\Sigma_p}^\epsilon(\Omega \mathfrak{p}^\perp(p) \otimes A^{\otimes p}, A) \right|^t,$$

where Hom^ϵ denotes the graded mixed complex of morphisms.

Example 7.7. Consider the case of crystalline 2-Poisson structures on a classifying stack $X = BG$ for G a reductive group. By degree considerations, the graded Lie algebra $\Gamma(BG; F_{\mathcal{D}}^{-\infty} \text{Pol}_{BG}^{\text{crys}}(2))$ is formal, and one verifies

$$\pi_0 \text{Pois}^{\text{crys}}(BG, 2) \simeq (\text{Sym}_k^2(\mathfrak{g}))^G.$$

Fix $\kappa \in (\text{Sym}_k^2(\mathfrak{g}))^G$. As explained in [Cal+17, Sec. 3.6.2], the $\mathfrak{p}_3$ -algebra structure on $\mathcal{B}_{BG}^*$ defined by κ is equivalent to a G_{dR} -equivariant $\mathfrak{p}_3$ -algebra structure on

$$\mathcal{B}_{BG}^* \simeq F_H^* \text{dR}_{\text{pt}/BG} \simeq C^*(\mathfrak{g}).$$

The Chevalley–Eilenberg complex

$$(\text{Sym}_k^*(\mathfrak{g}^\vee[-1]), \partial_{\text{CE}})$$

is a dg $\mathbb{D}^+$ -algebra model for $C^*(\mathfrak{g})$, and an explicit $\mathfrak{p}_3^{\text{gr}}$ -algebra structure on it is uniquely specified by

$$\kappa: \text{Sym}_k^2(\mathfrak{g}^\vee[-1]) \rightarrow k.$$

The key point is this is a Lie^{gr} bracket which is *itself* of weight -1 , and therefore defines an element of

$$\pi_0 F^{-1} \text{Map}_{\text{Op}(\mathcal{C})}(\mathfrak{p}_3^{\text{gr}}, \text{end}(A)),$$

which is not present in

$$\text{Map}_{\text{Op}(\mathcal{C})}(\mathfrak{p}_3^{\text{gr}}, \text{end}(A)) \simeq F^0 \text{Map}_{\text{Op}(\mathcal{C})}(\mathfrak{p}_3^{\text{gr}}, \text{end}(A)).$$

We refer the reader to [Saf21, Sec. 1.5] for a detailed discussion of crystalline polyvectors on BG .

7.3. Crystals and D-modules. With a view towards the main theorem, we will now explain how to compare the category of modules over the crystalline principal parts to quasicoherent and ind-coherent sheaves on the original stack. We will first need a precise statement which expresses the relative de Rham cohomology of a formal moduli problem in terms of Lie algebroid cohomology. See also [Tom25, Thm. 2.40].

Proposition 7.8. *Let $S = \text{Spec } A$ be an affine derived scheme and $\mathcal{X} \in \mathcal{FMP}_{S/}$ given by a Lie algebroid L over S . Then there is an equivalence of complete filtered commutative algebras*

$$C^*(L) \simeq F_H^* \text{dR}_{S/\mathcal{X}}.$$

Proof. Consider an affine $\mathrm{Spec} R$ and an R -point $\mathrm{Spec} R \rightarrow \mathcal{X}$. Let $S_R = \mathrm{Spec} R \times_{\mathcal{X}} S$, and let $\mathrm{Spec} B \rightarrow S_R$ be an affine over S_R :

$$\begin{array}{ccccc} \mathrm{Spec} B & \longrightarrow & S_R & \longrightarrow & S \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & \mathrm{Spec} R & \longrightarrow & \mathcal{X}. \end{array}$$

By definition of the de Rham algebra,

$$F_H^* dR_{S/\mathcal{X}} \simeq \lim_{\mathrm{Spec} R \rightarrow \mathcal{X}} \lim_{\mathrm{Spec} B \rightarrow S_R} F_H^* dR_{B/R}.$$

On the other hand, by [Fu24, Thm. 4.26] and [TV22, Prop. 1.2.3], we may in fact write

$$C^*(L) \simeq \lim_{\mathrm{Spec} R \rightarrow \mathcal{X}} \lim_{\mathrm{Spec} B \rightarrow S_R} C^*(T_{B/R})$$

in $\widehat{\mathrm{Fil}}\mathcal{A}lg_R$, where $T_{B/R}$ is the tangent Lie algebroid. By the universal property of Hodge-filtered de Rham cohomology, there is a canonical map

$$F_H^* dR_{B/R} \rightarrow C^*(T_{B/R}).$$

In [CG19, Lemma A.4], this map is proven to be an equivalence. We use the descent statements above to conclude. $\square$

Recall the categories $\mathrm{Mod}_{C^*(L)}^{\mathrm{in}}$ and $\mathrm{IndCoh}_0^X(\mathcal{Z})$ defined in Section 3.7. The following proposition answers the question of what is the role played by the complete filtration in the definition of crystalline principal parts.

Proposition 7.9. *Let $S = \mathrm{Spec} A$ be an affine derived scheme and let $f: S \rightarrow \mathcal{Z}$ be an object of $\mathcal{FMP}_{S/}$. Assume $\mathcal{Z}$ admits a perfect cotangent complex. There is an equivalence of ∞ -categories*

$$\mathcal{C}: \mathrm{IndCoh}_0^S(\mathcal{Z}) \xrightarrow{\simeq} \mathrm{Mod}_{F_H^* dR_{S/\mathcal{Z}}}^{\mathrm{in}}(\widehat{\mathrm{Fil}}^+ \mathcal{QCoh}(\mathcal{Z})),$$

such that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}_0^S(\mathcal{Z}) & \xrightarrow{\mathcal{C}} & \mathrm{Mod}_{F_H^* dR_{S/\mathcal{Z}}}^{\mathrm{in}}(\widehat{\mathrm{Fil}}^+ \mathcal{QCoh}(\mathcal{Z})) \\ \mathrm{pr} \downarrow & & \downarrow \mathrm{gr} \\ \mathcal{QCoh}(S) & \xrightarrow{-\otimes_A(\mathrm{gr}_H^* dR_{S/\mathcal{Z}})} & \mathrm{Mod}_{\mathrm{gr}_H^* dR_{S/\mathcal{Z}}}(\mathrm{Gr}^+ \mathcal{QCoh}(\mathcal{Z})) \end{array}$$

commutes.

Proof. Using Theorem 3.27, let $L \in \mathcal{LA}_A$ be a Lie algebroid such that $B_A L \simeq \mathcal{Z}$. By Theorem 3.34, we get an equivalence

$$\nu_L: \mathrm{IndCoh}_0^S(\mathcal{Z}) \simeq \mathrm{Mod}_{C^*(L)}^{\mathrm{in}}(\widehat{\mathrm{Fil}}^+ \mathcal{QCoh}(\mathcal{Z})).$$

Applying Theorem 7.8, we obtain a composed equivalence of categories

$$\mathcal{C}: \mathrm{IndCoh}_0^S(\mathcal{Z}) \simeq \mathrm{Mod}_{F_H^* dR_{S/\mathcal{Z}}}^{\mathrm{in}}(\widehat{\mathrm{Fil}}^+ \mathcal{QCoh}(\mathcal{Z}))$$

with the desired commutativity. $\square$

Corollary 7.10. *Let $R \in \mathfrak{aCAlg}_k^{\text{fp}}$ and consider the natural embedding $i: \text{Spec } R^{\text{red}} \rightarrow \text{Spec } R$; note that i exhibits $\text{Spec } R$ as an object of $\mathcal{FMP}_{\text{Spec } R^{\text{red}}/}$. Then the diagram*

$$\begin{array}{ccc} \text{Mod}_R & & \\ \Upsilon_R \downarrow & \searrow -\otimes_R \mathcal{D}^*(R) & \\ \text{IndCoh}_0(R) & \xrightarrow[\mathcal{C}]{} & \text{Mod}_{\mathcal{D}^*(R)}^{\text{in}}(\widehat{\text{Fil}}^+ \text{Mod}_R) \end{array}$$

commutes.

Proof. By compact generation and continuity of the functors involved, it suffices to restrict to $\text{Perf}(R) \subset \text{Mod}_R$ and verify commutativity on the unit R . Using conservativity of gr^* , we pass to $\text{Mod}_{\text{gr}^* \mathcal{D}(R)}$. We may conclude using Theorem 7.8 and the commutativity of the square in Theorem 7.9. $\square$

Remark 7.11. Theorem 7.10 confirms the conjecture made in [Cal+17, Remark 2.2.6]. Note that one must restrict to ind-coherent sheaves which are quasicoherent over $\text{Spec } R^{\text{red}}$ because the crystalline perspective introduces boundedness issues.

8. IDENTIFICATION OF DEFORMATION THEORIES

In this section, we finally state and prove the main theorem: the equivalence between crystalline and infinitesimal polyvectors.

Theorem 8.1. *Let X be a finite type derived stack with perfect cotangent complex and let $n \in \mathbf{Z}$. Then there is a natural equivalence of Lie algebra objects*

$$\mathbf{F}_{\mathcal{D}}^{-\infty} \text{Pol}_X^{\text{crys}}(n) \simeq \text{Pol}_X^{\text{inf}}(n)$$

in $\text{GrQCoh}(X_{\text{dR}}) \simeq \text{GrIndCoh}(X_{\text{dR}})$.

The proof will occupy the rest of the section. It will be useful to ponder the following diagram:

$$\begin{array}{ccccc} & & \widehat{\mathbf{T}}^*[n+1](X_R^\wedge/R) & \longrightarrow & \widehat{\mathbf{T}}^*[n+1]X \\ & \nearrow j & \downarrow & \nearrow & \downarrow \\ \text{Spec } R^{\text{red}} & \xrightarrow{\bar{i}} & X_R^\wedge & \xrightarrow{\bar{u}} & X \\ & \searrow \bar{p} & \downarrow \bar{p} & \lrcorner & \downarrow p \\ & & \text{Spec } R & \xrightarrow{u} & X_{\text{dR}} \end{array}$$

We note the following:

- The maps $i, \bar{i}, j$ are objects of $\mathcal{FMP}_{R^{\text{red}}/}$;
- The map $\bar{p}$ is an object of $\mathcal{FMP}_{/R}$;
- All four downward maps are nil-isomorphisms.

8.1. Conceptual proof outline. The proof is somewhat delicate, so we will attempt to give the reader a guide here, highlighting the main objects involved. We will give more precise definitions in the main body of the proof. The construction of crystalline polyvectors uses strict differential graded models in an essential way: the Lie bracket of multiderivations is defined by explicit operadic formulas. Thus, the proof must use cofibrant models in order to construct a Lie algebra map $\text{Pol}_X^{\text{crys}}(n) \rightarrow \text{Pol}_X^{\text{inf}}(n)$ in the first place.

It suffices to replace $X \rightarrow X_{\text{dR}}$ with $X_R^\wedge \rightarrow \text{Spec } R$. Besides $\mathcal{D}^*(R)$ and $\mathcal{B}^*(R)$, we will crucially use

$$\mathcal{A}^*(R) = \text{Sym}_{\mathcal{B}^*(R)}(\mathbb{T}_{\mathcal{B}^*(R)/\mathcal{D}^*(R)}[-n-1]),$$

which we think of as a model for $\widehat{\mathbb{T}}^*[n+1](X_R^\wedge/R)$. In particular, it has an exact $(n+1)$ -symplectic form ω . The proof consists of the following sequence of steps:

1. Define a new Lie algebra

$$\text{Der}_0(\mathcal{A}^*(R)/\mathcal{D}^*(R)),$$

which models the *homotopy stabilizer* of ω in $|\Omega_{\mathcal{A}^*(R)/\mathcal{D}^*(R)}^{2,\text{ex}}|$ under the natural action of $\text{Der}(\mathcal{A}^*(R)/\mathcal{D}^*(R))$.

2. Construct a Lie algebra map

$$\beta: \text{Pol}_X^{\text{crys}}(n) \rightarrow \text{Der}(\mathcal{A}^*(R)/\mathcal{D}^*(R)),$$

which we think of as a *Hamiltonian vector field map*. Cofibrant models are especially taken advantage of at this step: we use *strict nondegeneracy* of ω , as well as a *strict Cartan calculus*.

3. Show that β lifts to

$$\kappa: \text{Pol}_X^{\text{crys}}(n) \rightarrow \text{Der}_0(\mathcal{A}^*(R)/\mathcal{D}^*(R)),$$

and that κ is an equivalence of Lie algebra objects. This comes down to the *Cartan homotopy formula* from step 2.

4. Show that there is an equivalence of Lie algebra objects $\text{Der}_0(\mathcal{A}^*(R)/\mathcal{D}^*(R)) \simeq \text{Pol}_X^{\text{inf}}(n)$ by identifying the Lie algebra actions defining each side. The key to making this work is Theorem 7.9.

The next subsection will describe the cofibrant models in some more detail. The four subsections which follow correspond to steps 1 through 4 in sequence.

A warning about notation: we will be using $\text{dg } \mathbb{D}^+$ -algebras (aka graded mixed algebras), and we will write $|-|$ for the total object functor which models $F^{-\infty}$. This is *not* what is denoted $|-|$ in [Cal+17], and rather is identified with the functor $|-|^t$ of loc. cit. The latter is the one which is a model for the underlying object of a filtered object in an ∞ -category, thus we reward it with the more natural notation.

8.2. Cofibrant models. Fix once and for all a cofibrant model for $R \rightarrow R^{\text{red}}$ in non-positively graded commutative dg algebras; without loss of generality, we assume that R^{red} is a semi-free dg algebra, and R is a semi-free subalgebra on a subset of the generators of R^{red} . Then the $\text{dg } \mathbb{D}^+$ -algebra of relative differential forms

$$\mathcal{D} = (\text{Sym}_{R^{\text{red}}}^\bullet(\mathbb{L}_{R^{\text{red}}/R}[-1]), \epsilon = \text{d})$$

models the complete filtered algebra $\mathcal{D}^*(R)$. Choose a cofibration

$$\mathcal{D} \rightarrow \mathcal{B}$$

of $\text{dg } \mathbb{D}^+$ -algebras modeling the natural map $\mathcal{D}^*(R) \rightarrow \mathcal{B}^*(R)$.

Define a new commutative $\mathbb{D}^+$ -algebra $\mathcal{A}$ over R by the formula

$$\mathcal{A} = \text{Sym}_{\mathcal{B}} \left(\text{Hom}_{\mathcal{B}}(\Omega_{\mathcal{B}/\mathcal{D}}^1[n+1], \mathcal{B}) \right).$$

By cofibrancy of $\mathcal{D} \rightarrow \mathcal{B}$, it follows that $\mathcal{A}$ is a cofibrant model for the (complete) symmetric algebra over $\mathcal{B}^*(R)$ generated by $\mathbb{T}_{\mathcal{B}^*(R)/\mathcal{D}^*(R)}$. Moreover,

$$\mathcal{B} \rightarrow \mathcal{A}$$

is a cofibration of $\mathbb{D}^+$ -algebras. By definition, we have

$$\mathrm{Pol}_{X_R^\wedge}^{\mathrm{crys}}(n) \simeq \mathrm{MDer}(\mathcal{B}/\mathcal{D}, n)$$

as graded $\mathfrak{p}_{n+2}$ -algebras.

We will denote the $\mathrm{Sym}_{\mathcal{B}}$ -grading of $\mathcal{A}$ by $\mathrm{gr}_{\mathrm{rel}}^*\mathcal{A}$, for “relative”. It coincides with the fiberwise grading of the shifted cotangent stack. By its definition, the tautological 1-form on $\widehat{\mathbb{T}}^*[n+1]X_R^\wedge/R$ admits a strict model in terms of $\mathcal{A}$. Indeed, it is the $\mathrm{gr}_{\mathrm{rel}}^1\mathcal{A}$ -linear section of

$$\mathcal{A} \otimes_{\mathcal{B}} \Omega_{\mathcal{B}/\mathcal{D}}^1[-n-1]$$

corresponding to the duality pairing

$$(21) \quad \Omega_{\mathcal{B}/\mathcal{D}}^1 \otimes_{\mathcal{B}} \mathrm{gr}_{\mathrm{rel}}^1\mathcal{A} \rightarrow \mathcal{B}$$

in the 1-category of $\mathbb{D}^+$ -dg-modules over $\mathcal{B}$. We denote the resulting $(n+1)$ -shifted 1-form by $\theta_{\mathcal{A}}$. Since (21) is a strictly nondegenerate pairing, the induced exact 2-form

$$\lambda = d\theta_{\mathcal{A}}$$

is also strictly nondegenerate. We write

$$(22) \quad \omega^{-1}: \Omega_{\mathcal{A}/\mathcal{D}}^1 \rightarrow T_{\mathcal{A}/\mathcal{D}}[-n-1]$$

for the equivalence resulting from “raising indices” by λ .

We will now introduce the Cartan calculus defined by classical formulas which will be useful in the second and third steps of the proof.

Definition 8.2. Let $\mathbb{k}, A$ be cofibrant commutative $\mathbb{D}^+$ -algebras over R and let $\mathbb{k} \rightarrow A$ be a cofibration. We define a morphism of A -modules

$$\iota: \mathrm{Der}(A/\mathbb{k}) \rightarrow \mathrm{Der}\left(\mathrm{Sym}_A^\bullet(\Omega_{A/\mathbb{k}}^1[-1])\right)-1$$

as the unique extension of the duality pairing

$$\mathrm{Der}(A/\mathbb{k}) \otimes_A \Omega_{A/\mathbb{k}}^1[-1] \simeq T_{A/\mathbb{k}} \otimes_A \Omega_{A/\mathbb{k}}^1[-1] \rightarrow A[-1]$$

satisfying the Leibniz rule. Note that the weight shift (-1) occurs because it by construction decreases symmetric power degree by 1. Define also

$$(23) \quad \mathcal{L} = [d, \iota]: \mathrm{Der}(A/\mathbb{k}) \rightarrow \mathrm{Der}\left(\mathrm{Sym}^*(\Omega_{A/\mathbb{k}}^1[-1])\right).$$

We observe that $\mathcal{L}$ factors through endomorphisms of the commutative $\mathbb{D}^+$ -algebra $\Omega_{A/\mathbb{k}}^\bullet$, i.e. commutes with the de Rham differential d .

In particular, specializing now to $\mathbb{k} = \mathcal{D}, A = \mathcal{A}$, we get a Lie algebra representation

$$\mathrm{Der}(\mathcal{A}/\mathcal{D}) \rightarrow \mathrm{End}(\mathrm{Sym}_{\mathcal{A}}(\Omega_{\mathcal{A}/\mathcal{D}}^1[-1]), d).$$

Let $\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}$ denote the $\mathbb{D}^+$ -dg-subalgebra

$$\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]} = \left(\mathcal{A} \oplus \Omega_{\mathcal{A}/\mathcal{D}}^1[-1](1), d \right) \subset (\mathrm{Sym}_{\mathcal{A}}(\Omega_{\mathcal{A}/\mathcal{D}}^1[-1]), d).$$

This is an explicit model for the complex of $(n+1)$ -shifted exact 2-forms on $\mathcal{A}$. We get a representation

$$(24) \quad \mathrm{Der}(\mathcal{A}/\mathcal{D}) \rightarrow \mathrm{End}(\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1], d),$$

and therefore a representation on the underlying object:

$$(25) \quad a: \mathrm{Der}(\mathcal{A}/\mathcal{D}) \rightarrow \mathrm{End}_{\mathcal{D}}\left(|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|\right).$$

Notice also that we have

$$I: \mathrm{Der}(\mathcal{A}/\mathcal{D}) \rightarrow \mathrm{End}_{\mathcal{A}}\left(\mathrm{gr}\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]\right)-1,$$

which provides a nullhomotopy for the action map a by the Cartan formula (23).

As a summary, the three distinguished dg $\mathbb{D}^+$ -algebras are:

- $\mathcal{D}$, which represents $F_H^* dR_{R^{\mathrm{red}}/R}$ and is thought of as the structure sheaf (perhaps more accurately, the dualizing sheaf) of $\mathrm{Spec} R$;
- $\mathcal{B}$, which represents $F_H^* dR_{R^{\mathrm{red}}/X_R^\wedge}$, is an algebra over $\mathcal{D}$ and plays the role of the structure sheaf of $X_R^\wedge$;
- $\mathcal{A} = \mathrm{Sym}_{\mathcal{B}}(\mathbb{T}_{\mathcal{B}/\mathcal{D}}[-n-1])$, which is equipped with an extra grading $\mathrm{gr}_{\mathrm{rel}}\mathcal{A}$ by the symmetric algebra degree and plays the role of the cotangent stack $\hat{T}^*[n+1]X_R^\wedge/R$. It is also equipped with an exact $(n+1)$ -symplectic form $\lambda \in \pi_0|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|$ which has rel-degree 1.

The algebra $\mathcal{D}$ is the “base ring” for all constructions, and in particular we keep in mind that there may be multiple $\mathbb{D}^+$ -structures on one object. The derivations of $\mathcal{A}$ over $\mathcal{D}$ are thought of as vector fields on $\hat{T}^*[n+1]X_R^\wedge/R$, and the multiderivations $\mathrm{MDer}(\mathcal{B}/\mathcal{D}, n+1)$ are thought of as polyvectors on $X_R^\wedge$.

As a last remark, note that we do *not* choose any kind of generators over $\mathcal{D}$ for our algebras — indeed, we will only need that the relevant cotangent complexes are *dualizable*. In this way, we do not introduce any more formulas than are already required by the operadic construction of $\mathrm{Pol}^{\mathrm{crys}}$.

8.3. Hamiltonian derivations. We are now ready to begin the proof itself. In this subsection, we define the dg Lie algebra $\mathrm{Der}_0(\mathcal{A}/\mathcal{D})$. Recall that $\mathcal{A}$ is equipped with a “tautological” $(n+1)$ -shifted exact 2-form

$$\lambda \in \pi_0|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|,$$

which is nondegenerate. We now study the interaction of λ with the action of derivations.

Using the action (25), consider the dg Lie algebra over $\mathcal{D}$ defined by

$$\mathrm{Der}(\mathcal{A}/\mathcal{D}) \ltimes |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]| = \mathrm{Der}(\mathcal{A}/\mathcal{D}) \oplus |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|,$$

with Lie bracket

$$[(\xi, \alpha), (\xi', \alpha')] = ([\xi, \xi'], \mathcal{L}(\xi)\alpha' - \mathcal{L}(\xi')\alpha).$$

Given a class $\alpha \in \pi_0|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|$, the action morphism induces a dg Lie algebra morphism

$$\rho_\alpha: \mathrm{Der}(\mathcal{A}/\mathcal{D}) \rightarrow \mathrm{Der}(\mathcal{A}/\mathcal{D}) \ltimes |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|$$

defined by

$$\rho(\xi) = (\xi, \mathcal{L}(\xi)\alpha).$$

We use the map ρ_λ to give a definition of $\text{Der}_0(\mathcal{A}/\mathcal{D})$ as derivations which preserve the exact $(n+1)$ -symplectic form λ up to homotopy. Precisely, it is the following.

Definition 8.3. The dg Lie algebra $\text{Der}_0(\mathcal{A}/\mathcal{D})$ over $\mathcal{D}$ is defined to be the homotopy fiber product

$$\begin{array}{ccc} \text{Der}_0(\mathcal{A}/\mathcal{D}) & \longrightarrow & \text{Der}(\mathcal{A}/\mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \rho_0 \\ \text{Der}(\mathcal{A}/\mathcal{D}) & \xrightarrow{\rho_\lambda} & \text{Der}(\mathcal{A}/\mathcal{D}) \times |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]| \end{array}$$

in $\text{Alg}_{\text{Lie}}^{\text{gr}}(\text{Mod}_{\mathcal{D}})$.

We observe that the underlying $\mathcal{D}$ -module of $\text{Der}_0(\mathcal{A}/\mathcal{D})$ is equivalent to the homotopy fiber

$$\text{Fib} \left(\mathcal{L}(-)\lambda: \text{Der}(\mathcal{A}/\mathcal{D}) \rightarrow |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]| \right).$$

The construction is motivated by the fact that it provides a homotopy-invariant definition of the *stabilizer* of λ under the action of the dg Lie algebra $\text{Der}(\mathcal{A}/\mathcal{D})$. Thus, the Lie algebra $\text{Der}_0(\mathcal{A}/\mathcal{D})$ is a cochain-level definition of Hamiltonian derivations. It should be thought of as the affine analogue of $\text{Pol}_X^{\text{inf}}(n)$; step 4 will turn this comparison into an equivalence of Lie algebra objects.

8.4. The derivation on $\mathcal{A}$ associated to a multiderivation. We now proceed with step 2: the construction of a dg Lie algebra map

$$\beta: \text{MDer}(\mathcal{B}/\mathcal{D}, n+1) \rightarrow \text{Der}(\mathcal{A}/\mathcal{D})$$

over $\mathcal{D}$. First, by definition of multiderivations, there is a provided (strict) isomorphism

$$\sigma: \text{MDer}(\mathcal{B}/\mathcal{D}, n+1) \simeq \mathcal{A}$$

of commutative dg algebras over $\mathcal{D}$. Recall also the isomorphism (22)

$$\omega^{-1}: \Omega_{\mathcal{A}/\mathcal{D}}^1 \simeq T_{\mathcal{A}/\mathcal{D}}[-n-1].$$

Definition 8.4. We define β to be the composition

$$\text{MDer}(\mathcal{B}/\mathcal{D}, n+1) \xrightarrow{\sigma} \mathcal{A} \xrightarrow{\text{d}} \Omega_{\mathcal{A}/\mathcal{D}}^1 \xrightarrow{\omega^{-1}} \text{Der}(\mathcal{A}/\mathcal{D}).$$

Since ω^{-1} is linear with respect to gr_{rel}^* , we see that β intertwines multivector grading with the grading on $\text{Der}(\mathcal{A}/\mathcal{D})$ induced by $\text{gr}_{\text{rel}}^*\mathcal{A}$.

The following lemma is key to the existence of the map in the main theorem.

Lemma 8.5. *The $\mathcal{D}$ -module morphism β is a morphism of (graded) dg Lie algebras.*

Proof. This comes down to the classical calculation. The domain Lie algebra is determined by the bracket in weight 1, i.e. on $\text{Der}(\mathcal{B}/\mathcal{D})$. For $\xi, \xi' \in \text{Der}(\mathcal{B}/\mathcal{D})$,

$$\iota_{[\beta(\xi), \beta(\xi')]} \omega = [\mathcal{L}_{\beta(\xi)}, \iota_{\beta(\xi')}] \omega.$$

It is quick to see that $\iota_{\beta(\xi)} \omega = \text{d}\sigma(\xi) \in \Omega_{\mathcal{A}/\mathcal{D}}^1$ — indeed, λ itself is defined via the duality pairing ι . But then we see that

$$\iota_{[\beta(\xi), \beta(\xi')]} \omega = \mathcal{L}_{\beta(\xi)} \iota_{\beta(\xi')} \omega = \text{d}[\sigma(\xi), \sigma(\xi')],$$

which implies the lemma. $\square$

8.5. Preservation of the exact symplectic structure. We have constructed the graded dg Lie algebra $\mathrm{Der}_0(\mathcal{A}/\mathcal{D})$, as well as a graded dg Lie algebra morphism to it from $\mathrm{MDer}(\mathcal{B}/\mathcal{D}, n+1)$. We arrive at step 3, which is precisely the content of the following proposition.

Proposition 8.6. *The map β factors through a morphism*

$$(26) \quad \kappa: \mathrm{MDer}(\mathcal{B}/\mathcal{D}, n+1) \rightarrow \mathrm{Der}_0(\mathcal{A}/\mathcal{D})$$

in the homotopy category of dg Lie algebras over $\mathcal{D}$. Moreover, κ is an equivalence.

Proof. Recall the morphism ρ_λ from step 1 and consider the composition $\mathrm{pr}_2 \circ \rho_\lambda \circ \beta$, where pr_2 is the projection onto $|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}|$. Explicitly,

$$\mathrm{pr}_2 \rho_\lambda(\beta(\xi)) = \mathcal{L}_{\omega^{-1}d\sigma(\xi)}\lambda = d\sigma(\xi) + \iota_{\omega^{-1}d\sigma(\xi)}\lambda.$$

Let

$$h: |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}| \rightarrow |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}|[-1]$$

be given by

$$h(\xi) = \sigma^{-1}(\xi) + \iota_{\omega^{-1}d\sigma(\xi)}\lambda$$

for $\xi \in \mathcal{A}$ and $h(\eta) = 0$ for $\eta \in \Omega_{\mathcal{A}/\mathcal{D}}^1$. By the Cartan identity⁶

$$[\mathcal{L}_v, \iota_w] = \iota_{[v, w]},$$

h is equivariant for the $\mathrm{Der}(\mathcal{A}/\mathcal{B})$ -action. We thus get a homotopy

$$\rho_\lambda \circ \beta \Rightarrow \rho_0 \circ \beta$$

between Lie algebra morphisms. By the construction of $\mathrm{Der}_0(\mathcal{A}/\mathcal{D})$, h induces a dg Lie algebra map κ as indicated.

To prove κ is an equivalence, it suffices to show it is a homotopy equivalence of $\mathcal{D}$ -modules. The underlying module of $\mathrm{Der}_0(\mathcal{A}/\mathcal{D})$ is given by

$$(\mathrm{Der}(\mathcal{A}/\mathcal{D}) \oplus |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}|[-1], \partial_c),$$

where

$$\partial_c(v, \eta) = (\partial v, \mathcal{L}_v\lambda + \partial\eta - d\eta_0)$$

is the cocone differential. Here, ∂ denotes internal differentials, and η_0 is the projection of η onto the 0th associated graded of $\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}$. Define the $\mathcal{D}$ -module map

$$\theta: (\mathrm{Der}(\mathcal{A}/\mathcal{D}) \oplus |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}|[-1], \partial_c) \rightarrow \mathcal{A}$$

by the formula

$$\theta(v, \eta) = -\iota_v\lambda + \eta_0.$$

By the Cartan homotopy formula, we get $\theta \circ \kappa = \mathrm{id}$. On the other hand,

$$\kappa(\theta(v, \eta)) = (v - \omega^{-1}(\mathcal{L}_v\lambda - d\eta_0), \eta_0 - \iota_{\omega^{-1}(\mathcal{L}_v\lambda - d\eta_0)}\lambda).$$

For $v \in \mathrm{Der}(\mathcal{A}/\mathcal{D})$ and $\eta \in |\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}|$, let

$$h'(v, \eta) = (\omega^{-1}\eta_1, \iota_{\omega^{-1}\eta_1}\lambda),$$

⁶Note that this classical equation shows that the Lie bracket of vector fields is a derived bracket, which means precisely that it is an appropriate homotopy fiber.

where η_1 is the projection of η onto $\Omega_{\mathcal{A}/\mathcal{D}}^1[n+1]$. Noting that h' commutes with internal differentials, we have

$$\begin{aligned} (\partial_c h' + h' \partial_c)(v, \eta) &= (\omega^{-1}(\mathcal{L}_v \lambda - d\eta_0), \iota_{\omega^{-1}(\mathcal{L}_v \lambda - d\eta_0)} \lambda) \\ &= (\kappa \circ \theta - \text{id})(v, \eta). \end{aligned}$$

This concludes the proof of the proposition. $\square$

8.6. Lifting back to formal moduli. We have now completed steps 1 through 3. In short, we have constructed an equivalence of dg Lie algebras

$$\kappa: \text{MDer}(\mathcal{B}/\mathcal{D}, n+1) \rightarrow \text{Der}_0(\mathcal{A}/\mathcal{D}),$$

where the target is defined on the cochain level using the natural action of $\text{Der}(\mathcal{A}/\mathcal{D})$ on the complex of exact 2-forms. We will now complete the final step: providing an equivalence of dg Lie algebras $\text{Der}_0(\mathcal{A}/\mathcal{D}) \simeq \text{Pol}_X^{\text{inf}}(n)$.

Observe that both $\text{Der}_0(\mathcal{A}/\mathcal{D})$ and $\text{Pol}_X^{\text{inf}}(n)$ are given by similar-looking constructions: one is a homotopy pullback defined from a Lie algebra representation on the cochain level, the other is a pullback in an ∞ -category arising from an abstractly defined action of a Lie algebra object. To identify them, we will invoke Theorem 7.9 to import $\text{Pol}_X^{\text{inf}}(n)$ into the realm of model categories, and identify the constituents of the two constructions. Thus, our strategy will be to identify the $\text{Der}(\mathcal{A}/\mathcal{D})$ -action on $|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]|$ with a representative of the $\text{VF}(\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R)$ -module structure on $\Omega_{\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R}^{2,\text{ex}}[n+1]$.

First, using Theorem 7.6 and Theorem 7.9, $\mathcal{C}\Omega_{\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R}^{2,\text{ex}}[n+1]$ is equivalent to

$$|\Omega_{\mathcal{A}/\mathcal{D}}^{[0,1]}[n+1]| \in \text{GrMod}_{\mathcal{D}}.$$

Moreover, the underlying graded ind-coherent sheaf over R of $\text{VF}(\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R)$ is $\bar{p}_* \mathbb{T}_{\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R}$, which we know is modeled by $\mathbb{T}_{\mathcal{A}/\mathcal{D}} \in \text{GrMod}_{\mathcal{D}}$. By the identification of the $\mathbf{T}$ -action with the de Rham differential and using Theorem 5.7 and Eq. (23) respectively, we deduce that it will suffice to show that the Lie bracket on $\text{Der}(\mathcal{A}/\mathcal{D})$ is a model of the Lie algebra structure on $\bar{p}_* \mathbb{T}_{\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R}$.

Thus, we have reduced the main theorem to the following.

Proposition 8.7. *There is an equivalence*

$$\text{Der}(\mathcal{A}/\mathcal{D}) \simeq \mathcal{C}(\text{VF}(\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R))$$

in the ∞ -category $\text{Alg}_{\text{Lie}}^{\text{gr}}(\text{Mod}_{\mathcal{D}})$.

Proof. Fix an animated commutative R -algebra A with $A^{\text{red}} \simeq R^{\text{red}}$ and let

$$M_A \in \mathcal{L}\mathcal{A}_{R^{\text{red}}}$$

denote a dg Lie algebroid representing (by Theorem 3.27) the formal moduli problem

$$\widehat{\mathbf{T}}^*[n+1](X_R^\wedge/R) \times_R A \in \mathcal{FMP}_{\text{Spec } R^{\text{red}}/\cdot}.$$

Moreover, by Theorem 3.27, we have

$$\widehat{\text{Aut}}(\widehat{\mathbf{T}}^*[n+1]X_R^\wedge/R)(A) \simeq \text{End}_{\mathcal{L}\mathcal{A}/R^{\text{red}}}(M_A) \times_{\text{End}_{\mathcal{L}\mathcal{A}/R^{\text{red}}}(\mathbb{T}_{R^{\text{red}}/A})} \text{pt}.$$

Now,

$$\mathrm{End}_{\mathcal{L}\mathcal{A}/R^{\mathrm{red}}}(\mathrm{M}_A) \simeq \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathrm{C}^*(\mathrm{M}_A)) \simeq \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathrm{F}_H^* \mathrm{dR}_{R^{\mathrm{red}}/\widehat{\mathrm{T}}^*[n+1](X_R^\wedge/R) \times_R A}),$$

where the first equivalence follows from Theorem 3.30, and the second is by Theorem 7.8. But base change for Hodge-filtered de Rham cohomology implies

$$\Gamma_{\mathrm{BG}_m}(\widehat{\mathrm{T}}^*[n+1](X_R^\wedge/R) \times_R A; \mathcal{O}) \simeq \mathcal{D} \otimes_R A,$$

and so

$$\mathrm{End}_{\mathcal{L}\mathcal{A}/R^{\mathrm{red}}}(\mathrm{M}_A) \simeq \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{A} \otimes_R A).$$

By the same considerations, we also have

$$\mathrm{End}_{\mathcal{L}\mathcal{A}/R^{\mathrm{red}}}(\mathrm{T}_{R^{\mathrm{red}}/A}) \simeq \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{D} \otimes_R A).$$

We will now use this to define a map of formal groups over R

$$\phi: \exp_R(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D})) \rightarrow \widehat{\mathrm{Aut}}(\widehat{\mathrm{T}}^*[n+1]X_R^\wedge/R).$$

By the identifications we have already made, $\mathrm{Der}(\mathcal{A}/\mathcal{D})$ is an induced $\mathcal{D}$ -module, so the domain is well-defined. On the level of functors of points, the map

$$\phi: \exp_R(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D}))(A) \rightarrow \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{A} \otimes_R A) \times_{\mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{D} \otimes_R A)} \mathrm{pt}$$

is defined as follows. By definition of $\exp_R$, the domain is identified with

$$\mathrm{Map}_{\mathcal{FMP}/R} \left(\mathrm{Spec} A, \mathrm{Spec}^{\mathrm{inf}}(\mathrm{U}(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D}))) \right).$$

Ignoring the formal group structure, we have

$$\begin{aligned} \exp_R(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D}))(A) &\simeq \mathrm{Map}_{\mathcal{FMP}/R} \left(\mathrm{Spec} A, \mathrm{Spec}^{\mathrm{inf}}(\mathrm{cSym}(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D}))) \right) \\ &\simeq \mathrm{Map}_{\widehat{\mathrm{Fil}}\mathrm{Mod}_R}(\mathcal{D} \otimes_R A, \mathrm{Der}(\mathcal{A}/\mathcal{D})) \times_{\mathrm{Map}_{\widehat{\mathrm{Fil}}\mathrm{Mod}_R}(A, \mathrm{Der}(\mathcal{A}/\mathcal{D}))} \mathrm{pt}. \end{aligned}$$

We remark that $\mathcal{D} \otimes_R A \simeq \mathrm{F}_H^* \mathrm{dR}_{R^{\mathrm{red}}/A}$ by base change. Consider the map

$$\begin{aligned} \phi_0: \mathrm{Map}_{\widehat{\mathrm{Fil}}\mathrm{Mod}_R}(\mathcal{D} \otimes_R A, \mathrm{Der}(\mathcal{A}/\mathcal{D})) &\simeq \mathrm{Map}_{\mathrm{Mod}_R}(A, \mathrm{Der}(\mathcal{A}/\mathcal{D})) \\ &\longrightarrow \mathrm{Map}_{\mathrm{Mod}_R}(\mathcal{A} \otimes_R A, \mathcal{A}) \end{aligned}$$

arising from the construction of $\mathrm{Der}(\mathcal{A}/\mathcal{D})$. The Lie bracket of derivations is induced by composition of maps in the codomain of ϕ_0 , so we get a map of monoid objects

$$\mathrm{Map}_{\mathcal{FMP}/R} \left(\mathrm{Spec} A, \mathrm{Spec}^{\mathrm{inf}}(\mathrm{U}(\mathcal{C}^{-1} \mathrm{Der}(\mathcal{A}/\mathcal{D}))) \right) \rightarrow \mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{A} \otimes_R A),$$

and it lies over the identity in $\mathrm{End}_{\widehat{\mathrm{Fil}}\mathcal{C}\mathrm{Alg}_A}(\mathcal{D} \otimes_R A)$ as the derivations fix $\mathcal{D}$. We thus obtain a map ϕ as above. The tangent map of ϕ , after applying $\mathcal{C}$, is the inverse of the canonical equivalence

$$T_{\mathcal{A}/\mathcal{D}} \simeq \mathrm{Der}(\mathcal{A}/\mathcal{D}).$$

It follows that ϕ is an equivalence of Lie algebra objects, as needed. $\square$

APPENDIX A. ACTIONS OF SEMIDIRECT PRODUCTS

As it will suffice for our application, we assume throughout this section that $\mathcal{C}$ is a full subcategory of an ∞ -topos and assume that the requisite mapping objects exist; the reader is free to simply take $\mathcal{C} = \mathcal{FMP}_{/X_{\text{dR}}}$ for X a (left) derived stack over k . We write

$$\underline{\text{Map}}_{\mathcal{C}}(A, B) \in \mathcal{C}$$

for the internal mapping object of $A, B \in \mathcal{C}$. This appendix is dedicated to proving the following theorem.

Theorem A.1. *Let A, B be two objects of $\mathcal{C}$. Write*

$$E(A) = \underline{\text{End}}_{\mathcal{C}}(A) \in \mathcal{C}, \quad E(A, B) = \underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \in \mathcal{C}$$

and equip $E(A), E(A, B)$ with their natural monoid structures. Write $M = \underline{\text{Map}}_{\mathcal{C}}(A, B)$.

Suppose $E(A)$ is a group object. Then there exists an object $\mathcal{S} \in \mathcal{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{M\}$ equipped with morphisms

$$\begin{array}{ccc} (E(A), M) & & (E(A, B), M) \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array}$$

whose underlying monoid is $E(A) \ltimes E(A, B)$.

The proof will occupy the rest of the subsection. Our strategy will be to leverage the constructions of [Lur, Sec. 4.7.1], to which we refer the reader for the notation (with exact references to definitions).

We work in the formalism of $\mathbb{A}_{\infty}$ -monoidal ∞ -categories [Lur, p. 4.1.3.6] and for simplicity refer to them simply as “monoidal ∞ -categories”. Similarly, we refer to $\mathbb{A}_{\infty}$ -algebra objects as “algebra objects”. Suppose $p: \mathcal{M}^{\otimes} \rightarrow \Delta^1 \times \Delta^{\text{op}}$ exhibits $\mathcal{M}$ as weakly enriched over the planar ∞ -operad $\mathcal{C}^{\otimes}$. We will assume throughout that p is a coCartesian fibration and refer to $\mathcal{M}$ simply as *tensoring* over $\mathcal{C}^{\otimes}$. Write $\tilde{\mathcal{C}}^{\otimes} \rightarrow \Delta^{\text{op}} \times \Delta^1$ for the coCartesian fibration defining the canonical tensoring of $\mathcal{C}$ over itself. Write also

$$\text{Arr } \mathcal{C} = \text{Fun}(\Delta^1, \mathcal{C})$$

for the arrow category of $\mathcal{C}$. Observe that $\text{Arr } \mathcal{C}$ is tensoring over $\mathcal{C}$ by the Cartesian product. This is obtained by starting with the self-tensoring $\widetilde{\text{Arr } \mathcal{C}}$ and using the map

$$\mathcal{C}^{\otimes} \rightarrow \text{Arr } \mathcal{C}^{\otimes},$$

given by inclusion of the identity arrows.

Let $\mathcal{M}$ be an ∞ -category tensoring over $\mathcal{C}$, and let $M \in \mathcal{M}$. We briefly recall the definition of the monoidal ∞ -category $\mathcal{C}[M]$ used in the construction of endomorphism objects; see [Lur, pp. 4.7.1.28, 4.7.1.17] for precise definitions. The objects of $\mathcal{C}[M]$ are pairs of an object $C \in \mathcal{C}$ and a morphism $\eta: C \otimes M \rightarrow M$, which are referred to as *enriched morphisms*. The monoidal structure can be thought of as a coherent extension of the formula

$$(C, \eta) \otimes (C', \eta') = (C \otimes C', (1 \otimes \eta') \circ \eta).$$

Namely, there is a coCartesian fibration

$$q: \text{Str } \mathcal{M}^{\text{en}} \rightarrow \Delta^{\text{op}}$$

such that the fiber over $[n]$ is a suitably defined simplicial set of n -fold compositions of enriched morphisms. Moreover, q factors through $\mathrm{Str}\mathcal{M} \rightarrow \Delta^{\mathrm{op}}$, where $\mathrm{Str}\mathcal{M}$ is defined such that there are canonical isomorphisms $\mathrm{Str}_{[n]} \simeq \mathcal{M}^n$. There is a section $\Delta^{\mathrm{op}} \rightarrow \mathrm{Str}\mathcal{M}$ such that its image under the latter isomorphism is $(M, \dots, M)$. Then the monoidal structure on $\mathcal{C}[M]$ is defined by

$$\mathcal{C}[M]^{\otimes} = \Delta^{\mathrm{op}} \times_{\mathrm{Str}\mathcal{M}} \mathrm{Str}\mathcal{M}^{\mathrm{en}}.$$

We write

$$\underline{\mathrm{Map}}_{\mathcal{C}}(A, B) \in \mathcal{C}$$

for the internal mapping object. The main result of Section 4.7.1 of [Lur] constructs a canonical algebra structure on

$$\underline{\mathrm{End}}_{\mathcal{C}}(M) = \underline{\mathrm{Map}}_{\mathcal{C}}(M, M),$$

together with an action on M . In particular, $\underline{\mathrm{End}}_{\mathcal{C}}(M)$ is a final object of $\mathcal{C}[M]$.

Recall that we are assuming $\mathcal{C}$ is a symmetric monoidal full subcategory of an ∞ -topos together with its Cartesian monoidal structure. Fix $A, B \in \mathcal{C}$ from now on.

Definition A.2. Write

$$\mathrm{Arr}\mathcal{C} = \mathrm{Fun}(\Delta^1, \mathcal{C})$$

for the arrow category of $\mathcal{C}$. We define the ∞ -category

$$\mathrm{Arr}\mathcal{C}[A, B] = \mathrm{Arr}(\mathcal{C})[\mathrm{pr}: A \times B \rightarrow A],$$

equipped with the monoidal structure defined above, denoted $\mathrm{Arr}\mathcal{C}[A, B]^{\otimes}$.

Lemma A.3. *There is a monoidal functor*

$$\mathcal{C}[A]^{\otimes} \rightarrow \mathrm{Arr}\mathcal{C}[A, B]^{\otimes}$$

taking the object $a: C \times A \rightarrow A$ to

$$\begin{array}{ccc} C \times A \times B & \xrightarrow{a \times \mathrm{id}} & A \times B \\ \downarrow & & \downarrow \\ C \times A & \xrightarrow{a} & A. \end{array}$$

Proof. By definition of $\mathrm{Str}\mathcal{C}^{\mathrm{en}}$ [Lur, p. 4.7.1.17], the product-preserving functor

$$(- \times B \rightarrow -): \mathcal{C} \rightarrow \mathrm{Arr}\mathcal{C},$$

given by taking A to the projection

$$\mathrm{pr}: A \times B \rightarrow A,$$

induces a morphism of coCartesian fibrations

$$\mathrm{Str}\mathcal{C}^{\mathrm{en}} \rightarrow \mathrm{Str}(\mathrm{Arr}\mathcal{C})^{\mathrm{en}}$$

over Δ^{op} . Taking fibered products, it yields a monoidal functor

$$\mathcal{C}[A]^{\otimes} \rightarrow \mathrm{Arr}\mathcal{C}[A, B]^{\otimes},$$

which on objects behaves as claimed. $\square$

It follows that there is an induced action of

$$\underline{\text{End}}_{\mathcal{C}}(A)$$

on the object $A \times B \rightarrow A$ of the arrow category. Let

$$\varepsilon_A: \underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times (\underline{\text{End}}_{\mathcal{C}}(A) \times A) \rightarrow \underline{\text{Map}}_{\mathcal{C}}(A \times B, B) \times A$$

denote the map given by the canonical equivalence in the first factor and evaluation in the second. Write also

$$\varepsilon_B: \underline{\text{Map}}_{\mathcal{C}}(A \times B, B) \times A \times B \rightarrow A \times B$$

for the product of the projection to A and the evaluation morphism to B .

Proposition A.4. *Let ε denote the composite*

$$\underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times \underline{\text{End}}_{\mathcal{C}}(A) \times A \times B \xrightarrow{\varepsilon_B \circ (\varepsilon_A \times \text{pr}_B)} A \times B$$

and let

$$\gamma: \underline{\text{End}}_{\mathcal{C}}(A) \times A \rightarrow A$$

be the canonical action map. Then the pair (ε, γ) exhibits the projection

$$\underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times \underline{\text{End}}_{\mathcal{C}}(A) \rightarrow \underline{\text{End}}_{\mathcal{C}}(A)$$

as an endomorphism object for $(A \times B \rightarrow A) \in \text{Arr}(\mathcal{C})$.

Proof. Using the assumption that $\mathcal{C}$ is a full subcategory of an ∞ -topos, the following identification of internal mapping objects follows from applying the corresponding result to mapping spaces pointwise:

$$\begin{aligned} & \underline{\text{Map}}_{\text{Arr}(\mathcal{C})}(A \times B \rightarrow A, A \times B \rightarrow A) \\ & \simeq \text{eq} \left(\underline{\text{Map}}_{\mathcal{C}}(A \times B, A \times B) \times \underline{\text{Map}}_{\mathcal{C}}(A, A) \rightrightarrows \underline{\text{Map}}_{\mathcal{C}}(A \times B, A) \right) \\ & \simeq \underline{\text{Map}}_{\mathcal{C}}(A \times B, B) \times \text{eq} \left(\underline{\text{Map}}_{\mathcal{C}}(A \times B, A) \times \underline{\text{Map}}_{\mathcal{C}}(A, A) \rightrightarrows \underline{\text{Map}}_{\mathcal{C}}(A \times B, A) \right) \\ & \simeq \underline{\text{Map}}_{\mathcal{C}}(A \times B, B) \times \underline{\text{Map}}_{\mathcal{C}}(A, A) \\ & \simeq \underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times \underline{\text{End}}_{\mathcal{C}}(A). \end{aligned}$$

Tracing through the composition pointwise, the evaluation of endomorphisms is identified with (ε, γ) . This implies (ε, γ) is a final object of $\text{Arr } \mathcal{C}[A, B]$, from which we conclude. $\square$

In particular, by [Lur, Cor. 4.7.1.40], the arrow

$$\underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times \underline{\text{End}}_{\mathcal{C}}(A) \rightarrow \underline{\text{End}}_{\mathcal{C}}(A)$$

acquires the structure of a monoid object in $\text{Arr } \mathcal{C}$ in an essentially unique way.

Corollary A.5. *There exists a monoid object $\mathcal{F}(A, B)$ of $\mathcal{C}$ whose underlying object is*

$$\underline{\text{Map}}_{\mathcal{C}}(A, \underline{\text{End}}_{\mathcal{C}}(B)) \times \underline{\text{End}}_{\mathcal{C}}(A)$$

and such that $A \times B$ is canonically equipped with an action of $\mathcal{F}(A, B)$. Moreover, there is a morphism of monoid objects

$$\mathcal{F}(A, B) \rightarrow \underline{\text{End}}_{\mathcal{C}}(A)$$

and an equivariant structure on the projection $A \times B \rightarrow A$.

Proposition A.6. *Suppose $\underline{\mathcal{E}nd}_{\mathcal{C}}(A)$ is a group object. Then there exists*

$$\mathcal{T} \in \mathcal{LMod}(\mathcal{C}) \times_{\mathcal{C}} \{\underline{\mathcal{M}ap}_{\mathcal{C}}(A, B)\}$$

whose image in $\mathcal{Mon}(\mathcal{C})$ is equivalent to $\mathcal{F}(A, B)$.

Proof. We have

$$(27) \quad \underline{\mathcal{M}ap}_{\mathcal{C}}(A, B) \simeq \text{Fib} \left(c: \underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B) \rightarrow \underline{\mathcal{M}ap}_{\mathcal{C}}(A, A) \right),$$

where the map c is post-composition with the projection $A \times B \rightarrow A$, and the fiber is taken at the identity⁷. First, we claim $\underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B)$ has the structure of a module over $\underline{\mathcal{E}nd}_{\mathcal{C}}(A)$. Indeed, there are functors

$$\mathcal{C}[A] \longrightarrow \mathcal{C}[A \times \underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B)] \longrightarrow \mathcal{C}[\underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B)],$$

where the first corresponds to the inclusion of enhanced strings [Lur, p. 4.7.1.6], and the second is induced by the evaluation map. By construction of the monoidal structure [Lur, p. 4.7.1.17], the composite lifts to a monoidal functor. This in turn implies that

$$\underline{\mathcal{E}nd}_{\mathcal{C}}(A) \in \mathcal{LMod}(\mathcal{C}) \times_{\mathcal{Mod}(\mathcal{C})} \{\underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B)\},$$

as claimed.

Since we assume $\underline{\mathcal{E}nd}_{\mathcal{C}}(A)$ is a group object, we can convert the left action to a *right* action by inversion. Combining it with the action of Theorem A.5, we lift $\underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B)$ to an object of

$${}_{\mathcal{F}(A, B)}\mathcal{BMod}(\mathcal{C})_{\underline{\mathcal{E}nd}_{\mathcal{C}}(A)}.$$

Using the map $\mathcal{F}(A, B) \rightarrow \underline{\mathcal{E}nd}_{\mathcal{C}}(A)$, we get

$$\underline{\mathcal{M}ap}_{\mathcal{C}}(A, A \times B) \in {}_{\mathcal{F}(A, B)}\mathcal{BMod}(\mathcal{C})_{\mathcal{F}(A, B)},$$

and take the diagonal $\mathcal{F}(A, B)$ -module structure. This yields an action of the monoid object

$$\mathcal{F}(A, B) \rightarrow \underline{\mathcal{E}nd}_{\mathcal{C}}(A)$$

on the arrow c in $\text{Arr } \mathcal{C}$, which factors through

$$\mathcal{F}(A, B) \rightarrow \text{pt}.$$

We get an induced action on the fiber (27), as needed. $\square$

Corollary A.7. *Suppose both $\underline{\mathcal{E}nd}_{\mathcal{C}}(A)$ and $\underline{\mathcal{E}nd}_{\mathcal{C}}(B)$ are group objects. Then*

$$\mathcal{F}(A, B) \simeq \underline{\mathcal{E}nd}_{\mathcal{C}}(A) \times \underline{\mathcal{M}ap}_{\mathcal{C}}(A, \underline{\mathcal{E}nd}_{\mathcal{C}}(B)).$$

Proof. Follows from the proof of the previous proposition and the universal property of the semidirect product. $\square$

⁷More precisely, by our assumption on $\mathcal{C}$ we may take this fiber in some ∞ -category of presheaves to obtain the internal limit.

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