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THETA BASES FOR RANK-TWO QUANTUM CLUSTER ALGEBRAS OF GEOMETRIC TYPE

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Abstract. We prove that the coefficient-free quantum theta and greedy bases coincide for all positive exchange parameters. Their homogeneous lifts form an integral theta basis for arbitrary geometric coefficients and compatible integral quantizations, with frozen variables inverted. For nonnegative initial coefficient blocks, they also form a basis with polynomial frozen coefficients. A signed coefficient example separates the ordinary and upper algebras.

1. Introduction

Throughout, $\mathbb{N} = \mathbb{Z}_{\geq 0}$, v is an indeterminate, and $[r]_+ = \max(r, 0)$ for an integer r. Let b, c be positive integers. The coefficient-free quantum cluster algebra $\mathcal{A}_v(b, c)$ is generated by variables X_j, $j \in \mathbb{Z}$, with

$$X_{j+1}X_{j-1} = \begin{cases} v^b X_j^b + 1, & j \text{ odd}, \\ v^c X_j^c + 1, & j \text{ even}, \end{cases} \qquad X_{j+1}X_j = v^2 X_j X_{j+1}. \tag{1}$$

We use the normalization of Lee–Li–Rupel–Zelevinsky [6]. Their quantum greedy basis is indexed by $a = (a_1, a_2) \in \mathbb{Z}^2$; its element $G[a]$ has a finite expansion

$$G[a] = \sum_{p,q \in \mathbb{N}} e_a(p, q) X^{(-a_1+bp, -a_2+cq)}, \qquad e_a(0, 0) = 1, \quad X^{(r,s)} = v^{rs} X_1^r X_2^s.$$

This is the meaning of pointedness used below. The coefficients are specified by (8).

The classical greedy and theta bases agree in rank two [3]. Burcroff–Lee give quantum-weighted bijections in type A_2 and the Kronecker case [2]. Davison–Mandel construct quantum theta bases in the skew-symmetric setting [4]; their Remark 4.4 also discusses the skew-symmetrizable construction and the expected comparison with quantum greedy elements. Gao's thesis [5], Theorems 3.12 and 3.15, states the comparison in the symmetric and finite rank-two cases. Its Conjecture 5.2 asks for the comparison when one exchange parameter divides the other.

We use the following scattering convention. In the lattice $M = \mathbb{Z}^2$, the initial walls of the g-diagram carry

$$\Psi_{v^b}(X^{(-b,0)}), \qquad \Psi_{v^c}(X^{(0,c)}) \tag{2}$$

on the horizontal and vertical axes, respectively. Here $\Psi_w(z) = \exp\big(\sum_{j\geq 1}(-1)^{j+1} z^j/(j(w^j - w^{-j}))\big)$ is a formal series over $\mathbb{Q}(w)$. A broken line carries a based monomial on each straight segment; its velocity is the negative exponent, and at a wall it selects a term of the wall-crossing action. The theta function is the sum of its terminal monomials for a fixed initial exponent and endpoint. Section 3 defines the completion, crossing signs, and genericity used in this construction. For $g = (g_1, g_2)$, set

$$T(g) = (g_1 + b\min(g_2, 0), g_2), \qquad \rho(g) = -T(g). \tag{3}$$

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Theorem 1.1. *For all $b, c \in \mathbb{Z}_{>0}$ and $g \in \mathbb{Z}^2$, the broken-line theta function $\vartheta_{g,Q}$ of the diagram (2), with Q generic in the first quadrant, is a Laurent polynomial and*

$$\vartheta_{g,Q} = G[\rho(g)].$$

Equivalently, put $g(a) = \big(b[a_2]_+ - a_1, -a_2\big)$; then $\vartheta_{g(a),Q} = G[a_1, a_2]$. The family of these theta functions is a $\mathbb{Z}[v^{\pm 1}]$-basis of $\mathcal{A}_v(b, c)$.

We next allow arbitrary geometric coefficients in the sense of Berenstein–Zelevinsky and Davison–Mandel: a finite frozen lattice, arbitrary integral frozen exchange rows, and a compatible quantum form. Write

$$B_0 = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} B_0 \\ C \end{pmatrix}, \qquad \widetilde{B}^{\mathsf{T}}\widetilde{\Lambda} = (D \;\; 0), \qquad D = \operatorname{diag}(d_1, d_2) > 0,$$

where $C \in \mathrm{Mat}_{f\times 2}(\mathbb{Z})$, $\widetilde{\Lambda}$ is integral and skew, and $R = \mathbb{Z}[v^{\pm 1}]$. Full based monomials multiply by $X^m X^{m'} = v^{\widetilde{\Lambda}(m,m')} X^{m+m'}$. Compatibility forces $\ell = d_1/c = d_2/b \in \mathbb{Q}_{>0}$; this scale need not be integral. Here $f \in \mathbb{N}$ is the number of frozen variables, and the first two coordinates of $m \in \mathbb{Z}^{2+f}$ are mutable. The frozen quantum Laurent torus is $\mathcal{R}_F = \bigoplus_{z\in\mathbb{Z}^f} RX^{(0,z)}$, with multiplication inherited from the full torus. A balanced Laurent coefficient means the coefficient of a based monomial X^m. The bar operation is the anti-involution fixing every X^m and sending v to v^{-1}. The ordinary algebra is generated by the mutable cluster variables and the chosen frozen coefficient ring; the upper algebra is the intersection of the corresponding seed Laurent charts, all inside the same skew field. The initial ice walls are $\{x_i = 0\}$, $i = 1, 2$, with factors $\Psi_{v^{d_i}}(X^{\widetilde{B}e_i})$, where e_i is the i-th coordinate vector. Their consistent completion defines Θ_m, with initial exponent m, as in Section 7. We evaluate it at a generic endpoint whose mutable projection lies in the initial first quadrant.

Theorem 1.2. *For every such ice seed, the following statements hold.*

1. *All broken-line theta functions Θ_m, $m \in \mathbb{Z}^{2+f}$, are finite integral Laurent polynomials. They form a seed-independent R-basis of the ordinary quantum cluster algebra, which equals its upper algebra, when all frozen variables are inverted. The functions $\Theta_{(g,0)}$, $g \in \mathbb{Z}^2$, form both a left and a right $\mathcal{R}_F$-basis.*
2. *The theta functions equal the homogeneous lifts of the quantum greedy elements given in (32). Their balanced Laurent coefficients lie in $\mathbb{Z}[v^{\pm \gcd(d_1,d_2)}]$, and every theta function is bar invariant.*
3. *If $C \geq 0$ entrywise, including $C = I_2$, the ordinary and upper algebras also agree without inverting any frozen variable. Their R-basis is $\{\Theta_{(g,z)} : g \in \mathbb{Z}^2,\ z \in \mathbb{N}^f\}$, and $\{\Theta_{(g,0)}\}$ is a left and right basis over the frozen quantum polynomial algebra.*

The first two assertions are proved in Section 7, and the third in Theorem 9.3. Proposition 9.1 treats the disconnected principal part. For signed coefficient rows, Proposition 9.4 gives an example whose ordinary and upper algebras differ with polynomial frozen coefficients.

The coefficient-free proof uses the denominator-coordinate geometry of [3, Sections 4–5]. We verify the quantum coordinate change and bound broken lines by their angular momentum. Grouping lines at their last axis crossing gives Gaussian divisibility and strict bounds on the quotient degrees. These bounds imply the greedy recurrences.

For geometric coefficients, projection identifies the broken lines with their rank-two counterparts. An orthogonal finite lattice extension separates the mutable and frozen factors; taking the original lattice degree and descending the parameter ring give the integral basis. A frozen-valuation argument with acyclic standard monomials treats nonnegative coefficient blocks.

2. Quantum tori and support rigidity

2.1. **Normalization.** Put $R = \mathbb{Z}[v^{\pm 1}]$, $K = \mathbb{Q}(v)$, and

$$\mathcal{T}_R = R\langle X_1^{\pm 1}, X_2^{\pm 1} \mid X_2X_1 = v^2 X_1X_2\rangle.$$

For $u = (u_1, u_2) \in M$, define

$$X^u = v^{u_1u_2} X_1^{u_1} X_2^{u_2}, \qquad \Lambda(u, w) = u_2w_1 - u_1w_2.$$

Then

$$X^u X^w = v^{\Lambda(u,w)} X^{u+w}. \tag{4}$$

The anti-involution determined by $\overline{v} = v^{-1}$, $\overline{X_i} = X_i$ fixes every X^u. For example, the exchange matrix

$$B_0 = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$$

is compatible with the matrix of Λ, since $B_0^{\mathsf{T}}\Lambda = \operatorname{diag}(c, b)$. Changing the orientation changes the accompanying sign convention.

For $n \geq 0$, write

$$P_{n,w}(z) = \prod_{j=1}^{n}(1 + w^{n+1-2j}z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_w z^k. \tag{5}$$

The coefficient of z^k in this product defines the symmetric quantum binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_w$ for $0 \leq k \leq n$; we set it to zero for $k < 0$ or $k > n$, with $n \geq 0$. In particular, $P_{0,w} = 1$. We shall use

$$P_{n,w}(z)^{-1} = \sum_{k\geq 0}(-1)^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_w z^k \quad (n > 0). \tag{6}$$

When $n = 0$, the inverse is 1; coefficients of positive powers on the right are interpreted as zero.

2.2. **The greedy input.** A finite element of $\mathcal{T}_K$ is pointed at $a \in \mathbb{Z}^2$ if it has the form

$$F = \sum_{p,q\geq 0} e(p, q) X^{(-a_1+bp, -a_2+cq)}, \qquad e(0, 0) = 1. \tag{7}$$

The existence and basis theorems in [6, Theorems 1.7 and 1.9] give integral elements $G[a]$ characterized by the following recurrences, imposed only for $p + q > 0$:

$$e(p, q) = \begin{cases} \displaystyle\sum_{k=1}^{p}(-1)^{k-1} \begin{bmatrix} [a_2 - cq]_+ + k - 1 \\ k \end{bmatrix}_{v^b} e(p-k, q), & ca_1q \leq ba_2p, \\ \displaystyle\sum_{\ell=1}^{q}(-1)^{\ell-1} \begin{bmatrix} [a_1 - bp]_+ + \ell - 1 \\ \ell \end{bmatrix}_{v^c} e(p, q-\ell), & ca_1q \geq ba_2p. \end{cases} \tag{8}$$

On the equality line both formulas apply. These elements form an R-basis of $\mathcal{A}_v(b, c)$, are bar invariant, contain the normalized cluster monomials, and specialize to the classical greedy basis. The basis is independent of the initial cluster.

For a pointed element define row and column polynomials

$$E_q(z) = \sum_{p\geq 0} e(p, q)z^p, \qquad F_p(z) = \sum_{q\geq 0} e(p, q)z^q.$$

The divisibility conditions are

$$P_{a_2-cq,v^b}(z) \mid E_q(z) \qquad (q \geq 0,\ cq < a_2), \tag{9}$$

$$P_{a_1-bp,v^c}(z) \mid F_p(z) \qquad (p \geq 0,\ bp < a_1). \tag{10}$$

Divisibility is in $K[z]$, where z is an auxiliary commuting indeterminate. The degree of a nonzero polynomial is its largest exponent with nonzero coefficient; we set $\deg 0 = -\infty$.

These are also the conditions for Laurentness in the adjacent tori; see [6, Lemma 2.2]. For our application we prove the conditions directly from broken lines, before asserting membership in a cluster algebra.

2.3. **A support bound sufficient for uniqueness.** Assume $a_1, a_2 > 0$, and let

$$O = (0,0), \quad A = (a_2, 0), \quad B = (a_1/b, a_2/c), \quad C = (0, a_1).$$

Define Ω_a to be the union of the closed triangles OAB and OBC, with the segments AB and BC removed and the points A, C restored. Thus B is excluded and the two coordinate-axis edges are included. This definition also covers degenerate triangles.

Proposition 2.1 (Support rigidity)**.** *Let $a_1, a_2 > 0$. A finite pointed element (7) satisfying (9)–(10) and*

$$\{(p,q) : e(p,q) \neq 0\} \subseteq \Omega_a$$

equals $G[a]$.

Proof. Set $\gamma = ca_1/(ba_2)$. For $0 < q < a_2/c$, every point of Ω_a on row q satisfies

$$p < a_2 - cq + \gamma q. \tag{11}$$

Indeed, this is the strict bound from AB on the first triangle; on the second triangle $p \leq \gamma q$, which is smaller than the right side. If $q \geq a_2/c$, every point on that row satisfies $p < \gamma q$. At equality the first triangle could contribute only the excluded point B, and the second triangle lies on the other side of OB.

For $0 < q < a_2/c$, put $n = a_2 - cq$. By divisibility, $E_q = P_{n,v^b} H_q$, where $H_q \in K[z]$. Since P_{n,v^b} is monic of degree n, (11) implies

$$\deg H_q \leq \lceil \gamma q \rceil - 1.$$

The coefficient of z^p in $H_q = E_q / P_{n,v^b}$ is consequently zero for every $p \geq \gamma q$. Formula (6) gives the first recurrence in (8). For $q \geq a_2/c$ and $p \geq \gamma q$, the coefficient $e(p,q)$ is zero by the support bound, and the same recurrence is zero because $[a_2 - cq]_+ = 0$.

The row $q = 0$ has degree at most a_2. Its divisibility and constant coefficient imply $E_0 = P_{a_2,v^b}$, which gives the first recurrence on that axis for $p > 0$.

The corresponding column bounds are

$$q < a_1 - bp + \gamma^{-1} p \quad (0 < p < a_1/b), \qquad q < \gamma^{-1} p \quad (p \geq a_1/b).$$

The column $p = 0$ is the segment $0 \leq q \leq a_1$. The same division argument, now using (10), gives the second recurrence, including the column axis.

In particular, a nonzero lattice point on $p = \gamma q$ satisfies both recurrences. Below B, both quotient-degree bounds are strict. At or beyond B, the point is outside Ω_a and both positive parts in the recurrences vanish. Thus the equality line contributes no free coefficient. Each recurrence expresses a coefficient in terms of smaller $p+q$. Induction, with $e(0,0) = 1$, proves equality with the greedy family. □

Remark 2.2. The sharper support characterization in [6, Proposition 3.7] already implies Gao's Proposition 3.11. The proposition above also permits a larger support region in the two trapezoid cases. The comparison in Section 5 uses degree bounds on the last-axis quotients directly.

3. Rank-two quantum scattering

3.1. **Completions and wall crossings.** For a strongly convex monoid $P \subset M$, write $P^+ = P \setminus \{0\}$. We use either

$$P_d = \mathbb{Z}_{\geq 0}(b,0) + \mathbb{Z}_{\geq 0}(0,c), \qquad P_g = \mathbb{Z}_{\geq 0}(-b,0) + \mathbb{Z}_{\geq 0}(0,c).$$

Both have the degree $p+q$ on their displayed generators. The Lie algebra

$$\mathfrak{g}_P = \bigoplus_{n\in P^+} KX^n, \qquad [X^n, X^m] = (v^{\Lambda(n,m)} - v^{-\Lambda(n,m)})X^{n+m}$$

has a degree completion $\widehat{\mathfrak{g}}_P$. The Lie algebra quotients discard homogeneous terms of degree greater than N; they are finite dimensional and nilpotent. For the action on series supported in $m+P$, truncation instead means retaining X^{m+n} with the increment $n \in P$ of degree at most N. Products use the induced increment filtration on $(m+m')+P$. This defines the inverse limits on translated supports; it does not require an ideal filtration of the full Laurent torus. The exponential group $G_P = \exp(\widehat{\mathfrak{g}}_P)$ acts by conjugation on the completed quantum Laurent algebra consisting of formal sums whose support lies in a finite union of translates of P. Strong convexity makes multiplication of these sums well defined.

Define

$$\Psi_w(z) = \exp\left(\sum_{j\geq 1} \frac{(-1)^{j+1}}{j(w^j - w^{-j})} z^j\right). \tag{12}$$

This is a formal series over a rational function field. It satisfies

$$\Psi_w(w^2 z) = (1+wz)\Psi_w(z). \tag{13}$$

The infinite-product notation $\prod_{j\geq 0}(1+w^{2j+1}z)^{-1}$ denotes this same formal series.

A wall is a line through the origin or a ray from the origin, parallel to a nonzero $n \in P$, together with

$$f \in \exp\left(\prod_{j\geq 1} KX^{jn_0}\right),$$

where n_0 is a primitive lattice vector along n; only exponents belonging to P are used. A diagram has finitely many nontrivial walls at every degree. Coincident walls are combined by multiplying their commuting factors. A crossing with velocity ξ acts by

$$\mathrm{Ad}_{f^\epsilon}, \qquad \epsilon = -\operatorname{sgn}\Lambda(n,\xi). \tag{14}$$

Path-ordered products compose these automorphisms in the order of crossing. A generic path avoids the origin and crosses each relevant wall transversely. All such statements are interpreted in each finite-degree truncation. A diagram is consistent if every closed generic path has trivial product.

Lemma 3.1. *Let $n=(b,0)$, $u=(r,s)$, and $N=|s|$. When a broken line with exponent u crosses the horizontal initial wall $\Psi_{v^b}(X^n)$, its possible next exponents and coefficients are*

$$u+kn, \qquad \begin{bmatrix} N \\ k \end{bmatrix}_{v^b}, \qquad 0 \leq k \leq N.$$

For the vertical wall $\Psi_{v^c}(X^{(0,c)})$, replace N by $|r|$, n by $(0,c)$, and v^b by v^c.

Proof. Write $\lambda = \Lambda(n,u)$. Commutation gives

$$\mathrm{Ad}_{\Psi_{v^b}(X^n)} X^u = X^u \frac{\Psi_{v^b}(v^{2\lambda}X^n)}{\Psi_{v^b}(X^n)}.$$

Since the broken-line velocity is $-u$, the crossing exponent in (14) is $\operatorname{sgn}\lambda$. If $\lambda = bN > 0$, the quotient is $\prod_{j=1}^{N}(1+v^{b(2j-1)}X^n)$. Its k-th coefficient is $v^{bNk}\begin{bmatrix} N \\ k \end{bmatrix}_{v^b}$, and multiplication by X^u contributes v^{-bNk}. The factors cancel. If $\lambda = -bN < 0$, the inverse adjoint produces the analogous product with negative odd powers, with the same cancellation. The vertical calculation uses $|\Lambda((0,c),u)| = c|r|$. □

3.2. **The denominator diagram.**

Proposition 3.2. *There is a unique consistent central diagram $\mathfrak{D}^d(b,c)$, up to equivalence, with initial walls*

$$(\mathbb{R}(1,0),H),\quad (\mathbb{R}(0,1),V),\qquad H=\Psi_{v^b}(X^{(b,0)}),\quad V=\Psi_{v^c}(X^{(0,c)}),$$

and with all other walls on rays $\mathbb{R}_{\le 0}(bp,cq)$, $p,q>0$.

Proof. In the degree completion for P_d, form

$$C=H^{-1}VHV^{-1}.$$

Its logarithm has no pure horizontal or vertical term: projection to either axis makes this commutator equal to 1. Factor C as an ordered product of elements supported on individual mixed rays, in increasing order of their geometric slopes $cq/(bp)$.

This factorization exists and is unique by induction on degree. After degrees below N have been fixed, the remaining degree-N error is a finite sum of monomials. At this degree these corrections commute with all positive-degree terms. Assign each correction to its ray and continue. This also proves finiteness modulo every degree.

Put the resulting factor on the negative ray parallel to its exponents. A counterclockwise circuit, with the crossing convention (14), has trivial product precisely when

$$VH=HCV,$$

which holds by construction. At each finite degree, all walls are central and the only joint is the origin. Cancellation of opposite crossings and the circuit identity prove consistency for every closed path. The ordered factorization also proves uniqueness. □

3.3. **Broken lines.** For a generic endpoint Q away from the walls, a broken line with initial exponent $m\in M$ is a continuous piecewise affine map $\gamma:(-\infty,0]\to M_{\mathbb{R}}\setminus\{0\}$ ending at Q. It has finitely many bends; inserting or removing a subdivision with unchanged attached monomial does not give a different line. Each segment carries aX^u, with velocity $-u$. The initial segment carries X^m. At a crossing, the next monomial is a term of the crossing automorphism applied to the incoming monomial. The unchanged term is allowed. A bend means a change of exponent. Set $\vartheta^d_{0,Q}=1$, and for other m set

$$\vartheta^d_{m,Q}=\sum_{\gamma}\mathrm{Mono}(\gamma),$$

where the summand is the last attached monomial.

For the quantum-torus Lie algebra above, formal broken-line sums are well defined degree by degree and satisfy endpoint transport [7, Example 2.1(iii), Proposition 2.13, and Theorem 2.14]. We check the structural hypotheses of that result. Commutators are derivations of the quantum torus, preserve the exponent grading, and satisfy

$$[X^n,X^u]=0\quad\text{if }\Lambda(n,u)=0.$$

A wall with direction n has normal functional $u\mapsto\Lambda(n,u)$. For two directions n,n', the normal in the bracket condition of that reference is

$$\Lambda(n,n')\Lambda(n+n',\,\cdot\,).$$

It is parallel to the normal of the bracket degree $n+n'$; if $\Lambda(n,n')=0$, the bracket itself is zero. Thus its wall-compatible Lie-action condition holds. For the unsaturated monoids P_d,P_g, extend the grading to the saturated lattice cone by assigning zero Lie components to missing degrees. The action on each translated support is completed in increment degree as above.

For fixed initial exponent and total increment, there are finitely many relevant walls and bend sequences: each bend increases degree. For a generic endpoint, reconstructing a prescribed sequence backward intersects a straight segment with a specified wall at each step, so gives at most one path. Endpoint transport identifies the sums by the path-ordered

product. In particular they are constant in a chamber. We apply the theorem to the consistent completion of the initial walls.

We may choose Q away from the countably many exceptional directions that occur in the construction. In particular, Q_1/Q_2 may be chosen irrational.

4. Geometric support and polynomiality

Every bend in $\mathfrak{D}^d$ adds a nonzero exponent in P_d. This follows from the ray subgroup containing its wall factor, and holds for a factor or its inverse. Consequently both exponent coordinates are nondecreasing along a broken line. The unbent line is the unique contribution with final exponent m. Thus

$$\vartheta^d_{m,Q} = \sum_{p,q\geq 0} e(p,q) X^{m+(bp,cq)}, \qquad e(0,0) = 1 \tag{15}$$

as a formal series.

Lemma 4.1. *If Q lies in the first quadrant, then*

$$\vartheta^d_{m,Q} = \begin{cases} X^m, & m_1, m_2 \geq 0, \\ \displaystyle\sum_{k=0}^{-m_2} \begin{bmatrix} -m_2 \\ k \end{bmatrix}_{v^b} X^{m+(bk,0)}, & m_1 \geq 0 > m_2, \\ \displaystyle\sum_{k=0}^{-m_1} \begin{bmatrix} -m_1 \\ k \end{bmatrix}_{v^c} X^{m+(0,ck)}, & m_2 \geq 0 > m_1. \end{cases}$$

Proof. If $m_1 \geq 0$, the first exponent stays nonnegative and the spatial first coordinate is nonincreasing. A line ending at $Q_1 > 0$ therefore stays to the right of the vertical axis. It meets neither that axis nor an additional wall in the third quadrant. If $m_2 \geq 0$, it remains in the first quadrant. If $m_2 < 0$, it crosses only the horizontal wall, and Lemma 3.1 gives the second formula. Interchanging the coordinates gives the third formula. This argument includes zero coordinates of m. □

It remains to consider $m_1, m_2 < 0$. For a segment with position z and exponent u, put

$$L = z_2 u_1 - z_1 u_2.$$

Lemma 4.2. *The number L is constant along a broken line. A line starting in the third quadrant and ending in the first quadrant enters the first quadrant through the positive horizontal axis if $L > 0$, and through the positive vertical axis if $L < 0$. For generic Q, these are the only possibilities.*

Proof. On a segment, $z' = -u$, so the derivative of L vanishes. At a bend, u changes to $u + kn$ and z is parallel to n. The change in L is $k(z_2 n_1 - z_1 n_2) = 0$.

If a spatial coordinate is nonpositive, its exponent must still be negative for the line eventually to reach the corresponding positive coordinate of Q. Otherwise the exponent can never decrease, and that spatial coordinate can never increase again. The line therefore passes through quadrant II or quadrant IV before entering quadrant I. At a positive horizontal crossing, $L = -z_1 u_2 > 0$; at a positive vertical crossing, $L = z_2 u_1 < 0$. After entering quadrant I there are no further walls, and the line cannot leave and return. Finally $L = Q_2 u_1 - Q_1 u_2$ for the final exponent u. Genericity excludes zero unless $u = 0$. A zero final exponent would give a stationary final segment, forcing Q to be its last bend point on a wall, which is impossible. □

Lemma 4.3. *Let $m_1, m_2 < 0$, and let $e = (e_1, e_2)$ be the final exponent of a broken line ending in the first quadrant. If $L > 0$, then*

$$m_2 \leq e_2 < 0, \qquad m_1 \leq e_1 \leq (m_1/m_2 - b) e_2. \tag{16}$$

Equality in the last bound is possible only for $e = (m_1 - bm_2, m_2)$. *If* $L < 0$, *then*

$$m_1 \le e_1 < 0, \qquad m_2 \le e_2 \le (m_2/m_1 - c)e_1, \tag{17}$$

with equality in the last bound only for $e = (m_1, m_2 - cm_1)$.

Proof. Suppose $L > 0$. Before the final horizontal crossing, both exponent coordinates are negative. They must be negative in quadrant III; the crossing into quadrant IV changes only the second exponent, and this exponent must remain negative in order to reach the upper half-plane.

Every earlier bend takes place at $z = -tn$, $t > 0$, and changes u to $u' = u + kn$, $k > 0$. Then

$$u_1u_2' - u_2u_1' = k(u_1n_2 - u_2n_1) = -kL/t < 0.$$

As $u_1, u_1' < 0$, this implies $u_2'/u_1' < u_2/u_1$. If u is the exponent immediately before the last horizontal crossing, it follows that

$$u_2/u_1 \le m_2/m_1,$$

and equality is possible only if there has been no earlier bend. The last crossing has

$$e = (u_1 + bk, u_2), \qquad 0 \le k \le -u_2.$$

Hence

$$e_1 \le u_1 - bu_2 = (u_1/u_2 - b)e_2 \le (m_1/m_2 - b)e_2.$$

The monotonicity of exponent coordinates gives the lower bounds. Equality requires $u = m$ and $k = -m_2$. The argument for $L < 0$ interchanges the coordinates. □

Proposition 4.4. *For every* $m \in M$, $\vartheta^d_{m,Q}$ *is a Laurent polynomial over* K.

Proof. The cases in Lemma 4.1 are finite. In the remaining case, (16) and (17) are alternatives. Their union is bounded: in the first alternative e_2 lies in a bounded interval and both bounds on e_1 are finite; the second alternative has the transposed property. There are therefore only finitely many possible final lattice exponents. Each such exponent has a fixed finite degree increment from m, so its coefficient is a finite broken-line sum. Thus the theta series is a Laurent polynomial. □

5. Divisibility from the final axis crossing

Proposition 5.1. *The coefficients of* $\vartheta^d_{m,Q}$ *in* (15) *satisfy*

$$P_{-m_2-cq,v^b}(z) \mid E_q(z) \quad (m_2 + cq < 0), \qquad P_{-m_1-bp,v^c}(z) \mid F_p(z) \quad (m_1 + bp < 0).$$

Proof. Fix a row with $s = m_2 + cq < 0$. By chamber independence we may choose a generic Q satisfying

$$Q_1 > \max(0, -m_1)Q_2. \tag{18}$$

Every broken line in this row has negative second exponent on every segment, since exponents are nondecreasing and its final second exponent is $s < 0$. Its vertical coordinate increases strictly. It crosses the horizontal axis exactly once.

Its final first exponent r is at least m_1, and $-s \ge 1$. Its angular momentum therefore satisfies

$$L = Q_2r - Q_1s \ge Q_2m_1 + Q_1 > 0.$$

Entry into the first quadrant through the positive vertical axis would give $L < 0$. Thus the horizontal crossing is on the positive axis and is the last wall crossing. This also excludes a horizontal crossing on the negative axis, which would have to be followed by a positive vertical crossing.

Cut such a line immediately before the horizontal crossing. Write its terminal prefix monomial as $CX^{(r_0,s)}$, with $r_0 = m_1 + bp_0$. Positive dilation of space, with the corresponding time change, preserves the condition $\gamma' = -u$, all central walls, and all attached coefficients. Normalize the prefix endpoint to $(1, 0)$.

Conversely, fix such a normalized prefix and let $N = -s$. For every $0 \le k \le N$, set

$$x_k = Q_1 - \frac{Q_2(r_0 + bk)}{s}.$$

It is positive by (18), since $r_0 + bk \ge m_1$ and $-s \ge 1$. Dilate the prefix to end at $(x_k, 0)$, choose the k-th term in the horizontal expansion of Lemma 3.1, and attach the straight segment to Q. That segment lies entirely in the first quadrant after the crossing. It meets no further wall. Cutting and this extension are inverse constructions.

Each normalized prefix therefore contributes the whole polynomial

$$Cz^{p_0} P_{N,v^b}(z)$$

to the row. The sum of prefix monomials is a polynomial: the choice $k = 0$ extends every prefix to a valid full broken line, so the individual bounds in Lemma 4.3, or the easy cases in Lemma 4.1, bound p_0; degreewise finiteness bounds the number of contributions for each p_0. Consequently

$$E_q(z) = P_{N,v^b}(z) \sum_{\text{normalized prefixes}} Cz^{p_0}$$

in $K[z]$.

For the column assertion choose $Q_2 > \max(0, -m_2)Q_1$, cut before the positive vertical crossing, and normalize to $(0, 1)$. The possible increments are $(0, c\ell)$, $0 \le \ell \le -r$, for $r = m_1 + bp < 0$, with coefficients $\begin{bmatrix} -r \\ \ell \end{bmatrix}_{v^c}$. This gives the stated column divisor. Chamber independence identifies the polynomials obtained from the separate row and column endpoint choices. □

Remark 5.2. The rows $q = 0$ and columns $p = 0$ are included. If the relevant exponent is zero, its putative divisor is $P_0 = 1$.

Proposition 5.3 (Quotient-degree bounds)**.** *Suppose $m = -a$, $a_1, a_2 > 0$, and put $\gamma = ca_1/(ba_2)$. The quotients in Proposition 5.1 satisfy*

$$E_0 = P_{a_2,v^b}, \qquad E_q = P_{a_2-cq,v^b} H_q, \quad \deg H_q < \gamma q \quad (0 < q < a_2/c),$$

and

$$F_0 = P_{a_1,v^c}, \qquad F_p = P_{a_1-bp,v^c} J_p, \quad \deg J_p < \gamma^{-1} p \quad (0 < p < a_1/b).$$

Also $e(p, q) = 0$ whenever $bp \ge a_1$ and $cq \ge a_2$.

Proof. Consider a prefix occurring in the row packet construction. Immediately before its positive horizontal crossing its exponent is (r_0, s), with both entries negative. The slope calculation in the proof of Lemma 4.3, applied before this last crossing, gives

$$\frac{r_0}{s} \ge \frac{m_1}{m_2} = \frac{a_1}{a_2}.$$

Equality is possible only if the prefix has no bend. For $q > 0$ there has been a positive change in the second exponent, so at least one bend has occurred. Multiplication by $s < 0$ therefore gives

$$-a_1 + bp_0 = r_0 < \frac{a_1}{a_2}(-a_2 + cq), \qquad p_0 < \gamma q.$$

This bounds every individual monomial of the prefix quotient H_q, hence its degree.

When $q = 0$, a prefix cannot bend: an additional wall or a vertical-axis bend would increase its second exponent, while its unique horizontal crossing has not yet occurred. The normalized unbent prefix is unique, has coefficient 1, and has $p_0 = 0$. Thus $H_0 = 1$. For the column construction a normalized vertical prefix has exponent (r, s_0), $r < 0$, and negative angular momentum. The coordinate-interchanged slope calculation gives

$$\frac{s_0}{r} \ge \frac{a_2}{a_1},$$

strictly if $p > 0$. Therefore

$$-a_2 + cq_0 = s_0 < \frac{a_2}{a_1}(-a_1 + bp), \qquad q_0 < \gamma^{-1}p.$$

For $p = 0$ the same no-bend argument gives the unique prefix of coefficient 1, hence $J_0 = 1$.

Finally a line starting in quadrant III enters quadrant I through one of its positive axes. If it enters horizontally, its final second exponent is negative; if it enters vertically, its final first exponent is negative. It cannot have both final coordinates nonnegative. This proves the last assertion. □

Theorem 5.4. *For every $b, c > 0$, $m \in \mathbb{Z}^2$, and generic first-quadrant endpoint Q,*

$$\vartheta^d_{m,Q} = G[-m_1, -m_2].$$

Proof. Suppose both coordinates of m are negative and set $a = -m$. For $0 < q < a_2/c$, Proposition 5.3 makes the coefficient of z^p in $E_q/P_{a_2-cq,v^b}$ zero whenever $p \geq \gamma q$. Expanding the inverse by (6) gives the first greedy recurrence in (8). The row $q = 0$ follows from $E_0 = P_{a_2,v^b}$. For $q \geq a_2/c$ and $p \geq \gamma q$, one also has $p \geq a_1/b$. The last assertion of Proposition 5.3 makes the coefficient zero, as is the recurrence with $[a_2 - cq]_+ = 0$. The column bounds give the second recurrence for $p \leq \gamma q$. On equality both recurrences hold. Induction on $p + q$, starting at $e(0, 0) = 1$, identifies the coefficients with those of $G[a]$.

If one coordinate of m is nonnegative, Lemma 4.1 gives the elementary greedy formulas. For example, when $m_1 \geq 0 > m_2$, the only coefficients are $\begin{bmatrix} -m_2 \\ p \end{bmatrix}_{v^b}$ on $q = 0$; these satisfy (8) with the prescribed constant coefficient. The other mixed quadrant is the column version, and when both coordinates are nonnegative the element is X^m. □

6. The change from g- to denominator coordinates

We now identify the preceding construction with the quantum cluster diagram specified in (2). Put

$$A = \Psi_{v^b}(X^{(-b,0)}), \qquad B = \Psi_{v^c}(X^{(0,c)}).$$

Use P_g, and complete the two initial full axes by outgoing rays in quadrant IV. The same ordered-factorization construction as in Proposition 3.2 defines a unique consistent diagram $\mathfrak{D}^g$. With its factors ordered by increasing p/q, the circuit identity reads

$$ABA^{-1} = B \prod_i C_i, \tag{19}$$

where C_i is supported in positive multiples of $(-bp_i, cq_i)$, $p_i, q_i > 0$. Its theta functions use the same broken-line definition as before, with $\vartheta_{0,Q} = 1$.

Lemma 6.1. *Every nontrivial outgoing factor of $\mathfrak{D}^g$ has $0 < p_i \leq cq_i$. The factor on the extremal ray is*

$$\Psi_{v^c}(X^{(-bc,c)}).$$

Proof. The adjoint calculation gives

$$\mathrm{Ad}_A\, X^{(0,c)} = \sum_{k=0}^{c} \begin{bmatrix} c \\ k \end{bmatrix}_{v^b} X^{(-bk,c)}. \tag{20}$$

Conjugation commutes with products and formal logarithms. Thus $\mathrm{Ad}_A \log B = \log(ABA^{-1})$ is supported in the cone generated by $(0, c)$ and $(-bc, c)$. The degree-induction factorization stays in this cone, which proves $p_i \leq cq_i$. On its extremal ray, only the $k = c$ term of (20) can contribute. Its coefficient is 1, and repeated products of this monomial have no additional commutation factor. The restriction of the logarithm to that ray is therefore $\log \Psi_{v^c}(X^{(-bc,c)})$. □

Let $H_+ = \{y \geq 0\}$, $H_- = \{y \leq 0\}$, and let T be the piecewise linear map in (3). Its restrictions are $T_+ = \mathrm{id}$ and

$$T_- = S, \qquad S(x, y) = (x + by, y).$$

Both preserve Λ. Their monomial maps are therefore quantum torus automorphisms.

Lemma 6.2. *Let $H = \Psi_{v^b}(X^{(b,0)})$, and let S also denote the monomial automorphism $X^u \mapsto X^{Su}$. Then, as automorphisms of the quantum fraction field,*

$$\mathrm{Ad}_H \circ S = \mathrm{Ad}_{A^{-1}}. \tag{21}$$

Proof. Both sides fix X_1. On X_2, they equal

$$X_2(1 + v^b X_1^{-b})^{-1} = X^{(b,1)}(1 + v^{-b} X_1^b)^{-1}.$$

The equality uses $X^{(b,1)} = v^b X_1^b X_2 = v^{-b} X_2 X_1^b$. Since the fraction field is generated by X_1, X_2, this proves the identity. □

Proposition 6.3. *The map T gives a coefficient-preserving correspondence between broken lines of $\mathfrak{D}^g$ with initial exponent g and endpoint Q, and broken lines of $\mathfrak{D}^d$ with initial exponent $T(g)$ and endpoint $T(Q)$. In particular, if Q is in the first quadrant,*

$$\vartheta_{g,Q} = \vartheta^d_{T(g),Q}.$$

Proof. Transform wall supports and exponent labels by T_+ in the upper half-plane and by S in the lower half-plane. Lemma 6.1 sends every outgoing exponent to

$$(-bp + bcq, cq) = \big(b(cq - p), cq\big),$$

which has nonnegative coordinates. The extremal outgoing ray becomes the negative vertical half-line with label $\Psi_{v^c}(X^{(0,c)})$. It joins the unchanged positive vertical half-line to form the full vertical initial wall. The old negative vertical half-line becomes an outgoing ray. Replace the horizontal label A by H.

Write the ordered product in (19) as $\prod_i C_i = C_< C_*$, where $C_* = \Psi_{v^c}(X^{(-bc,c)})$ is the extremal factor and $C_<$ is the product of the remaining factors. Put $W = S(B) = \Psi_{v^c}(X^{(bc,c)})$. Applying S to (19) gives

$$AWA^{-1} = WS(C_<)B,$$

since $S(A) = A$ and $S(C_*) = B$. In particular, S commutes with Ad_A. The cut identity implies $\mathrm{Ad}_A \circ S = \mathrm{Ad}_{H^{-1}}$, hence $AWA^{-1} = H^{-1}BH$. Therefore

$$BH = H\big(WS(C_<)\big)B.$$

This is the denominator-diagram circuit identity. Its outgoing factors are ordered by increasing slope: W has slope $1/b$, and the transformed mixed ray has slope $cq/[b(cq - p)]$, strictly increasing in $p/q < c$. Interpret these infinite products in the quantum skew power-series algebra in X_2^c over $K(X_1)$. All nonhorizontal factors have positive X_2-degree. The wedge bound gives finitely many horizontal powers at each such degree. The maps $S, \mathrm{Ad}_A, \mathrm{Ad}_H$ act continuously on this algebra, and (21) holds there. The transformed labels lie in the P_d-completion. Moreover a bound on the new degree $p' + q$, where $p' = cq - p \geq 0$, bounds both q and $p \leq cq$; degreewise local finiteness is preserved. The transformed diagram is therefore the consistent completion of the two positive initial axes. Uniqueness in Proposition 3.2 identifies it with $\mathfrak{D}^d$.

Within either open half-plane the coordinate map is linear and preserves the quantum torus product, so it preserves broken-line velocities and attached coefficients. Only a horizontal crossing needs a separate check. For a crossing from H_- to H_+, write the incoming exponent as $u = (u_1, -N)$, $N > 0$. In $\mathfrak{D}^g$, the possible exponents after the crossing are

$$u - k(b, 0), \qquad 0 \leq k \leq N,$$

with coefficients $\begin{bmatrix}N\\k\end{bmatrix}_{v^b}$. The transformed incoming exponent is $Su = u - N(b,0)$. In $\mathfrak{D}^d$ the possible exponents are

$$Su + j(b,0) = u - (N-j)(b,0), \qquad 0 \le j \le N.$$

Taking $j = N - k$ gives the same outgoing exponent and the same coefficient, by quantum-binomial symmetry. For a crossing from H_+ to H_-, write $u = (u_1, N)$, $N > 0$. The g-crossing has outgoing exponent $u - k(b,0)$ and coefficient $\begin{bmatrix}N\\k\end{bmatrix}_{v^b}$. Its transformed outgoing exponent is

$$S(u - k(b,0)) = u + (N-k)(b,0).$$

The d-crossing from incoming exponent u has outputs $u + j(b,0)$ and coefficients $\begin{bmatrix}N\\j\end{bmatrix}_{v^b}$. Again $j = N - k$ matches both exponents and coefficients. This is the downward cut identity $\mathrm{Ad}_{H^{-1}} = S \circ \mathrm{Ad}_A$. Crossings without a bend are included: their transformed images can bend at the cut, and vice versa. If $g_2 = 0$, the initial exponent is fixed by both branches. The constructions are inverse on all segments and crossings. □

Proof of Theorem 1.1. Proposition 6.3 and Theorem 5.4 give

$$\vartheta_{g,Q} = \vartheta^d_{T(g),Q} = G[-T(g)].$$

The map T is an integral piecewise linear bijection, with inverse

$$T^{-1}(m_1, m_2) = (m_1 - b\min(m_2, 0), m_2).$$

Thus $-T$ indexes all greedy elements exactly once. Their integral basis theorem proves the stated basis assertion. □

Corollary 6.4. *The theta basis in Theorem 1.1 is bar invariant and independent, as a set of elements of the cluster algebra, of the initial cluster. Every theta element is universally Laurent and specializes at $v = 1$ to the corresponding classical theta element.*

Proof. Bar invariance, seed independence, universal Laurentness, and specialization to the classical greedy element follow from the corresponding properties of the quantum greedy basis. The classical identification with theta functions, in precisely the coordinates T, is [3, Theorems 4.3 and 5.1]. □

7. Arbitrary geometric coefficients

7.1. **Compatible ice seeds.** Let $f \ge 0$, let $\widetilde{M} = \mathbb{Z}^2 \oplus F$, where $F = \mathbb{Z}^f$, and write $\pi : \widetilde{M} \to \mathbb{Z}^2$ for projection. Fix an extended exchange matrix and an integral skew form

$$\widetilde{B} = \begin{pmatrix} B_0 \\ C \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \qquad \widetilde{B}^{\mathsf{T}}\widetilde{\Lambda} = (D \;\; 0), \quad D = \mathrm{diag}(d_1, d_2), \quad d_i > 0. \tag{22}$$

Here $b, c > 0$ and C is an arbitrary integral $f \times 2$ matrix. Write $\beta_i = \widetilde{B}e_i$ and $C_i = Ce_i$. The based quantum torus has multiplication

$$X^m X^{m'} = v^{\widetilde{\Lambda}(m,m')} X^{m+m'}.$$

Its frozen subalgebra is the quantum Laurent torus $\mathcal{R}_F = \bigoplus_{z \in F} RX^{(0,z)}$. It need not be commutative or central.

We use the ordinary and upper quantum cluster algebras $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{U}}$ of [1, Definition 4.12 and Section 5]. In particular, every frozen generator is inverted. This is also the convention in [4, Section 4.2]. For a seed toric frame M_t, mutation at $i \in \{1, 2\}$ replaces its i-th variable by

$$M_t\big(-e_i + [\widetilde{b}_{t,i}]_+\big) + M_t\big(-e_i + [-\widetilde{b}_{t,i}]_+\big), \tag{23}$$

where positive parts are taken coordinatewise. The algebra $\widetilde{\mathcal{A}}$ is generated by all mutable cluster variables and $\mathcal{R}_F$; the upper algebra is the intersection of all seed quantum Laurent tori in their common skew field.

Compatibility gives

$$\widetilde{\Lambda}(\beta_i, m) = d_i \pi_i(m). \tag{24}$$

Since DB_0 is skew, $bd_1 = cd_2$. Put

$$\ell = \frac{d_1}{c} = \frac{d_2}{b} = \frac{\delta}{\kappa}, \qquad \kappa = \gcd(b, c), \quad \delta = \gcd(d_1, d_2). \tag{25}$$

The number ℓ can be rational. We temporarily adjoin a suitable root of v and use $w = v^\ell$ for the coefficient-free comparison. Let $\Lambda_0(u, u') = u_2 u_1' - u_1 u_2'$. Then

$$\widetilde{\Lambda}(\widetilde{B}n, m) = \ell \Lambda_0(B_0 n, \pi m), \tag{26}$$

$$\widetilde{\Lambda}(\widetilde{B}n, \widetilde{B}n') = \ell \Lambda_0(B_0 n, B_0 n'). \tag{27}$$

These identities hold for all integral arguments and extend rationally.

The initial quantum scattering walls in $\widetilde{M}_\mathbb{R}$ are

$$\big(\{x_1 = 0\}, \Psi_{v^{d_1}}(X^{\beta_1})\big), \qquad \big(\{x_2 = 0\}, \Psi_{v^{d_2}}(X^{\beta_2})\big). \tag{28}$$

The positive monoid is $\widetilde{B}\mathbb{N}^2$, graded by the sum of the two coordinates in $\mathbb{N}^2$. Completions use the degree $n_1 + n_2$ of the wall exponent $\widetilde{B}n$. In this higher-dimensional setting a wall support is a rational polyhedral cone of codimension one, possibly with lineality, contained in $\{x : \widetilde{\Lambda}(\widetilde{B}n, x) = 0\}$. At velocity ξ, its factor f acts by Ad_{f^ϵ}, with $\epsilon = -\operatorname{sgn} \widetilde{\Lambda}(\widetilde{B}n, \xi)$. Coincident factors are multiplied. Joints are boundaries and intersections of distinct wall supports of codimension at least two; generic paths avoid these and meet walls transversely. Broken lines have velocity equal to the negative full exponent. For the cylindrical diagram below the joint is $\pi_\mathbb{R}^{-1}(0)$. Equation (24) makes the wall actions the quantum mutations with compatibility diagonal D.

7.2. **The diagram and its broken lines.**

Lemma 7.1. *Let $\mathfrak{D}_0$ be the coefficient-free g-diagram at parameter $w = v^\ell$. Its consistent completion lifts to the completion of (28) by replacing every wall support σ by $\pi_\mathbb{R}^{-1}(\sigma)$ and every wall monomial $X_0^{B_0 n}$ by $X^{\widetilde{B}n}$. For every $m \in \widetilde{M}$, the linear map*

$$\iota_m(X_0^{\pi m + B_0 n}) = X^{m + \widetilde{B}n} \tag{29}$$

intertwines all wall actions on the corresponding completed cosets.

Proof. The map $B_0 n \mapsto \widetilde{B}n$ is an isomorphism of the wall monoids because B_0 is invertible. Equation (27) identifies their quantum products, Lie brackets, exponentials, and degree completions. It takes the initial factors to (28). Equation (26) proves the claimed intertwining first for homogeneous brackets and then by exponentiation.

If a wall has exponent $B_0 n$, its inverse image lies in the $\widetilde{\Lambda}$-orthogonal hyperplane of $\widetilde{B}n$. The inverse image of an outgoing ray is outgoing, since both the wall direction and its support project to the corresponding base data. Lineality in $F_\mathbb{R}$ is allowed in a wall support. Crossing signs and path-ordered products are preserved by projection. Hence the lifted diagram is consistent and has only outgoing added walls. Uniqueness of consistent completion identifies it with the required diagram. □

Fix $Q \in \widetilde{M}_\mathbb{R}$ such that $Q_0 = \pi Q$ is generic in the first quadrant. Write

$$\vartheta^0_{g,Q_0} = \sum_{n \in \mathbb{N}^2} A_{g,n}(w) X_0^{g + B_0 n}$$

for the coefficient-free theta function.

Proposition 7.2 (Theta lifting formula)**.** *The broken-line theta function of* (28) *is*

$$\Theta_{m,Q} = \sum_{n\in\mathbb{N}^2} A_{\pi m,n}(v^\ell) X^{m+\widetilde{B}n}. \tag{30}$$

It is a finite Laurent polynomial. Every balanced Laurent coefficient belongs to $\mathbb{Z}[v^{\pm\delta}]$*, and* $\Theta_{m,Q}$ *is bar invariant.*

Proof. Suppose first that $\pi m \neq 0$. A base broken line has segment exponents $a(t) = \pi m + B_0 n(t)$, with $n(t) \in \mathbb{N}^2$. Its unique lift with endpoint Q is

$$\widetilde{\gamma}(t) = Q + \int_t^0 \big(m + \widetilde{B}n(s)\big)\, ds. \tag{31}$$

Its derivative is the negative of the full segment exponent, and its projection is the original broken line. Since every wall and joint is a cylinder, the crossing times, transversality conditions, and avoidance of joints agree. Lemma 7.1 preserves the coefficient of each chosen monomial. Conversely, projection recovers $n(t)$ uniquely, and (31) recovers the entire lifted line. A projected exponent cannot first become zero: a bend $a \mapsto a + ku = 0$ would have $\Lambda_0(u, a) = 0$, so the wall action would fix X_0^a and that nonconstant term would vanish. This proves a coefficient-preserving bijection.

If $\pi m = 0$, the initial monomial commutes with all wall factors. Only the straight frozen-direction line contributes, giving $\Theta_m = X^m$; for $m = 0$ this is the convention $\Theta_0 = 1$. Thus (30) holds in this case as well.

Theorem 1.1 makes the sum finite. Its greedy coefficients use only Gaussian parameters w^b and w^c. The recurrence therefore puts them in $\mathbb{Z}[w^{\pm\kappa}]$, which becomes $\mathbb{Z}[v^{\pm\delta}]$ by (25). Bar invariance follows from the same property of the greedy coefficients and of based monomials. □

In denominator coordinates, retain $g(a) = (b[a_2]_+ - a_1, -a_2)$, and let $e_a(p, q; w)$ be the coefficient in (7) for the quantum greedy element at parameter w. Then

$$\boxed{\Theta_{(g(a),z),Q} = \sum_{p,q\geq 0} e_a(p, q; v^\ell) X^{(-a_1+bp,\, -a_2+cq,\, z+qC_1+([a_2]_+-p)C_2)}.} \tag{32}$$

Indeed, the wall-monoid coordinates are $n = (q, [a_2]_+ - p)$. Whenever a coefficient is nonzero, $p \leq [a_2]_+$, as follows from the coefficient-free g-pointed expansion. The sum in (32) thus uses nonnegative wall coordinates.

The coefficients satisfy (8) with Gaussian parameters v^{d_2} and v^{d_1}, respectively. These recurrences characterize the homogeneous greedy lift on its projected support.

7.3. **Membership and the integral basis.** We prove algebra membership by an orthogonal lattice extension. This is a rank-two instance of coefficient variation; see [10, Section 4, Lemmas 4.7, 4.9, and 4.10].

Write m_F for the frozen component of m. Define rational linear maps $J : \mathbb{Q}^2 \to F_{\mathbb{Q}}$, $s : \mathbb{Q}^2 \to \widetilde{M}_{\mathbb{Q}}$, and $h : \widetilde{M}_{\mathbb{Q}} \to F_{\mathbb{Q}}$ by

$$J = CB_0^{-1}, \qquad s(u) = (u, Ju), \qquad h(m) = m_F - J\pi m. \tag{33}$$

Thus $m = s(\pi m) + (0, h(m))$ and $h(\widetilde{B}n) = 0$. Let Ω be the restriction of $\widetilde{\Lambda}$ to $F_{\mathbb{Q}}$. Compatibility implies

$$\widetilde{\Lambda}(s(u), (0, z)) = 0, \tag{34}$$

$$\widetilde{\Lambda}(m, m') = \ell\Lambda_0(\pi m, \pi m') + \Omega(h(m), h(m')). \tag{35}$$

For the first identity write $s(u) = \widetilde{B}B_0^{-1}u$; the second follows from (27) and the first.

Set

$$F' = F + J\mathbb{Z}^2 \subset F_{\mathbb{Q}}, \qquad L' = s(\mathbb{Z}^2) \oplus (0, F'), \qquad \Gamma = L'/\widetilde{M}.$$

The lattice F' contains F with finite index, so Γ is finite. The Γ-degree of X^m, $m \in L'$, is the class of m modulo $\widetilde{M}$. Choose $N > 0$ clearing ℓ and all pairings on L', and put $R' = \mathbb{Z}[v^{\pm 1/N}]$. This is a finite free, faithfully flat extension of R. Write $\mathcal{T}_{F'}$ for the frozen quantum torus over R', with monomials $Z^z = X^{(0,z)}$. Equations (34)–(35) give

$$R'[L']_{\widetilde{\Lambda}} \simeq \mathcal{T}^0_{v^\ell,R'} \otimes_{R'} \mathcal{T}_{F'}, \qquad X^m = U^{\pi m} Z^{h(m)}, \quad U^u = X^{s(u)}. \tag{36}$$

The two tensor factors commute with one another. The degree-zero part for the Γ-grading is precisely the original lattice torus over R'.

Lemma 7.3. *Let $\mathcal{A}^0$ be the coefficient-free rank-two quantum cluster algebra at parameter v^ℓ, with scalars extended to R'. The algebra generated by $R' \otimes_R \widetilde{\mathcal{A}}$ and $\mathcal{T}_{F'}$ inside (36) is*

$$\mathcal{A}^{\mathrm{ext}} = \mathcal{A}^0 \otimes_{R'} \mathcal{T}_{F'}. \tag{37}$$

Moreover,

$$(\mathcal{A}^{\mathrm{ext}})_0 = R' \otimes_R \widetilde{\mathcal{A}}. \tag{38}$$

Proof. At every seed t, its toric frame has a factorization

$$M_t(a) = U_t(a_{\mathrm{uf}}) Z^{h_t(a)}, \qquad h_t : \mathbb{Z}^{2+f} \longrightarrow F', \qquad h_t \widetilde{B}_t = 0, \tag{39}$$

where U_t is the coefficient-free toric frame for the same mutation sequence. Initially this is (36), with $h_0(a) = a_F - J a_{\mathrm{uf}}$. Assume (39) at t. The two exponent vectors in (23) have equal h_t-image, since their difference is the mutated column $\widetilde{b}_{t,i}$. Their mutable parts are the two coefficient-free exchange vectors. Thus the new variable is the new coefficient-free variable times their common frozen monomial.

The seed form is the sum of the form of U_t and the form $(a, a') \mapsto \Omega(h_t(a), h_t(a'))$. Hence the normalized ordered product factors into the two normalized ordered products, giving (39). If E has columns e_j for $j \neq i$ and $-e_i + [\widetilde{b}_{t,i}]_+$ for $j = i$, then $h_{t'} = h_t E$. Matrix mutation has the form $\widetilde{B}_{t'} = E \widetilde{B}_t H$, where H is the identity except in row i, with $H_{ii} = -1$ and $H_{ij} = [-b_{t,ij}]_+$ for $j \neq i$. Since $E^2 = 1$, $h_{t'} \widetilde{B}_{t'} = 0$. This completes the induction.

Each mutable variable in either algebra is therefore a variable in the other times an invertible element of $\mathcal{T}_{F'}$, proving (37). Every Z^z, $z \in F'$, normalizes every cluster variable of $\widetilde{\mathcal{A}}$ by a scalar unit, by (39). Consequently any word in the extended generators can be written as a scalar times a word in the generators of $\widetilde{\mathcal{A}}$ times one Z^z. The word in $\widetilde{\mathcal{A}}$ has degree zero in Γ, so the product has degree zero precisely when $(0, z) \in \widetilde{M}$, or equivalently $z \in F$. Taking homogeneous components of a finite sum proves (38). □

Theorem 7.4 (Integral coefficient theorem)**.** *For the ice seed (22), all $\Theta_{m,Q}$, $m \in \widetilde{M}$, belong to $\widetilde{\mathcal{A}}$ and form an R-basis. Moreover,*

$$\widetilde{\mathcal{A}} = \widetilde{\mathcal{U}}.$$

The family $\{\Theta_{(g,0),Q} : g \in \mathbb{Z}^2\}$ is both a left and a right $\mathcal{R}_F$-basis. The full R-basis is independent of the initial seed and contains every normalized extended cluster monomial, with arbitrary integral frozen exponents.

Proof. Theorem 1.1 gives an R'-basis $\{\vartheta^0_g(U) : g \in \mathbb{Z}^2\}$ of $\mathcal{A}^0$. Tensoring with $\mathcal{T}_{F'}$ gives the basis

$$\{\vartheta^0_g(U) Z^z : g \in \mathbb{Z}^2,\ z \in F'\}$$

of $\mathcal{A}^{\mathrm{ext}}$. Each element is homogeneous for Γ, because its exponent differences are $\widetilde{B}n \in \widetilde{M}$. It has degree zero exactly when $s(g) + (0, z) \in \widetilde{M}$. Proposition 7.2 identifies these degree-zero elements with the Θ_m. By (38), they form an R'-basis of $R' \otimes_R \widetilde{\mathcal{A}}$.

Proposition 7.2 already puts every Θ_m in the original torus over R. Faithful flatness descends its membership from $R' \otimes \widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{A}}$: apply it to the quotient of the original torus by the R-submodule $\widetilde{\mathcal{A}}$. Applying it next to the kernel and cokernel of the map $\bigoplus_m R \to \widetilde{\mathcal{A}}$, $e_m \mapsto \Theta_m$, descends the basis assertion.

To prove the upper-algebra equality, let $x \in \widetilde{\mathcal{U}}$. Equation (39) puts its image in $\mathcal{T}_t^0 \otimes \mathcal{T}_{F'}$ for every seed t. Expand it in the finitely many frozen monomials occurring in its initial torus expression. These monomials are linearly independent over the coefficient-free skew field: multiplying a proposed finite relation by a common Ore denominator reduces it to monomial independence in the split torus. Their coefficients therefore lie in every $\mathcal{T}_t^0$. Coefficient-free rank two is acyclic, so $\bigcap_t \mathcal{T}_t^0 = \mathcal{A}^0$ by [1, Theorem 7.5], applied over R' to the integral form $N\ell\Lambda_0$ in parameter $v^{1/N}$. Hence $x \in \mathcal{A}^{\mathrm{ext}}$, and (38) followed by the same scalar descent gives $x \in \widetilde{\mathcal{A}}$. The reverse inclusion is the quantum Laurent phenomenon.

For $z \in F$, compatibility gives the normalized shift identity

$$\Theta_{m+(0,z)} = v^{-\widetilde{\Lambda}((0,z),m)} X^{(0,z)} \Theta_m = v^{-\widetilde{\Lambda}(m,(0,z))} \Theta_m X^{(0,z)}. \tag{40}$$

The R-basis assertion therefore gives the left and right $\mathcal{R}_F$-basis assertions.

Finally, (39) identifies a theta function computed from any seed with a coefficient-free theta function times one frozen monomial in the extension. The coefficient-free basis is seed independent. If a seed has the opposite ordered orientation, interchange its mutable indices before applying the rank-two statement; the skew form and parameters are relabelled at the same time. The resulting coefficient-free factor is one original ϑ_g^0. Membership, already proved for the seed in question, and the Laurent phenomenon put the product in the original lattice torus. Since B_0 is injective and the coefficient-free expansion has leading coefficient one, its leading full exponent $s(g) + (0, z)$ cannot cancel and belongs to $\widetilde{M}$. The lifting formula identifies the product with $\Theta_{s(g)+(0,z)}$. Applying the inverse mutation gives equality of the basis sets. A normalized cluster monomial has the analogous factorization with a coefficient-free cluster monomial; this is a coefficient-free theta function and gives the last assertion. □

The full R-basis in the theorem is seed independent. A chosen set of zero-frozen-index representatives is a basis over $\mathcal{R}_F$; a change of seed can change those representatives by the normalized frozen units in (40).

7.4. **Multiplication and changes of coefficients.** The arbitrary frozen form affects multiplication through an explicit scalar. Write

$$\vartheta_g^0 \vartheta_{g'}^0 = \sum_k \alpha_{g,g'}^k(w) \vartheta_k^0.$$

The coefficient-free algebra is graded by $\mathbb{Z}^2/B_0\mathbb{Z}^2$, and its theta basis is homogeneous. Thus $\alpha_{g,g'}^k \neq 0$ implies $k - g - g' \in B_0\mathbb{Z}^2$.

Proposition 7.5 (Multiplication formula)**.** *Let $g = \pi m$, $g' = \pi m'$, and, for every nonzero term, set*

$$m_k = m + m' + \widetilde{B}B_0^{-1}(k - g - g') \in \widetilde{M}.$$

Then

$$\Theta_m \Theta_{m'} = \sum_k v^{\Omega(h(m),h(m'))} \alpha_{g,g'}^k(v^\ell) \Theta_{m_k}. \tag{41}$$

The displayed coefficients belong to R. Their signs, and the signs of Laurent coefficients in corresponding seed charts, are the same as in the coefficient-free case. In particular, strong positivity is independent of the geometric coefficient matrix and of the additional compatible frozen form.

Proof. In (36) one has $\Theta_m = \vartheta_g^0(U) Z^{h(m)}$. Multiply these factorizations and use the quantum product in $\mathcal{T}_{F'}$. The resulting term has frozen degree $h(m)+h(m')$, which is exactly the degree of m_k. This proves (41); different k give different m_k. The integral basis theorem puts each coefficient in R, although the two factors in its displayed expression may separately involve fractional powers. An injective positive rational rescaling of exponents and multiplication by a monomial preserve every scalar coefficient sign. Formula (30), applied in each seed using

(39), proves the assertion for Laurent expansions. Taking all m, m' proves both directions of the strong positivity statement. □

Here positivity means nonnegative integral coefficients in the quantum parameter, and strong positivity refers to multiplication in the full R-basis. The smaller ring $\mathbb{Z}[v^{\pm\delta}]$ in Proposition 7.2 need not contain the multiplication coefficients: even two spectator frozen monomials can multiply with an extra factor v.

There is a version of (41) with manifestly integral exponents. Since the theta coefficients belong to $\mathbb{Z}[w^{\pm\kappa}]$, every monomial coefficient of $\vartheta^0_g\vartheta^0_{g'}$ lies in $w^{\Lambda_0(g,g')}\mathbb{Z}[w^{\pm\kappa}]$: shifting either exponent by $B_0\mathbb{Z}^2$ changes its pairing by a multiple of κ. Separating quantum powers modulo κ in the unique theta basis expansion gives

$$\alpha^k_{g,g'}(w) = w^{\Lambda_0(g,g')}P^k_{g,g'}(w^\kappa), \qquad P^k_{g,g'}(t) \in \mathbb{Z}[t^{\pm1}].$$

Thus the corresponding coefficient in (41) is

$$v^{\widetilde{\Lambda}(m,m')}P^k_{g,g'}(v^\delta). \tag{42}$$

This also proves its integrality directly from (35).

Suppose two seeds with the same B_0, D admit a lattice map $L : \widetilde{M} \to \widetilde{M}'$ satisfying

$$\pi' L = \pi, \qquad L\widetilde{B} = \widetilde{B}', \qquad \widetilde{\Lambda}'(Lm, Lm') = \widetilde{\Lambda}(m, m').$$

Then the based quantum-torus homomorphism $X^m \mapsto (X')^{Lm}$ sends Θ_m to Θ'_{Lm}, directly by (30). Dropping frozen coordinates is such a homomorphism only when it preserves the skew forms. In particular, setting a noncentral frozen variable equal to 1 is generally not a permissible quantum coefficient specialization.

Proposition 7.6 (Adapted principal specialization)**.** *Every seed (22) is obtained from a compatible seed with principal coefficients and f additional frozen variables by a central coefficient quotient preserving theta functions.*

Proof. On $\widetilde{M}^\sharp = \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus F$, set

$$\widetilde{B}^\sharp = \begin{pmatrix} B_0 \\ I_2 \\ 0 \end{pmatrix}, \qquad L(u, n, z) = (u, Cn + z), \qquad \widetilde{\Lambda}^\sharp(x, y) = \widetilde{\Lambda}(Lx, Ly) \quad (x, y \in \widetilde{M}^\sharp).$$

Then $L\widetilde{B}^\sharp = \widetilde{B}$, and the pullback form is integral and compatible with the same D. The map L is onto and satisfies the preceding conditions, so it sends $\Theta^\sharp_m$ to Θ_{Lm}. The integral basis theorem makes the induced ordinary-algebra homomorphism surjective.

Every $k \in \ker L$ is frozen and lies in the radical of the pullback form. Thus X^k is central and $\Theta^\sharp_{m+k} = X^k\Theta^\sharp_m$. Grouping a finite theta expansion by fibers of L, independence of the target basis shows that the kernel is generated by $X^k - 1$, $k \in \ker L$. This proves the assertion. □

With signed coefficient rows, a cluster generator can map to a target generator times a frozen unit, while the displayed theta mapping remains exact.

7.5. **Principal coefficients.** For $C = I_2$ and $z = 0$, formula (32) reads

$$\Theta^{\mathrm{prin}}_{g(a)} = \sum_{p,q\ge0} e_a(p, q; v^\ell)X^{(-a_1+bp,-a_2+cq,q,[a_2]_+-p)}. \tag{43}$$

Both frozen exponents are nonnegative. These functions are polynomials in the frozen variables in the initial chart, and their full frozen translates form the basis of the cluster algebra with inverted frozen variables. For example, the first mutations are

$$\Theta^{\mathrm{prin}}_{(-1,0)} = X^{(-1,0,0,0)} + X^{(-1,c,1,0)}, \qquad \Theta^{\mathrm{prin}}_{(b,-1)} = X^{(b,-1,0,0)} + X^{(0,-1,0,1)}.$$

A useful compatible principal form is

$$\widetilde{\Lambda}_{\mathrm{prin}} = \begin{pmatrix} 0 & -D \\ D & -DB_0 \end{pmatrix}. \tag{44}$$

Its mutable variables commute; the frozen directions supply the required compatibility pairings. When $b = c = 2$ and $D = I_2$, this integral form has $\ell = 1/2$. All coefficients in (43) nevertheless lie in R, since the Gaussian parameters are both v.

For this example, write $Y_1 = X^{(0,0,1,0)}$ for the first frozen generator. The relation $X_1Y_1 = v^{-2}Y_1X_1$ prevents a homomorphism setting $Y_1 = 1$ while retaining the indeterminate v and an invertible X_1. An integral monomial rescaling from the coefficient-free torus would require an integral frozen correction matrix satisfying $SB_0 = I_2$. Here B_0^{-1} has half-integral entries, so such a rescaling exists over the extended lattice L'.

8. Finite type and positivity

8.1. **Finite rank two.** If $bc \le 3$, all greedy elements are normalized cluster monomials. For $b = 1$, their denominator fans are as follows. The other orientation interchanges the coordinates.

A_2	$(-1,0),(0,-1),(1,0),(1,1),(0,1)$
B_2	$(-1,0),(0,-1),(1,0),(1,1),(1,2),(0,1)$
G_2	$(-1,0),(0,-1),(1,0),(1,1),(2,3),(1,2),(1,3),(0,1)$.

Consecutive rays a_j, a_{j+1} are unimodular and their closed cones cover the plane. Index each displayed list starting at $a_1 = (-1,0)$, so that a_j is the denominator vector of X_j, and extend cyclically. For $a = ra_j + sa_{j+1}$, $r, s \in \mathbb{N}$, the corresponding element is

$$G[a] = v^{rs} X_j^r X_{j+1}^s.$$

On a common ray one exponent is zero, so neighboring cone descriptions agree. The origin gives 1.

The finite quantum scattering identities can also be checked directly. For example, with our convention and $\psi_h(u) = \Psi_{v^h}(X^u)$, the B_2 identity is

$$\psi_2(0,2)\psi_1(1,0) = \psi_1(1,0)\psi_2(2,2)\psi_1(1,2)\psi_2(0,2).$$

For G_2 it is

$$\begin{aligned}\psi_3(0,3)\psi_1(1,0) = \psi_1(1,0)\psi_3(3,3)\psi_1(2,3)\\ \cdot \psi_3(3,6)\psi_1(1,3)\psi_3(0,3).\end{aligned}$$

Here $\psi_h(r,s)$ abbreviates $\psi_h((r,s))$. These are the identities of [8, Section 3.2] in the present normalization; they follow from the pentagon relation and

$$\Psi_w(z) = \prod_{j=1}^{r} \Psi_{w^r}(w^{r+1-2j}z).$$

8.2. **Positivity and negative coefficients.**

Example 8.1. In $\mathcal{A}_v(2,3)$, the coefficient of $X^{(1,-1)}$ in $G[3,4]$ is

$$e(2,1) = v^2 - 1 + v^{-2}.$$

This follows directly from (8):

$$e(2,1) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{v^2} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{v^3} = v^2 - 1 + v^{-2}.$$

Consequently the same coefficient occurs in $\vartheta_{(5,-4),Q}$.

There is also a negative multiplication coefficient. The product $X^{(3,4)}G[3,4]$ has only nonnegative exponent vectors $(2p, 3q)$, so its initial-cluster monomial expansion is already an expansion in the theta basis. Its coefficient at $X^{(4,3)}$ is

$$v^7(v^2 - 1 + v^{-2}) = v^9 - v^7 + v^5.$$

Thus this basis is not strongly positive.

Example 8.2 (A principal-coefficient product)**.** Take $b = 2, c = 3$, $C = I_2$, $D = \mathrm{diag}(3, 2)$, and the compatible form (44). Then $\Theta_{(3,4,0,0)} = X^{(3,4,0,0)}$. In its product with $\Theta_{(5,-4,0,0)}$, the coefficient of $\Theta_{(4,3,1,2)}$ is

$$v^{-25}(v^2 - 1 + v^{-2}) = v^{-23} - v^{-25} + v^{-27}.$$

Indeed, the contributing term of the second factor has exponent $(1, -1, 1, 2)$, and

$$\widetilde{\Lambda}\big((3, 4, 0, 0), (1, -1, 1, 2)\big) = -25.$$

Every product exponent has the form $(2p, 3q, q, 4 - p) \geq 0$, so each term is already a theta monomial. The coefficient is thus a genuine negative structure constant. The two factors and all these monomials belong to the polynomial-frozen theta basis of Theorem 9.3 as well.

For $b = c$, take the common dilogarithm parameter $t = v^b$ and the ambient skew form Λ/b. Then $t^{(\Lambda/b)(u,u')} = v^{\Lambda(u,u')}$, so the skew-symmetric strong positivity theorem of Davison–Mandel [4] applies to the present torus and diagram. Under Theorem 1.1, both Laurent positivity and positivity of multiplication coefficients transfer to the quantum greedy basis in this range. Proposition 7.5 then transfers strong positivity to every compatible geometric coefficient system with that principal part. For nondividing b, c, nonpositive theta coefficients are also established in [8, Theorem 5.10 and Proposition 5.11].

Remark 8.3. When coefficients of a quantum theta element are nonnegative, specialization at $v = 1$ preserves its exact support. Without nonnegativity, specialization alone gives only an inclusion: a nonzero Laurent polynomial may vanish at 1.

9. Disconnected rank two and frozen regularity

9.1. **The disconnected principal part.** An exchange matrix with two mutable indices has either the connected form considered above, up to reversing orientation, or zero principal part.

Proposition 9.1. *Let $(\widetilde{B}, \widetilde{\Lambda})$ be an integral compatible ice seed with two mutable indices and $\widetilde{B} = (0; C)$. Let $\widetilde{B}^{\mathsf{T}}\widetilde{\Lambda} = (D \;\; 0)$, and let $d_1, d_2 > 0$. Then C has rank two. Put $\beta_i = (0, C_i)$, $r_i = [-C_i]_+$, and $N_i = [-a_i]_+$. For $m = (a_1, a_2, z)$, the theta function is*

$$\Theta_m = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} \begin{bmatrix} N_1 \\ k_1 \end{bmatrix}_{v^{d_1}} \begin{bmatrix} N_2 \\ k_2 \end{bmatrix}_{v^{d_2}} X^{m + k_1\beta_1 + k_2\beta_2}. \tag{45}$$

These functions form a bar-invariant, universally Laurent positive, strongly positive R-basis of the ordinary quantum cluster algebra with inverted frozen variables. The ordinary and upper algebras agree. If frozen variables are instead not inverted, the theta functions with

$$z \geq N_1 r_1 + N_2 r_2 \tag{46}$$

coordinatewise form an R-basis of that ordinary algebra. In this noninverted case the ordinary and upper algebras also agree, and equal the intersection of the frozen-localized algebra with the initial mixed Laurent chart.

Proof. Compatibility gives $\widetilde{\Lambda}(\beta_i, m) = d_i a_i$. Pairing a proposed linear dependence between β_1, β_2 with the two mutable basis vectors proves their independence. Moreover, $\widetilde{\Lambda}(\beta_1, \beta_2) = 0$, and each β_i is orthogonal to the frozen lattice. The initial dilogarithms therefore commute, so their two hyperplanes already form a consistent diagram. Every bend changes only frozen

exponents. A broken line crosses the i-th hyperplane once if $a_i < 0$, and never otherwise. The two independent Gaussian choices give (45).

The only other variable in direction i is

$$X_i' = X^{-e_i+(0,r_i)} + X^{-e_i+(0,r_i)+\beta_i}.$$

The two directions mutate independently, giving four clusters. For fixed a, let $Z_i(a_i) = X_i^{a_i}$ when $a_i \geq 0$, and $Z_i(a_i) = (X_i')^{-a_i}$ otherwise. Put $\mu_i = a_i e_i + (0, N_i r_i)$. Quantum binomial expansion and $\widetilde{\Lambda}(\beta_i, \mu_j) = 0$ for $i \neq j$ show that

$$S_a := v^{-\widetilde{\Lambda}(\mu_1,\mu_2)} Z_1(a_1) Z_2(a_2) = \Theta_{(a,N_1r_1+N_2r_2)}.$$

Frozen monomials and factors from different mutable directions quasi-commute. In a fixed direction, if $\eta_i = \widetilde{\Lambda}(e_i, (0, r_i))$, then

$$X_i X_i' = v^{\eta_i} X^{(0,r_i)} + v^{\eta_i - d_i} X^{(0,r_i)+\beta_i},$$

and the reversed product has the reversed scalar exponents. Repeated use of these relations reduces the number of mutable factors and expresses every word as a linear combination of $X^{(0,z)} S_a$. These elements are independent: different a have disjoint mutable-exponent supports, and at fixed a a relation is $PS_a = 0$ in a domain, with P in the frozen quantum torus. The normalized frozen shifts identify this basis with (45).

The displayed Gaussian coefficients are positive, and the same straightening uses only positive two-term relations and quantum monomial factors. It proves strong positivity. The four seed descriptions prove universal Laurent positivity. Bar invariance is immediate in based monomials; ordinary equals upper also follows from [1, Theorem 7.5].

If the frozen variables are not inverted, the straightening relations still use only nonnegative frozen exponents. Hence $X^{(0,z)} S_a$, $z \in \mathbb{N}^f$, give an independent spanning set. Its normalized labels are exactly (46). Equivalently, the minimum j-th frozen exponent in (45) is $z_j - N_1(r_1)_j - N_2(r_2)_j$; endpoint Gaussian coefficients show that this minimum is attained.

To prove the last equality, expand an element of the localized algebra uniquely as $\sum_a P_a S_a$, with frozen Laurent coefficients P_a. The least exponent in each frozen coordinate of S_a is zero by the endpoint calculation. Least coordinate degrees add under multiplication: the two least-degree components have nonzero product in the quantum Laurent domain. Moreover, different a have disjoint mutable supports. Hence $\sum_a P_a S_a$ lies in the initial mixed chart precisely when every P_a is a frozen polynomial. This proves that the intersection is the ordinary polynomial coefficient algebra. The mixed quantum Laurent phenomenon [9, Definition 2.14 and Theorem 2.15], together with the localized equality already proved, then gives equality with the noninverted upper algebra. □

9.2. **Polynomial frozen coefficients with nonnegative initial rows.** We now allow frozen variables to be noninvertible and assume that the initial coefficient block is entrywise nonnegative.

Let $\mathcal{R}_F^+ = \bigoplus_{z\in\mathbb{N}^f} RX^{(0,z)}$ be the frozen quantum polynomial algebra, and let $\mathcal{T}_0^+$ be the initial chart with mutable variables inverted and frozen variables not inverted. Write $\mathcal{A}_{\mathrm{pol}}$ for the algebra generated by all mutable cluster variables and $\mathcal{R}_F^+$, and $\mathcal{U}_{\mathrm{pol}}$ for the intersection of the analogous mixed charts in all seeds. For a finite Laurent polynomial x, let $\nu_j(x)$ be its least exponent in the j-th frozen coordinate, with $\nu_j(0) = +\infty$.

Lemma 9.2. *Let $H_i \in \mathcal{T}_0^+$ be a family whose components of frozen degree zero, denoted H_i^0, are R-linearly independent. For a finite sum with $P_i \in \mathcal{R}_F$,*

$$\nu_j\left(\sum_i P_i H_i\right) = \min_i \nu_j(P_i). \tag{47}$$

Zero coefficients are assigned valuation $+\infty$; for the all-zero family the minimum is $+\infty$. The same formula holds with the coefficients on the right.

Proof. If every $P_i = 0$, the assertion is immediate. Otherwise expand every P_i in based frozen monomials. Order their finitely many exponent vectors lexicographically, placing coordinate j first, and choose the least exponent α that occurs with nonzero coefficient. A nonzero nonnegative frozen exponent from H_i can contribute to the α-layer only from a strictly smaller coefficient exponent, which is absent. That layer is consequently

$$X^{(0,\alpha)} \sum_i p_{i,\alpha} H_i^0, \qquad p_{i,\alpha} \in R.$$

It is nonzero by the asserted independence. No exponent in the sum has j-th coordinate smaller than α_j, proving (47). For right coefficients the common frozen monomial is factored on the right. □

Theorem 9.3 (Nonnegative coefficient rows)**.** *Assume* (22) *and* $C \geq 0$ *entrywise. Then*

$$\mathcal{A}_{\mathrm{pol}} = \mathcal{U}_{\mathrm{pol}} = \widetilde{\mathcal{A}} \cap \mathcal{T}_0^+. \tag{48}$$

The family $\{\Theta_{(g,0)} : g \in \mathbb{Z}^2\}$ *is both a left and a right* $\mathcal{R}_F^+$*-basis. Equivalently,*

$$\{\Theta_{(g,z)} : g \in \mathbb{Z}^2,\ z \in \mathbb{N}^f\} \tag{49}$$

is an R*-basis.*

Proof. Let X_1', X_2' denote the first mutations of the initial variables. For $a \in \mathbb{Z}^2$, put $\xi(a) = (a_1 + b[-a_2]_+, a_2)$. Let E_a be the multiple of

$$X_1^{[a_1]_+} X_2^{[a_2]_+} (X_1')^{[-a_1]_+} (X_2')^{[-a_2]_+}$$

by a power of v for which the coefficient of $X^{(\xi(a),0)}$ is one. This normalization exists: since $C \geq 0$,

$$X_1' = X^{(-1,0,0)} + X^{(-1,c,C_1)}, \qquad X_2' = X^{(b,-1,0)} + X^{(0,-1,C_2)}.$$

Multiplying these expansions gives

$$E_a = X^{(\xi(a),0)} + \sum_{0 \neq n \in \mathbb{N}^2} e_{a,n} X^{(\xi(a)+B_0 n, Cn)}, \qquad e_{a,n} \in R. \tag{50}$$

The sum is finite and $a \mapsto \xi(a)$ is a bijection. The normalized standard monomials form a left $\mathcal{R}_F$-basis of $\widetilde{\mathcal{A}}$, by [1, Theorems 7.3 and 7.5]. Their exponent differences are exchange-lattice vectors, which are orthogonal to the frozen lattice. They therefore quasi-commute with every frozen monomial by a fixed scalar unit, giving a right basis as well.

We check the hypothesis of Lemma 9.2 even when C has zero columns. On taking frozen degree zero in (50), only n with $Cn = 0$ survive. The linear functional

$$\omega(u) = -u_1/b + u_2/c$$

satisfies $\omega(B_0 n) = n_1 + n_2 > 0$ for $n \neq 0$. In a proposed finite relation among the E_a^0, choose the smallest $\omega(\xi(a))$ among its nonzero summands. At that weight only their distinct leading monomials occur, so they cannot cancel. Thus the E_a^0 are independent. Exactly the same argument applies to $\Theta_{(g,0)}^0$, using (30) and its leading coefficient one.

Apply Lemma 9.2 to the unique finite standard-monomial expansion of an element of $\widetilde{\mathcal{A}}$. It belongs to $\mathcal{T}_0^+$ exactly when all its frozen coefficients belong to $\mathcal{R}_F^+$. Hence

$$\widetilde{\mathcal{A}} \cap \mathcal{T}_0^+ = \bigoplus_a \mathcal{R}_F^+ E_a \subseteq \mathcal{A}_{\mathrm{pol}}.$$

The quantum Laurent phenomenon allowing noninverted frozen variables, [9, Definition 2.14 and Theorem 2.15], gives

$$\mathcal{A}_{\mathrm{pol}} \subseteq \mathcal{U}_{\mathrm{pol}} \subseteq \widetilde{\mathcal{U}} \cap \mathcal{T}_0^+.$$

In this application its field is $\mathbb{Q}(v)$, its ground subring is R, its quantum commutation units lie in $R^\times$, and its chosen set of inverted frozen variables is empty. Theorem 7.4 identifies $\widetilde{\mathcal{U}} = \widetilde{\mathcal{A}}$; the two inclusions prove (48).

Apply the same valuation lemma to the left and right frozen theta expansions from Theorem 7.4. This proves both polynomial module basis assertions. The normalized frozen shift formula (40) then gives (49). □

This R-basis is the intersection of the full theta basis with $\mathcal{A}_{\mathrm{pol}}$, so it is intrinsic to that algebra. The hypothesis on C may be imposed at any seed; thus it suffices that a seed with nonnegative coefficient block exists.

If only a chosen subset of the frozen variables is inverted, the same proof imposes the valuation conditions only on the remaining coordinates. For $C \geq 0$, it gives the ordinary and upper algebra equality and the corresponding theta basis for every such partial choice of frozen inverses.

9.3. **Signed coefficient rows: a counterexample.** For signed coefficient rows, the ordinary algebra with polynomial frozen coefficients can be strictly smaller than its upper algebra.

Proposition 9.4. *There is a compatible acyclic rank-two quantum seed with one central frozen variable Y such that*

$$\mathcal{A}_{\mathrm{pol}} \subsetneq \mathcal{U}_{\mathrm{pol}}.$$

Here $\mathcal{A}_{\mathrm{pol}}$ is generated by all mutable cluster variables and Y, and $\mathcal{U}_{\mathrm{pol}}$ is the intersection of seed charts with mutable variables inverted and Y not inverted. A theta function of the frozen-localized algebra lies in $\mathcal{U}_{\mathrm{pol}} \setminus \mathcal{A}_{\mathrm{pol}}$, while its product with Y belongs to $\mathcal{A}_{\mathrm{pol}}$.

Proof. Take

$$\widetilde{B} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \widetilde{\Lambda} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\widetilde{B}^{\mathsf{T}}\widetilde{\Lambda} = (2I_2 \;\; 0)$. Mutation at the first mutable index, followed by interchange of the two mutable indices, returns these matrices. Label the consecutive clusters by (X_m, X_{m+1}), $m \in \mathbb{Z}$. Their relations are

$$X_{m+1}X_m = v^2 X_m X_{m+1}, \qquad X_{m+1}X_{m-1} = v^2 X_m^2 + Y. \tag{51}$$

Put $M_m(r,s) = v^{rs} X_m^r X_{m+1}^s$, and define

$$Z_m = M_m(-1,1) + M_m(1,-1) + Y M_m(-1,-1).$$

The neighboring mutations are

$$X_{m+2} = M_m(-1,2) + Y M_m(-1,0), \qquad X_{m-1} = M_m(2,-1) + Y M_m(0,-1).$$

Using based multiplication gives

$$Z_m X_{m+1} = Z_{m+1} X_{m+1} = v X_{m+2} + v^{-1} X_m.$$

Cancellation proves that all Z_m are the same element Z. Its displayed formula in every seed has frozen exponents $0, 0, 1$, so $Z \in \mathcal{U}_{\mathrm{pol}}$. The identities

$$Z X_m = v X_{m+1} + v^{-1} X_{m-1}$$

also express all cluster variables recursively in any one mixed chart, in both directions. Thus $\mathcal{A}_{\mathrm{pol}} \subseteq \mathcal{U}_{\mathrm{pol}}$ here follows directly.

Give the initial mixed Laurent chart the direct-sum grading $\deg X_1 = \deg X_2 = 1$, $\deg Y = 2$, and $\deg v = 0$. The three terms of Z have degree zero, and the preceding linear recurrence proves that every X_m has degree one. Therefore the zero-degree component of the algebra generated by all X_m and Y is exactly R. Since Z is a nonconstant sum of three distinct based monomials, $Z \notin \mathcal{A}_{\mathrm{pol}}$.

Finally, direct multiplication in the initial torus gives

$$X_0 X_3 = v^{-2} X_1 X_2 + v Y Z, \qquad Y Z = v^{-1} X_0 X_3 - v^{-3} X_1 X_2 \in \mathcal{A}_{\mathrm{pol}}.$$

Formula (32), with $a = (1,1)$, $C = (-1,1)$, and $z = 0$, identifies $Z = \Theta_{(1,-1,0)}$. This proves all assertions. □

In this example the least nonnegative power of Y that makes the theta function regular in all mixed seed charts is zero, whereas the least power that puts it in the ordinary polynomial coefficient algebra is one.

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