

The Anti-Arithmetic Product of Symmetric Functions

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Abstract

The *arithmetic product* $\square$ is a bilinear operation on the ring of symmetric functions over $\mathbb{Q}$ defined by

$$p_\lambda \square p_\mu = \prod_{i,j} p_{\text{lcm}(\lambda_i, \mu_j)}^{\text{gcd}(\lambda_i, \mu_j)}$$

on the power-sum basis (p_λ) . It is known that this operation preserves the subring $\mathbf{Symm}_{\mathbb{Z}}$ of symmetric functions with integer coefficients, and is Schur-positive on pairs of Schur functions. Indeed, it corresponds to induction of representations from $S_n \times S_m$ to S_{nm} .

In this work, we consider the *anti-arithmetic product* $\diamond$, which is defined by the same formula but with gcd and lcm interchanged. We show that it, too, preserves $\mathbf{Symm}_{\mathbb{Z}}$, even though it lacks the Schur positivity property. This was conjectured on MathOverflow in 2014.

Our method is a more intricate variant of the representation-theoretical interpretation of $\square$. The anti-arithmetic product does not correspond to an operation on actual representations; but its adjoint can still be described as a map on characters, which (as we show) is a λ -ring morphism from the character ring of S_{mn} to that of $S_m \times S_n$. Now, a 1975 theorem of Boorman shows that the representation ring of S_n is generated by the natural permutation representation $M_n = \mathbb{Q}^n$ as a λ -ring. Thus, the proof of integrality boils down to only computing the image of M_{mn} under this adjoint, which can be done using Möbius inversion in the representation ring.

This work is aimed at readers familiar with representation rings and the representation theory of symmetric groups. No prior knowledge of λ -rings is presumed. Three self-contained proofs of Boorman's theorem are given in the appendices, two of them lifting the theorem to the noncommutative symmetric functions (or Solomon's descent algebra), recovering a result of Schocker.

Manifest. This work was written by GPT-6 based on a proof found by GPT-5.5. It has then been extensively edited and fully proofread by myself.

This work is in the public domain. –DG

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1 Definitions and statements

1.1 Integral and rational symmetric functions

Let $\mathbf{Symm}_{\mathbb{Z}}$ denote the ring of symmetric functions over $\mathbb{Z}$, and let $\mathbf{Symm}_{\mathbb{Q}}$ be the analogous ring over $\mathbb{Q}$. Thus,

$$\mathbf{Symm}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{Symm}_{\mathbb{Z}}.$$

We write $\mathbf{Symm}_{\mathbb{Z},n}$ and $\mathbf{Symm}_{\mathbb{Q},n}$ for the homogeneous components of degree n of $\mathbf{Symm}_{\mathbb{Z}}$ and $\mathbf{Symm}_{\mathbb{Q}}$. Our conventions for symmetric functions are the standard ones; see, for example, Sagan [36, Chapter 4], Stanley [41, Chapter 7], or Grinberg–Reiner [21, Section 2].

For $r \geq 1$, let p_r be the r -th power-sum symmetric function, and for a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ let

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

Each p_λ belongs to $\mathbf{Symm}_{\mathbb{Z}}$, but the family $(p_\lambda)_\lambda$ is only a $\mathbb{Q}$ -basis of $\mathbf{Symm}_{\mathbb{Q}}$, not a $\mathbb{Z}$ -basis of $\mathbf{Symm}_{\mathbb{Z}}$. For instance,

$$h_2 = \frac{p_1^2 + p_2}{2}.$$

Equivalently,

$$\mathbf{Symm}_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, p_3, \dots],$$

whereas the integral ring is

$$\mathbf{Symm}_{\mathbb{Z}} = \mathbb{Z}[h_1, h_2, h_3, \dots] = \mathbb{Z}[e_1, e_2, e_3, \dots].$$

Consequently, it is very easy to define multilinear operations on $\mathbf{Symm}_{\mathbb{Q}}$ by prescribing their values on power sums, but it is a separate question whether such operations preserve $\mathbf{Symm}_{\mathbb{Z}}$. Many familiar operations do: the Kronecker product and plethysm are standard examples¹. This phenomenon, and several operations conveniently defined on the power-sum basis, are discussed for example in [21, Exercise 2.9.4]; see also the symmetric-function chapters of Stanley [41, Chapter 7] and Macdonald [28, Chapter I]. The two operations considered below provide another illustration of precisely this integrality issue.

1.2 The arithmetic and anti-arithmetic products

Since the p_λ form a $\mathbb{Q}$ -basis of $\mathbf{Symm}_{\mathbb{Q}}$, the following definitions make sense over $\mathbb{Q}$.

Definition 1.1. *The arithmetic product $\square$ is the $\mathbb{Q}$ -bilinear operation on $\mathbf{Symm}_{\mathbb{Q}}$ defined by*

$$p_\lambda \square p_\mu = \prod_{i=1}^r \prod_{j=1}^s p_{\text{lcm}(\lambda_i, \mu_j)}^{\text{gcd}(\lambda_i, \mu_j)} \quad (1)$$

for all pairs of partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ (written without trailing zeroes).

¹Admittedly, plethysm is not multilinear, but the same construction principle applies with some complications.

The anti-arithmetic product $\diamond$ is the $\mathbb{Q}$ -bilinear operation on $\mathbf{Symm}_{\mathbb{Q}}$ defined by

$$p_{\lambda} \diamond p_{\mu} = \prod_{i=1}^r \prod_{j=1}^s p_{\gcd(\lambda_i, \mu_j)}^{\text{lcm}(\lambda_i, \mu_j)} \quad (2)$$

for all pairs of partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ (written without trailing zeroes).

If $|\lambda| = m$ and $|\mu| = n$, then both right-hand sides have degree mn , because each pair (λ_i, μ_j) contributes total degree $\lambda_i \mu_j$. Thus, for $m, n \geq 0$, we have

$$\square, \diamond : \mathbf{Symm}_{\mathbb{Q}, m} \times \mathbf{Symm}_{\mathbb{Q}, n} \longrightarrow \mathbf{Symm}_{\mathbb{Q}, mn}.$$

The arithmetic product is the symmetric-function shadow of the arithmetic product of combinatorial species of Maia and Méndez [29]; its integrality is discussed in [21, Exercise 4.4.9] and in the MathOverflow question [17]. For a further development of the species-theoretic side, Li [27] introduced an exponential composition based on the arithmetic product and used it to study prime graphs under Cartesian product. The anti-arithmetic product was introduced in the MathOverflow question [16], where its integrality was asked for. Unlike the arithmetic product, it is not Schur-positive on pairs of Schur functions; for example, $s_{(2,1,1)} \diamond s_{(2,1,1)}$ has Schur coefficients of both signs [16].²

Our main result is the following.

Theorem 1.2 (Integrality). *For all $m, n \geq 0$, the following statements hold.*

(a) We have

$$\mathbf{Symm}_{\mathbb{Z}, m} \square \mathbf{Symm}_{\mathbb{Z}, n} \subseteq \mathbf{Symm}_{\mathbb{Z}, mn}.$$

(b) If $f \in \mathbf{Symm}_{\mathbb{Z}, m}$ and $g \in \mathbf{Symm}_{\mathbb{Z}, n}$ are Schur-positive, then $f \square g$ is Schur-positive.

(c) We have

$$\mathbf{Symm}_{\mathbb{Z}, m} \diamond \mathbf{Symm}_{\mathbb{Z}, n} \subseteq \mathbf{Symm}_{\mathbb{Z}, mn}.$$

Parts (a) and (b) are known. We include them because the proof of (a) is parallel to our proof of part (c), except at exactly one point. Both proofs proceed using the classical Frobenius correspondence between symmetric functions and representations/class functions of symmetric groups. For the arithmetic product $\square$, a genuine group homomorphism³

$$S_m \times S_n \longrightarrow S_{mn}$$

provides the required pullback on representations. For the anti-arithmetic product no such homomorphism is available; instead, we construct a certain *virtual* representation by Adams operations and Möbius inversion.

Before we present these proofs, we shall develop the machinery needed for them. Very little of it is new, but not all of it is found in textbooks.

²More explicitly, its coefficients of $s_{(12,4)}$ and $s_{(14,1,1)}$ are -3 and 1 , respectively.

³We let S_n denote the n -th symmetric group; this group consists of the permutations of the set $[n] := \{1, 2, \dots, n\}$.

Remark 1.3 (Degree zero). *Theorem 1.2 is easy when one of m and n is 0. Indeed, by the empty product convention, each partition λ satisfies*

$$1 \sqcup p_\lambda = p_\lambda \sqcup 1 = 1 \diamond p_\lambda = p_\lambda \diamond 1 = 1 = p_\lambda(1, 0, 0, 0, \dots).$$

Thus, by linearity, for any $f \in \mathbf{Symm}_{\mathbb{Z}}$, we have

$$1 \sqcup f = f \sqcup 1 = 1 \diamond f = f \diamond 1 = f(1, 0, 0, 0, \dots) \in \mathbb{Z} \subseteq \mathbf{Symm}_{\mathbb{Z}}.$$

In particular, if λ is a partition, then $1 \sqcup s_\lambda = s_\lambda \sqcup 1$ is 1 if λ is a one-row partition and is 0 otherwise.

Thus, in proving Theorem 1.2, we can focus on the case when $m, n \geq 1$. We thus WLOG assume that $m, n \geq 1$ from now on.

Remark 1.4. *Both operations $\sqcup$ and $\diamond$ are easily seen to be commutative and associative. Moreover, p_1 is a neutral element for $\sqcup$, whereas $\diamond$ has no neutral element.*

2 Exterior powers, Adams operations, and class functions

We will use the basic language of λ -rings and their Adams operations. The words “ λ -ring” and “Adams operation” can suggest a good deal more machinery than we need. In this section we develop the small part of the theory used later, starting from exterior powers and the standard relation between elementary symmetric functions and power sums. For systematic treatments, see Knutson [25, Chapter I, especially §§1 and 4], Yau [45, Chapters 1 and 3], or Hazewinkel [24, Section 16]. None of the general structure theorems from these references will be used. For the symmetric-group representation rings specifically, Thibon [42] and Scharf–Thibon [38] develop Adams operations and inner plethysm directly in symmetric-function language. A modern structural perspective, emphasizing Adams operations as natural transformations of the representation-ring functor, is given by Meir–Szymik [31].

2.1 Representation rings and class functions

For every finite group G below, we work with finite-dimensional representations over $\mathbb{Q}$. The constructions in this section do not require $\mathbb{Q}$ to be a splitting field. For symmetric groups and their finite direct products, however, $\mathbb{Q}$ is a splitting field; this will be used in Section 3.

Let $\mathrm{Cl}_{\mathbb{Q}}(G)$ be the $\mathbb{Q}$ -algebra of $\mathbb{Q}$ -valued class functions on G , with pointwise addition and multiplication. In particular, the character χ_V of any finite-dimensional representation V of G over $\mathbb{Q}$ belongs to $\mathrm{Cl}_{\mathbb{Q}}(G)$. Let $R(G)$ be the Grothendieck ring of finite-dimensional $\mathbb{Q}G$ -modules (see, e.g., [40, Chapter 9]). Thus $R(G)$ is a $\mathbb{Z}$ -algebra: addition comes from direct sum and multiplication from tensor product. We write $[V]$ for the class of a representation V (although we will occasionally drop the brackets and just write V). Maschke’s theorem and ordinary character theory give an injective ring homomorphism

$$\chi : R(G) \hookrightarrow \mathrm{Cl}_{\mathbb{Q}}(G), \quad [V] \longmapsto \chi_V, \tag{3}$$

which we shall call the *character embedding*. We shall usually identify $R(G)$ with its image under this map. Standard background on these facts may be found, for example, in Etingof–Golberg–Hensel–Liu–Schwendner–Vaintrob–Yudovina [9, §4.2], Serre [40, Chapter 2], or the lecture notes of Lassueur [26, §9].

2.2 A weak notion of λ -ring

Our notion of a λ -ring is a weak one, imposing axioms for $\lambda^0(x)$ and $\lambda^1(x)$ and $\lambda^n(x+y)$. Many authors know such λ -rings under the name of *pre- λ -rings*, and only deem them worthy of the name “ λ -ring” if they satisfy additional axioms for $\lambda^n(xy)$ and $\lambda^n(\lambda^m(y))$. To us, these extra axioms are unnecessary.

Definition 2.1. *A λ -ring in this note (often called a pre- λ ring) is a commutative unital ring A equipped with (usually non-linear) maps*

$$\lambda^r : A \longrightarrow A \quad \text{for all } r \geq 0$$

such that all $x, y \in A$ satisfy the axioms

$$\lambda^0(x) = 1, \tag{4}$$

$$\lambda^1(x) = x, \tag{5}$$

$$\lambda^n(x+y) = \sum_{i=0}^n \lambda^i(x) \lambda^{n-i}(y). \tag{6}$$

We can encode these maps λ^r into a single generating function

$$\lambda_t(x) = \sum_{r \geq 0} \lambda^r(x) t^r \in A[[t]] \tag{7}$$

defined for all $x \in A$ (that is, into a map $\lambda_t : A \rightarrow A[[t]]$); then the axioms (4) and (5) say that

$$\lambda_t(x) = 1 + xt + (\text{higher powers of } t),$$

whereas the axiom (6) can be equivalently rewritten as

$$\lambda_t(x+y) = \lambda_t(x) \lambda_t(y). \tag{8}$$

In other words, λ_t must be a group homomorphism from the additive group of A to the multiplicative group of formal power series with constant term 1 over A , and it must have the property that $\frac{d}{dt} \lambda_t(x) |_{t=0} = x$ for each $x \in A$.

A λ -ring morphism is a unital ring homomorphism $f : A \rightarrow B$ between two λ -rings that commutes with every λ^r (that is, satisfies $f \circ \lambda^r = \lambda^r \circ f$ for each $r \geq 0$). A λ -subring of a λ -ring A is a unital subring closed under every λ^r . The λ -subring of A generated by a subset $X \subseteq A$ is the smallest λ -subring of A containing X .

Note that the operations λ^r on a λ -ring A , taken in combination, carry the same information as their generating series λ_t . In particular, a λ -ring can be defined by providing λ_t instead of the λ^r 's.

Example 2.2 (The binomial λ -ring). The ring $\mathbb{Z}$ is a λ -ring under

$$\lambda^r(a) = \binom{a}{r}, \quad \lambda_t(a) = (1+t)^a.$$

The binomial theorem shows that these two equalities fit together with (7); the equality (6) is the Chu–Vandermonde convolution.

Example 2.3 (Representation rings). Let G be a group, and consider its representation ring $R(G)$. For an actual G -representation V , put

$$\lambda_t([V]) = \sum_{r \geq 0} [\Lambda^r V] t^r \in R(G)[[t]]. \quad (9)$$

The canonical decomposition

$$\bigwedge^r (V \oplus W) \cong \bigoplus_{i+j=r} \bigwedge^i V \otimes \bigwedge^j W$$

gives

$$\lambda_t([V \oplus W]) = \lambda_t([V]) \lambda_t([W]).$$

Thus, (9) defines a monoid homomorphism $x \mapsto \lambda_t(x)$ from the additive monoid of actual representations of G to the multiplicative group $1 + tR(G)[[t]]$ of power series with constant term 1. Since $R(G)$ is the Grothendieck completion of the former monoid, this homomorphism extends uniquely to a group homomorphism

$$\lambda_t : R(G) \longrightarrow 1 + tR(G)[[t]]$$

by the rule

$$\lambda_t(x - y) = \frac{\lambda_t(x)}{\lambda_t(y)}.$$

Thus $R(G)$ is a λ -ring (the proof of (5) is easy).

2.3 ψ -rings

Adams operations are even easier to axiomatize.

Definition 2.4. A ψ -ring is a commutative unital ring A equipped with unital ring endomorphisms

$$\psi^r : A \longrightarrow A \quad \text{for all } r \geq 1$$

such that

$$\psi^1 = \text{id} \quad \text{and} \quad \psi^{rs} = \psi^r \circ \psi^s \quad \text{for all } r, s \geq 1.$$

The endomorphisms ψ^r are known as the Adams operations of A . A morphism of ψ -rings is a unital ring homomorphism commuting with all ψ^r .

Proposition 2.5. *For every finite group G , the class-function algebra $\text{Cl}_{\mathbb{Q}}(G)$ is a ψ -ring under the ψ -operations ψ^r defined by*

$$(\psi^r f)(g) = f(g^r) \quad \text{for all } r \geq 1 \text{ and } f \in \text{Cl}_{\mathbb{Q}}(G) \text{ and } g \in G. \quad (10)$$

Proof. The function $\psi^r f$ defined in (10) is a class function, because if g and h are two conjugate elements of G , then their r -th powers g^r and h^r are also conjugate. The map ψ^r preserves sums, products, and the constant function 1, because all operations on class functions are pointwise. Moreover, for all $r, s \geq 1$ and all $f \in \text{Cl}_{\mathbb{Q}}(G)$ and all $g \in G$, we have

$$\psi^r(\psi^s f)(g) = (\psi^s f)(g^r) = f((g^r)^s) = f(g^{rs}) = (\psi^{rs} f)(g).$$

Thus, $\psi^{rs} = \psi^r \circ \psi^s$. □

2.4 From ψ to λ over $\mathbb{Q}$

Here is the only general construction we need from the theory of λ -rings. The qualification “over $\mathbb{Q}$ ” matters: over an arbitrary ring the formula below can introduce denominators, and their cancellation is a genuine integrality question.

Proposition 2.6. *Let A be a ψ -ring over $\mathbb{Q}$ (that is, a $\mathbb{Q}$ -algebra equipped with a ψ -ring structure). For any $x \in A$, define $\lambda_t(x) \in A[[t]]$ by*

$$\lambda_t(x) = \exp \left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(x) \frac{t^k}{k} \right). \quad (11)$$

Then the following statements hold.

- (a) The operations λ^r defined by (7) in terms of this λ_t make A into a λ -ring.
- (b) Every morphism of ψ -rings over $\mathbb{Q}$ is a morphism of the resulting λ -rings.

The familiar symmetric-function identity

$$\sum_{r \geq 0} e_r t^r = \exp \left(\sum_{k \geq 1} (-1)^{k-1} p_k \frac{t^k}{k} \right) \quad (12)$$

is one way to remember the formula (11); comparing the two identities shows that the $\mathbb{Q}$ -algebra homomorphism $\mathbf{Sym}_{\mathbb{Q}} \rightarrow A$ that sends all power-sums p_k to $\psi^k(x)$ (for a given $x \in A$) will send all e_r to $\lambda^r(x)$.

Proof of Proposition 2.6. (a) Since $\psi^1 = \text{id}$, the coefficient of t in (11) is x , while the constant coefficient is 1. Since every ψ^k is additive,

$$\begin{aligned} \lambda_t(x + y) &= \exp \left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(x) + \psi^k(y)) \frac{t^k}{k} \right) \\ &= \exp \left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(x) \frac{t^k}{k} \right) \cdot \exp \left(\sum_{k \geq 1} (-1)^{k-1} \psi^k(y) \frac{t^k}{k} \right) \\ &= \lambda_t(x) \lambda_t(y). \end{aligned}$$

This proves the λ -ring axioms (4), (5) and (8), thus showing that A is indeed a λ -ring.

(b) If a ring homomorphism $\varphi : A \rightarrow B$ commutes with all ψ^k , then applying φ to (11) shows that it commutes with every λ^r . $\square$

Remark 2.7. *The proof of Proposition 2.6 uses much less than a ψ -ring structure. To construct the weak λ -ring structure (11), it is enough that A be a commutative $\mathbb{Q}$ -algebra and that we be given additive maps*

$$\psi^r : A \longrightarrow A \quad \text{for all } r \geq 1$$

with $\psi^1 = \text{id}$. Neither multiplicativity of the ψ^k nor the relations $\psi^{rs} = \psi^r \circ \psi^s$ enter the proof. (Additivity already implies $\mathbb{Q}$ -linearity here.) These stronger properties are part of the definition of a ψ -ring because they hold for the Adams operations that interest us and are what lead to the usual special λ -ring structure; they are not needed for the weak λ -ring axioms used in Proposition 2.6.

Differentiating the logarithm of (11) gives

$$\left(\frac{d}{dt} \lambda_t(x) \right) / \lambda_t(x) = \sum_{k \geq 1} (-1)^{k-1} \psi^k(x) t^{k-1}, \quad (13)$$

that is,

$$\frac{d}{dt} \lambda_t(x) = \lambda_t(x) \cdot \sum_{k \geq 1} (-1)^{k-1} \psi^k(x) t^{k-1}. \quad (14)$$

Comparing t^{r-1} -coefficients, we obtain the Newton recurrence

$$r \lambda^r(x) = \sum_{k=1}^r (-1)^{k-1} \lambda^{r-k}(x) \psi^k(x) \quad (15)$$

for each $r \geq 1$ and $x \in A$. This equation can be solved for $\psi^r(x)$, yielding

$$\psi^r(x) - \lambda^1(x) \psi^{r-1}(x) + \lambda^2(x) \psi^{r-2}(x) - \cdots + (-1)^r r \lambda^r(x) = 0. \quad (16)$$

Thus, $\psi^r(x)$ can be expressed as a polynomial in the inputs $\psi^1(x), \psi^2(x), \dots, \psi^{r-1}(x)$ and $\lambda^1(x), \lambda^2(x), \dots, \lambda^r(x)$. This shows (by induction) that each $\psi^r(x)$ can be expressed as an *integral* polynomial in the elements

$$\lambda^1(x), \lambda^2(x), \dots, \lambda^r(x)$$

(a version of the usual Newton identities for power-sum symmetric functions). For example,

$$\psi^1(x) = \lambda^1(x), \quad \psi^2(x) = \lambda^1(x)^2 - 2\lambda^2(x).$$

The absence of denominators in this direction will be useful below, as it shows that the λ -ring structure on A in Proposition 2.6 uniquely determines the ψ -ring structure it originates from.

2.5 Characters respect exterior powers

Consider again a finite group G . The class function algebra $\text{Cl}_{\mathbb{Q}}(G)$ of G is a ψ -ring over $\mathbb{Q}$, and thus (by Proposition 2.6) becomes a λ -ring. We shall now show that $R(G)$ is a λ -subring of this λ -ring $\text{Cl}_{\mathbb{Q}}(G)$; that is, the λ -ring structure on $R(G)$ is a restriction of that induced by the ψ -ring $\text{Cl}_{\mathbb{Q}}(G)$.

Proposition 2.8. (a) *The character embedding*

$$R(G) \hookrightarrow \text{Cl}_{\mathbb{Q}}(G)$$

is a morphism of λ -rings.

(b) *The operations ψ^r of (10) preserve $R(G)$.*

(c) *For every virtual representation $x \in R(G)$ and every $g \in G$, we have*

$$\chi_{\psi^r(x)}(g) = \chi_x(g^r), \quad (17)$$

where χ_y denotes the image of any $y \in R(G)$ under the character embedding.

Proof. (a) Let V be an actual representation of G , and let A be the matrix by which a given element $g \in G$ acts on V . On one hand, by the classical formula for the characteristic polynomial of a matrix in terms of its principal minors⁴, we have

$$\det(I + tA) = \sum_{j \geq 0} \text{tr}(\wedge^j A) t^j. \quad (18)$$

On the other hand, if the eigenvalues of A in an algebraic closure are $\alpha_1, \dots, \alpha_d$, then

$$\begin{aligned} \det(I + tA) &= \prod_{i=1}^d (1 + \alpha_i t) = \sum_{r \geq 0} e_r(\alpha_1, \alpha_2, \dots, \alpha_d) t^r \\ &= \exp \left(\sum_{k \geq 1} (-1)^{k-1} \left(\sum_i \alpha_i^k \right) \frac{t^k}{k} \right) \quad (\text{by (12)}) \\ &= \exp \left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right), \end{aligned} \quad (19)$$

since each $k \geq 1$ satisfies $\sum_i \alpha_i^k = \text{tr}(A^k) = \chi_V(g^k) = (\psi^k(\chi_V))(g)$. Comparing this with (18), we find

$$\sum_{j \geq 0} \text{tr}(\wedge^j A) t^j = \exp \left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right).$$

On the other hand, applying (11) to $x = \chi_V$, and evaluating at g , we obtain

$$\sum_{j \geq 0} (\lambda^j(\chi_V))(g) t^j = \exp \left(\sum_{k \geq 1} (-1)^{k-1} (\psi^k(\chi_V))(g) \frac{t^k}{k} \right)$$

⁴To be fully precise, the characteristic polynomial of A is $\det(tI - A)$ rather than $\det(I + tA)$. But these two polynomials have the same coefficients up to sign and order. Alternatively, Proposition 11 in [6, §III.8.5], applied with $u = A$, $\xi = 1$, and $\eta = t$, gives (18) directly.

(since the algebra structure on $\text{Cl}_{\mathbb{Q}}(G)$ is pointwise). Comparing these two equalities, we find

$$\sum_{j \geq 0} (\lambda^j(\chi_V))(g) t^j = \sum_{j \geq 0} \text{tr}(\Lambda^j A) t^j.$$

Thus, for each $j \geq 0$, we obtain $\lambda^j(\chi_V)(g) = \text{tr}(\Lambda^j A) = \chi_{\Lambda^j V}(g) = \chi_{\lambda^j([V])}(g)$. Since g was arbitrary, this proves that the class-function $\lambda^j(\chi_V)$ equals $\chi_{\lambda^j([V])}$.

This shows that the character embedding $R(G) \rightarrow \text{Cl}_{\mathbb{Q}}(G)$ commutes with λ^j (and thus with λ_t) at least on actual representations. The fact that λ_t is a group homomorphism extends this to all virtual representations (since $R(G)$ is generated as an abelian group by the actual representations). Thus the character embedding is a λ -ring morphism.

(b) Let $x \in R(G)$. Then, by (16), we can write $\psi^r(x)$ as an integral polynomial in the exterior-power operations on x . Hence $\psi^r(x) \in R(G)$.

(c) The character embedding is a λ -ring morphism by part (a), and thus commutes with the ψ^r operations. Thus, for any $x \in R(G)$ and $r \geq 1$, we have $\chi_{\psi^r(x)} = \psi^r(\chi_x)$. Evaluating this at a $g \in G$ yields (17). $\square$

Thus we may regard $R(G)$ simultaneously as a λ -subring and a ψ -subring of $\text{Cl}_{\mathbb{Q}}(G)$.

2.6 A word about special λ -rings

In the usual terminology, one often reserves “ λ -ring” for a structure satisfying additional universal identities governing $\lambda^r(xy)$ and $\lambda^r(\lambda^s(x))$; such rings are often called *special λ -rings*. Representation rings are special, and a ψ -ring over $\mathbb{Q}$ as above yields a special λ -ring because its Adams operations are commuting ring endomorphisms. These stronger axioms play no role in our proof, so we do not state them. See Knutson [25, Chapter I], Yau [45, Chapters 1–3], and Hazewinkel [24, Section 16] for the full theory.

3 The representation ring of the symmetric group

3.1 Frobenius characteristic

For any partition $\lambda = (1^{m_1} 2^{m_2} \dots) \vdash n$, set

$$z_{\lambda} = \prod_{i \geq 1} i^{m_i} m_i!.$$

For a class function $f \in \text{Cl}_{\mathbb{Q}}(S_n)$, write $f(\lambda)$ for its value on the conjugacy class of cycle type λ . The *Frobenius characteristic* (also known as the *characteristic map* in [36, §4.7], or as the *characteristic isomorphism* in [43, (5.5)]) is the $\mathbb{Q}$ -linear map

$$\text{ch}_n : \text{Cl}_{\mathbb{Q}}(S_n) \longrightarrow \mathbf{Symm}_{\mathbb{Q},n}, \quad \text{ch}_n(f) = \sum_{\lambda \vdash n} f(\lambda) \frac{p_{\lambda}}{z_{\lambda}}. \quad (20)$$

It is an isomorphism of $\mathbb{Q}$ -vector spaces, since the p_{λ} form a basis of $\mathbf{Symm}_{\mathbb{Q},n}$. The basic theorem of Frobenius (see, e.g., [36, just before Proposition 4.7.2]) says that

$$\text{ch}_n(\chi^{\lambda}) = s_{\lambda}, \quad (21)$$

where χ^λ is the irreducible character of the Specht module S^λ . Since the Specht modules are defined over $\mathbb{Q}$ and form a complete set of absolutely irreducible $\mathbb{Q}S_n$ -modules, restriction of (20) gives an isomorphism of abelian groups

$$R(S_n) \xrightarrow{\sim} \mathbf{Symm}_{\mathbb{Z},n}. \quad (22)$$

Thus we have a commutative square

$$\begin{array}{ccc} R(S_n) & \hookrightarrow & \mathrm{Cl}_{\mathbb{Q}}(S_n) \\ \downarrow \mathrm{ch}_n & & \downarrow \mathrm{ch}_n \\ \mathbf{Symm}_{\mathbb{Z},n} & \hookrightarrow & \mathbf{Symm}_{\mathbb{Q},n}. \end{array}$$

This is the bridge between the integrality problem for symmetric functions in $\mathbf{Symm}_{\mathbb{Q},n}$ and the integral lattice of virtual characters in $\mathrm{Cl}_{\mathbb{Q}}(S_n)$. Note that the product on $R(S_n)$ and $\mathrm{Cl}_{\mathbb{Q}}(S_n)$ corresponds to the so-called *Kronecker product* (also known as the *internal product*) on the symmetric functions; but we will not gain anything from this fact.

3.2 The natural representation generates as a λ -ring

Let

$$M_n = \mathbb{Q}^n$$

be the natural permutation representation of S_n , with basis $e_1, \dots, e_n$. Its action is given by $\sigma(e_i) = e_{\sigma(i)}$ for any $\sigma \in S_n$ and $i \in [n]$. Let V_n be the reflection representation of S_n , that is, the subrepresentation of M_n consisting of vectors whose coordinates sum to zero. For $n \geq 2$, this is the Specht module $S^{(n-1,1)}$; for $n = 1$, it is the zero representation. Then

$$M_n \cong \mathbf{1} \oplus V_n. \quad (23)$$

We will need a 1975 result of Evelyn Boorman [3]: the representation M_n generates $R(S_n)$ as a λ -ring. Marin later proved the equivalent statement that the exterior powers of V_n generate $R(S_n)$ as a ring [30]; his proof uses a formula of Dvir and points out earlier equivalent symmetric-function results of Butler [7] and Boorman. We record the theorem in the form needed later. Three self-contained proofs are given in the appendices.

Theorem 3.1 (Boorman; Marin). *For every $n \geq 1$, the λ -subring of $R(S_n)$ generated by $[M_n]$ is all of $R(S_n)$.*

Boorman's proof is a more general categorical argument, based on a backward induction on stabilizers of tuples and applicable also to Burnside rings. Marin's proof filters by the number $n - \lambda_1$ of boxes below the first row and uses Dvir's formula to identify the top-depth multiplication. We will give three further proofs in the appendices below. The first proof (Appendix A) uses triangularity with respect to the same natural depth filtration as Marin, but replaces Dvir's formula by an elementary Schur-functor filtration and Kostka triangularity. For generalizations of hook-type generating sets to wreath products, see Harman [23]. Appendix B gives a second and a third proof, obtained from stronger integral generation theorems for Solomon's descent algebra (or, equivalently, for noncommutative symmetric functions). For a readable introduction to Solomon's descent algebra, including its realization through the face semigroup algebra of the braid arrangement, see Saliola [37, §2, especially §2.1].

3.3 The scalar product and integral self-duality

For any finite group G , we define the bilinear scalar product $\langle \cdot, \cdot \rangle_G$ on $\text{Cl}_{\mathbb{Q}}(G)$ by setting

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x)g(x^{-1}) \quad \text{for all } f, g \in \text{Cl}_{\mathbb{Q}}(G). \quad (24)$$

For symmetric groups, every conjugacy class is invariant under inversion, so this is simply the usual character scalar product without any need for complex conjugation. For $G = S_n$, the irreducible characters χ^λ , with $\lambda \vdash n$, are an orthonormal basis of $\text{Cl}_{\mathbb{Q}}(S_n)$. Indeed, for any finite group G , it is well-known that any two finite-dimensional G -representations U and V satisfy

$$\langle \chi_U, \chi_V \rangle_G = \dim \text{Hom}_G(U, V), \quad (25)$$

which (by Schur's lemma) is 0 when U and V are non-isomorphic irreducibles and is positive when U and V are isomorphic irreducibles. For $G = S_n$, all Specht modules S^λ satisfy $\text{Hom}_{S_n}(S^\lambda, S^\lambda) \cong \mathbb{Q}$ (see, e.g., [9, last paragraph of §5.13]) and therefore $\langle \chi^\lambda, \chi^\lambda \rangle_{S_n} = 1$.

The Hall scalar product on $\mathbf{Symm}_{\mathbb{Q},n}$ is characterized by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda. \quad (26)$$

From (20) one checks immediately that Frobenius characteristic is an isometry:

$$\langle \text{ch}_n(f), \text{ch}_n(g) \rangle = \langle f, g \rangle_{S_n} \quad \text{for all } f, g \in \text{Cl}_{\mathbb{Q}}(S_n). \quad (27)$$

Equivalently, the Schur functions form an orthonormal basis:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

In particular, the lattice $\mathbf{Symm}_{\mathbb{Z},n}$ is self-dual:

Lemma 3.2. *If $F \in \mathbf{Symm}_{\mathbb{Q},n}$ satisfies*

$$\langle F, H \rangle \in \mathbb{Z} \quad \text{for every } H \in \mathbf{Symm}_{\mathbb{Z},n},$$

then $F \in \mathbf{Symm}_{\mathbb{Z},n}$.

Proof. Write $F = \sum_{\lambda \vdash n} c_\lambda s_\lambda$. Then, if $\langle F, H \rangle \in \mathbb{Z}$ for every $H \in \mathbf{Symm}_{\mathbb{Z},n}$, then in particular $\langle F, s_\lambda \rangle \in \mathbb{Z}$ for every $\lambda \vdash n$; but the orthonormality of the Schur functions yields $\langle F, s_\lambda \rangle = c_\lambda$, so that we obtain $c_\lambda \in \mathbb{Z}$ for every λ , and therefore $F \in \mathbf{Symm}_{\mathbb{Z},n}$. $\square$

We note one simple fact connecting the scalar product with the Frobenius characteristic ch_n : If $h \in \text{Cl}_{\mathbb{Q}}(S_n)$ is a class function, then

$$\langle p_\nu, \text{ch}_n(h) \rangle = h(\nu) \quad (28)$$

for every partition ν of n . This follows from (20) and (26).

3.4 Products of symmetric groups

For any finite groups G and H , the map

$$\mathrm{Cl}_{\mathbb{Q}}(G) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(H) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}(G \times H), \quad f \otimes h \longmapsto ((g, k) \mapsto f(g)h(k)) \quad (29)$$

is an isomorphism of $\mathbb{Q}$ -algebras, since conjugacy classes in $G \times H$ are pairs of conjugacy classes (and since $\mathbb{Q}^X \otimes_{\mathbb{Q}} \mathbb{Q}^Y \cong \mathbb{Q}^{X \times Y}$ for any two finite sets X and Y).

Thus, in particular, there is a ring isomorphism

$$\mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n) \xrightarrow{\cong} \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n) \quad (30)$$

which sends every $f \otimes g$ to the class function on $S_m \times S_n$ given by $(\sigma, \tau) \mapsto f(\sigma)g(\tau)$; we shall denote the latter class function by $f \boxtimes g$. It is easy to see that this isomorphism preserves scalar products, in the sense that all $f_1, f_2 \in \mathrm{Cl}_{\mathbb{Q}}(S_m)$ and $g_1, g_2 \in \mathrm{Cl}_{\mathbb{Q}}(S_n)$ satisfy

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{S_m \times S_n} = \langle f_1, f_2 \rangle_{S_m} \langle g_1, g_2 \rangle_{S_n}. \quad (31)$$

This follows from the definition of the scalar product using a simple double-sum computation.

On the other hand, if U is a $\mathbb{Q}[S_m]$ -module and V is a $\mathbb{Q}[S_n]$ -module, then there is a $\mathbb{Q}[S_m \times S_n]$ -module $U \boxtimes V$ called the *external tensor product* of U and V ; it is simply the tensor product $U \otimes V$ on which S_m acts on the first factor while S_n acts on the second. Thus we easily obtain a ring homomorphism

$$R(S_m) \otimes_{\mathbb{Z}} R(S_n) \rightarrow R(S_m \times S_n) \quad (32)$$

that sends each $[U] \otimes [V]$ to $[U \boxtimes V]$.

There is furthermore a canonical ring homomorphism

$$R(S_m) \otimes_{\mathbb{Z}} R(S_n) \rightarrow \mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Z}} \mathrm{Cl}_{\mathbb{Q}}(S_n) = \mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n) \quad (33)$$

obtained by tensoring the character embeddings of S_m and S_n (note that tensoring two $\mathbb{Q}$ -vector spaces over $\mathbb{Z}$ is the same as tensoring them over $\mathbb{Q}$). We claim the following:

Proposition 3.3. *The ring homomorphism (32) is an isomorphism, and the ring homomorphism (33) is injective. These two homomorphisms as well as the isomorphism (30) and the character embedding $R(S_m \times S_n) \rightarrow \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n)$ fit together into a commutative diagram*

$$\begin{array}{ccc} R(S_m) \otimes_{\mathbb{Z}} R(S_n) & \xrightarrow{\sim} & R(S_m \times S_n) \\ \downarrow & & \downarrow \\ \mathrm{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \mathrm{Cl}_{\mathbb{Q}}(S_n) & \xrightarrow{\sim} & \mathrm{Cl}_{\mathbb{Q}}(S_m \times S_n). \end{array} \quad (34)$$

Proof. Any $\mathbb{Q}[S_m]$ -module U and any $\mathbb{Q}[S_n]$ -module V satisfy

$$\chi_{U \boxtimes V} = \chi_U \boxtimes \chi_V.$$

Thus, the diagram (34) is commutative.

Moreover, this shows that any partitions λ, ν of m and μ, ρ of n satisfy

$$\begin{aligned} \langle \chi_{S^\lambda \boxtimes S^\mu}, \chi_{S^\nu \boxtimes S^\rho} \rangle_{S_m \times S_n} &= \langle \chi^\lambda \boxtimes \chi^\mu, \chi^\nu \boxtimes \chi^\rho \rangle_{S_m \times S_n} = \langle \chi^\lambda, \chi^\nu \rangle_{S_m} \langle \chi^\mu, \chi^\rho \rangle_{S_n} \quad (\text{by (31)}) \\ &= \delta_{\lambda\nu} \delta_{\mu\rho} \quad (\text{since the } \chi^\lambda \text{ are orthonormal}). \end{aligned}$$

In particular, each $S^\lambda \boxtimes S^\mu$ is irreducible (since a group representation whose character has norm⁵ 1 is always irreducible⁶), and these irreducible $\mathbb{Q}[S_m \times S_n]$ -modules $S^\lambda \boxtimes S^\mu$ are non-isomorphic for distinct pairs (λ, μ) (since their scalar products with each other are 0). Thus, we have found $p(m)p(n)$ mutually orthonormal vectors $\chi_{S^\lambda \boxtimes S^\mu}$ in the $\mathbb{Q}$ -vector space $\text{Cl}_{\mathbb{Q}}(S_m \times S_n)$ (where $p(k)$ denotes the number of all partitions of k). Since $p(m)p(n)$ is the dimension of this $\mathbb{Q}$ -vector space (because $S_m \times S_n$ has $p(m)p(n)$ conjugacy classes), these $p(m)p(n)$ orthonormal vectors must form an orthonormal basis of $\text{Cl}_{\mathbb{Q}}(S_m \times S_n)$; hence, there cannot be any further irreducible $\mathbb{Q}[S_m \times S_n]$ -modules (because (25) shows that any such module would have a character orthogonal to all the $\chi_{S^\lambda \boxtimes S^\mu}$, thus causing $\text{Cl}_{\mathbb{Q}}(S_m \times S_n)$ to have dimension larger than $p(m)p(n)$). In other words, our external tensor products $S^\lambda \boxtimes S^\mu$ are a complete enumeration of all the irreducible representations of $S_m \times S_n$ (with no duplicates, since they are pairwise non-isomorphic). This shows that (32) is an isomorphism.

So the top row of the diagram (34) is an isomorphism, while the right column is injective (being the character embedding). Hence, the left column must be injective as well. In other words, (33) is injective. This completes the proof. $\square$

Note that we could have saved ourselves some trouble in the above proof if we recalled the general result that if G and H are two finite groups and K is a field of characteristic 0 over which both groups are split (in particular, if $G = S_m$ and $H = S_n$, then K can be any field of characteristic 0), then the irreducible representations of $G \times H$ over K are precisely the external tensor products $U \boxtimes V$ where U is an irreducible representation of G and V is an irreducible representation of H . (See, e.g., [9, Theorem 3.10.2].) But we chose the above proof for its self-containedness.

Proposition 3.3 allows us to move freely between representations/class functions of $S_m \times S_n$ and tensor products of representations/class functions on the two factors.

4 Arithmetic and anti-arithmetic products

We now prove Theorem 1.2. The argument is most transparent after passing to the adjoints of the two products.

4.1 Two pullback maps on conjugacy classes

First, we define two maps from $S_m \times S_n$ to S_{mn} : the *arithmetic rule* $\mathcal{B}$ and the *anti-arithmetic rule* $\mathcal{A}$.

To define them, we let $\sigma \in S_m$ and $\tau \in S_n$.

⁵By “norm” we mean its scalar product with itself.

⁶In fact, if it were not, then it would break into a direct sum of two nontrivial subrepresentations (by Maschke), and thus its character would have norm ≥ 2 by (25).

For the arithmetic rule, let $\mathcal{B}(\sigma, \tau)$ be the permutation of $[m] \times [n]$ defined by

$$\mathcal{B}(\sigma, \tau)(i, j) = (\sigma(i), \tau(j)). \quad (35)$$

This permutation $\mathcal{B}(\sigma, \tau)$ is also known as $\sigma \times \tau$, and its cycle type can be easily described: Each pair consisting of an a -cycle $(x_1, x_2, \dots, x_a)$ of σ and a b -cycle $(y_1, y_2, \dots, y_b)$ of τ induces

$$\gcd(a, b) \text{ cycles of length } \text{lcm}(a, b) \quad (36)$$

in the cycle decomposition of $\mathcal{B}(\sigma, \tau)$ (their union is the whole Cartesian product of the two chosen cycles)⁷. Thus the map

$$\mathcal{B} : S_m \times S_n \longrightarrow S_{mn} \quad (37)$$

is a genuine group homomorphism (after choosing an identification $[m] \times [n] \cong [mn]$), and its cycle rule is exactly (1).

For the anti-arithmetic rule, we define $\mathcal{A}(\sigma, \tau) \in S_{mn}$ only up to conjugacy, by specifying the cycle type of $\mathcal{A}(\sigma, \tau)$ rather than the permutation itself: Namely, for every pair consisting of an a -cycle of σ and a b -cycle of τ , the permutation $\mathcal{A}(\sigma, \tau)$ shall get

$$\text{lcm}(a, b) \text{ cycles of length } \gcd(a, b). \quad (38)$$

The total number of letters contributed by this pair is again ab , so these data define a partition of mn , hence a conjugacy class of S_{mn} . The specific value of $\mathcal{A}(\sigma, \tau)$ in this conjugacy class can be chosen arbitrarily; thus, $\mathcal{A}$ will not (usually) be a group homomorphism.

The two class maps define pullbacks

$$\Delta_{m,n}^{\square} : \text{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \text{Cl}_{\mathbb{Q}}(S_m \times S_n), \quad (39)$$

$$\Delta_{m,n}^{\diamond} : \text{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \text{Cl}_{\mathbb{Q}}(S_m \times S_n) \quad (40)$$

by

$$(\Delta_{m,n}^{\square} f)(\sigma, \tau) = f(\mathcal{B}(\sigma, \tau)),$$

$$(\Delta_{m,n}^{\diamond} f)(\sigma, \tau) = f(\mathcal{A}(\sigma, \tau)).$$

Both $\Delta_{m,n}^{\square}$ and $\Delta_{m,n}^{\diamond}$ are unital algebra homomorphisms because multiplication of class functions is pointwise.

⁷The reason for this is pretty simple: The orbit of any pair $(x_i, y_j) \in \{x_1, x_2, \dots, x_a\} \times \{y_1, y_2, \dots, y_b\}$ under the permutation $\mathcal{B}(\sigma, \tau) = \sigma \times \tau$ has size $\text{lcm}(a, b)$ (because $\sigma^k(x_i) = x_i$ holds only when k is a multiple of a , whereas $\tau^k(y_j) = y_j$ holds only when k is a multiple of b). Thus, the ab -element set $\{x_1, x_2, \dots, x_a\} \times \{y_1, y_2, \dots, y_b\}$ is partitioned into orbits of size $\text{lcm}(a, b)$ each. Of course, the number of these orbits must thus be $\frac{ab}{\text{lcm}(a, b)} = \gcd(a, b)$.

4.2 These pullbacks are the adjoints of the two products

Recall that the Frobenius characteristic is an isomorphism $\text{ch}_k : \text{Cl}_{\mathbb{Q}}(S_k) \longrightarrow \mathbf{Symm}_{\mathbb{Q},k}$ for each $k \geq 0$. Thus, we can transport the maps (39) and (40) through the Frobenius characteristic. That is, we define two $\mathbb{Q}$ -linear maps

$$\Delta_{m,n}^{\square} : \mathbf{Symm}_{\mathbb{Q},mn} \longrightarrow \mathbf{Symm}_{\mathbb{Q},m} \otimes_{\mathbb{Q}} \mathbf{Symm}_{\mathbb{Q},n}, \quad (41)$$

$$\Delta_{m,n}^{\diamond} : \mathbf{Symm}_{\mathbb{Q},mn} \longrightarrow \mathbf{Symm}_{\mathbb{Q},m} \otimes_{\mathbb{Q}} \mathbf{Symm}_{\mathbb{Q},n} \quad (42)$$

(we use the same symbols as for the original two maps (39) and (40)) so that the following diagram commutes for each $\circ \in \{\square, \diamond\}$:

$$\begin{array}{ccc} \text{Cl}_{\mathbb{Q}}(S_{mn}) & \xrightarrow{\Delta_{m,n}^{\circ}} & \text{Cl}_{\mathbb{Q}}(S_m \times S_n) \\ \text{ch}_{mn} \downarrow \cong & & \downarrow \cong \\ \mathbf{Symm}_{\mathbb{Q},mn} & \xrightarrow{\Delta_{m,n}^{\circ}} & \mathbf{Symm}_{\mathbb{Q},m} \otimes_{\mathbb{Q}} \mathbf{Symm}_{\mathbb{Q},n}, \end{array} \quad (43)$$

where the right vertical arrow is the composition of the inverse of (30) with $\text{ch}_m \otimes \text{ch}_n : \text{Cl}_{\mathbb{Q}}(S_m) \otimes_{\mathbb{Q}} \text{Cl}_{\mathbb{Q}}(S_n) \rightarrow \mathbf{Symm}_{\mathbb{Q},m} \otimes_{\mathbb{Q}} \mathbf{Symm}_{\mathbb{Q},n}$. All tensor products of rational symmetric-function spaces below are over $\mathbb{Q}$. Their scalar product is specified by

$$\langle F_1 \otimes G_1, F_2 \otimes G_2 \rangle = \langle F_1, F_2 \rangle \langle G_1, G_2 \rangle. \quad (44)$$

Proposition 4.1. *For $F \in \mathbf{Symm}_{\mathbb{Q},m}$, $G \in \mathbf{Symm}_{\mathbb{Q},n}$, and $H \in \mathbf{Symm}_{\mathbb{Q},mn}$, the following statements hold.*

(a) We have

$$\langle F \square G, H \rangle = \langle F \otimes G, \Delta_{m,n}^{\square} H \rangle. \quad (45)$$

(b) We have

$$\langle F \diamond G, H \rangle = \langle F \otimes G, \Delta_{m,n}^{\diamond} H \rangle. \quad (46)$$

Proof. (a) By bilinearity, it suffices to take $F = p_{\lambda}$ and $G = p_{\mu}$ for two partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_s)$. Thus,

$$F \square G = p_{\lambda} \square p_{\mu} = \prod_{i=1}^r \prod_{j=1}^s p_{\text{lcm}(\lambda_i, \mu_j)}^{\text{gcd}(\lambda_i, \mu_j)} = p_{\omega},$$

where ω is the partition obtained by writing down $\text{gcd}(\lambda_i, \mu_j)$ copies of $\text{lcm}(\lambda_i, \mu_j)$ for each pair (i, j) . By (36), this partition ω is precisely the cycle type of $\mathcal{B}(\sigma, \tau)$ when σ is a permutation with cycle type λ and τ is a permutation with cycle type μ . Consequently,

$$(\Delta_{m,n}^{\square} f)(\lambda, \mu) = f(\omega) \quad \text{for any } f \in \text{Cl}_{\mathbb{Q}}(S_{mn}) \quad (47)$$

(by the definition of $\Delta_{m,n}^{\square}$).

Now, let $h \in \text{Cl}_{\mathbb{Q}}(S_{mn})$ be the class function with $\text{ch}_{mn}(h) = H$. Then, from (28) we have

$$\langle p_{\nu}, H \rangle = h(\nu) \quad (48)$$

for every partition ν of mn . Applying this to $\nu = \omega$, we obtain

$$\langle F \boxdot G, H \rangle = h(\omega) \quad (49)$$

(since $F \boxdot G = p_\omega$). Thus we have expressed the left-hand side of (45) as $h(\omega)$.

To compute the right-hand side, write $k = \Delta_{m,n}^\square h \in \text{Cl}_\mathbb{Q}(S_m \times S_n) = \text{Cl}_\mathbb{Q}(S_m) \otimes_\mathbb{Q} \text{Cl}_\mathbb{Q}(S_n)$, where the last equality sign is really the isomorphism (30) used as an identification. Thus, its Frobenius characteristic (i.e., the image of k under $\text{ch}_m \otimes \text{ch}_n$) is

$$\Delta_{m,n}^\square H = \sum_{\alpha \vdash m} \sum_{\beta \vdash n} k(\alpha, \beta) \frac{p_\alpha \otimes p_\beta}{z_\alpha z_\beta}$$

(by the definitions of ch_m and ch_n). Taking the scalar product of this equality with $F \otimes G = p_\lambda \otimes p_\mu$, we thus obtain

$$\begin{aligned} \langle F \otimes G, \Delta_{m,n}^\square H \rangle &= \left\langle p_\lambda \otimes p_\mu, \sum_{\alpha \vdash m} \sum_{\beta \vdash n} k(\alpha, \beta) \frac{p_\alpha \otimes p_\beta}{z_\alpha z_\beta} \right\rangle \\ &= \sum_{\alpha \vdash m} \sum_{\beta \vdash n} \frac{k(\alpha, \beta)}{z_\alpha z_\beta} \langle p_\lambda, p_\alpha \rangle \cdot \langle p_\mu, p_\beta \rangle \quad (\text{by (44)}) \\ &= \sum_{\alpha \vdash m} \sum_{\beta \vdash n} \frac{k(\alpha, \beta)}{z_\alpha z_\beta} \delta_{\lambda\alpha} z_\lambda \cdot \delta_{\mu\beta} z_\mu \quad (\text{by (26)}) \\ &= \frac{k(\lambda, \mu)}{z_\lambda z_\mu} z_\lambda z_\mu = k(\lambda, \mu) = (\Delta_{m,n}^\square h)(\lambda, \mu) = h(\omega) \quad (\text{by (47)}). \end{aligned}$$

Comparing this with (49), we obtain $\langle F \boxdot G, H \rangle = \langle F \otimes G, \Delta_{m,n}^\square H \rangle$, and part (a) is proved.

(b) The anti-arithmetic case is identical, with the anti-arithmetic cycle type in place of the arithmetic one. $\square$

Thus integrality of the products follows from integrality of their adjoints.

Proposition 4.2. *Let $\circ$ denote either $\boxdot$ or $\diamond$. If*

$$\Delta_{m,n}^\circ(\text{Symm}_{\mathbb{Z},mn}) \subseteq \text{Symm}_{\mathbb{Z},m} \otimes_\mathbb{Z} \text{Symm}_{\mathbb{Z},n}, \quad (50)$$

then

$$\text{Symm}_{\mathbb{Z},m} \circ \text{Symm}_{\mathbb{Z},n} \subseteq \text{Symm}_{\mathbb{Z},mn}.$$

Proof. Take $F \in \text{Symm}_{\mathbb{Z},m}$ and $G \in \text{Symm}_{\mathbb{Z},n}$. For every $H \in \text{Symm}_{\mathbb{Z},mn}$, Proposition 4.1 gives

$$\begin{aligned} \langle F \circ G, H \rangle &= \langle F \otimes G, \Delta_{m,n}^\circ H \rangle \\ &\in \langle \text{Symm}_{\mathbb{Z},m} \otimes_\mathbb{Z} \text{Symm}_{\mathbb{Z},n}, \text{Symm}_{\mathbb{Z},m} \otimes_\mathbb{Z} \text{Symm}_{\mathbb{Z},n} \rangle \quad (\text{by (50)}) \\ &\subseteq \mathbb{Z} \end{aligned}$$

because the Schur bases on the two tensor factors are orthonormal. Lemma 3.2 now implies $F \circ G \in \text{Symm}_{\mathbb{Z},mn}$. Thus, $\text{Symm}_{\mathbb{Z},m} \circ \text{Symm}_{\mathbb{Z},n} \subseteq \text{Symm}_{\mathbb{Z},mn}$. $\square$

In view of the commutative diagram (43), condition (50) can be rewritten as

$$\Delta_{m,n}^\circ(R(S_{mn})) \subseteq R(S_m \times S_n). \quad (51)$$

Indeed, the left vertical isomorphism in (43) sends $R(S_{mn})$ precisely onto $\mathbf{Symm}_{\mathbb{Z},mn}$, by (22). Since the diagram (43) is commutative, this entails that the right vertical isomorphism in (43) sends $\Delta_{m,n}^\circ(R(S_{mn}))$ precisely onto $\Delta_{m,n}^\circ(\mathbf{Symm}_{\mathbb{Z},mn})$. On the other hand, this right vertical isomorphism also sends $R(S_m \times S_n)$ precisely onto $\mathbf{Symm}_{\mathbb{Z},m} \otimes_{\mathbb{Z}} \mathbf{Symm}_{\mathbb{Z},n}$: indeed, by Proposition 3.3, the external tensor products $S^\lambda \boxtimes S^\mu$ form a $\mathbb{Z}$ -basis of $R(S_m \times S_n)$, and their images are $s_\lambda \otimes s_\mu$, the $\mathbb{Z}$ -basis of this tensor-product lattice. Hence, we conclude that the right vertical isomorphism in (43) transforms both sides of (51) into the respective sides of (50). Thus, in order to prove (50), we need only verify (51). We thus aim to do this for both of our pullbacks.

4.3 Power compatibility

The first decisive observation is that both class maps $\mathcal{B}$ and $\mathcal{A}$ commute (up to conjugacy) with raising permutations to powers.

For the arithmetic map $\mathcal{B}$, this is immediate from the fact that it is a group homomorphism:

$$\mathcal{B}(\sigma^r, \tau^r) = \mathcal{B}((\sigma, \tau)^r) = \mathcal{B}(\sigma, \tau)^r. \quad (52)$$

For the anti-arithmetic map $\mathcal{A}$, it is not automatic, but it is still true at the level of conjugacy classes. The verification relies on the following elementary number-theoretic identity:

Lemma 4.3 (A gcd identity). *For all positive integers a, b, r , one has*

$$\gcd\left(\frac{a}{\gcd(a, r)}, \frac{b}{\gcd(b, r)}\right) = \frac{\gcd(a, b)}{\gcd(\gcd(a, b), r)}. \quad (53)$$

Proof. A straightforward proof can be done using p -valuations: Fix a prime p , and put

$$\alpha = v_p(a), \quad \beta = v_p(b), \quad \rho = v_p(r).$$

It is classical that any two positive integers s and t satisfy $v_p(\gcd(s, t)) = \min(v_p(s), v_p(t))$. Thus, every positive integer m satisfies

$$v_p(m / \gcd(m, r)) = (v_p(m) - \rho)_+,$$

where $x_+ = \max(x, 0)$. Hence, the p -adic valuation of the left-hand side of (53) is⁸

$$\min((\alpha - \rho)_+, (\beta - \rho)_+) = (\min(\alpha - \rho, \beta - \rho))_+ = (\min(\alpha, \beta) - \rho)_+.$$

The valuation of the right-hand side is

$$(v_p(\gcd(a, b)) - \rho)_+ = (\min(\alpha, \beta) - \rho)_+.$$

Thus the two sides have the same p -adic valuation for every prime p , so they are identical. $\square$

⁸We are using the fact that $\min(x_+, y_+) = (\min(x, y))_+$ for any reals x and y .

Lemma 4.4. For all $r \geq 1$, we have

$$\mathcal{A}(\sigma^r, \tau^r) \sim \mathcal{A}(\sigma, \tau)^r, \quad (54)$$

where $\sim$ denotes conjugacy in S_{mn} .

Proof. Consider one a -cycle of σ and one b -cycle of τ . We shall show that they contribute the same amount of cycles to $\mathcal{A}(\sigma^r, \tau^r)$ as they do to $\mathcal{A}(\sigma, \tau)^r$, and that these cycles all have the same length.

Set

$$g = \gcd(a, b), \quad c = \gcd(g, r).$$

The definition of $\mathcal{A}(\sigma, \tau)$ shows that our two chosen cycles of σ and τ produce $\text{lcm}(a, b)$ cycles of length g in $\mathcal{A}(\sigma, \tau)$. Taking the r -th power splits each such g -cycle into c cycles of length g/c (since the r -th power of a g -cycle is a permutation of cycle type $(g/c, g/c, \dots, g/c)$). Thus the permutation $\mathcal{A}(\sigma, \tau)^r$ gains

$$\text{lcm}(a, b)c \quad \text{cycles of length } g/c \quad (55)$$

from our two cycles.

Now take the r -th powers of σ and τ first. The a -cycle of σ splits into $\gcd(a, r)$ many cycles of σ^r , each having length $a/\gcd(a, r)$. The b -cycle of τ splits into $\gcd(b, r)$ many cycles of τ^r , each having length $b/\gcd(b, r)$. Each pair consisting of one of the $a/\gcd(a, r)$ -cycles of σ^r and one of the $b/\gcd(b, r)$ -cycles of τ^r then gives rise to a number of cycles of $\mathcal{A}(\sigma^r, \tau^r)$, each having length

$$\begin{aligned} \gcd\left(\frac{a}{\gcd(a, r)}, \frac{b}{\gcd(b, r)}\right) &= \frac{\gcd(a, b)}{\gcd(\gcd(a, b), r)} \quad (\text{by Lemma 4.3}) \\ &= \frac{g}{\gcd(g, r)} = \frac{g}{c}. \end{aligned}$$

Since the block coming from our initial two cycles of σ and τ has ab letters in total, the number of such cycles is

$$\frac{ab}{g/c} = \frac{ab}{g}c = \frac{ab}{\gcd(a, b)}c = \text{lcm}(a, b)c.$$

So $\mathcal{A}(\sigma^r, \tau^r)$ receives

$$\text{lcm}(a, b)c \quad \text{cycles of length } g/c$$

from our two cycles. This agrees with (55). So the permutations $\mathcal{A}(\sigma^r, \tau^r)$ and $\mathcal{A}(\sigma, \tau)^r$ have the same cycles of each size, and consequently are conjugate. $\square$

Corollary 4.5. On rational class functions, the following statements hold.

- (a) The map $\Delta_{m,n}^{\square} : \text{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \text{Cl}_{\mathbb{Q}}(S_m \times S_n)$ is a morphism of ψ -rings, and hence of λ -rings.
- (b) The map $\Delta_{m,n}^{\diamond} : \text{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \text{Cl}_{\mathbb{Q}}(S_m \times S_n)$ is a morphism of ψ -rings, and hence of λ -rings.

Proof. For either part, let $\mathcal{C}$ denote the relevant class map ($\mathcal{B}$ for part (a), and $\mathcal{A}$ for part (b)). Then, any $\sigma \in S_m$ and $\tau \in S_n$ satisfy

$$\mathcal{C}(\sigma, \tau)^r \sim \mathcal{C}(\sigma^r, \tau^r) \quad (56)$$

(where $\sim$ means conjugacy in S_{mn}). Indeed, for part (a), this follows from (52); for part (b), it follows from Lemma 4.4.

Let Δ denote the map $\Delta_{m,n}^\square$ in part (a), or the map $\Delta_{m,n}^\diamond$ in part (b). We must show that Δ is a ψ -ring morphism and hence a λ -ring morphism.

For any class function $f \in \text{Cl}_{\mathbb{Q}}(S_{mn})$ and any $r \geq 1$ and any $\sigma \in S_m$ and $\tau \in S_n$, we have

$$\begin{aligned} (\Delta \psi^r f)(\sigma, \tau) &= (\psi^r f)(\mathcal{C}(\sigma, \tau)) && \text{(by the definition of } \Delta) \\ &= f(\mathcal{C}(\sigma, \tau)^r) && \text{(by the definition of } \psi^r) \\ &= f(\mathcal{C}(\sigma^r, \tau^r)) && \text{(by (56))} \\ &= (\Delta f)(\sigma^r, \tau^r) && \text{(by the definition of } \Delta) \\ &= (\psi^r \Delta f)(\sigma, \tau) && \text{(by the definition of } \psi^r). \end{aligned}$$

Thus, $\Delta \psi^r f = \psi^r \Delta f$ for any f and r . Hence, Δ is a ψ -ring morphism (since Δ is a ring morphism). Consequently, Proposition 2.6(b) makes Δ a λ -ring morphism. $\square$

At this point, Theorem 3.1 reduces the entire integrality problem to one calculation: we need only check that the pullback $\Delta([M_{mn}])$ (where Δ is $\Delta_{m,n}^\square$ or $\Delta_{m,n}^\diamond$) of the natural representation M_{mn} is an integral virtual representation of $S_m \times S_n$. Once this is proved, it will follow by Corollary 4.5 that Δ (being a λ -ring morphism, with $R(S_m \times S_n)$ a λ -subring of its target) sends the entire λ -subring of $R(S_{mn})$ generated by $[M_{mn}]$ to $R(S_m \times S_n)$; but Theorem 3.1 shows that this λ -subring is the whole $R(S_{mn})$, and thus it will follow that $\Delta(R(S_{mn})) \subseteq R(S_m \times S_n)$. That is, (51) will follow. This, in turn, will yield (50) (since (51) is just a restatement of (50)), and therefore (by Proposition 4.2) we will obtain $\text{Symm}_{\mathbb{Z},m} \circ \text{Symm}_{\mathbb{Z},n} \subseteq \text{Symm}_{\mathbb{Z},mn}$, which will prove parts (a) and (c) of Theorem 1.2.

4.4 The arithmetic product: an actual pullback

For the arithmetic map there is nothing to construct: Since the map $\mathcal{B}$ is a group homomorphism, its pullback $\Delta_{m,n}^\square$ is just restriction of characters/representations along this homomorphism. Restricting the natural permutation representation M_{mn} of S_{mn} along the group homomorphism (37) (and then identifying $[mn]$ with $[m] \times [n]$) gives the permutation representation of $S_m \times S_n$ on $[m] \times [n]$ (or, rather, an isomorphic copy thereof). On the basis vector $e_i \otimes e_j$ of $M_m \boxtimes M_n$, the element $(\sigma, \tau) \in S_m \times S_n$ acts by

$$e_i \otimes e_j \longmapsto e_{\sigma(i)} \otimes e_{\tau(j)},$$

which is exactly the product action on $[m] \times [n]$. Therefore

$$\Delta_{m,n}^\square(M_{mn}) = M_m \boxtimes M_n \in R(S_m \times S_n). \quad (57)$$

Corollary 4.5 says that $\Delta_{m,n}^\square$ is a λ -ring morphism on rational class functions. Since $R(S_m \times S_n)$ is a λ -subring of $\text{Cl}_\mathbb{Q}(S_m \times S_n)$, equation (57), together with the fact that M_{mn} generates $R(S_{mn})$ as a λ -ring (by Theorem 3.1), gives

$$\Delta_{m,n}^\square(R(S_{mn})) \subseteq R(S_m \times S_n).$$

That is, (51) holds for $\circ = \square$. Hence, (50) holds for $\circ = \square$ (since (51) is just a restatement of (50)). Proposition 4.2 thus proves the arithmetic integrality assertion

$$\text{Symm}_{\mathbb{Z},m} \square \text{Symm}_{\mathbb{Z},n} \subseteq \text{Symm}_{\mathbb{Z},mn}.$$

In other words, Theorem 1.2 (a) is proved.

There is also a stronger conclusion. Since $\mathcal{B} : S_m \times S_n \hookrightarrow S_{mn}$ is a genuine group embedding, $\Delta_{m,n}^\square$ on representation rings is ordinary restriction. Its adjoint is induction. That is, the map $\square$ on the class function algebras is induction (since (45) shows that this map is the adjoint of $\Delta_{m,n}^\square$). Thus, for $\lambda \vdash m$ and $\mu \vdash n$, we have

$$s_\lambda \square s_\mu = \text{ch}_{mn} \left(\text{Ind}_{S_m \times S_n}^{S_{mn}} (S^\lambda \boxtimes S^\mu) \right), \quad (58)$$

where $S_m \times S_n$ is embedded by the product action. Hence, $s_\lambda \square s_\mu$ is Schur positive. This yields Theorem 1.2 (b).

The same product-action restriction $S_{mn} \downarrow S_m \times S_n$ has recently been studied by Ryba [35] from the viewpoint of stable symmetric-group characters; his Kronecker-comultiplication formulas and stability results give a close representation-theoretic companion to the arithmetic side of the present note.

4.5 Detecting exact cycle lengths by Adams operations

For the anti-arithmetic map, the pullback of M_{mn} is not supplied by an actual restriction functor. We now construct it as a virtual representation.

For any $1 \leq a \leq m$, define

$$C_{m,a} = \sum_{d|a} \mu(a/d) \psi^d(M_m) \in R(S_m), \quad (59)$$

where μ is the number-theoretic Möbius function. This is an integral virtual representation because the Adams operations preserve $R(S_m)$ by Proposition 2.8(b).

If $\sigma \in S_m$ is any permutation, we shall write $m_a(\sigma)$ for its number of a -cycles.

Lemma 4.6. *Let $\sigma \in S_m$ and $1 \leq a \leq m$ be arbitrary. Then,*

$$\chi_{C_{m,a}}(\sigma) = a m_a(\sigma). \quad (60)$$

Proof. The character χ_{M_m} of the natural permutation representation M_m of S_m is the class function that sends each permutation to its number of fixed points. Hence, each $d \geq 1$ satisfies $\chi_{M_m}(\sigma^d) = |\text{Fix}(\sigma^d)|$. Since (17) yields $\chi_{\psi^d(M_m)}(\sigma) = \chi_{M_m}(\sigma^d)$, we can rewrite this as

$$\chi_{\psi^d(M_m)}(\sigma) = |\text{Fix}(\sigma^d)| = \sum_{b|d} b m_b(\sigma). \quad (61)$$

Here, the last equality sign is because the fixed points of σ^d are precisely the points that lie on b -cycles of σ that satisfy $b \mid d$, and because each such b -cycle contributes exactly b fixed points for σ^d .

Taking characters in (59) gives

$$\begin{aligned}\chi_{C_{m,a}}(\sigma) &= \sum_{d \mid a} \mu(a/d) \chi_{\psi^d(M_m)}(\sigma) \\ &= \sum_{d \mid a} \mu(a/d) \sum_{b \mid d} b m_b(\sigma) \quad (\text{by (61)}) \\ &= \sum_{b \mid a} b m_b(\sigma) \sum_{d: b \mid d \mid a} \mu(a/d).\end{aligned}\tag{62}$$

The inner sum is 1 for $b = a$ and 0 otherwise, by Möbius inversion. Thus, (62) simplifies to $\chi_{C_{m,a}}(\sigma) = a m_a(\sigma)$, which is precisely (60). $\square$

Thus $C_{m,a}$ is a virtual representation whose character counts the letters lying in cycles of *exactly* length a .

Example 4.7. For the first few values of a , formula (59) gives (whenever the displayed indices are at most m)

$$C_{m,1} = M_m, \quad C_{m,2} = \psi^2(M_m) - M_m,$$

and

$$C_{m,6} = \psi^6(M_m) - \psi^3(M_m) - \psi^2(M_m) + M_m.$$

Accordingly, their characters are respectively $m_1(\sigma)$, $2m_2(\sigma)$, and $6m_6(\sigma)$.

Remark 4.8 (Prior work on $C_{m,a}$). The Frobenius characteristic ch_m sends the virtual character $C_{m,a}$ to the symmetric function $h_{m-a}p_a$. This is not hard to prove using the Murnaghan–Nakayama rule. It also follows easily from Giannelli–Law–Long–Vallejo [15, Definition 3.7 and Theorem 3.9]. For a partition λ and a positive integer e , they define a virtual character $V^\lambda[e]$ by a signed sum over all ways of adding an e -hook to λ . The sign is exactly the usual Murnaghan–Nakayama sign $(-1)^\ell$, where ℓ is the leg length of the added hook; indeed, their proof starts from the power-sum Murnaghan–Nakayama identity

$$s_\lambda p_e = \sum_{\alpha} (-1)^{\ell(\alpha/\lambda)} s_\alpha.$$

Their Theorem 3.9 says that, if a permutation σ has exactly k cycles of length e , then

$$V^\lambda[e](\sigma) = ke \chi^\lambda(\tau),$$

where τ is obtained from σ by deleting one such e -cycle (and the value is 0 when $k = 0$). Taking $\lambda = (m - a)$ and $e = a$ (with λ the empty partition when $a = m$) gives

$$V^{(m-a)}[a](\sigma) = a m_a(\sigma).$$

Hence Lemma 4.6 shows that

$$C_{m,a} = V^{(m-a)}[a] \quad \text{in } R(S_m).$$

Thus the virtual character $C_{m,a}$ has appeared before; formula (59) gives a different Adams–Möbius expression for this special case.

Remark 4.9 (Cycle-counting functions and character polynomials). *The functions*

$$X_a(\sigma) = m_a(\sigma)$$

are the classical cycle-counting functions that (taken over all $a \in \{1, 2, \dots, n\}$ together) can serve as an alternative encoding of the cycle type of σ . For each partition μ , Garsia and Goupil [12, equation I.2] express the character value $\chi^{(n-|\mu|, \mu)}(\sigma)$ (where $n \geq \mu_1 + |\mu|$ is arbitrary) as a polynomial $q_\mu(X_1(\sigma), X_2(\sigma), \dots, X_n(\sigma))$ in these functions. (They write a_i for $X_i(\sigma)$.) They give an explicit umbral formula for q_μ in [12, Proposition I.1].

A particularly close symmetric-function precedent for the calculation above is the evaluation at permutation eigenvalues used by Orellana–Zabrocki [34]. Their Section 2.1 defines Ξ_μ as the multiset of eigenvalues of a permutation matrix of cycle type μ . Their Section 8, equation (66), gives exactly

$$p_d[\Xi_\mu] = \sum_{b|d} b m_b(\mu)$$

[34, Section 8, equation (66)]. This is equation (61) in the present note. Their Section 5, equation (22), defines character polynomials in the cycle-counting variables, and their Proposition 12 identifies them with symmetric functions under the mutually inverse substitutions

$$p_k \mapsto \sum_{d|k} d X_d, \quad X_k \mapsto \frac{1}{k} \sum_{d|k} \mu(k/d) p_d$$

[34, Section 5, equation (22) and Proposition 12]. The inverse substitution is also written in Section 8, in the discussion between equations (67) and (69).⁹

Thus both the cycle-counting identity and its Möbius inversion occur explicitly in this literature. Lemma 4.6 applies that inversion to the Adams operations of M_m , while Proposition 2.8 ensures that the resulting function aX_a is an integral virtual character.

4.6 The anti-arithmetic substitute for the natural representation

Define

$$W_{m,n} = \sum_{\substack{1 \leq a \leq m, 1 \leq b \leq n; \\ \gcd(a,b)=1}} C_{m,a} \boxtimes C_{n,b} \in R(S_m \times S_n) \quad (63)$$

(this holds because $C_{m,a} \in R(S_m)$ and $C_{n,b} \in R(S_n)$).

Proposition 4.10. *Using the character embedding to identify each representation ring $R(G)$ with a subring of $\text{Cl}_{\mathbb{Q}}(G)$, the following statements hold.*

(a) *In $\text{Cl}_{\mathbb{Q}}(S_m \times S_n)$, we have*

$$\Delta_{m,n}^\diamond(M_{mn}) = W_{m,n}. \quad (64)$$

(b) *We have $\Delta_{m,n}^\diamond(M_{mn}) \in R(S_m \times S_n)$.*

⁹All section, proposition, and equation numbers cited here for Orellana–Zabrocki refer to arXiv:1605.06672v5.

Proof. For $(\sigma, \tau) \in S_m \times S_n$, Lemma 4.6 gives

$$\chi_{W_{m,n}}(\sigma, \tau) = \sum_{\substack{1 \leq a \leq m, 1 \leq b \leq n; \\ \gcd(a,b)=1}} a m_a(\sigma) b m_b(\tau). \quad (65)$$

Indeed, characters multiply under external tensor products:

$$\chi_{C_{m,a} \boxtimes C_{n,b}}(\sigma, \tau) = \chi_{C_{m,a}}(\sigma) \chi_{C_{n,b}}(\tau),$$

and Lemma 4.6 evaluates these two factors as $a m_a(\sigma)$ and $b m_b(\tau)$, respectively.

On the other hand, the character of M_{mn} at a permutation is its number of fixed points. Hence, $\chi_{M_{mn}}(\mathcal{A}(\sigma, \tau))$ is the number of fixed points of $\mathcal{A}(\sigma, \tau)$. Let us compute this number. A pair consisting of an a -cycle of σ and a b -cycle of τ contributes, under the anti-arithmetic rule, $\text{lcm}(a, b)$ cycles of length $\gcd(a, b)$ to $\mathcal{A}(\sigma, \tau)$. These are fixed points exactly when $\gcd(a, b) = 1$; in that case there are

$$\text{lcm}(a, b) = ab$$

of them. Thus, the given pair of cycles contributes exactly ab fixed points to $\mathcal{A}(\sigma, \tau)$ if $\gcd(a, b) = 1$; otherwise it contributes none. Summing over all pairs of cycles gives

$$\chi_{M_{mn}}(\mathcal{A}(\sigma, \tau)) = \sum_{\substack{1 \leq a \leq m, 1 \leq b \leq n; \\ \gcd(a,b)=1}} m_a(\sigma) m_b(\tau) \cdot ab = \sum_{\substack{1 \leq a \leq m, 1 \leq b \leq n; \\ \gcd(a,b)=1}} a m_a(\sigma) b m_b(\tau).$$

Comparing this with (65), we find

$$\chi_{W_{m,n}}(\sigma, \tau) = \chi_{M_{mn}}(\mathcal{A}(\sigma, \tau)) = \left(\Delta_{m,n}^\diamond(\chi_{M_{mn}}) \right)(\sigma, \tau).$$

Since σ and τ were arbitrary, this shows that $\chi_{W_{m,n}} = \left(\Delta_{m,n}^\diamond(\chi_{M_{mn}}) \right)$, which proves part (a). Part (b) follows from part (a) and the fact that $W_{m,n} \in R(S_m \times S_n)$ by its definition (63). $\square$

We can now finish the proof of Theorem 1.2 in a few lines of λ -ring reasoning.

Proposition 4.11. *For all $m, n \geq 1$, we have*

$$\Delta_{m,n}^\diamond(R(S_{mn})) \subseteq R(S_m \times S_n). \quad (66)$$

Proof. By Corollary 4.5 (b), the map $\Delta_{m,n}^\diamond$ is a λ -ring morphism

$$\text{Cl}_{\mathbb{Q}}(S_{mn}) \longrightarrow \text{Cl}_{\mathbb{Q}}(S_m \times S_n).$$

By Proposition 4.10 (b), it sends the natural representation M_{mn} into the λ -subring $R(S_m \times S_n) \subseteq \text{Cl}_{\mathbb{Q}}(S_m \times S_n)$. By Theorem 3.1 (applied to mn instead of n), this representation M_{mn} generates $R(S_{mn})$ as a λ -ring. Therefore the whole of $R(S_{mn})$ is sent into $R(S_m \times S_n)$. $\square$

Proof of Theorem 1.2. We have already proved parts (a) and (b) in Subsection 4.4.

(c) Proposition 4.11 shows that (51) holds for $\circ = \diamond$. Thus, (50) holds for $\circ = \diamond$ (since (51) is just a restatement of (50)). Consequently, by Proposition 4.2, we find $\mathbf{Symm}_{\mathbb{Z},m} \diamond \mathbf{Symm}_{\mathbb{Z},n} \subseteq \mathbf{Symm}_{\mathbb{Z},mn}$, and Theorem 1.2 (c) is proved. $\square$

4.7 What the proof is really using

It may be useful to isolate the short core of the above argument for the anti-arithmetic product. There are four ingredients.

- (i) The anti-arithmetic map on conjugacy classes is compatible with powers:

$$\mathcal{A}(\sigma^r, \tau^r) \sim \mathcal{A}(\sigma, \tau)^r.$$

Therefore its pullback is a ψ -ring morphism (i.e., a ring morphism respecting the Adams operations) and hence a λ -ring morphism (i.e., a ring morphism respecting the exterior-power operations) on rational class functions.

- (ii) The Adams operations of the natural S_m -representation detect fixed points of powers:

$$\chi_{\psi^d(M_m)}(\sigma) = |\text{Fix}(\sigma^d)|.$$

Möbius inversion therefore produces the exact-cycle virtual representations $C_{m,a}$.

- (iii) Coprime pairs of cycle lengths are exactly the pairs that produce fixed points under the anti-arithmetic rule. This gives the virtual representation $W_{m,n}$ in (63), which is the anti-arithmetic pullback of M_{mn} .

- (iv) The single representation M_{mn} generates $R(S_{mn})$ as a λ -ring.

Some of these ingredients have their natural habitat in the Burnside ring of G more than in the representation ring $R(G)$ (or, to stay categorical, in the category of G -sets rather than of representations). However, the Frobenius characteristic is an isomorphism from $R(S_n)$ rather than from the Burnside ring, and so we would not have had much of an advantage by working in the Burnside ring.

4.8 A general class-map criterion

The proof also isolates a general mechanism that may be useful elsewhere. Let G and H be finite groups. We shall call any map

$$a : \{\text{conjugacy classes of } G\} \longrightarrow \{\text{conjugacy classes of } H\}$$

a *class map*. Pullback gives an algebra homomorphism

$$a^* : \text{Cl}_{\mathbb{Q}}(H) \longrightarrow \text{Cl}_{\mathbb{Q}}(G).$$

For a conjugacy class C of H and $r \geq 1$, write $C^{[r]}$ for the conjugacy class containing h^r , where h is any element of C . If

$$a([g^r]) = a([g])^{[r]} \quad \text{for all } g \in G \text{ and } r \geq 1, \tag{67}$$

then a^* commutes with all Adams operations, hence is a λ -ring morphism on rational class functions. To restrict this morphism to the integral representation rings, one needs the additional arithmetic condition

$$a^*(R(H)) \subseteq R(G).$$

The anti-arithmetic proof establishes this condition by checking the image of one λ -generator.

This viewpoint explains both the similarity and the difference between the two products. The arithmetic class map comes from a group homomorphism, so integrality of pullback is automatic. The anti-arithmetic class map only has the weaker power-compatibility property (67); integrality has to be manufactured separately, and Proposition 4.10 is exactly the missing step.

Example 4.12. A basic non-homomorphic example is the power class map

$$[g] \longmapsto [g^q] \quad \text{for a given } q \geq 1$$

from the conjugacy classes of a finite group G to themselves. It satisfies (67), and its pullback on class functions is exactly the Adams operation ψ^q . Proposition 2.8(b) says that this pullback preserves $R(G)$.

Remark 4.13. Single λ -generation of the representation ring (i.e., it being generated as a λ -ring by a single element) is not entirely peculiar to symmetric groups, but (particularly because of the integral structure) is less common than it may appear. For instance, it can be shown that the representation ring $R(S_2 \times S_2)$ of the Klein four-group $S_2 \times S_2$ is not generated by a single element as a λ -ring. One may wonder what groups G have the property. To avoid field-of-definition issues in this remark, let $R_{\mathbb{C}}(G)$ denote the complex representation ring of a finite group G . If $G = C_m$ is cyclic, then a faithful one-dimensional character generates $R_{\mathbb{C}}(G)$ already as a ring.

More generally, let V be a finite-dimensional complex representation of G , and let A_V be the λ -subring of $R_{\mathbb{C}}(G)$ generated by V . Then

$$A_V \otimes_{\mathbb{Z}} \mathbb{C} = R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C}$$

if and only if the characteristic polynomials

$$\det(t - V(g)) \quad (g \in G)$$

separate the conjugacy classes of G . Indeed, their coefficients are, up to signs, the characters of the exterior powers $\wedge^j V$, while $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{Cl}_{\mathbb{C}}(G)$ is the algebra of all complex-valued functions on the finite set of conjugacy classes. Thus the algebra generated by these coefficients is the whole function algebra exactly when they separate its points. Integral λ -generation asks in addition that this full-rank subring have index 1 in $R_{\mathbb{C}}(G)$.

For the necessity in the separation assertion, suppose $V(g)$ and $V(h)$ have the same characteristic polynomial. Their eigenvalue multisets, and thus those of $V(g^r)$ and $V(h^r)$, agree for every $r \geq 1$. By (11), the class functions constant on each such spectral equivalence class form a λ -subring containing V . Thus every element of A_V takes the same value at g and h , so $A_V \otimes \mathbb{C}$ cannot be the full class-function algebra unless spectral equivalence separates conjugacy classes.

For comparison, Adams and Conway found a related phenomenon for compact, simply connected Lie groups: along each arm of the Dynkin diagram, the fundamental representations can be recovered successively from exterior powers of the representation at the end of the arm. Guillot [22] gives an elementary proof. Thus his construction replaces the usual set of fundamental representations by a smaller set indexed by the arms of the Dynkin diagram; for example, his discussion of E_6 uses three λ -generators. What is particularly convenient for S_n is that the very elementary permutation representation M_n already works integrally.

4.9 Further precedents and nearby literature

Several parts of the proof have close relatives in the literature, although we do not know a previous occurrence of the anti-arithmetic integrality argument itself.

First, the use of Adams operations on $R(S_n)$ is classical. In symmetric-function language it is precisely inner plethysm by a power sum: under the Frobenius characteristic, the r -th Adams operation corresponds to

$$f \longmapsto p_r\{f\},$$

where $\{ \}$ denotes inner plethysm (with the power sum in the outer slot). Thibon [42] and Scharf–Thibon [38] use the Hopf algebra of symmetric functions to study precisely these Adams operators and to recover Littlewood’s formulas for inner plethysm. The present proof uses only the easiest part of that theory, namely the power-trace identity $\chi_{\psi^r V}(g) = \chi_V(g^r)$ and Newton’s formulas. Meir–Szymik [31] give a useful modern account of Adams operations on finite group representation rings and characterize them as natural operations on the representation-ring functor.

Second, the cycle-length arithmetic behind the ordinary arithmetic product belongs to a broader gcd/lcm circle of ideas. The necklace ring of Metropolis–Rota [32] has multiplication whose structure constants involve gcd and lcm, and Dress–Siebeneicher [8] identify closely related necklace and Burnside-ring constructions with the big Witt vectors and λ -rings. These works are not needed for the proof above, but they provide a conceptual home for the same arithmetic on cycle lengths. On the species side, Maia–Méndez [29] and Li [27] show how the ordinary arithmetic product interacts with Dirichlet series, Cartesian products, and prime decompositions of combinatorial structures. Ryba [35] studies the corresponding restriction $S_{mn} \downarrow S_m \times S_n$ in the stable-character basis and proves stability results for its multiplicities.

Finally, the generation result for $R(S_n)$ has a substantial history. Murnaghan [33] studied generation of irreducible representations under Kronecker products already in 1955. Butler [7] and especially Boorman [3] obtained forms of the one-generator result in the language of S -operations and λ -rings; Marin [30] later gave a short proof based on Dvir’s formula. Harman [23] extends this circle of results to representation rings of certain wreath products.

A A triangular proof of Theorem 3.1

As we promised, we shall now give three proofs of Theorem 3.1, both to keep this paper self-contained and to explore the “roads less traveled” around this result.

A.1 Depth and tail

For a partition $\lambda \vdash n$, define its *depth* by

$$\text{depth}(\lambda) = n - \lambda_1. \tag{68}$$

Thus, if $d = \text{depth}(\lambda)$, we can write uniquely¹⁰

$$\lambda = (n - d, \alpha), \quad \text{where } \alpha \vdash d \text{ with } \alpha_1 \leq n - d.$$

¹⁰If a partition β is empty, then we interpret its first part β_1 as 0.

Here $(n - d, \alpha)$ means the partition consisting of $n - d$ followed by the entries of α . We call α the *tail* of λ . Let

$$\alpha' = (c_1, c_2, \dots, c_r) \quad (69)$$

be the conjugate partition of α , and define

$$T_\lambda = \bigotimes_{j=1}^r \bigwedge^{c_j} M_n. \quad (70)$$

Clearly $[T_\lambda]$ belongs to the λ -subring of $R(S_n)$ generated by M_n .

We shall prove that T_λ contains S^λ once, and that all other constituents S^μ of T_λ are triangularly smaller: either they have smaller depth, or they have the same depth and a strictly smaller tail in dominance order.

A.2 Reminder on Schur functors

We will use Schur functors to create new representations of S_n from old. We first explain the two definitions of Schur functors that we shall use. Let V be a $\mathbb{Q}$ -vector space and $\beta \vdash d$. The tensor power $V^{\otimes d}$ carries the right S_d -action given by

$$(v_1 \otimes \cdots \otimes v_d) \cdot g = v_{g(1)} \otimes \cdots \otimes v_{g(d)} \quad \text{for all } g \in S_d$$

(that is, the place-permutation action). It also carries the corresponding left action $g \cdot w = w \cdot g^{-1}$. Using the left action, we can define the Schur functor as a multiplicity space:

$$\mathbb{S}_{\text{Hom}}^\beta(V) := \text{Hom}_{S_d}(S^\beta, V^{\otimes d}). \quad (71)$$

Using the right action, we can instead define it by a balanced tensor product:

$$\mathbb{S}_\otimes^\beta(V) := V^{\otimes d} \otimes_{\mathbb{Q}[S_d]} S^\beta. \quad (72)$$

The latter is the construction used by Fulton [11, §8.3]. Both constructions are functorial in V , and thus carry any action on V that acts diagonally on the tensor power (and thus commutes with place permutations). In particular, when $V = M_n$, they are S_n -representations. The following lemma explains their equivalence, including the role of duality.

Lemma A.1 (Multiplicity spaces and balanced tensor products).

- (a) Let G be a finite group, and let W and E be two left $\mathbb{Q}[G]$ -modules, where E is finite-dimensional. Give W the right G -action $w \cdot g = g^{-1}w$. There is a canonical isomorphism

$$W \otimes_{\mathbb{Q}[G]} E^* \longrightarrow \text{Hom}_G(E, W), \quad w \otimes \varphi \longmapsto \left(e \longmapsto \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}e) gw \right). \quad (73)$$

It is natural in W and E , and respects every action on W commuting with G .

(b) Consequently, a choice of G -equivariant isomorphism $E \cong E^*$ gives an isomorphism $W \otimes_{\mathbb{Q}[G]} E \cong \text{Hom}_G(E, W)$ that is natural in W .

(c) In particular, if $G = S_d$, then, after choosing an isomorphism $S^\beta \cong (S^\beta)^*$, we have

$$\mathbb{S}_\otimes^\beta(V) \cong \mathbb{S}_{\text{Hom}}^\beta(V)$$

naturally in V .

Proof. (a) This is a combination of two standard isomorphisms in group representation theory (over fields of characteristic 0).

For any G -representation U , we define its *invariant space*

$$U^G := \{u \in U \mid gu = u \text{ for all } g \in G\}$$

and its *coinvariant space*

$$U_G := U / \text{span}_{\mathbb{Q}} \{gu - u \mid u \in U \text{ and } g \in G\}.$$

Then, the *averaging operator* $P_U : U \rightarrow U$ given by

$$P_U(u) = \frac{1}{|G|} \sum_{g \in G} gu$$

is a projection onto U^G ; furthermore it kills every difference $gu - u$ and thus factors through the coinvariant space U_G . Thus, it induces a linear map

$$\bar{P}_U : U_G \rightarrow U^G,$$

which is easily seen to be a vector space isomorphism (its inverse simply sends each $u \in U^G$ to its projection onto U_G). This isomorphism is the first ingredient we need.

The second is even more basic (and holds over any field): The ordinary tensor product $W \otimes_{\mathbb{Q}} E^*$ has the diagonal left G -action, where the G -action on E^* is given by $(g\varphi)(e) = \varphi(g^{-1}e)$. The Hom-space $\text{Hom}_{\mathbb{Q}}(E, W)$ also has a canonical left G -action, given by $(gf)(e) = gf(g^{-1}e)$ for all $f \in \text{Hom}_{\mathbb{Q}}(E, W)$ and $g \in G$ and $e \in E$. The standard vector-space isomorphism

$$Q : W \otimes_{\mathbb{Q}} E^* \longrightarrow \text{Hom}_{\mathbb{Q}}(E, W), \quad w \otimes \varphi \longmapsto (e \mapsto \varphi(e)w)$$

is G -equivariant. Thus, it restricts to a vector space isomorphism

$$Q^G : (W \otimes_{\mathbb{Q}} E^*)^G \longrightarrow (\text{Hom}_{\mathbb{Q}}(E, W))^G$$

on the invariant spaces.

Now, we combine the two ingredients. Applying the above-constructed isomorphism $\bar{P}_U : U_G \rightarrow U^G$ to $U = W \otimes_{\mathbb{Q}} E^*$, and composing it with the isomorphism Q^G , we obtain an isomorphism

$$(W \otimes_{\mathbb{Q}} E^*)_G \xrightarrow[\cong]{\bar{P}_{W \otimes_{\mathbb{Q}} E^*}} (W \otimes_{\mathbb{Q}} E^*)^G \xrightarrow[\cong]{Q^G} (\text{Hom}_{\mathbb{Q}}(E, W))^G.$$

But the coinvariant space $(W \otimes_{\mathbb{Q}} E^*)_G$ is precisely the balanced tensor product $W \otimes_{\mathbb{Q}[G]} E^*$ (since the balancing relations $(g^{-1}w) \otimes \varphi = w \otimes (g\varphi)$ in the definition of $W \otimes_{\mathbb{Q}[G]} E^*$ are precisely the coinvariant relations $gu = u$, after applying them to pure tensors $u = (g^{-1}w) \otimes \varphi$), whereas the invariant space $(\text{Hom}_{\mathbb{Q}}(E, W))^G$ is precisely the space $\text{Hom}_G(E, W)$ of G -equivariant maps (since an $f \in \text{Hom}_{\mathbb{Q}}(E, W)$ is G -invariant if and only if $gf(g^{-1}e) = f(e)$ for all $g \in G$ and $e \in E$, but this is equivalent to f being G -equivariant). Hence, the isomorphism we just obtained is an isomorphism $W \otimes_{\mathbb{Q}[G]} E^* \longrightarrow \text{Hom}_G(E, W)$. Moreover, it is given by the exact formula (73) (this follows from the definitions of P_U and Q).

These constructions are natural and commute with any additional action on W commuting with G .

(b) This is automatic.

(c) Rational Specht modules are self-dual. One can see this directly by taking a positive definite rational bilinear form on S^β and averaging it over S_d : the resulting form remains positive definite and is S_d -invariant, so it yields an S_d -equivariant isomorphism from S^β to $(S^\beta)^*$.¹¹ Thus, setting $G = S_d$, $E = S^\beta$ and $W = V^{\otimes d}$ in part (b), we obtain $V^{\otimes d} \otimes_{\mathbb{Q}[S_d]} S^\beta \cong \text{Hom}_{S_d}(S^\beta, V^{\otimes d})$. That is, $\mathbb{S}_{\otimes}^\beta(V) \cong \mathbb{S}_{\text{Hom}}^\beta(V)$. $\square$

We write $\mathbb{S}^\beta(V)$ for the multiplicity-space Schur functor $\mathbb{S}_{\text{Hom}}^\beta(V)$ defined in (71), and use Lemma A.1 (c) to pass to the tensor definition (72) when convenient. The isomorphism between them is natural in V once the self-duality of S^β has been fixed; the isomorphism with the dual in (73) requires no such choice.

A.3 A triangular decomposition of $\mathbb{S}^\beta(M_n)$

Lemma A.2. *Let $\beta \vdash d$, and assume $\beta_1 \leq n - d$. Then, in $R(S_n)$, we have*

$$[\mathbb{S}^\beta(M_n)] = [S^{(n-d, \beta)}] + \sum_{\substack{\mu \vdash n; \\ \text{depth}(\mu) < d}} a_{\beta\mu} [S^\mu] \quad (74)$$

for some nonnegative integers $a_{\beta\mu}$.

Proof. We will use the notation $\text{Inj}(X, Y)$ for the set of all injections from a set X to a set Y . This set has a left action by the symmetric group S_Y (by composition) and a right action by the symmetric group S_X (also by composition).

Step 1: the lower part of the filtration. Filter $M_n^{\otimes d}$ by the number of distinct basis vectors that occur in a pure tensor. More precisely, let F_r be the span of the basis tensors

$$e_{i_1} \otimes \cdots \otimes e_{i_d} \quad \text{for which } |\{i_1, \dots, i_d\}| \leq r.$$

This span F_r is stable under the commuting actions of S_n and S_d . Thus, we obtain a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d = M_n^{\otimes d}$$

of $M_n^{\otimes d}$ by S_n -subrepresentations.

¹¹See [18, Theorem 5.19.35] for a proof under more minimalistic assumptions.

For a tuple $\mathbf{i} = (i_1, \dots, i_d) \in [n]^d$, let $\pi(\mathbf{i})$ be the set partition of $[d]$ whose blocks are the fibers of the map $j \mapsto i_j$; equivalently, j and k lie in the same block of $\pi(\mathbf{i})$ if and only if $i_j = i_k$. We call $\pi(\mathbf{i})$ the *kernel partition* of $\mathbf{i}$. For example, the tuple $(3, 7, 3, 3, 7)$ has kernel partition $\{\{1, 3, 4\}, \{2, 5\}\}$.

Now fix a set partition π of $[d]$ with exactly r blocks. The basis tensors $e_{i_1} \otimes \dots \otimes e_{i_d}$ with $\pi(\mathbf{i}) = \pi$ are naturally in S_n -equivariant bijection with the injections from the r -element set of blocks of π into $[n]$: an injection records the common value i_j on each block. This S_n -set $\text{Inj}(\pi, [n])$ is transitive, and the stabilizer of one such injection is isomorphic to S_{n-r} (since a permutation in the stabilizer must fix all the r elements of the image of the injection, but can permute the remaining $n - r$ elements of $[n]$ arbitrarily). Hence its permutation representation $\mathbb{Q}[\text{Inj}(\pi, [n])]$ is

$$\text{Ind}_{S_{n-r}}^{S_n} \mathbf{1}.$$

By the branching rule¹², every Specht module S^μ occurring in this representation satisfies

$$\mu_1 \geq n - r,$$

and hence $\text{depth}(\mu) \leq r$.

Thus, as an S_n -representation,

$$F_r = \bigoplus_{\substack{\pi \text{ a set partition of } [d]; \\ |\pi| \leq r}} \mathbb{Q}[\text{Inj}(\pi, [n])],$$

where $|\pi|$ is understood in the literal sense (i.e., it is the number of blocks of π). The summand indexed by π is isomorphic, as an S_n -representation, to

$$\text{Ind}_{S_{n-|\pi|}}^{S_n} \mathbf{1}.$$

By the branching rule, every Specht module S^μ occurring in this summand satisfies

$$\text{depth}(\mu) \leq |\pi| \leq r.$$

Hence every S_n -constituent of F_r has depth at most r (where the depth of a Specht module S^ν is defined to be the depth of the partition ν). It follows that every S_n -constituent of F_{d-1} has depth at most $d - 1$. Consequently, every S_n -constituent of $\text{Hom}_{S_d}(S^\beta, F_{d-1})$ has depth at most $d - 1$ as well (since $\text{Hom}_{S_d}(S^\beta, F_{d-1})$ is an S_n -subrepresentation of $\text{Hom}_{\mathbb{Q}}(S^\beta, F_{d-1}) \cong F_{d-1}^{\oplus \dim S^\beta}$).

Step 2: the top quotient. The top quotient F_d/F_{d-1} of our filtration has a basis consisting of tensors with pairwise distinct indices. Thus, a variant of our above reasoning shows that

$$F_d/F_{d-1} \cong \mathbb{Q}[\text{Inj}([d], [n])] \tag{75}$$

equivariantly for both commuting actions of S_n and S_d . We compute its S^β -multiplicity space using the tensor definition. Define the subgroup $H = S_d \times S_{n-d} \subseteq S_n$, where S_d

¹²In terms of symmetric functions, this is just the Pieri rule. Under the Frobenius correspondence ch , the representation $\text{Ind}_{S_{n-r}}^{S_n} \mathbf{1}$ corresponds to the symmetric function $h_{n-r} h_1^r = s_{(n-r)} h_1^r$, which (by repeated application of the Pieri rule) is obtained from $s_{(n-r)}$ by adding r cells to the Young diagram. Obviously, adding cells cannot make the first row any shorter, so the resulting Schur functions s_μ all satisfy $\mu_1 \geq n - r$.

permutes $1, \dots, d$ and S_{n-d} permutes $d+1, \dots, n$. The injection $f_0 : [d] \hookrightarrow [n]$ given by $f_0(i) = i$ identifies $\text{Inj}([d], [n])$ with the left S_n -set S_n/S_{n-d} via $gS_{n-d} \mapsto gf_0$. Hence there is an isomorphism of (S_n, S_d) -bimodules

$$\mathbb{Q}[S_n] \otimes_{\mathbb{Q}[H]} \mathbb{Q}[S_d] \cong \mathbb{Q}[\text{Inj}([d], [n])], \quad g \otimes a \mapsto gf_0a, \quad (76)$$

where $(b, c) \in H$ acts on $\mathbb{Q}[S_d]$ by left multiplication by b , and S_d acts on it by right multiplication. (The map is well-defined since $(b, c)f_0 = f_0b$. It is an isomorphism because coset representatives for S_n/H , together with $a \in S_d$, parametrize the cosets of S_{n-d} in S_n .) We transform the right S_d -action on $\mathbb{Q}[\text{Inj}([d], [n])]$ into a left S_d action by the familiar rule $g \cdot f = f \cdot g^{-1}$ for all $g \in S_d$ and $f \in \mathbb{Q}[\text{Inj}([d], [n])]$.

Now, we have a chain of left S_n -module isomorphisms

$$\begin{aligned} \text{Hom}_{S_d}(S^\beta, F_d/F_{d-1}) &\cong \text{Hom}_{S_d}(S^\beta, \mathbb{Q}[\text{Inj}([d], [n])]) && \text{(by (75))} \\ &\cong \mathbb{Q}[\text{Inj}([d], [n])] \otimes_{\mathbb{Q}[S_d]} S^\beta && \text{(by Lemma A.1 (b))} \\ &\cong (\mathbb{Q}[S_n] \otimes_{\mathbb{Q}[H]} \mathbb{Q}[S_d]) \otimes_{\mathbb{Q}[S_d]} S^\beta && \text{(by (76))} \\ &\cong \mathbb{Q}[S_n] \otimes_{\mathbb{Q}[H]} (\mathbb{Q}[S_d] \otimes_{\mathbb{Q}[S_d]} S^\beta) && \text{(by associativity of } \otimes) \\ &\cong \mathbb{Q}[S_n] \otimes_{\mathbb{Q}[H]} (S^\beta \boxtimes \mathbf{1}) \\ &\quad \left(\begin{array}{l} \text{where } \mathbf{1} \text{ is the trivial representation of } S_{n-d}, \\ \text{so that } S^\beta \boxtimes \mathbf{1} \text{ is a representation of } S_d \times S_{n-d} = H \end{array} \right) \\ &= \text{Ind}_{S_d \times S_{n-d}}^{S_n} (S^\beta \boxtimes \mathbf{1}). \end{aligned} \quad (77)$$

Here, the last $\cong$ sign relied on the H -equivariant isomorphism $\mathbb{Q}[S_d] \otimes_{\mathbb{Q}[S_d]} S^\beta \cong S^\beta \boxtimes \mathbf{1}$ given by $a \otimes s \mapsto as \otimes 1$; the S_{n-d} -factor acts trivially. By Pieri's rule, the right-hand side of (77) is the multiplicity-free sum of S^ν over partitions $\nu \vdash n$ such that ν/β is a horizontal strip of size $n-d$.

The strip has $n-d$ cells in distinct columns, and ν has ν_1 columns. Thus every such ν satisfies $\nu_1 \geq n-d$, hence $\text{depth}(\nu) \leq d$. Suppose equality holds. Then $\nu_1 = n-d$. Together with $\beta \subseteq \nu$, the horizontal-strip condition is equivalent to the interlacing inequalities

$$\nu_{i+1} \leq \beta_i \quad \text{for all } i \geq 1.$$

Since

$$\sum_{i \geq 1} \nu_{i+1} = n - \nu_1 = d = |\beta| = \sum_{i \geq 1} \beta_i,$$

all these inequalities must be equalities. Hence

$$\nu = (n-d, \beta).$$

Thus, $S^{(n-d, \beta)}$ occurs with multiplicity 1 in the S_n -representation

$$\text{Hom}_{S_d}(S^\beta, F_d/F_{d-1}),$$

whereas all other constituents of this multiplicity space have $\text{depth} < d$. (Here it is important to take the S^β -multiplicity space: in the whole quotient F_d/F_{d-1} , the multiplicity of $S^{(n-d, \beta)}$ is $\dim S^\beta$, which can exceed 1.)

Step 3: take the S^β -multiplicity space. We have

$$M_n^{\otimes d} = F_d \cong F_{d-1} \oplus (F_d/F_{d-1})$$

as $S_n \times S_d$ -representations, by Maschke's theorem. Applying the functor $\text{Hom}_{S_d}(S^\beta, -)$ (which clearly respects direct sums) to this decomposition, we see that

$$\text{Hom}_{S_d}(S^\beta, M_n^{\otimes d}) \cong \text{Hom}_{S_d}(S^\beta, F_{d-1}) \oplus \text{Hom}_{S_d}(S^\beta, F_d/F_{d-1}).$$

The first addend here consists entirely of S_n -constituents of depth at most $d-1$ (by Step 1), while the second addend contains $S^{(n-d, \beta)}$ with multiplicity 1 and, apart from that, only constituents of depth $< d$ as well (by Step 2). Thus, altogether, $\text{Hom}_{S_d}(S^\beta, M_n^{\otimes d})$ is a direct sum of $S^{(n-d, \beta)}$ with a number of Specht modules of depth $< d$. Since $\mathbb{S}^\beta(M_n) = \text{Hom}_{S_d}(S^\beta, M_n^{\otimes d})$, this proves (74). $\square$

Lemma A.2 is a small triangular piece of the classical restriction problem from GL_n to S_n . Indeed, regarding S_n as the group of permutation matrices in GL_n , the S_n -representation $\mathbb{S}^\beta(M_n)$ is the restriction of the polynomial GL_n -representation $\mathbb{S}^\beta(\mathbb{Q}^n)$. Orellana–Zabrocki [34, Introduction] discuss this restriction problem and encode such restrictions by evaluating symmetric functions at the eigenvalues of permutation matrices. We only need the top-depth triangular statement above.

A.4 A triangular decomposition of T_λ

We now use the standard Kostka-triangular decomposition of a tensor product of exterior powers. Recall (69). The symmetric-function identity

$$e_{\alpha'} = \prod_{j=1}^r e_{c_j} = \sum_{\beta \vdash d} K_{\beta', \alpha'} s_\beta \quad (78)$$

(where $K_{\lambda, \mu}$ are the Kostka numbers)¹³ translates, by Schur–Weyl theory, to

$$T_\lambda \cong \bigoplus_{\beta \vdash d} K_{\beta', \alpha'} \mathbb{S}^\beta(M_n) \quad (79)$$

(for an explicit reference, apply [11, §8.3, Corollary 2(b)] to α' instead of μ and restrict to the symmetric group $S_n \subseteq GL_n$). Thus, in $R(S_n)$, we have

$$[T_\lambda] = \sum_{\beta \vdash d} K_{\beta', \alpha'} [\mathbb{S}^\beta(M_n)]. \quad (80)$$

However, it is well-known (see, e.g., [41, Proposition 7.10.5]) that the Kostka number $K_{\beta', \alpha'}$ is nonzero only if $\beta' \trianglerighteq \alpha'$ (the symbol $\trianglerighteq$ means “dominates or equals”), equivalently $\beta \trianglelefteq \alpha$. In particular, whenever a nonzero addend occurs in (80), we have $\beta \trianglelefteq \alpha$ and thus $\beta_1 \leq \alpha_1 \leq n-d$, so Lemma A.2 applies. Furthermore, the addend for $\beta = \alpha$ occurs with coefficient 1, since (again by [41, Proposition 7.10.5]) we have

$$K_{\alpha', \alpha'} = 1.$$

We obtain the promised triangularity:

¹³See, e.g., [41, Corollary 7.15.3] (setting $\nu = \emptyset$) for a proof of (78).

Proposition A.3. Let $\lambda = (n - d, \alpha) \vdash n$. Then

$$[T_\lambda] = [S^\lambda] + \sum_{\beta \triangleleft \alpha} K_{\beta', \alpha'} [S^{(n-d, \beta)}] + \sum_{\substack{\mu \vdash n; \\ \text{depth}(\mu) < d}} b_{\lambda\mu} [S^\mu], \quad (81)$$

where the $b_{\lambda\mu}$ are nonnegative integers, and $\beta \triangleleft \alpha$ means strict dominance.

Proof. As we observed, all the nonzero addends in (80) satisfy $\beta \trianglelefteq \alpha$ and $\beta_1 \leq n - d$, so that we can rewrite them using Lemma A.2. This yields

$$\begin{aligned} [T_\lambda] &= \sum_{\substack{\beta \vdash d; \\ \beta \trianglelefteq \alpha}} K_{\beta', \alpha'} \left([S^{(n-d, \beta)}] + \sum_{\substack{\mu \vdash n; \\ \text{depth}(\mu) < d}} a_{\beta\mu} [S^\mu] \right) \\ &= \sum_{\substack{\beta \vdash d; \\ \beta \trianglelefteq \alpha}} K_{\beta', \alpha'} [S^{(n-d, \beta)}] + \sum_{\substack{\mu \vdash n; \\ \text{depth}(\mu) < d}} b_{\lambda\mu} [S^\mu] \end{aligned}$$

for some nonnegative integers $b_{\lambda\mu}$. In light of the fact that $K_{\alpha', \alpha'} = 1$, we can rewrite the first sum $\sum_{\substack{\beta \vdash d; \\ \beta \trianglelefteq \alpha}} K_{\beta', \alpha'} [S^{(n-d, \beta)}]$ as $[S^{(n-d, \alpha)}] + \sum_{\substack{\beta \vdash d; \\ \beta \triangleleft \alpha}} K_{\beta', \alpha'} [S^{(n-d, \beta)}]$; and the proposition is proved (since $(n - d, \alpha) = \lambda$). $\square$

A.5 First proof of Theorem 3.1

First proof of Theorem 3.1. Let A_n be the λ -subring of $R(S_n)$ generated by M_n . We prove $[S^\lambda] \in A_n$ for every $\lambda \vdash n$, by induction first on $d = \text{depth}(\lambda)$ and then, for fixed d , upward along dominance of the tail α in $\lambda = (n - d, \alpha)$.

The element $[T_\lambda]$ belongs to A_n (by its construction). In (81), every constituent in the last sum has smaller depth, so belongs to A_n by the outer induction. Every constituent in the middle sum has the same depth but strictly smaller tail, so belongs to A_n by the inner induction. Subtracting these already-known classes from $[T_\lambda]$ gives $[S^\lambda] \in A_n$. This completes the induction.

Since the Specht classes $[S^\lambda]$ form a $\mathbb{Z}$ -basis of $R(S_n)$, this proves $A_n = R(S_n)$. $\square$

Remark A.4. The two parts of the triangular order can be seen concretely already for $n = 6$ and $d = 3$. First consider the Schur functor corresponding to $\beta = (3)$. Sorting the basis tensors of $\text{Sym}^3(M_6)$ according to whether they involve three, two, or one distinct basis vectors (as in the proof of Lemma A.2) gives

$$\mathbb{S}^{(3)}(M_6) = \text{Sym}^3(M_6) \cong S^{(3,3)} \oplus \underbrace{S^{(4,1,1)} \oplus 2S^{(4,2)} \oplus 4S^{(5,1)} \oplus 3S^{(6)}}_{\text{constituents of depth} < 3}.$$

Thus the lower-depth terms need not be lower in dominance order. For instance, $(3, 3)$ and $(4, 1, 1)$ are incomparable: the first partial sum favors $(4, 1, 1)$, whereas the first two partial sums favor $(3, 3)$. This is why dominance order alone is not sufficient.

The second part of the induction—dominance among tails of the same depth—is visible if we take $\lambda = (3, 2, 1)$, so that $d = 3$, $\alpha = (2, 1)$, and

$$T_\lambda = \bigwedge^2 M_6 \otimes M_6.$$

Since $e_2 e_1 = s_{(2,1)} + s_{(1,1,1)}$, we have

$$T_\lambda \cong \mathbb{S}^{(2,1)}(M_6) \oplus \mathbb{S}^{(1,1,1)}(M_6),$$

and the same calculation gives

$$T_{(3,2,1)} \cong S^{(3,2,1)} \oplus \underbrace{S^{(3,1,1,1)}}_{\substack{\text{same depth 3,} \\ (1,1,1) \triangleleft (2,1)}} \oplus \underbrace{3S^{(4,1,1)} \oplus 2S^{(4,2)} \oplus 3S^{(5,1)} \oplus S^{(6)}}_{\text{constituents of depth } < 3}.$$

Thus this one example displays exactly the two kinds of already-known terms that occur in (81): smaller-depth constituents, and same-depth constituents whose tails are strictly smaller in dominance order.

Remark A.5 (Relation with Marin’s formulation). *Marin states his theorem [30, Theorem 1.1] by saying that the exterior powers of the standard representation V_n generate $R(S_n)$ as a ring (not just as a λ -ring). This implies Theorem 3.1. Conversely, the triangular proof above establishes this ordinary-ring statement directly: every T_λ is a tensor product of exterior powers of M_n , and its triangular elimination uses only addition, subtraction, and tensor product. Thus the exterior powers of M_n generate $R(S_n)$ as a ring. To pass between the two families of exterior powers, use $M_n = \mathbf{1} \oplus V_n$, and therefore*

$$\bigwedge^k M_n \cong \bigwedge^k V_n \oplus \bigwedge^{k-1} V_n.$$

This decomposition expresses the exterior powers of M_n in terms of those of V_n . Solving it recursively for $\bigwedge^k V_n$ expresses those of V_n in terms of those of M_n . Hence these two families generate the same ordinary subring. Merely knowing that a λ -subring contains V_n would not, by itself, prove ordinary ring generation.

B Two descent-algebra lifts of Boorman–Marin

Theorem 3.1 says that the representation ring $R(S_n)$ is λ -generated by the natural permutation representation M_n . If we replace M_n by the reflection representation $V_n = M_n - \mathbf{1}$, then the representations obtained immediately by applying the λ -operations are

$$\bigwedge^k V_n \cong S^{(n-k, 1^k)} \quad \text{for } 0 \leq k \leq n-1$$

(see, e.g., [19, Proposition 2] for a proof). Thus the most direct ordinary-ring form of the Boorman–Marin theorem says that the hook representations $S^{(n-k, 1^k)}$ generate $R(S_n)$ as a $\mathbb{Z}$ -algebra. Schocker [39] found a remarkably literal lift of this statement to Solomon’s descent algebra: certain exact-descent-class sums attached to hook shapes already generate the whole descent algebra. In fact, his theorem uses only about half of the hooks.

We begin by recalling the noncommutative-symmetric-function language in which this lift is especially transparent. We then translate Schocker’s theorem carefully into our conventions and give an independent proof. First Solomon’s Mackey formula proves generation by all hooks; then multiplication by the longest permutation reduces this family to its upper

half. Afterwards we establish a second, different generation theorem: the complete functions $H_k H_{n-k}$ also generate the descent algebra. Their commutative images are not hook characters, but rather the permutation characters on k -subsets. Thus the two results lift two different generating families in $R(S_n)$.

B.1 Complete and ribbon bases, and Solomon's Mackey formula

Let $\mathbf{NSym}_{\mathbb{Z}}$ be the free associative $\mathbb{Z}$ -algebra on generators $H_1, H_2, \dots$, graded in such a way that each H_i is homogeneous with $\deg H_i = i$. Also, put $H_0 = 1$. If $I = (i_1, \dots, i_r)$ is a composition, write

$$H_I = H_{i_1} \cdots H_{i_r}.$$

The elements H_I , for $I \models n$, form a $\mathbb{Z}$ -basis of the homogeneous component $\mathbf{NSym}_{\mathbb{Z},n}$. The algebra $\mathbf{NSym}_{\mathbb{Z}}$ is known as the algebra of *noncommutative symmetric functions* over $\mathbb{Z}$; it was first introduced in [14] (see [21, §5.4] for a modern introduction).

We shall also use the *ribbon basis* (see [14, §3.2, §4.4]). If $I \models n$, define

$$R_I = \sum_{J \text{ coarsens } I} (-1)^{\ell(I) - \ell(J)} H_J. \quad (82)$$

Here a *coarsening* of I is obtained by repeatedly replacing two consecutive parts by their sum. Möbius inversion on the refinement poset gives

$$H_I = \sum_{J \text{ coarsens } I} R_J. \quad (83)$$

For example,

$$R_{(2,1,1)} = H_{(2,1,1)} - H_{(3,1)} - H_{(2,2)} + H_{(4)}.$$

For $w \in S_n$, write

$$\text{Des}(w) = \{i \in [n-1] : w(i) > w(i+1)\}.$$

For $I = (i_1, \dots, i_r) \models n$, put

$$D(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}, \quad x_I = \sum_{\substack{w \in S_n; \\ \text{Des}(w) \subseteq D(I)}} w \in \mathbb{Z}[S_n].$$

The (*integral*) *Solomon descent algebra* is

$$\mathcal{D}(S_n) := \bigoplus_{I \models n} \mathbb{Z} x_I \subseteq \mathbb{Z}[S_n], \quad (84)$$

with the usual group-algebra product. Solomon's Mackey formula shows that this span is closed under multiplication (see, e.g., [2, Theorem 3.8.1], [37, proof of Theorem 2.1], [20, Theorem 4.12], or many other places). The *descent-class correspondence* is the $\mathbb{Z}$ -linear map

$$\iota : \mathbf{NSym}_{\mathbb{Z},n} \longrightarrow \mathcal{D}(S_n), \quad H_I \longmapsto x_I. \quad (85)$$

It is an isomorphism of $\mathbb{Z}$ -modules. By Möbius inversion,

$$\iota(R_I) = \sum_{\substack{w \in S_n; \\ \text{Des}(w)=D(I)}} w =: \Delta_I,$$

the so-called *exact-descent-class sum* corresponding to I . So the ribbon basis of $\mathbf{NSym}$ corresponds to the exact-descent-class basis of $\mathcal{D}(S_n)$.

Besides its usual product, $\mathbf{NSym}_{\mathbb{Z},n}$ carries an *internal product* $*$, corresponding to multiplication in $\mathcal{D}(S_n)$. We use the row-reading matrix convention below. With the usual composition of permutations, $(uv)(i) = u(v(i))$, the map ι is an anti-isomorphism:

$$\iota(F * G) = \iota(G)\iota(F).$$

Thus our internal product corresponds to the opposite of the usual group-algebra product. This distinction does not affect generation, but fixes the order in the hook identity below. See, for example, Blessenohl–Schocker [5, Introduction and Chapter 14].

We shall use the standard matrix form of Solomon’s Mackey formula, due in this language to Garsia–Remmel ([13, Proposition 1.1], [37, Corollary 2.4], [4, Proposition 4.3], [44, Theorem 2], [5, Remark B.5]). If $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are compositions of n , then

$$H_I * H_J = \sum_M H_{\text{comp}(M)}, \quad (86)$$

where $M = (m_{uv})$ ranges over all $p \times q$ matrices with nonnegative integer entries whose row-sum vector is I and whose column-sum vector is J ; explicitly, this means that

$$\sum_{v=1}^q m_{uv} = i_u \quad \text{for all } 1 \leq u \leq p, \quad \sum_{u=1}^p m_{uv} = j_v \quad \text{for all } 1 \leq v \leq q.$$

The composition $\text{comp}(M)$ is obtained by reading the entries of M row by row and deleting the zero entries; we call it the *row-reading composition* of M .

Example B.1. Take $I = (3, 1)$ and $J = (2, 2)$. There are exactly two nonnegative 2×2 matrices with row-sum vector I and column-sum vector J :

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Their row-reading compositions are $(1, 2, 1)$ and $(2, 1, 1)$. Thus (86) gives

$$H_{(3,1)} * H_{(2,2)} = H_{(1,2,1)} + H_{(2,1,1)}.$$

Finally, recall the abelianization map

$$\pi : \mathbf{NSym}_{\mathbb{Z}} \longrightarrow \mathbf{Symm}_{\mathbb{Z}}, \quad H_r \longmapsto h_r. \quad (87)$$

It satisfies

$$\pi(R_I) = r_I, \quad (88)$$

where r_I is the ribbon Schur function of ribbon shape I . On each homogeneous component, π also respects the internal products: applying π to (86) gives the corresponding matrix rule for the Kronecker product of complete symmetric functions (see, e.g., [28, §I.7, Example 23 (e)] or [5, Appendix B, proof of the $\xi^r \xi^q$ formula]; this is essentially an application of the Mackey formula for tensor products of permutation characters).

In these conventions, *Solomon's epimorphism* is the surjective ring homomorphism

$$\theta : \mathcal{D}(S_n) \longrightarrow R(S_n), \quad x_I \longmapsto [\text{Ind}_{S_I}^{S_n} \mathbf{1}], \quad (89)$$

where (for any composition $I = (i_1, i_2, \dots, i_r)$ of n) we let $S_I = S_{i_1} \times \dots \times S_{i_r}$ denote the Young subgroup permuting the consecutive blocks of sizes $i_1, \dots, i_r$ (alternatively, $\text{Ind}_{S_I}^{S_n} \mathbf{1}$ can be described as the Young permutation module of tabloids for a Young diagram with rows of lengths $i_1, \dots, i_r$). Equivalently, if π_n denotes the degree- n restriction of π , then

$$\theta \circ \iota = \text{ch}^{-1} \circ \pi_n. \quad (90)$$

Indeed, $\text{ch}([\text{Ind}_{S_I}^{S_n} \mathbf{1}]) = h_I = \pi(H_I)$, so that $\text{ch} \circ \theta \circ \iota = \pi_n$. The compatibility of π_n with internal products, together with the commutativity of $R(S_n)$, shows that θ is a ring homomorphism even though ι reverses products. It is surjective because the h_λ for $\lambda \vdash n$ form a $\mathbb{Z}$ -basis of $\mathbf{Sym}_{\mathbb{Z},n}$. One can equally view its values as virtual characters; in particular,

$$\theta(\iota(R_I)) = \text{ch}^{-1}(r_I).$$

B.2 Schocker's hook generators: the direct lift

For $0 \leq k \leq n-1$, define the *noncommutative hook*

$$Q_{n,k} := R_{(n-k, 1^k)} \in \mathbf{NSym}_{\mathbb{Z},n}. \quad (91)$$

Thus, under the correspondence ι of $\mathbf{NSym}_{\mathbb{Z},n}$ with Solomon's descent algebra, $Q_{n,k}$ becomes the sum of all permutations whose descent set is

$$\{n-k, n-k+1, \dots, n-1\}$$

(this is the empty set when $k=0$). The associated ribbon is the 180° -rotated ordinary hook shape $(n-k, 1^k)$. Hence (88) gives

$$\pi(Q_{n,k}) = s_{(n-k, 1^k)} \quad (92)$$

(since 180° -rotation of skew Young diagrams does not change the Schur function). Under the Frobenius characteristic, this is the character of $\wedge^k V_n$. Thus the elements $Q_{n,k}$ are exactly the descent-algebra lifts one would naturally expect from the λ -ring formulation of the Boorman–Marin theorem.

Schocker's theorem [39, p. 152, Theorem] is stated using the exact-descent-class sums $\Delta_{\{1, \dots, k\}}$ for $(n-1)/2 \leq k \leq n-1$. Conjugation by the longest permutation w_0 sends a descent set D to $\{n-i : i \in D\}$, and hence sends $\Delta_{\{1, \dots, k\}}$ to the exact-descent-class sum with descent set $\{n-k, \dots, n-1\}$, which is $\iota(Q_{n,k})$ in our conventions. Thus Schocker proves the following stronger statement.

Theorem B.2 (Schocker). *Fix $n \geq 1$. The integral Solomon descent algebra $\mathcal{D}(S_n)$ is generated under its product by the elements*

$$\iota(Q_{n,k}) \quad \text{for all } k \text{ satisfying } \left\lceil \frac{n-1}{2} \right\rceil \leq k \leq n-1.$$

Under Solomon's epimorphism, these generators map to the irreducible hook characters $\chi^{(n-k, 1^k)}$; see [39, p. 152, Corollary 1].

We first prove the weaker statement that all the $Q_{n,k}$ generate, directly and integrally from Solomon's Mackey formula. We then recover the full theorem by multiplying hook descent classes by the longest permutation.

Proposition B.3. *Fix $n \geq 1$. Under the internal product, the $\mathbb{Z}$ -algebra $\mathbf{NSym}_{\mathbb{Z},n}$ is generated by*

$$Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-1}.$$

Proof. Let B_n be the $\mathbb{Z}$ -subalgebra generated by the $Q_{n,k}$. We prove that every complete-basis element H_I belongs to B_n . We use a finite double induction: first downward on the first part of I , and, among compositions having the same first part, downward on the length.

The composition (n) causes no difficulty, since $H_{(n)} = R_{(n)} = Q_{n,0}$. Now let

$$I = (m, a, a_3, \dots, a_r), \quad r \geq 2,$$

and assume that $H_L \in B_n$ whenever either¹⁴ $L_1 > m$, or $L_1 = m$ and $\ell(L) > r$. Set

$$I' = (m + a, a_3, \dots, a_r).$$

Since the first part of I' is greater than m , we have $H_{I'} \in B_n$ by induction. We shall multiply it by the hook generator

$$Q_{n,a} = R_{(n-a, 1^a)}.$$

Step 1: locate the possible leading parts. By (82), $Q_{n,a}$ is an alternating sum of H_J , where $J = (j_1, \dots, j_q)$ runs over the coarsenings of $(n-a, 1^a)$. Every such J satisfies

$$j_1 \geq n-a, \quad \text{hence} \quad \sum_{v=2}^q j_v \leq a. \quad (93)$$

Consider a matrix $M = (m_{uv})$ occurring in the Mackey formula for $H_{I'} * H_J$. Its first row has sum $m+a$. Its first entry therefore satisfies

$$\begin{aligned} m_{11} &= m + a - \sum_{v=2}^q m_{1v} \\ &\geq m + a - \sum_{v=2}^q j_v \quad \left(\text{since each } v \geq 2 \text{ satisfies } m_{1v} \leq \sum_{u=1}^{r-1} m_{uv} = j_v \right) \\ &\geq m \quad \left(\text{since } \sum_{v=2}^q j_v \leq a \right). \end{aligned}$$

¹⁴We write L_1 for the first entry of the composition L .

In particular, $m_{11} > 0$, so m_{11} is the first part of $\text{comp}(M)$. Thus every complete-basis term occurring in $H_{I'} * Q_{n,a}$ has first part at least m .

Step 2: analyze the equality case. Suppose that $m_{11} = m$. Then equality must hold throughout the preceding inequalities. Consequently

$$j_1 = n - a, \quad \sum_{v=2}^q j_v = a,$$

and every entry below the first row in columns $2, \dots, q$ is zero. Hence J has the form

$$J = (n - a, b_1, \dots, b_s) \quad \text{with} \quad b_1 + \dots + b_s = a,$$

and there is exactly one matrix M with first entry m , namely

$$\begin{pmatrix} m & b_1 & b_2 & \dots & b_s \\ a_3 & 0 & 0 & \dots & 0 \\ a_4 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_r & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Its row-reading composition is

$$(m, b_1, \dots, b_s, a_3, \dots, a_r),$$

which has length $r + s - 1 \geq r$. Equality of lengths occurs only for $s = 1$. In that case $J = (n - a, a)$ and the row-reading composition is exactly I . The coefficient of $H_{(n-a,a)}$ in $Q_{n,a} = R_{(n-a,1^a)}$ is

$$(-1)^{(a+1)-2} = (-1)^{a-1}.$$

Therefore

$$H_{I'} * Q_{n,a} = (-1)^{a-1} H_I + \sum_L c_L H_L, \quad (94)$$

where $c_L \in \mathbb{Z}$, and every L in the sum satisfies either $L_1 > m$, or $L_1 = m$ and $\ell(L) > r$.

All terms in the sum in (94) belong to B_n by induction, as does the left-hand side. Since the coefficient of H_I is ± 1 , we conclude that $H_I \in B_n$. This completes the double induction. $\square$

Remark B.4. *The proof is an integral triangular elimination. The relevant order on compositions is deliberately simple: first compare the first parts, and if they agree, compare the lengths. Formula (94) expresses H_I , up to sign, in terms of elements that are earlier in this induction.*

Proof of Theorem B.2. Let $w_0 \in S_n$ be the longest permutation, given by $w_0(i) = n + 1 - i$. The only permutation with every position a descent is w_0 , so

$$\iota(Q_{n,n-1}) = w_0.$$

For any $w \in S_n$, right multiplication by w_0 reverses its one-line notation. In particular,

$$\text{Des}(ww_0) = \{n - i : i \in [n - 1] \setminus \text{Des}(w)\}.$$

If $\text{Des}(w) = \{n - k, \dots, n - 1\}$, then this set is $\{k + 1, \dots, n - 1\}$, the descent set defining $Q_{n,n-1-k}$. Since $w \mapsto ww_0$ is a bijection, summing over this exact descent class gives

$$\iota(Q_{n,k})w_0 = \iota(Q_{n,n-1-k}).$$

That is, $\iota(Q_{n,k})\iota(Q_{n,n-1}) = \iota(Q_{n,n-1-k})$ (since $\iota(Q_{n,n-1}) = w_0$). Using the anti-isomorphism (85), this yields

$$Q_{n,n-1} * Q_{n,k} = Q_{n,n-1-k} \quad \text{for all } 0 \leq k \leq n - 1. \quad (95)$$

Now let B_n^+ be the $\mathbb{Z}$ -subalgebra generated by the hooks $Q_{n,k}$ with $\lceil (n-1)/2 \rceil \leq k \leq n-1$. For any $j < \lceil (n-1)/2 \rceil$, set $k = n - 1 - j$. Then $k \geq \lceil (n-1)/2 \rceil$, so both $Q_{n,k}$ and $Q_{n,n-1}$ belong to B_n^+ . Equation (95) therefore puts $Q_{n,j}$ in B_n^+ as well (since $j = n - 1 - k$). Thus the subalgebra B_n^+ contains each of $Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-1}$. By Proposition B.3, it thus equals all of $\mathbf{NSym}_{\mathbb{Z},n}$. In other words, $\mathbf{NSym}_{\mathbb{Z},n}$ under the internal product is generated by the $Q_{n,k}$ with $\lceil (n-1)/2 \rceil \leq k \leq n-1$. By applying ι , this entails that the descent algebra $\mathcal{D}(S_n)$ is generated by the $\iota(Q_{n,k})$ with $\lceil (n-1)/2 \rceil \leq k \leq n-1$. This proves the full upper-half generation theorem (Theorem B.2) over $\mathbb{Z}$, and hence over any commutative coefficient ring by base change. $\square$

B.3 Second proof of Theorem 3.1

Proposition B.3 now gives a direct integral descent-algebra proof of Boorman–Marin.

Second proof of Theorem 3.1. Applying the surjective ring morphism π to Proposition B.3 and recalling (92), we see that the hook Schur functions

$$s_{(n-k, 1^k)} \quad \text{for all } 0 \leq k \leq n - 1$$

generate $(\mathbf{Sym}_{\mathbb{Z},n}, *)$. Subsequently applying the inverse Frobenius characteristic map ch^{-1} , we conclude that the hook Specht modules $\Lambda^k V_n$ generate $R(S_n)$ under tensor product. Since $V_n = M_n - \mathbf{1}$ belongs to the λ -subring generated by M_n , so does $\lambda^k(V_n) = \Lambda^k V_n$ for every k . Therefore that λ -subring contains a set of ordinary ring generators of $R(S_n)$, and must be all of $R(S_n)$. This proves Theorem 3.1. $\square$

B.4 A second lift: two-block complete functions

The hook ribbons above are the natural lifts of the exterior powers appearing in Boorman–Marin. There is, however, another pleasantly small generating family upstairs. The following theorem is independent of Schocker’s theorem and will lead to a different family of generators downstairs, as well as eventually to a different – third – proof of Theorem 3.1.

Theorem B.5. *Fix $n \geq 1$. Under the internal product, the $\mathbb{Z}$ -algebra $\mathbf{NSym}_{\mathbb{Z},n}$ is generated by the n elements*

$$H_{(k,n-k)} = H_k H_{n-k} \quad \text{with } 1 \leq k \leq n, \quad (96)$$

where a zero part in a composition is understood to be omitted by default.

Proof. Let A_n be the $\mathbb{Z}$ -subalgebra of $\mathbf{NSym}_{\mathbb{Z},n}$ (with the internal product) generated by the elements in (96). We prove that $H_I \in A_n$ for every composition $I \models n$, by downward induction on the first part of I . The initial case is $I = (n)$, for which $H_I = H_{(n,0)}$ is one of the generators.

Now let

$$I = (m, a_2, a_3, \dots, a_r), \quad r \geq 2,$$

and assume that $H_K \in A_n$ for every composition $K \models n$ whose first part is strictly larger than m . Set

$$I' = (m + a_2, a_3, \dots, a_r) \quad \text{and} \quad J = (n - a_2, a_2).$$

The first part of I' is $m + a_2 > m$, so $H_{I'} \in A_n$ by induction, while H_J is one of the generators and thus belongs to A_n as well. Thus, $H_{I'} * H_J \in A_n$.

Apply (86) to $H_{I'} * H_J$. Every matrix M that occurs has two columns. Write its first row as

$$(m + a_2 - d, d).$$

Since the second column has total sum a_2 , we have $0 \leq d \leq a_2$. If $d < a_2$, then the first part of $\text{comp}(M)$ is $m + a_2 - d > m$. If $d = a_2$, all later entries in the second column must vanish, and there is exactly one possible matrix, namely

$$\begin{pmatrix} m & a_2 \\ a_3 & 0 \\ a_4 & 0 \\ \vdots & \vdots \\ a_r & 0 \end{pmatrix}.$$

Its row-reading composition is I . Consequently

$$H_{I'} * H_J = H_I + \sum_{\substack{K \models n; \\ K_1 > m}} c_K H_K \tag{97}$$

for some $c_K \in \mathbb{N}$, where K_1 denotes the first entry of K . Every term in the sum belongs to A_n by induction, as does $H_{I'} * H_J$; so (97) yields $H_I \in A_n$. This completes the induction. $\square$

Remark B.6. *The proof is triangular in a precise elementary sense: when (97) is solved for H_I , every other basis element that occurs has first part strictly larger than I_1 . Thus the downward induction eliminates the standard basis elements one first-part level at a time.*

Under the descent-algebra identification ι , the element $H_{(k,n-k)} \in \mathbf{NSym}_{\mathbb{Z},n}$ corresponds to

$$a_k = x_{(k,n-k)} = \sum_{\substack{w \in S_n; \\ \text{Des}(w) \subseteq \{k\}}} w = \sum_{\substack{w \in S_n; \\ w(1) < \dots < w(k); \\ w(k+1) < \dots < w(n)}} w \in \mathcal{D}(S_n),$$

with empty chains omitted; this is a sum over the so-called *Grassmannian permutations* (or *inverse riffle shuffles*) with their only possible descent (if any) at k . Thus, Theorem B.5 takes the following equivalent form:

Corollary B.7. *For every $n \geq 1$, the integral Solomon descent algebra $\mathcal{D}(S_n)$ is generated by $a_1, a_2, \dots, a_n$.*

This is somewhat similar, but not identical, to Theorem B.2. Schocker's generators are exact descent-class sums and map to irreducible hook characters. The elements a_k are sums over all descent sets contained in a one-element set and map to Young permutation characters.

B.5 The two-block generators downstairs

Applying the abelianization map π to Theorem B.5 gives another integral generation result for the representation ring.

Corollary B.8. *For every $n \geq 1$, the ring $(\mathbf{Symm}_{\mathbb{Z},n}, *)$ is generated by the n elements*

$$h_k h_{n-k} \quad \text{with } 1 \leq k \leq n.$$

Equivalently, $R(S_n)$ is generated under tensor product by the n permutation representations

$$P_{n,k} := \mathbb{Q} \left[\binom{[n]}{k} \right] \quad \text{with } 1 \leq k \leq n \quad (98)$$

on the k -subsets of $[n]$.

Proof. Under the Frobenius characteristic,

$$\text{ch}_n(P_{n,k}) = h_k h_{n-k},$$

since $P_{n,k} \cong \text{Ind}_{S_k \times S_{n-k}}^{S_n} \mathbf{1}$. Equivalently, its character is the two-part Young character of type $(k, n-k)$. In the notation of Blessenohl–Schocker [5, Chapter 12 and Appendix B], this is $\xi^{(k, n-k)}$: their ordered set partitions of type $(k, n-k)$ are simply pairs (A, A^c) , hence are naturally identified with the k -subsets A of $[n]$. $\square$

B.6 Third proof of Theorem 3.1

For curiosity's sake, we shall finally outline how these permutation modules $P_{n,k}$ can themselves be manufactured from the single λ -generator $M_n = P_{n,1}$. This gives a third proof of Theorem 3.1, logically independent of Schocker's hook-generation theorem once Theorem B.5 is known, although it may be argued that it is merely a remix of the first two proofs.

We extend the definition (98) to $k = 0$. Of course, $P_{n,0}$ and $P_{n,n}$ are trivial S_n -modules.

For positive integers $c_1, \dots, c_r$ with $d = c_1 + \dots + c_r \leq n$, define the permutation module

$$Y_{c_1, \dots, c_r} = \mathbb{Q} \left[\{(A_1, \dots, A_r) : A_i \subseteq [n], |A_i| = c_i \text{ and the } A_i \text{ are pairwise disjoint}\} \right]. \quad (99)$$

This is isomorphic to a Young permutation module of tabloids (encoding each A_i as the i -th row of a tabloid, and the complement $[n] \setminus (A_1 \cup A_2 \cup \dots \cup A_r)$ as the $(r+1)$ -th row of this tabloid). We call d the *support size* of this permutation module. Note that $Y_k = P_{n,k}$.

Lemma B.9. *Let L_n be the λ -subring of $R(S_n)$ generated by M_n . Suppose that $P_{n,j} \in L_n$ for every $j < d$. Then every $Y_{c_1, \dots, c_r}$ of support size d with $r \geq 2$ belongs to L_n .*

Proof sketch. We prove the assertion simultaneously for all support sizes d by strong induction on d . For $d = 1$ there is nothing to prove, since $r \geq 2$. Fix $d \geq 2$, assume the assertion known for all smaller support sizes, and assume, as in the statement, that $P_{n,j} \in L_n$ for every $j < d$.

Since $r \geq 2$, each $c_i < d$, so each P_{n,c_i} belongs to L_n . Their tensor product $P_{n,c}^\otimes := P_{n,c_1} \otimes P_{n,c_2} \otimes \dots \otimes P_{n,c_r}$ is the permutation representation on all tuples $(A_1, \dots, A_r)$ of subsets of $[n]$ with $|A_i| = c_i$, with no disjointness condition. Let $\mathfrak{P}_{n,c}$ be the S_n -set consisting of these tuples. For each nonempty $B \subseteq [r]$, define the *membership region*

$$R_B = \left(\bigcap_{i \in B} A_i \right) \setminus \left(\bigcup_{i \notin B} A_i \right)$$

(so that the R_B for different B 's are disjoint and their union is $A_1 \cup A_2 \cup \dots \cup A_r$; visually speaking, they are the regions of the Venn diagram of $A_1, A_2, \dots, A_r$). Decompose the S_n -set $\mathfrak{P}_{n,c}$ into orbits according to the cardinalities $|R_B|$ of all these membership regions. Exactly one orbit is the pairwise-disjoint locus (99). Each of the other orbits has union of size strictly smaller than d , and is itself a permutation module of the form $Y_{b_1, \dots, b_s}$ with support size $< d$, after the nonempty membership regions are listed as labeled blocks.

For every such smaller-support orbit module, let $e < d$ be its support size. All $P_{n,j}$ with $j < e$ belong to L_n because $j < e < d$. If an orbit has at least two nonempty membership regions, the induction hypothesis puts its module in L_n . If it has exactly one such region, its module is $Y_e = P_{n,e}$, which belongs to L_n by the assumption $e < d$. Hence all these other orbit modules belong to L_n . Subtracting them from $P_{n,c_1} \otimes \dots \otimes P_{n,c_r}$ leaves $Y_{c_1, \dots, c_r}$. $\square$

Proposition B.10. *For every $0 \leq k \leq n$, one has $P_{n,k} \in L_n$.*

Proof sketch. We use strong induction on k . The cases $k = 0$ and $k = 1$ are clear. Assume $k \geq 2$ and that $P_{n,j} \in L_n$ for all $j < k$.

Step 1: decompose the symmetric power into permutation orbits. The symmetric power $\text{Sym}^k(M_n)$ has a basis consisting of the monomials of total degree k in the basis vectors $e_1, \dots, e_n$, and S_n permutes this basis. Its orbits are indexed by partitions $\lambda \vdash k$: the partition records the positive multiplicities of the basis vectors occurring in a monomial. If $m_r(\lambda)$ denotes the number of parts of λ equal to r , then the orbit of type λ affords the permutation module

$$Y_{m_1(\lambda), m_2(\lambda), \dots},$$

with zero entries omitted. Its support size is $\ell(\lambda) = \sum_r m_r(\lambda)$.

For $\lambda = (1^k)$ this orbit module is $P_{n,k}$. For each of the remaining partitions $\lambda \vdash k$, we have $\ell(\lambda) < k$, so Lemma B.9 puts its orbit module in L_n when there are at least two nonzero multiplicities $m_r(\lambda)$. If there is only one, the orbit module is $P_{n,\ell(\lambda)}$ and belongs to L_n directly by the induction hypothesis. Hence

$$[\text{Sym}^k(M_n)] = [P_{n,k}] + \sum_{\substack{\lambda \vdash k; \\ \lambda \neq (1^k)}} [Y_{m_1(\lambda), m_2(\lambda), \dots}], \quad (100)$$

and all addends in the sum belong to L_n .

Step 2: isolate the k -subset orbit. The left-hand side belongs to L_n . Indeed, in the ring $R(S_n)[[t]]$, we have (see, e.g., [10, Lemma 3.29 and the preceding derivation] for an elementary proof using eigenvalues¹⁵)

$$\sum_{j \geq 0} [\text{Sym}^j(M_n)] t^j = \frac{1}{\lambda_{-t}([M_n])} = \lambda_{-t}(-[M_n]),$$

so each symmetric-power class $[\text{Sym}^j(M_n)]$ belongs to the λ -subring L_n . All terms in the sum in (100) belong to L_n , so $[P_{n,k}] \in L_n$ as well. $\square$

Third proof of Theorem 3.1 (sketched). By Corollary B.8, the classes $[P_{n,k}]$ generate $R(S_n)$ as an ordinary ring. By Proposition B.10, every one of these classes belongs to the λ -subring L_n generated by M_n . Hence $R(S_n) \subseteq L_n$. The reverse inclusion is tautological, so $L_n = R(S_n)$. $\square$

Remark B.11. Let us summarize the differences between our three proofs of Theorem 3.1:

1. The first proof worked directly in $R(S_n)$, showing that the hook Specht modules $\wedge^k V_n$ generate $R(S_n)$. As an intermediate step, it used the $\mathbb{S}^\beta(M_n)$ (Schur functors evaluated on M_n) and the T_λ (tensor products of exterior powers of M_n).
2. The second proof worked in the algebra $\mathbf{NSym}_{\mathbb{Z},n}$, showing that the noncommutative hooks $Q_{n,k}$ (which are Schocker's hook descent classes, conjugated by the longest permutation) generate $\mathbf{NSym}_{\mathbb{Z},n}$ under the Kronecker product $*$. Upon projecting down onto $R(S_n)$ via the chain of surjective ring morphisms

$$\mathbf{NSym}_{\mathbb{Z},n} \xrightarrow{\pi} \mathbf{Sym}_{\mathbb{Z},n} \xrightarrow{\text{ch}^{-1}} R(S_n),$$

these hooks became the hook Specht modules $\wedge^k V_n$.

3. The third proof again worked in $\mathbf{NSym}_{\mathbb{Z},n}$. But instead of lifting the hook Specht modules $\wedge^k V_n$, it showed that the two-block complete functions $H_k H_{n-k}$ generate $\mathbf{NSym}_{\mathbb{Z},n}$ under $*$. Under the same ring surjections as before, these became the k -subset permutation modules $P_{n,k}$. Then, another triangularity argument was needed to show that these $P_{n,k}$ belong to the λ -subring generated by M_n , so that Theorem 3.1 again followed.

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¹⁵A more conceptual proof can be found in [1, Corollary 2.14 (applied to $A = \text{Sym}(V)$ and $A^\dagger = \wedge(V^*)$)].

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