hexagon:2610.00025v3

MINIMALITY OF THE RANK-TWO QUANTUM GREEDY BASIS

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Abstract. We prove that the rank-two quantum greedy basis is the least strongly positive basis when $b \mid c$ or $c \mid b$, resolving Conjecture 14 of Lee–Li–Rupel–Zelevinsky. Every strongly positive basis has nonnegative expansion coefficients in the greedy basis.

1. Introduction

Lee–Li–Rupel–Zelevinsky ask whether the quantum greedy basis is the least strongly positive basis when one exchange parameter divides the other [1, Conjecture 14]. Universal Laurent positivity alone does not give this comparison: one must control the coefficients of an arbitrary strongly positive basis in the greedy basis. We obtain this control from positive multiplication and bar invariance. First, a degree bound forces every fixed finite collection of lower coefficients to vanish along suitable growing labels. These labels then provide multipliers which turn a prescribed scattering coefficient into an actual positive structure constant.

Lemma 3.1 holds for all positive b, c. Theorem 1.2 uses the scattering realization in Proposition 1.1, obtained from [6, Proposition 5.1 and Section 6].

AI contribution. The mathematical argument and manuscript were developed through several rounds of conversations with GPT-6-Astra. The manuscript subsequently underwent GPT-based review, including adversarial checks of the proof and its dependencies.

1.1. **Conventions.** Fix positive integers b, c and an indeterminate v. Put $R = \mathbb{Z}[v^{\pm 1}]$ and $R_+ = \mathbb{Z}_{\geq 0}[v^{\pm 1}]$, with zero included in R_+. Let $\mathcal{T}$ be the quantum torus over R generated by invertible X_1, X_2. For $m = (m_1, m_2), n = (n_1, n_2) \in \mathbb{Z}^2$, set $\det(m, n) = m_1 n_2 - m_2 n_1$. For $r, s \in \mathbb{Z}$, our normalization is

$$X_2 X_1 = v^2 X_1 X_2, \qquad X^{(r,s)} = v^{rs} X_1^r X_2^s, \qquad X^m X^n = v^{-\det(m,n)} X^{m+n}.$$

The exchange relations are

$$X_{m+1} X_{m-1} = 1 + v^{d_m} X_m^{d_m}, \qquad d_m = b \ (m \text{ odd}), \quad d_m = c \ (m \text{ even}).$$

The algebra $\mathcal{A}_v(b, c)$ is the R-subalgebra of the fraction skew field of $\mathcal{T}$ generated by the variables X_m, $m \in \mathbb{Z}$. Its clusters are (X_m, X_{m+1}); their normalized cluster monomials are $v^{rs} X_m^r X_{m+1}^s$, for $r, s \in \mathbb{Z}_{\geq 0}$. For an integer a write $[a]_+ = \max(a, 0)$. The bar anti-involution sends v to v^{-1}, fixes each X_m, and reverses multiplication. It fixes every normalized torus monomial.

For $F = \sum_m f_m X^m$ write $[X^m]F = f_m$ and $\operatorname{supp} F = \{m : f_m \neq 0\}$. Missing coefficients are zero. The notation $F \leq H$ means $[X^m](H - F) \in R_+$ for all m. For a Laurent polynomial $f(v)$, $\deg_v f$ is its largest exponent with nonzero coefficient, and $\deg_v 0 = -\infty$. Likewise $[T^j]$ extracts the coefficient of T^j in a polynomial or formal power series in a commuting variable T. For $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$ define

$$[n]_w = \frac{w^n - w^{-n}}{w - w^{-1}}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_w = \prod_{r=1}^{k} \frac{[n - r + 1]_w}{[r]_w}.$$

The empty product is one; the Gaussian coefficient is zero for $k < 0$ or $k > n$. These are Laurent polynomials in the indeterminate w.

2020 *Mathematics Subject Classification.* Primary 13F60; Secondary 16T20.

Keywords. Quantum cluster algebra, quantum greedy basis, strongly positive basis, scattering diagram.

An element F is pointed at $(A,B)\in\mathbb{Z}^2$ if

$$F=\sum_{p,q\geq 0} e_F(p,q)X^{(bp-A,cq-B)},\qquad e_F(0,0)=1,$$

with finitely many nonzero $e_F(p,q)\in R$. Set $e_F(p,q)=0$ when either index is negative. The quantum greedy element with this label is denoted $G[A,B]$. We use its finite basis theorem and greedy recurrence [2, Theorems 1.7 and 1.9, Section 3].

A strongly positive R-basis has all four properties in [1, Section 5]: its elements are bar invariant, it is independent of the initial cluster, it contains every normalized cluster monomial, and all its multiplication structure constants belong to R_+. Basis expansions in $\mathcal{A}_v(b,c)$ are finite.

1.2. **Scattering realization.** Put

$$P=\mathbb{Z}_{\geq 0}(-b,0)+\mathbb{Z}_{\geq 0}(0,c)\subset L=\mathbb{Z}^2,\qquad \deg(-bk,cj)=k+j\quad (k,j\in\mathbb{Z}_{\geq 0}).$$

A wall is a line or ray through the origin carrying an automorphism of a completed quantum torus. A broken line is a piecewise linear path ending at a point Q off the walls, decorated on each segment by a monomial. A segment decorated by aX^m has velocity $-m$. At a bend, its next monomial is a term of the oriented wall action on the previous monomial. Every nontrivial bend adds an element of $P\setminus\{0\}$ to its exponent. The theta function labeled by h is the sum of the terminal decorations of broken lines with initial monomial X^h. The unbent line contributes X^h. Supports of completed expressions lie in finitely many translates of P; truncation through $K\in\mathbb{Z}_{\geq 0}$ retains increments of degree at most K. Each such set of increments is finite.

Proposition 1.1. *Suppose $b\mid c$. The central rank-two quantum scattering diagram with incoming degrees $(-b,0)$ and $(0,c)$ has increments in P and finitely many wall directions in each degree truncation. Every oriented wall action used by a broken line has coefficients in R_+ on the full lattice L. In the initial chart, its normalized theta functions satisfy*

$$\vartheta_{\gamma(A,B)}=G[A,B],\qquad \gamma(A,B)=(-A+b[B]_+,-B),\qquad (A,B)\in\mathbb{Z}^2. \tag{1.1}$$

Proof. The diagram and its finite truncations are constructed in [6, Section 6.2]. Proposition 5.1 there proves positivity of outgoing wall actions on $\mathbb{Z}^2$; the incoming actions are products of positive quantum binomials. Sections 6.1–6.3 identify the theta function with $v^sG[A,B]$ for some integer s. Its coefficient at $X^{\gamma(A,B)}$ is one: the unbent line contributes one, and every bend adds a nonzero element of the pointed monoid P. For $B>0$, the greedy axis recurrence gives $e_G(B,0)=\begin{bmatrix}B\\B\end{bmatrix}_{v^b}=1$; for $B\leq 0$, the pointed coefficient is one. Thus $G[A,B]$ has coefficient one at the same exponent, so $v^s=1$. □

The map γ is a bijection, with inverse

$$A=b[-h_2]_+-h_1,\qquad B=-h_2.$$

We use the formal theta construction, pointedness, and algebra transport from [4, Proposition 3.1 and Theorem 3.2]. They give charts ι_Q for generic endpoints Q, with

$$\iota_Q(\vartheta_h)=X^h+\sum_{\rho\in P\setminus\{0\}} a_{h,\rho,Q}X^{h+\rho}. \tag{1.2}$$

Each coefficient is well-defined by finite-degree truncation. We take generic endpoints off every rational line through the origin; in particular they are nonzero and off all walls. Positive wall actions give $a_{h,\rho,Q}\in R_+$. The maps ι_Q are algebra homomorphisms into these completed tori, compatible with path transport. An element with expansion $X^h+\sum_{0\neq\rho\in P}a_\rho X^{h+\rho}$ is called P-pointed at h.

Theorem 1.2. *Suppose $b \mid c$ or $c \mid b$. For every strongly positive basis $\mathcal{B}$ and every $(A, B) \in \mathbb{Z}^2$, write its uniquely pointed element as $B[A, B]$. Then*

$$B[A, B] = G[A, B] + \sum_{i,j \geq 1} t_{i,j} G[A - bi, B - cj], \qquad t_{i,j} \in R_+,$$

where the sum is finite. Consequently $B[A, B] - G[A, B]$ is Laurent nonnegative in every cluster. The greedy basis is strongly positive and is the unique least strongly positive basis in the sense of LLRZ Conjecture 14.

2. Pointed elements and positive sections

The pointed parametrization below is from [7]. The acyclic standard monomial basis theorem [3] gives the basis

$$M[A, B] = v^{AB} X_3^{[A]_+} X_1^{[-A]_+} X_2^{[-B]_+} X_0^{[B]_+}.$$

It is pointed at (A, B): its support has the form $(-A + bp, -B + cq)$, $p, q \geq 0$, and the coefficient at $(-A, -B)$ is one. If $A \leq 0$ or $B \leq 0$, it is a normalized cluster monomial.

Lemma 2.1. *Every strongly positive basis has exactly one element pointed at every $(A, B) \in \mathbb{Z}^2$, and these exhaust the basis. Every such element is Laurent nonnegative in every cluster. For $A, B > 0$,*

$$B[A, B] \leq M[A, B] \tag{2.1}$$

coefficientwise in the initial Laurent torus. On the boundary $A \leq 0$ or $B \leq 0$, $B[A, B] = G[A, B] = M[A, B]$.

Proof. For $Y \in \mathcal{B}$, multiply by a sufficiently large normalized initial cluster monomial to clear both Laurent denominators. The resulting Laurent expansion is an expansion in distinct initial cluster monomials, hence in elements of $\mathcal{B}$. Uniqueness and positive multiplication imply that every Laurent coefficient of Y is nonnegative. This argument works in every cluster.

Each $M[A, B]$ is a power of v times a product of cluster variables, so it has a finite positive $\mathcal{B}$-expansion. There is no Laurent cancellation. All appearing elements have support within the support of $M[A, B]$. At its corner the coefficient identity is $1 = \sum_Y h_Y k_Y$ with $h_Y, k_Y \in R_+$. There is exactly one nonzero summand, and it is one. Its factors are inverse powers of v. Bar invariance of Y, and invariance of normalized monomials under bar, force its corner coefficient to be one. Select this element for each label.

To show that the selected elements exhaust $\mathcal{B}$, in any finite signed expansion in pointed standard monomials choose the largest first label coordinate, breaking ties by the largest second. Its corner cannot cancel. Reversing the coordinate priority gives the corresponding bound on the second label coordinate. Thus an element supported in $r \geq -A$, $s \geq -B$ has standard expansion using only labels $a \leq A$, $d \leq B$. Each selected element consequently equals $M[A, B]$ plus finitely many standard terms with coordinatewise smaller labels, at least one strict. On the boundary the selected element is already $M[A, B]$. Induction on $A + B$ in the positive quadrant shows that the selected elements span every standard monomial. A spanning subset of a basis is the whole basis. The two coordinate minima determine a pointed label uniquely. Finally the coefficient of the selected $B[A, B]$ in the positive expansion of $M[A, B]$ is one, proving (2.1). □

For $A, B > 0$, let $e_F(p, q)$ be the normalized coefficients of $F = B[A, B]$. The quantum binomial theorem gives

$$X_3^A = \sum_q \begin{bmatrix} A \\ q \end{bmatrix}_{v^c} X^{(-A, cq)}, \qquad X_0^B = \sum_p \begin{bmatrix} B \\ p \end{bmatrix}_{v^b} X^{(bp, -B)}.$$

In their product the torus phase is $v^{-AB+bcpq}$. The prefactor v^{AB} in $M[A,B]$ cancels its first term. The standard coefficient and its degree capacity are

$$[X^{(bp-A,cq-B)}]M[A,B] = v^{bcpq}\begin{bmatrix}B\\p\end{bmatrix}_{v^b}\begin{bmatrix}A\\q\end{bmatrix}_{v^c}, \tag{2.2}$$

$$D_{A,B}(p,q) = bp(B-p) + cq(A-q) - bcpq. \tag{2.3}$$

In particular $e_F = 0$ outside $0 \le p \le B,\ 0 \le q \le A$. If $e_F(p,q) \neq 0$, bar invariance and (2.2) give

$$\deg_v e_F(p,q) \le D_{A,B}(p,q), \qquad D_{A,B}(p,q) < 0 \implies e_F(p,q) = 0. \tag{2.4}$$

Indeed the smallest degree in the standard coefficient is $-D_{A,B}(p,q)$. Bar invariance makes the smallest degree of $e_F(p,q)$ equal to $-\deg_v e_F(p,q)$. By (2.1), this is at least $-D_{A,B}(p,q)$. A nonzero bar-invariant Laurent polynomial has nonnegative highest degree, so a negative capacity forces vanishing.

For integers $n \ge 0$ and $d > 0$, define the symmetric Gaussian polynomial

$$P_n^{(d)}(T) = \prod_{\ell=0}^{n-1}(1 + v^{d(n-1-2\ell)}T), \qquad P_0^{(d)} = 1.$$

It is monic with constant term one, with coefficients in R_+. For integer labels (A,B) and indices $k, \ell \ge 0$, define column and row quotients

$$C^F_{A,B;k}(T) = \sum_{q\ge 0} e_F(k,q)T^q, \qquad Q^F_{A,B;k}(T) = \frac{C^F_{A,B;k}(T)}{P^{(c)}_{[A-bk]_+}(T)}, \tag{2.5}$$

$$E^F_{A,B;\ell}(T) = \sum_{p\ge 0} e_F(p,\ell)T^p, \qquad W^F_{A,B;\ell}(T) = \frac{E^F_{A,B;\ell}(T)}{P^{(b)}_{[B-c\ell]_+}(T)}. \tag{2.6}$$

For a strongly positive basis element these quotients belong to $R_+[T]$. This follows from adjacent-cluster Laurent positivity, separately for each fixed power of the mutated variable. Explicitly, for $n \ge 0$ and $s \in \mathbb{Z}$,

$$v^{sn}X_2^sX_3^n = \sum_{q=0}^{n}\begin{bmatrix}n\\q\end{bmatrix}_{v^c}X^{(-n,s+cq)}.$$

The analogous identity with X_0, X_1 gives the row formula. The constant term one of the divisor and the lower support bound show that the quotient has no negative powers of T. Zero divisor length means division by one.

For the row $q = 0$, the standard rectangle gives degree at most B in T, while the divisor $P_B^{(b)}$ has degree B and is monic. The quotient is therefore constant; its constant term is $e_F(0,0) = 1$. The column $p = 0$ gives the identical conclusion with A, c. Consequently

$$e_F(p,0) = \begin{bmatrix}B\\p\end{bmatrix}_{v^b}, \qquad e_F(0,q) = \begin{bmatrix}A\\q\end{bmatrix}_{v^c}.$$

The greedy element has the same axes. The finite greedy basis theorem and a corner argument as in Lemma 2.1 therefore give the unique finite expansion

$$B[A,B] = G[A,B] + \sum_{i,j\ge 1} t_{i,j}G[A-bi, B-cj], \qquad t_{i,j} \in R. \tag{2.7}$$

The corner argument first bounds all lower labels coordinatewise. Splitting the expansion into exponent classes modulo $b\mathbb{Z}\oplus c\mathbb{Z}$ and using independence eliminates every other class. Hence its labels are $(A-bi, B-cj)$ with $i,j \ge 0$ and root coefficient one. On $q = 0$, only terms with $j = 0$ contribute. If one of their nonroot coefficients survived, the smallest such i would be the coefficient of $p = i$ in the difference of the two axes, since every lower greedy element has constant coefficient one. This contradicts equality of the axes. The $p = 0$ column similarly excludes $i = 0, j > 0$. The coefficients $t_{i,j}$ may at this stage have either sign.

Lemma 2.2. *For (2.7),*

$$Q^F_{A,B;k} - Q^G_{A,B;k} = \sum_{1\le i\le k,\ j\ge 1} t_{i,j} T^j Q^G_{A-bi,B-cj;k-i}. \tag{2.8}$$

The row identity is

$$W^F_{A,B;\ell} - W^G_{A,B;\ell} = \sum_{i\ge 1,\, 1\le j\le \ell} t_{i,j} T^i W^G_{A-bi,B-cj;\ell-j}.$$

Proof. Compare the complete k-th columns before division. Each translated greedy term contributes $T^j C^G_{A-bi,B-cj;k-i}$. Its divisor is exactly the same divisor as that of the root column, since

$$[(A-bi)-b(k-i)]_+ = [A-bk]_+.$$

Division by this common nonzero polynomial proves the identity in $R[T]$. For rows the common length is $[(B-cj)-c(\ell-j)]_+ = [B-c\ell]_+$. For negative lower labels, Laurentness in the adjacent torus still gives polynomial division; constancy of the divisor at $T=0$ excludes negative powers of the quotient. Thus the identities also apply to signed coefficients and boundary labels. □

3. Finite rigidity

The greedy recurrence [2] has the following consequence: for $i, j > 0$ and positive A, B,

$$cAj \ge bBi \implies [T^j] Q^G_{A,B;i} = 0, \tag{3.1}$$

$$cAj \le bBi \implies [T^i] W^G_{A,B;j} = 0. \tag{3.2}$$

For example the column recurrence is precisely the vanishing of the coefficient in the product of the column with

$$\big(P^{(c)}_n(T)\big)^{-1} = \sum_{\ell\ge 0} (-1)^\ell \begin{bmatrix} n+\ell-1 \\ \ell \end{bmatrix}_{v^c} T^\ell, \qquad n = A-bi > 0.$$

We will only use these equations where the divisor length is positive. Both recurrences hold at equality.

Lemma 3.1. *Fix $b, c > 0$, integers $I, J > 0$, positive integers a, d, and a finite set of offsets $(a_0, d_0) \in \mathbb{Z}^2$. Suppose*

$$bdi - caj \ne 0 \qquad (1 \le i \le I,\ 1 \le j \le J). \tag{3.3}$$

For $A_N = Na + a_0$, $B_N = Nd + d_0$, every strongly positive basis satisfies

$$t^{(N)}_{i,j} = 0 \qquad (1 \le i \le I,\ 1 \le j \le J)$$

in (2.7), for all sufficiently large integers N. One bound on N works for every offset in the finite set and every strongly positive basis.

Proof. Put

$$K_{i,j} = bcij - bi^2 - cj^2, \qquad \Delta_N = biB_N - cjA_N.$$

The two exact capacity evaluations are

$$D_{A_N,B_N}(i, A_N - bi + j) = \Delta_N + K_{i,j}, \tag{3.4}$$

$$D_{A_N,B_N}(B_N - cj + i, j) = -\Delta_N + K_{i,j}. \tag{3.5}$$

Choose N so that $A_N > bI$, $B_N > cJ$, and, for every indicated pair and offset,

$$N|bdi - caj| > |bid_0 - cja_0| + \max(K_{i,j}, 0).$$

These are finitely many strict inequalities with positive slopes, so such an integer exists. They ensure $|\Delta_N| > \max(K_{i,j}, 0)$ and $\Delta_N \ne 0$.

Induct on $i+j$ inside the rectangle. Suppose all preceding coefficients there have vanished. If $\Delta_N < 0$, the greedy column quotient coefficient at T^j vanishes by (3.1). Equation (3.4) is negative, so $e_F(i, A_N - bi + j) = 0$. Put $n = A_N - bi > 0$. In

$$C^F_{A_N,B_N;i} = P^{(c)}_n Q^F_{A_N,B_N;i},$$

all coefficients of both factors are nonnegative. The monic term T^n shows that $[T^j]Q^F$ is one of the nonnegative summands of the zero coefficient at T^{n+j}. It is therefore zero. The coefficient multiplying $t^{(N)}_{i,j}$ in (2.8) is the constant coefficient of the zeroth column quotient of $G[A_N - bi, B_N - cj]$, which is one. Taking T^j therefore gives $t^{(N)}_{i,j} = 0$. Every other term that could contribute has indices $i' \le i,\ j' \le j$ with $i' + j' < i + j$; terms outside this rectangle cannot contribute.

If $\Delta_N > 0$, use (3.5), (3.2), and the row identity instead. The monic end of $P^{(b)}_{B_N - cj}$ gives the same conclusion. □

4. Monomial truncations

In this section and the next, suppose $b \mid c$ and use the diagram of Proposition 1.1.

Relabel the arbitrary basis by

$$\widehat{B}_h = B[b[-h_2]_+ - h_1, -h_2].$$

Both $\widehat{B}_h$ and ϑ_h are P-pointed in the initial chart, with coefficient one at X^h. For positive labels this follows from the standard rectangle and the axis endpoint $p = B, q = 0$; boundary labels are cluster monomials.

Lemma 4.1. *The finite expansion of $\widehat{B}_h$ in the theta basis has the form*

$$\widehat{B}_h = \vartheta_h + \sum_{\rho \in P \setminus \{0\}} T_{h,\rho} \vartheta_{h+\rho}. \tag{4.1}$$

In every chart Q, its support lies in $h + P$, with coefficient one at h. If

$$\widehat{B}_h \widehat{B}_s = \sum_{\ell} c^{\ell}_{h,s} \widehat{B}_{\ell},$$

then $c^{\ell}_{h,s} \neq 0$ implies $\ell \in h + s + P$.

Proof. For a finite signed expansion in elements supported in their label plus P, the largest first coordinate of a label, with the smallest second coordinate breaking ties, has a corner that cannot cancel. Choosing the smallest second coordinate first gives the other support bound. The torus grading gives the congruences defining P. The unique root corner has coefficient one. This proves (4.1). Equation (1.2) gives its support assertion in every chart. The initial-chart product is supported in $h + s + P$; the same finite corner argument gives the assertion about output labels. □

Write $F = \mathcal{O}_h(> K)$ if its support lies in $\{h + \rho : \rho \in P,\ \deg \rho > K\}$. Degrees add under multiplication.

Lemma 4.2. *Let $K \ge 1$, let $S \subset L$ be finite, and let C be a nonempty open two-dimensional cone. There is $u \in C \cap L$, $u \neq 0$, such that, for all sufficiently large N and every $s \in S$,*

$$\widehat{B}_{Nu+s} - \vartheta_{Nu+s} = \sum_{\substack{\rho \in P \\ \deg \rho > K}} T_{Nu+s,\rho} \vartheta_{Nu+s+\rho}.$$

The bound is uniform over s and over strongly positive bases.

Proof. Choose an integral direction u avoiding $u_2 = 0$, the line $u_1 + bu_2 = 0$, and the finite additional lines described below. An open cone contains rational directions outside any finite union of lines; scaling gives an integral direction. If $u_2 > 0$, the second denominator coordinate of $Nu + s$ is negative for large N, so $\widehat{B}_{Nu+s} = \vartheta_{Nu+s}$ exactly. If $u_2 < 0$, put

$d = -u_2 > 0$ and $a = bd - u_1$. When $a < 0$ the first denominator coordinate is eventually negative, with the same conclusion.

It remains to consider $a, d > 0$. Exclude the finitely many lines

$$bdi - caj = 0 \qquad (1 \le i \le cK,\ 1 \le j \le K).$$

For $s = (s_1, s_2)$, the denominator label is

$$A_N = Na - s_1 - bs_2, \qquad B_N = Nd - s_2.$$

Apply Lemma 3.1 to this finite set of offsets. If a theta correction has increment $\rho = (-bk, cj)$ with $k + j \le K$, its second denominator coordinate is $B_N - cj > 0$ for large N, and its first one is

$$A_N - b(cj - k).$$

Thus its denominator change has indices $i = cj - k,\ j$. The strict triangular expansion (2.7) rules out $i \le 0$ or $j = 0$, except for the root term. All remaining indices satisfy $1 \le i \le cK,\ 1 \le j \le K$, so Lemma 3.1 makes their coefficients zero. □

Lemma 4.3. *Fix a generic endpoint Q, a finite set $S \subset L$, and $K \ge 1$. Choose an open convex cone C, of angle less than π, containing Q and disjoint from all walls relevant through degree K. For any $u \in C \cap L$, $u \ne 0$, all sufficiently large N satisfy*

$$\iota_Q(\vartheta_{Nu+s}) = X^{Nu+s} + \mathcal{O}_{Nu+s}(> K) \qquad (s \in S).$$

Proof. Such a cone exists because the diagram is central and the truncation has finitely many wall directions. For sufficiently large N, all vectors

$$Nu + s + \rho, \qquad s \in S, \quad \rho \in P, \quad \deg \rho \le K,$$

lie in C. Suppose a broken line ending at Q has one of these final exponents and a nonzero total increment of degree at most K. Trace its final straight segment backwards:

$$Q + t(Nu + s + \rho), \qquad t \ge 0.$$

Convexity keeps this ray in C. It cannot meet a wall of the truncation, so the last nontrivial bend cannot exist. A higher-degree wall cannot supply an increment of total degree at most K. The unique unbent contribution has coefficient one. □

Corollary 4.4 (Uniform monomial truncation)**.** *For fixed Q, S, K, one may choose a single integral u and a single integer N_0, independent of the strongly positive basis, such that*

$$\iota_Q(\widehat{B}_{Nu+s}) = X^{Nu+s} + \mathcal{O}_{Nu+s}(> K) \qquad (s \in S,\ N \ge N_0). \tag{4.2}$$

Proof. Choose u in the cone of Lemma 4.3, also avoiding the finite lines of Lemma 4.2. Take the larger of the two uniform bounds on N. The higher theta corrections cannot return to smaller degree, since every theta function has support in its label plus P. □

5. Structure constants and minimality

Theorem 5.1 (Structure-constant detector)**.** *Fix $h \in L$, a generic endpoint Q, and $m = h + \delta$, $\delta \in P$. There exist $u \in L$ and N_0 such that, for every strongly positive basis and every $N \ge N_0$,*

$$[\widehat{B}_{Nu+m}](\widehat{B}_h \widehat{B}_{Nu}) = v^{-N \det(m,u)} [X^m] \iota_Q(\widehat{B}_h). \tag{5.1}$$

The left bracket denotes the coefficient in the actual $\mathcal{B}$-basis expansion. In particular every $\widehat{B}_h$ is nonnegative in every scattering chart.

Proof. Set $K = \max(1, \deg \delta)$, and use Corollary 4.4 with

$$S = \{0\} \cup \{h + \rho : \rho \in P,\ \deg \rho \le K\}.$$

This is a fixed finite set, independent of N and of the basis. Strong positivity gives the finite expansion

$$\widehat{B}_h\widehat{B}_{Nu} = \sum_{\ell} c_N(\ell)\widehat{B}_\ell, \qquad c_N(\ell) \in R_+.$$

Lemma 4.1 restricts its labels to $Nu + h + P$. An output label whose increment from $Nu + h$ has degree greater than K cannot contribute to the monomial X^{Nu+m} in any chart. All other output labels are $Nu + s$ with $s \in S$, and their expansions are single monomials to the required order by (4.2). Thus the coefficient at X^{Nu+m} on the right is exactly $c_N(Nu + m)$.

On the left $\iota_Q(\widehat{B}_{Nu}) = X^{Nu} + \mathcal{O}_{Nu}(> K)$, while $\iota_Q(\widehat{B}_h)$ is supported in $h + P$. Only its coefficient at X^m can contribute. The normalized multiplication law supplies the factor $v^{-\det(m,Nu)}$, proving (5.1). The factor is a positive unit and $c_N(Nu + m) \in R_+$. The desired scattering coefficient therefore belongs to R_+. All coefficients outside $h + P$ vanish. □

Lemma 5.2. *In a finite expansion $F = \sum_\ell a_\ell\vartheta_\ell$ supported in theta labels $h + P$, every coefficient a_m, $m \in h + P$, equals $[X^m]\iota_Q(F)$ for a suitable generic endpoint Q.*

Proof. Only labels ℓ with $m - \ell \in P$ can contribute; their required increments have degree at most $\deg(m - h)$. If $m \neq 0$, choose Q sufficiently near the direction m, on one side if that direction is a wall, so that the backwards ray $Q + tm$, $t \geq 0$, meets no wall of this finite truncation. The last-bend argument then rules out every contribution except the unbent line with initial label m, whose coefficient is one. If $m = 0$, a final segment with zero velocity cannot start on a wall and end at a generic point off the walls. Again only $\vartheta_0 = 1$ contributes. This coefficient extraction also appears in [5] and [4, Theorem 3.16]. □

Proof of Theorem 1.2. First reduce to $b \mid c$. Let Y_m denote the cluster variables of $\mathcal{A}_v(c, b)$, with $Y^{(r,s)} = v^{rs}Y_1^rY_2^s$. The assignment $X_m \mapsto Y_{3-m}$, fixing v, defines an R-linear anti-isomorphism: reversing products carries each exchange relation to the corresponding relation with b, c interchanged. It sends normalized monomials $X^{(r,s)}$ to $Y^{(s,r)}$. The greedy recurrence therefore sends $G[A, B]$ to the greedy element with label (B, A) in $\mathcal{A}_v(c, b)$. It also preserves bar invariance, cluster monomials, and nonnegative structure constants, with the multiplication order reversed. Consequently the asserted basis comparison and leastness for (c, b) imply those for (b, c). It remains to treat $b \mid c$.

Apply Lemma 5.2 to the finite expansion (4.1). Theorem 5.1 makes every extracted coefficient nonnegative. Under the bijection γ, these are exactly the coefficients $t_{i,j}$ of (2.7). The root coefficient is one.

Each term in the broken-line product formula [4, Proposition 3.3] is a product of positive wall contributions and a power of v. For a prescribed output label the total increment is fixed, so only finitely many contributions occur. The ordinary greedy basis theorem already gives a finite algebraic expansion of a product of two greedy elements. The exact identification (1.1) and uniqueness in the pointed completion identify its coefficients with those in the broken-line formula. Thus only finitely many output labels occur and all structure constants are in R_+. Bar invariance, cluster monomials, and seed independence are the greedy basis properties proved in [2].

The companion input also gives universal Laurent positivity of every greedy element. Therefore the finite positive expansion of $B[A, B] - G[A, B]$ is positive in every cluster. If another strongly positive basis had the same least property, comparison with the greedy basis in both directions would give equality of every pointed element. □

Remark 5.3. For $b = c = 2$, put

$$Z = X^{(-1,-1)} + X^{(1,-1)} + X^{(-1,1)}.$$

Then $G[2, 2] = Z^2 - 2$, while the triangular basis element is $Z^2 - 1$. Its unit basis coordinate is zero but its initial Laurent constant is one. For the constant coefficient, one may multiply by $X^{(N,N)}$ with N large enough to clear all denominators. The resulting coefficient of the initial cluster monomial $X^{(N,N)}$ is one, as in (5.1).

Appendix A. Second-column quotients and a positive remainder

Throughout this appendix $A, B > 0$; translated lower labels may be arbitrary integers.

Proposition A.1. *Let F be bar invariant, (A,B)-pointed, positive in the initial cluster and its two adjacent clusters, and satisfy $F \le M[A,B]$. For an integer $q \ge 0$ with $cAq \le 2bB$, suppose $e_F(1,q) - e_G(1,q) \in R_+$. Then $e_F(2,q) - e_G(2,q) \in R_+$.*

Proof. The axes of F equal the greedy axes. For fixed q, put $n = B - cq > 0$ and write the row as $\sum_p e_F(p,q)T^p = P_n^{(b)}(T)V_{F,q}(T)$. Multiplication of coefficients through degree two gives

$$[T^2]V_{F,q} = e_F(2,q) - [n]_{v^b} e_F(1,q) + \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{v^b} e_F(0,q).$$

The Gaussian identity used here is $[n]^2 - \begin{bmatrix} n \\ 2 \end{bmatrix}_w = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_w$; it follows from $[n-1]+[n+1] = [2][n]$. By (3.2), the greedy row quotient has coefficient zero at degree two. Hence

$$e_F(2,q) - e_G(2,q) = [n]_{v^b}(e_F(1,q) - e_G(1,q)) + [T^2]V_{F,q} \in R_+.$$

If $n \le 0$ in the same row branch, the greedy coefficient is zero. The equality boundary uses both valid greedy recurrences; $q = 0$ follows from the axes. Points outside the standard rectangle have zero coefficients. □

Proposition A.2. *For $A, B > 0$ and a strongly positive basis $\mathcal{B}$, the difference*

$$B[A, B-1]X_0 - v^{-A}B[A,B]$$

has a finite expansion

$$\sum_{i,j \ge 1} s_{i,j} B[A - bi, B - cj], \qquad s_{i,j} \in R_+.$$

Proof. The element $B[A, B-1]X_0$ is Laurent positive and has a positive $\mathcal{B}$-expansion. Its support is contained in $(-A,-B) + \mathbb{Z}_{\ge 0}(b,0) + \mathbb{Z}_{\ge 0}(0,c)$, and its corner coefficient is v^{-A}. The corner argument bounds its basis labels by (A,B) and shows that the coefficient of $B[A,B]$ is exactly v^{-A}. After removing this single basis term all remaining structure constants are nonnegative. On the column $p = 0$ the product coefficients are $v^{-A}\begin{bmatrix} A \\ q \end{bmatrix}_{v^c}$, and on the row $q = 0$ they are $v^{-A}\begin{bmatrix} B \\ p \end{bmatrix}_{v^b}$. The latter follows from

$$\begin{bmatrix} B \\ p \end{bmatrix}_{v^b} = v^{bp}\begin{bmatrix} B-1 \\ p \end{bmatrix}_{v^b} + v^{b(p-B)}\begin{bmatrix} B-1 \\ p-1 \end{bmatrix}_{v^b}.$$

Both axes of the difference vanish. The finite axis argument used for (2.7), now in the pointed $\mathcal{B}$-basis, makes both label decreases strict. Thus

$$R^0_{A,B} = B[A,B-1]X_0 - v^{-A}B[A,B] = \sum_{i,j \ge 1} s_{i,j} B[A - bi, B - cj], \qquad s_{i,j} \in R_+. \qquad \square$$

Let $H_k(T)$ be its k-th column relative to (A,B). Lemma 2.2, applied to the pointed $\mathcal{B}$-basis, gives

$$\frac{H_k(T)}{P^{(c)}_{[A-bk]_+}(T)} = \sum_{1 \le i \le k,\ j \ge 1} s_{i,j} T^j Q^{\mathcal{B}}_{A-bi, B-cj; k-i}(T) \in R_+[T].$$

It holds for every integer $k \ge 0$, including negative lower labels and zero divisor length, by division of the complete translated columns. The phase in forming the remainder itself is

$$H_k = v^{bk-A} C^{\mathcal{B}}_{A,B-1;k} + v^{bk-A-bB} C^{\mathcal{B}}_{A,B-1;k-1}(v^{bc}T) - v^{-A} C^{\mathcal{B}}_{A,B;k}.$$

For $n = A - bk \geq 0$, the extra Gaussian ratio in its second term is

$$\frac{P^{(c)}_{n+b}(v^{bc}T)}{P^{(c)}_{n}(T)} = P^{(c)}_{b}(v^{c(n+b)}T) \in R_+[T].$$

If $n < 0$, the denominator is one and the numerator is $P^{(c)}_{[n+b]_+}(v^{bc}T)$.

For $F = B[A, B]$, (2.8) gives the second-layer quotient

$$U_2 = Q^F_{A,B;2} - Q^G_{A,B;2} - \sum_{j\geq 1} t_{1,j}T^j Q^G_{A-b,B-cj;1} = \sum_{j\geq 1} t_{2,j}T^j.$$

Theorem 1.2 gives $t_{i,j} \in R_+$, hence $U_2 \in R_+[T]$. Together with the positive greedy column quotients, (2.8) also gives nonnegativity of the column quotient difference and the column difference.

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