

REFLEXIVE EQUIVALENCE AND REFLEXIVE-MINIMAL ALGEBRAS

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ABSTRACT. Let A be a finite-dimensional algebra over a field. A finite-dimensional algebra B over the same field is called reflexively equivalent to A if their categories of reflexive modules are equivalent. We prove that there is a basic algebra $A_{\min}$, unique up to isomorphism, such that B is reflexively equivalent to A if and only if $B \cong \text{End}_{A_{\min}}(A_{\min} \oplus X)$ for some reflexive $A_{\min}$ -module X . Moreover, we compute $A_{\min}$ explicitly. We extend these results to module-finite algebras over henselian local rings of dimension at most one under a dominant dimension condition at minimal primes. We also introduce reflexive modules over additive categories and prove that taking reflexive modules is idempotent: the reflexive modules over the category of reflexive modules form an equivalent category. As an application, we show that reflexive equivalence classes of algebras with finitely many indecomposable reflexive modules correspond bijectively to Morita equivalence classes of algebras whose reflexive modules are projective.

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1. INTRODUCTION

Let A be a finite-dimensional algebra over a field k . For a right A -module M , put $M^* = \text{Hom}_A(M, A)$. We say that M is *reflexive* if the canonical evaluation $M \rightarrow M^{**}$ is an isomorphism [AB], and we denote by $\text{ref } A$ the additive category of finite-dimensional reflexive right A -modules. Two finite-dimensional k -algebras A and B are called *reflexively equivalent* if $\text{ref } A \simeq \text{ref } B$ as k -linear additive categories.

Morita equivalent algebras are clearly reflexively equivalent, and there are many more examples. For instance, take any $X \in \text{ref } A$. Then A and $\text{End}_A(A \oplus X)$ are reflexively equivalent (Corollary 2.15), and we call $\text{End}_A(A \oplus X)$ an *enlargement* of A . If X is indecomposable and nonprojective, then

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$\text{End}_A(A \oplus X)$ has one more simple module than A , so it is not Morita equivalent to A . Thus it is natural to ask the following.

Question 1.1. *When are two finite-dimensional k -algebras reflexively equivalent? How can one describe all algebras in a reflexive equivalence class?*

In this paper, we give a complete answer to these questions. To state it, we use the order on Morita equivalence classes given by enlargements. We denote by $[A]$ the Morita equivalence class of A , and write

$$[A] \leq_{\text{ref}} [B] \iff B \text{ is Morita equivalent to } \text{End}_A(A \oplus X) \text{ for some } X \in \text{ref } A.$$

Then $\leq_{\text{ref}}$ is a partial order on Morita equivalence classes of finite-dimensional k -algebras, and two algebras are reflexively equivalent if and only if they have a common upper bound:

$$\text{ref } A \simeq \text{ref } B \iff \text{there is a finite-dimensional } C \text{ with } [A] \leq_{\text{ref}} [C] \text{ and } [B] \leq_{\text{ref}} [C]$$

(Propositions 2.19 and 2.20). We show that each reflexive equivalence class has the least element with respect to $\leq_{\text{ref}}$, which can be computed explicitly, and that every algebra in the class is an enlargement of it. We call A *reflexive-minimal* if $[A]$ is minimal in its reflexive equivalence class with respect to $\leq_{\text{ref}}$.

The least element is a corner algebra of A determined by simple modules. Let A be a basic finite-dimensional k -algebra, and take a complete set $e_1, \dots, e_n$ of primitive orthogonal idempotents of A . We write $S_i = e_i A / e_i \text{rad } A$ and $L_i = A e_i / (\text{rad } A) e_i$ for the simple right and left modules, and denote by $I_{\text{ref}}(A)$ the set of indices i such that at least one of

$$\text{Hom}_A(S_i, A), \quad \text{Ext}_A^1(S_i, A), \quad \text{Hom}_{A^{\text{op}}}(L_i, A), \quad \text{Ext}_{A^{\text{op}}}^1(L_i, A)$$

is nonzero. Equivalently, S_i or L_i is a direct summand of the socle of the first two terms of the minimal injective coresolution of A_A or of ${}_A A$, respectively (Remark 3.6). Now we put $e_{\min} = \sum_{i \in I_{\text{ref}}(A)} e_i$ and $A_{\min} = e_{\min} A e_{\min}$. If A is not basic, we put $A_{\min} = B_{\min}$ for a basic algebra B Morita equivalent to A . Up to isomorphism, $A_{\min}$ does not depend on these choices (Corollary 3.12(2)). Our first main result is the following.

Theorem A (= Corollary 3.12(2) and Theorem 3.11(1),(2)). *Let A be a basic finite-dimensional k -algebra.*

- (1) *The Morita equivalence class $[A_{\min}]$ is the least element of the reflexive equivalence class of A under $\leq_{\text{ref}}$. In particular, $A_{\min}$ is the unique basic reflexive-minimal algebra in this class, up to isomorphism.*
- (2) *For a basic finite-dimensional k -algebra B , we have $\text{ref } A \simeq \text{ref } B$ if and only if $A_{\min} \cong B_{\min}$.*
- (3) *A finite-dimensional k -algebra B is reflexively equivalent to A if and only if $B \cong \text{End}_{A_{\min}}(A_{\min} \oplus X)$ for some $X \in \text{ref } A_{\min}$.*

Thus reflexive equivalence classes are parametrized by basic reflexive-minimal algebras, and the class of A consists of the algebras $\text{End}_{A_{\min}}(A_{\min} \oplus X)$ for $X \in \text{ref } A_{\min}$. This is an analogue of Morita theory, where the Morita equivalence class of A consists of the algebras $\text{End}_B(P)$ for progenerators P over a basic algebra B Morita equivalent to A . In particular, $A_{\min}$ appears as a corner algebra of every algebra in its class, and it is the unique basic algebra in the class with the smallest number of simple modules (Corollary 3.12(2)). Let us see an example.

Example 1.2. Multiply paths from left to right, and let $A = kQ/(bc, ca)$ and $B = kQ'/(bc, da)$, where

$$Q: \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ c \uparrow & & \swarrow b \\ 3 & & \end{array} \quad Q': \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ d \uparrow & & \downarrow b \\ 4 & \xleftarrow{c} & 3 \end{array}$$

We have $I_{\text{ref}}(A) = I_{\text{ref}}(B) = \{1, 3\}$ and

$$A_{\min} \cong B_{\min} \cong k \left(1 \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{v} \end{array} 3 \right) / (uv, vu).$$

The arrows (u, v) are (ab, c) in $A_{\min}$ and (ab, cd) in $B_{\min}$. Thus Theorem A(2) gives $\text{ref } A \simeq \text{ref } B$, although A and B are not Morita equivalent: they have three and four simple modules, respectively. Examples 3.13 and 3.14 give the computations.

Let us explain the idea of the proof. For a subset T of $\{1, \dots, n\}$, put $e_T = \sum_{i \in T} e_i$. Then $(-)_T$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(e_T A e_T)$ if and only if the maps $A \rightarrow \text{End}_{e_T A e_T}(A e_T)$ and $A^{\text{op}} \rightarrow \text{End}_{(e_T A e_T)^{\text{op}}}(e_T A)$ given by multiplication are bijective, and we show that this happens exactly when $I_{\text{ref}}(A) \subseteq T$ (Theorem 3.5). Hence $[A_{\min}] \leq_{\text{ref}} [A]$. To show that $[A_{\min}]$ is the least element, we prove that every equivalence $F: \text{ref } B \rightarrow \text{ref } A$ satisfies $e_{\min} A \in \text{add } F(B)$. For this, we use simple functors on $\text{ref } A$ (Section 3.4).

In this second step, we regard $\text{ref } A$ as a ring with several objects. In general, for an additive category $\mathcal{C}$, a right $\mathcal{C}$ -module is an additive functor $\mathcal{C}^{\text{op}} \rightarrow \text{Ab}$; we write $\text{Mod } \mathcal{C}$ for the category of right $\mathcal{C}$ -modules and $\text{mod } \mathcal{C}$ for its subcategory of finitely presented ones. The conjugate of a right module F is the left module $F^*(X) = \text{Hom}_{\text{Mod } \mathcal{C}}(F, \mathcal{C}(-, X))$, and the conjugate of a left module is defined dually. We say that F is reflexive if the canonical map $F \rightarrow F^{**}$ is invertible, and define the *reflexive completion* $\text{ref } \mathcal{C}$ to consist of the reflexive functors F such that both F and F^* are finitely presented. Representables give a fully faithful Yoneda functor $\mathcal{C} \rightarrow \text{ref } \mathcal{C}$, and evaluation at A identifies $\text{ref}(\text{proj } A)$ with $\text{ref } A$. We call two additive categories *reflexively equivalent* if their reflexive completions are equivalent. Our second main result is that reflexive completion is idempotent.

Theorem B (= Theorem 2.5). *For every essentially small idempotent-complete additive category $\mathcal{C}$, the Yoneda functor induces an equivalence*

$$\text{ref } \mathcal{C} \xrightarrow{\sim} \text{ref}(\text{ref } \mathcal{C}).$$

We call $\mathcal{C}$ *reflexive-rigid* if its Yoneda functor $\mathcal{C} \rightarrow \text{ref } \mathcal{C}$ is an equivalence. By Theorem B, reflexive completions are exactly reflexive-rigid categories up to equivalence, and $\mathcal{C} \mapsto \text{ref } \mathcal{C}$ induces a bijection (Corollary 2.6)

$$\frac{\{\text{additive categories}\}}{\text{reflexive equivalence}} \longleftrightarrow \frac{\{\text{reflexive-rigid additive categories}\}}{\text{additive equivalence}}. \quad (1.1)$$

where all categories are essentially small, additive and idempotent complete. In contrast, $\mathcal{C} \mapsto \text{mod } \mathcal{C}$ is not idempotent: for $\mathcal{C} = \text{proj } \Delta$ with $\Delta = k[x]/(x^2)$, the category $\text{mod } \mathcal{C} \simeq \text{mod } \Delta$ has one simple object, whereas $\text{mod}(\text{mod } \mathcal{C})$ has two (Example 2.9).

This categorical viewpoint allows us to describe all algebras of a reflexive equivalence class at once, as subcategories of one additive category $\mathcal{E}$. A subcategory $\mathcal{C} \subseteq \mathcal{E}$ is called *reflexively dense* if $X \mapsto \mathcal{E}(-, X)|_{\mathcal{C}}$ induces an equivalence $\mathcal{E} \simeq \text{ref } \mathcal{C}$, and we denote by $\text{ref-dense}(\mathcal{E})$ the poset of reflexively dense subcategories of $\mathcal{E}$ under inclusion. For $\mathcal{E} = \text{ref } A$ with A basic, the greatest element of $\text{ref-dense}(\mathcal{E})$ is $\mathcal{E}$ itself, and the least element is $\text{add}(e_{\min} A)$ (Theorem 3.9). Morita equivalence classes in the reflexive equivalence class of A correspond to the orbits of those reflexively dense subcategories of $\text{ref } A$ that have an additive generator, under k -linear autoequivalences of $\text{ref } A$ (Proposition 2.22).

Next we consider when $\text{ref } A \simeq \text{proj } \Gamma$ holds for some finite-dimensional k -algebra Γ . We say that A has *finite reflexive type* if there are only finitely many isomorphism classes of indecomposable reflexive A -modules; this is exactly the case when such Γ exists (Proposition 2.8(2)). We call A *reflexive-rigid* if $\text{ref } A = \text{proj } A$. If A has finite reflexive type and $\text{add } M = \text{ref } A$, then $\text{ref } \text{End}_A(M) = \text{proj } \text{End}_A(M) \simeq \text{ref } A$ by Theorem B. Thus $A \mapsto \text{End}_A(M)$ induces a bijection (Corollary 5.3)

$$\frac{\left\{ \begin{array}{c} \text{finite-dimensional } k\text{-algebras} \\ \text{of finite reflexive type} \end{array} \right\}}{\text{reflexive equivalence}} \longleftrightarrow \frac{\left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{reflexive-rigid algebras} \end{array} \right\}}{\text{Morita equivalence}}. \quad (1.2)$$

Moreover, A has finite reflexive type if and only if its reflexive equivalence class contains only finitely many Morita equivalence classes (Proposition 5.2), and the class consists of a single Morita equivalence class exactly when $A_{\min}$ is reflexive-rigid (Corollary 5.6). The classification of reflexive-rigid algebras and the recognition of finite reflexive type are left as Problems 5.11 and 5.12.

Our construction of the reflexive-minimal algebra also works over commutative Noetherian base rings of dimension at most one. For a two-sided Noetherian ring Λ , we denote by $\text{ref } \Lambda$ the category of finitely generated reflexive right Λ -modules. Over a commutative Noetherian ring R , we use R -linear functors and equivalences, and define $\leq_{\text{ref}}$ and reflexive-minimality for module-finite R -algebras as above. Since module-finite algebras over a henselian local ring are semiperfect [Hash, p. 88], $\leq_{\text{ref}}$ is a partial order on their Morita equivalence classes (Proposition 2.19(3)). For a two-sided Noetherian ring B , we write $\text{dom. dim } B \geq 2$ if the first two terms of the minimal injective coresolution of B_B

are projective, and for a prime ideal $\mathfrak{p}$ of R , we put $\Lambda_{\mathfrak{p}} = \Lambda \otimes_R R_{\mathfrak{p}}$. Our third main result is the following.

Theorem C (= Theorem 4.5(3),(4)). *Let $(R, \mathfrak{m})$ be a commutative henselian Noetherian local ring with $\dim R \leq 1$, and let $\Lambda \neq 0$ be a module-finite R -algebra. Assume that $\dim \Lambda_{\mathfrak{p}} \geq 2$ for every minimal prime $\mathfrak{p} \neq \mathfrak{m}$ of R with $\Lambda_{\mathfrak{p}} \neq 0$. Then the following hold, with R -linear reflexive equivalence.*

- (1) *The reflexive equivalence class of Λ has a least Morita equivalence class under $\leq_{\text{ref}}$, represented by a unique basic reflexive-minimal algebra $\Lambda_{\min}$ up to R -algebra isomorphism. If Λ is basic, then $\Lambda_{\min} \cong e_{\min} \Lambda e_{\min}$, where $I_{\text{ref}}(\Lambda)$ is defined by the same Hom and Ext^1 conditions on simple modules as $I_{\text{ref}}(A)$ above.*
- (2) *For every module-finite R -algebra Γ , we have*

$$\text{ref } \Gamma \simeq \text{ref } \Lambda \iff \Gamma \cong \text{End}_{\Lambda_{\min}}(\Lambda_{\min} \oplus X) \text{ for some } X \in \text{ref } \Lambda_{\min}.$$

The assumption is automatically satisfied if R is Artinian, and also if $\Lambda_{\mathfrak{p}}$ is self-injective for every minimal prime $\mathfrak{p} \neq \mathfrak{m}$. In particular, if R is a one-dimensional domain with fraction field K , Theorem C applies to every R -order in a self-injective K -algebra, that is, whenever $\Lambda \otimes_R K$ is self-injective (Corollary 4.6). Moreover, the assumption is preserved by R -linear reflexive equivalence (Proposition 4.4). We prove Theorem C as Theorem 4.5 for R of arbitrary dimension, but there the assumption forces $\dim(R/\text{ann}_R \Lambda) \leq 1$.

The situation is different in higher dimensions. For $R = k[[x, y, z]]$, the reflexive equivalence class of R contains R and $\text{End}_R(M)$ for some reflexive R -module M . These are nonisomorphic local algebras, and both are basic and reflexive-minimal, so the class has no least Morita equivalence class; correspondingly, $\text{ref-dense}(\text{ref } R)$ has two minimal elements (Example 4.11). Moreover, for a Hom-finite additive category, minimal reflexively dense subcategories may not exist at all (Theorem A.1).

Remark 1.3. Let us compare our results with previous work.

- For an idempotent $e \in A$, Fuller [Ful, Theorem 4] and Cunningham, Rutter and Turnidge [CRT, Theorem 2.6] characterize the bijectivity of the map $A \rightarrow \text{End}_{eAe}(Ae)$ by the vanishing of Hom and first extensions from the simple right modules annihilated by e into A_A , and Fuller identifies the smallest corner for which this map is bijective [Ful, p. 661]. Our set $I_{\text{ref}}(A)$ is the union of the index sets given by this criterion for A and for A^{op} . Our new point is that $A_{\min}$ is the least among all algebras reflexively equivalent to A , not only among the corner algebras of A .
- The Morita–Tachikawa correspondence [Mor, Tac], in the form recalled in [Han, Theorem 4.1 and Proposition 4.2], describes the finite-dimensional algebras A with $\dim A \geq 2$ as $\text{End}_B(G)$ for generator-cogenerators $G \in \text{mod } B$, with $\text{ref } A \simeq \text{mod } B$. We recover this description from Theorem A: in this case, $A_{\min}$ is the endomorphism algebra of a basic additive generator of $\text{add}(B \oplus DB)$ (Corollary 4.3).
- Hanihara [Han, Theorem 6.3] describes reflexively equivalent algebras as endomorphism algebras of generator-cogenerators in the maximal exact structure on the category of reflexive modules, under the assumption that the first two terms of the minimal injective coresolutions of the right and left regular modules have projective dimension at most one. Theorem A does not need such an assumption.
- Over a normal domain R , Iyama and Reiten describe the equivalences between the categories of those modules over R -algebras that are reflexive as R -modules [IR, Proposition 2.4(2)]. There reflexivity is defined by $\text{Hom}_R(-, R)$, while ours is defined by $\text{Hom}_A(-, A)$.
- Theorem B is an additive analogue of the idempotence of Isbell’s reflexive completion of a category, which uses all presheaves [AL, Corollary 9.4]; in our setting we require F and F^* to be finitely presented.

Organization. Section 2 proves Theorem B and studies reflexively dense subcategories and the order $\leq_{\text{ref}}$. Section 3 proves Theorem A and gives worked computations. Section 4 describes the reflexive equivalence classes whose reflexive completions are module categories (Corollary 4.3), proves Theorem C, and computes examples over discrete valuation rings. It also gives examples in which least or minimal reflexively dense subcategories do not exist. Section 5 relates finite reflexive type to reflexive-rigidity and formulates Problems 5.11 and 5.12. Appendix A proves Theorem A.1, which is used in Example 4.12.

Conventions and notation. Categories are essentially small, additive and idempotent complete. Subcategories are full, replete, additive and closed under direct summands. Modules are right modules unless specified otherwise, and a *finite* module is a finitely generated module. We use the notation $\text{Mod } \mathcal{C}$, $\text{mod } \mathcal{C}$, $\text{ref } \mathcal{C}$ and $\text{ref } A$ of Section 2.1. Linear structures and finiteness hypotheses are stated in each section; equivalences preserve the specified scalars. We reserve $D = \text{Hom}_k(-, k)$ for vector space duality, which differs from the conjugate $(-)^*$. Write $\text{add } X$ for finite direct sums and summands, and $\text{ind } \mathcal{C}$ for the set of isomorphism classes of indecomposable objects. We identify indecomposable objects with their isomorphism classes when using set notation. Endomorphisms are multiplied by composition, $fg = f \circ g$. Products of paths in quiver presentations are written from left to right. In module diagrams, the label i denotes S_i , and radical layers are displayed from top to bottom.

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2. REFLEXIVE COMPLETION

In this section R denotes a commutative ring. For R -linear categories, all functors and equivalences between them are R -linear; the general additive case is $R = \mathbb{Z}$. In applications to k -algebras we use $R = k$. We construct $\text{ref } \mathcal{C}$ and prove that the Yoneda functor $\text{ref } \mathcal{C} \rightarrow \text{ref}(\text{ref } \mathcal{C})$ is an equivalence. For a subcategory $\mathcal{C} \subseteq \mathcal{U} \subseteq \text{ref } \mathcal{C}$, we give a criterion for the functor $Y \mapsto \text{Hom}_{\text{ref } \mathcal{C}}(-, Y)|_{\mathcal{U}}$ to induce an equivalence $\text{ref } \mathcal{C} \rightarrow \text{ref } \mathcal{U}$.

2.1. Conjugate modules. A *right $\mathcal{C}$ -module* is an additive functor $\mathcal{C}^{\text{op}} \rightarrow \text{Ab}$, and $\text{Mod } \mathcal{C}$ denotes the abelian category of right $\mathcal{C}$ -modules. If $\mathcal{C}$ is R -linear, every additive functor has the canonical R -action $rx = F(r1_X)(x)$ on $F(X)$; with these actions it is an R -linear functor to $\text{Mod } R$. Natural transformations respect these actions. Thus additive functors to abelian groups and R -linear functors to R -modules give the same module category. Left $\mathcal{C}$ -modules are right $\mathcal{C}^{\text{op}}$ -modules. For $X \in \mathcal{C}$ put $P_X = \mathcal{C}(-, X) \in \text{Mod } \mathcal{C}$ and $P^X = \mathcal{C}(X, -) \in \text{Mod } \mathcal{C}^{\text{op}}$. By the Yoneda lemma $\text{Hom}_{\text{Mod } \mathcal{C}}(P_X, F) \cong F(X)$, and since $\mathcal{C}$ is idempotent complete, $X \mapsto P_X$ identifies $\mathcal{C}$ with the category of finitely generated projective right $\mathcal{C}$ -modules. A right $\mathcal{C}$ -module F is *finitely generated* if there is an epimorphism $P_{X_0} \rightarrow F$, and *finitely presented* if there is an exact sequence $P_{X_1} \rightarrow P_{X_0} \rightarrow F \rightarrow 0$, and $\text{mod } \mathcal{C}$ denotes the category of finitely presented right $\mathcal{C}$ -modules. We use three standard facts: direct summands of finitely presented modules are finitely presented; the cokernel of a morphism between finitely presented modules is finitely presented; and the kernel of an epimorphism from a finitely presented module onto a finitely presented module is finitely generated.

For $F \in \text{Mod } \mathcal{C}$ define the *conjugate* left $\mathcal{C}$ -module F^* by

$$F^*(X) = \text{Hom}_{\text{Mod } \mathcal{C}}(F, P_X),$$

and for $G \in \text{Mod } \mathcal{C}^{\text{op}}$ define $G^* \in \text{Mod } \mathcal{C}$ by $G^*(X) = \text{Hom}_{\text{Mod } \mathcal{C}^{\text{op}}}(G, P^X)$. The Yoneda lemma gives $(P_X)^* \cong P^X$ and $(P^X)^* \cong P_X$. The *evaluation* $\delta_F: F \rightarrow F^{**}$ sends $x \in F(X)$ to the morphism $F^* \rightarrow P^X$ whose component at Y sends $\varphi \in F^*(Y) = \text{Hom}_{\text{Mod } \mathcal{C}}(F, P_Y)$ to $\varphi_X(x) \in \mathcal{C}(X, Y)$. For $F \in \text{Mod } \mathcal{C}$ and $G \in \text{Mod } \mathcal{C}^{\text{op}}$, both $\text{Hom}_{\text{Mod } \mathcal{C}}(F, G^*)$ and $\text{Hom}_{\text{Mod } \mathcal{C}^{\text{op}}}(G, F^*)$ are identified with the families of bilinear maps $F(X) \times G(Y) \rightarrow \mathcal{C}(X, Y)$ natural in X and Y . The resulting natural isomorphism

$$\text{Hom}_{\text{Mod } \mathcal{C}}(F, G^*) \cong \text{Hom}_{\text{Mod } \mathcal{C}^{\text{op}}}(G, F^*) \quad (2.1)$$

says that the conjugation functor

$$(-)^*: \text{Mod } \mathcal{C} \rightarrow (\text{Mod } \mathcal{C}^{\text{op}})^{\text{op}}$$

is left adjoint to the conjugation functor $(-)^*: (\text{Mod } \mathcal{C}^{\text{op}})^{\text{op}} \rightarrow \text{Mod } \mathcal{C}$. The unit at F is δ_F , and the counit at G is δ_G , read in the opposite category. In particular the triangle identity $(\delta_F)^* \circ \delta_{F^*} = 1_{F^*}$ holds for every $F \in \text{Mod } \mathcal{C}$.

Definition 2.1. A right $\mathcal{C}$ -module F is *reflexive* if δ_F is an isomorphism. The *reflexive completion* of $\mathcal{C}$ is the full subcategory

$$\text{ref } \mathcal{C} = \{F \in \text{mod } \mathcal{C} \mid F^* \in \text{mod } \mathcal{C}^{\text{op}}, \delta_F \text{ is an isomorphism}\}.$$

For a ring A , let $M^* = \text{Hom}_A(M, A)$. We write $\text{ref } A$ for the finitely presented right A -modules M such that M^* is finitely presented as a left module and $M \rightarrow M^{**}$ is invertible.

The following lemma collects the basic properties of $\text{ref } \mathcal{C}$ and identifies $\text{ref}(\text{proj } A)$ with $\text{ref } A$ for a ring A . Part (4) records the algebras for which every finite module is reflexive; it is used in Example 2.9 and in Appendix A.

Lemma 2.2. *Let $\mathcal{C}$ be a category.*

- (1) *The category $\text{ref } \mathcal{C}$ is essentially small and idempotent complete, and $X \mapsto P_X$ is a fully faithful functor $\mathcal{C} \rightarrow \text{ref } \mathcal{C}$. If $\mathcal{C}$ is R -linear over a commutative coherent ring and $\mathcal{C}(X, Y)$ is finitely presented over R for all $X, Y \in \mathcal{C}$, then $\text{Hom}_{\text{Mod } \mathcal{C}}(F, G)$ is finitely presented over R for all $F, G \in \text{ref } \mathcal{C}$. In particular, if $R = k$ and $\mathcal{C}$ is Hom-finite, then $\text{ref } \mathcal{C}$ is Hom-finite.*
- (2) *Conjugation restricts to a duality $(-)^*: \text{ref } \mathcal{C} \rightarrow \text{ref}(\mathcal{C}^{\text{op}})$ whose quasi-inverse is again conjugation.*
- (3) *Let A be a ring. Evaluation at the object A of $\text{proj } A$ induces an equivalence $\text{ref}(\text{proj } A) \simeq \text{ref } A$.*
- (4) *For a finite-dimensional k -algebra A , we have $\text{ref } A = \text{mod } A$ if and only if A is self-injective.*

Proof. (1) Finitely presented modules form a set up to isomorphism. They are closed under finite direct sums and summands, as is the condition that evaluation is invertible. The same applies to their conjugates. Thus $\text{ref } \mathcal{C}$ is essentially small and idempotent complete. The Yoneda lemma gives $(P_X)^* \cong P_X^X$ and δ_{P_X} invertible, so $P_X \in \text{ref } \mathcal{C}$. Over R , presentations show that values of finitely presented functors are finitely presented R -modules. A presentation of F expresses $\text{Hom}(F, G)$ as a kernel between two such values of G ; coherence of R gives the assertion.

(2) Let $F \in \text{ref } \mathcal{C}$. Then $F^* \in \text{mod } \mathcal{C}^{\text{op}}$ and $F^{**} \cong F \in \text{mod } \mathcal{C}$. Since δ_F is invertible, so is $(\delta_F)^*$, and the identity $(\delta_F)^* \circ \delta_{F^*} = 1_{F^*}$ shows that δ_{F^*} is invertible. Thus $F^* \in \text{ref}(\mathcal{C}^{\text{op}})$. The evaluations give natural isomorphisms from the identity functors to the composites of the two conjugations.

(3) Since $\text{End}_A(A) \cong A$ by left multiplication, evaluation at A is an equivalence $\text{Mod}(\text{proj } A) \rightarrow \text{Mod } A$ sending P_X to X ; under it, finitely presented modules correspond to finitely presented A -modules. The same holds for left modules. For $F \in \text{Mod}(\text{proj } A)$ we have $F^*(A) = \text{Hom}(F, P_A) \cong \text{Hom}_A(F(A), A)$, and under these identifications δ_F at A is the evaluation map $F(A) \rightarrow F(A)^{**}$. Hence $F \in \text{ref}(\text{proj } A)$ if and only if $F(A) \in \text{ref } A$.

(4) Suppose that A is self-injective. The modules A_A and ${}_A A$ are injective, so $M \mapsto M^*$ is exact on right and on left modules, and $M \mapsto M^{**}$ is exact. Let $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a projective presentation. The evaluations of P_0 and P_1 are isomorphisms, so the five lemma shows that δ_M is an isomorphism. Conversely, suppose that $\text{ref } A = \text{mod } A$, and let I be an indecomposable injective module. Choose a surjection $Q \rightarrow I^*$ from a finitely generated projective left module Q . Applying $(-)^*$ gives an injection $I \cong I^{**} \rightarrow Q^*$ into a projective module. This embedding splits, so every indecomposable injective is projective. There are equally many isomorphism classes of indecomposable injectives and projectives, namely as many as simple modules, so the indecomposable injectives are all the indecomposable projectives; hence A is self-injective. $\square$

2.2. Restriction and idempotence. Let $\mathcal{X}$ be an additive category and $\mathcal{V} \subseteq \mathcal{X}$ a subcategory. For $Y \in \mathcal{X}$ write

$$\Phi_{\mathcal{V}}(Y) = \mathcal{X}(-, Y)|_{\mathcal{V}}, \quad \Phi^{\mathcal{V}}(Y) = \mathcal{X}(Y, -)|_{\mathcal{V}}.$$

We call $\Phi_{\mathcal{V}}: \mathcal{X} \rightarrow \text{Mod } \mathcal{V}$ the *restricted Yoneda functor*. Section 2.3 uses this notation for subcategories of an arbitrary category. We identify $\mathcal{C}$ with its Yoneda image in $\mathcal{E} = \text{ref } \mathcal{C}$. For $\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{E}$, the next theorem describes $\text{ref } \mathcal{U}$ as a full subcategory of $\mathcal{E}$. Taking $\mathcal{U} = \mathcal{E}$ will prove idempotence.

Theorem 2.3. *Let $\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{E} = \text{ref } \mathcal{C}$.*

- (1) *The functors $\Phi_{\mathcal{U}}: \mathcal{E} \rightarrow \text{Mod } \mathcal{U}$ and $\Phi^{\mathcal{U}}: \mathcal{E}^{\text{op}} \rightarrow \text{Mod } \mathcal{U}^{\text{op}}$ are fully faithful. There are natural isomorphisms $\Phi_{\mathcal{U}}(Y)^* \cong \Phi^{\mathcal{U}}(Y)$ and $\Phi^{\mathcal{U}}(Y)^* \cong \Phi_{\mathcal{U}}(Y)$, and $\delta_{\Phi_{\mathcal{U}}(Y)}$ is an isomorphism for every $Y \in \mathcal{E}$.*
- (2) *For every $F \in \text{ref } \mathcal{U}$, the restriction $F|_{\mathcal{C}}$ lies in $\mathcal{E}$ and $F \cong \Phi_{\mathcal{U}}(F|_{\mathcal{C}})$.*
- (3) *The functor $\Phi_{\mathcal{U}}$ restricts to an equivalence from the full subcategory of those $Y \in \mathcal{E}$ with $\Phi_{\mathcal{U}}(Y) \in \text{mod } \mathcal{U}$ and $\Phi^{\mathcal{U}}(Y) \in \text{mod } \mathcal{U}^{\text{op}}$ onto $\text{ref } \mathcal{U}$.*

Proof. Let $\iota^*: \text{Mod } \mathcal{U} \rightarrow \text{Mod } \mathcal{C}$ be restriction. Its right adjoint is $(\iota_* G)(X) = \text{Hom}_{\text{Mod } \mathcal{C}}(X, G)$, where $X \in \mathcal{U}$ is viewed as a $\mathcal{C}$ -module via $\mathcal{U} \subseteq \text{ref } \mathcal{C}$. Yoneda gives $\iota^* \iota_* G \cong G$. The adjunction

isomorphism is

$$\mathrm{Hom}_{\mathrm{Mod}\mathcal{U}}(F, \iota_* G) \cong \mathrm{Hom}_{\mathrm{Mod}\mathcal{C}}(\iota^* F, G). \quad (2.2)$$

Similarly, since $\mathcal{U}(X, c) \cong X^*(c)$ for $X \in \mathcal{U}$ and $c \in \mathcal{C}$, restriction $\mathrm{Mod}\mathcal{U}^{\mathrm{op}} \rightarrow \mathrm{Mod}\mathcal{C}^{\mathrm{op}}$ has the right adjoint ι'_* given by

$$(\iota'_* G')(X) = \mathrm{Hom}_{\mathrm{Mod}\mathcal{C}^{\mathrm{op}}}(X^*, G'),$$

and its counit is invertible.

(1) We have $\Phi_{\mathcal{U}}(Y) = \iota_* Y$. The duality of Lemma 2.2(2) gives $\mathcal{E}(Y, X) \cong \mathrm{Hom}_{\mathrm{Mod}\mathcal{C}^{\mathrm{op}}}(X^*, Y^*)$, so $\Phi^{\mathcal{U}}(Y) = \iota'_* Y^*$. Since their counits are invertible, the functors ι_* and ι'_* are fully faithful, and conjugation $\mathcal{E}^{\mathrm{op}} \rightarrow \mathrm{ref}(\mathcal{C}^{\mathrm{op}})$ is an equivalence by Lemma 2.2(2). Hence $\Phi_{\mathcal{U}}$ and $\Phi^{\mathcal{U}}$ are fully faithful. Consequently, for $X \in \mathcal{U}$,

$$\Phi_{\mathcal{U}}(Y)^*(X) = \mathrm{Hom}(\Phi_{\mathcal{U}}(Y), \Phi_{\mathcal{U}}(X)) \cong \mathcal{E}(Y, X) = \Phi^{\mathcal{U}}(Y)(X).$$

The opposite calculation gives $\Phi^{\mathcal{U}}(Y)^* \cong \Phi_{\mathcal{U}}(Y)$. At $X \in \mathcal{U}$, evaluation sends $y \in \mathcal{E}(X, Y)$ to the natural transformation whose component at $Z \in \mathcal{U}$ is

$$\mathcal{E}(Y, Z) \longrightarrow \mathcal{E}(X, Z), \quad a \longmapsto a \circ y.$$

Under the isomorphism $\mathrm{Hom}(\Phi^{\mathcal{U}}(Y), \Phi^{\mathcal{U}}(X)) \cong \mathcal{E}(X, Y)$ given by full faithfulness of $\Phi^{\mathcal{U}}$, this transformation corresponds to y . Hence $\delta_{\Phi_{\mathcal{U}}(Y)}$ is invertible.

(2) Put $G = F|_{\mathcal{C}}$. Since $\mathcal{U}(-, c) = \iota_* P_c$ for $c \in \mathcal{C}$, (2.2) identifies $F^*|_{\mathcal{C}}$ with G^* . Since $\mathcal{U}(c, -) = \iota'_* P^c$ for $c \in \mathcal{C}$, the adjunction for left modules identifies $F^{**}|_{\mathcal{C}}$ with G^{**} . These identifications are restriction of natural transformations, so they identify $\delta_F|_{\mathcal{C}}$ with δ_G . Hence δ_G is invertible. Restricting presentations of F and F^* gives exact sequences

$$U_1 \longrightarrow U_0 \longrightarrow G \longrightarrow 0, \quad W_1^* \longrightarrow W_0^* \longrightarrow G^* \longrightarrow 0$$

where $U_i, W_i \in \mathcal{U}$ are regarded as right $\mathcal{C}$ -modules via $\mathcal{U} \subseteq \mathrm{ref}\mathcal{C}$, and W_i^* are their conjugates over $\mathcal{C}$. Cokernels of maps between finitely presented modules are finitely presented, so $G \in \mathrm{ref}\mathcal{C}$. Finally, naturally in $X \in \mathcal{U}$,

$$F(X) \cong \mathrm{Hom}(F^*, \Phi^{\mathcal{U}}(X)) \cong \mathrm{Hom}_{\mathrm{Mod}\mathcal{C}^{\mathrm{op}}}(G^*, X^*) \cong \mathcal{E}(X, G).$$

Here the three isomorphisms use reflexivity of F , the adjunction for left modules, and the conjugate duality of Lemma 2.2(2), respectively. Thus $F \cong \Phi_{\mathcal{U}}(G)$.

(3) By (1), finite presentation of both functors gives $\Phi_{\mathcal{U}}(Y) \in \mathrm{ref}\mathcal{U}$. By (2), every object of $\mathrm{ref}\mathcal{U}$ arises in this way. $\square$

We now apply Theorem 2.3 with $\mathcal{U} = \mathrm{ref}\mathcal{C}$. We first define reflexive equivalence and reflexive-rigidity for categories.

Definition 2.4. Two categories $\mathcal{C}$ and $\mathcal{C}'$ are *reflexively equivalent* if $\mathrm{ref}\mathcal{C} \simeq \mathrm{ref}\mathcal{C}'$. A category $\mathcal{C}$ is *reflexive-rigid* if its Yoneda functor $\mathcal{C} \rightarrow \mathrm{ref}\mathcal{C}$ is an equivalence, that is, every object of $\mathrm{ref}\mathcal{C}$ is representable. Two rings A and B are *reflexively equivalent* if $\mathrm{ref}A \simeq \mathrm{ref}B$, and a ring A is *reflexive-rigid* if $\mathrm{ref}A = \mathrm{proj}A$.

By Lemma 2.2(3), two rings A and B are reflexively equivalent if and only if the categories $\mathrm{proj}A$ and $\mathrm{proj}B$ are, and A is reflexive-rigid if and only if $\mathrm{proj}A$ is.

Theorem 2.5. *For every category $\mathcal{C}$, the Yoneda functor $\mathrm{ref}\mathcal{C} \rightarrow \mathrm{ref}(\mathrm{ref}\mathcal{C})$ is an equivalence.*

Proof. Take $\mathcal{U} = \mathcal{E} = \mathrm{ref}\mathcal{C}$ in Theorem 2.3. Then $\Phi_{\mathcal{U}}(Y)$ and $\Phi^{\mathcal{U}}(Y)$ are representable, hence finitely presented, and $\Phi_{\mathcal{U}}$ is the Yoneda functor. Theorem 2.3(3) shows that it is an equivalence onto $\mathrm{ref}\mathcal{E}$. $\square$

Corollary 2.6. *A category is equivalent to the reflexive completion of some category if and only if it is reflexive-rigid. Reflexive completion induces a bijection*

$$\frac{\{\text{R-linear additive categories}\}}{\text{reflexive equivalence}} \longleftrightarrow \frac{\{\text{reflexive-rigid R-linear additive categories}\}}{\text{R-linear equivalence}}.$$

The forward map is $[\mathcal{C}] \mapsto [\mathrm{ref}\mathcal{C}]$, and its inverse sends a reflexive-rigid category to its reflexive equivalence class. For $R = k$, the bijection restricts to Hom-finite categories.

Proof. Theorem 2.5 shows that $\text{ref } \mathcal{C}$ is reflexive-rigid and reflexively equivalent to $\mathcal{C}$. Conversely, a reflexive-rigid category $\mathcal{E}$ is equivalent to $\text{ref } \mathcal{E}$. If $\mathcal{E}$ and $\mathcal{E}'$ are reflexive-rigid, then $\text{ref } \mathcal{E} \simeq \text{ref } \mathcal{E}'$ if and only if $\mathcal{E} \simeq \mathcal{E}'$. This proves that the two maps are mutually inverse. The last assertion follows from Lemma 2.2(1). $\square$

Remark 2.7. The first assertion of Corollary 2.6 has the following analogue for Isbell's reflexive completion [Isb, AL], which is built from all presheaves: a category is equivalent to such a completion exactly when every functor on it that is reflexive in Isbell's sense is representable [AL, Corollary 9.5]. Avery and Leinster call these categories *reflexively complete*. In our additive setting, we require finite presentation of both F and F^* . Finite presentation of F makes $\text{ref } \mathcal{C}$ essentially small (Lemma 2.2(1)), and finite presentation of F^* makes conjugation a duality between $\text{ref } \mathcal{C}$ and $\text{ref } (\mathcal{C}^{\text{op}})$ (Lemma 2.2(2)).

An *additive generator* of $\mathcal{E}$ is an object G with $\mathcal{E} = \text{add } G$. Its existence decides whether $\mathcal{E}$ is equivalent to $\text{proj } \Gamma$ for an R -algebra Γ .

Proposition 2.8. *Let $\mathcal{E} = \text{ref } \mathcal{C}$.*

- (1) *The category $\mathcal{E}$ has an additive generator if and only if $\mathcal{E} \simeq \text{proj } \Gamma$ for a reflexive-rigid R -algebra Γ . In this case Γ is unique up to R -linear Morita equivalence, and one can take $\Gamma = \text{End}_{\mathcal{E}}(G)$ for any additive generator G .*
- (2) *Suppose $R = k$ and $\mathcal{C}$ is Hom-finite. The conditions in (1) hold if and only if $\text{ind } \mathcal{E}$ is finite. Every algebra Γ with $\mathcal{E} \simeq \text{proj } \Gamma$ is then finite-dimensional over k .*

Proof. (1) If $\mathcal{E} = \text{add } G$, the functor $\text{Hom}_{\mathcal{E}}(G, -)$ gives an equivalence $\mathcal{E} \simeq \text{proj } \Gamma$ for $\Gamma = \text{End}_{\mathcal{E}}(G)$. Under this equivalence and Lemma 2.2(3), the Yoneda functor $\mathcal{E} \rightarrow \text{ref } \mathcal{E}$ corresponds to the inclusion $\text{proj } \Gamma \subseteq \text{ref } \Gamma$. Theorem 2.5 therefore gives $\text{ref } \Gamma = \text{proj } \Gamma$. Conversely, $\text{proj } \Gamma = \text{add } \Gamma$ has an additive generator. Two such algebras have equivalent projective categories, hence are Morita equivalent.

(2) By Lemma 2.2(1), $\mathcal{E}$ is Hom-finite and idempotent complete, hence Krull–Schmidt. It has an additive generator precisely when it has finitely many indecomposable isomorphism classes. For any equivalence $\text{proj } \Gamma \simeq \mathcal{E}$, the image G of Γ_{Γ} satisfies $\Gamma \cong \text{End}_{\mathcal{E}}(G)$, which is finite-dimensional. $\square$

Example 2.9. Let $\Delta = k[x]/(x^2)$, $S = \Delta/(x)$ and $\mathcal{E} = \text{mod } \Delta$. Every finite Δ -module is a direct sum of copies of Δ and S . Thus $\mathcal{E} = \text{add}(\Delta \oplus S)$, and evaluation at $\Delta \oplus S$ gives

$$\text{mod } \mathcal{E} \simeq \text{mod } \text{End}_{\Delta}(\Delta \oplus S).$$

The endomorphism algebra has two simple modules, one for each indecomposable summand, whereas the abelian category $\mathcal{E}$ has only one simple object. Hence $\text{mod } \mathcal{E}$ is not equivalent to $\mathcal{E}$. Thus applying $\mathcal{C} \mapsto \text{mod } \mathcal{C}$ to $\mathcal{C} = \text{proj } \Delta$ twice changes the category, since $\text{mod}(\text{proj } \Delta) \simeq \mathcal{E}$. Since Δ is self-injective, Lemma 2.2(4) gives $\text{ref } \Delta = \mathcal{E}$, so $\text{ref}(\text{proj } \Delta) \simeq \mathcal{E}$ by Lemma 2.2(3), and Theorem 2.5 gives $\text{ref } \mathcal{E} \simeq \text{ref}(\text{ref}(\text{proj } \Delta)) \simeq \text{ref}(\text{proj } \Delta) \simeq \mathcal{E}$.

Example 2.10. Even for a Hom-finite category $\mathcal{C}$, a module F with F and F^* finitely generated need not be finitely presented. Thus using finite generation in Definition 2.1 would give a different category. Let $\mathcal{C}$ be the additive hull of the k -linear category

$$c_i \longrightarrow a \longrightarrow b \longrightarrow d \quad (i \geq 1),$$

with all paths of length two zero. Each vertex has endomorphism ring k , so this hull is Krull–Schmidt. Let F take the value k at a and zero at every other vertex. Then

$$0 \longrightarrow \bigoplus_{i \geq 1} P_{c_i} \longrightarrow P_a \longrightarrow F \longrightarrow 0.$$

Thus F is finitely generated but not finitely presented. A morphism $F \rightarrow P_X$ is an element of $\mathcal{C}(a, X)$ annihilated by composition with every arrow $c_i \rightarrow a$, and up to scalars the only nonzero such element is the arrow $a \rightarrow b$. Hence F^* is the left simple functor at b and has the presentation $0 \rightarrow P^d \rightarrow P^b \rightarrow F^* \rightarrow 0$. Its conjugate is F , and the evaluation of F is invertible. Both F and F^* are therefore finitely generated, but $F \notin \text{ref } \mathcal{C}$.

2.3. Reflexively dense subcategories. For a fixed category $\mathcal{E}$, we consider subcategories $\mathcal{C}$ for which the restricted Yoneda functor $\Phi_{\mathcal{C}}$ induces an equivalence $\mathcal{E} \rightarrow \text{ref } \mathcal{C}$. For $\mathcal{E} = \text{ref } A$, these include the essential images of $\text{proj } B$ under equivalences $\text{ref } B \simeq \text{ref } A$, for the rings B reflexively equivalent to A . Section 2.4 uses these images to compare such rings B .

Definition 2.11. Let $\mathcal{E}$ be a category. A subcategory $\mathcal{C} \subseteq \mathcal{E}$ is *reflexively dense* in $\mathcal{E}$ if the restricted Yoneda functor induces an equivalence $\Phi_{\mathcal{C}}: \mathcal{E} \xrightarrow{\sim} \text{ref } \mathcal{C}$. These subcategories form a poset $\text{ref-dense}(\mathcal{E})$ under inclusion.

The following lemma records the basic properties of reflexively dense subcategories.

Lemma 2.12. *Let $\mathcal{E}$ be a category.*

- (1) *For every category $\mathcal{C}$, both the Yoneda image of $\mathcal{C}$ and $\text{ref } \mathcal{C}$ are reflexively dense subcategories of $\text{ref } \mathcal{C}$. In particular $\text{ref } \mathcal{C}$ is the greatest element of $\text{ref-dense}(\text{ref } \mathcal{C})$.*
- (2) *If $\mathcal{C} \in \text{ref-dense}(\mathcal{E})$, then $\Phi^{\mathcal{C}}$ is an equivalence $\mathcal{E}^{\text{op}} \rightarrow \text{ref}(\mathcal{C}^{\text{op}})$. In particular $\mathcal{C}^{\text{op}} \in \text{ref-dense}(\mathcal{E}^{\text{op}})$.*
- (3) *An equivalence $\Psi: \mathcal{E} \rightarrow \mathcal{E}'$ induces an isomorphism of posets $\text{ref-dense}(\mathcal{E}) \rightarrow \text{ref-dense}(\mathcal{E}')$ sending $\mathcal{C}$ to the subcategory of objects isomorphic to objects of $\Psi(\mathcal{C})$.*
- (4) *Two categories are reflexively equivalent if and only if they are equivalent to two reflexively dense subcategories of one category.*
- (5) *For every ring A , $\text{proj } A \in \text{ref-dense}(\text{ref } A)$.*

Proof. (1) By the Yoneda lemma, the functor $\Phi_{\mathcal{C}}: \text{ref } \mathcal{C} \rightarrow \text{Mod } \mathcal{C}$ is isomorphic to the inclusion of $\text{ref } \mathcal{C}$ into $\text{Mod } \mathcal{C}$. For $\text{ref } \mathcal{C}$ itself use Theorem 2.5.

(2) Under $\mathcal{E} \simeq \text{ref } \mathcal{C}$, Theorem 2.3(1) identifies $\Phi^{\mathcal{C}}$ with $\Phi_{\mathcal{C}}$ followed by conjugation.

(3) Transport of functors along an equivalence preserves conjugation, finite presentation, and evaluation, and hence reflexive density.

(4) Transport the Yoneda image of $\mathcal{C}'$ into $\text{ref } \mathcal{C}$ using the given equivalence. Conversely, two reflexively dense subcategories have equivalent completions by definition.

(5) Apply (1) to $\text{proj } A$ and use Lemma 2.2(3). $\square$

Theorem 2.3(3) gives the following criterion for $\mathcal{U} \in \text{ref-dense}(\mathcal{E})$ when $\mathcal{U}$ contains a reflexively dense subcategory.

Proposition 2.13. *Let $\mathcal{C} \in \text{ref-dense}(\mathcal{E})$ and let $\mathcal{U}$ be a subcategory with $\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{E}$. Then $\mathcal{U} \in \text{ref-dense}(\mathcal{E})$ if and only if $\Phi_{\mathcal{U}}(Y) \in \text{mod } \mathcal{U}$ and $\Phi^{\mathcal{U}}(Y) \in \text{mod } \mathcal{U}^{\text{op}}$ for every $Y \in \mathcal{E}$.*

Proof. The equivalence $\Phi_{\mathcal{C}}: \mathcal{E} \rightarrow \text{ref } \mathcal{C}$ identifies $\mathcal{U}$ with a subcategory of $\text{ref } \mathcal{C}$ containing the image of the Yoneda functor, and $\Phi_{\mathcal{U}}$ with the functor $\Phi_{\mathcal{U}'}$ of Theorem 2.3, where $\mathcal{U}'$ is the image of $\mathcal{U}$. Apply Theorem 2.3(3) to $\mathcal{U}'$. $\square$

We next show that $\text{add}(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m)$ is reflexively dense whenever each $\mathcal{C}_i$ is reflexively dense. Under additional finiteness hypotheses, adjoining finitely many arbitrary objects also preserves reflexive density; this gives Corollary 2.15.

Proposition 2.14. *Let $\mathcal{C}_1, \dots, \mathcal{C}_m \in \text{ref-dense}(\mathcal{E})$, where $m \geq 1$.*

- (1) *The subcategory $\text{add}(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m)$ is reflexively dense.*
- (2) *Suppose $\mathcal{E}$ is linear over a commutative coherent ring R and $\mathcal{E}(X, Y)$ is finitely presented over R for all $X, Y \in \mathcal{E}$. For any $X_1, \dots, X_r \in \mathcal{E}$, the subcategory $\text{add}(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m \cup \{X_1, \dots, X_r\})$ is reflexively dense.*

Proof. (1) Put $\mathcal{U} = \text{add}(\bigcup_i \mathcal{C}_i)$. A right $\mathcal{U}$ -module is finitely generated whenever its restrictions to all $\mathcal{C}_i$ are: take the sum of their finite sets of generators. For $H = \Phi_{\mathcal{U}}(Y)$ each restriction is finitely presented, so choose an epimorphism $\mathcal{U}(-, W) \rightarrow H$. Its kernel restricts to a finitely generated module on every $\mathcal{C}_i$, since both terms restrict to finitely presented modules. Hence the kernel is finitely generated and H is finitely presented. The argument for left modules is the same. Apply Proposition 2.13.

(2) Put $\mathcal{U}' = \text{add}(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m \cup \{X_1, \dots, X_r\})$. A right $\mathcal{U}'$ -module H' is finitely generated whenever its restrictions to all $\mathcal{C}_i$ are finitely generated and every $H'(X_j)$ is a finitely generated R -module: finitely many R -generators of $H'(X_j)$ define a morphism from a finite direct sum of copies of $\mathcal{U}'(-, X_j)$ to H'

which is surjective at X_j . For $H = \Phi_{\mathcal{U}}(Y)$ the R -modules $H(X_j) = \mathcal{E}(X_j, Y)$ are finitely presented, so we can choose an epimorphism $\mathcal{U}'(-, W) \rightarrow H$. Let K be its kernel. As in (1), each restriction $K|_{\mathcal{C}_i}$ is finitely generated. Each $K(X_j)$ is the kernel of an R -linear map $\mathcal{E}(X_j, W) \rightarrow \mathcal{E}(X_j, Y)$ between finitely presented R -modules, hence is finitely generated because R is coherent. Thus K is finitely generated and H is finitely presented. Apply the same argument to left modules and use Proposition 2.13. $\square$

For finite-dimensional k -algebras, the next corollary can also be derived from results of Ma and Sauter [MS, Lemmas 2.1, 2.2, 2.9 and 3.3].

Corollary 2.15. *Let R be commutative coherent and A an R -algebra finitely presented as an R -module. For $X \in \text{ref } A$, the functor $\text{Hom}_A(A \oplus X, -)$ induces an equivalence*

$$\text{ref } A \simeq \text{ref } \text{End}_A(A \oplus X).$$

In particular this holds for finite-dimensional k -algebras and for module-finite algebras over commutative Noetherian rings.

Proof. Finite presentations over A give finite presentations over R . For finitely presented A -modules M, N , an A -presentation of M expresses $\text{Hom}_A(M, N)$ as a kernel between finite sums of N ; hence it is finitely presented over R . Since $\text{proj } A \in \text{ref-dense}(\text{ref } A)$ by Lemma 2.12(5), Proposition 2.14(2) gives $\text{add}(A \oplus X) \in \text{ref-dense}(\text{ref } A)$. Finally, evaluation at $A \oplus X$ gives $\text{add}(A \oplus X) \simeq \text{proj } \text{End}_A(A \oplus X)$, and Lemma 2.2(3) identifies $\text{ref}(\text{proj } \text{End}_A(A \oplus X))$ with $\text{ref } \text{End}_A(A \oplus X)$. $\square$

When $\mathcal{E}$ has weak kernels and weak cokernels, reflexive density can be tested by approximations. Recall that a *weak kernel* of a morphism $f: X \rightarrow Y$ in $\mathcal{E}$ is a morphism $K \rightarrow X$ for which $\mathcal{E}(-, K) \rightarrow \mathcal{E}(-, X) \rightarrow \mathcal{E}(-, Y)$ is exact; weak cokernels are defined dually. As for rings, a category $\mathcal{C}$ is *two-sided coherent* if it has weak kernels and weak cokernels; for $\mathcal{C} = \text{proj } A$ this means that the ring A is left and right coherent. A subcategory is *functorially finite* if every object has a right approximation by it and a left approximation by it. A right approximation $U \rightarrow Y$ means that all maps from objects of the subcategory to Y factor through it; the left condition is dual.

Proposition 2.16. *Let $\mathcal{C} \in \text{ref-dense}(\mathcal{E})$ and $\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{E}$. If $\mathcal{E}$ is two-sided coherent, the following are equivalent.*

- (1) $\mathcal{U} \in \text{ref-dense}(\mathcal{E})$.
- (2) $\mathcal{U}$ is functorially finite in $\mathcal{E}$.
- (3) $\mathcal{U}$ is two-sided coherent.

Proof. (1) $\Rightarrow$ (2): Finite generators of $\Phi_{\mathcal{U}}(Y)$ and $\Phi^{\mathcal{U}}(Y)$ give the two approximations.

(2) $\Rightarrow$ (1): Choose a right approximation $U_0 \rightarrow Y$, a weak kernel $W \rightarrow U_0$ in $\mathcal{E}$, and a right approximation $U_1 \rightarrow W$. Then $\mathcal{U}(-, U_1) \rightarrow \mathcal{U}(-, U_0) \rightarrow \Phi_{\mathcal{U}}(Y) \rightarrow 0$ is a presentation. The opposite argument gives a presentation of $\Phi^{\mathcal{U}}(Y)$. Apply Proposition 2.13.

(2) $\Rightarrow$ (3): Approximate in $\mathcal{U}$ a weak kernel or weak cokernel computed in $\mathcal{E}$.

(3) $\Rightarrow$ (1): Identify $\mathcal{E}$ with $\text{ref } \mathcal{C}$. Write $Y \in \mathcal{E}$ as the kernel of a map between representable $\mathcal{C}$ -modules by applying $(-)^*$ to a presentation of Y^* . It is also a kernel in $\mathcal{E}$, and the two representing objects belong to $\mathcal{U}$. Thus $\Phi_{\mathcal{U}}(Y)$ is a kernel between representable $\mathcal{U}$ -modules. Two successive weak kernels in $\mathcal{U}$ give a finite presentation of a kernel between representables, so $\Phi_{\mathcal{U}}(Y)$ is finitely presented. Use weak cokernels for $\Phi^{\mathcal{U}}(Y)$ and apply Proposition 2.13. $\square$

We next give sufficient conditions for a subcategory $\mathcal{C} \subseteq \mathcal{V}$ to be reflexively dense in $\mathcal{V}$. Part (2) will be applied with $\mathcal{V}$ abelian in Proposition 4.2 and Theorem A.1.

Proposition 2.17. *Let $\mathcal{V}$ be a category and $\mathcal{C} \subseteq \mathcal{V}$ a subcategory. Suppose that the functors $\Phi_{\mathcal{C}}: \mathcal{V} \rightarrow \text{Mod } \mathcal{C}$ and $\Phi^{\mathcal{C}}: \mathcal{V}^{\text{op}} \rightarrow \text{Mod } \mathcal{C}^{\text{op}}$ are fully faithful and take values in $\text{mod } \mathcal{C}$ and $\text{mod } \mathcal{C}^{\text{op}}$, respectively.*

- (1) *The functor $\Phi_{\mathcal{C}}$ takes values in $\text{ref } \mathcal{C}$.*
- (2) *If moreover every morphism of $\mathcal{C}$ has a kernel in $\mathcal{V}$, then $\Phi_{\mathcal{C}}: \mathcal{V} \rightarrow \text{ref } \mathcal{C}$ is an equivalence, that is, $\mathcal{C} \in \text{ref-dense}(\mathcal{V})$.*

Proof. (1) Full faithfulness identifies $\Phi_{\mathcal{C}}(Y)^* \cong \Phi^{\mathcal{C}}(Y)$ and $\Phi^{\mathcal{C}}(Y)^* \cong \Phi_{\mathcal{C}}(Y)$ by composition. Under these identifications evaluation sends $y: X \rightarrow Y$ to $(a: Y \rightarrow Z) \mapsto a \circ y$, as in the proof of Theorem 2.3(1), and hence is invertible. The assumed finite presentations therefore give $\Phi_{\mathcal{C}}(Y) \in \text{ref } \mathcal{C}$. (2) For $F \in \text{ref } \mathcal{C}$, apply $(-)^*$ to a presentation $P^{W_1} \rightarrow P^{W_0} \rightarrow F^* \rightarrow 0$. If $g: W_0 \rightarrow W_1$ is the corresponding morphism, this gives $0 \rightarrow F \rightarrow P_{W_0} \xrightarrow{P_g} P_{W_1}$. For its kernel Z in $\mathcal{V}$ we have $F \cong \Phi_{\mathcal{C}}(Z)$. Thus $\Phi_{\mathcal{C}}$ is essentially surjective as well as fully faithful. $\square$

2.4. An order on Morita equivalence classes. We define a relation on Morita equivalence classes by requiring an equivalence $F: \text{ref } A \rightarrow \text{ref } B$ to satisfy $F(\text{proj } A) \subseteq \text{proj } B$. Proposition 2.14(1) will give a common upper bound for any two reflexively equivalent rings. For module-finite algebras over a Noetherian ring, Proposition 2.20(2) shows that $[A] \leq_{\text{ref}} [B]$ holds if and only if B is Morita equivalent to $\text{End}_A(A \oplus X)$ for some $X \in \text{ref } A$, as in the introduction.

Definition 2.18. For R -algebras A and B , write $[A] \leq_{\text{ref}} [B]$ if there is an R -linear equivalence $F: \text{ref } A \rightarrow \text{ref } B$ such that $F(P) \in \text{proj } B$ for every $P \in \text{proj } A$. Here $[A]$ denotes the R -linear Morita equivalence class of A .

Recall that a ring A is semiperfect precisely when $\text{proj } A$ is Krull–Schmidt. In this case $\text{proj } A$ has finitely many indecomposable objects up to isomorphism. We use this finiteness to prove that $\leq_{\text{ref}}$ is antisymmetric for semiperfect rings.

Proposition 2.19. *Let A and B be R -algebras.*

- (1) *The relation $\leq_{\text{ref}}$ is a preorder on R -linear Morita equivalence classes.*
- (2) *The categories $\text{ref } A$ and $\text{ref } B$ are R -linearly equivalent if and only if there is an R -algebra C such that $[A] \leq_{\text{ref}} [C]$ and $[B] \leq_{\text{ref}} [C]$. Given $F: \text{ref } A \xrightarrow{\sim} \text{ref } B$, one can take*

$$C = \text{End}_B(B \oplus F(A)).$$

If A and B are semiperfect, then this C is semiperfect.

- (3) *The relation $\leq_{\text{ref}}$ restricts to a partial order on Morita equivalence classes of semiperfect R -algebras.*

Proof. (1) Morita equivalences induce equivalences of reflexive categories carrying projectives onto projectives. Thus the relation is well defined on Morita equivalence classes. Identity functors and composition give reflexivity and transitivity.

(2) Suppose $F: \text{ref } A \rightarrow \text{ref } B$ is an equivalence. By Lemma 2.12(3),(5), both $\text{proj } B$ and $F(\text{proj } A)$ are reflexively dense in $\text{ref } B$. Proposition 2.14(1) shows that $\mathcal{U} = \text{add}(B \oplus F(A))$ is reflexively dense. Evaluation at $B \oplus F(A)$ identifies $\text{ref } \mathcal{U}$ with $\text{ref } C$, and the restricted Yoneda functor $\Phi_{\mathcal{U}}$ sends both subcategories into $\text{proj } C$. Hence $[A] \leq_{\text{ref}} [C]$ and $[B] \leq_{\text{ref}} [C]$. If A and B are semiperfect, then A and B are finite direct sums of objects with local endomorphism rings. Full faithfulness of F gives such a decomposition of $F(A)$, hence of $B \oplus F(A)$. Its endomorphism ring C is therefore semiperfect. Conversely, a common upper bound gives $\text{ref } A \simeq \text{ref } C \simeq \text{ref } B$.

(3) For semiperfect A and B , let $F: \text{ref } A \rightarrow \text{ref } B$ be an equivalence with $F(\text{proj } A) \subseteq \text{proj } B$. Then F induces an injection $\text{ind}(\text{proj } A) \rightarrow \text{ind}(\text{proj } B)$ between finite sets. If $[B] \leq_{\text{ref}} [A]$ also holds, the two sets have the same cardinality. Thus $F(\text{proj } A) = \text{proj } B$ up to isomorphism, and A and B are Morita equivalent. $\square$

The next proposition characterizes $[A] \leq_{\text{ref}} [B]$ using idempotents and endomorphism rings.

Proposition 2.20. *The relation in Definition 2.18 has the following descriptions.*

- (1) *For semiperfect R -algebras A and B , the condition $[A] \leq_{\text{ref}} [B]$ is equivalent to the existence of an idempotent $e \in B$ such that A is R -linearly Morita equivalent to eBe and $(-)e$ restricts to an equivalence $\text{ref } B \rightarrow \text{ref}(eBe)$.*
- (2) *Suppose that R is Noetherian. For module-finite R -algebras A and B , the condition $[A] \leq_{\text{ref}} [B]$ is equivalent to B being R -linearly Morita equivalent to $\text{End}_A(A \oplus X)$ for some $X \in \text{ref } A$. The R -algebra C in Proposition 2.19(2) is then module-finite; in particular it is finite-dimensional if $R = k$.*

Proof. (1) Suppose $F: \text{ref } A \rightarrow \text{ref } B$ witnesses $[A] \leq_{\text{ref}} [B]$. The subcategory $F(\text{proj } A)$ is generated by finitely many indecomposable projectives. Since B is semiperfect, their sum is isomorphic to eB for some idempotent $e \in B$. Hence $F(\text{proj } A) = \text{add}(eB)$ up to isomorphism, and A is Morita equivalent to eBe . This subcategory is reflexively dense by Lemma 2.12(3),(5). Evaluating the restricted Yoneda functor at eB gives $\text{Hom}_B(eB, -) = (-)e$, and $\text{add}(eB) \simeq \text{proj}(eBe)$, so Lemma 2.2(3) shows that this functor induces the required equivalence. Conversely, a quasi-inverse of $(-)e$ sends $\text{proj}(eBe)$ to $\text{add}(eB) \subseteq \text{proj } B$. Composing with the Morita equivalence for A proves $[A] \leq_{\text{ref}} [B]$.

(2) For an equivalence F witnessing $[A] \leq_{\text{ref}} [B]$, put $M = F^{-1}(B) \in \text{ref } A$. Since $F(A) \in \text{add } B$, we have $A \in \text{add } M$. Choose n with $M^{\oplus n} \cong A \oplus X$; then $X \in \text{ref } A$. Since F is fully faithful and R -linear, $\text{End}_A(M) \cong \text{End}_B(B) \cong B$ as R -algebras. Hence $\text{End}_A(A \oplus X) \cong M_n(B)$, which is Morita equivalent to B . Conversely, Corollary 2.15 gives an equivalence $\text{Hom}_A(A \oplus X, -): \text{ref } A \rightarrow \text{ref } \text{End}_A(A \oplus X)$ sending $\text{proj } A$ into projectives. Finally, $B \oplus F(A)$ is a finite B -module, so its endomorphism algebra C is module-finite over R . $\square$

We can now define minimality using this partial order.

Definition 2.21. A semiperfect R -algebra A is *reflexive-minimal* if $[A]$ is a minimal element under $\leq_{\text{ref}}$ among semiperfect R -algebras. Equivalently, every idempotent $e \in A$ for which $(-)e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(eAe)$ is full, that is, $AeA = A$.

The equivalent form follows from Proposition 2.20(1), since for an idempotent e of a semiperfect algebra A , the corner eAe is Morita equivalent to A if and only if eA has every indecomposable projective A -module as a direct summand, that is, if and only if e is full. Since $[B] \leq_{\text{ref}} [A]$ implies $\text{ref } B \simeq \text{ref } A$, a semiperfect R -algebra A is reflexive-minimal exactly when $[A]$ is minimal among the Morita equivalence classes of semiperfect R -algebras reflexively equivalent to A . Extending the last assertion of Proposition 2.20(2), if R is Noetherian and A is module-finite, then every R -algebra B with an R -linear equivalence $F: \text{ref } B \rightarrow \text{ref } A$ is module-finite, since $B \cong \text{End}_A(F(B))$ and $F(B)$ is a finite A -module. Thus the reflexive equivalence class of a module-finite algebra consists of module-finite algebras, and that of a finite-dimensional k -algebra consists of finite-dimensional algebras.

To compare the algebras in a reflexive equivalence class inside one category, note that an equivalence $F: \text{ref } B \rightarrow \mathcal{E}$ sends $\text{proj } B$ to the reflexively dense subcategory $\text{add } F(B)$. If $G: \text{ref } B \rightarrow \mathcal{E}$ is another equivalence, the autoequivalence GF^{-1} of $\mathcal{E}$ sends $\text{add } F(B)$ to $\text{add } G(B)$.

Proposition 2.22. Let A be a finite-dimensional k -algebra and $\mathcal{E} = \text{ref } A$. There is a bijection

$$\frac{\{\mathcal{C} \in \text{ref-dense}(\mathcal{E}) \mid \mathcal{C} \text{ has an additive generator}\}}{\text{autoequivalences of } \mathcal{E}} \longleftrightarrow \frac{\{B \mid B \text{ is finite-dimensional, } \text{ref } B \simeq \mathcal{E}\}}{\text{Morita equivalence}},$$

sending $\text{add } M$ to $[\text{End}_{\mathcal{E}}(M)]$. For $\mathcal{C} = \text{add } M$ and $\mathcal{D} = \text{add } N$ in $\text{ref-dense}(\mathcal{E})$,

$$[\text{End}_{\mathcal{E}}(M)] \leq_{\text{ref}} [\text{End}_{\mathcal{E}}(N)] \iff \Psi(\mathcal{C}) \subseteq \mathcal{D} \text{ for some autoequivalence } \Psi \text{ of } \mathcal{E}.$$

Proof. For $\mathcal{C} = \text{add } M \in \text{ref-dense}(\mathcal{E})$, evaluation at M identifies $\text{ref } \mathcal{C}$ with $\text{ref } \text{End}_{\mathcal{E}}(M)$, and $\text{End}_{\mathcal{E}}(M)$ is finite-dimensional. Every algebra B with $\text{ref } B \simeq \mathcal{E}$ arises by transporting $\text{proj } B$ into $\mathcal{E}$. Two such subcategories give Morita equivalent algebras precisely when they are equivalent. An equivalence $\mathcal{C} \rightarrow \mathcal{D}$ induces $\text{ref } \mathcal{C} \simeq \text{ref } \mathcal{D}$; composing with the equivalences $\Phi_{\mathcal{C}}$ and $\Phi_{\mathcal{D}}$ gives an autoequivalence of $\mathcal{E}$ carrying $\mathcal{C}$ onto $\mathcal{D}$. This proves the bijection. Under the same identifications, an equivalence preserving projectives in the sense of Definition 2.18 corresponds exactly to an autoequivalence Ψ with $\Psi(\mathcal{C}) \subseteq \mathcal{D}$. $\square$

3. REFLEXIVE-MINIMAL ALGEBRAS AND EQUIVALENCE CRITERIA

We prove Theorem A. By Proposition 2.22, the order $\leq_{\text{ref}}$ on Morita equivalence classes in the reflexive equivalence class of A corresponds to inclusion of reflexively dense subcategories of $\text{ref } A$, up to autoequivalences of $\text{ref } A$. So we look for a least reflexively dense subcategory. Section 3.1 identifies indecomposable objects that every reflexively dense subcategory of a Krull–Schmidt category must contain. Sections 3.2 and 3.3 determine, for an idempotent e of a module-finite algebra A over a commutative Noetherian ring, when $\text{add}(eA)$ is reflexively dense in $\text{ref } A$. Section 3.4 proves that $\text{add}(e_{\min}A)$ is the least reflexively dense subcategory and recovers all algebras in the reflexive equivalence class of A from $A_{\min}$. Its arguments are written for module-finite algebras over a

henselian local ring, so that they also apply in the proof of Theorem 4.5. Section 3.5 computes examples over a field.

3.1. Objects contained in every reflexively dense subcategory. Let $\mathcal{E}$ be a Krull–Schmidt category. For an indecomposable object U the algebra $\text{End}_{\mathcal{E}}(U)$ is local, and a morphism $X \rightarrow U$ lies in $\text{rad}_{\mathcal{E}}(X, U)$ if and only if it is not a split epimorphism; dually, a morphism $U \rightarrow X$ lies in $\text{rad}_{\mathcal{E}}(U, X)$ if and only if it is not a split monomorphism. The *simple functors* at U are

$$S_U = \mathcal{E}(-, U) / \text{rad}_{\mathcal{E}}(-, U) \in \text{Mod } \mathcal{E}, \quad S^U = \mathcal{E}(U, -) / \text{rad}_{\mathcal{E}}(U, -) \in \text{Mod } \mathcal{E}^{\text{op}}.$$

For $X \in \mathcal{E}$ we have $S_U(X) \neq 0$ if and only if U is isomorphic to a direct summand of X .

Definition 3.1. For a Krull–Schmidt category $\mathcal{E}$, let $\Sigma(\mathcal{E}) \subseteq \text{ind } \mathcal{E}$ consist of the indecomposable objects U for which there are $Y \in \mathcal{E}$ and $j \in \{0, 1\}$ such that

$$\text{Ext}_{\text{Mod } \mathcal{E}}^j(S_U, P_Y) \neq 0 \quad \text{or} \quad \text{Ext}_{\text{Mod } \mathcal{E}^{\text{op}}}^j(S^U, P^Y) \neq 0.$$

Here Ext^0 means Hom , and the extension groups are taken in the abelian categories $\text{Mod } \mathcal{E}$ and $\text{Mod } \mathcal{E}^{\text{op}}$.

We show that every reflexively dense subcategory of $\mathcal{E}$ contains the objects in $\Sigma(\mathcal{E})$. For $\mathcal{E} = \text{ref } A$, Theorem 3.9(2) will identify these objects with the projectives $e_i A$ indexed by $I_{\text{ref}}(A)$ (Construction 3.4) when $(-)_e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(e_{\min} A e_{\min})$, for example when A is Artinian.

Lemma 3.2. *Let $\mathcal{E}$ be a Krull–Schmidt category.*

- (1) *Every equivalence $F: \mathcal{E} \rightarrow \mathcal{E}'$ induces a bijection $\Sigma(\mathcal{E}) \rightarrow \Sigma(\mathcal{E}')$.*
- (2) *If $\mathcal{C} \in \text{ref-dense}(\mathcal{E})$, then $\Sigma(\mathcal{E}) \subseteq \text{ind } \mathcal{C}$.*

Proof. (1) An equivalence transports representable and simple functors, and preserves their Hom and Ext^1 groups.

(2) Let $\iota^*: \text{Mod } \mathcal{E} \rightarrow \text{Mod } \mathcal{C}$ be restriction. Its right adjoint satisfies

$$(\iota_* G)(X) = \text{Hom}_{\text{Mod } \mathcal{C}}(\Phi_{\mathcal{C}}(X), G), \quad \iota_* \Phi_{\mathcal{C}}(Y) \cong P_Y.$$

If $\iota^* T = 0$, adjunction gives $\text{Hom}(T, P_Y) = 0$. Every extension $0 \rightarrow P_Y \xrightarrow{a} W \rightarrow T \rightarrow 0$ splits: restriction makes $\iota^* a$ invertible, and adjunction sends its inverse to a retraction of a . Hence $\text{Ext}^1(T, P_Y) = 0$ as well. If $U \notin \mathcal{C}$, no object of $\mathcal{C}$ has U as a summand, so $\iota^* S_U = 0$, and hence

$$\text{Hom}_{\text{Mod } \mathcal{E}}(S_U, P_Y) = \text{Ext}_{\text{Mod } \mathcal{E}}^1(S_U, P_Y) = 0.$$

Applying the same argument to $\mathcal{C}^{\text{op}} \in \text{ref-dense}(\mathcal{E}^{\text{op}})$ gives

$$\text{Hom}_{\text{Mod } \mathcal{E}^{\text{op}}}(S^U, P^Y) = \text{Ext}_{\text{Mod } \mathcal{E}^{\text{op}}}^1(S^U, P^Y) = 0.$$

Thus $U \notin \Sigma(\mathcal{E})$. □

3.2. Double centralizers. To determine which projectives generate a reflexively dense subcategory, we study restriction to a corner. Throughout Sections 3.2 and 3.3, let R be a commutative Noetherian ring, A a module-finite R -algebra, and $e \in A$ an idempotent. The module Ae is a right eAe -module and eA is a left eAe -module, and we consider the maps

$$\lambda: A \rightarrow \text{End}_{eAe}(Ae), \quad \lambda(a)(p) = ap, \quad \rho: A^{\text{op}} \rightarrow \text{End}_{(eAe)^{\text{op}}}(eA), \quad \rho(a)(q) = qa.$$

A module M over a ring Λ has the *double centralizer property* if the map $\Lambda \rightarrow \text{End}_{\text{End}_{\Lambda}(M)}(M)$ given by the action is bijective. Thus λ is bijective exactly when the left A -module Ae has the double centralizer property, and ρ is bijective exactly when the right A -module eA has it. In Section 3.3 we show that $(-)_e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(eAe)$ exactly when λ and ρ are bijective. For Artinian rings, the criterion using simple modules in the next lemma is due to Fuller [Ful, Theorem 4]; Cunningham, Rutter and Turnidge [CRT, Theorem 2.6] give a version for right perfect rings. The proof below uses an adjunction to obtain the form over R for all finite modules.

Lemma 3.3. *The map λ is bijective if and only if*

$$\mathrm{Hom}_A(Y, A) = \mathrm{Ext}_A^1(Y, A) = 0 \quad \text{for every finite right } A\text{-module } Y \text{ with } Ye = 0. \quad (3.1)$$

The map ρ is bijective if and only if

$$\mathrm{Hom}_{A^{\mathrm{op}}}(Z, A) = \mathrm{Ext}_{A^{\mathrm{op}}}^1(Z, A) = 0 \quad \text{for every finite left } A\text{-module } Z \text{ with } eZ = 0.$$

If A is Artinian, these equivalences become

$$\begin{aligned} \lambda \text{ is bijective} &\iff \mathrm{Hom}_A(S, A) = \mathrm{Ext}_A^1(S, A) = 0 \\ &\quad \text{for every simple right } S \text{ with } Se = 0, \\ \rho \text{ is bijective} &\iff \mathrm{Hom}_{A^{\mathrm{op}}}(L, A) = \mathrm{Ext}_{A^{\mathrm{op}}}^1(L, A) = 0 \\ &\quad \text{for every simple left } L \text{ with } eL = 0. \end{aligned}$$

Proof. The second statement is the first one for A^{op} , so we prove the first. Let $\mathcal{S}$ be the full subcategory of $\mathrm{mod} A$ consisting of the finite modules Y with $Ye = 0$. The exact functor $F = (-)e: \mathrm{mod} A \rightarrow \mathrm{mod} eAe$ has the right adjoint $G = \mathrm{Hom}_{eAe}(Ae, -)$, where $(h \cdot a)(p) = h(ap)$. The counit $(GZ)e \rightarrow Z$, $h \mapsto h(e)$, is an isomorphism, and the unit is

$$\eta_N: N \rightarrow G(Ne), \quad \eta_N(n)(p) = np.$$

By the triangle identity, $F(\eta_N)$ composed with the counit at Ne is the identity of Ne , so $F(\eta_N)$ is an isomorphism. Since F is exact, $\mathrm{Ker} \eta_N$ and $\mathrm{Coker} \eta_N$ are annihilated by e . They are finite, since N and $G(Ne) = \mathrm{Hom}_{eAe}(Ae, Ne)$ are finite over R . Hence they lie in $\mathcal{S}$. For $Y \in \mathcal{S}$ and $Z \in \mathrm{mod} eAe$ we have $\mathrm{Hom}_A(Y, GZ) \cong \mathrm{Hom}_{eAe}(Ye, Z) = 0$. Moreover $\mathrm{Ext}_A^1(Y, GZ) = 0$: if $0 \rightarrow GZ \xrightarrow{a} W \rightarrow Y \rightarrow 0$ is exact, then Fa is an isomorphism, and the morphism $We \rightarrow Z$ obtained from $(Fa)^{-1}$ and the counit corresponds to a morphism $W \rightarrow GZ$ which is a retraction of a , as in the proof of Lemma 3.2(2).

Suppose that $\mathrm{Hom}_A(Y, N) = 0 = \mathrm{Ext}_A^1(Y, N)$ for every $Y \in \mathcal{S}$. Then $\mathrm{Ker} \eta_N = 0$, and the exact sequence $0 \rightarrow N \rightarrow G(Ne) \rightarrow \mathrm{Coker} \eta_N \rightarrow 0$ splits. Hence $\mathrm{Coker} \eta_N$ is isomorphic to a submodule of $G(Ne)$ lying in $\mathcal{S}$, so it is zero, and η_N is an isomorphism. Conversely, if η_N is an isomorphism then $N \cong G(Ne)$ satisfies these vanishing conditions. For $N = A$ the unit η_A is λ . This proves that λ is bijective if and only if (3.1) holds. If A is Artinian, every module in $\mathcal{S}$ has a finite filtration with simple factors in $\mathcal{S}$, and the long exact sequences show that the vanishing for all $Y \in \mathcal{S}$ is equivalent to the vanishing for the simple modules S with $Se = 0$. $\square$

3.3. Reflexive equivalence induced by corners. We prove that $(-)e$ restricts to an equivalence $\mathrm{ref} A \rightarrow \mathrm{ref}(eAe)$ if and only if the maps λ and ρ of Section 3.2 are bijective. For Artinian algebras, we use Lemma 3.3 to express this condition through Hom and first extensions from simple modules.

Construction 3.4. Suppose that A is basic and semiperfect. Choose primitive orthogonal idempotents $e_1, \dots, e_n$ of A whose sum is 1, let J be the radical of A , and put $S_i = e_i A / e_i J$ and $L_i = Ae_i / Je_i$. Let $I_{\mathrm{ref}}(A)$ consist of the indices i for which at least one of

$$\mathrm{Hom}_A(S_i, A), \quad \mathrm{Ext}_A^1(S_i, A), \quad \mathrm{Hom}_{A^{\mathrm{op}}}(L_i, A), \quad \mathrm{Ext}_{A^{\mathrm{op}}}^1(L_i, A)$$

is nonzero. Put $e_T = \sum_{i \in T} e_i$ for $T \subseteq \{1, \dots, n\}$, and set $e_{\min} = e_{I_{\mathrm{ref}}(A)}$ and $A_{\min} = e_{\min} A e_{\min}$.

Construction 3.4 applies to basic Artinian algebras and to basic module-finite algebras over a henselian local ring, since these algebras are semiperfect. For a basic finite-dimensional k -algebra, Corollary 3.12(2) shows that $A_{\min}$ is the unique basic reflexive-minimal algebra in its class; in particular it is independent of the choices up to isomorphism. By Theorem 4.5(3), the same holds for the algebras of that theorem. For an Artinian algebra, the next theorem shows that $I_{\mathrm{ref}}(A)$ is the smallest set T for which $(-)e_T$ restricts to an equivalence $\mathrm{ref} A \rightarrow \mathrm{ref}(e_T A e_T)$.

Theorem 3.5. *Let R be a commutative Noetherian ring, A a module-finite R -algebra, and $e \in A$ an idempotent. The following conditions are equivalent.*

- (1) *The functor $(-)e$ restricts to an equivalence $\mathrm{ref} A \rightarrow \mathrm{ref}(eAe)$.*
- (2) *The subcategory $\mathrm{add}(eA)$ is reflexively dense in $\mathrm{ref} A$.*
- (3) *The maps λ and ρ of Section 3.2 are bijective.*

If A is basic and Artinian and $e = e_T$, these conditions are equivalent to $I_{\mathrm{ref}}(A) \subseteq T$.

Proof. (1) $\Rightarrow$ (2): Evaluation at eA identifies modules over $\text{add}(eA)$ with modules over eAe , and restricts to $\text{add}(eA) \simeq \text{proj}(eAe)$ on representables. It sends the restricted Yoneda functor to $X \mapsto \text{Hom}_A(eA, X) = Xe$, and identifies the reflexive subcategories by Lemma 2.2(3). Thus (1) says that $\Phi_{\text{add}(eA)}: \text{ref } A \rightarrow \text{ref } \text{add}(eA)$ is an equivalence, which is (2).

(2) $\Rightarrow$ (3): Put $\mathcal{C} = \text{add}(eA)$. By (2) and Lemma 2.12(2), the functors $\Phi_{\mathcal{C}}$ and $\Phi^{\mathcal{C}}$ are fully faithful on $\text{ref } A$, and in particular at the object A . Under evaluation at eA , the map $\text{End}_A(A) \rightarrow \text{Hom}(\Phi_{\mathcal{C}}(A), \Phi_{\mathcal{C}}(A))$ becomes $\text{End}_A(A) \rightarrow \text{End}_{eAe}(Ae)$, which is λ under the identification $\text{End}_A(A) = A$ by left multiplication. Similarly, $\Phi^{\mathcal{C}}$ sends A to $\text{Hom}_A(A, eA) = eA$, and its map on $\text{End}_A(A)$ is ρ . Hence λ and ρ are bijective.

(3) $\Rightarrow$ (1): Put $B = eAe$. The right B -module Ae decomposes as $Ae = B \oplus (1 - e)Ae$, and $\lambda(e)$ is the projection onto B . Hence $\text{Hom}_B(Ae, B) = \lambda(e) \text{End}_B(Ae)$, and λ restricts to a bijection $eA \rightarrow \text{Hom}_B(Ae, B)$. Similarly, ρ restricts to a bijection $Ae \rightarrow \text{Hom}_{B^{\text{op}}}(eA, B)$. Under these bijections the evaluation map of Ae is the identity, so $Ae \in \text{ref } B$, and λ identifies A with $\text{End}_B(B \oplus (1 - e)Ae)$. Corollary 2.15 therefore gives an equivalence $\text{Hom}_B(Ae, -): \text{ref } B \rightarrow \text{ref } A$. Since $\text{Hom}_B(Ae, N)e \cong \text{Hom}_B(B, N) \cong N$ naturally in N , the functor $(-)_e$ sends $\text{ref } A$ into $\text{ref } B$ and is a quasi-inverse of this equivalence.

If A is Artinian and $e = e_T$, then a simple right module S_i satisfies $S_i e = 0$ exactly when $i \notin T$, and similarly for L_i . By Lemma 3.3, condition (3) is equivalent to

$$\text{Hom}_A(S_i, A) = \text{Ext}_A^1(S_i, A) = \text{Hom}_{A^{\text{op}}}(L_i, A) = \text{Ext}_{A^{\text{op}}}^1(L_i, A) = 0 \quad (i \notin T),$$

that is, $I_{\text{ref}}(A) \subseteq T$. $\square$

Remark 3.6. Let A be as in Construction 3.4, and let $0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ be a minimal injective coresolution of A_A . For a simple module S the maps $\text{Hom}_A(S, I^j) \rightarrow \text{Hom}_A(S, I^{j+1})$ vanish by minimality, so $\text{Ext}_A^j(S, A) \cong \text{Hom}_A(S, I^j)$. Hence $\text{Hom}_A(S_i, A)$ or $\text{Ext}_A^1(S_i, A)$ is nonzero exactly when S_i is a direct summand of $\text{soc}(I^0 \oplus I^1)$, and similarly for the left simple modules and the minimal injective coresolution of ${}_A A$. Thus $I_{\text{ref}}(A)$ is determined by the socles of the first two terms of these two coresolutions.

3.4. The least reflexively dense subcategory and reconstruction. We show that $\text{add}(e_{\min} A)$ is contained in every reflexively dense subcategory of $\text{ref } A$. By Lemma 3.2(2), it suffices to prove that $e_i A \in \Sigma(\text{ref } A)$ for $i \in I_{\text{ref}}(A)$. We then recover every algebra reflexively equivalent to A from $A_{\min}$. We work in the following setting, which includes basic finite-dimensional k -algebras and the algebras of Theorem 4.5.

Setting 3.7. Let R be a commutative henselian Noetherian local ring and A a basic module-finite R -algebra. Equivalences are R -linear. The algebra A is two-sided Noetherian, and it is semiperfect because every module-finite algebra over a henselian local ring is semiperfect [Hash, p. 88]. The endomorphism ring of a finite A -module is again module-finite over R , hence semiperfect; the category of finite A -modules is therefore Krull–Schmidt. We use the idempotents e_i , the simple modules S_i and L_i , the set $I_{\text{ref}}(A)$ and the idempotent $e_{\min}$ of Construction 3.4. The case $R = k$ is that of basic finite-dimensional k -algebras.

Lemma 3.8. *Let A be as in Setting 3.7, $\mathcal{E} = \text{ref } A$, $i \in \{1, \dots, n\}$ and $U = e_i A$.*

- (1) *If $\text{Hom}_A(S_i, A) \neq 0$ or $\text{Ext}_A^1(S_i, A) \neq 0$, then $\text{Hom}_{\text{Mod } \mathcal{E}}(S_U, P_A) \neq 0$ or $\text{Ext}_{\text{Mod } \mathcal{E}}^1(S_U, P_A) \neq 0$.*
- (2) *If $\text{Hom}_{A^{\text{op}}}(L_i, A) \neq 0$ or $\text{Ext}_{A^{\text{op}}}^1(L_i, A) \neq 0$, then $\text{Hom}_{\text{Mod } \mathcal{E}^{\text{op}}}(S^U, P^A) \neq 0$ or $\text{Ext}_{\text{Mod } \mathcal{E}^{\text{op}}}^1(S^U, P^A) \neq 0$.*

In particular $e_i A \in \Sigma(\text{ref } A)$ for every $i \in I_{\text{ref}}(A)$.

Proof. (1) Identify $\text{Mod}(\text{proj } A)$ with $\text{Mod } A$ by evaluation at A . Then restriction $\iota^*: \text{Mod } \mathcal{E} \rightarrow \text{Mod } A$ is $T \mapsto T(A)$, and its right adjoint, described in the proof of Lemma 3.2(2), is $\iota_* N = \text{Hom}_A(-, N)|_{\mathcal{E}}$ for $N \in \text{Mod } A$. In particular $\iota_* Y = P_Y$ for $Y \in \mathcal{E}$. A morphism $A \rightarrow e_i A$ is a split epimorphism if and only if it is surjective, that is, if and only if its image is not contained in $e_i J$. Hence $\iota^* S_U = e_i A / e_i J = S_i$, and adjunction gives $\text{Hom}_{\text{Mod } \mathcal{E}}(S_U, P_A) \cong \text{Hom}_A(S_i, A)$.

Suppose that $\text{Ext}_A^1(S_i, A) \neq 0$, and choose a nonsplit exact sequence $0 \rightarrow A \rightarrow N \xrightarrow{q} S_i \rightarrow 0$ in $\text{mod } A$. Applying the left exact functor ι_* gives an exact sequence $0 \rightarrow P_A \rightarrow \iota_* N \xrightarrow{\iota_* q} \iota_* S_i$. Let

$\pi: e_i A \rightarrow S_i$ be the canonical surjection. Composition with π defines a morphism $P_U \rightarrow \iota_* S_i$, which vanishes on $\text{rad}_{\mathcal{E}}(-, U)$ because radical morphisms $X \rightarrow e_i A$ have image in $e_i J$. Let $\sigma: S_U \rightarrow \iota_* S_i$ be the induced morphism, and let W be the pullback of $\iota_* q$ along σ . This gives a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_A & \longrightarrow & W & \longrightarrow & S_U \\ & & \parallel & & \downarrow & & \downarrow \sigma \\ 0 & \longrightarrow & P_A & \longrightarrow & \iota_* N & \xrightarrow{\iota_* q} & \iota_* S_i. \end{array}$$

The morphism $W \rightarrow S_U$ is surjective: an element of $S_U(X)$ is represented by some $f: X \rightarrow e_i A$; since $e_i A$ is projective, $\pi = q\tilde{\pi}$ for some $\tilde{\pi}: e_i A \rightarrow N$, and $\tilde{\pi}f \in (\iota_* N)(X)$ is mapped to $\pi f = \sigma(f)$. Thus the upper row is an exact sequence $0 \rightarrow P_A \rightarrow W \rightarrow S_U \rightarrow 0$ in $\text{Mod } \mathcal{E}$. Evaluated at A , the morphism σ becomes the identity of S_i and $\iota_* q$ becomes q , so $W(A) \cong N$ and the evaluated upper row is the chosen nonsplit sequence. Since evaluation at A is exact, the upper row does not split, and $\text{Ext}_{\text{Mod } \mathcal{E}}^1(S_U, P_A) \neq 0$.

(2) The duality $(-)^*: \mathcal{E} \rightarrow \text{ref}(A^{\text{op}})$ of Lemma 2.2(2) identifies left $\mathcal{E}$ -modules with right $\text{ref}(A^{\text{op}})$ -modules. It sends $e_i A$ to Ae_i , the functor S^U to the simple functor at Ae_i , and P^A to the representable functor of $A^* \cong {}_A A$. Since the simple right A^{op} -module at i is L_i , part (2) is part (1) for A^{op} . $\square$

The next theorem assumes that $(-)e_{\min}$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(e_{\min} A e_{\min})$. By Theorem 3.5, this holds whenever A is Artinian; Theorem 4.5(2) proves it under a hypothesis on the localizations of A .

Theorem 3.9. *Let A be as in Setting 3.7, and put $e = e_{\min}$. Suppose that $(-)e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(eAe)$.*

- (1) *The subcategory $\text{add}(eA)$ is the least element of $\text{ref-dense}(\text{ref } A)$.*
- (2) $\Sigma(\text{ref } A) = \{e_i A \mid i \in I_{\text{ref}}(A)\}$.

Proof. (1) Theorem 3.5 gives $\text{add}(eA) \in \text{ref-dense}(\text{ref } A)$. By Lemma 3.8, every $e_i A$ with $i \in I_{\text{ref}}(A)$ lies in $\Sigma(\text{ref } A)$, so by Lemma 3.2(2) every reflexively dense subcategory contains $\text{add}(eA)$.

(2) Lemma 3.2(2) applied to $\text{add}(eA)$ shows that every object of $\Sigma(\text{ref } A)$ is isomorphic to some $e_i A$ with $i \in I_{\text{ref}}(A)$, and the converse is Lemma 3.8. $\square$

If $\text{add}(eA)$ is the least element of $\text{ref-dense}(\text{ref } A)$, we can describe every algebra reflexively equivalent to A as an endomorphism algebra over eAe . The following proposition applies both over fields and over the local rings of Theorem 4.5.

Proposition 3.10. *Let A be as in Setting 3.7. Suppose that $\text{add}(eA)$ is the least element of $\text{ref-dense}(\text{ref } A)$ for an idempotent e . Put $\Gamma = eAe$.*

- (1) *The algebra Γ is basic and reflexive-minimal, and $\text{proj } \Gamma$ is the least element of $\text{ref-dense}(\text{ref } \Gamma)$. Its Morita equivalence class $[\Gamma]$ is the least element under $\leq_{\text{ref}}$ among module-finite R -algebras reflexively equivalent to A .*
- (2) *A module-finite R -algebra B is reflexively equivalent to A if and only if $B \cong \text{End}_{\Gamma}(\Gamma \oplus X)$ for some $X \in \text{ref } \Gamma$. For every equivalence $F: \text{ref } B \rightarrow \text{ref } \Gamma$, the module Γ is a direct summand of $F(B)$.*
- (3) *Up to R -algebra isomorphism, Γ is the unique basic reflexive-minimal algebra in the reflexive equivalence class of A .*

Proof. (1) By Theorem 3.5, the functor $(-)e$ gives an equivalence $\text{ref } A \rightarrow \text{ref } \Gamma$ sending $\text{add}(eA)$ to $\text{proj } \Gamma$. By Lemma 2.12(3), $\text{proj } \Gamma$ is the least element of $\text{ref-dense}(\text{ref } \Gamma)$. The module eA is a summand of the basic module A , so its endomorphism algebra Γ is basic. It is module-finite over R , hence semiperfect, so Definition 2.21 applies to it. If $f \in \Gamma$ is an idempotent and $(-)f$ restricts to an equivalence $\text{ref } \Gamma \rightarrow \text{ref}(f\Gamma f)$, then $\text{add}(f\Gamma)$ is reflexively dense by Theorem 3.5. Since $\text{proj } \Gamma$ is the least element of $\text{ref-dense}(\text{ref } \Gamma)$, we get $\text{add}(f\Gamma) = \text{proj } \Gamma$, so $\Gamma f\Gamma = \Gamma$. For any module-finite R -algebra B and any equivalence $F: \text{ref } B \rightarrow \text{ref } \Gamma$, the subcategory $\text{add } F(B)$ is reflexively dense by Lemma 2.12(3),(5), and therefore contains $\text{proj } \Gamma$. Thus F^{-1} sends $\text{proj } \Gamma$ into $\text{proj } B$, proving $[\Gamma] \leq_{\text{ref}} [B]$.

- (2) For an equivalence $F: \text{ref } B \rightarrow \text{ref } \Gamma$, the subcategory $\text{add } F(B)$ is reflexively dense by Lemma 2.12(3),(5). It contains $\text{proj } \Gamma$ by (1). Since Γ is basic and the categories are Krull–Schmidt, $F(B) \cong \Gamma \oplus X$ for some $X \in \text{ref } \Gamma$. Full faithfulness gives $B \cong \text{End}_\Gamma(\Gamma \oplus X)$. Conversely, every such endomorphism algebra is reflexively equivalent to Γ by Corollary 2.15, and hence to A .
- (3) Suppose that B is basic and reflexive-minimal and choose F as in (2). The projection of $F(B) = \Gamma \oplus X$ onto Γ corresponds to an idempotent $f \in B$. The equivalence F identifies $\text{add}(fB)$ with $\text{proj } \Gamma$, so $\text{add}(fB)$ is reflexively dense. By Theorem 3.5 and reflexive-minimality, f is full. Thus B is Morita equivalent to $fBf \cong \Gamma$. Basicness gives $B \cong \Gamma$. $\square$

For a basic finite-dimensional k -algebra A , Theorem 3.5 shows that $(-)_e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref } A_{\min}$. Thus Theorem 3.9 and Proposition 3.10 apply and give the following description of its reflexive equivalence class.

Theorem 3.11. *Let A and B be finite-dimensional k -algebras, and suppose that A is basic.*

- (1) *If B is basic, then A and B are reflexively equivalent if and only if $A_{\min} \cong B_{\min}$.*
- (2) *Put $\Lambda = A_{\min}$. Then B is reflexively equivalent to A if and only if $B \cong \text{End}_\Lambda(\Lambda \oplus X)$ for some $X \in \text{ref } \Lambda$. Moreover, for every equivalence $F: \text{ref } B \rightarrow \text{ref } \Lambda$ the module Λ is isomorphic to a direct summand of $F(B)$.*

Proof. (1) By Theorem 3.9 and Proposition 3.10(1),(3), $A_{\min}$ is the unique basic reflexive-minimal algebra in the reflexive equivalence class of A , and likewise for B . Thus $\text{ref } A \simeq \text{ref } B$ implies $A_{\min} \cong B_{\min}$. Conversely, this isomorphism and Theorem 3.5 give $\text{ref } A \simeq \text{ref } A_{\min} \simeq \text{ref } B_{\min} \simeq \text{ref } B$.

(2) Apply Proposition 3.10(2). $\square$

For an arbitrary finite-dimensional k -algebra A we put $A_{\min} = B_{\min}$ for a basic algebra B Morita equivalent to A . By Proposition 3.10(3), $A_{\min}$ is well defined up to isomorphism, independently of the choice of B and of its idempotents, and A is reflexively equivalent to $A_{\min}$ by Theorem 3.5.

Corollary 3.12. *Let A be a basic finite-dimensional k -algebra.*

- (1) *Let n be the number of simple A -modules up to isomorphism. The following conditions are equivalent.*
 - (a) *A is reflexive-minimal.*
 - (b) *$I_{\text{ref}}(A) = \{1, \dots, n\}$.*
 - (c) *$A \cong A_{\min}$.*
- (2) *The class $[A_{\min}]$ is the least element under $\leq_{\text{ref}}$ in the reflexive equivalence class of A . Up to isomorphism, $A_{\min}$ is the unique basic reflexive-minimal algebra in that class. It is also the unique basic algebra in that class with the least number of simple modules.*
- (3) *If A is local, then $A \cong A_{\min}$.*
- (4) *$I_{\text{ref}}(A_{\min})$ consists of all indices of the simple modules of $A_{\min}$, that is, $(A_{\min})_{\min} \cong A_{\min}$.*

Proof. (1) (a) $\Leftrightarrow$ (b): For every idempotent $e \in A$ there is a subset T with $eA \cong e_TA$, since eA is isomorphic to a direct sum of pairwise nonisomorphic summands e_iA of A . The ideal AeA and the subcategory $\text{add}(eA)$ depend only on the isomorphism class of eA . Hence, by Definition 2.21 and Theorem 3.5, the algebra A is reflexive-minimal if and only if e_T is full for every T such that $(-)_e$ restricts to an equivalence $\text{ref } A \rightarrow \text{ref}(e_TAe_T)$. Since A is basic, e_T is full exactly when $T = \{1, \dots, n\}$. By Theorem 3.5, $(-)_e$ restricts to such an equivalence exactly when $I_{\text{ref}}(A) \subseteq T$. Thus A is reflexive-minimal if and only if $I_{\text{ref}}(A) = \{1, \dots, n\}$.

(b) $\Rightarrow$ (c): In this case $e_{\min} = 1$ and $A_{\min} = A$.

(c) $\Rightarrow$ (b): The algebra $A_{\min}$ has $|I_{\text{ref}}(A)|$ simple modules up to isomorphism, so $A \cong A_{\min}$ forces $|I_{\text{ref}}(A)| = n$.

(2) By Proposition 3.10(1),(3), the class $[A_{\min}]$ is the least element and $A_{\min}$ is the unique basic reflexive-minimal algebra. By Theorem 3.11(2), every algebra in the class is isomorphic to $\text{End}_\Lambda(\Lambda \oplus X)$ with $\Lambda = A_{\min}$. Its number of simple modules is the number of isomorphism classes of indecomposable direct summands of $\Lambda \oplus X$. This number is at least that of Λ , with equality if and only if $X \in \text{proj } \Lambda$, in which case $\text{End}_\Lambda(\Lambda \oplus X)$ is Morita equivalent to Λ and is isomorphic to Λ if it is basic.

(3) The module A_A has a simple submodule, which is isomorphic to S_1 , so $I_{\text{ref}}(A) = \{1\}$.

(4) The algebra $A_{\min}$ is reflexive-minimal by (2). Apply (1) to $A_{\min}$. $\square$

3.5. Computing reflexive-minimal algebras. We compute reflexive-minimal algebras over an arbitrary field k . We obtain $I_{\text{ref}}(A)$ from the minimal injective coresolutions of A_A and ${}_A A$ using Remark 3.6, and then apply Theorem 3.11 to decide reflexive equivalence.

Example 3.13. Let $A = kQ/(bc, ca)$, where

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ \uparrow c & & \searrow b \\ & 3 & \end{array}$$

A minimal injective coresolution of A_A begins

$$0 \longrightarrow A \longrightarrow \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \oplus \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{\oplus 2} \longrightarrow \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \longrightarrow \cdots.$$

The assignment $e_1 \leftrightarrow e_3$, $e_2 \mapsto e_2$, $a \leftrightarrow b$, $c \mapsto c$ defines an anti-automorphism of A , so the minimal injective coresolution of ${}_A A$ is obtained by interchanging labels 1 and 3. Thus $I_{\text{ref}}(A) = \{1, 3\}$. With $e_{\min} = e_1 + e_3$, $u = ab$ and $v = c$, we obtain

$$A_{\min} \cong \Lambda := k \left(\begin{array}{ccc} 1 & \xrightarrow{u} & 3 \\ & \xleftarrow{v} & \end{array} \right) / (uv, vu). \quad (3.2)$$

Thus $\text{ref } A \simeq \text{ref } \Lambda$.

Example 3.14. Let $B = kQ'/(bc, da)$, where

$$\begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ \uparrow d & & \downarrow b \\ 4 & \xleftarrow{c} & 3 \end{array}$$

A minimal injective coresolution of B_B begins

$$0 \longrightarrow B \longrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}^{\oplus 2} \longrightarrow \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 1 \end{smallmatrix} \longrightarrow \cdots.$$

The assignment $e_1 \leftrightarrow e_3$, $e_2 \mapsto e_2$, $e_4 \mapsto e_4$, $a \leftrightarrow b$, $c \leftrightarrow d$ defines an anti-automorphism of B , so interchanging labels 1 and 3 again gives the minimal injective coresolution of ${}_B B$. Reading the socles gives $I_{\text{ref}}(B) = \{1, 3\}$. The corner on these vertices is $B_{\min} \cong \Lambda$ from (3.2), with arrows $u = ab$ and $v = cd$. Therefore

$$\text{ref } A \simeq \text{ref } B$$

for the algebra A of Example 3.13. The algebras A and B are not Morita equivalent, since they have three and four simple modules, respectively.

The following hereditary example shows that the last assertion of Theorem 3.5 fails if the Ext^1 conditions are omitted from the definition of $I_{\text{ref}}(A)$.

Example 3.15. Let H be the path algebra of $1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Its minimal injective coresolution is

$$0 \longrightarrow H \longrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{\oplus 3} \longrightarrow 1 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \longrightarrow 0.$$

All three indices occur in the socles of these two terms, so $I_{\text{ref}}(H) = \{1, 2, 3\}$ and $H_{\min} = H$. In particular, $\text{Ext}_H^1(S_2, H) \cong k$, although both $\text{Hom}_H(S_2, H)$ and $\text{Hom}_{H^{\text{op}}}(L_2, H)$ vanish. The Hom tests alone select $T = \{1, 3\}$, since the right socle of H is a sum of copies of S_3 and the left socle is a sum of copies of L_1 . By Theorem 3.5, the functor $(-)_e T$ with $T = \{1, 3\}$ does not restrict to an equivalence $\text{ref } H \rightarrow \text{ref}(e_T H e_T)$.

The next example distinguishes reflexive-minimality from reflexive-rigidity within one reflexive equivalence class.

Example 3.16. Let $\Delta = k[x]/(x^2)$, $S = \Delta/(x)$ and $\Gamma = \text{End}_\Delta(\Delta \oplus S)$, as in Example 2.9. With e_1, e_2 corresponding to Δ, S , respectively, Γ has quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 1$ and relation $ba = 0$. A minimal injective coresolution of Γ_Γ begins

$$0 \longrightarrow \Gamma \longrightarrow \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^{\oplus 2} \longrightarrow \frac{1}{1} \longrightarrow \cdots.$$

The left coresolution has the same layers, since interchanging a and b defines an anti-automorphism fixing the vertices. Hence

$$I_{\text{ref}}(\Gamma) = \{1\}, \quad \Gamma_{\min} = e_1 \Gamma e_1 \cong \Delta.$$

Since Δ is local, $\Delta_{\min} = \Delta$ by Corollary 3.12(3), and since Δ is self-injective, $\text{ref } \Delta = \text{mod } \Delta = \text{add}(\Delta \oplus S)$ by Lemma 2.2(4) and Example 2.9. Theorem 3.11 therefore gives $\text{ref } \Gamma \simeq \text{ref } \Delta$ and shows that Δ and Γ are the only basic algebras in this class. By Theorem 3.9(1), every reflexively dense subcategory contains $\text{add } \Delta$, and $\text{ref } \Delta$ itself is reflexively dense by Lemma 2.12(1). Thus $\text{ref-dense}(\text{ref } \Delta)$ is the two-element chain

$$\text{add } \Delta \subset \text{add}(\Delta \oplus S).$$

These subcategories are equivalent to $\text{proj } \Delta$ and $\text{proj } \Gamma$, respectively, and Γ is reflexive-rigid by Proposition 2.8(1). Thus Δ is reflexive-minimal but is not reflexive-rigid, whereas Γ is reflexive-rigid but is not reflexive-minimal.

4. MODULE CATEGORIES AND LOCAL BASE RINGS

We first describe reflexive equivalence classes whose completions are module categories; Proposition 4.1, which says when $\text{ref } B$ is abelian, is also used in the proof of Theorem C. We then prove Theorem C, which extends Theorem A to module-finite algebras over local rings, compute examples over discrete valuation rings, and give examples in which least or minimal reflexively dense subcategories do not exist.

4.1. Module categories. Dominant dimension characterizes when the category of reflexive modules is abelian and, for finite-dimensional algebras, when it is a module category. For a two-sided Noetherian ring B , the condition $\text{dom. dim } B \geq 2$ means that the first two terms in the minimal injective coresolution of B_B are projective. This condition is left-right symmetric [Hos, p. 275, Theorem].

Proposition 4.1 ([Han, Theorem 3.19, Theorem 4.1 and Proposition 4.2]). *For a two-sided Noetherian ring B , the category $\text{ref } B$ is abelian if and only if $\text{dom. dim } B \geq 2$. Such a ring is Artinian by a result of Sumioka, recalled in [IS, Proposition 7]. For a finite-dimensional k -algebra A , the following are equivalent:*

- (1) $\text{dom. dim } A \geq 2$;
- (2) $\text{ref } A \simeq \text{mod } B$ for some finite-dimensional k -algebra B ;
- (3) $A \cong \text{End}_B(M)$ for a generator-cogenerator $M \in \text{mod } B$ over some finite-dimensional k -algebra B .

If $A \cong \text{End}_B(M)$ as in (3), then $\text{Hom}_B(M, -)$ induces an equivalence $\text{mod } B \rightarrow \text{ref } A$.

Hanihara's reflexive modules are those of Definition 2.1. The first assertion is [Han, Theorem 3.19]. The equivalence (1) $\Leftrightarrow$ (3) is the Morita–Tachikawa correspondence [Han, Theorem 4.1], and (3) $\Rightarrow$ (2) with the last sentence is [Han, Proposition 4.2]. Finally, (2) $\Rightarrow$ (1) follows from the first assertion, since $\text{mod } B$ is abelian.

For a finite-dimensional algebra B , the next proposition describes the reflexively dense subcategories of $\text{mod } B$ by functorial finiteness. Unlike Proposition 2.16, it does not assume that the subcategory contains a given reflexively dense subcategory.

Proposition 4.2. *Let B be a finite-dimensional k -algebra. A subcategory $\mathcal{C} \subseteq \text{mod } B$ is reflexively dense in $\text{mod } B$ if and only if $\text{add}(B \oplus DB) \subseteq \mathcal{C}$ and $\mathcal{C}$ is functorially finite in $\text{mod } B$.*

Proof. Suppose that $\mathcal{C}$ contains B and DB and is functorially finite. We verify the hypotheses of Proposition 2.17(2) for $\mathcal{V} = \text{mod } B$; kernels exist in $\text{mod } B$. Let $\theta: \Phi_{\mathcal{C}}(Y) \rightarrow \Phi_{\mathcal{C}}(Z)$ be a morphism. Its component at B is a B -linear map $g: Y \rightarrow Z$, by naturality with respect to $\text{End}_B(B) = B$. For $X \in \mathcal{C}$ choose an epimorphism $p: B^m \rightarrow X$; naturality with respect to p and injectivity of

$\text{Hom}_B(p, Z)$ show that $\theta_X(f) = g \circ f$ for every $f: X \rightarrow Y$. Hence $\Phi_{\mathcal{C}}$ is full, and it is faithful by evaluation at B . The dual argument, using monomorphisms into $(DB)^m$, shows that $\Phi_{\mathcal{C}}$ is fully faithful. A right $\mathcal{C}$ -approximation $U_0 \rightarrow Y$ is surjective because $B \in \mathcal{C}$; choosing a right $\mathcal{C}$ -approximation U_1 of its kernel gives a presentation $P_{U_1} \rightarrow P_{U_0} \rightarrow \Phi_{\mathcal{C}}(Y) \rightarrow 0$. Dually a left $\mathcal{C}$ -approximation is injective because $DB \in \mathcal{C}$, and approximating its cokernel gives a presentation of $\Phi_{\mathcal{C}}(Y)$. Hence $\mathcal{C} \in \text{ref-dense}(\text{mod } B)$.

Conversely, let $\mathcal{C} \in \text{ref-dense}(\text{mod } B)$ and $Y \in \text{mod } B$. A presentation $P_{U_1} \rightarrow P_{U_0} \rightarrow \Phi_{\mathcal{C}}(Y) \rightarrow 0$ comes from morphisms $U_1 \rightarrow U_0 \xrightarrow{q} Y$, and q is a right $\mathcal{C}$ -approximation. For $Z \in \text{mod } B$, full faithfulness of $\Phi_{\mathcal{C}}$ identifies $\text{Hom}_B(Y, Z)$ with the morphisms $U_0 \rightarrow Z$ vanishing on U_1 , so q is a cokernel of $U_1 \rightarrow U_0$ in $\text{mod } B$. In particular q is surjective, and if Y is projective then q splits and $Y \in \mathcal{C}$. By Lemma 2.12(2) the same argument applies to $Y \mapsto \text{Hom}_B(Y, -)|_{\mathcal{C}}$, giving left approximations and showing that $\mathcal{C}$ contains the injective modules. Thus $\mathcal{C}$ is functorially finite and contains $\text{add}(B \oplus DB)$. $\square$

The next corollary describes the reflexive equivalence classes whose reflexive completions are module categories. By Proposition 4.1, these are the classes of the finite-dimensional algebras A with $\text{dom. dim } A \geq 2$.

Corollary 4.3. *Let B be a finite-dimensional k -algebra, let G be a basic additive generator of $\text{add}(B \oplus DB)$, and put $\Gamma = \text{End}_B(G)$. Then $\text{Hom}_B(G, -)$ induces an equivalence $\text{mod } B \rightarrow \text{ref } \Gamma$, the algebra Γ is reflexive-minimal with $\Gamma \cong \Gamma_{\min}$, and a finite-dimensional k -algebra is reflexively equivalent to Γ if and only if it is isomorphic to $\text{End}_B(M)$ for a generator-cogenerator $M \in \text{mod } B$.*

Proof. By Proposition 4.2, the functor $\text{Hom}_B(G, -)$ induces an equivalence $\text{mod } B \rightarrow \text{ref } \Gamma$, and $\text{add } G$ is the least reflexively dense subcategory of $\text{mod } B$. The equivalence sends $\text{add } G$ to $\text{proj } \Gamma$, so $\text{proj } \Gamma$ is the least element of $\text{ref-dense}(\text{ref } \Gamma)$ by Lemma 2.12(3). Since G is basic, Γ is basic, so Proposition 3.10(1), applied to Γ with $e = 1$, shows that Γ is reflexive-minimal, and $\Gamma \cong \Gamma_{\min}$ by Corollary 3.12(1). By Theorem 3.11, the algebras reflexively equivalent to Γ are exactly the algebras $\text{End}_{\Gamma}(\Gamma \oplus X')$ with $X' \in \text{ref } \Gamma$. Transporting along $\text{Hom}_B(G, -)$, which sends G to Γ , these are the algebras $\text{End}_B(G \oplus X)$ with $X \in \text{mod } B$, that is, the endomorphism algebras of the generator-cogenerators of $\text{mod } B$. $\square$

4.2. Module-finite algebras over local rings. We prove Theorem C. Its hypothesis concerns the localizations $\Lambda_{\mathfrak{p}} = \Lambda \otimes_R R_{\mathfrak{p}}$ at the nonmaximal primes $\mathfrak{p}$ of the base ring R . We first show that an R -linear equivalence $\text{ref } \Lambda \simeq \text{ref } \Gamma$ induces equivalences after localization at every prime of R . Consequently, the hypothesis depends only on the reflexive equivalence class of Λ .

Proposition 4.4. *Let R be a commutative Noetherian ring and Λ, Γ module-finite R -algebras. An R -linear equivalence $\text{ref } \Lambda \simeq \text{ref } \Gamma$ induces an $R_{\mathfrak{p}}$ -linear equivalence $\text{ref } \Lambda_{\mathfrak{p}} \simeq \text{ref } \Gamma_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R . Consequently*

$$\text{dom. dim } \Lambda_{\mathfrak{p}} \geq 2 \iff \text{dom. dim } \Gamma_{\mathfrak{p}} \geq 2$$

whenever these localized algebras are nonzero; one is zero if and only if the other is zero.

Proof. Let $F: \text{ref } \Lambda \rightarrow \text{ref } \Gamma$ be an equivalence and put $\Omega = \text{End}_{\Gamma}(\Gamma \oplus F\Lambda)$. The projection idempotents onto Γ and $F\Lambda$ have corners isomorphic to Γ and Λ , respectively. By Proposition 2.14(1) and Lemma 2.12(3),(5), the subcategory $\text{add}(\Gamma \oplus F\Lambda)$ is reflexively dense in $\text{ref } \Gamma$, and under the resulting equivalence $\text{ref } \Gamma \simeq \text{ref } \Omega$ the subcategories $\text{proj } \Gamma$ and $F(\text{proj } \Lambda)$ correspond to the subcategories $\text{add}(e\Omega)$ for the two idempotents e . Both are reflexively dense in $\text{ref } \Omega$. The two maps λ and ρ of Section 3.2 for each idempotent are therefore isomorphisms by Theorem 3.5. Since Hom between finite modules commutes with localization, all four maps remain isomorphisms after localization at $\mathfrak{p}$. Apply Theorem 3.5 over $R_{\mathfrak{p}}$ to obtain the claimed equivalence. The assertion on dominant dimension follows from Proposition 4.1, since being abelian is preserved by equivalence. A ring B satisfies $\text{ref } B = 0$ if and only if $B = 0$, which proves the last assertion. $\square$

Let Λ be a basic module-finite algebra over a henselian Noetherian local ring. By Lemma 3.3 and Theorem 3.5, whether $(-)_eT$ restricts to an equivalence $\text{ref } \Lambda \rightarrow \text{ref}(e_T\Lambda e_T)$ depends on Hom and Ext^1 into Λ from all finite modules annihilated by e_T , on both sides. The condition $I_{\text{ref}}(\Lambda) \subseteq T$ tests only the simple modules among them. If Λ is Artinian, this suffices because these modules have

finite length. Part (2) of the next theorem proves that it also suffices under a hypothesis on the nonmaximal localizations of Λ ; parts (3) and (4) then follow from Theorem 3.9 and Proposition 3.10.

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a commutative henselian Noetherian local ring and $\Lambda \neq 0$ a module-finite R -algebra. Suppose $\text{dom. dim } \Lambda_{\mathfrak{p}} \geq 2$ whenever $\mathfrak{p} \neq \mathfrak{m}$ and $\Lambda_{\mathfrak{p}} \neq 0$. Equivalences are R -linear.*

- (1) $\dim(R/\text{ann}_R \Lambda) \leq 1$.
- (2) *Suppose Λ is basic. Choose primitive orthogonal idempotents $e_1, \dots, e_n$ with sum 1, and define $I_{\text{ref}}(\Lambda)$ as in Construction 3.4. For $T \subseteq \{1, \dots, n\}$, the functor $(-)_e T$ restricts to an equivalence $\text{ref } \Lambda \rightarrow \text{ref}(e_T \Lambda e_T)$ if and only if $I_{\text{ref}}(\Lambda) \subseteq T$.*
- (3) *Suppose Λ is basic. Put $e = e_{\min}$ and $\Lambda_{\min} = e \Lambda e$. Then $\text{add}(e \Lambda)$ is the least element of $\text{ref-dense}(\text{ref } \Lambda)$. The class $[\Lambda_{\min}]$ is the least Morita equivalence class under $\leq_{\text{ref}}$ in the reflexive equivalence class of Λ , and $\Lambda_{\min}$ is the unique basic reflexive-minimal algebra in that class, up to R -algebra isomorphism. A module-finite R -algebra B is reflexively equivalent to Λ if and only if $B \cong \text{End}_{\Lambda_{\min}}(\Lambda_{\min} \oplus X)$ for some $X \in \text{ref } \Lambda_{\min}$.*
- (4) *Suppose Λ is not basic, and let Λ' be a basic algebra Morita equivalent to Λ . Then Λ' satisfies the hypothesis and is reflexively equivalent to Λ . Put $\Lambda_{\min} = \Lambda'_{\min}$, which is well defined up to R -algebra isomorphism. Then $[\Lambda_{\min}]$ is the least Morita equivalence class under $\leq_{\text{ref}}$ in the reflexive equivalence class of Λ , $\Lambda_{\min}$ is the unique basic reflexive-minimal algebra in that class, and a module-finite R -algebra B is reflexively equivalent to Λ if and only if $B \cong \text{End}_{\Lambda_{\min}}(\Lambda_{\min} \oplus X)$ for some $X \in \text{ref } \Lambda_{\min}$.*
- (5) *The hypothesis on the localizations $\Lambda_{\mathfrak{p}}$ is invariant under R -linear reflexive equivalence. If a nonzero module-finite R -algebra B also satisfies it, then $\text{ref } B \simeq \text{ref } \Lambda$ if and only if $\Lambda_{\min} \cong B_{\min}$ as R -algebras.*

Proof. (1) By Proposition 4.1, each nonzero $\Lambda_{\mathfrak{p}}$ with $\mathfrak{p} \neq \mathfrak{m}$ is Artinian. Put $\bar{R} = R/\text{ann}_R \Lambda$. For a nonmaximal prime $\mathfrak{q}$ of $\bar{R}$, the algebra $\Lambda_{\mathfrak{q}}$ is Artinian and finite faithful over $\bar{R}_{\mathfrak{q}}$. Let $\mathfrak{n}$ be the maximal ideal of $\bar{R}_{\mathfrak{q}}$. The descending sequence $\mathfrak{n}^r \Lambda_{\mathfrak{q}}$ stabilizes. Nakayama's lemma gives $\mathfrak{n}^r \Lambda_{\mathfrak{q}} = 0$ for some r , and faithfulness gives $\mathfrak{n}^r = 0$. Every nonmaximal prime of $\bar{R}$ therefore has height zero, proving the assertion.

(2) Replace R by $\bar{R}$, which is again a henselian Noetherian local ring; the hypothesis on Λ is unchanged, since the localizations of Λ at primes of $\bar{R}$ are those at the corresponding primes of R . By Lemma 3.3 and Theorem 3.5, $(-)_e T$ restricts to an equivalence $\text{ref } \Lambda \rightarrow \text{ref}(e_T \Lambda e_T)$ if and only if

$$\text{Hom}_{\Lambda}(Y, \Lambda) = \text{Ext}_{\Lambda}^1(Y, \Lambda) = 0 \quad \text{for every finite } Y \text{ with } Y e_T = 0, \quad (4.1)$$

and the analogous condition holds for left modules. Necessity of $I_{\text{ref}}(\Lambda) \subseteq T$ follows by taking simple modules. For sufficiency assume this inclusion. If $\dim R = 0$, all finite Λ -modules have finite length, so Lemma 3.3 applies.

Suppose $\dim R = 1$. Choose $t \in \mathfrak{m}$ with $\sqrt{tR} = \mathfrak{m}$. The assumed equalities $\text{Hom}_{\Lambda}(S_i, \Lambda) = \text{Ext}_{\Lambda}^1(S_i, \Lambda) = 0$ for $i \notin T$ extend by induction to every finite-length module annihilated by e_T . For a finite module Y with $Y e_T = 0$, let Y_{tor} be its t -power torsion submodule and put $Y' = Y/Y_{\text{tor}}$. Both Y_{tor} and Y'/tY' have finite length and are annihilated by e_T . Applying $\text{Hom}_{\Lambda}(-, \Lambda)$ to

$$0 \rightarrow Y' \xrightarrow{t} Y' \rightarrow Y'/tY' \rightarrow 0$$

shows that t acts surjectively on $\text{Hom}_{\Lambda}(Y', \Lambda)$ and injectively on $\text{Ext}_{\Lambda}^1(Y', \Lambda)$. Nakayama's lemma gives $\text{Hom}_{\Lambda}(Y', \Lambda) = 0$.

Write $R_t = R[t^{-1}]$. This is a zero-dimensional Noetherian ring, hence an Artinian ring. Its finitely many local factors are $R_{\mathfrak{p}}$ for the primes not containing t , all of which are nonmaximal in R . Consequently Λ_t is a finite product of the nonzero $\Lambda_{\mathfrak{p}}$ occurring in the hypothesis. Each factor has dominant dimension at least two by hypothesis, so the minimal injective coresolution $0 \rightarrow \Lambda_t \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ has $I^0, I^1 \in \text{proj } \Lambda_t$. Localization gives $\text{Hom}_{\Lambda_t}(Y'_t, \Lambda_t) = 0$, so

$$\text{Hom}_{\Lambda_t}(Y'_t, I^0) = \text{Hom}_{\Lambda_t}(Y'_t, I^1) = 0, \quad \text{Ext}_{\Lambda_t}^1(Y'_t, \Lambda_t) = 0.$$

Since $\text{Ext}_{\Lambda}^1(Y', \Lambda)_t \cong \text{Ext}_{\Lambda_t}^1(Y'_t, \Lambda_t)$, the finite R -module $\text{Ext}_{\Lambda}^1(Y', \Lambda)$ is thus both t -power torsion and t -torsionfree, so it vanishes. The sequence $0 \rightarrow Y_{\text{tor}} \rightarrow Y \rightarrow Y' \rightarrow 0$ proves (4.1). Since the condition $\text{dom. dim } \Lambda_{\mathfrak{p}} \geq 2$ is left-right symmetric, the same argument applies to left modules.

- (3) By (2), $(-)_e$ restricts to an equivalence $\text{ref } \Lambda \rightarrow \text{ref } \Lambda_{\min}$. Theorem 3.9 shows that $\text{add}(e\Lambda)$ is the least element of $\text{ref-dense}(\text{ref } \Lambda)$. The remaining assertions are Proposition 3.10(1)–(3).
- (4) Since Λ is semiperfect, a basic algebra Λ' Morita equivalent to Λ exists, and it is module-finite over R . A Morita equivalence induces an R -linear equivalence $\text{ref } \Lambda \simeq \text{ref } \Lambda'$, as in the proof of Proposition 2.19(1). By Proposition 4.4, Λ' satisfies the hypothesis. The last three assertions of (3) for Λ' concern the reflexive equivalence class of Λ' , which is that of Λ ; they give the three assertions of (4). Since $\Lambda'_{\min}$ is the unique basic reflexive-minimal algebra in this class, it does not depend on the choice of Λ' .
- (5) Proposition 4.4 gives invariance of the hypothesis. The equivalence criterion follows from (3) and (4), since an equivalence identifies least reflexively dense subcategories and hence the endomorphism algebras of their basic generators. $\square$

The hypothesis of Theorem 4.5 can be checked directly in two cases: over an Artinian local base and over a one-dimensional domain. Over a nonlocal Artinian base, part (3) of the next corollary obtains the least reflexively dense subcategory by decomposing the base into local rings. For a one-dimensional domain R with fraction field K , we write $\Lambda_K = \Lambda \otimes_R K$ for the *generic algebra* of Λ .

Corollary 4.6. *Let R be a commutative Noetherian ring and $\Lambda \neq 0$ a module-finite R -algebra.*

- (1) *If R is an Artinian local ring, then Λ satisfies the hypothesis of Theorem 4.5.*
- (2) *If R is a one-dimensional henselian local domain with fraction field K , then Λ satisfies the hypothesis of Theorem 4.5 if and only if $\Lambda_K = 0$ or $\text{dom. dim } \Lambda_K \geq 2$.*
- (3) *If R is Artinian and Λ is basic, write $R = \prod_j R_j$ with R_j local and $\Lambda_j = R_j \otimes_R \Lambda$, and let $e \in \Lambda$ be the sum of the idempotents $e_{\min}$ of the nonzero Λ_j . Then $\text{add}(e\Lambda)$ is the least element of $\text{ref-dense}(\text{ref } \Lambda)$, and $e\Lambda e \cong \prod_j (\Lambda_j)_{\min}$ is the unique basic reflexive-minimal algebra in the reflexive equivalence class of Λ .*

Proof. (1) An Artinian local ring is henselian and has no nonmaximal prime.

(2) The only nonmaximal prime of R is (0) , and $\Lambda_{(0)} = \Lambda_K$.

(3) The ring Λ is the product of the R_j -algebras Λ_j . Since R -linear functors commute with the idempotents of R , the category $\text{ref } \Lambda$, its reflexively dense subcategories and its R -linear equivalences are products over the factors. Apply Theorem 4.5(3) to each nonzero Λ_j , using (1). $\square$

For algebras that are free over a discrete valuation ring, the computation of $I_{\text{ref}}(\Lambda)$ simplifies as follows.

Corollary 4.7. *Let R be a henselian discrete valuation ring with uniformizer π and fraction field K . Let Λ be a nonzero basic R -algebra that is finite free as an R -module, and assume $\text{dom. dim } \Lambda_K \geq 2$. Then Theorem 4.5 applies, and $I_{\text{ref}}(\Lambda)$ is the set of indices i such that S_i occurs in the right socle or L_i occurs in the left socle of $\Lambda/\pi\Lambda$. In particular, these conclusions hold if Λ_K is self-injective.*

Proof. For a simple right Λ -module S , we have $\pi S = 0$ and $\text{Hom}_{\Lambda}(S, \Lambda) = 0$, since π is regular on Λ . Multiplication by π also kills $\text{Ext}_{\Lambda}^1(S, \Lambda)$. Applying $\text{Hom}_{\Lambda}(S, -)$ to $0 \rightarrow \Lambda \xrightarrow{\pi} \Lambda \rightarrow \Lambda/\pi\Lambda \rightarrow 0$ gives

$$\text{Hom}_{\Lambda}(S, \Lambda) = 0, \quad \text{Ext}_{\Lambda}^1(S, \Lambda) \cong \text{Hom}_{\Lambda/\pi\Lambda}(S, \Lambda/\pi\Lambda). \quad (4.2)$$

Apply the same argument to left modules. Thus the four Hom and Ext^1 conditions defining $I_{\text{ref}}(\Lambda)$ reduce to the socle conditions in the statement. Since Λ is nonzero and free, $\Lambda_K \neq 0$, and Theorem 4.5 applies by Corollary 4.6(2). A self-injective algebra has dominant dimension at least two. $\square$

In the following example, Corollary 4.7 selects every index, so a nonlocal order is its own reflexive-minimal algebra.

Example 4.8. Let R, π, K be as in Corollary 4.7, and put

$$\Lambda = \begin{bmatrix} R & R \\ \pi R & R \end{bmatrix}.$$

Then Λ is basic and finite free of rank four over R , and $\Lambda_K = M_2(K)$ is semisimple. The algebra $\Lambda/\pi\Lambda$ is the radical-square-zero algebra of the oriented two-cycle: the classes of E_{12} and πE_{21} give the arrows. Both right simple modules occur in its socle, so $I_{\text{ref}}(\Lambda) = \{1, 2\}$ and $\Lambda_{\min} = \Lambda$.

The next example gives two algebras with semisimple generic algebras which are reflexively equivalent but not Morita equivalent.

Example 4.9. Let R, π, K be as in Corollary 4.7, with residue field κ . Put

$$\Lambda = R[u]/(u^2 - \pi u), \quad X = \Lambda/(u), \quad \Gamma = \text{End}_\Lambda(\Lambda \oplus X).$$

The algebra Λ is local and finite free of rank two over R , and $\Lambda_K \cong K \times K$, so Theorem 4.5 applies. Since Λ is local, its only idempotents are 0 and 1. Hence Λ is reflexive-minimal and $\Lambda_{\min} = \Lambda$. Moreover,

$$\text{Hom}_\Lambda(X, \Lambda) = R(u - \pi), \quad \text{Hom}_\Lambda(\Lambda, X) \cong R, \quad \text{End}_\Lambda(X) \cong R.$$

Consequently Γ is finite free of rank five over R , and $\Gamma_K \cong M_2(K) \times K$, which is semisimple. Put $\Delta = \Lambda/\pi\Lambda = \kappa[u]/(u^2)$, so that $X/\pi X = \kappa$. Each of the four R -modules $\text{Hom}_\Lambda(X, \Lambda)$, $\text{Hom}_\Lambda(\Lambda, X)$, $\text{End}_\Lambda(X)$ and $\text{End}_\Lambda(\Lambda) = \Lambda$ is free, and reduction modulo π maps it isomorphically onto the corresponding Hom space over Δ . For $\text{End}_\Lambda(\Lambda)$ this is $\Lambda/\pi\Lambda = \Delta$. The other three modules have rank one, the corresponding Hom spaces over Δ are one-dimensional, and the generators reduce to nonzero maps; for instance, $u - \pi$ reduces to the embedding $\kappa \rightarrow \Delta$ with image κu . Hence

$$\Gamma/\pi\Gamma \cong \text{End}_\Delta(\Delta \oplus \kappa).$$

The algebra $\Gamma/\pi\Gamma$ is basic, and π lies in the radical of Γ , so Γ is basic. Label the summands Λ and X by 1 and 2. As in Example 3.16, over any field, the right and left regular modules of $\text{End}_\Delta(\Delta \oplus \kappa)$ embed into sums of copies of the indecomposable projective-injective module at 1, so both socles are sums of copies of the simple module at 1. Corollary 4.7 therefore gives

$$I_{\text{ref}}(\Gamma) = \{1\}, \quad \Gamma_{\min} = e_1 \Gamma e_1 \cong \Lambda = \Lambda_{\min}.$$

Thus Theorem 4.5(5) gives $\text{ref } \Gamma \simeq \text{ref } \Lambda$. The algebras Λ and Γ have one and two simple modules, respectively, so they are not Morita equivalent.

Example 4.10. Theorem 4.5 also applies, by Corollary 4.6(2), when the generic algebra is not semisimple. For $R = k[[t]]$ and $\Lambda = R[\varepsilon]/(\varepsilon^2)$, the generic algebra $k((t))[\varepsilon]/(\varepsilon^2)$ is self-injective, so its minimal injective coresolution is $0 \rightarrow \Lambda_K \rightarrow \Lambda_K \rightarrow 0$ and $\text{dom. dim } \Lambda_K \geq 2$. The algebra Λ is local, so it is reflexive-minimal.

4.3. Nonexistence of least and minimal reflexively dense subcategories. We show that the hypothesis of Theorem 4.5 cannot be omitted, and that minimal reflexively dense subcategories need not exist. For the regular local ring $R = k[[x, y, z]]$ of dimension three, $\text{ref-dense}(\text{ref } R)$ has distinct minimal elements and no least element.

Example 4.11. Let $R = k[[x, y, z]]$ and $M = \text{coker}(R \xrightarrow{(x, y, z)^t} R^3)$. The module M has depth two and is free away from the maximal ideal. Thus it satisfies Serre's condition $\text{depth } M_{\mathfrak{p}} \geq \min\{2, \dim R_{\mathfrak{p}}\}$ at every prime $\mathfrak{p}$, which characterizes reflexivity over a normal domain [IR, Section 2]. It has rank two and needs three generators, hence is not free. It is indecomposable: an endomorphism lifts to a matrix T with $T(x, y, z)^t = b(x, y, z)^t$ for some $b \in R$. Comparing linear terms gives $T \bmod \mathfrak{m} = (b \bmod \mathfrak{m})I_3$. Thus the endomorphism or one minus that endomorphism is surjective and hence invertible. Therefore $\text{End}_R(M)$ is local.

We use the following results of Iyama and Reiten for a module-finite algebra Λ over a normal Noetherian domain R , where a finite Λ -module is called reflexive over R if it is reflexive as an R -module [IR, Proposition 2.4]. First, $\text{Hom}_\Lambda(X, Y)$ is reflexive over R for finite Λ -modules X and Y with Y reflexive over R . Second, suppose that Λ is reflexive over R , and let N be a finite Λ -module, reflexive over R , such that $N_{\mathfrak{p}}$ is a progenerator over $\Lambda_{\mathfrak{p}}$ for every prime $\mathfrak{p}$ of height one. Then $\text{Hom}_\Lambda(N, -)$ is an equivalence from the finite Λ -modules reflexive over R to the finite $\text{End}_\Lambda(N)$ -modules reflexive over R , and if $\text{Hom}_R(\Lambda, R) \cong \Lambda$ as Λ -bimodules, then the same holds for $\text{End}_\Lambda(N)$.

Let N be a nonzero reflexive R -module, and put $\Gamma = \text{End}_R(N)$. We apply these results with $\Lambda = R$. The R -module Γ is reflexive by the first result. For each prime $\mathfrak{p}$ of height one, $N_{\mathfrak{p}}$ is a nonzero finite torsionfree module over the discrete valuation ring $R_{\mathfrak{p}}$, hence free and a progenerator. Thus $\text{Hom}_R(N, -)$ is an equivalence from $\text{ref } R$ to the category of finite Γ -modules which are reflexive

over R , and $\text{Hom}_R(\Gamma, R) \cong \Gamma$ as Γ -bimodules. This bimodule isomorphism and adjunction give natural isomorphisms

$$\text{Hom}_\Gamma(X, \Gamma) \cong \text{Hom}_\Gamma(X, \text{Hom}_R(\Gamma, R)) \cong \text{Hom}_R(X, R)$$

for right and for left Γ -modules X , so a finite Γ -module is reflexive over Γ if and only if it is reflexive over R . Thus $\text{Hom}_R(N, -): \text{ref } R \rightarrow \text{ref } \Gamma$ is an equivalence. It identifies $\text{add } N$ with $\text{proj } \Gamma$, so $\text{add } N$ is reflexively dense by Lemma 2.12(3),(5). Taking $N = R$ and $N = M$ gives the reflexively dense subcategories $\text{add } R$ and $\text{add } M$. Each is minimal since its generator has local endomorphism ring, and their intersection is zero. Thus $\text{ref-dense}(\text{ref } R)$ has no least element. On the level of algebras, R and $\text{End}_R(M)$ are reflexively equivalent local algebras, hence basic, and reflexive-minimal because their only idempotents are 0 and 1. They are not isomorphic, since $\text{End}_R(M)$ has rank four over R . Hence they are not Morita equivalent, and the reflexive equivalence class of R has no least Morita equivalence class. The hypothesis of Theorem 4.5 fails at a prime $\mathfrak{p}$ of height one: the ring $R_{\mathfrak{p}}$ is a discrete valuation ring, and $\text{ref } R_{\mathfrak{p}}$ consists of the finite free modules. This category is not abelian, since multiplication by a uniformizer is both monic and epic there but is not invertible; so $\text{dom. dim } R_{\mathfrak{p}} < 2$ by Proposition 4.1.

In Example 4.11 minimal reflexively dense subcategories exist but are not unique. In the next example, whose proof is given in Appendix A, they do not exist at all.

Example 4.12. Let $\mathcal{E}$ be the category of finite-dimensional $k[t]$ -modules on which t acts nilpotently, put $M_s = k[t]/(t^s)$, and for $T \subseteq \mathbb{N}_{>0}$ put $\mathcal{C}_T = \text{add}\{M_s \mid s \in T\}$. By Theorem A.1(1) and (3), the reflexively dense subcategories of $\mathcal{E}$ are exactly the $\mathcal{C}_T$ with T unbounded, and $\text{ref-dense}(\mathcal{E})$ has no minimal element. Since $\mathcal{E} \simeq \text{ref } \mathcal{C}_T$ for unbounded T , the category $\mathcal{E}$ is a Hom-finite reflexive completion without a minimal reflexively dense subcategory.

5. FINITE REFLEXIVE TYPE AND REFLEXIVE-RIGID ALGEBRAS

For a basic finite-dimensional k -algebra A , Theorem 3.9(1) identifies $\text{add}(e_{\min}A)$ as the least element of $\text{ref-dense}(\text{ref } A)$, and the greatest element is $\text{ref } A$ itself (Lemma 2.12(1)). The least element always has an additive generator. We now determine when the greatest element $\text{ref } A$ has one. This relates finite reflexive type to reflexive-rigidity and to the number of Morita equivalence classes within a reflexive equivalence class. Throughout this section algebras are finite-dimensional over k , and categories and equivalences are k -linear. The categories under consideration are Hom-finite.

5.1. Finite reflexive type. For a finite-dimensional k -algebra A , we show that $\text{ref } A$ has an additive generator if and only if there are finitely many Morita equivalence classes of finite-dimensional k -algebras reflexively equivalent to A .

Definition 5.1. An algebra A has *finite reflexive type* if $\text{ind}(\text{ref } A)$ is finite, equivalently, by Proposition 2.8(2), if $\text{ref } A$ has an additive generator.

Proposition 5.2. Let A be a finite-dimensional k -algebra. The following are equivalent.

- (1) A has finite reflexive type.
- (2) There are finitely many Morita equivalence classes of finite-dimensional algebras reflexively equivalent to A .
- (3) The set

$$\{ |\text{ind}(\text{proj } B)| \mid B \text{ is finite-dimensional, } \text{ref } B \simeq \text{ref } A \}$$

is bounded.

- (4) A is reflexively equivalent to a reflexive-rigid finite-dimensional algebra.

When these conditions hold, the algebra in (4) is unique up to Morita equivalence, and one may take $\text{End}_A(M)$ for any M with $\text{add } M = \text{ref } A$.

Proof. (1) $\Rightarrow$ (2): Put $\mathcal{E} = \text{ref } A$. Under an equivalence $\text{ref } B \simeq \mathcal{E}$, the image of $\text{proj } B$ is the additive closure of a subset of the finite set $\text{ind } \mathcal{E}$. There are finitely many such subcategories, and each of their additive equivalence classes determines a Morita equivalence class of B .

(2) $\Rightarrow$ (3): The number $|\text{ind}(\text{proj } B)|$ is the number of simple B -modules up to isomorphism, and is Morita invariant.

(3) $\Rightarrow$ (1): Suppose that $\text{ind}(\text{ref } A)$ is infinite. For each $m \geq 1$, choose pairwise nonisomorphic indecomposable nonprojective reflexive modules $X_1, \dots, X_m$. By Corollary 2.15,

$$B_m = \text{End}_A(A \oplus X_1 \oplus \dots \oplus X_m)$$

is reflexively equivalent to A . Its indecomposable projectives correspond to the distinct indecomposable summands of $A \oplus X_1 \oplus \dots \oplus X_m$, so $|\text{ind}(\text{proj } B_m)| = |\text{ind}(\text{proj } A)| + m$, contradicting (3).

(1) $\Rightarrow$ (4): Take M with $\text{add } M = \text{ref } A$ and put $\Gamma = \text{End}_A(M)$. Proposition 2.8 gives $\text{ref } \Gamma = \text{proj } \Gamma \simeq \text{ref } A$.

(4) $\Rightarrow$ (1): If $\text{ref } B = \text{proj } B$ and $\text{ref } B \simeq \text{ref } A$, then $\text{ind}(\text{ref } A)$ is finite.

For uniqueness, two reflexive-rigid algebras B and B' in the class satisfy $\text{proj } B \simeq \text{ref } A \simeq \text{proj } B'$, so they are Morita equivalent. $\square$

Proposition 5.2 gives the following bijection for finite-dimensional k -algebras, analogous to the categorical bijection in Corollary 2.6.

Corollary 5.3. *There is a bijection*

$$\frac{\left\{ \begin{array}{c} \text{finite-dimensional } k\text{-algebras} \\ \text{of finite reflexive type} \end{array} \right\}}{\text{reflexive equivalence}} \longleftrightarrow \frac{\left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{reflexive-rigid algebras} \end{array} \right\}}{\text{Morita equivalence}}.$$

It sends the reflexive equivalence class of A to the Morita equivalence class of $\text{End}_A(M)$, where $\text{add } M = \text{ref } A$. Its inverse sends a reflexive-rigid algebra to its reflexive equivalence class.

Proof. Proposition 5.2 shows that each class on the left contains exactly one Morita equivalence class of reflexive-rigid algebras. Every reflexive-rigid algebra has finite reflexive type, since its reflexive modules are projective. $\square$

By Theorem 3.11, the classes on the left can also be identified with the isomorphism classes of basic reflexive-minimal algebras of finite reflexive type. Starting with a reflexive-rigid algebra Γ , this identification gives $\Gamma_{\min}$; see Example 3.16.

Hanahara [Han, Remark 6.10] asks for categorical characterizations of categories of reflexive modules. For categories with finitely many indecomposable objects up to isomorphism, Theorem 2.5 gives the following criterion.

Corollary 5.4. *Let $\mathcal{C}$ be a Hom-finite k -linear category with finitely many indecomposable isomorphism classes, choose an additive generator G , and put $\Gamma = \text{End}_{\mathcal{C}}(G)$. The following are equivalent.*

- (1) $\mathcal{C} \simeq \text{ref } A$ for some finite-dimensional k -algebra A .
- (2) $\mathcal{C}$ is reflexive-rigid.
- (3) Γ is reflexive-rigid.

Proof. (1) $\Rightarrow$ (2): Apply Theorem 2.5 and Lemma 2.2(3).

(2) $\Leftrightarrow$ (3): The equivalence $\mathcal{C} \simeq \text{proj } \Gamma$ identifies the Yoneda functor of $\mathcal{C}$ with the inclusion $\text{proj } \Gamma \subseteq \text{ref } \Gamma$, as in Proposition 2.8(1).

(3) $\Rightarrow$ (1): Take $A = \Gamma$, since $\mathcal{C} \simeq \text{proj } \Gamma = \text{ref } \Gamma$. $\square$

5.2. Reflexive equivalence classes consisting of one Morita equivalence class. We determine when every algebra reflexively equivalent to A is Morita equivalent to A . For a basic algebra A , the least element of $\text{ref-dense}(\text{ref } A)$ is $\text{add}(e_{\min}A)$ by Theorem 3.9(1), and the greatest is $\text{ref } A$ by Lemma 2.12(1). We show that the reflexive equivalence class of A is a single Morita equivalence class exactly when

$$\text{add}(e_{\min}A) = \text{proj } A = \text{ref } A,$$

that is, when A is both reflexive-minimal and reflexive-rigid.

Theorem 5.5. *Let A be a basic finite-dimensional k -algebra with simple modules indexed by $\{1, \dots, n\}$. The following conditions are equivalent.*

- (1) Every finite-dimensional k -algebra reflexively equivalent to A is Morita equivalent to A .
- (2) The poset $\text{ref-dense}(\text{ref } A)$ has exactly one element.

(3) $\text{ref } A = \text{proj } A$ and $I_{\text{ref}}(A) = \{1, \dots, n\}$.

Proof. (3) $\Rightarrow$ (2): Since $I_{\text{ref}}(A) = \{1, \dots, n\}$, $\text{proj } A$ is the least element of $\text{ref-dense}(\text{ref } A)$ by Theorem 3.9(1), and since $\text{ref } A = \text{proj } A$ it is also the greatest element.

(2) $\Rightarrow$ (1): Let $F: \text{ref } B \rightarrow \text{ref } A$ be an equivalence. By Lemma 2.12(3) and (5), $\text{add } F(B) \in \text{ref-dense}(\text{ref } A)$, hence $\text{add } F(B) = \text{proj } A$. Thus $\text{proj } B \simeq \text{proj } A$, and B is Morita equivalent to A .

(1) $\Rightarrow$ (3): If X is an indecomposable nonprojective reflexive A -module, then $\text{End}_A(A \oplus X)$ is reflexively equivalent to A by Corollary 2.15 and has $n + 1$ simple modules, so it is not Morita equivalent to A . If $I_{\text{ref}}(A) \neq \{1, \dots, n\}$, then $A_{\min}$ is reflexively equivalent to A by Theorem 3.5 and has fewer than n simple modules, so it is not Morita equivalent to A . $\square$

Corollary 5.6. *Let A be a finite-dimensional k -algebra. Every finite-dimensional k -algebra reflexively equivalent to A is Morita equivalent to A if and only if $A_{\min}$ is reflexive-rigid.*

Proof. The algebras A and $A_{\min}$ have the same reflexive equivalence class, and $I_{\text{ref}}(A_{\min})$ consists of all indices of its simple modules by Corollary 3.12(4). Apply Theorem 5.5. $\square$

For a local algebra A we have $I_{\text{ref}}(A) = \{1\}$ by Corollary 3.12(1),(3), so the reflexive equivalence class of A is a single Morita equivalence class exactly when $\text{ref } A = \text{proj } A$.

5.3. Algebras whose reflexive modules are projective. We turn to structural conditions for reflexive-rigidity. A module M is a *second syzygy* if there is an exact sequence $0 \rightarrow M \rightarrow P_1 \rightarrow P_0$ with P_0, P_1 projective. Every reflexive module is a second syzygy: if $Q_1 \rightarrow Q_0 \rightarrow M^* \rightarrow 0$ is a projective presentation of the left module M^* , then $0 \rightarrow M^{**} \rightarrow Q_0^* \rightarrow Q_1^*$ is exact. If $\text{gl. dim } A \leq 2$, then every second syzygy is projective, and therefore $\text{ref } A = \text{proj } A$. The converse holds under the following assumption on the injective envelope of the regular module.

Proposition 5.7. *Let A be a finite-dimensional k -algebra such that the injective envelope of the left regular module ${}_A A$ is projective. Then every second syzygy is reflexive. Consequently $\text{ref } A = \text{proj } A$ if and only if $\text{gl. dim } A \leq 2$.*

Proof. Let I be the injective envelope of ${}_A A$. For every $X \in \text{mod } A$ there is an isomorphism

$$\text{Hom}_{A^{\text{op}}}(\text{Ext}_A^2(X, A), I) \cong \text{Tor}_2^A(X, I). \quad (5.1)$$

Indeed, let $P_\bullet$ be a projective resolution of X . Since I is injective, $\text{Hom}_{A^{\text{op}}}(-, I)$ is exact, so the left side is the second homology of $\text{Hom}_{A^{\text{op}}}(P_\bullet, I) \cong P_\bullet \otimes_A I$. As I is projective, the right side of (5.1) vanishes, and since A embeds into I we get $\text{Hom}_{A^{\text{op}}}(\text{Ext}_A^2(X, A), A) = 0$.

Now let $0 \rightarrow M \xrightarrow{i} P_1 \xrightarrow{f} P_0$ be exact with P_0, P_1 projective, and put $X = \text{Coker } f$ and $N = \text{Im } f$. Applying $(-)^*$ to $0 \rightarrow M \rightarrow P_1 \rightarrow N \rightarrow 0$ gives an exact sequence

$$P_1^* \xrightarrow{i^*} M^* \rightarrow \text{Ext}_A^1(N, A) \rightarrow 0,$$

and $\text{Ext}_A^1(N, A) \cong \text{Ext}_A^2(X, A)$. Applying $(-)^*$ again, we see that the kernel of $i^{**}: M^{**} \rightarrow P_1^{**}$ is isomorphic to $\text{Hom}_{A^{\text{op}}}(\text{Ext}_A^2(X, A), A)$, which is zero. As $f^{**}i^{**} = 0$ and the evaluations of P_0 and P_1 are isomorphisms, i^{**} induces an injective map $\theta: M^{**} \rightarrow \text{Ker } f = M$, and naturality of evaluation gives $\theta\delta_M = 1_M$. Hence θ is bijective, and so is δ_M .

Thus $\text{ref } A$ is the category of second syzygies. If $\text{gl. dim } A > 2$, choose X with projective dimension at least three; then the second syzygy of X in a minimal projective resolution is reflexive and not projective. $\square$

The first assertion of Proposition 5.7 also follows from [Han, Lemma 3.8], which shows that every second syzygy is reflexive if $\text{Hom}_{A^{\text{op}}}(\text{Ext}_A^2(X, A), A) = 0$ for all $X \in \text{mod } A$; the first paragraph of the proof verifies this condition.

Corollary 5.8. *Let A be a Nakayama algebra. Then $\text{ref } A = \text{proj } A$ if and only if $\text{gl. dim } A \leq 2$.*

Proof. By Proposition 5.7 it suffices to show that the injective envelope I of an indecomposable projective left module Ae_i is projective. Since Ae_i is uniserial, it has a simple socle and I is indecomposable. Indecomposable injective left modules are of the form $D(e_j A)$, and $e_j A$ is uniserial, so I is uniserial and has a projective cover $p: Q \rightarrow I$ with Q indecomposable. Let $U = p^{-1}(Ae_i)$. The restriction $U \rightarrow Ae_i$ is surjective, hence split, and U is uniserial, hence indecomposable. As

$Ae_i \neq 0$, the map $U \rightarrow Ae_i$ is an isomorphism. Its kernel is $\text{Ker } p$, so p is injective and $I \cong Q$ is projective. $\square$

The following example combines Corollary 5.8 with Theorem 5.5.

Example 5.9. Let $A = k(1 \xrightarrow{a} 2 \xrightarrow{b} 3)/(ab)$. Its indecomposable right projectives are

$$\frac{1}{2}, \quad \frac{2}{3}, \quad 3.$$

This is a Nakayama algebra of global dimension two, so $\text{ref } A = \text{proj } A$ by Corollary 5.8. The right socle contains S_2, S_3 , and the left socle contains L_1 . Hence $I_{\text{ref}}(A) = \{1, 2, 3\}$, and Theorem 5.5 shows that every algebra reflexively equivalent to A is Morita equivalent to A .

Without the assumption of Proposition 5.7, global dimension does not decide whether $\text{ref } A = \text{proj } A$. The following example occurs in [Asg, Discussion 4.2]; its reflexive-rigidity also follows from [Ram, Corollaries 2.5 and 3.3].

Example 5.10. Let $A = k[x, y]/(x, y)^2$, with radical J spanned by x and y . Then $\text{gl. dim } A = \infty$, since the first syzygy of k is $J \cong k^2$. Let M be a reflexive A -module. It is a second syzygy, so $M \cong M' \oplus P$ with P projective and M' the second syzygy of some module in a minimal projective resolution. Then M' is contained in the radical of the projective module in degree one. Since $J^2 = 0$, we have $M'J = 0$. If $M' \neq 0$, then k is a direct summand of M and hence reflexive. But the left module $k^* \cong \text{soc}({}_A A) = J$ is isomorphic to k^2 , so $k^{**} \cong \text{Hom}_{A^{\text{op}}}(k, A)^2$ has dimension four. Hence $M' = 0$ and $\text{ref } A = \text{proj } A$. As A is local, the reflexive equivalence class of A is a single Morita equivalence class, by Corollaries 3.12(3) and 5.6. The algebra A does not satisfy the hypothesis of Proposition 5.7: since $\text{soc}({}_A A) = J$ has dimension two, A is not self-injective, so the indecomposable injective left module $D(A_A)$ is not projective, and the injective envelope of ${}_A A$ is a direct sum of copies of $D(A_A)$.

Ramras asked when every finitely generated reflexive module over a two-sided Noetherian ring is projective; see Huang and Qin [HQ, Section 4] for this question and its homological reformulations. For split local algebras with radical cube zero, Ringel and Zhang [RZ] give necessary conditions for the existence of nonprojective reflexive modules. Ringel [Rin] characterizes the existence of such modules for local algebras of dimension six with radical cube zero over an algebraically closed field. For finite-dimensional k -algebras, Ramras's question takes the following form.

Problem 5.11. *Characterize the finite-dimensional k -algebras A with $\text{ref } A = \text{proj } A$.*

By Corollary 5.3, a classification in Problem 5.11, up to Morita equivalence, classifies algebras of finite reflexive type up to reflexive equivalence. For a given algebra A , constructing the corresponding reflexive-rigid algebra requires an additive generator of $\text{ref } A$. This leads to a separate recognition problem.

Problem 5.12. *Find structural criteria for a finite-dimensional k -algebra A to have finite reflexive type, and methods for constructing an additive generator of $\text{ref } A$ when A has finite reflexive type.*

APPENDIX A. NILPOTENT MODULES AND NONEXISTENCE OF MINIMAL ELEMENTS

We prove the theorem used in Example 4.12, which shows that a Hom-finite reflexive completion need not have a minimal reflexively dense subcategory. Let $\mathcal{E}$, M_s and $\mathcal{C}_T$ be as in that example.

Theorem A.1. *Let $\mathcal{E}$ and $\mathcal{C}_T$ be as above.*

- (1) *The reflexively dense subcategories of $\mathcal{E}$ are exactly the subcategories $\mathcal{C}_T$ with T unbounded.*
- (2) *Every category $\mathcal{C}$ with $\text{ref } \mathcal{C} \simeq \mathcal{E}$ is equivalent to $\mathcal{C}_T$ for exactly one unbounded set T .*
- (3) *The poset $\text{ref-dense}(\mathcal{E})$ has no minimal element.*

Proof. (1) Let T be unbounded. We verify the hypotheses of Proposition 2.17(2) for $\mathcal{V} = \mathcal{E}$ and $\mathcal{C} = \mathcal{C}_T$. Kernels exist since $\mathcal{E}$ is abelian.

To see that $\Phi_{\mathcal{C}}$ is fully faithful, let $Y, Z \in \mathcal{E}$ and choose $l \in T$ with $t^l Y = 0 = t^l Z$. Evaluation at 1 identifies $\mathcal{E}(M_l, Y)$ with Y and $\mathcal{E}(M_l, Z)$ with Z . A morphism $\theta: \Phi_{\mathcal{C}}(Y) \rightarrow \Phi_{\mathcal{C}}(Z)$ therefore gives a linear map $g = \theta_{M_l}: Y \rightarrow Z$, which is $k[t]$ -linear by naturality with respect to multiplication by t on M_l . For $s \in T$ choose $l' \in T$ with $l' \geq l$ and $l' \geq s$. Naturality with respect to the surjections

$M_{l'} \rightarrow M_l$ and $M_{l'} \rightarrow M_s$ shows that $\theta_{M_s}(f) = g \circ f$ for every $f: M_s \rightarrow Y$. Hence $\theta = \Phi_C(g)$, and Φ_C is full; it is faithful because $\Phi_C(Y)(M_l) \cong Y$. The duality D preserves $\mathcal{E}$ and $\mathcal{C}_T$, since $DM_s \cong M_s$, and $\mathcal{E}(Y, X) \cong \mathcal{E}(DX, DY)$ identifies $\Phi^C(Y)$ with $\Phi_C(DY)$ composed with D ; so Φ^C is fully faithful as well.

Next, we prove finite presentation of $\Phi_C(Y)$ and $\Phi^C(Y)$. For $m \geq 1$ let l be the least element of T with $l \geq m$. Every morphism $M_s \rightarrow M_m$ with $s \in T$ and $s \geq m$ factors through the surjection $M_l \rightarrow M_m$, since $s \geq l$. Adding a basis of $\mathcal{E}(M_s, M_m)$ for the finitely many $s \in T$ with $s < m$, we obtain a right $\mathcal{C}_T$ -approximation of M_m , and it is surjective. Taking direct sums, every $Y \in \mathcal{E}$ has a surjective right $\mathcal{C}_T$ -approximation $U_0 \rightarrow Y$. Choosing a right $\mathcal{C}_T$ -approximation U_1 of its kernel gives an exact sequence $P_{U_1} \rightarrow P_{U_0} \rightarrow \Phi_C(Y) \rightarrow 0$ in $\text{Mod } \mathcal{C}_T$, so $\Phi_C(Y) \in \text{mod } \mathcal{C}_T$. Applying D , we see that $\Phi^C(Y) \in \text{mod } \mathcal{C}_T^{\text{op}}$. Now Proposition 2.17(2) shows that $\mathcal{C}_T \in \text{ref-dense}(\mathcal{E})$.

Conversely, let $\mathcal{C} \in \text{ref-dense}(\mathcal{E})$. Then $\mathcal{C} = \mathcal{C}_T$ for $T = \{s \mid M_s \in \mathcal{C}\}$, and $T \neq \emptyset$ since $\text{ref } 0 = 0$. Suppose that T is bounded, with largest element s_0 . Put $R_0 = k[t]/(t^{s_0})$ and $X = \bigoplus_{s \in T} M_s$. Then $\mathcal{C}$ is equivalent to $\text{proj End}_{R_0}(X)$, and X has $M_{s_0} = R_0$ as a direct summand. Since R_0 is self-injective, all finite R_0 -modules are reflexive by Lemma 2.2(4), and Corollary 2.15 gives $\text{ref } \mathcal{C} \simeq \text{ref End}_{R_0}(X) \simeq \text{ref } R_0 = \text{mod } R_0$, which has only s_0 indecomposable objects, contradicting $\text{ref } \mathcal{C} \simeq \mathcal{E}$.

(2) A category $\mathcal{C}$ with $\text{ref } \mathcal{C} \simeq \mathcal{E}$ is equivalent to a reflexively dense subcategory of $\mathcal{E}$ by Lemma 2.12(1) and (3), hence to some $\mathcal{C}_T$ with T unbounded. The set T is determined by the equivalence class of $\mathcal{C}_T$, since $\dim_k \text{End}(M_s) = s$.

(3) Removing one element from an unbounded set leaves it unbounded. $\square$

REFERENCES

- [Asg] M. Asgharzadeh, *Reflexivity revisited*, preprint, arXiv:1812.00830v7.
- [AB] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. **94** (1969).
- [AL] T. Avery and T. Leinster, *Isbell conjugacy and the reflexive completion*, Theory Appl. Categ. **36** (2021), No. 12, 306–347.
- [CRT] R. S. Cunningham, E. A. Rutter, Jr., and D. R. Turnidge, *Rings of quotients of endomorphism rings of projective modules*, Pacific J. Math. **41** (1972), 647–668.
- [Ful] K. R. Fuller, *Double centralizers of injectives and projectives over artinian rings*, Illinois J. Math. **14** (1970), 658–664.
- [Han] N. Hanihara, *Reflexive modules and Auslander-type conditions*, preprint, arXiv:2412.19625v2.
- [Hash] M. Hashimoto, *Homological aspects of equivariant modules: Matijevic–Roberts and Buchsbaum–Rim*, Sūrikaiseikikenkyūsho Kōkyūroku **997** (1997), 66–108.
- [Hos] M. Hoshino, *On dominant dimension of Noetherian rings*, Osaka J. Math. **26** (1989), no. 2, 275–280.
- [HQ] Z. Huang and H. Qin, *Homological behavior of Auslander’s k -Gorenstein rings*, Algebr. Represent. Theory **15** (2012), 835–853.
- [Isb] J. R. Isbell, *Adequate subcategories*, Illinois J. Math. **4** (1960), 541–552.
- [IS] Y. Iwanaga and H. Sato, *Minimal injective resolutions of Gorenstein rings*, Comm. Algebra **18** (1990), no. 11, 3835–3856.
- [IR] O. Iyama and I. Reiten, *Fomin–Zellevinsky mutation and tilting modules over Calabi–Yau algebras*, Amer. J. Math. **130** (2008), 1087–1149.
- [MS] B. Ma and J. Sauter, *On faithfully balanced modules, F -cotilting and F -Auslander algebras*, J. Algebra **556** (2020), 1115–1164.
- [Mor] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.
- [Ram] M. Ramras, *Betti numbers and reflexive modules*, in *Ring theory (Proc. Conf., Park City, Utah, 1971)*, Academic Press, New York, 1972, 297–308.
- [Rin] C. M. Ringel, *The short local algebras of dimension 6 with non-projective reflexive modules*, Commun. Math. Stat. **11** (2023), 195–227.
- [RZ] C. M. Ringel and P. Zhang, *Gorenstein-projective modules over short local algebras*, J. Lond. Math. Soc. (2) **106** (2022), no. 2, 528–589.
- [Tac] H. Tachikawa, *Quasi-Frobenius rings and generalizations. QF-3 and QF-1 rings*, notes by C. M. Ringel, Lecture Notes in Math. **351**, Springer, Berlin, 1973.

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